Solutions to problems of Chapter 5

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Problem 5.1 (Independence of mechanisms): Let P_X be the mixture of k sharp Gaussian peaks at positions $s_1, \dots, s_k$ as shown in Figure 5.5, left. Let Y be obtained from X by adding some Gaussian noise N with zero mean and a width σ_N such that the separate peaks remain visible as in Figure 5.5, right.

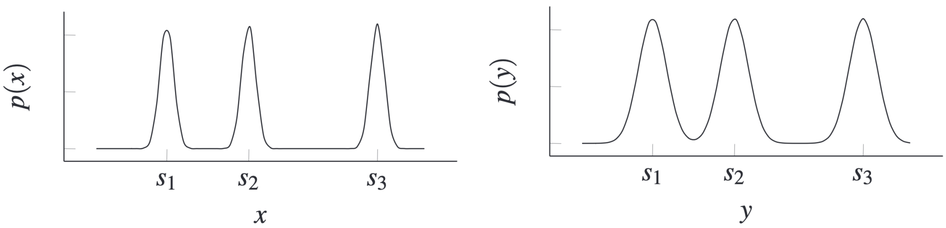

Figure 5.5: Example where X causes Y and, as a result, P_Y and $P_{X|Y}$ contain information about each other. Left: P_X is a mixture of sharp peaks at the positions s_1, s_2, s_3. Right: P_Y is obtained from P_X by convolution with Gaussian noise with zero mean and thus consists of less sharp peaks at the same positions s_1, s_2, s_3. Then $P_{X|Y}$ also contains information about s_1, s_2, s_3 (see Problem 5.1).

(a) Argue intuitively why $P_{X|Y}$ also contains information about the positions $s_1, \dots, s_k$ of the peaks and thus $P_{X|Y}$ and P_Y share this information.

(b) The transition between P_X and P_Y can be described by convolution (from P_X to P_Y) and deconvolution (from P_Y to P_X). If $P_{Y|X}$ is considered as the linear map converting the input P_X to the output P_Y then $P_{Y|X}$ coincides with the convolution map. Argue why $P_{X|Y}$ does not coincide with the deconvolution map (as one may think at first glance).

Solution:

(a) It is easy to see that for values of Y far from $s_1, ..., s_k$ the value of X being close $s_1, ..., s_k$ is low. Moreover, when Y obtains value close to $s_1, ..., s_k$ that highly increases the probability of X obtaining value close to $s_1, ..., s_k$. Hence, $P_{X|Y}$ contains information about $s_1, ..., s_k$.

(b) Truly $P_{Y|X}$ is the convolution map from P_X to P_Y as $P_Y = \int P_X P_{Y|X} \,\mathrm{d}\, x$. However, $P_{X|Y}$ is not the deconvolution map from P_Y to P_X as it is rather the convolution map from P_Y to P_X which can be seen from $P_X = \int P_Y P_{X|Y} \,\mathrm{d}\, y$.