Solutions to problems of Chapter 3

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Problem 3.6: Show that $P^{\mathfrak{C}}_{C|E=2}$ in

$$C := N_C$$
$$E := 4 \cdot C + N_E$$

where $N_C, N_E \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$, is a Gaussian distribution:

$$C \mid E = 2 \sim \mathcal{N}\left(\frac{8}{17}, \sigma^2 = \frac{1}{17}\right)$$

Solution: First note that $P(C, E)$ is a multivariate Gaussian with $\mu = [0, 0]^T$ and $\Sigma = \begin{bmatrix} 1 & 4 \\ 4 & 17 \end{bmatrix}$. Then using the result for bivariate Gaussian distributions we get:

$$C \mid E = 2 \sim \mathcal{N}\left(\mu_C + \frac{\sigma_C}{\sigma_E}\rho\left(2 - \mu_E\right), \left(1 - \rho^2\right)\sigma_C^2\right)$$

where $\sigma_C = \sqrt{\Sigma_{1,1}} = 1$, $\mu_C = \mu_E = 0$, $\sigma_E = \sqrt{\Sigma_{2,2}} = \sqrt{17}$, and $\rho = \frac{4}{\sqrt{17}}$. Plugging in the values gives:

$$C \mid E = 2 \sim \mathcal{N}\left(\frac{8}{17}, \frac{1}{17}\right)$$

Problem 3.7: Assume that we know that a process either follows the SCM

$$X := Y + N_X$$
$$Y := N_Y,$$

where $N_X \sim \mathcal{N}\left(\mu_X, \sigma_X^2\right)$ and $N_Y \sim \mathcal{N}\left(\mu_X, \sigma_Y^2\right)$ with unknown μ_X, μ_Y and $\sigma_X, \sigma_Y > 0$, or it follows the SCM

$$X := M_X$$
$$Y := X + M_Y,$$

where $M_X \sim \mathcal{N}\left(v_X, \tau_X^2\right)$ and $M_Y \sim \mathcal{N}\left(v_Y, \tau_Y^2\right)$ with unknown v_X, v_Y and $\tau_X, \tau_Y > 0$. Is there a single intervention distribution that lets you distinguish between the two SCMs?

Solution: An intervention do($X = 100$) will change the mean of Y if the second model is the true one otherwise the mean will remain the same. The difference between pre- and post- interventional means of Y can be detected via *t-test* using samples from observational and interventional distributions. Please note that larger values at which X is controlled make the detection easier under the second model. The algorithm of identification is given below:

1. Gather samples from the observational distribution P_Y.
2. Set X to some non-zero (preferably large) value x.
3. Gather samples from the interventional distribution $P_Y^{\mathfrak{C};\mathrm{do}(X:=x)}$.
4. Perform *t-test* on the two groups of samples. If difference is significant choose the second model as the true model otherwise pick the first one.

Problem 3.8 (Cyclic SCMs) We have mentioned that if the assignments inherit a cyclic structure, the SCM does not necessarily induce a unique distribution over the observed variables. Sometimes there is no solution and sometimes it is not unique. a) We first look at an example that induces a unique solution. Consider the *SCM*

$$X := 2 \cdot Y + N_X$$
$$Y := 2 \cdot X + N_Y$$

with $(N_X, N_Y) \sim P$ for an arbitrary distribution P. Compute $\alpha, \beta, \gamma, \delta$ such that

$$X := \alpha N_X + \beta N_Y$$
$$Y := \gamma N_X + \delta N_Y$$

yields a solution (X, Y, N_X, N_Y) of the SCM; that is, the vector satisfies Equa tions (3.12) and (3.13). The solution can be seen as a special case of Equa tion (6.2).
b) Consider the SCM

$$X := Y + N_X$$
$$Y := X + N_Y$$

with $(N_X, N_Y) \sim P$. Show that if P allows for a density with respect to Lebesgue measure and factorizes, that is, $N_X \perp\!\!\!\perp N_Y$, then there is no solution (X, Y, N_X, N_Y) of the SCM. Furthermore, construct a distribution P, and a vector (X, Y, N_X, N_Y) that solves the SCM.
Solution: a) Consider the first equation where $X := 2 \cdot Y + N_X$, plugging in the corresponding dependencies of X, Y in terms of N_X and N_Y, we get:

$$\alpha N_X + \beta N_Y = 2\gamma N_X + 2\delta N_Y + N_X$$
$$N_X(\alpha - 2\gamma - 1) = N_Y(2\delta - \beta)$$

Coefficients in the brackets should be 0 as N_X and N_Y are independent and no linear equality can exist between them.

$$\alpha = 2\gamma + 1$$
$$\beta = 2\delta$$

Now, consider the equation for Y where $Y := 2 \cdot X + N_Y$. Similarly, plugging in dependencies of X, Y in terms of N_X and N_Y, we get:

$$\gamma N_X + \delta N_Y = 2\alpha N_X + 2\beta N_Y + N_Y$$
$$N_X(\gamma - 2\alpha) = N_Y(2\beta - \delta + 1)$$

by the same logic as before:

$$\gamma = 2\alpha$$
$$\delta = 2\beta + 1$$

Solving the four equations above we get that $\delta = \alpha = -\frac{1}{3}, \beta = \gamma = -\frac{2}{3}$.
b) Plugging in the equation for Y into the equation of X yields that $N_X = -N_Y$ which contradicts the independence $N_X \perp\!\!\!\perp N_Y$. Furthermore, for $\forall c \in \mathbb{R}$ the values of $N_X = c, N_Y = -c, X = c/2, Y = -c/2$ solve the SCM. Consequently, any distribution of the form $P = P(X)\delta(X + Y)$ solves the SCM, where $\delta(\cdot)$ denoted Dirac's delta function.