ENCYCLOPEDIA
OF
MATHEMATICAL SCIENCES

INCLUDING THEIR APPLICATIONS

VOLUME ONE:
ARITHMETIC AND ALGEBRA

FELIX KLEIN

Translated from the original German work
"Encyklopädie der Mathematischen Wissenschaften mit
Einschluss ihrer Anwendungen"

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ENCYCLOPEDIA
OF
MATHEMATICAL SCIENCES

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VOLUME ONE IN TWO PARTS

ARITHMETIC AND ALGEBRA

EDITED BY

WILHELM FRANZ MEYER

IN KÖNIGSBERG I. PR.

PART ONE

LEIPZIG, PRINTED AND PUBLISHED BY B. G. TEUBNER

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Introductory Report on the Enterprise of Publishing the Encyclopedia of Mathematical Sciences

In September of 1894, *Felix Klein* and *Heinrich Weber* met with *Franz Meyer*, then professor at the Mining Academy in Clausthal, on a journey to the Harz Mountains. There, the first plan for the Encyclopedia of Mathematical Sciences was drafted. *Franz Meyer* developed his idea of composing a *dictionary of pure and applied mathematics.*

The ending century has, as in many areas of human knowledge, given rise to the desire for a comprehensive presentation of the scientific work accomplished during its course, which should also include the manifold applications to natural science and technology. Exhaustive, of course, in the sense of a complete presentation delving into all details of the widely branched structure, indicating all paths in both historical and methodological directions, such a work could not be planned, given the lack of comprehensive preliminary work, if one did not want to jeopardize its implementation. Thus, it was initially the intention to compile and characterize only the "most necessary", the fundamental "concepts" of our mathematical knowledge in the form of a *Lexicon.*

"It should" — as *Franz Meyer* explained in a first draft — "provide the explanation of the concept falling under a given keyword in the form in which it first appeared, along with indication of the literary source, as far as possible. While this was mainly intended for newer concepts, the old and even obsolete expressions should nevertheless be mentioned, to preserve them as in a museum. This would be followed by the historical development of the concept follow up to the most recent times. Almost every concept differentiates and splits over time, takes on different nuances and applications, branches according to the uses made of it, deepens and generalizes. The respective technical term undergoes corresponding changes, additions, and compositions. The most important sections in this concept's career should again be provided with evidence." Thus, the developmental history of each individual concept should, in its part, provide a picture of progressive science.

The plan found full approval from *Klein* and *Weber*.

Fresh courage to execute it might strengthen during the wandering through mountain and forest. A great goal had been brought before their eyes, worth investing the effort and enduring the difficulties that the path would present. The enterprise exceeded the power of the individual; it was to become a collective effort of our German mathematicians, to which each would contribute according to their special field of work, and beyond that, where development brought it with it, researchers from abroad were to be recruited as well.

At that time, the *Cartel of German Academies* had just been formed, determined to implement and promote large scientific enterprises in collaborative work. The task set here appeared genuinely as a task for the Cartel. Through the academies, not only financial support should be offered, but also in scientific terms, the progress of work that would not be completed quickly — at that time, they thought of implementation in six to seven years — should be secured.

The *German Mathematical Association*, however, should primarily make the enterprise their own through the cooperation of their members. For them, the successfully begun plan of large detailed scientific reports on all current areas of mathematics, which were to be recorded in the annual reports, was complemented by this new comprehensive task, for which preliminary work could be drawn from those, at least in part.*)

*) Already at the first meeting of the German Mathematical Association in Halle, autumn 1891, *Felix Müller* during the discussion of "literary under-

Thus, the importance and need for comprehensive presentation of widely branched knowledge was naturally accompanied by the necessity of uniting their representatives for collaborative work.

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At the *Natural Scientists' Meeting* in *Vienna* in September 1894, the *German Mathematical Association* decided to adopt the plan of composing a dictionary of pure and applied mathematics and commissioned *Franz Meyer* to seek scientific and financial support from the academies and learned societies of *Göttingen*, *Leipzig*, *Munich*, and *Vienna* united in the Cartel.

At the beginning of 1895, the first draft of the book, combined with a preliminary financing plan (which was established with the involvement of *B. G. Teubner* in *Leipzig*) was presented to the academies and received principal approval from *Göttingen*, *Munich*, and *Vienna*, while the *Society of Sciences* in *Leipzig*, due to lack of available funds, found itself compelled to abstain from participation in the enterprise for the time being.

The learned societies commissioned *F. Klein* (*Göttingen*), *W. v. Dyck* (*Munich*), *G. v. Escherich* (*Vienna*) to initiate discussions with the editorial board and with a publisher to be considered, and to draft a detailed plan of the enterprise regarding both its scientific and financial aspects. This academic commission subsequently stood as a permanent institution alongside the editorial board. It strengthened itself right at the beginning through *H. Weber* (*Strasbourg*) as representative of the *German Mathematical Association* and *L. Boltzmann* (*Vienna*) as advisor in scientific matters. Later, *H. v. Seeliger* (*Munich*) and more recently *O. Hölder* (*Leipzig*), who will be mentioned later, joined as well.

In detailed preliminary work, which concerned the organization of the material and its classification into larger comprehensive as well as smaller

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takings suitable for facilitating the study of mathematics" (1st Annual Report of the *G.M.A.*, p. 59), in connection with the presentation of the draft of his (meanwhile published) mathematical vocabulary, pointed to such an alphabetically arranged mathematical encyclopedia.

individual articles, and then the anticipated scope of the entire work, occupied the summer of 1895. The decision about the feasibility of the enterprise, however, came from a conference of the *academy delegates* with *Franz Meyer* in September 1896 in Leipzig, in which *A. Wangerin*, in place of *H. Weber*, as well as publisher *Alfred Ackermann-Teubner* participated. Alongside a first draft of a keyword-based content arrangement, the manuscript of *Felix Müller's* previously mentioned lexicon of mathematical terminology was present — and it became apparent that for the intended purposes of an encyclopedia, an alphabetical arrangement could not be maintained. If one wanted to link the presentation of our current mathematical knowledge to individual concepts and technical terms and their transformation, as was the original plan mentioned at the beginning, the proper selection of keywords to be included, free from unnecessary ballast, around which the entire presentation would have to be grouped, would already present considerable difficulties. Nevertheless, such an arrangement would result in extensive fragmentation of the content, while on the other hand, particularly in presenting research results and methods, repetitions would hardly be avoidable. The lexicon would moreover acquire a completely inhomogeneous character, because alongside coherent developments about individual areas, very short sections, mere explanations, and countless cross-references would have to be inserted.

Thus in Leipzig, upon *Dyck's* proposal, the decision was made to abandon the idea of a proper *lexicon* and replace the *artificial* system of alphabetical ordering with the *natural* system of a purely subject-based arrangement and presentation of mathematical fields of knowledge. Even in such an arrangement, the manifold connections between individual disciplines are often enough severed, the mutual interweaving in subject matter or methodological terms can only partially be expressed, and the sequential presentation must completely replace the simultaneity of facts. But it is still possible to follow the main thread of guiding thoughts in the simply laid out presentation and incorporate into it the development of individual areas with their further elaboration.

Based on this new principle, the arrangement for the volumes dedicated to pure mathematics was first established. For its development, as well as for the preparation of two articles on "Surfaces of Third Order" and "Potential Theory," they succeeded in gaining *Heinrich Burkhardt*, then lecturer at the University of Göttingen, alongside *Franz Meyer*, and persuaded the former to join the editorial board, for it became apparent from the start that the editorial task could not be managed by a single person. Specifically, *Franz Meyer* later took on the editorship of Volume I (Arithmetic and Algebra) and Volume III (Geometry), while *Heinrich Burkhardt* took that of Volume II (Analysis).

It cannot be denied that with the change in the system of presentation, there was also a shift in content or at least a different emphasis of the same. Not the individual concept, but the structure of content in the results and methods of mathematical research forms the principle of grouping. Thus, the following was established as the task of the "*Encyclopedia of Mathematical Sciences*," as the work was called from then on:

> "The task of the *Encyclopedia* shall be to provide, in a concise form suitable for quick orientation, but with the greatest possible completeness, a comprehensive presentation of the mathematical sciences according to their current content of established results, and at the same time to demonstrate through careful literature references the historical development of mathematical methods since the beginning of the 19th century. It shall not limit itself to so-called pure mathematics, but shall also consider applications to mechanics and physics, astronomy and geodesy, the various branches of technology and other fields, and thereby give an overall picture of the position that mathematics holds within today's culture."

A further difficulty now lay in measuring the scope of the entire work and in a proper distribution of space across the individual areas. Comparisons with earlier works of similar nature, with analogous ones from other disciplines, offered only slight guidance. Here, an initial approach could only be established as a desirable limitation, not as a reliable norm, yet such an estimate had to form the basis for measuring the resources to be contributed by the academies as well as for negotiations with the publishing house.

It was agreed to set six large octavo volumes of forty sheets each as the starting point for space allocation. Three volumes were to serve pure mathematics, two applied mathematics, and another was to be dedicated to historical, philosophical, and didactic questions. Each volume was to be provided with its own index. The final volume should also contain a comprehensive overview and, to make the work usable as a reference work, a detailed alphabetically arranged *index*.

For the entire implementation of the enterprise, the editorial board was to work together with the commission appointed by the academies:

The *editorial board* was tasked with structuring the material in detail based on the work's arrangement established in joint consultations with the commission; to gain contributors, to reach understanding with them about the distribution of areas and to mediate the mutual reference of reviewers concerning neighboring areas; to ensure a unified character of the various articles; to oversee the printing; to compile the indexes; finally, through the commission, to provide regular reports to the participating academies about the work's progress.

The *academic commission* was to be responsible for maintaining the special interest of the academies in the work's prosperity and providing vigorous scientific support to the editorial board. In particular, this commission's approval should be required for any changes proving necessary in the work's plan or in the composition of the editorial board, as well as for the selection of contributors.

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In spring 1896, the presented plans and proposals of the commission and editorial board received the approval of the academies of *Göttingen, Munich, and Vienna*, and the contract for publication was concluded with *B. G. Teubner* publishing house in Leipzig.

And now the work began — under favorable auspices, for right from the start, the editorial board succeeded in securing a large, significant circle of contributors, ready to put their work in service of the common cause, setting aside their special interests. *"General Principles"* had been issued, which were intended to ensure as much as possible a common basis for the structure of articles and uniform treatment of the material, without overly restricting the scientific freedom and individuality of the individual who bears full responsibility for their presentation.*)

Regarding the arrangement of individual volumes, as it gradually took shape based on these foundations, this will be reported in the special introductions by the editorial board. Here it should only be emphasized how the establishment and gradual completion of the comprehensive arrangement, the mutual alignment of the content of individual essays, and the clarification of their mutual relationships were particularly promoted in the frequent personal conferences between contributors, editors, and commission members. They represent a sacrifice by all participants that must be acknowledged with the utmost gratitude, but also a lasting

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*) We believe we should reproduce them at this point with those modifications and additions which they later received, particularly when undertaking the volumes of applied mathematics.

General Principles for the Preparation of Articles.

1. Within each article, the mathematical *concepts* peculiar to the respective field, their most important *properties*, the most fundamental *theorems*, and the *investigation methods* that have proven fruitful are presented.

2. The execution of *proofs* of the communicated theorems must be omitted; only where principally important proof methods are concerned can a brief indication of them be given.

3. The parts of the work relating to *applications* should fulfill a dual purpose: they should, *on one hand*, orient the mathematician about what questions the applications pose to them, and *on the other hand*, inform the astronomer, physicist, engineer about what answer mathematics gives to these questions. Accordingly, they limit themselves to the mathematical side of applications; instrumentation, observation techniques, collection of constants, regulations fall outside the framework of the work.

benefit for the entire work and for all who have participated in it. The Natural Scientists' Meetings of the last decade, starting from the Vienna meeting of 1894, where the foundation stone of the work was laid, the national Mathematics Congress in Zurich (1897), as well as other conferences of the academic commission and editorial board, which were almost regularly combined with the annual meetings of the Cartel of German Academies, offered important opportunities for joint consultation about the work's progress and exchange of ideas about its detailed development.

The necessity of personal discussion became particularly apparent when, in 1897, after the most essential steps for the arrangement and implementation of the first three volumes of pure mathematics had been taken, it was time to approach the volumes dedicated to applied mathematics. From the outset, it had become clear that only an expansion of the editorial board could ensure the implementation of the enterprise and likewise, that — if one did not want to delay completion

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If the first of these goals is to be achieved, it will be necessary to: briefly indicate the considerations that led to the mathematical formulation of the problem in question; explicitly establish this formulation; indicate the limits within which the occurring constants lie in practical cases; indicate the degree of accuracy up to which the formulation in question is to be considered correct.

If the second goal is also to be achieved, one must not limit oneself to mere references to those places in the first three volumes where the problem in question is treated; one must briefly state the result of the required mathematical operations (equation solving, geometric construction, integration). However, repetition of literature references is not necessary.

4. Strictly chronological arrangement of the material would necessitate many repetitions for which there is no space; but the *gradual development of concepts and methods* will be explained at appropriate points and documented through *precise literature references.*

5. The existing historical monographs and bibliographic resources will provide good initial orientation services to the contributors; however, the first principle of all historical criticism requires that the presentation ultimately be based on personal *study of the original works.*

6. While results from *older developmental periods* should be included, specific proof of their origin will have to be omitted; otherwise, following principle (5) would delay the completion of the work beyond measure, as the required orienting works are still lacking, especially for the 18th and partly also for the 17th century

of the whole into the distant future — it was necessary to tackle the work from all sides. The academic commission hoped to persuade *F. Klein* to join the editorial board and specifically to take charge of the volume relating to mechanics, and likewise to gain *A. Sommerfeld* (then private lecturer in Göttingen) for the editing of the mathematical-physical part. Initially, *Klein* undertook several major journeys (to England, France, Holland, Italy, and Austria), for which the academies had granted the necessary means in a liberal way, to make the necessary preliminary work for the arrangement, development, and collaboration on these volumes. While the participation of non-German authors had already become of essential importance for the character of the reports in the first volumes, here, with the volumes dedicated to applied mathematics, it is particularly important to be able to count on the collaboration of non-German authors according to the development of individual areas.

As much as we want to claim the entire enterprise as German in its foundation and execution, it is from

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lacking. Accordingly, the historical presentation will generally begin with the start of the nineteenth century. Insofar as citations to earlier times are given at all, they are to be understood in the sense that no guarantee is provided whether an even earlier source could have been cited.

7. The individual mathematical subjects are not considered as isolated from each other; on the contrary, it is one of the main tasks of the work to bring general awareness to the manifold *interweaving and overlapping of the most diverse areas*.

8. One-sided emphasis of a particular school standpoint runs counter to the work's purpose. The most desirable would be if everywhere it were possible to integrate the results obtained by different paths into an *objective presentation*; where this appears unfeasible, at least each of the opposing views should be given a voice.

9. The Encyclopedia is not called upon to decide pending *disputes*, particularly those about priority.

10. If concepts or theorems belonging to another area are used in one field, reference is simply made to the section treating the *latter area* (using the signature used in the arrangement), even if it appears at a later point in the Encyclopedia. Moreover, things about which one can doubt whether they belong in an earlier or later section will generally be included at the earlier point.

11. As far as it can be done without compromising principles (7) and (10), the requirements for readers' prior knowledge will be kept such that

of utmost importance, if it is not to represent a one-sided viewpoint, that in the conception and presentation of individual areas, all voices that have contributed to the uniqueness of their development are heard. The permanent holdings of any science are an international good, gained from the collective work of scholars of all times and all countries. But in different directions, with varying emphasis and appreciation of individual areas, with characteristic differences in methods and form of presentation, different nations and different epochs have participated in this work. This must be expressed in the Encyclopedia in the presentation of content according to its historical development as well as in the recruitment of contributors. Indeed, the enterprise today counts, alongside its foundation of German authors, scholars from America, Belgium, England, France, Holland, Italy, from Norway, Austria, Russia, Sweden among its contributors.

In the years 1898 and 1899, particularly through the personal efforts and connections of *F. Klein*, the implementation

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the work can also be useful to someone who seeks orientation about only a specific area.

12. Bibliographic completeness of *literature references* is neither possible nor even desirable, just as exhaustive enumeration of all main proposed theorems or suggested technical terms.

13. However, all important *technical terms* actually in use should appear and find explanation, so they can later be included in the index. Cases should be noted particularly where the same term or symbol is used by different authors with different meanings, especially those where the meaning of a term has imperceptibly expanded over time. Among obsolete terms, a sparing selection should be made.

14. Wherever necessary for understanding, *figures* will be included *in the text.*

15. The enterprise does not have the means for including extensive *collections of formulas*, or similar *tables of numerical* values of the functions treated — which should not be copied from other works without prior verification anyway. However, information about where such can be found is desirable, if necessary with a warning against uncritical use. — Very small tables can find space, which illustrate the behavior of a function through a few appropriately chosen numerical values; often a graphical representation will serve the same purpose even better.

16. *Citations* to frequently used journals will be given in uniform abbreviated

of the volumes dedicated to applied mathematics could be secured and a first arrangement of the same could be drafted. In doing so, it proved necessary to distribute the entire abundant material of applications across three volumes instead of two as planned, of which the fourth would encompass mechanics, the fifth mathematical physics, and the sixth geodesy, geophysics and astronomy, while a seventh volume was reserved for historical, philosophical and didactic questions.

In 1899, *Klein* definitively took over the editorship of the volume dedicated to mechanics, soon after *Sommerfeld* took the editorship of the fifth volume, mathematical physics.

An arrangement of the sixth volume could only be approached after multiple preliminary negotiations in 1900. It was undertaken by *E. Wiechert* in Göttingen for geodesy and geophysics, and by *R. Lehmann-Filhés* in Berlin for astronomy, both thereby joining the encyclopedia's editorial board. Unfortunately, the latter found himself compelled to step down from the editorial board in 1902, where he had conducted the initial negotiations with the selected contributors in a most commendable way. In his

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form (according to a specially established scheme); books are cited, where they appear in an article for the first time, with family name and abbreviated first name of the author, main part of the title, place and year, in case of multiple occurrences the later times in shorter form. Where the matter doesn't seem important enough for more detailed information, mere enumerations of names usually have little use for the reader.

17. Generally meaningless *ornamental epithets*, such as groundbreaking, ingenious, magnificent, classical etc. should be avoided. Instead, it will be indicated in which direction progress lies in each case: whether in finding *new results* — or in *rigorous foundation* of previously only conjecturally proposed or insufficiently proven theorems — or in shortening cumbersome developments through the use of *new tools* — or finally in *systematic arrangement* of an entire theory.

Specifically for the preparation of volumes for applied mathematics, the following remarks apply:

1. Since the encyclopedia essentially addresses a mathematical audience, it must place emphasis on the *mathematical* side of theories. This will include, on one hand, the mathematical *formulation* of the tasks under consideration, and on the other hand, their mathematical *implementation*. The latter viewpoint, which often recedes in specifically physical and scientific books, will be essentially kept in mind here. On the other hand, however, in contrast to the presentation in the majority of mathematical works, the experimental foundation of individual

place, *K. Schwarzschild*, who had just been appointed to the Göttingen Observatory, joined in 1903.

At Easter 1904, *Conrad H. Müller*, who had been involved in the editorial work for some time, was appointed by the academic commission as co-editor of the fourth volume to support *F. Klein*; finally in July 1904, *Ph. Furtwängler* (in Potsdam) was called to edit the first part of Volume VI (Geodesy and Geophysics) in collaboration with *E. Wiechert*.

Meanwhile, on November 7, 1898, the first issue of the first volume was published, containing *H. Schubert*'s report on the foundations of arithmetic, *E. Netto*'s report on combinatorics, and *A. Pringsheim*'s extensive work on irrational numbers and convergence of infinite processes. In August 1899, the publication of the second volume then began with *Pringsheim*'s foundations of general function theory, followed by *A. Voss*'s essay on differential and

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areas should be described, namely to the extent that the reader gains a general judgment about the foundation and accuracy limit of the mathematical theory.

2. The general plan of the encyclopedia corresponds, as emphasized in the writing, to the *historical arrangement of material* and provision of the main moments of *historical development*. However, for the present volumes in this regard, it should be noted that the results of applied mathematics become outdated more quickly than those of pure mathematics, and therefore the historical development here does not have the same importance for understanding the current state of theory as there. Nevertheless, historical presentation will generally be desirable in the following volumes as well, insofar as it is compatible with systematics and clarity.

3. In the fields of applied mathematics, the literature is often very scattered and disconnected. The editorial board has therefore made it their concern to establish connections in advance in as many directions as possible, with mathematicians, physicists, technicians, ... astronomers, as well as in different countries; they will be gladly ready to communicate or at least point out otherwise difficult-to-obtain literature to the contributors based on these connections.

4. Finally, it does not seem necessary that each article be prepared by a single author. Rather, *smaller contributions* that cover only part of the material to be treated in the respective article are sometimes desirable. Such contributions can be printed as an appendix to the comprehensive article or, if the author agrees, be made available to the main reviewer of the area and incorporated by them. The authorship of such contributions will be appropriately expressed in the article's heading.

integral calculus, as well as that of the late *G. Brunel* on definite integrals. In October 1902, the first issue of the third volume appeared with essays by *H. v. Mangoldt* and *R. v. Lilienthal* on differential geometry. The publication of parts dedicated to applied mathematics began in June 1901 with the fourth volume with *M. Abraham's* presentation of geometric concepts for mechanics of deformable bodies and two treatises by *A.E.H. Love* on hydrodynamics. At Easter 1903, Volume V (Mathematical Physics) followed, introduced by *C. Runge's* essays on measurement and measuring and *J. Zenneck's* on gravitation, to which *G. H. Bryant's* general foundation of dynamics is joined. Still in the course of this year, the publication of the first issues of both parts of the sixth volume will begin. They will contain, on one hand, essays by *C. Reinhertz* and *P. Pizzetti* on geodesy, by *S. Finsterwalder* on photogrammetry, and on the other hand (in the astronomical part) treatises by *E. Anding* and *F. Cohn* on the theory of coordinates.

Not everyone has approved of this activity beginning on all sides of the work, fearing that the completion of individual volumes might be delayed too much. Also, the reader currently receives a not easily overlooked patchwork of individual issues, which libraries are also reluctant to release for use. However, it must be said that a delay in publication due to the broad scope of editorial activity does not occur, because it almost always involves different editors and contributors; on the contrary, the uniform progress of the whole is of essential importance for utilizing the mutual relationships between individual volumes and individual essays. On the other hand, the publishing house has recently accounted for the ease of use of individual issues through special equipment and binding of the issues.

Here is the place to emphasize with special thanks the extremely great accommodation of the publishing house *B. G. Teubner*. On one hand, the firm has most willingly fulfilled all extensive wishes and requirements of the editors and authors regarding printing, and on the other hand, through its own commitment to the honoraria to be expended, has made it possible to meet the increasingly strong need arising in the course of development to carry out the work in a scope considerably expanded compared to the initial plan.

One may regret that the original approach of presenting a very concise overview of our current mathematical knowledge in six manageable volumes has been abandoned, and may not without concern see how from volume to volume the work extends beyond the boundaries drawn at the beginning. However, the striving for greatest possible completeness in individual sections and the desire to be clear and comprehensible, even at the expense of brevity — a desire that has repeatedly been expressed to us from reader circles — form the immediate reason for the growth in scope. But the essential reason lies perhaps deeper: The work is a *first* according to its task, so it cannot be a complete one in fulfilling it. Only when the vast field it encompasses lies before us in this first version as a whole, when the circle of problems it has to present has been measured once, will one be able to see how much remains to be done for deepening its content, for simplifying and making the presentation more concise, for aligning and interconnecting all individual parts.

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Two circumstances significant for the recognition of the scientific work accomplished so far are still to be mentioned in our historical report:

The first is the publication of a French adaptation of the Encyclopedia, for which the publishers *B. G. Teubner* in Leipzig and *Gauthier-Villars et fils* in Paris received authorization from the academies in 1900. *J. Molk*, Professor at the Faculty of Sciences in Nancy, was entrusted with directing this edition initially for the volumes dedicated to pure mathematics, while for publishing the volumes of applied mathematics he collaborated with *P. Appell*, member of the Institut de France (mechanics), as well as with *A. Potier* (physics), *Ch. Lallemand* (geodesy and geophysics) and *H. Andoyer* (astronomy).

It is not merely a translation, but an adaptation envisioned, in which the leading French scholars have promised their participation have promised. While fully preserving the character of the German original, this edition shall take into account the usage of French circles and, on the other hand, through joint collaboration of authors and editors, the individual articles shall experience manifold additions, especially regarding literature citations.*)

Thus the German work in its French edition will be made accessible to and appreciated by even wider circles.

We may see a further recognition of the work's implementation so far, which we welcome with particular joy, in the fact that recently the Royal Saxon Society of Sciences in *Leipzig* has also made it possible to participate in the publication of the Encyclopedia. They have delegated *O. Hölder* to the academic commission on their part.

Thus the publication now appears as a joint enterprise of the learned societies united in the Cartel of German Academies in Göttingen, Leipzig, Munich and Vienna, and it also demonstrates the significance of this union for the implementation of tasks that are only possible in united work; at the same time, however, the authority of the academies, which have made the enterprise their own, provides the guarantee that future development, completion of the whole, as well as later revisions are placed in the best hands and secured in their scientific foundation.

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*) The prospectus of the French edition characterizes the nature of the adaptation in the following way:

In the French edition, efforts have been made to reproduce the essential features of the articles from the German edition; however, in the adopted mode of presentation, French customs and habits have been extensively taken into account.

This French edition will offer a very particular character through the collaboration of German and French mathematicians. The author of each article in the German edition has, in fact, indicated the modifications they deemed appropriate to introduce in their article and, on the other hand, the French editing of each article has given rise to an exchange of views in which all interested parties have participated; additions due particularly to French collaborators will be placed between two asterisks. The importance of such collaboration, of which the French edition of the Encyclopedia will offer the first example, will escape no one.

Thus may the Encyclopedia under this friendly aspect, under the protection of the united Academies, serve the sciences in its part:

Pure mathematical research, by preparing the old furrowed ground for new sowing and harvest, and opening up conquered land to fertilizing intellectual work;

The *applied branches of knowledge*, by bringing together the often separated paths of mathematical and scientific consideration, and preparing the foundation and method for their further development;

The *totality of all intellectual work*, by designating and circumscribing the position that belongs to the mathematical sciences in the realm of human knowledge.

Munich, July 30, 1904

Walther von Dyck,
as Chairman of the academic commission
for the publication of the Encyclopedia.

Preface to the First Volume

The present first volume of the Encyclopedia of Mathematical Sciences encompasses *Arithmetic, Algebra, Number Theory, Probability Calculation* (with applications to adjustment and interpolation, statistics and life insurance), as well as some adjacent disciplines: *Difference Calculation, Numerical Computation, Mathematical Games, and Mathematical Economics.*

These are approximately those parts of pure mathematics that are not specifically of an analytical or geometrical character.

This separation from *Analysis* (Volume II) and *Geometry* (Volume III) could naturally not be entirely rigid; in many cases, an overlap into these two large areas was inevitable, and it was equally impossible to maintain the concept of "pure" mathematics everywhere. This may be elaborated in more detail, following the main features and progression of Volume I.

In Arithmetic (Section A), the irrational and the concept of limits (A 3) simultaneously form the foundation of contemporary analysis; for the convergence and divergence of infinite series, products, fractions, and determinants, analytical and analytic functions therefore had to be drawn upon as evidence. The theory of simple and higher complex quantities (A 4) necessitated an examination of the metric properties of the simplest continuous transformation groups. Set theory (A 5) has become significant as a fundamental classification principle for functions in general, and recently also for the foundations of geometry and analysis situs.

In Algebra (Section B), some of the richest applications of invariant and group theory (B 2, B 3 c, d, B 3 f), especially the theory of finite linear groups, relate to significant geometric configurations; conversely, the doctrine of algebraic formations and transformations is equally closely linked with the development of algebraic functions, as well as algebraic geometry.

In the section (C) on Number Theory, it is primarily the approximation methods of analytic number theory (C 3) that substantially originated from analysis and, conversely, have promoted it; a main application to geometry appears to be the impossibility of squaring the circle. The article (C 6) on complex multiplication could with equal justification be viewed as an integral component of the theory of elliptical functions.

Finally, reference should be made to the manifold relationships between probability and difference calculus, along with their applications (D, E), to the approximation of certain integrals, the intervention of the theory of general systems and methods of descriptive geometry in numerical computation (F).

Precisely these final sections (D, E, F, G) simultaneously teach the extent to which principles originally drawn from pure mathematics prove their power in solving the most diverse technical problems.

We now turn to the systematic division of sections into individual articles and can be briefer in doing so, as the detailed overall table of contents in any case permits an external orientation about the distribution of material.

The section (A) on *Arithmetic* begins with the elements (A 1, *H. Schubert*), the arithmetic basic operations and their applications to positive and negative, whole and fractional numbers; connected to this naturally is combinatorics (A 2, *E. Netto*), whose most essential offshoot is the theory of determinants.

From the elements, one can proceed within arithmetic in four different directions.

Either one expands (A 3, *A. Pringsheim*) the domain of rational numbers by incorporating the irrational, and simultaneously transfers the arithmetic basic operations to an unlimited number of objects. From this emerges the concept of the limit of a number sequence, and from this again through specification the theory of convergence and divergence of infinite series, products, continued fractions, and determinants. In this context, the theory of finite continued fractions could also be included.

Secondly, one can drop the restriction to the real and expand the domain of arithmetic quantities through the creation of expand the domain of general and higher complex quantities (A 4, *E. Study*); suitable classification principles enable the organic integration of particularly significant complex quantities, especially quaternions.

Thirdly, the natural number sequence can be continued beyond itself (A 5, *A. Schoenflies*) and one arrives at the various modifications of sets and transfinite numbers.

Or finally, one builds (A 6, *H. Burkhardt*), in connection with combinatorics, on the basis of the permutation process, the doctrine of substitutions of a number of elements. As the most far-reaching type of summarizing substitutions proves to be the "group", initially for a finite, subsequently for an unlimited series of elements or operations.

By borrowing from analysis the concept of a continuously variable quantity, one enters the domain of *Algebra*, as treated in Section B. With the help of the first three or four arithmetic species, the entire or fractional rational functions of one and several variables arise (B 1 a, b, *E. Netto*). The investigations belonging here group themselves around two main problems, first the formal elimination of unknowns from equation systems, which reaches a certain conclusion in the theory of modular systems, then the proof of existence for solutions of algebraic equations and equation systems.

The theory of entire functions experiences a sharper formulation on the basis of the concept of the "rationality domain", whereby the coefficients of the functions are themselves conceived as entire functions of a number of original variables, but with only integer coefficients (B 1 c, *G. Landsberg*). Thereby it succeeds in subordinating the properties of algebraic formations, especially with respect to rational transformations, to the various operations of a fundamental process, the reduction of infinite function or form systems to a finite number. These developments therefore simultaneously serve as the algebraic foundation of higher number theory (Section C).

From this point, a branching into subspecialties of specific character occurs again in Section B. Apart from the article B 3 a (*C. Runge*), which discusses more practical questions of how to enclose equation roots within suitable limits and approximate them using numerically usable algorithms, the group concept emerges as the predominant one. Among the rational transformations of variables of entire functions (forms) are primarily the linear to be considered (B 2, *W. Fr. Meyer*). If one subjects those variables to any linear group, the coefficients of the given forms are likewise subject to a certain linear group, and the task of linear invariant theory is to establish the invariants of this latter group, or more generally, of any linear group, to classify them appropriately, and to represent the limited series of these as entire or rational functions of a finite number among them. A special interest is claimed by the groups composed of a finite number of substitutions, due to their relationships to theory, analysis, and geometry, to which therefore a special article (B3f, *A. Wiman*) is dedicated.

As the actual carrier of the entire section may be considered the *Galois* theory of equation groups (B 3 c, d, *O. Holder*), already touched upon in B 1 c, which, originally proceeding from the special question of the solvability of certain equations by root signs, has in its further development subordinated the theories of rational as well as arithmetic and geometric rationality domains (as well as the formal integration theories of differential equations). An introduction to this theory is formed by the doctrine (B 3 b, *K. Th. Vahlen*) of one- and multi-valued algebraic functions of one or more quantity series, especially the roots of equations and equation systems.

Number theory, or the explicit execution of the characteristics of individual arithmetic rationality domains, could be directly connected to the article B 1 c from today's standpoint. For historical reasons, however, the formation of a separate section (C) was recommended.

Following the presentation of the elementary divisibility laws of natural numbers (C 1, *F. Bachmann*), the treatment of linear, bilinear, quadratic, and certain higher forms, equations, and congruences follows (C 2, *K. Th. Vahlen*); the concepts of elementary divisors and the rank of a matrix, already emerging in algebra, serve here as fundamental principles.

The following article (C 3, *P. Bachmann*) on analytic theory will once do justice to the scattered methods of additive composition of numbers, whose systematic treatment is still pending. On the other hand, it goes into the approximate determination of mean values of number-theoretic functions.

Finally, in connection with the periodic properties of algebraic numbers, discusses the recently emerged questions of the transcendence of specific irrationalities, such as e and π.

The main goal of contemporary systematic number theory, the extension of the divisibility laws of natural numbers, as well as subsequently the reciprocity laws of power residues to algebraic number fields, that is, to the rational functions of algebraic numbers, especially quadratic ones, is pursued in C 4 a, b (*D. Hilbert*). While the task in the case of a circular field was solved by introducing ideal numbers, in the general case the creations of field ideals or field forms take their place.

The special case of quadratic class fields occurring in the complex multiplication of elliptic functions finds its resolution in C 6 (*H. Weber*).

Of the lower and higher arithmetic and algebra treated up to this point, it can be said that they constitute a closed unity.

This is not the same for the following sections, already mentioned above; they satisfy more of *a negative definition, belonging neither to the preceding nor to the next two volumes.*

Section D is essentially dominated by probability calculation; although, for example, the adjustment calculation (D 2) can be theoretically constructed without the help of specific probability concepts, the relevant concepts and methods have gradually developed through probability calculation.

Probability calculation (D 1, *E. Czuber*), initially only an application of combinatorics to some hazard games, has, namely after the adoption of infinitesimal and geometric concepts, extended its scope so extensively that it lies at the basis of a considerable number of mathematical approximation methods, explicitly or implicitly. From an epistemological perspective, it has the service of resolving the concept of chance, or rather the circumstances accompanying it, to a certain degree into mathematical approaches and laws.

The most productive source of *adjustment calculation* (D 2, *J. Sehinger*) is the principle of the minimum sum of squares, which in a modified form, as the principle of least constraint, can also serve as the source of entire dynamics.

In *interpolation calculation* (D 3, *J. Bauschinger*) any arbitrary function determined by certain data through an entire rational function or also through a finite trigonometric series; both in the practical elaboration of the required algorithms and in the more theoretical establishment of remainder terms, difference calculus renders full service, which is independently treated in Article E (*D. Seliwanoff*).

The tasks of *statistics* (D 4 a, *L. v. Bortkiewicz*) and *life insurance* (D4b, *G. Bohlmann*) speak for themselves.

The article F (*R. Mehmke*), on *numerical computation*, contains more than its title indicates. Starting from techniques and tables that serve to facilitate practical calculation with rational and other numbers, the material expands through the use of the most diverse types of computing apparatus and machines into a wide mathematical-technical discipline; also, it becomes a kind of geometric arithmetic through the inclusion of fruitful graphical methods.

The articles G 1 (*W. Ahrens*), G 2 (*L. Pareto*) on *games* and *economic theor* may be viewed as an appendix.

The article G 3 (*A. Pringsheim*) on *infinite processes with complex terms*, which represents a direct supplement to both A 3 and A 4, was originally intended for the second volume. With regard to its close relationship to these essays and in accordance with the arrangement in the French edition of the Encyclopedia, it has subsequently been assigned to the first volume as the concluding article.

To enable the reader a more convenient handling of the volume, it has been divided into two parts. For internal and external reasons, it was recommended to conclude the first part with the last article (B 3 f) of Section B (Algebra).

May the Encyclopedia, which presents the mathematical inventions of a century in historical development, also enliven the epistemological study of the fundamental question of what should actually be considered "new" in mathematics! Does the new lie in an expansion and deepening of a stock of a priori knowledge gained through inner intuition, or does it merely come down to a different grouping of existing empirical facts?

Then it shall still be permitted to set forth the guiding perspectives according to which the *register* has been prepared. It was to be a word and subject register. In the *word register*, only expressions are included in the present volume; if this inclusion has also extended to a considerable number of terms that are either outdated, or less in circulation, and may often have sprung from a momentary idea of the author, this was done to satisfy to a certain degree those who are interested in the philological side of word formations. The selection was less straightforward with the considerable treasure of technical expressions, as found primarily in the articles on statistics and life insurance, numerical computation, games, and economic theory. The editor has endeavored to extract expressions for such subjects that still retain a certain measure of mathematical thinking.

Regarding the subject *register*, two difficulties should be pointed out at the outset, which could only be approximately resolved.

First is the question of which authors to cite. With some exceptions, only those authors who no longer belong to the present are considered here under the keywords, and even these mostly only when their name — often merely by chance or even improperly — has become a kind of common currency associated with a specific concept, sentence, or specific method. On the other hand, there was a temptation, when an author was mentioned, to strive for a certain completeness with respect to their outstanding achievements. The justified wish that some might have harbored, that this principle should have been extended to as many or even all authors, could not be satisfied already with regard to the available space. Where, conversely, the name of a researcher was mentioned within the register text, often only for greater clarity, it was not highlighted by print, as such emphasis was reserved for other purposes (see below).

The other difficulty lay in the *word formations* themselves. The large number of contributors makes it understandable that the same word does not always denote the same concept, and conversely, that the same concept — not to mention sentences — is labeled with the most diverse names.

The editor has spared no effort to unite factually related citations at a single location, but is very well aware that many gaps may still remain in this regard, and conversely, that something superfluous may have been found. That for keywords indicating more extensive categories, details had to be suppressed as much as possible, hardly requires justification. That, conversely, the clarity of the register text has suffered from frequent interlocking, shall be readily admitted.

It may perhaps raise concerns that a series of *adjectives* have been incorporated into the keywords. This was done, however, for such adjectives that are more than mere epithets, that express a particularly characteristic property of the noun. Since, on the other hand, the noun as a word must not be missing, the register in various respects resembles a mathematical table with double, sometimes even multiple, entries.

Regarding emphasis through print, it is to be noted that the keywords, as well as references to such, are rendered by spaced letters. The *italic* print is related to the marking of subsections within larger word articles, whether these sections relate to characteristic areas of the volume, or also to derivations and compositions of the word. It appeared expedient to name those characteristic areas briefly in part. For example, the articles B 1 c and C 4 a, b were summarized under the generic name "Body Theory", furthermore the graphical methods occupying a large space under "Graphics", among others. A specific note at the head of the first register page provides information about the specific use of semicolons, as well as round and square brackets.

From the original intention of giving each citation with mention of the article and its author, abstention was made with consideration of the scope. Precisely regarding the scope, the editor has received the most diverse wishes, ranging approximately from two to ten sheets. The actual scope of not quite $3\frac{1}{2}$ sheets results in a percentage of somewhat over 6% in relation to the $70\frac{1}{2}$ sheets of text in the volume.

For the assessment of the register, reference should be made to its origin; only after a separate register had been prepared for each article were these, as best as possible, drawn together into a whole.

Originally there was an intention to add corrections and supplements to the volume. Meanwhile, what has previously been in this respect so inhomogeneous that abstention was taken from it, in the hope that it will later be possible, which is also entirely desirable for other reasons, to allow supplementary volumes to follow the respective completed volumes of the Encyclopedia from time to time.

In conclusion, the editor gladly fulfills the duty to express his special thanks in various directions: first and foremost to the Academies of Göttingen, Leipzig, Munich, and Vienna, as well as the commission appointed by them; then to all the gentlemen contributors of the first volume, who have expended their notable time and effort; furthermore to colleague *H. Burkhardt*, who subjected every article to multiple corrections and contributed an extensive series of critical and historical remarks; and not least to the *Teubner* publishing house for its far-reaching accommodation during the arduous and lengthy printing process.

Königsberg i/Pr., April 1904.

W. Franz Meyer
as Editor.

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1. Foundations of Arithmetic. By H. Schubert in Hamburg

(Completed in July 1898.)

2. Combinatorics. By E. Netto in Giessen

(Completed in October 1898.)

3. Irrational Numbers and Convergence of Infinite Processes.

By A. Pringsheim in Munich

(Completed in September 1898.)

4. Theory of Common and Higher Complex Quantities.
By E. Study in Greifswald (now Bonn)

(Completed in November 1898.)

5. Set Theory. By A. Schönflies in Göttingen (now Königsberg)

(Completed in November 1898.)

6. Finite Discrete Groups. By H. Bürkhardt in Zürich

(Completed in November 1898.)

B. Algebra

1a. Rational Functions of One Variable; their Zeros. By E. Netto in Giessen

(Completed in May 1899.)

1b. Rational Functions of Several Variables. By E. Netto in Giessen

(Completed in July 1899.)

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By G. Landsberg in Heidelberg

(Completed in August 1899.)

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3b. Rational Functions of Roots; Symmetric and Affect Functions.
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(Completed in October 1899.)

3e. Systems of Equations. By E. Netto in Giessen and K. Th. Vahlen in Königsberg (See: B 1b and B 3b.)

3f. Finite Groups of Linear Substitutions. By A. Wiman in Lund

(Completed in December 1899.)

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I A 1. FOUNDATIONS OF ARITHMETIC

(THE FOUR BASIC OPERATIONS; INTRODUCTION OF NEGATIVE AND FRACTIONAL NUMBERS; THIRD-LEVEL OPERATIONS FROM A FORMAL PERSPECTIVE)

BY

H. SCHUBERT

IN HAMBURG

Table of Contents

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1. Counting and Numbers. *Counting* things[1)] means viewing them as similar[2)], comprehending them together, and to them individually other things

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1) That *non-physical* things can also be counted was emphasized by *G. F. Leibniz* against the Scholastics in "De arte combinatoria" (1666), as well as by *J. Locke* in his work "An Essay concerning human understanding" (1690, Book II). In contrast, *J. St. Mill* (Logic, Book III, 26) sees the fact stated in the definition of a number as *physical*, similar to *G. Frege* in "Foundations of Arithmetic" (Breslau 1884).

2) *E. Schröder* in his "Textbook of Arithmetic" (Leipzig 1878) p. 4 emphasizes that the counting process must be preceded by both a combining of the things to be counted and the recognition of the similarity of the things to be counted, as does *E. Mach* in his book "The Principles of Heat Theory" (Leipzig 1896) (No. 7).

to assign[3], which are also viewed as similar[4]. Each of the things to which other things are assigned during counting is called

3) That counting is not possible without a more or less conscious *assignment* or mapping was emphasized by *K. Weierstrass* in the introduction to his lectures on the theory of analytical functions. (See *E. Kossak* "The Elements of Arithmetic", Berlin, Progr. Friedr. Werder-Gymn., 1872). This is similarly emphasized by *E. Schröder* in his textbook, *L. Kronecker* in his essay "On the Concept of Number" (Philosophical Essays dedicated to Zeller, Leipzig 1887, Journ. f. Math. 101), *R. Dedekind* in his work "What are Numbers and What Should They Be?" (Braunschweig 1887 and 1893), in which "assignment" is thoroughly analyzed through a series of definitions (chain) and theorems.

4) In this definition of number, all philosophers and mathematicians essentially agree. However, opinions differ on which psychological moments enable the formation of the number concept. Following *I. Kant's* example, *W. R. Hamilton* emphasizes *time* perception as the foundation of the concept. For him, algebra is "Science of Order in Progression" or "Science of Pure Time". He first expresses this in his paper published in Dubl. Trans. 17, II (1835) "Theory of Conjugate Functions or Algebraic Couples with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time". Later he repeats this view in the preface to his work "Lectures on Quaternions" (Dublin 1853). *H. Helmholtz* takes the same position in his paper "Counting and Measuring" in the philosophical essays dedicated to *Eduard Zeller* (Leipzig 1887); as does *W. Brix* in his paper "The Mathematical Concept and its Development Forms" (Volumes V and VI of *Wundt's* Philosophical Studies; also published as dissertation, Leipzig 1889). In contrast, *J. F. Herbart* in his "Psychology as Science" (Königsberg 1824, Volume II) says that number has no more to do with time than many other types of ideas. *J. J. Baumann* and *F. A. Lange* believe that number aligns far better with *spatial representation* than with time representation, specifically *J. J. Baumann* in his work "The Theory of Space, Time and Mathematics in Modern Philosophy" (Berlin 1869) and *F. A. Lange* in his "Logical Studies" of 1877. Similarly, *F. G. Husserl* opposes efforts to base the concept of number on the idea of time, specifically in Volume I of his work "Philosophy of Arithmetic", Halle 1891.

Aristotle is often presented as the first to define number through the concept of time. This is incorrect. *Aristotle* conversely defines time through the concept of number in his "Physics" (Book IV, Chapter 11, p. 219 B or German translation by Prantl, Leipzig 1854, p. 207 ff.), where it states: "Time is the number of movement according to before and after" and further: "Time is the number of circular motion". *Euclid's* often repeated statement (Elements, Book VII) "Number is a multitude of units" can hardly be considered a definition.

Analyses of the number concept can be found, besides in the already cited writings, particularly in the following recent papers and books:

W. Wundt, Logic, Volume I;

G. Frege, Foundations of Arithmetic, Breslau 1884;

R. Lipschitz, Foundations of Analysis, Bonn 1877 (§ 1);

U. Dini, Foundations for a Theory of Functions of a Variable Quantity, translated by J. Lüroth (Leipzig 1892);

a unit[5]; each of the things that are assigned to other things during counting is called a one[5]. The result of counting is called a *number*. Due to the similarity of the units among themselves and the ones among themselves, the number is independent of the order in which the ones are assigned to the units[6].

When one reminds of a number through an added concept of how the units were viewed as similar, one expresses a *denominate number*. By completely disregarding the nature of the counted things, one arrives from the concept of denominate number to the concept of *undenominate number*. By number alone, an undenominate number is always to be understood.

To *communicate* numbers, one can choose any similar things as ones (fingers, counting beads, chalk marks). Primitive peoples who have no writing use stones or shells as ones when they want to communicate numbers. If one assigns similar written characters to the things to be counted, one obtains the usual *numerical symbols*[7]. Thus, in ancient times, the Romans represented the numbers from one to nine through a sequence of strokes, the Aztecs

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G. Peano, Arithmetices principia nova methodo exposita, Torino 1889;

K. Th. Michaelis, On Kant's Concept of Number, Progr. Charlottenburg, Berlin, at Gärtner, 1884;

E. Knoch, On the Concept of Number and Elementary Instruction in Arithmetic, Program of the Realprogymnasium in Jenkau, 1892;

G. F. Lipps, Investigations on the Foundations of Mathematics in Wundt's Philosophical Studies (from Volume X onwards); the fourth chapter (in Volume XI) contains the logical development of the number concept.

Of these authors, *G. Peano* and *E. Knoch* build upon Dedekind's foundations in his already cited work.

The laws of arithmetic can be built upon the concept of number without any axiom, as *K. Weierstrass* emphasized in his lectures, among others.

5) The distinction between units and ones in this sense comes from *E. Schröder* (Textbook of Arithmetic and Algebra, Leipzig 1873, p. 5 or Outline of Arithmetic and Algebra, first issue, p. 1).

6) *H. Helmholtz* and *L. Kronecker*, in the already cited essays for Zeller's jubilee, conceive the assignment as if the numbers One, Two, Three, etc. are assigned to the units to be counted. This view is opposed by *E. G. Husserl* in the appendix to the first part of his "Philosophy of Arithmetic" (Halle 1891).

7) On *number communication* and *number representation* through word or writing, one finds detailed information in the following writings:

represented the numbers from one to nineteen through combinations of individual circles. Modern civilized peoples still have natural symbols only on dice, dominoes, and playing cards. If one assigns similar sounds to the things to be counted, one obtains the natural number sounds, as heard, for example, from the chimes of clocks. Instead of such natural number symbols and number sounds, one usually uses symbols and words that are methodically composed of a few elementary symbols and word stems[7)]. The modern numeral writing, which is based on the principle of place value and the introduction of a symbol for nothing, was invented by Indian Brahma priests, became known to the Arabs around 800 and reached Christian Europe around 1200, where over the course of the following centuries the new numeral writing and new calculation gradually displaced calculation with Roman numerals[7)].

The study of relationships between numbers is called *arithmetic* ($\alpha\rho\iota\theta\mu\acute{o}\varsigma$ = number). *Calculation* means methodically deriving sought numbers from given numbers. In arithmetic, it is customary to express any number through a *letter*, whereby it must only be noted that within one and the same consideration

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M. Cantor, Mathematical Contributions to Cultural Life of Peoples, Halle 1863;

G. Friedlein, The Numerals and Elementary Calculation of Greeks and Romans and of Christian Occident from 7th to 13th Century, Erlangen 1869; Gerbert, The Geometry of Boethius and the Indian Numerals, Erlangen 1861.

H. Hankel, On the History of Mathematics in Antiquity and Middle Ages, Leipzig 1874;

M. Cantor, Lectures on History of Mathematics, Leipzig 1880, Volume I;

A. F. Pott, The Quinary and Vigesimal Counting Methods Among Peoples of All Parts, Halle 1847;

A. F. Pott, The Language Diversity in Europe, Demonstrated by Numerals, Halle 1868;

K. Fink, Brief Outline of a History of Elementary Mathematics, Tübingen 1890;

A. von Humboldt, On the Systems of Numerals Common Among Different Peoples and on the Origin of Place Value in Indian Numbers (J. f. Math., Volume 4);

P. Treutlein, Progr. Gymn. Karlsruhe 1875.

M. Charles, On the Passage of the First Book of Boethius' Geometry, Brux. 1836; Historical Overview..., Brux. 1837; Par. C. R. 4, 1836; 6, 1838; 8, 1839.

F. Unger, The Methodology of Practical Arithmetic in Historical Development, Leipzig 1888;

H. G. Zeuthen, History of Mathematics in Antiquity and Middle Ages, Copenhagen 1896;

H. Schubert, Counting and Number, a Cultural-Historical Study, in Virchow-Holtzendorff's Collection of Popular Scientific Lectures, Hamburg 1887.

the same letter must always represent one and the same number[8].

Two numbers a and b are called *equal*[9] when the units of a and those of b can be assigned to each other in such a way that all units of a and b participate in this assignment. Two numbers a and b are called *unequal*[9] when such an assignment is not possible. Since during counting the units are considered as similar, it is irrelevant for determining whether a and b are equal or unequal which units of a and b are assigned to each other. When two numbers are unequal, one is called the *greater*, the other the *lesser*[9]. a is called greater than b when the units of a and b can be assigned to each other in such a way that while all units of b participate in this assignment, not all units of a do. The judgment that two numbers are equal or unequal is called an *equation* or *inequality*, respectively. For equal, greater, lesser, arithmetic uses the three symbols $=, >, <$,[9] which are placed between the compared numbers. When drawing a conclusion from multiple comparisons, this is indicated by a horizontal line. The most fundamental conclusions of arithmetic are:

$$\frac{a=b}{b=a}; \qquad \frac{a>b}{b<a}; \qquad \frac{a<b}{b>a}.$$

These conclusions refer to only two compared numbers. The following conclusions refer to three numbers:

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8) The first seeds of arithmetic letter calculation can already be found among the Greeks (*Nikomachos* around 100 AD, *Diophantos* around 300 AD), even more among the Indians and Arabs (*Alchwarizmî* around 800 AD, *Alkalsâdî* around 1450). However, proper letter calculation using the symbols $=, >, <$; and operation symbols was only developed in the 16th century (*Vieta* † 1603), primarily in Germany and Italy. The currently common equality sign first appears in *M. Recorde* (1556). More details on this in:

L. Matthiessen, Principles of Ancient and Modern Algebra of Literal Equations, 1878;

P. Treutlein, The German Coss, Zeitschr. f. Math., Volume 24;

S. Günther, History of Mathematical Education in Medieval Germany until 1525, Berlin 1887; Contributions to the Invention of Continued Fractions, Progr., Weissenburg 1872; Mixed Investigations on the History of Mathematical Sciences, Leipzig 1876.

Only through *L. Euler* (from 1707-1783) did the arithmetic symbolic language acquire its current more fixed form.

9) A more precise analysis of the concepts equal, more, less, greater and smaller can be found in *E. G. Husserl's* Philosophy of Arithmetic, Volume I, Chapters 5 and 6 (Halle 1891), where additional philosophical literature can also be found.

$$\frac{\begin{array}{c}a = m\\ b = m\end{array}}{a = b};\qquad \frac{\begin{array}{c}a > m\\ b = m\end{array}}{a > b};\qquad \frac{\begin{array}{c}a < m\\ b = m\end{array}}{a < b};\qquad \frac{\begin{array}{c}a > m\\ m > b\end{array}}{a > b}.$$

2. Addition[10]. When one has two groups of units, and indeed in such a way that not only are all units of each group similar, but also each unit of one group is similar to each unit of the other group, one can do two things: either one can count each group individually and interpret each of the two counting results as a number, or one can extend the counting over both groups and interpret the counting result as a number. In the former case, one obtains two numbers, in the latter case only one number. One then says of this

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10) The logically precise *construction* of the four fundamental operations of arithmetic carried out here in the text was most thoroughly implemented by *E. Schröder* in his textbook (Leipzig 1873, Volume I: The Seven Algebraic Operations). Besides him, the following have contributed to such a construction:

1) *M. Ohm*, Attempt at a Completely Consistent System of Arithmetic (2 volumes, 2nd edition, Berlin 1829);

2) *W. R. Hamilton*, Preface to the Lectures on Quaternions, Dublin 1853;

3) *M. Cantor*, Foundation of Elementary Arithmetic, Heidelberg 1855;

4) *H. Grassmann*, Textbook of Arithmetic, Berlin 1861;

5) *H. Hankel*, Theory of Complex Number Systems, Leipzig 1867 (Section I, I, II);

6) *J. Bertrand*, Treatise on Arithmetic, 4th edition, Paris 1867;

7) *R. Baltzer*, The Elements of Mathematics, Volume I, last (7th) edition, Leipzig 1885;

8) *O. Stolz*, Lectures on General Arithmetic, Leipzig 1885.

A combination of consistent construction with didactic considerations for beginners was first attempted by *E. Schröder* in his Outline of Arithmetic and Algebra, Part I, Leipzig 1874, then more extensively by *H. Schubert* in his textbooks (Collection of Arithmetic and Algebraic Questions and Problems, four editions, Potsdam 1883 to 1896, System of Arithmetic, Potsdam 1885, Arithmetic and Algebra in Göschen Collection (Leipzig 1896,1898).

In earlier centuries, there was even uncertainty about which operations should be considered as basic arithmetic operations, as in the middle of the 15th century with *J. Regiomontanus*, *G. v. Peurbach*, *Lucius Pacioli* and in the *Bamberg arithmetic book*. *Peurbach's* Algorithm, for example, knows eight basic operations, namely Numeratio, Additio, Subtractio, Mediatio, Duplatio, Multiplicatio, Divisio, Progressio.

The operations and laws of arithmetic appear in connection with more general viewpoints in *formal arithmetic*, in *logic calculus*, and in *conceptual notation*. Formal arithmetic studies the relationships of quantities without regard to these quantities being numbers. In particular, one should read on this subject on one hand *H. Grassmann's* Theory of Extension, 1844 and 1878, on the other hand *H. Hankel's* Theory of Complex Number Systems, Leipzig 1867. Regarding logic calculus and conceptual notation, cf. Vol. VI.

latter case obtained number that it is the sum of the two numbers obtained in the former case, and these two numbers are called the *summands* of the sum. The transition from two numbers to a single one just described is called *addition*. Counting and adding thus differ only in that counting deals with a single group, while adding deals with two groups of units. To indicate that from two numbers a and b a third number s has emerged through addition, one places the symbol + (plus) between the two summands. From the definitions of being greater and of addition follows: A sum is greater than one of its summands, namely by the other summand. When $a > b$, then a can be the sum of two summands, one of which is b.

From the concept of counting follows that there can always be only one number which is the sum of any two numbers, and conversely that there can also be only one number which, when combined with a given number through addition, leads to a *greater* given number as sum.

Since the result of counting is independent of the order in which one counts, it must be:

$$a + b = b + a$$

One calls the law expressed herein the *commutation law*[11)] of addition. Despite this law, one can conceptually distinguish the two

11) If + is the symbol for a very generally conceived connection of two quantities, then for this connection the *commutative* law applies when $a + b = b + a$, the *associative* law applies when $(a + b) + c = a + (b + c)$. Furthermore, if $\times$ is the symbol for a second generally conceived connection, then the *distributive* law applies to both when $(a + b) \times c = (a \times c) + (b \times c)$, or when $(a \times b) + c = (a + c) \times (b + c)$. The distinction of these three basic laws and their names are first found in Germany in *H. Hankel* (Theory of Complex Number Systems, Leipzig 1867), in England already since about 1840, and (according to Hankel) the terms commutative and distributive were first used by *Servois* (Gergonne's Ann., Vol. V, 1814, p. 93), associative probably first by *Hamilton*. These three basic laws also play a fundamental role in more general relationships than arithmetic operations, such as in formal mathematics, logic calculus and conceptual notation. (Cf. here Vol. VI.) In logic calculus, $a + b$ means everything that is either a or b, $a{\cdot}b$ everything that is both a and b. For each of these operations, the commutative and associative laws apply. Additionally, the distributive law applies in both forms, as $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ and furthermore $(a \cdot b) + c = (a + c) \cdot (b + c)$. *E. Schröder* drew attention to this in his work "Operation Circle of Logic Calculus" (Leipzig 1877).

summands, by conceiving in the operation of addition one as passive[12], the other as active[12]. The passive summand could be called *augend*[12], the active *auctor*[12] (increment[12]). This conceptually possible distinction is arithmetically unnecessary due to the commutation law.

From the concept of counting further follows:

$$(a + b) + c = a + (b + c)$$

The expressed law is called the *association law*[11] of addition.

From the uniqueness of addition alone follows the law about combining two equations through addition, whereby from $a = b$ and $c = d$ follows: $a + c = b + d$. However, how an equation and an inequality or two inequalities are to be combined can only be recognized through application of the association law. The combined effect of both basic laws of addition yields:

When any number of numbers are combined through addition in any order such that the sum of two numbers always appears again as a summand of a new addition, the number that represents the final result is always the same, regardless of the order in which the given numbers are combined by addition.

This truth justifies calling the final result the *sum of all* given numbers, and thereby extending the concept of sum such that it may have not only two, but any number of summands.

3. Subtraction[13]. In addition, two numbers, the two summands a and b, are given, and from them emerges a third number, the sum s. If one now conversely considers the sum s and one summand as given, then the other summand emerges from this as a number which (according to No. 2) is completely determined. Finding this number from the sum s and the given summand is called *subtraction*[13] The number s, which was previously sum

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12) The distinction of the two numbers connected by an arithmetic operation through the terms *passive* and *active* was first given by *E. Schröder* in his "Outline of Arithmetic and Algebra" (Leipzig 1874). In addition, he calls the passive number augend, the active increment. *H. Schubert* distinguishes between augend and auctor, for example in his "Arithmetic and Algebra" in "Göschen Collection" (Leipzig).

13) Addition, multiplication and exponentiation are usually called direct operations and their inversions indirect operations. In *H. Hankel* (Theory of complex number systems,

is called *minuend* in this just defined *inverse*[13)] calculation method. The given summand could be called "minutor" as the active number. However, the name *subtrahend* is customary for this. The result of subtraction is called difference. The sign of subtraction is a horizontal line (minus), before which one places the minuend and after which one places the subtrahend. Thus it is:

$$(s-a)+a=s$$

the *definition formula* of subtraction. From the uniqueness of subtraction follows secondly:

$$(s+a)-a=s$$

The *transposition rule of first degree* is also based on the definition of subtraction, whereby a subtrahend (summand) on one side of an equation may be omitted there to appear on the other side as summand (subtrahend). Through transposition, an unknown summand or an unknown minuend can be *isolated*, i.e., it can be arranged that it stands alone on one side of an equation.

An equation is called *identical* if it remains correct regardless of what numbers one may substitute for the letters appearing in it. An identical equation is called a *formula* if it serves to express a truth in arithmetic symbolic language. An equation is called a *determining equation* if it becomes correct only when the letters appearing in it are replaced by specific (not by all) numbers. If in a determining equation all occurring numbers except one are known, one usually denotes the still unknown number with x[14)], and then arises the task of *solving the equation*, i.e., finding the number that must be substituted for x for the equation to become correct. An equation in which x is summand or minuend is solved by the transposition rule of first degree

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Leipzig 1867) the direct operations of arithmetic belong to the *thetic*, the indirect to the lytic types of connections. From the thetic connection type $a \times b = c$ follow the two lytic $c \top a = b$ and $c \perp b = a$.

14) In *Diophantos*, the unknown is denoted by a final sigma as the last letter of $\alpha\rho\iota\theta\mu\acute{o}\varsigma$. On the origin of the designation x for an unknown number, one should read: *Treutlein*, The German Coss, Zeitschr. f. Math. Vol. 24 and the multiply doubted view of *P. A. de Lagarde*, Where does the mathematicians' x come from? (Gött. Nachr., 1882).

4. Combination of Addition and Subtraction. From the definition of subtraction follow, when applying the commutation law and the association law of addition, the four formulas:

$$\begin{gathered} a+(b-c)=(a+b)-c \\ a-(b+c)=(a-b)-c \\ a-(b-c)=(a-b)+c \\ a-b=(a-n)-(b-n) \end{gathered}$$

Since in each of these formulas there stands a difference on at least one of the two sides, the *proof* of the same only concerns checking whether the other side fulfills the property prescribed by the definition of subtraction, in which checking only the two basic laws of addition or a formula may be applied that precedes the one to be proved here and so can already be considered as proved.

Since from $p = q$ according to No.1 follows $q = p$, every arithmetic formula contains two truths, which one obtains depending on whether one interprets the formula from left to right or from right to left. When one reads the association law of addition and the first three of the four above formulas from left to right, they yield rules about adding and subtracting sums and differences. When one reads them from right to left, they yield rules about increasing and decreasing sums and differences. These formulas provide the proof that summands and subtrahends can be brought into any order, and the rules according to which from an equation and an inequality or from two inequalities through subtraction of the right and left sides a new inequality can be concluded.

The arithmetic symbolic language[8)] has developed such that in the operations of addition and subtraction the parentheses[15)] around the *preceding* calculation type may be omitted, but must be placed around the following one, so that the association law of addition and the first three of the above formulas may also be written as:

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15) *E. Schröder* first states in his textbook the following general rule about placing parentheses in arithmetic: An expression that is part of a new expression is enclosed in parentheses. Gradually it has become customary to omit these parentheses in two cases, first when of two *same-level* operations the *precedin* one should be executed first, second when of two *different-level* operations the one of higher level should be executed first.

$$a + (b + c) = a + b + c; \quad a + (b - c) = a + b - c;$$
$$a - (b + c) = a - b - c; \quad a - (b - c) = a - b + c;$$

5. Zero[16]. Since according to the definition of subtraction the minuend is a sum whose one summand is the subtrahend, the connection of two equal numbers by the minus sign has no meaning. Such a connection has indeed the form of a difference but represents no number in the sense of No. 1. Now arithmetic follows a principle that one calls the *principle of permanence*[17] or of exceptionlessness and that consists in four things:

first, in giving every sign connection that represents none of the numbers defined up to then such a meaning that the connection may be treated according to the same rules as if it represented one of the numbers defined up to then;

second, in defining such a connection as a number in the extended sense of the word and thereby extending the concept of number;

third, in proving that for numbers in the extended sense the same theorems hold as for numbers in the not yet extended sense;

fourth, in defining what equal, greater and lesser means in the extended number domain[19].

Accordingly, the sign connection a - a is subjected to the two basic laws of addition and the definition formula of subtraction, whereby it is achieved that the formulas of No. 4 must also hold for the sign connection a - a. Through

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16) *Zero* appears as the common symbol for all difference forms in which minuend and subtrahend are equal only since the 17th century. Originally, zero was only a vacat sign for a missing level number in the *place-value* numeral writing invented by the Indians. (Cf. the literature indicated in note 7.) It is called "tziphra" in the arithmetic book of the monk *Maximus Planudes* living in the 14th century (German by *H. Waeschke*, Halle 1878), from which the English cypher and the French zero for zero originated. The German word Ziffer, also coming from tziphra, has gained a more general meaning in German. Other numeral writings, like the additive of the Romans or the multiplicative of the Chinese, have no sign for zero.

17) The principle of *permanence*, which is given here in the text the form suitable for the extension of the number concept, was first expressed in most general form by *H. Hankel* (§ 3 of the Theory of Complex Number Systems, Leipzig 1867), after *G. Peacock* had already emphasized the necessity of a purely formal mathematics and in connection with it a principle from which through extension that of permanence emerges (*G. Peacock* in Brit. Ass. III, London 1834, Symbolical Algebra, Cambridge 1845).

application of the formula $a - b = (a - n) - (b - n)$ to $a - a$, one then recognizes that all difference forms in which the minuend equals the subtrahend are to be set equal to each other. This justifies introducing a common fixed symbol for all these equal sign connections. This is the symbol 0 (zero)[16]. Furthermore, one calls what this symbol expresses a *"number"*, which one also calls zero. But since zero is not a result of counting (No. 1), the concept of number has experienced an extension through the inclusion of zero in the language of arithmetic. From the definition $a - a = 0$ follows how to proceed with zero in addition and subtraction, namely: $p + 0 = p$, $0 + p = p$, $p - 0 = p$, $0 + 0 = 0$, $0 - 0 = 0$.

6. Negative Numbers[18]. When in $a - b$ the minuend a is smaller than the subtrahend b, then $a - b$ represents no number in the sense of No. 1. According to the principle of permanence[17] introduced in No. 5, the difference form $a - b$ must then be subjected to the definition formula of subtraction $a - b + b = a$, from which follows that the formulas treated in No. 4 become applicable to $a - b$ also in the case where $a < b$. Hereby one recognizes that all difference forms can be set equal[19] to each other in which the subtrahend is greater than the minuend by the same amount. It is therefore natural to express all difference forms $a - b$, in which b is greater than a by p, through p. Finally, by calling such difference forms also *"numbers"*, one extends

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18) Although in a logical construction of arithmetic the introduction of *negative numbers* must precede the introduction of fractional numbers, historically negative numbers came into use much later than fractional numbers. The Greek arithmeticians calculated only with differences in which the minuend was greater than the subtrahend. The first traces of calculating with negative numbers are found with the Indian mathematician *Bhāskara* (born 1114), who distinguishes between the negative and positive value of a square root. The Arabs also recognized negative roots of equations. *L. Pacioli* at the end of the 15th century and *Cardano*, whose Ars magna appeared in 1550, know something of negative numbers but attach no independent meaning to them. *G. Cardano* calls them aestimationes falsae or fictae, *Michael Stifel* (in his Arithmetica integra appearing in 1544) calls them numeri absurdi. Only *T. Harriot* (around 1600) considers negative numbers for themselves and lets them form one side of an equation. The actual calculation with negative numbers, however, begins only with *R. Descartes* († 1650), who assigned to one and the same letter sometimes a positive, sometimes a negative numerical value.

19) That the extension of the concepts equal, greater and lesser brought about by an extension of the number concept requires closer discussion is emphasized in newer textbooks.

the number concept and arrives at the introduction of negative numbers. Accordingly, the definition formula of the *negative number* -p (minus p) reads:

$$-p = b - (b + p)$$

In contrast to negative numbers, the results of counting defined in No. 1 are called *positive* numbers. From the definition formula of the negative number -p follows for $p = 0$ that -$p = 0$ - p, and since also $p = 0 + p$, it is natural to set $+p$ for p. The plus and minus signs placed before a number (in the sense of No. 1) are called *signs*. Negative numbers thus have the sign minus, positive ones the sign plus. Of the two signs, each is called the *inverse* of the other. Numbers provided with signs are called *relative*. If one omits the sign from a relative number, there results a number in the sense of No. 1, which one calls the *absolute value*[20] of the relative number. From these definitions follows how relative numbers are to be connected through addition and through subtraction. As result always appears a relative number or zero.

The introduction of relative numbers makes it possible to conceive any parentheses-free sequence of additions and subtractions as a *"sum"* of purely relative numbers. One calls a sum conceived in this way *algebraic* and the relative numbers that compose it its *terms*. If an algebraic sum stands in parentheses before which a plus sign or minus sign stands, the same may be omitted if one retains or reverses all the signs of the terms contained in it.

Through the introduction of the number zero (No. 5) and negative numbers, the comparison conclusions indicated in No. 2 and No. 4 receive a more extended meaning when one applies greater and lesser also to the newly introduced numbers. One calls, regardless of whether a and b are zero, positive or negative, $a > b$ when a - b is positive, $a < b$ when a - b is negative[19].

Finally, the newly introduced numbers also make such equations solvable that according to the original number concept had to be considered unsolvable. Thus the equation $x + 5 = 5$ is unsolvable according to No. 1, but solvable according to No. 5. Thus further the equation $x + 5 = 3$ is unsolvable according to No. 1, but solvable according to this number[17].

7. Multiplication[21]. Since due to the basic laws of addition the order in which additions are performed, the result leaves unchanged, so that at the

20) The expression "absolute value" for the modulus of any complex number has become customary through *K. Weierstrass'* lectures.

end of No. 2 the result of any number of successive additions could be conceived as a *sum of many* summands. If the latter now all represent one and the same number a, it is natural to set this number only once and add a sign which indicates *how many* summands a the sum should contain. One thereby arrives at a *new* connection of two numbers, namely the number a, which is thought of as summand and the number p, which counts *how often* this summand is thought. One calls this new connection *multiplication* and designates it as an operation of *second degree*, while one calls addition and its inverse, subtraction, operations of *first degree*. To multiply a number a (passive) with a number p (active) thus means to calculate a sum of p summands, each of which is a. The number a, *which* appears as summand thereby, is called *multiplicand*, the number p, which counts *how often* the summand is thought, is called *multiplier*. The result is called product. Due to No. 5 and No. 6 the multiplicand can be positive, zero or negative. The multiplier however, which indicates *how many* summands are meant, can only be a result of counting, thus only a number in the sense of No. 1. By conceiving a number a also as a sum of a single summand, the multiplier may also be the number 1. The sign of multiplication is a point set between the multiplicand a and the multiplier p. The definition formula of multiplication accordingly reads:

$$a \cdot p = \overset{1)}{a} + \overset{2)}{a} + \overset{3)}{a} + \dots + \overset{p)}{a}$$

where the number set above each summand indicates which summand it is. Earlier one set instead of the point the sign $\times$.

From the uniqueness of addition follows the uniqueness of multiplication, and from that follows that equal multiplied with equal yields equal. From the definition formula of multiplication follow through the formulas of No. 3 and No. 4 the four *distribution laws*[11)]:

I. $a \cdot p + a \cdot q = a \cdot (p + q)$;

IIa. $a \cdot p - a \cdot q = a \cdot (p - q)$, when $p > q$;

IIb. $a \cdot p - a \cdot q = 0$, when $p = q$;

IIc. $a \cdot p - a \cdot q = -[a \cdot (q - p)]$, when $q > p$;

III. $a \cdot p + b \cdot p = (a + b)p$;

IV. $a \cdot p - b \cdot p = (a - b)p$.

[About omitting the parentheses on the left sides of these formulas read note 15.]

Formulas I and II show, read forward, how a common multiplicand is *separated*, read backward, how to multiply *with* a sum or difference. Formulas III and IV show, read forward, how a common multiplier is *separated*, read backward, how a sum or difference is multiplied. From the distribution laws follows how an equation and an inequality or two inequalities are to be combined through multiplication, if the four compared numbers are positive.

How a product is to be treated whose multiplicand is zero or negative follows from No. 5 and No. 6. But when in a product the *multiplier is zero or negative*, this initially represents a meaningless sign connection. According to the principle of permanence[17)], it is now to be given a meaning that permits calculating with it according to the same rules as if the multiplier were a difference that represents a number in the sense of No. 1. Therefore in formula II the restriction $p > q$ is to be lifted, to derive from it how to multiply *with* zero and negative numbers. Thus it follows that $a{\cdot}0 = 0$ and $a{\cdot}(\text{-}w) = \text{-}(a{\cdot}w)$. From this then also follows how relative numbers are to be connected through multiplication.

From the distribution laws also follows that for multiplication the *commutation law*[11)] and the *association law*[11)] are correct.

The commutation law of multiplication eliminates the necessity of distinguishing between multiplicand and multiplier for pure arithmetic. One therefore designates both with the common name *factor* and writes them in any order. One calls the product a *multiple* of each of its factors and each factor a *divisor* of the product. Furthermore, one calls in a product each factor the *coefficient* of the other.

With denominate numbers the distinction between multiplicand and multiplier emerges through the fact that the former can be denominate but the latter must be undenominate. Therefore with denominate multiplicand the commutation law is meaningless.

For the arithmetic of undenominate numbers follows from the combined effect of the commutation law and the association law that the order in which multiplications follow each other is irrelevant regarding the final result. This justifies extending the concept of product such that it may have not just two, but any *number of factors*.

Through the fact that these factors can all represent the same number, the possibility of defining a direct operation of third degree - exponentiation[21] - is given.

An algebraic sum whose terms can also be products is called a *polynomial*. One multiplies two polynomials by multiplying each term of one with each term of the other and combining the obtained products again into an algebraic sum. Each term becomes positive or negative depending on whether it arises through multiplication from terms with equal or unequal signs. The proof of this rule follows from formulas I to IV.

From the definitions and results developed so far, one can conclude that when any number of numbers that are zero or relative are connected in any way through addition, subtraction and multiplication, the final result must always be zero or relative, thus one of the numbers defined so far.

8. Division[10]. Division emerges from multiplication through inversion[13], namely through considering the product and one factor as given, the other factor as sought. Thereby the number that was previously product receives the name *dividend*, the given factor the name *divisor*, the sought factor the name *quotient*. The sign of division is a colon (read: divided by), before which one places the dividend and after which one places the divisor. Accordingly, the *definition formula of division* reads:

$$(p : a) \cdot a = p$$

Instead of $p{:}a$ one also writes $\frac{p}{a}$, more rarely p/a. Like subtraction, division also has conceptually two inversions, since both the passive factor, the multiplicand, and the active factor, the multiplier, can be sought. If the dividend is a denominate number, then finding the multiplicand is called *partition*, finding the multiplier *measurement*. Due to the commutation law, however, with undenominate numbers it is unnecessary to distinguish between the two inversions of multiplication. For $p{:}a$ to have meaning, p must be able to be a product whose one factor is a, i.e., p must be a multiple of a, or, what is the same, a must be a divisor of p.

From the fact that $0 \cdot m = 0$ follows two things:

21) Cf. here No. 11.

1) Zero divided by zero is to be set equal to any arbitrary number. Therefore one calls the sign connection 0:0 *ambiguous*.

2) Zero divided by any arbitrary number always yields the number zero.

But when the divisor is zero and the dividend is not zero but any relative number p, then arises the question which number, multiplied by zero, leads to the relative number p. Since none of the numbers defined so far has the required property, the principle of permanence[17)] is to be applied. But the investigation of *what* meaning is then to be assigned to p:0 when p is not zero belongs in another chapter of mathematics (cf. I A 3).

Since division leads uniquely to one of the already defined numbers only when the divisor is not zero, one may conclude a third equation from two equations through division only *when the divisors are different from zero.* Many fallacies of elementary arithmetic as well as higher analysis are based on disregarding this restriction.

How relative numbers are divided follows from the corresponding rules for the multiplication of relative numbers.

From the definition formula of division also follows:

$$(p \cdot a) : a = p, \text{ if } a \text{ is not zero}$$

This formula yields in conjunction with the definition of division the rule that multiplication and division with the same number cancel each other out, *if this number is not zero.*

From the fact that the two equations $x \cdot b = p$ and $x = p:b$ mutually condition each other, if b is not zero, follows the *transposition rule of second degree.* Through second-degree transposition one can either isolate an unknown factor or an unknown divisor and accomplish the solution of determining equations.

9. Combination of Division with Addition, Subtraction and Multiplication. With help of the definition formula of division (No. 8) one can recognize the correctness of the following formulas:

I. $(a+b) : m = a : m + b : m$,

II. $(a-b) : m = a : m - b : m$,

III. $a \cdot (b : c) = a \cdot b : c$,

IV. $(a : b) \cdot c = a : b : c$,

V. $a : (b : c) = a : b \cdot c$,

VI. $a : b = (a \cdot m) : (b \cdot m)$,

VII. $a : b = (a : n) : (b : n)$

[About omitting the parentheses on the right sides of F. I to V read note 15.]

Here the occurring divisors are naturally to be understood as divisors of the associated dividend. In particular, none of the divisors may be zero. Formulas III, IV, V, VII correspond in the second degree exactly to the four formulas established in No. 4 for the first degree.

The two distribution formulas III and IV in No. 7, as well as the formulas designated here with I and II teach, read in one direction, how a sum or difference is multiplied or divided, read in the other direction, how products with equal factor or quotients with equal divisor are added or subtracted. In the first case parentheses are dissolved, in the second set.

From formulas I and II also follows how an algebraic sum is divided by a number, and how conversely any algebraic sum of quotients with common divisor can be transformed into a quotient whose divisor is the common divisor of all terms. When in an algebraic sum of quotients the *divisors are different*, one can transform these quotients through formula VI into other quotients that all have the same divisor (general divisor), and then apply the rule just mentioned.

The association law of multiplication and the above formulas III, IV, V teach, depending on whether one reads them in one direction or the other, both how to multiply or divide with products or quotients, and how products or quotients are multiplied or divided. In the first case parentheses are dissolved, in the second set. Furthermore, these forms show that factors and divisors can be brought into any order without the final result thereby changing.

When two quotients have equal positive divisor, that one represents the larger number which has the larger dividend. But when two quotients, whose dividend and divisor are positive, have equal dividend, that one which has the larger divisor represents the *smaller* number. These rules follow from the established formulas and yield how an inequality and an equation or two inequalities are to be combined through division when the divisors are positive and divisors of the associated dividends.

10. Fractional Numbers[21]. In §5 and §6 the principle of permanence[17] created from a - b, where a is not greater than b, zero and negative numbers. In the same way arise from $a{:}b$, where a is not a multiple of b, the *fractional numbers*, namely through transferring the definition formula of division

$$(a : b) \cdot b = a$$

to $a{:}b$, if a is not a multiple of b. Thus one also achieves the transfer of all definitions and formulas established so far to the quotient form $a{:}b$ and the lifting of the restriction expressed in No. 8 and No. 9, "if the divisor is a divisor of the dividend". In particular, the equation $b{\cdot}x = a$ now appears solvable even when b is not a divisor of a.

By calling the quotient form $a{:}b$, where b is not a divisor of a, a *"number"*, one extends anew the concept of number, enlarges the field of investigation of arithmetic and perfects the means[22] with which it works. In contrast to the fractional numbers thus arising, all numbers defined so far (Nos. 1, 5, 6) are called *whole numbers*. The dividend a of a fraction $a{:}b$ is called its *numerator*, the divisor b its *denominator*. One designates a fraction by a horizontal line[21], a whole number set above it, which is its numerator, and a whole number set below it, which is its denominator.

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21) Calculation with *fractions* was already done in antiquity. Indeed, the oldest mathematical manual, the *Rhind Papyrus* in the British Museum, already contains a peculiar fraction calculation (details in *M. Cantor's* History of Mathematics, Volume I), in which each fraction is written as a sum of different *unit fractions*. The *Greeks* distinguished numerator and denominator in their letter-numeral writing through different stroking of the letters, but preferred unit fractions. The *Romans* sought to represent fractions as multiples of $\frac{1}{12}$, $\frac{1}{24}$, etc. up to $\frac{1}{288}$, in connection with their coin division. The *Indians* and *Arabs* knew unit fractions and derived fractions, but preferred, just like the ancient *Babylonians* and, following them, the *Greek astronomers*, *sexagesimal fractions* (cf. note 24). The *fraction line* and today's writing of fractions comes from *Leonardo of Pisa* (called *Fibonacci*), whose liber abaci (around 1220) became the source for the arithmetic books of the next centuries.

22) That the introduction of negative and fractional numbers can be dispensed with by algebra, and that those numbers are only symbols that facilitate calculation, *L. Kronecker* expounded in his treatise "On the Concept of Number" (J. f. Math. 101, 1887). According to *Kronecker*, therefore, the extensions of the number concept serve only for what *E. Mach* calls "economy of science". (Cf. *E. Mach*, Mechanics, Leipzig 1883; Popular Scientific Lectures, Leipzig 1896; The Principles of Heat Theory, Leipzig 1896, p. 391 ff.)

To compare fractions, one brings them to a common denominator through application of formula VI in No. 9 and calls a fraction equal to another, greater or smaller[19], than the other, when its numerator is *equal* to the numerator of the other fraction, *greater* or *smaller* than its numerator. One calls a fraction $\frac{a}{b}$ greater or smaller than the whole number c, depending on whether $a > b \cdot c$ or $a < b \cdot c$. A fraction that is smaller than 1 is called *proper*, a fraction that is greater than 1 *improper*. Through application of the considerations in No. 5 and No. 6 to fractions, one arrives at the concepts of negative fraction, positive fraction, relative fraction and absolute value[20] of a relative fraction. Each of the numbers defined so far is thus zero or positive-whole or negative-whole or positive-fractional or negative-fractional. One combines all numbers that have one of these characteristics, thus all numbers defined so far, through the word *"rational"*[23], in contrast to the later defined irrational numbers (cf. IA 3). The rules for how rational numbers are to be connected through addition, subtraction, multiplication and division follow from the formulas established in the earlier paragraphs.

According to formula VII in No. 9, a fraction whose numerator and denominator have a common whole-number divisor can be set equal to that fraction which arises when one divides numerator and denominator by this divisor. This procedure is called *reducing* the fraction. The first-degree analogue to reducing fractions is the reduction of the minuend and subtrahend of a difference by one and the same whole number. While however in this procedure any arbitrary whole number can be achieved as minuend and as subtrahend from any arbitrary difference of whole numbers, through reducing any arbitrary fraction not any arbitrary whole number can be achieved as numerator

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23) The distinction between rational and irrational quantities appears among the Greeks in geometric form already before *Euclid* (around 300 BC), first probably with *Pythagoras* (around 500), who recognized that the hypotenuse of an isosceles right triangle is unspeakable ($\alpha\rho\rho\eta\tau o\varsigma$) when the catheti are speakable. *Plato* (429-348) recognized the irrationality in the diagonal of the square over five (Plato's Republic, VII 546). Even more extensively *Euclid* treated the irrational ($\alpha\lambda o\gamma o\nu$) in the 10th book of his "Elements", and indeed in geometric form, distinguishing whether two lines are commensurable or incommensurable ($\alpha\sigma\upsilon\mu\mu\epsilon\tau\rho o\varsigma$). *Archimedes* (287-212) in his calculation of the number π enclosed the square root of three and of other numbers in very close rational bounds. About the irrational in modern times cf. here IA 3.

or denominator. One must be content to reduce until numerator and denominator no longer have a common divisor. If the numerator of a fraction is a divisor of the denominator, then through reduction a fraction can be achieved whose numerator is 1. Such fractions with numerator 1 are called *unit fractions*[21)]. Any arbitrary fraction can be conceived as the product of its numerator with its unit fraction, i.e., with that fraction which has the same denominator. When $a > b$ and a and b are whole numbers, then $a = m \cdot b + v$ can be set where m is a quite definite whole number and $v < b$, from which follows that every fraction is greater than a whole number by a proper fraction, or that every rational number can be enclosed[19)] between two bounds that are whole numbers, differ by 1, and of which one is greater, the other smaller[19)], than the rational number.

Through continuation of the place-value principle, on which our numeral writing is based, to the right one arrives at *decimal fractions*[24)], i.e. fractions of which only the numerator needs to be written because the denominator is ten or hundred or thousand etc. Which of these numbers is meant as denominator is indicated by the position of a comma[24)]. (Cf. Numerical Calculation in IF.)

The introduction of relative numbers transforms every subtraction into an addition, namely through reversal of a sign. Similarly, the introduction of fractional numbers transforms every division into a multiplication. If one understands by *reciprocal value* of a fraction the fraction whose numerator and denominator are denominator and numerator of the original fraction, then one recognizes that the result of division by a fraction agrees with the result of multiplication with its reciprocal value.

The arithmetic which comprises only the operations of first and second degree concludes, as follows from the above, with the following two final results:

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24) *Decimal fractions* arose during the 16th century. *Johann Kepler* (1571-1630) introduced the decimal comma. The principle underlying decimal fraction notation was already used in antiquity with *sexagesimal fractions*. In these one lets multiples of $\frac{1}{60}$ and then of $\frac{1}{3600}$ follow the wholes. That these are of *Babylonian* origin has become undoubted through the discovery of a sexagesimal place-value numeral writing (with 59 different numerals, but without a sign for nothing) used by Babylonian astronomers. The Greek astronomers too (*Ptolemy*, around 150 AD) calculated with sexagesimal fractions. For example, *Ptolemy* set $\pi = 3 \mathbin{..} 8 \mathbin{..} 30 = 3 + \frac{8}{60} + \frac{30}{3600}$. Our sixty-division of the hour and degree, as well as the expressions *minute* (pars *minuta* prima) and *second* (pars minuta *secunda*) are remnants of the old sexagesimal fractions.

1) When general number symbols (letters) are connected in any way through operations of first and second degree, the result can always be represented as a quotient whose dividend and divisor is an algebraic sum of products;

2) When any number of rational numbers are connected in any way through operations of first and second degree, the result is always a rational number, provided division by zero does not occur.

New extensions of the number domain become necessary only when one connects the numbers defined so far through operations of third degree. (Cf. No. 11 as well as IA 3 and IA 4.)

11. The Three Operations of Third Degree. In No. 7 the definition of product is extended such that it may contain any number of factors. The case where these are all equal leads to the direct operation of third degree, *exponentiation*[25)]. To raise a number a (passive) to the power of a number p (active) thus means to form a product of p factors, each of which is a. The number a, *which* is set as factor of a product, is called *base*, the number p, which indicates *how often* the other number a is to be set as factor of a product, is called *exponent*[25)]. The result of exponentiation, which one writes a^p and reads "a to the power p", is called power. Insofar as one conceives a as product of one factor, one sets $a^1 = a$. Through the definition of exponentiation,

25) Powers with exponents 1 to 6 were already designated in abbreviated form by *Diophantus.* He calls the second power $\delta\upsilon\nu\alpha\mu\iota\varsigma$, a word to which through the Latin translation potentia the word "power" is to be traced back. In the 14th to 16th centuries there are already traces of calculation with powers and roots, as with *Oresme* (†1382), *Adam Riese* (†1559), *Christoff Rudolf* (around 1530) and notably with *Michael Stifel* in his Arithmetica integra (Nuremberg 1544). More details about this in *M. Cantor's* History of Mathematics. But only the invention of logarithms at the beginning of the 17th century procured full citizenship in arithmetic for operations of third degree. The deeper understanding of their connection however belongs to an even later time. The inventors of logarithms are *Jost Bürgi* (†1632) and *John Napier* (†1617). *Kepler* (†1630) also has great merits regarding the spread of knowledge of logarithms, *Henry Briggs* (†1630) introduced base ten and published a collection of logarithms of this base. (Cf. Numerical Calculation, IF.) The word logarithm ($\lambda o\gamma o\nu\ \alpha\rho\iota\theta\mu o\varsigma$ - number of a ratio) is explained by the fact that one sought to relate two ratios by raising one to a power to obtain the other. Thus one called 8 to 27 the third ratio of 2 to 3. Also the expression "numerus rationem exponens" occurs for logarithm, from which perhaps the word "exponent" derives.

the following laws of exponentiation arise from the laws of multiplication:

$$\left.\begin{array}{ll} \text{I.} & a^p \cdot a^q = a^{p+q}; \\ \text{IIa.} & a^p : a^q = a^{p-q}, \text{ if } p > q \text{ is;} \\ \text{IIb.} & a^p : a^q = 1, \text{ if } p = q \text{ is;} \\ \text{IIc.} & a^p : a^q = 1 : a^{q-p}, \text{ if } p < q \text{ is;} \end{array}\right\} \begin{array}{l}\text{(Distribution} \\ \text{formulas with} \\ \text{same base.)}\end{array}$$

$$\left.\begin{array}{ll} \text{III.} & a^q \cdot b^q = (a \cdot b)^q; \\ \text{IV.} & a^q : b^q = (a : b)^q; \end{array}\right\} \begin{array}{l}\text{(Distribution formulas with} \\ \text{same exponent.)}\end{array}$$

$$\text{V. } (a^p)^q = a^{p \cdot q} = (a^q)^p; \quad \text{(Association formula.)}$$

According to the definition of exponentiation, the base can be any number; the exponent however must be a result of counting, thus a positive whole number. For "positive-whole" one also says *"natural"*; accordingly a power with such an exponent is called a *"natural"* one.

Due to a geometric application, powers with exponent 2 are also called squares, with exponent 3 also cubes.

If the base is a sum, a difference, a product, a quotient or a power, it is to be enclosed in parentheses. On the other hand, the higher position of the exponent makes parentheses around it superfluous.

According to the definition of exponentiation, a^0 and a^{-n}, where -n is a negative whole number, are initially meaningless signs. Also products whose multiplier is zero or negative were, according to the original definition of multiplication, meaningless signs. Yet such signs received meaning, according to the principle of permanence, through the desire to be able to multiply with such differences just as with differences that represent a positive number. In the same way one proceeds with the power forms

$$a^0 \quad and \quad a^{-n}.$$

One thus sets $a^0 = a^{p-p}$, lifts the restriction $p > q$ in formula IIa, and applies the same, read backwards. Then comes:

$$a^0 = a^{p-p} = a^p : a^p = 1$$

Similarly one sets $a^{-n} = a^{p-(p+n)}$, lifts the restriction $p > q$ in formula IIa, finds thereby $a^p : a^{p+n}$, now applies formula IIc and obtains $1 : a^n$.

The extension of the concept of exponentiation to the case where the exponent is a fractional number can only be accomplished after the laws of radicalization, one of the two inversions of exponentiation, are established.

Since in exponentiation the commutation law does not hold, because in general b^n is not equal to n^b, the two inversions of exponentiation must be *distinguished* not only logically but also *arithmetically*. The operation which in $b^n = a$ considers the base, thus the passive number, as sought, but a and n as given, is called *radicalization*[25]; the operation which in $b^n = a$ considers the exponent, thus the active number, as sought, but b and a as given, is called *logarithmization*[25].

"nth root of a", written: $\sqrt[n]{a}$, [25] is thus the number which, raised to the nth power, yields a. Accordingly, $(\sqrt[n]{a})^n = a$ is the definition formula of radicalization. The number which was originally power is called *radicand* in radicalization, the number which was power exponent is called *root exponent* and the number which was base is called *root*. Through the definition of radicalization arise from the laws of exponentiation the following laws of radicalization:

$$\left.\begin{array}{l} \text{I. } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}\ ; \\ \text{II. } \sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{a : b}\ ; \end{array}\right\} \text{(Distributive formulas.)}$$

$$\left.\begin{array}{l} \text{III. } \sqrt[p]{a^q} = (\sqrt[p]{a})^q\ ; \\ \text{IV. } \sqrt[p]{\sqrt[q]{a}} = \sqrt[pq]{a} = \sqrt[q]{\sqrt[p]{a}}\ ; \end{array}\right\} \text{(Associative formulas.)}$$

$$\text{V. } \sqrt[np]{a^{nq}} = \sqrt[p]{a^q}\ .$$

Through radicalization, powers with *fractional exponents* can be defined. Since $\frac{p}{q} \cdot q = p$ is the definition formula of the fractional number $\frac{p}{q}$, and since $a^{\frac{p}{q} \cdot q}$ equals $(a^{\frac{p}{q}})^q$, under $a^{\frac{p}{q}}$ is to be understood a number which, raised to the qth power, yields a^p, and this is $\sqrt[q]{a^p}$.

Similarly one recognizes that $a^{-\frac{p}{q}} = 1 : \sqrt[q]{a^p}$. Furthermore formula V shows how $\sqrt[n]{a}$, where n is a positive or negative fractional number, can be represented as a power whose base is a and whose exponent is rational. Every root can thus be represented as a power whose base is the radicand of the root, just as every quotient can be represented as a product whose multiplicand is the dividend of the quotient.

When a is any rational number and n is a whole number, then a^n represents a rational number. But when with rational a the number n is indeed rational but not whole-numbered, then there exists only a rational number that may be set equal to a^n when a is the qth power of a rational number, where q is the denominator of the number n. In all other cases a^n, where n is not whole-numbered, represents a sign connection which still needs to be given meaning. (Cf. IA 3 and IA 4)

"Logarithm of a to base b"[25)], written: $\log_b a$, is the exponent with which b must be raised to yield a. Accordingly:

$$b^{\log_b a} = a$$

is the definition formula of *logarithmization*. The number which was originally power is called *logarithmand* in logarithmization, the number which was base is called *logarithm base*, and the number which was exponent is called *logarithm*. Through the definition of logarithmization arise from the laws of exponentiation the following laws of logarithmization:

I. $\log_b(p \cdot q) = \log_b p + \log_b q$;

II. $\log_b(p : q) = \log_b p - \log_b q$;

III. $\log_b(p^m) = m \cdot \log_b p$;

IV. $\log_b a = \frac{\log_c a}{\log_b a}$.

When b is any rational number, then $\log_b a$ represents a rational number only when a equals a power whose base is b and whose exponent is a rational number. [This is for example the case when $b = \frac{9}{4}$ and $a = \frac{8}{27}$, then $(\frac{9}{4})^{-\frac{3}{2}} = (\frac{4}{9})^{+\frac{3}{2}} = (\sqrt[2]{\frac{4}{9}})^3 = (\frac{2}{3})^3 = \frac{8}{27}$. Therefore $\log_{\frac{9}{4}} \frac{8}{27}$ equals the rational number $-\frac{3}{2}$.] In all other cases $\log_b a$ represents a sign connection which still needs to be given meaning. (Cf. IA 3 and IA 4.)

In the following table of arithmetic operations, 16 is always considered as passive, 2 as active number.

Table of 7 Operations

Operation	Example	The passive number, here 16, is called:	The active number, here 2, is called:	Result is called:
Addition	$16 + 2 = 18$	Augend (Summand)	Addend (Summand)	Sum
Subtraction	$16 - 2 = 14$	Minuend	Subtrahend	Difference
Multiplication	$16 \cdot 2 = 32$	Multiplicand (Factor)	Multiplier (Factor)	Product
Division	$16 : 2 = 8$	Dividend	Divisor	Quotient
Exponentiation	$16^2 = 256$	Base	Exponent	Power
Radicalization	$\sqrt[2]{16} = 4$	Radicand	Root Exponent	Root
Logarithmization	$\log_2 16 = 4$	Logarithmand	Logarithm Base	Logarithm

How from each of the three direct operations addition, multiplication, exponentiation their two inversions follow, shows the following table:

Degree:	Direct Operation:	Indirect Operation:	Sought is:
I	Addition: $5 + 3 = 8$	Subtraction: $8 - 3 = 5$	Augend
		Subtraction: $8 - 5 = 3$	Addend
II	Multiplication: $5 \cdot 3 = 15$	Division: $15 : 3 = 5$	Multiplicand
		Division: $15 : 5 = 3$	Multiplier
III	Exponentiation: $5^3 = 125$	Radicalization: $\sqrt[3]{125} = 5$	Base
		Logarithmization: $\log_5 125 = 3$	Exponent

In the same way as multiplication arises from addition, exponentiation from multiplication, one could also derive from exponentiation as the direct operation of third degree a direct *operation of fourth degree*[26)], from this one of fifth degree and so on derive.

26) Of treatises that relate to operations of fourth or higher degree, mentioned here are: those by *H. Gerlach* in Zeitschr. f. math. nat. Unterr. Vol. 13, Issue 6, by *F. Wöpcke* in J. f. Mat. 42, by *E. Schulze*

Yet already the definition of a direct operation of fourth degree, while logically justified, is unimportant because already at the third degree the commutation law loses its validity. To arrive at a direct operation of fourth degree, one has to consider a^a as exponent of a, consider the power thus created again as exponent of a and continue so until a is set b times. If one calls the result then $(a;b)$, then $(a;b)$ represents the result of the direct operation of fourth degree. For this holds e.g.: $(a;b)^{(a;c)} = (a;c+1)^{(a;b-1)}$.

in Arch. f. Math. (2) III (1886). *G. Eisenstein* investigated through series expansions the function $x^{y^{\frac{1}{y}}}$ as inversion of $y = (x;\infty)$ in J. f. Math. Vol. 28. The textbooks by *Hankel, Grassmann, H. Scheffler, E. Schröder, O. Schloemilch, Schubert* mention the direct operation of fourth degree without going into it in more detail.

I A 2. COMBINATORICS

BY

E. NETTO

IN GIESSEN

Table of Contents

The monographs are given in No. 1 and No. 35.

1. Combinatorics; Historical Appreciation. Combinatorics has developed neither in its elementary nor in its higher analytical domains as was hoped for in an exuberant manner at the beginning of the century by representatives of the *"combinatorial school"*. Beginnings of combinatorics can be traced far back; as a branch of science it may be considered only from *Bl. Pascal*[1)], *G. W. Leibnitz*[2)], *J. Wallis*[3)], but especially from *Jac. Bernoulli I.* and *A. de Moivre*[4)] onwards. The basic features of the elementary parts have passed into every textbook; the analytical applications recede very much. Thus the more comprehensive monographs all come from earlier times[5)], and more deeply penetrating treatises exist only in small number[6)].

2. Combinatorial Operations. Definitions. Of the infinitely many possible combinatorial operations, three have gained principal validity as equal (despite logical concerns): permutations (P.), combinations[7)] (C.) and variations (V.). We call any arrangement of n elements a complexion (Cp.) of the same. — P. of n elements are called the Cp. which deliver all given elements in all possible sequences. If the elements are different from each other, then there are $n!$, if among them a equal ones of one kind, b equal ones of another kind etc. occur, then there are $n! : (a!\, b! \ldots)$.

C. of n elements to the kth class are all Cp. of k each of those n elements without consideration of the arrangement; if each element may be taken only once, then they are C. without repetition (w/o r.), otherwise with repetition (w/ r.). There are in kth class

$$\text{C. w/o r.} \quad \tfrac{n!}{k!(n-k)!}, \qquad \text{C. w/ r.} \quad \tfrac{(n+k-1)!}{k!(n-1)!}\ .$$

1) *Pascal*, Traité du triangle arithmétique. Paris 1665, written 1664 (Op. posth.).

2) *Leibnitz*, Dissertatio de arte Combinatoria. Lipsiae 1668. Opp. II, T. I, p. 339.

3) *Wallis*, Treatise of algebra. Lond. 1673 and 1685.

4) *Bernoulli*, Ars conjectandi. Basil.1713 (Op. posth.). *Moivre*, Probabilities. Lond.1718.

5) *K. F. Hindenburg*, Nov. Syst. Permutationum etc. Lips. 1781. — *J. Weingärtner*, Lehrb. d. combinator. Analysis. Leipz. 1800. — *Knr. Stahl*, Grundrifs d. Kombin.-Lehre. Jena 1800. — *Bernh. Thibaut*, Grundr. d. allgem. Arithm. od. Analysis. Götting. 1809. — *Chr. Kramp*, Elem. d'Arithm. Cologne 1808. — *Fr. W. Spehr*, Lehrbegr. d. rein. Kombin.-Lehre. Braunschw. 1824. — *A. v. Ettingshausen*, D. kombinat. Analysis. Wien 1826. — *L. Öttinger*, Lehre v. d. Kombinat. Freiburg 1837.

6) *Hessel*, Arch. f. Math. 7 (1845), p. 395. — *Öttinger*, ib. 15 (1850), p. 241.

7) *Hindenburg* also writes "Komplexionen"; these break down into proper combinations, conternations, conquaternations etc.

V. arise when one permutes the elements in the C. There are in kth class

$$\text{V. w/o r.} \quad \tfrac{n!}{(n-k)!}, \qquad \text{V. w/ r.} \quad n^k \ .$$

3. Inversion; Transposition. Since the elements are valid only insofar as they are identical or different, their designation can be made arbitrarily e.g. through digits 1, 2, 3, ... or through letters a, b, c, ... Thereby a new agent enters the consideration, which can now be used in different directions. A Cp. is called *well-ordered* if always the higher digit (the alphabetically later letter) stands behind the lower (the earlier). Any deviation from this is called *inversion*[8)]. For counting the number of inversions in extensive Cp., *P. Gordan* gives a rule[9)]. A Cp. of different elements belongs to the *first* or *second* class (is *even* or *odd*), depending on whether it contains an even or odd number of inversions[10)]. Through a *transposition*, i.e. rearrangement of two elements, the class is changed[11)].

4. Permutations with Restricted Position Occupation. P. with *restricted position* occupation are those where either a prescribed number of elements maintain their positions, or where certain positions may only be occupied by certain elements.

L. Euler[12)] introduces a function $f(n)$ which gives the number of P. where each element changes its original position. Connected with this is an $F(n, m)$ which indicates in how many P. of n elements exactly m keep their position[13)]. It is:

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8) *G. Cramer*, Introduct. à l'Analyse des lignes courbes (1750); Genève. Appendice p. 658. — *T. P. Kirkman*, Cambr. a. Dubl. J. 2 (1847), p. 191.

9) Vorles. üb. Invarianten-Theor., herausgeg. v. *G. Kerschensteiner* (1885), Leipz. I, p. 2.

10) *E. Bezout*, Mém. Paris (1764), p. 292. — *A. L. Cauchy*, J. d. l'Éc. pol. cah. 10 (1815), p. 41. — *K. G. Jacobi*, Werke 3, p. 359 = J. f. M. 22 (1841), p. 285.

11) *P. S. Laplace*, Mém. Paris (1772), p. 294.

12) Mém. Pétersb. 3 (1809), p. 57. — *Öttinger*, Lehre v. d. Kombin. Freiburg 1837. — *M. Cantor*, Z. f. Math. 2 (1857), p. 65. — *R. Baltzer*, Leipz. Ber. (1873), p. 534. — *S. Kantor*, Z. f. Math. 15 (1870), p. 361. — *A. Cayley*, Edinb. Proceed. 9 (1878), p. 338 u. 388.

13) *M. Cantor*, Z. f. Math. 2 (1857), p. 410. — *J. J. Weyrauch*, J. f. M. 74 (1872), p. 273

$$f(n) = nf(n-1) + (-1)^n = (n-1)[f(n-1) + f(n-2)] \;;$$
$$f(0) = 1,\ f(1) = 0 \;.$$
$$F(n,m) = \binom{n}{m} f(n-m) \;.$$

Another restriction of position occupation is that certain positions may only be occupied by certain elements[14], e.g. the even positions only by even numbers[15]; or, if equal elements occur, that not two such follow each other[16].

Restrictions of position occupation also lie therein that the P. themselves enter into different arrangements, e.g. such that n P. of n elements should be placed under each other so that in no column do equal elements occur[17] (Latin square).

5. Related Permutations. *Related permutations* according to *H. A. Rothe*[18] are two P. when the position order and the position element (as number) of one are exchanged against those of the other; it will be to determine how many P. are related to themselves[19]. *P. A. Mac-Mahon* gives extensions of these concepts[19].

6. Sequences. *D. André* has introduced the concept of *sequence* for P. and investigated it in a whole series of treatises[20]. Consecutive number elements of a P. form a sequence if each following is larger (smaller) than the preceding. Every P. breaks down into sequences. The number of occurring sequences determines the "type" of the P. It is investigated how many P. belong to a certain type. It shows that the number of P. with even sequence number equals that of P. with odd sequence number. Geometric representations are given, etc.

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14) *C. W. Baur*, Z. f. Math. 2 (1857), p. 267.

15) *A. Laisant*, C. R. 112 (1891), p. 1047.

16) *O. Terquem*, J. d. Math. 4 (1839), p. 177. — Further restrictions in position occupation investigated by *Th. Muir*, Edinb. Proceed. 10 (1881), p. 187. *A. Holtze*, Arch. f. Math. (2), 11 (1892), p. 284.

17) *A. Cayley*, Messenger (2), 19 (1890), p. 135. *M. Frolov*, J. m. spec. (3), 4 (1890), p. 8 u. 25. *J. Bourget*, J. de Math. (3), 8 (1882), p. 413. *P. Seelhof*, Arch. f. Math. (2), 1 (1884), p. 97.

18) Hindenb. Arch. f. M. (1795).

19) *P. A. Mac Mahon*, Messeng. (2), 24 (1894), p. 69.

20) C. R. 97 (1883), p. 1356; 115 (1892), p. 872; 118 (1894), p. 575. [*G. Darboux* Rapport; C. R. 118 (1894), p. 1026]. Soc. m. d. Fr. 21 (1893), p. 131; Ann. Éc. Norm. (3), 1 (1884), p. 121.

7. Application to Questions of Arithmetic. Connected with the P.-theorems are the questions of in how many ways one can arithmetically carry out sums or products of given elements, which are different or partly equal to each other, with or without rearrangement of these elements[21)].

8. Combinations to Specific Sum or Specific Product. Also with C. and V. the individual Cp. can be subject to restrictions. The most well-known and important consists in that the C. and V. of natural numbers are considered whose elements w/ r. or w/o r. have a *specific sum*, which is called the *weight* of the Cp. Their significance appears in invariant theory. *L. Euler* was the first who treated this question (Introd. in Anal. Lausanne 1748, § 299 ff.; Comm. Acad. Petr. 3 [1753], p. 159), who gave development coefficients of certain products for the number of these C. and thereby arrived at relations between C. w/ r. and C. w/o r. These questions were later pursued further in many ways[22)], and their answering particularly promoted by *Cayley*[23)] and *J. J. Sylvester*[24)] through establishment of tables and geometric representations. *Mac-Mahon* has carried these investigations further, which were then also extended to the decomposition of number pairs. We must content ourselves with these remarks, as the further applications to symmetric functions and invariant theory are no longer of combinatorial nature.

In similar manner *Möbius* has treated the C. where the elements of the Cp. have a *specific product*[25)]. They are ordered according to their classes, and the numbers of associated C. brought into connection with each other. Such relations also appear for the case that conditions are imposed on the Cp., e.g. that with the prescribed product $a^\alpha b^\beta$ in each Cp. each element has at least one factor b.

Ettingshausen has furthermore gone into treating each Cp. as a product

21) *E. Ch. Catalan*, J. d. Math. 6 (1874), p. 74. *E. Schröder*, Z. f. Math. 15 (1870), p. 361.

22) *M. Stern*, J. f. M. 21 (1840), p. 91 u. p. 177, ibid. 95 (1883), p. 102. *C. Wasmund*, Arch. f. Math. 21 (1853), p. 228, ibid. 34 (1860), p. 440.

23) Lond. Transact. 145 (1855), p. 127, ibid. 148 (1858), p. 47. Amer. J. 6 (1883), p. 63.

24) Quart. J. 1 (1855—57), p. 81 u. p. 141. Amer. J. 5 (1882), p. 251. C. R. 96 (1883). Vgl. auch *Mac Mahon*, Lond. Trans. 184 (1894), p. 835, sowie den Bericht über Combin. Analysis: Lond. M. S. Pr. 28 (1897), p. 5.

25) J. f. M. 9 (1832); 105.

to consider, and to sum all such products belonging to a C.-class[26]. Further the classes are divided according to specific moduli and numerical relations between them are determined[27].

And not only to the C. themselves do such investigations relate, but also to Cp. that are derived in various ways from the ordinary C. For example, to the first element of each Cp. 0 is added, to the second 1, ... to the nth $(n-1)$. Thus products arise between whose sums again remarkable relations can be specified[28]. Cf. also *Th. B. Sprague*, Edinb. Proc. 37 (1893), p. 399.

9. Combinations with Restricted Position Occupation. The path of investigation which relates to *restricted position occupation* branches here. First, similar to P., requirements are made that certain elements occur in a prescribed number of times[29], or that a maximum number for their occurrence is given[30].

10. Triple Systems. Another direction has proved particularly important for geometry, probability calculation, for algebra. Independent of each other, *T. P. Kirkman*[31] and *J. Steiner*[32] posed almost identical tasks; the First his "schoolgirl problem": Fifteen girls are taken out 35 times in rows of 3, so that not 2 go together twice; the Last the following: From N elements C. of the 3rd class (triples) should be selected so that each pair occurs once and only once; further C. of the 4th class (quadruples) so that in them each triple that did not occur among the previous ones occurs once and only once etc. *Cayley*[33] and *R. R. Anstice*[34] treated individual cases of the *"triple systems"*. A general rule for the formation of such systems, which require $N = 6n+1$, $6n+3$, was given by *M. Reiss*[35].

26) Die kombinatorische Analysis. Wien (1826).

27) *A. A. Cournot*, Bull. sci. m. (1829). *Ch. Ramus*, J. f. M. 11 (1834), p. 353.

28) *H. F. Scherk*, J. f. M. 3 (1828), p. 96; J. f. M. 4 (1829), p. 226.

29) *Ad. Weiss*, J. f. M. 34 (1847), p. 255.

30) *Öttinger*, Arch. f. M. 15 (1850), p. 241. *Baur*, Z. f. M. 2 (1857), p. 267. *Scherk*, Math. Abhandl. Berlin (1825), p. 67. *Andre*, Ann. Éc. norm. (2), 5 (1876), p. 155.

31) Cambr. a. Dubl. m. J. 7 (1852), p. 527 u. 8 (1853), p. 38; vgl. *T. Clausen*, Arch. f. M. 21 (1853), p. 93.

32) J. f. M. 45 (1853), p. 181 — Werke II, p. 435.

33) Phil. Mag. (3), 37 (1850), p. 50. — Ibid. (4), 25 (1862), p. 59.

34) Cambr. a. Dubl. m. J. 7 (1852), p. 279 u. 8 (1853), p. 149.

35) J. f. M. 56 (1859), p. 326.

In recent times, analytical representations were derived for prime numbers $6n+1$, further for numbers $6n+3$, if this is three times a prime number of the form $6n+5$, etc. Finally, analytical formation rules were given from which the construction follows for every possible N[36]. For $N = 13$ two triple systems are known[37]. The further parts of *Steiner*'s task have not yet been tackled. — *Cayley* draws attention to similar tasks[38]. Cf. IA6 No. 13 Note 67.

11. Extension of the Concept of Variation. The concept of V. has been extended in the direction that m rows of n elements each are given, and as V. mth class are then designated the Cp. which contain one element from each of the m rows. If the same position index of the elements may occur only once, then they are V. w/o r., otherwise V. w/ r.

12. Formulas. Between the various numbers discussed so far for P., C. and V. there exists an extraordinarily large number of connecting formulas. Here it must suffice to point to the main writers who have dealt with the derivation or compilation[39].

13. Binomial Coefficients. We have already mentioned that the proofs of the *binomial* and the *polynomial* theorem for whole positive exponents n

$$(z_1 + z_2 + \ldots + z_Q)^n = \sum \tfrac{n!}{x_1! x_2! \ldots x_Q!} z_1^{x_1} z_2^{x_2} \ldots z_Q^{x_Q}$$
$$(x_1 + x_2 + \ldots + x_Q = n)$$

are applications of combinatorial formulas. The binomial formula is found first in *H. Briggs*[40], then in *J. Newton*[41];

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36) *E. Netto*, Substit.-Theorie § 192ff. Leipz. (1882). Math. Ann. 42 (1892), p. 143. *E. H. Moore*, Math. Ann. 43 (1893), p. 271; N. Y. Bull. (2), 4 (1897), p. 11. *L. Heffter*, Math. Ann. 49 (1897), p. 101. *J. de Vries*, Rend. Palermo 8 (1894).

37) *K. Zulauf*, Dissert. Giessen (1897).

38) Phil. Mag. 30 (1865), p. 370.

39) *Hindenburg*, Nov. Syst. Permutationum, Combin. etc. primae lineae. Lips. (1781). — D. polynom. Lehrs., d. wichtigste Theorem d. ganzen Analysis, neu bearb. v. *J. N. Tetens*, *G. S. Klügel*, *A. Krauss*, *J. F. Pfaff* u. *Hindenburg*, herausgeg. v. *Hindenburg*. Leipz. 1796. *Hindenburg*, Infinitonomii dignitatum historia, leges etc. Vgl. auch *J. A. Grunert*, Arch. f. M. 1 (1841), p. 67; *Brianchon*, J. d. l'Éc. Polyt. t. 15 (1837), p. 159.

40) Arithmetica Logarithmica. London (1620).

41) Briefe an *Oldenburg* (1676) 13. Juni u. 24. Oktober.

The coefficients of the binomial expansion ($n = 2$), the *binomial coefficients* in their arrangement etc. as "arithmetic triangle"

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ . & . & & . & & . & . \end{array}$$

already appear in *Bl. Pascal*[42].

As extensions of the binomial theorem are to be mentioned, first the expansion of $a(a+b)(a+2b)...(a+nb)$ according to powers of a;[43] further the expansion $(x+a)^n = x^n + c_1(x+t_1)^{n-1} + c_2(x+t_1+t_2)^{n-2} + \dots$,
where the t_α are arbitrary quantities[44].

As extension of the binomial coefficients, expressions of the form

$$[n(n+k)(n+2k)...(n+(p-1)k)] : p!$$

have been introduced[45], whose numerators as faculties have been thoroughly investigated[46]. The analytical treatment does not belong here.

Between the binomial coefficients there exists an innumerable number of relations, whose classification has been initiated by *J. G. Hagen*[47]. Cf. also the "figurate numbers" of the ancients.

14. Applications. As already mentioned in No. 1, most applications of analytical nature of combinatorics offer only historical interest anymore. We limit ourselves to indicating the most important branches which combinatorics had undertaken to support. In first place belongs here probability calculation, in whose elementary parts combinatorial questions occur continuously, and from which conversely combinatorics has received many stimulations.

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42) Traité du triangle arithmet. Paris (1665) posth.; and earlier in *M. Stifel*, Arithm. integra. Norimb. (1544), p. 44.

43) *Pascal* "productum continuorum".

44) *N. H. Abel*, J. f. M. 1 (1826), p. 159 gives a special case; generally *A. v. Burg*, J. f. M. 1 (1826), p. 367. — *Cayley*, Phil. Mag. 6 (1853), p. 185 = Werke II, 102.

45) *Bl. Pascal*, see above.

46) *L. Euler*, Calc. diff. II. c. 16 u. 17. Berl. (1755). *Öttinger*, J. f. M. 33 (1846), p. 1, 117, 226, 329; further 35 (1847), p. 13 u. 38 (1849), p. 162, 216; finally 44 (1852), p. 26 u. 147, where historical information is also listed.

47) Synopsis. Berlin (1891), p. 64ff. Cf. also *G. Eisenstein*, Brief an *M. A. Stern*, Z. f. Math. 40 (1895), p. 198 of the hist. section.

It furthermore connects to the theory of series, formally determines the products, powers, quotients of series; the result of substituting a series for the variable z in a series that progresses according to powers of z; the formal inversion of series; the rationalization of such in which irrationalities enter; the general terms of recurring series; the logarithms of series and series of logarithms etc. Similarly it gives the form for the higher differentials of more complicated functions etc. For its purposes it had devised a complete notation system, which now is certainly entirely outdated[48)].

The entire theory of finite discrete groups (IA 6) can be directly connected to combinatorics.

Yet a second application, directed at solving linear equations, has developed in a surprising way. It has become the *theory of determinants*.

15. Determinants. Definition of the Concept. Let n^2 quantities a_{ik} $(i, k = 1, 2, ...n)$ be given; form all $n!$ products $a_{1i_1}a_{2i_2}...a_{ni_n}$ in which $i_1, i_2, ...i_n$ means a P. of $1, 2, ...n$ and give each the sign + or -, depending on whether this P. belongs to the first or second class. The sum of these $n!$ summands is the determinant of nth degree[49)]. *A. L. Cauchy*[50)] defines it also such that he develops the alternating product $\prod(a_i - a_k)$, $(i = 1, 2, ...n;\ k = i+1, ...n)$, and writes the exponents as second lower indices. *E. Schering*[51)] gives a geometric and an analytical explanation, *Kronecker* laid a function-theoretical one as foundation in his lectures.

The most common notations are[52)]

$$\begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix} = \sum \pm a_{11}a_{22}...a_{nn} = \|a_{h1}a_{h2}...a_{hn}\| = \|a_{hk}\| \ ;$$
$$(h, k = 1, 2, ...n) \ .$$

48) Cf. *Hindenburg*, Nov. Syst. etc. Leipz. (1781).

49) *Jacobi*, J. f. M. 22 (1841), p. 285 = Werke III, p. 355.

50) Analyse algébrique. Paris (1821).

51) Gött. Abh. 22 (1879), p. 102.

52) The third notation is often used by *L. Kronecker*; the last first by *St. Smith*, Brit. Ass. Rep. (1862) p. 504. As newly introduced *L. Kronecker* then has it, J. f. M. 68 (1868), p. 273. On further notations cf. *Cayley*, Phil. Mag. 21 (1861), p. 180. *Nanson*, Lond. phil. Mag. (5) 44 (1897), p. 396. *W. Schrader*, Determinanten. Halle 1887.

Historically it is to be noted that determinants were invented by *Leibnitz*[53)] and later independently by *Cramer*[54)] and were initially used for solving a system of linear equations. The first detailed theoretical expositions come from *J. Binet* and *Cauchy*; the theory of D. was generally introduced by *Jacobi*[55)]. Extensive literature references can be found in *Muir*[56)] continued from the beginning of the theory until 1885; Historical information also in *S. Günther*, Determinantentheorie, Erlangen (1875), in *Baltzer*, Determinanten, Leipzig (1881) and in *G. Salmon*, Modern higher algebra, Note I in *Baltzer*.

16. Definitions. The quantities a_{ik} are called the *elements* (El.) of the D.; the first (second) index gives the ordinal number of the row (column). The quantities a_{ii} form the *main diagonal*; the $a_{i,n+1-i}$ form the *secondary diagonal.* The term $a_{11}a_{22}...a_{nn}$ of D. is called its *principal term.* If one selects m values of the first and m of the second index from $1, 2, ...n$, the corresponding El. form a *subdeterminant* (Subd.) of mth degree[57)]. If the El. of its main diagonal are also El. of that of D., then the Subd. is called a *principal subdeterminant.* If the product of the principal terms of two Subd. is a term of D., then the Subd. are called *adjunct*, or also *complementary* Subd.

If $a_{ik} = a_{ki}$, then D. is called a *symmetric* D.

If $a_{ik} = a_{i+k-2}$, then D. is called a *recurrent* (one-sided, *orthosymmetric*) D. It is symmetric[58)].

If $a_{ik} = a_{i+1,k+1}$, where the indices are reduced mod n, then D. is called a *circulant*[59)], (also *negative-orthosymmetric* D.).

If $a_{ik} + a_{ki} = 0$, $a_{ii} = 0$, then D. is called a *half-symmetric.* If $a_{ik} + a_{ki} = 0$, for $i \neq k$, then D. is called a *skew*[60)].

53) Lettres à l'Hospital (1693). — Acta Erudit. Leipz. (1700), p. 206.

54) Introd. à l'anal. des courbes algebr. (1750). Genève. Appendice p. 656.

55) J. de l'Éc. Polytechn. Cah. 16 (1812), p. 280 u. Cah. 17 (1812), p. 29. — *Jacobi*, J. f. M. 22 (1841), p. 285 = Werke III, p. 355.

56) Quart. J. 18 (1882), p. 110; ibid. 21 (1886), p. 299. Edinb. Proc. 13 (1886), p. 547. In Phil. Mag. (5), 18 (1884), p. 416 *Muir* draws attention to *Ferd. Schweins* as a forgotten discoverer, "Theorie der Differenzen u. Differentiale" (1825). Heidelberg. Cap. IV, p. 317.

57) Also called Unterdeterminante, Partialdeterminante, Minor.

58) *H. Hankel*, Dissert. Leipz. (1861) Göttingen. — "Recurrierend" according to *G. Frobenius*; Berl. Ber. (1894), p. 253.

59) *Th. Muir*, Quart. J. (1882), p. 166. *Hankel* l. c.

60) *Jacobi*, J.f.M.2(1857), 354; ibid.29(1845), p.236. *Cayley*, J.f.M.38(1849), p.93; ibid.32(1846), p.119; ibid.50(1855), p.299. *Cayley* designates the half-symmetric D. as "skew-symmetric".

If $a_{ik} = a_{n+1-i,n+1-k}$, then D. is called a *centrosymmetric*.

17. Number Problems Regarding the Terms. The number of terms of a D. of nth degree is $n!$. Further questions connect to this: how many of the terms contain a prescribed number of El. of the main diagonal[61)]? How many terms has a D. whose main diagonal contains k El. 0 [62)]? How many different terms are there in symmetric, how many in half-symmetric D.[63)]?

18. Elementary Properties. The following properties of elementary nature show immediately: one can, without changing the value of D., make every αth R. into the αth C.[64)]. When transposing two parallel rows, the sign of D. changes; consequently a D. with two identical parallel rows equals zero[65)]. The D. can be represented as a linear, homogeneous function of the El. of each row[66)]. From this follows that one can pull out a common factor of all El. of a row before D. The degree of a D. can be increased by suitable *bordering*, i.e. addition of new R. and C. If two D. agree in $(n-1)$ rows, then they can be summed to a D. with the same $(n-1)$ rows. If $a_{ik} = b_{ik} + c_{ik}$ $(k = 1, ...n)$, then conversely the D. breaks down into individual summands. The linear, homogeneous representation delivers the partial derivative of D. with respect to a_{ik}. If we denote it with a'_{ki}, then follows $\sum_\lambda a_{i\lambda} a'_{\lambda k} = \bar{c}_{ik} D$ (i.e. $= D$ if $i = k$, otherwise $= 0$)[67)]. Like the a', so can also the higher Subd. be represented as partial derivatives of higher order[68)].

The D. does not change its value when to a row a parallel row is added or from it subtracted[69)].

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61) *Baltzer*, Determin. 4. Aufl. Leipz. 1875, p. 39. Leipz. Ber. (1873), p. 534. *C. J. Monro*, Messeng. (2), 2 (1872), p. 38.

62) *N. v. Szütz*, Math. Ann. 33 (1889), p. 477.

63) *J. J. Weyrauch*, J. f. M. 74 (1872), p. 273. *Cayley*, Monthly Not. of Astron. Soc. 34 (1873—74), p. 303 u. p.335. *G. Salmon*, Modern Algebra. Dublin (1885), p. 45.

64) *J. C. Becker*, Z. f. M. 16 (1871), p. 326. *Gordan*, Vorles. üb. Invar.-Th. (1885), p. 21. — The D. becomes "turned".

65) *Ch. A. Vandermonde*, Par. Acad. (1772), 2° part., p. 518, 522.

66) *Cramer*, l.c. *J. L. Lagrange*, Berl. Mem. (1773), p. 149, 153.

67) ϵ_{ik} introduced by *Kronecker*, J. f. Math. 68 (1868), p. 273. If one sets $a'_{ik}/D = \alpha_{ik}$, then *Kronecker* calls the systems a_{ik}, α_{ik} reciprocal systems.

68) *Jacobi*, J. f. Math. 22 (1841), p. 285, §10 = Werke III, p. 365.

69) *Jacobi*, J. f. Math. 22 (1841), p. 371 = Werke III, p. 452.

With help of the stated theorems, the calculation of D. with numerical El. can often be shortened, just as that of D. whose El. follow analytical laws. There exists an almost innumerable number of results; highlighting even just the most important would exceed the framework of this presentation[70].

19. Laplace's and Other Decomposition Theorems. From *P. S. Laplace*[71] comes an important theorem about the development of D. according to products of adjunct Subd. From the m first R. (C.) all possible Subd. of mth degree are formed, from the following R. (C.) all adjunct ones of $(n-m)$th degree. To the product of two adjunct ones such a sign is given that the product of their principal terms is a term of D. The sum of these products equals D. If one takes arbitrary m and $(n-m)$ R. (C.), then the sum $= 0$ if even just one common row occurs[72]. *Jacobi* draws from this a series of conclusions about D. with zero elements[73].

Very obvious is the extension of the theorem in the direction that the products consist of more than two factors[74].

Another extension uses the bordering of D. and indicates how from each result delivered by *Laplace*'s formula a new one about bordered D. can be derived[75]. To this extension another concerning adj. Subd. stands alongside[76].

20. Developments. Of further developments would still be to mention that of a D. where the diagonal terms $a_{ii} = b_{ii} + z$ are called; the development happens according to powers of z[77]. Also the development of a single-row bordered D. of $(n+1)$th degree according to the El. of the border is important[78].

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70) Cf. the examples in *Baltzer*, *S. Günther*, *R. F. Scott*, *Salmon* etc.

71) Recherches sur le calcul integral et sur le systeme du monde. Paris Ac. d. Sc. (1772) 2° part., p. 267. — *Cauchy*, l. c. p. 100. — *Jacobi*, l. c. Nr. 5.

72) *Cauchy*, l. c.

73) *Jacobi*, l. c. Nr. 5.

74) *Vandermonde*, l. c. p. 524. — *Laplace*, l. c. p. 294. — *Jacobi*, l. c. Nr. 8.

75) *Netto*, J. f. Math. 114 (1895), p. 345.

76) *E. Pascal*, Rend. Acc. d. Linc. (5) 5, (1896), p. 188. The theorem established there follows by the way from the previous one by means of a general theorem by *Th. Muir*, Edinb. Transact. 30 (1882), p. 1, through which one can transition from a formula about Subd. to another about adjunct Subd.

77) *Laplace*, Mécan. celeste, 1, liv. 2, Nr. 56. Paris (1799). — *Jacobi*, J. f. Math. 12 (1834), p. 15 = Werke III, p. 208.

78) *Cauchy*, l. c. p. 69.

From *O. Hesse* comes a theorem about decomposition of the bordered D. if the unbordered vanishes[79].

21. Composition and Product. The product of a D. of mth into a D. of nth degree can be easily represented as D. of $(m+n)$th degree by pushing together in diagonal direction (textitLaplace's theorem). *J. Ph. M. Binet* and *A. L. Cauchy* have represented the product of two D. of nth degree again as D. of nth degree[80]. Simultaneously they have given the following extension: From two systems a_{ik}, b_{ik} a third c_{ik} is formed, textitcomposed,

$$a_{ik}\ (i=1,...m; k=1,...n)\ b_{ik}\ (i=1,...n; k=1,...m)$$
$$c_{ik}=\sum a_{i\lambda}b_{\lambda k}\ (i=1,...m; k=1,...m; \lambda=1,...n);$$

then is $|c_{ik}|=0$ for $m>n$; further $|c_{ik}|=|a_{ik}||b_{ik}|$ for $m=n$; and finally $|c_{ik}|=\sum_t |a_{it}||b_{it}|$ for $m<n$, where t runs through all possible r-combinations of mth class from $1,2,...n$. The middle case gives the multiplication rule[81]; the different arrangement of El. in R. and C. delivers four different forms for the product[82]. To this representation connect analytically and number-theoretically important formulas[83].

22. Other Kind of Composition. *Kronecker*[84] has drawn attention to another kind of composition: a_{ik} $(i,k=1,...m)$ and b_{gh} $(g,h=1,...n)$ are composed to $c_{pq}=a_{ik}b_{gh}$ $(p=(i-1)n+g; q=(k-1)n+h; i,k=1,...m; g,h=1,...n)$. Then is $|c_{pq}|=|a_{ik}|^n\cdot|b_{gh}|^m$.

23. Compound Determinants. Detailed interest has turned to the question of *compound* D. (compound det.), i.e. to such whose elements are themselves D. formed according to certain laws. Most obvious is the investigation of the D. formed from the El. a'_{ik}, i.e. the adjuncts of a_{ik}. *Cauchy*[85] has for $|a'_{ik}|$ $(i,k=1,...n)$ indicated the given value;

79) J. f. Math. 69 (1868), p. 319.

80) J. de l'Éc. polyt. Cah. 16 (1812), p. 280; Cah. 17 (1812), p. 29.

81) Further proofs among others: *J. König*, Math. Ann. 14 (1879), p. 507. *M. Falk*, Brit. Ass. Rep. (1878), p. 473. *A. V. Jamet*, Nouv. Corresp. M. 3 (1877), p. 247.

82) *Cauchy*, l. c. p. 83.

83) *Ch. Hermite*, J. f. Math. 40 (1850), p. 297. *K. F. Gauss* Werke 3, p. 384. *Baltzer*, Leipz. Ber. (1873), p. 352. *S. Gundelfinger*, Z. f. Math. 18 (1873), p. 312.

84) Vorlesungen. *K. Hensel*, Acta mat.14 (1890—91), p.317. *Netto*, Acta mat.17 (1894), p.200. *B. Igel*, Monatsh. f. Math.3 (1892), p.55. *G. v. Escherich*, ib.3 (1892), p.68.

85) l. c. p. 82.

Jacobi[86) more generally for $|a'_{ik}|$ $(i, k = 1, 2, ...m; m < n)$. In the first case a power of D. appears, in the second one such, multiplied with a Subd. $|a_{ik}|$.

These theorems have been extended by *Franke*[87)]; instead of the a'_{ik}, the Subd. of mth degree p_{ik} $(i, k = 1, 2, ...\mu)$ are considered, where $\mu = \binom{n}{m}$, and the numbering extends to all μ Subd. of mth degree of D. Further shall p'_{ik} be adjunct to p_{ik}, i.e. p'_{ik} is a Subd. of $(n-m)$th degree of $|a_{ik}|$, and the product of the principal terms of p_{ik} and p'_{ik} is a term of $|a_{ik}|$. It results then

$$|p_{ik}| = D^{\binom{n-1}{m-1}}, \quad |p'_{ik}| = D^{\binom{n-1}{m}},$$

and here too one can represent the Subd. of $|p'_{ik}|$ in similar way as with *Jacobi* the Subd. of $|a'_{ik}|$.[88)]

Even more general is *Sylvester*'s theorem[89)], which we can briefly characterize as referring to bordering of the D. $|p_{ik}|$.

Other works concern themselves with composing D. from rows of two given D., and considering these new D. as elements of a D.[90)].

24. Rank of the Determinant. According to *Kronecker* one designates as rank r of a D. the largest number of the property that not all Subd. of rth degree vanish[91)]. Through exchange and through linear combinations of the rows r is not changed. If D. is of rank r, then its El. can be composed from two systems a_{ik} $(i = 1, ...n; k = 1, ...r)$ and b_{ik} $(i = 1, ...r; k = 1, ...n)$[92)]. Of importance is this concept for many questions of algebra, especially solution of linear equations (IB1b).

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86) l. c. §11. — *C. W. Borchardt*, Brief an Baltzer (1853).

87) J. f. Math. 61 (1863), p. 350.

88) *C. W. Borchardt*, J. f. Math. 61 (1863), p. 353, 355, draws attention that the theorem is a special case of the one given earlier by *Sylvester*; *Kronecker*, Berl. Ber. (1882), p. 822 proves its identity with the above one by *Jacobi.* — Cf. *Picquet*, C. R. 86 (1878), p. 310; J. de l'Éc. Pol. cah. 45 (1878), p. 201.

89) Phil. Mag. (4), 1 (1851), p. 415. Cf. *Frobenius*, J. f. Math. 86 (1879), p. 54); Berl. Ber. (1894), p. 242. — *Netto*, Acta mat. 17 (1894), p. 201; J. f. Math. 114 (1895), p. 345. *R. F. Scott*, Lond. Proceed. 14 (1883), p. 91. *C. A. v. Velzer*, Amer. J. 6 (1883), p. 164. *Em. Barbier*, C. R. 96 (1883), p. 1845; ib. 97 (1883), p. 82. *E. J. Nanson*, Lond. phil. Mag. (5) 44 (1897), p. 396.

90) *Picquet*, l. c. *G. Zehfuss*, Z. f. Math. 7 (1862), p. 496.

91) Berl. Ber. (1884), p. 1071.

92) *Kronecker*, J. f. Math. 72 (1870), p.152. *Baltzer*, Determinanten, 4. Aufl. Leipz. (1875), p. 53.

25. Here may still be mentioned a theorem by *Mac-Mahon* relating to general D. (Phil. Trans. 185 (1894), p.146). Between a D. and all the Subd. whose main diagonals fall into the main diagonal of D., there exist $2^n - n^2 + n - 2$ relations. Cf. also *Muir*, Phil. Mag. (1894), p. 537; Edinb. Proceed. 20 (1895), p. 300. *Cayley*, ibid. p. 306. *Nanson*, ibid. (1897), p. 362.

26. Symmetric Determinants. For *symmetric* D., i.e. such D. whose El. are symmetric in relation to the main diagonal, the a'_{ik} also form a symmetric D. — Every power of a symmetric D., and every even power of any D. is symmetric[93]. The product of a symmetric D. into the square of any D. can be represented as symmetric D.[94] If r is the rank of a symm. D., then it has a non-vanishing principal Subd. of degree r.[95] *H. G. Grassmann* had first indicated[96], that between the Subd. of symmetric D. linear relations exist; the same theorem was later rediscovered by *Kronecker*[97], and *C. Runge* has shown[98], that the relations given by him are the only existing ones. These have the following character:

$$|a_{gh}| = \sum |a_{ik}| \quad (g = 1, ...m; h = m+1, ...2m; i = 1, ...m-1, r; \\ k = m+1, ...r-1, m, r+1, ..., 2m)\ .$$

If one borders a symmetric, vanishing D. in symmetric way, then the resulting D. considered as function of the bordering elements is a square[99], as easily follows from Nr.19. If one enters $a_{ii} + z$ instead of the a_{ii} and sets the resulting symmetric D. equal to zero, then this equation has in x only real roots. The resulting equation is called the "secular equation"[100]. (Cf. Nr. 31.)

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93) *H. Seeliger*, Z. f. Math. 20 (1875), p. 468 - the El. of any power of a sym. D.

94) *O. Hesse*, J. f. Math. 49 (1853), p. 246. — Cf. about an extension *Muir*, Amer. J. 4 (1881), p. 351.

95) *S. Gundelfinger*, J. f. Math. 91 (1881), p. 235; cf. *Hesse*, analyt. Geom. d. Raumes, 3. Aufl. Leipz. (1881), p. 460. *Frobenius*, Berl. Ber. (1894), p. 245.

96) Ausdehnungslehre, Berlin (1862), p. 131. Cf. *Mehmke*, Math. Ann. 26 (1885), p. 209. The way how *Grassmann* uses instead of D. certain "combinatorial product formations" is most simply recognized from the "Overview" (Arch. f. Math. 6 [1845], p. 337). More details are found in the "Ausdehnungslehre" §37, §51ff., §63 ff. The D. appears thereby as a product $\prod(a_{i_1}e_1 + a_{i_2}e_2 + ...)$ of "extensive quantities", where $e_x^2 = 0$, $e_x e_\lambda = -e_\lambda e_x$ is.

97) Berl. Ber. (1882), p. 821. Cf. *Darboux*, J. d. Mat. (2) 19 (1874), p. 347.

98) J. f. Math. 93 (1882), p. 319.

99) *Cauchy*, l. c., p. 69.

100) *J. L. Lagrange*, Mém. de Berlin (1773), p. 108 for $n = 3$; generally

27. Recurrent Determinants. Circulants. The symmetry appears in *recurrent* D. $a_{ik} = a_{i+k-2}$ in even stronger measure. *Hankel* (l. c.), who designates them as orthosymmetric, represents them as $|\Delta_k|$, where the Δ_k are the initial terms of the difference series of a_{i+k}. These D. appear frequently in algebra[101)]; their rank becomes thereby of importance.

A special case of this form those recurrent D. of nth degree, where $a_{n+i} = a_i$[102)]; and with these closely connected are the *circulants* (cf. Nr. 16) ($a_{ik} = a_{i+1,k+1}$), which are symmetric in relation to the secondary diagonal, which can be transformed into those through exchange of R. A circulant can be decomposed into the product of the n factors

$$a_1 + \omega^\alpha a_2 + \omega^{2\alpha} a_3 + ... + \omega^{(n-1)\alpha} a_n \ , \quad (\alpha = 1, 2, ...n),$$

where ω means a primitive nth root of unity; from this follows immediately that a circulant of $2n$th degree can be represented as one of nth degree[103)]. A circulant of $2n$ degree can further be expressed as product of one of nth degree and a similarly formed one[104)].

28. Half-symmetric Determinants. For *half-symmetric* D.[105)] ($a_{ik} = -a_{ki}$, $a_{ii} = 0$) the following theorems hold: $A_{ik} = A_{ki}$; $D = 0$ for odd n; on the other hand $a'_{ik} = -a'_{ki}$; $\frac{\partial D}{\partial a_{ik}} = 0; D = 0$; is a square for even n. Every term of $\sqrt{D}$ is a product of $\frac{1}{2}n$ El. a_{ik}, whose indices are all different from each other, as e.g. the term appearing in $\sqrt{D}$ shows $a_{12}a_{34}...a_{n-1,n}$. $\sqrt{D}$ is set by *Cayley* (l. c.) $= \pm(1, 2, ...n)$ and designated as "Pfaffian".

Cauchy, Exerc. d. Math. 4 (1829), p. 140. *E. Kummer*, J. f. Math. 26 (1843), p. 268. *G. Bauer*, J. f. Math. 71 (1870), p. 46. *Sylvester*, Phil. Mag. 2 (1852), p. 138. *Borchardt*, J. de Math. 12 (1847), p. 50; J. f. Math. 30 (1846), p. 38.

101) *Jacobi*, J. f. Math. 15 (1836), p. 101. *Kronecker*, Berl. Ber. (1881), Juni; J. f. Math. 99 (1886), p. 346. *Frobenius*, Berl. Ber. (1894), p. 241.

102) "persymmetric D." according to *Muir*, Quart. J. 18 (1882), p. 264.

103) *J. W. L. Glaisher*, Quart. J. 15 (1878), p. 347; ibid. 16 (1878), p. 31. Cf. also IA 6, Nr. 28, 24.

104) *R. F. Scott*, Quart. J. 17 (1880), p. 129.

105) *Lagrange* and *S. D. Poisson* are probably, according to *Jacobi*, first encountered such D. Cf. *Jacobi*, J. f. Math. 2 (1827), p. 354. — *Cayley*, J. f. Math. 38 (1849), p. 93, calls them "skew-symmetric". He proves first that D is a square for even n. J. f. Math. 32 (1846), p. 119; ibid. 50 (1855), p. 299.

106) *Brioschi*, J. f. Math. 52 (1856), p. 133. *Cayley*, l. c. Cf. an extension by *Muir*, Phil. Mag. (5) 12 (1881), p. 391.

The square of every D. of even degree can be transformed into a half-symmetric D.[106], so that D. itself appears as Pfaffian. *Cayley* has further shown (l. c.), that when one borders a half-symmetric D. of odd degree arbitrarily by $a_{\alpha k}$, $a_{i\beta}$, the resulting D. breaks down into a product $\pm(\alpha, 2, ...n) \cdot (\beta, 2, ...n)$ of two Pfaffians. For $\alpha = \beta = 1$ the previous theorem follows this.

29. Skew Determinants. If one drops the condition $a_{ii} = 0$, one arrives at the *skew* D., whose treatment likewise goes back to *Cayley* (l. c.). The development of the skew D. according to the terms of the main diagonal (Nr. 20) delivers aggregates of half-symm. D. Thus if every $a_{ii} = z$, then in the development of D. according to powers of z only the terms with exponents n, $n-2$, $n-4$,... appear.

30. Centrosymmetric and Other Determinants. Finally let the *centrosymmetric* D. ($a_{ik} = a_{n+1-i,n+1-k}$) be briefly mentioned. Every such of even degree $2m$ can be represented as product of two D. of mth degree. Since now circulants (Nr. 27) can be made into centrosymmetric D. through rearrangement of the rows, the theorem mentioned (Nr. 27) follows this easily.

31. Further Determinant Formations. Besides the mentioned special formations many others have been investigated; thus for example the centroskew D. connect to the last discussed ones; further the *Vandermonde* or *power determinants* are to be mentioned, where $a_{ik} = a_i^{v_k}$, where the v_k mean arbitrary numbers. The continued fraction determinants[107], the *continuants* (*Sylvester*), deliver the representation of numerators and denominators of the approximation values of a continued fraction[108]. *Hermite* considers Par. C. R. 41 (1855), p. 181, J. f. Math. 52 (1856), p. 40 Det., in which a_{ik} and a_{ki} are complex conjugates. Extension of the secular equation.

To function theory connect formations like: 1) the *Wronskian* D.; 2) the *Jacobian* (functional) D.; 3) the *Hessian* D.

For 1) the a_{1i} are functions of x; the a_{xi} their $(x-1)$th derivatives[109].

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107) *Painvin*, J. d. Math. (2) 3 (1858), p. 41. *J. Sylvester*, Am. J. 1 (1878), p. 344.

108) *Sylvester*, Phil. Mag. 5 (1859), p. 458; 6 (1853), p. 297. *W. Spottiswoode*, J. f. Math. 51 (1856), p. 209. *E. Heine*, ibid. 57 (1860), p. 231. *S. Günther*, Erlangen (1873) u. Math. Ann. 7 (1874), p. 267. — Cf. II A 3.

109) *C. J. Malmsten*, J. f. Math. 39 (1850), p. 91. *Hesse*, ibid. 54 (1857), p. 249. *E. B. Christoffel*, ibid. 55 (1858), p. 281. *Frobenius*, ibid. 76 (1873), p. 236. *M. Pasch*, ibid. 80 (1875), p. 177.

For 2) a_i are functions of n variables $x_1, \ldots x_n$, and $a_{\chi i} = \frac{\partial a_i}{\partial x_\chi}$ [110].

For 3) a is a function of $x_1, \ldots x_n$, and $a_{\chi i} = \frac{\partial^2 a}{\partial x_\chi \partial x_i}$ [111].

To algebra connect formations like resultants and discriminants. We refer about this to IB1a and b.

32. Determinants of Higher Rank. Determinants of higher (νth) rank are formed from n^ν quantities $a_{h_1,\ldots h_\nu}$ in such a way that one exchanges the indices of equal position among themselves; then products of n of these quantities are formed, where never two factors at equal position have equal index, and finally according to the earlier sign rule the $\pm$ sign is prefixed. All these aggregates form the D. Of these hold a series of properties of ordinary Det.; others are to be modified; Det. of *even* and such of *odd* rank behave in some respects differently[112]. Here too a treatment in *Grassmann*'s sense is possible (*G. v. Escherich* l. c.), cf. note 96.

33. Infinite Determinants. If one considers a_{ik} $(i, k = 1, 2, \ldots \infty)$, one can understand $D_n = |a_{ik}|$ $(i, k = 1, 2, \ldots n)$ as function of n. If n grows, one arrives at the concept of *infinite* Det. Above all here the existence question is to be raised[113]. These formations are important for differential equations. Cf. IA3 Nr. 58, 59.

34. Matrices. A system of $m \cdot n$ quantities a_{ik} $(i = 1, 2, \ldots m; k = 1, 2, \ldots n)$ is called a *matrix*. To these structures connects a series of fundamental questions, whose treatment in IA 4 (bilinear forms) is given.

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110) *Jacobi*, J. f. Math. 12 (1834), p. 38 = Werke III, p. 233; J. f. Math. 22 (1841), p. 319 = Werke III, p. 393. *Sylvester*, Phil. Trans. (1854), p. 72. *Cayley*, J. f. Math. 52 (1856), p. 276. *Clebsch*, ibid. 69 (1868), p. 355. *Kronecker*, ibid. 72 (1870), p. 155 etc.

111) *Hesse*, J. f. Math. 28 (1844), p. 83; ibid. 42 (1851), p. 117; ibid. 56 (1859), p. 263. *Sylvester*, Cambr. a. Dubl. M. J. 6 (1851), p. 186.

112) Cubic D. were first treated by *A. de Gasparis* (1861). Following were: *Dahlander*, Oefvers. of K. Akad. Stockh. (1863). *G. Armenante*, Giorn. di Battagl. 6 (1868), p. 175. *E. Padova*, ibid. p. 182. *G. Zehfuss*, Frankf. (1868). *G. Garbieri*, Giorn. d. Batt. 15 (1877), p. 89. *H. W. L. Tanner*, Proceed. Lond. M.S. 10 (1879), p. 167. *R. F. Scott*, ib. 11 (1880), p. 17. *G. v. Escherich*, Wien. Denkschr. 43 (1882), p. 1. *L. Gegenbauer*, ib. 43 (1882), p. 17; 46 (1883), p. 291; 50 (1885), p. 145; 55 (1889), p. 39. Wien. Ber. 101 (1892), p. 425.

113) *G.W. Hill*, Acta Math. 8 (1886), p. 1, essentially reprint of a monograph Cambridge U.S.A. (1877). *H. Poincaré*, Bull. Soc. d. Fr. 13 (1884—85), p. 19; 15 (1885—86), p. 77. *Helge von Koch*, Acta math. 15 (1891), p. 53; ibid. 16 (1892—1893), p. 217.

The concept of rank as well as composition of matrices is to be established. From a matrix Det. can be formed in different ways. Their connection, as well as their invariant properties are to be investigated. Here belongs the case of *corresponding matrices*: a_{ik} $(i = 1, ...m; k = 1, ...\alpha)$ and b_{jl} $(j = 1...\beta; l = 1, 2, ...m)$, where $\alpha + \beta = m$, and the $\alpha \cdot \beta$ relations exist $\sum_{(q)} a_{qk} b_{jq} = c_{kj} = 0$, where proportionality of corresponding determinants occurs[114)].

35. Monographs. As textbooks about determinants we list, passing over those intended only for school use, as the main ones:

Brioschi, La teoria dei determinanti. Pavia (1854). German, Berlin (1856).

Spottiswoode, Elementary Theorems relating to Determinants, J. f. Math. 51 (1856), p. 209—271 and 328—381.

Baltzer, Theorie u. Anwendung der Determinanten. Leipzig (1857). Fifth Ed. (1881).

Salmon, Lessons introductory to the modern higher algebra. Dublin (1859). German Leipz. (1877) by *Fiedler*.

Hesse, Die Determinanten, elementar behandelt. Leipz. (1872).

Günther, Lehrbuch der Determinantentheorie. Erlangen (1875). Second Ed. (1877).

Scott, A treatise on the theory of determinants. Cambridge (1880).

P. Mansion, Eléments de la théorie des déterminants. Paris 4th ed. (1883).

L. Leboulleux, Traité élémentaire des déterminants. Genève (1884).

A. Sickenberger, Die Determinanten in genetischer Behandlung. München (1885).

Gordan, Vorlesungen über Invariantentheorie. I. Determinanten. Leipz. (1885).

Pascal, I determinanti. Milano (1897).

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114) The concept of matrix was introduced by *A. Cayley*, J. f. Math. 50 (1855), p. 282. *Cayley* wants to keep the theory of matrices separate from that of determinants.

I A 3. IRRATIONAL NUMBERS AND CONVERGENCE OF INFINITE PROCESSES

BY

ALFRED PRINGSHEIM

IN MUNICH

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Literature

Textbooks

Leonhard Euler, Introductio in analysin infinitorum. I. Lausannae 1748. German by *Michelsen* (Berlin 1788) and by *H. Maser* (Berlin 1885).

Augustin Cauchy, Cours d'analyse de l'école polytechnique. I. Analyse algébrique. Paris 1821. German by *C. Itzigsohn*, Berlin 1885.

M. A. Stern, Lehrbuch der algebraischen Analysis. Leipzig 1860.

Eugene Catalan, Traité élémentaire des séries. Paris 1860.

Oskar Schlömilch, Handbuch der algebraischen Analysis. Jena 1868 (4th Ed.).

Charles Méray, Nouveau Précis d'analyse infinitésimale. Paris 1872. Nouvelles leçons sur l'analyse infinitésimale. I. Paris 1894.

Karl Hattendorff, Algebraische Analysis. Hannover 1877.

Rudolf Lipschitz, Lehrbuch der Analysis. I: Grundlagen der Analysis. Bonn 1877.
Moritz Pasch, Einleitung in die Differential- und Integral-Rechnung. Leipzig 1882.
Max Simon, Elemente der Arithmetik als Vorbereitung auf die Functionentheorie. Strassburg 1884.
Otto Stolz, Vorlesungen über allgemeine Arithmetik. 2 Vols. Leipzig 1885-86.
Jules Tannery, Introduction à la théorie des fonctions d'une variable. Paris 1886.
Camille Jordan, Cours d'analyse de l'école polytechnique. 2nd ed. I. Paris 1893.
Ernesto Cesàro, Corso di analisi algebrica. Torino 1894.
Otto Biermann, Elemente der höheren Mathematik. Leipzig 1895.
Alfred Pringsheim, Vorlesungen über die element. Theorie der unendl. Reihen und der analyt. Functionen. I. Zahlenlehre. (Forthcoming from B. G. Teubner, Leipzig.)
Regarding irrational numbers compare also: *P. Bachmann*, Vorl. über die Natur der Irrationalzahlen, Leipzig 1892; regarding infinite series: *J. Bertrand*, Traité de calc. différentiel, Paris 1864[1].

Monographs

Siegm. Günther, Beiträge zur Erfindungsgeschichte der Kettenbrüche. School Program, Weissenburg 1872.
Paul du Bois-Reymond, Die allgemeine Functionentheorie. I (only). Tübingen 1882.
R. Reiff, Geschichte der unendlichen Reihen. Tübingen 1889.
Giulio Vivanti, Il concetto d'infinitesimo e la sua applicazione alla matematica. Mantova 1894.

Part One. Irrational Numbers and Concept of Limit
I. Irrational Numbers

1. Euclid's Ratios and Incommensurable Quantities. The *irrational numbers*, whose fundamental introduction forms one of the most essential foundations of *general arithmetic*, nevertheless first grew out of *geometric* needs: they originally appear as expression for the *ratio of incommensurable* (i.e. not measurable by any common measure) *pairs of line segments* (e.g. the diagonal and side of a square[2]). In this sense, Book 5 of *Euclid*, which develops the *general* theory of *"ratios"*, as well as Book 10 dealing with *incommensurable* quantities, can be viewed as the literary starting point for the theory of *irrational numbers*. Nevertheless, *Euclid* naturally treats only very specific quantities constructible with *compass* and *ruler* (thus, arithmetically speaking, representable by square roots representable) *irrationalities* in their property

1) The very extensive sections on *series* in *S. F. Lacroix's* large Traité de calc. diff. et intégr. (3 Vols., 2nd ed., Paris 1810-1819) contain little useful material about *elementary* series theory.

2) That the diagonal and side of a square are incommensurable is said to have been recognized already by *Pythagoras*; see *M. Cantor*, Gesch. der Math. 1 (Lpz. 1880), p. 130, 154.

as *incommensurable line segments*[3)]; the notion that the ratio of two such *special* or even two *entirely arbitrary* incommensurable line segments defines a *specific* (irrational) *number* remained foreign to him, as to all mathematicians of antiquity[4)].

2. Michael Stifel's Arithmetica Integra. But even for the arithmeticians and algebraists of the Middle Ages and Renaissance, the *irrationalities* inherited from geometry were still not *"real"*, but at best *improper* or *fictitious numbers*[5)], which were merely tolerated as a necessary evil. The first decisive step toward a more correct assessment of irrational numbers is probably owed to *Michael Stifel*, who in Book 2 of his *Arithmetica integra*[6)] deals extensively with *"Numeris irrationalibus"*[7)] following *Euclid*'s Book 10. Although he initially still adheres to the view abstracted from *Euclid* that *irrational* numbers *are not "real"* numbers[8)], this ultimately contains, as the

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3) More details about this (besides in *Euclid*): *Klügel*, Math. W.-B. 2, p. 949. *M. Cantor* l.c. p. 230. *Schlömilch*, Ztschr. f. M. 34 (1889), Hist.-lit. Abth. p. 201.

4) *Euclid* says (Elem. X, 7) quite explicitly: Incommensurable quantities *do not* relate to each other like numbers. *Jean Marie Constant Duhamel* (Des méthodes dans les sciences de raisonnement, Paris 1865-70) attempted (l.c. 2, p. 72-75) to make Euclidean ratio theory useful for founding the general concept of irrational numbers. But he ultimately spoils his initially correct method through unnecessary introduction of an unclear geometric limit concept. In contrast, *O. Stolz* (Allg. Arithm. 1, p. 35ff.), alongside a reproduction of Euclidean ratio theory adapted to today's presentation style, gives the necessary indications of how the latter could be developed into an unobjectionable theory of real numbers (particularly including irrational ones). Cf. also: *O. Stolz*, Grössen und Zahlen (Rectoral address of March 2, 1891, Lpz. 1891), p. 16; further: Nr. 13, footnote 84.

5) *"Numeri ficti"*, usually designated as *"Numeri surdi"*: this name attributed to *Leonardo of Pisa* (Liber abaci, 1202) persisted into the 18th century, in England (*"Surds"*) up to the present.

6) Nürnberg 1544. Fol. 103-223.

7) *Stifel* uses the designation *"radices surdae"* in a narrower sense l.c. Fol. 134.

8) The contrary statement in *C. J. Gerhardt* (Gesch. der Math. in Deutschl., Munich 1877, p. 69) seems incorrect to me. The relevant passage in *Stifel* (l.c. Fol. 103) reads quite unambiguously: "Non autem potest dici numerus verus, qui talis est, ut praecisione careat et ad numeros veros nullam cognitam habet proportionem. Sicut igitur infinitus numerus non est numerus: *sic irrationalis numerus non est verus numerus* atque lateat sub quadam infinitatis nebula."

relevant context teaches, only a different mode of *expression* from today's, which basically says nothing other than that *irrational* numbers are simply *not rational* ones. On the other hand, *Stifel* documents his understanding, essentially *agreeing* with modern views, through the statement that *every irrational* number, just like *every rational* one, has a *uniquely determined place* in the ordered number sequence[9)]. With this, indeed, the most essential aspect that characterizes irrationalities as *numbers* appears sharply emphasized for the first time. Of course, here under irrational numbers only certain simple *root quantities* are to be understood - a restriction that explains itself partly from the then still existing sole dominion of *Euclidean* methods in *geometry*, partly also from the circumstance that finding the nth root of a whole number lying between g^n and $(g+1)^n$ (g whole number) was the *only* task whose *insolubility* by a *rational* number one could really *prove* at that time[10)].

3. The Concept of Irrational Numbers in Analytic Geometry. Only the gradually occurring break with the geometry of the ancients, particularly the development of analytic-geometric method beginning with the appearance of *Descartes' Géométrie* (1637), then the invention of infinitesimal calculus by *Leibniz* and *Newton* (1684; 1687) created the need to further develop the equivalence between *line segments* and *numbers* and correspondingly complete the concept of irrational numbers. While *Descartes* had already designated arbitrary *line segment ratios* with *simple letters* and calculated *with them like numbers*, the statement that *every ratio* of two quantities corresponds to a *number* appears at the beginning of *Newton's Arithmetica universalis* (1707) directly as *definition* of number[11)]. And even more specifically tied to the geometric concept of *measurable quantity*, *Chr. Wolf*, whose textbooks,

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9) L.c. Fol. 103b, line 3 from bottom: "Item licet infiniti numeri *fracti* cadant inter quoslibet duos numeros immediatos, quemadmodum etiam infiniti numeri *irrationales* cadunt inter duos numeros integros immediatos. *Ex ordinibus tamen utrorumque facile est videre, ut nullus eorum ex suo ordine in alterum possit transmigrare.*"

10) *Stifel* l.c. Fol. 103b.

11) "*Numerum* non tam multitudinem unitatum quam abstractam quantitatis cujusvis ad aliam ejusdem generis quae pro unitate habetur rationem intelligimus." Of course, as *Stolz* aptly remarks (Allg. Arithm. 1, p. 94), this definition appears in *N.* only as a kind of showpiece: for a real development of irrational number theory based on Euclidean ratio

extremely widespread in the first half of the 17th century, despite their lack of originality and sharper criticism can still be considered as expression of the views then accepted by the great majority, defines: "*Number* is that which relates to unity as a straight line to a certain other straight line"[12]. The *number* thus appears as expression for the result of *measuring* one *line segment* by another which plays the role of the *unit segment* - a view that became the sole dominant one up into recent times and is still strictly maintained by individual mathematicians today. To every *line segment* (or also - by means of a simple and known modification - to every *point* on a line) there now corresponds a specific *number*, namely either a *rational* or an *irrational* one, i.e. initially an unboundedly continuable algorithm in rational numbers (infinite continued fraction; infinite decimal fraction) to be obtained according to suitable rules (*Euclidean* procedure for finding the greatest common measure[13] or unbounded subdivision of the measuring unit segment) to be obtained as an *unboundedly continuable algorithm in rational numbers* (infinite continued fraction; infinite decimal fraction); the *justification* for considering such an *unbounded system of rational numbers* as a *single specific number* is then seen exclusively in the fact that it is found as *arithmetic equivalent* of a *given line segment* using the *same measurement methods* which yield a *specific rational* number for *other* line segments. From this it *does not follow*, however, that one is conversely also justified in considering any *arbitrarily given arithmetic* structure of the indicated kind as an *irrational number* in the sense just defined, i.e. in regarding the existence of a *line segment* generating that structure under suitable measurement as *evident*[14]. This for the consistent development of

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theory it is by no means utilized. (Cf. also: *Stolz*, Zur Geometrie der Alten, Math. Ann. 22 [1888], p. 516.)

12) Elementa Matheseos universae. 1, Halae 1710: Elementa Arithmeticae, Art. 10. (I quote from the second edition of 1730 available to me.)

13) Eucl. Elem. X, 2, 3. *A. M. Legendre*, Geometrie, Livre III, Probl. 19.

14) The above cited *Chr. Wolf* knows only the following to say about this (l.c. Art. 296): "In geometria et analysi demonstrabitur, tales radices, quae actu dari non possunt, esse ad unitatem ut rectam lineam ad rectam aliam, consequenter numeros eosque irrationales, cum ex hypothesi rationales non sint." That ultimately comes down again to considering as *numbers* only those *arithmetically defined* irrationalities that are *geometrically constructible.* In doing so, *W.* handles the concept of *geometric constructibility* carelessly in such a way that he e.g. uses parabolas of arbitrarily high order without further ado as constructible

the concept of irrational numbers fundamental point was until recent times either passed over in silence, or dismissed with the help of alleged geometric evidences, or obscured rather than clarified through metaphysical phrases about continuity, concept of limit, and infinitesimals.

4. The Cantor-Dedekind Axiom and the Arithmetic Theories of Irrational Numbers. *G. Cantor* was probably the first to sharply emphasize that the assumption that to *every arithmetic structure* defined in the manner of an irrational number there must correspond a specific *line segment* appears neither *self-evident nor provable*, but rather involves an essential, purely *geometric axiom*[15]. And almost simultaneously *R. Dedekind* showed that the *axiom* in question (or, more precisely, one *equivalent* to it) first gives *tangible content to that* property which had previously been designated as *continuity* of the straight line *without any adequate definition*[16]. To make the foundations of general *arithmetic* completely independent of such a *geometric* axiom, each of the two named authors developed his own *purely arithmetic* theory of irrational numbers[17].

Another, likewise *purely arithmetic* method of introduction had already been used for some time by *K. Weierstrass* in his lectures on analytic functions[18].

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and then uses these for the alleged construction of $\sqrt[m]{x}$ (l.c. Elementa Analyseos, Art. 630).

15) Math. Ann. 5 (1872), p. 128.

16) *Stetigkeit und irrationale Zahlen.* Braunschweig 1872. The axiom in question appears there in the following formulation: "If all points of the line fall into two classes such that *every* point of the *first* class *lies* to the left of *every* point of the *second* class, then there exists *one and only one* point which produces this division..."

17) L.c. The *Cantor* theory was published approximately at the same time, as by its author himself, also by *E. Heine* (with explicit reference to verbal communications from *Cantor*) in a somewhat more detailed manner: J. f. Math. 74, p. 174 ff. On the other hand, *Ch. Méray independently* of *Cantor* likewise discovered the foundations of this theory and published them approximately simultaneously with *Cantor* and *Heine* in his: Nouveau Précis d'Analyse infinitésimale, Paris 1872.

18) The basic principles of *W.*'s theory were first briefly communicated by *H. Kossak* in a program treatise of the Werder Gymnasium, Berlin 1872 (p. 18 ff.). More details can be found in *S. Pincherle*, Giorn. di mat. 18 (1880), p. 185 ff.– *O. Biermann*, Theorie der analytischen Functionen, Leipzig 1887, p. 19 ff.

Cantor himself has subjected all *three* forms of definition to a critical comparison in volume 21 (1883) of Math. Ann. (p. 565 ff.) and on this occasion somewhat modified his first presentation (probably following that given by *Heine*), in such a way that the separation of the *irrational number* to be defined from any *notion of limit* is expressed even more sharply.

5. The Theories of Weierstrass and Cantor. The *Weierstrass* theory and the somewhat more convenient to handle *Cantor* theory, which one can appropriately describe with *Heine*[19] as a fortunate further development of the former, both connect to a specific *formal representation* of irrational numbers, as whose simplest and most familiar to everyone type appears that through *infinite decimal fractions*[20]. While *W.* retains from this the principle of *sum formation* as exclusive *generative element*, *C.* takes from that model the more general concept of the so-called *fundamental sequence*, i.e., a sequence of rational numbers a_v of the nature that $|a_{v+\varrho} - a_v|$ for a *sufficiently large* chosen value of v and *any* value of ϱ becomes *arbitrarily small.* It is then *essential* that the *general real* number to be defined (which according to circumstances can be a *rational* or *irrational* one) *is not* obtained as *sum* of an *"infinite"* number of elements or as *"infinitely distant"* term of a sequence through some nebulous *limit process.* It appears rather as a finished, *newly created object*, or, even more concretely according to *Heine*[21], as a *new number symbol*, whose *properties* are uniquely determined from those of the defining rational elements, to which a *uniquely determined place* within the domain of rational numbers is assigned, and with which one can *calculate* according to definite rules[22].

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19) L.c. p. 173.

20) A detailed presentation of the *Cantor* theory, which appropriately takes the theory of *systematic fractions* (generalization of decimal fractions) as starting point, can be found in *Stolz*, Allg. Arithm. 1, p. 97 ff.; another presentation, likewise not inappropriate for the beginner, in which *two monotonic* number sequences (see No. 13 of this article) serve for the definition of irrational numbers and their arithmetic operations, is given by *P. Bachmann*, Vorl. über die Natur der Irrationalzahlen (Lpz. 1892), p. 6 ff.

21) L.c. p. 173: "I place myself at the purely formal standpoint for the definition (of numbers), *in that I call certain tangible signs numbers*, so that the existence of these numbers is thus not in question." Differently *Cantor*: Math. Ann. 21 (1883), p. 553.

22) Cf. *Pincherle* l.c. Art. 18, 28. *Biermann* l.c. p. 24. *Heine* l.c. p. 177. *Cantor*, Math. Ann. 5, p. 125; 21, p. 568.

The *general real numbers* defined in this way are naturally *not* to be viewed initially as signs *for specific quantities* (countable or measurable magnitudes), and the concepts *"greater"* and *"smaller"* defined for them accordingly do not designate *quantity differences*, but merely *successions*. In particular, the concept of *rational* numbers also undergoes an *extension* in the sense that they appear as *signs* to which primarily only *a specific succession* belongs[23)], and which *can* represent specific *quantities*, but *need* not. If this decisive point is overlooked[24)], objections become understandable, such as those wrongly raised by *E. Hittgens* against the theories of *Weierstrass* and *Cantor*[25)]. That, moreover, the *Weierstrass-Cantor* numbers (including the *irrational* ones) can be used for representing specific *quantities*, e.g. *line segments*, has been explicitly shown by the authors of the theories in question[26)]: to *every line segment* corresponds (after fixing an arbitrary unit segment) *one* and *only one* specific *number*. The *converse* naturally holds again only for *rational* and *special irrational* numbers; for *arbitrary irrational numbers* only *if* one accepts the geometric *axiom* mentioned in Art. 4[27)].

6. The Theory of Dedekind. *Dedekind* defines the irrational number, without direct use of any arithmetic formalism, with the help of the concept of *"cut"* introduced by him[28)]; by this he understands a division of all rational numbers into two classes of individuals (a_1) and (a_2), such that throughout $a_1 < a_2$.

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23) One can, starting from this concept of *unique succession*, arrive at a *completely unified* construction of number theory if one introduces from the outset the *natural* numbers *not*, as usual, on the basis of the concept of cardinal number as *cardinal numbers*, but rather (following *H. Helmholtz* and *L. Kronecker*) as *ordinal numbers*. Cf. my essay Münch. Ber. 27 (1897), p. 325.

24) See e.g. *R. Lipschitz*, Grundl. d. Anal., Section I, and cf. my lecture: Über den Zahl- und Grenzbegriff im Unterricht. Jahresb. d. D. M.-V. 6 (1848), p. 78.

25) Math. Ann. 33 (1889), p. 155; likewise 35, p. 451. Reply by *Cantor*: ibid. 33, p. 476. Cf. also *Pringsheim*, Münch. Sitzber. 27, p. 322, footnote.

26) *Pincherle* l.c. Art. 19. *Cantor*, Math. Ann. 5, p. 127.

27) In *Pincherle*, whose presentation of *W.*'s theory certainly cannot be regarded as an authentic one, curiously that *axiom* (in *Dedekind*'s form) is again considered as a self-evident fact (l.c. Art. 20).

28) L.c. 4.

If then among the numbers a_1 there is a *greatest* or among the numbers a_2 a *smallest*, then the respective (*rational*) number is precisely the one which produces the *cut* in question. In the other case, a *newly created* individual α, an *irrational* number, is assigned to it and regarded as producing this *cut*. On the basis of this definition, the relationships of these new numbers α among themselves and to the rational numbers a, as well as the elementary arithmetic operations, can be uniquely determined, as *D.* himself has essentially carried out. A more detailed presentation in more *geometric* garb has been given by *M. Pasch* in his "Einleitung in die Differential- und Integral-Rechnung"[29] and later added some modifications[30] which make the in truth still essential *arithmetic* foundation of that theory appear more clearly.

Precisely because *Dedekind*'s method of introducing irrational numbers does not connect to any arithmetic algorithm, it gains the advantage of a very special brevity and conciseness. For the same reason, however, it also appears noticeably more abstract and adapts less conveniently to calculation than the *Cantor* theory. Not inappropriately, therefore, *J. Tannery* in his "Introduction à la Théorie des Fonctions"[31] has chosen a presentation which, starting from the *Dedekind* definition, subsequently gains connection to *Cantor*'s theory through inclusion of *Cantor*'s *fundamental sequences*.

7. Du Bois-Reymond's Fight Against the Arithmetic Theories. The separation of the concept of *number* from that of *measurable* magnitude, as it is established by the arithmetic theories of irrational numbers, with determination particularly by *P. Du Bois-Reymond*

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29) Leipzig 1882, §13.

30) Math. Ann. 40 (1892), p. 149.

31) Paris 1886, Chap. 1. Incidentally, *Tannery* makes an error when he (p. IX) attributes the actual basic idea of the *Dedekind* theory to *J. Bertrand* (Traité d'Arithmétique), as *Dedekind* has rightly emphasized in the preface to his work: "Was sind und was sollen die Zahlen?" (p. XIV). *Bertrand* actually uses the two classes designated in the text as $(a_1), (a_2)$ only, just like the older mathematicians, for the *approximate representation* of the *irrational number*; he does not at all connect its *definition* to the concept of the *cut*, which is foreign to him, but thoroughly to that of *measurable magnitude* (see l.c. 11th ed., 1895, Art. 270, 313), and he tacitly usurps for the foundation of addition and multiplication of irrational numbers (Art. 314, 315) the *axiom* of Art. 4.

has been opposed[32]. In his "Allgemeine Funktionentheorie" (Tübingen 1882) he rejects it as formalistic, degrading analysis to a mere play of symbols[33], and emphasizes for historical and philosophical reasons the inseparable connection of *number* with *measurable* or, as he calls it, *"linear" magnitude*. In doing so, he reduces the *requirement* contained in the *axiom* of Art. 4 to that of the *decimal fraction limit*, i.e., the *existence* of a specific *line segment* which (in the sense discussed more closely above in No. 3) corresponds to an arbitrarily given *infinite decimal fraction*[34]. He does not regard this statement without further ado as an *axiom*, but investigates to what extent it can be justified by considerations of an essentially psychological nature. The *epistemological* value of this discussion[35] will be reported at a later point[36]. For the *mathematician*, hardly anything else comes out of it in the end than that he must accept the requirement in question as an *axiom* if he wants to base the theory of *irrational numbers* on that of *measurable magnitudes*. This is the standpoint which *G. Ascoli* has recently emphasized as the only one appearing natural to him in contrast to the arithmetic irrational number theories[37]. Nevertheless, today the vast majority of scientific mathematicians might have

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32) *Herm. Hankel* (Theorie der complexen Zahlensysteme) wrote as early as 1867, thus at a time when at most the *Weierstrass* theory could be known to him through verbal communication, the following (l.c. p. 46): "Any attempt to treat irrational numbers formally and without the concept of magnitude must lead to highly abstruse and cumbersome artifices which, even if they could be carried out with perfect rigor, which we have just reason to doubt, would not have a higher scientific value." It appears extremely curious that precisely the creator of a *purely formal theory of rational numbers* has shown so little understanding for the corresponding further development of the concept of number.

33) L.c. Art. 18.

34) This requirement is indeed sufficient, since any arbitrarily arithmetically defined irrational number can be represented as a systematic fraction with arbitrary base, see No. 9, footnote 48.

35) L.c. p. 116ff.

36) VI A2 a, 3 a.

37) Rend. Ist. Lomb. (2) 28 (1895), p. 1060. *A.* formulates that axiom as follows: "If of the segments $\overline{a_1b_1}, \overline{a_2b_2}, \overline{a_3b_3}, \ldots$ each lies completely in the interior of the preceding one and $\lim_{n=\infty} \overline{a_nb_n} = 0$, then there always *exists one* and *only one* point that lies in the interior of all these segments."

joined one of the purely *arithmetic* forms of definition of irrational numbers and thus agree to a *separation* of *pure number theory* from the *actual theory of magnitude*[38]. The introduction of that *axiom* becomes necessary with this view only when it is a matter of transferring the results validly existing within pure *arithmetic* without its participation into *spatial intuition*[39].

8. Perfect Arithmetization in Kronecker's Sense. While the adherents of the just described *"arithmetizing"* direction content themselves with basing the *definition* of *irrational numbers* and the arithmetic operations to be performed with them on the theory of *rational*, thus ultimately of *whole* numbers, *Kronecker* has set forth a substantially higher degree of *"arithmetization"* of the entire number theory (arithmetic, analysis, algebra) as a desirable and, in his opinion, also attainable goal[40]. According to this, the arithmetic disciplines should *"strip off again all modifications and extensions of the concept of number* (except that of the natural number)"; in particular, therefore, *irrational numbers* should be definitively banished from them. That it will ever come to that does not seem very likely to me[41]. For if one observes what *Kronecker* proposes l.c. for the elimination of *negative* and *fractional*[42], as well as *algebraic* numbers, one gets the impression that the *perfect arithmetization* of those disciplines in question would amount to *dissolving* their well-tested mode of expression and symbolic language, which *summarizes* extremely *complicated* relations between *natural* numbers in the shortest and completely precise manner, into a most extensive and cumbersome formalism.

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38) Cf. also *Helmholtz*, Ges. Abh. 3, p. 359.

39) Cf. *F. Klein*, Math. Ann. 37 (1850), p. 572.

40) J. f. Math. 101, p. 338. The catchword *"Arithmetization"*, which has meanwhile become a technical term, was probably first used by *K.*

41) Cf. my above-cited essay: Münch. Sitzber. 27, p. 323, footnote. Further: *E. B. Christoffel*, Ann. di Mat. (2) 15 (1887), p. 253. (The content of this essay is, incidentally, essentially number-theoretical in nature.)

42) The method used by *Kronecker*, based on the arithmetic concept of *congruence*, is, incidentally, exactly the same as that already developed by *Cauchy* for the elimination of *imaginary* numbers: Exerc. d'anal. et de phys. math. 4 (1847), p. 87. Cf. I A 2, No. 3.

9. Different Representation Forms of Irrational Numbers and Irrationality of Certain Representation Forms. The simplest type of *number sequences* for *representing irrational numbers* are the infinite, i.e., unboundedly continuable *systematic fractions*[43]. Already in *Theon of Alexandria*[44] one finds a method for the approximate calculation of square roots using *sexagesimal* fractions. The latter remained exclusively in use even in the Middle Ages and were only gradually displaced by *decimal* fractions since the 16th century[45]. Instead of the *decimal* fractions now generally common in practice, the *dyadic* ones[46], due to their extraordinary formal simplicity and special geometric intuitiveness, prove to be preferentially suitable for the purposes of analytical proof.

The *non-periodic* infinite decimal fractions may be considered as the first arithmetic representation forms whose *irrationality* one has explicitly recognized (on the basis of the unique representability of every *rational* fraction by an always *periodic* infinite decimal fraction[47]). That conversely *every* irrational number is uniquely representable by an infinite decimal fraction (or systematic fraction with arbitrary base) was generally proven by *Stolz*[48].

A second fundamental representation form of irrational numbers, namely through infinite *continued fractions*[49] likewise connects to the problem of square root extraction. The calculation of a square root using an unboundedly continuable *regular continued fraction*[50] was first taught (admittedly only with *numerical* examples) by *Pietro*

43) A detailed theory of these in *Stolz*, Allg. Arithm. 1, p. 97 ff.

44) Around 360 A.D. *M. Cantor*, 1, p. 420.

45) Cf. *M. Cantor*, 2, p. 252, 565-569. *Siegm. Günther*, Verm. Unters. zur Gesch. der math. Wissensch. (Leipzig 1876), p. 97 ff.

46) *Leibniz* among others has particularly drawn attention to the advantages of the dyadic system: Mém. Par. 1703. (Opera omnia, Ed. Dutens, 3, p. 390.)

47) *Joh. Wallisii* de Algebra Tractatus (1693), Cap. 80.

48) L.c. p. 119.

49) In truth, this representation form would have been directly indicated by the geometric origin of the irrational number as ratio of *incommensurable* line segments and by the Euclidean method for establishing commensurability or incommensurability (Elem. X 2, 3). Historical development, however, has taken a different course.

50) I.e., one whose partial numerators are all = 1, whose partial denominators are natural numbers.

Cataldi[51] who accordingly is to be regarded as the *inventor of continued fractions*[52]. The *purely numerical* procedure discovered by *Cataldi* appears in the form of a *general analytical method* in *Leonhard Euler*, to whom we owe the first coherent theory of continued fractions. Already in his first treatise[53] on this subject he shows among other things the following: Every *rational* fraction can be represented by a *finite*, every *irrational* by an *infinite regular* continued fraction. In particular, the development of a *square root* always yields a *periodic* regular continued fraction; conversely, every convergent continued fraction of this type satisfies a *quadratic* equation with integer coefficients[54]. Then the numbers e, $\frac{e^2-1}{2}$ and others are represented by continued fractions[55] at first, of course, *purely numerically* (i.e., by setting approximately: $e = 2.71828182845904$). The law found on this path by mere *induction* for the formation of the *infinite* continued fractions is, however, then also *analytically* actually proven: with this, *Euler* has indeed established the *irrationality* of e and e^2 for the first time[56].

With the help of more general continued fraction developments, *Joh. Heinr. Lambert*[57] then succeeded in proving the *irrationality* of e^x, $\tan x$ and thus also of $\log x$, $\arctan x$ for *any* rational x, in particular[58] therefore that of π $(= 4\arctan 1)$[56]. An abbreviation of these proofs and at the same time a generally useful aid for recognizing the irrational was provided by *Legendre*'s

51) Trattato del modo brevissimo di trovare la radice quadra delli numeri. Bologna 1613.

52) Also the current *notation* of continued fractions (both the usual and the more compact one, cf. footnote 338) is already found in *C.*, with the only difference that he writes & instead of + (or .& instead of $\dot{+}$) (e.g., l.c. p. 70). The assumption that already the Greeks, especially *Archimedes* and *Theon of Smyrna* (around 130 A.D.), had known in principle the calculation of square roots using continued fractions is based solely on conjectures. Cf. *M. Cantor*, 1, p. 272, 369.

53) De fractionibus continuis. Comment. Petrop. 9 (1737), p. 98.

54) This theorem forms, as is well known, the basis of important investigations in the theory of quadratic forms (*Euler, Lagrange, Legendre, Dirichlet*, see I C 2) and the numerical solution of algebraic equations (*Lagrange*, see I B 3 a).

55) The notations e and π come from *Euler*, cf. *F. Rudio*, Archimedes, Huygens, Lambert, Legendre. Leipzig 1892, p. 53.

56) Cf. my note in the Münch. Sitzber. 1898, p. 325.

57) Hist. de l'Acad. de Berlin, Année 1761 (printed 1768), p. 265.

58) L.c. p. 297.

theorem on the *irrationality* of any infinite continued fraction:

$$\frac{a_1|}{|b_1} \pm \frac{a_2|}{|b_2} \pm \cdots \pm \frac{a_\nu|}{|b_\nu} \pm \cdots ,$$

for the case that the $\frac{a_\nu}{b_\nu}$ are ordinary proper fractions[59]. Using this theorem, *Legendre* first extended the irrationality proof to π^2. Also based on it is, for example, the proof given by *G. Eisenstein*[60] for the irrationality of certain series and products occurring in the theory of elliptic functions, such as:

$$\sum_{\nu=1}^{\infty} \frac{1}{p^{\nu^2}}, \quad \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{p^{\nu^2}}, \quad \sum_{\nu=1}^{\infty} \frac{r^\nu}{p^{\nu^2}}, \quad \prod_{\nu=1}^{\infty} (1 - \frac{1}{p^\nu})$$

(where p is a whole, r a rational positive number)[61].

10. Continuation. The extension of the *binomial* theorem to *fractional* exponents[62] taught how to represent roots of any degree by infinite series and thereby provided at the same time the first general series type of *immediately* recognizable irrationality. It seems to have remained the only one of this kind for a long time. The *direct* proof for the irrationality of the well-known *e-series* that has been included in most textbooks comes only from *J. Fourier*[63]. By applying a quite analogous proof method, *Stern*[64] showed the irrationality of the series: $\sum_\nu p^{-\nu} q^{-m_\nu}$, (where p, q are natural numbers, (m_v) an unbounded sequence of natural numbers, for which $m_{\nu+1} - m_\nu$ grows to infinity with ν) and: $\sum_\nu \pm (p_1 p_2 \cdots p_\nu)^{-1}$,(where p_1, p_2, p_3, ... is an unbounded sequence of increasing natural numbers), as well as some similar, somewhat more general series and infinite products equivalent to them.

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59) Eléments de géometrie (1794), Note IV. (Also reprinted in the above-cited work by *Rudio* p. 161.) Cf. No. 49.

60) J. f. Math. 27 (1843), p. 193; 28 (1844), p. 39.

61) The further investigations in this direction deal essentially with the separation of irrationalities into *algebraic* and *transcendental*. On this (specifically also on the *transcendence* of e and π) see I C 2.

62) Around 1666 by *Newton* (Letter to *Oldenburg* of Oct. 24, 1676 — see Opuscula, Ed. Castillioneus, 1 (1644), p. 328). *N.* found the theorem in question merely by induction. The first rigorous, purely elementary proof (i.e., without use of differential calculus) was given by *Euler*: Nov. Comment. Petrop. 19 (1774), p. 103.

63) According to *Stainville*, Mélanges d'analyse (1815), p. 339.

64) J. f. Math. 37 (1848), p. 95; 95 (1883), p. 197.

W. L. Glaisher[65)] pointed out that one recognizes quite immediately the irrationality of the series considered by *Eisenstein* $\sum p^{-\nu^2}$, $\sum(-1)^{\nu-1}p^{-\nu^2}$ and the more general: $\sum n_\nu \cdot p^{-m_\nu}$ (where m_ν, n_ν are natural numbers satisfying certain conditions) if one interprets them as *systematic* (obviously *non*-periodic) *fractions* with base p. He also proves, with the help of continued fraction developments, the irrationality of various other series, which essentially coincide with those treated by *Stern*.

A unique representation of *every* proper-fractional *irrational number*, modeled after the exponential series, by the series $\sum_{\nu=1}^{\infty} \frac{m_\nu}{\nu!}$ (where m_ν is a natural number $< \nu$) has been given by *Cyp. Stephanos*[66)]; the sum of the series yields a *rational* number if and only if from some specific ν onwards, throughout $m_\nu = \nu - 1$. Incidentally, this representation appears only as a special case of one given earlier by *G. Cantor*[67)]. Another likewise unique representation of *all* numbers lying between 0 and 1 by series of the form:

$$\frac{1}{m_1+1} + \sum_{\nu=1}^{\infty} \frac{1}{m_1(m_1+1)\cdots m_\nu(m_\nu+1)}$$

comes from *J. Lüroth*[68)]. The *rational* numbers always yield *periodic*, the *irrational* however *non-periodic* series of this kind - *vice versa*.

Finally, there belongs here also a unique representation communicated by *G. Cantor*[69)] of all numbers lying above 1 by infinite products of the form: $\prod_{\nu=1}^{\infty}(1+\frac{1}{m_\nu})$, where the m_ν are natural numbers and $m_{\nu+1} \geq m_\nu^2$. Here the *irrational* numbers are characterized by the fact that for infinitely many values of ν: $m_{\nu+1} > m_\nu^2$, while for every *rational* number, from a certain value ν onwards, throughout the relation $m_{\nu+1} = m_\nu^2$ holds[70)].

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65) Philosophical Magazine 45 (London 1873), p. 191.

66) Bull. de la S. M. d. F. 7 (1879), p. 81. A function-theoretical application of this representation method in *G. Darboux*, Ann. de l'École norm. (2), 7 (1879), p. 200.

67) Z. f. Math. 14 (1869), p. 124.

68) Math. Ann. 21 (1883), p. 411. There *L.* also gives one application each to function theory and set theory.

69) Z. f. Math. 14 (1869), p. 152.

70) A representation of *special* irrationalities by infinite products,

II. Concept of Limit.

11. The Geometric Origin of the Concept of Limit. The more general concept of *limit* or *limiting value* of a somehow defined, unbounded in number *set of numbers*, which stands in closest relation to the concept of irrational number, has emerged from the principle of *exhaustion*[71)] already used by *Euclid* and *Archimedes* in connection with the application of the *infinite* belonging only to more recent times. The *exhaustion* principle appears among the ancients in the form of a purely *apagogic* proof method useful for comparing surfaces and bodies, the core of which can be formulated as follows[72)]: "Two geometric magnitudes A, B are equal to each other if it can be shown that under the assumption $A > B$ the difference $A - B$, and under the assumption $A < B$ the difference $B - A$ would be smaller than any magnitude of the same kind as A, B." The conception of a spatial structure bounded by a continuously curved line or surface as a polygon or polyhedron with *"infinitely many"* and *"infinitely small"* sides is hardly found before the 16th century. Here too, the above-already cited *M. Stifel* may well be considered the first who defined the *circle* as an *infinite-polygon* and, even more precisely, in a sense as the *last* (thus in our terminology as the *"limit"*) of all possible polygons with finite number of sides[73)]. But while he concluded from this precisely the *impossibility* of representing the ratio of circumference and diameter by a *rational* or *irrational* number[73a)], *Joh. Kepler*, proceeding from analogous

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as the first example of which appears the well-known *Wallis* formula for π (see No. 41 Eq. [52]), *Ch. A. Vandermonde* gave, Mém. de l'Acad., Paris 1772. (In the German edition of *V.'s Abhandl.* aus der reinen Math. [Berlin 1888], p. 67.)

71) Cf. Art. *"Exhaustion"* in *Klügel's* W. B., 2, p. 152. A more critical presentation is given by *Hermann Hankel* in *Ersch* and *Gruber's* Encyklopädie, Sect. I, Vol. 90, Art. *"Grenze"*.

72) *Stolz*, Zur Geometrie der Alten. Math. Ann. 22 (1883), p. 514. Allg. Arithm. I, p. 24.

73) L.c. Fol. 224a. Def. 7. 8: "Recte igitur describitur circulus mathematicus esse polygonia infinitorum laterum. *Ante* circulum mathematicum sunt omnes polygoniae numerabilium laterum, quemadmodum ante numerum infinitum sunt omnes numeri dabiles."

73a) L.c. Fol. 224b. Def. 12. If one considers that *Stifel* did not yet have the *general* concept of irrational number (cf. No. 2), the above apparently false conclusion may be regarded not only as perfectly logical, but even as a characteristic sign of the (approaching modern conception) arithmetically-sharp thinking of *Stifel*.

views, arrived at useful formulas for the *cubature* of bodies of rotation[74]. Shortly thereafter appeared *Bonav. Cavalieri's Geometry of Indivisibles*[75], which, going considerably beyond *Kepler's* special investigations, notwithstanding the somewhat mystical nature of those "indivisibles", is usually regarded as the first fundamental presentation of *a general scientific method of exhaustion*[76].

12. The Arithmetization of the Concept of Limit. *John Wallis*[77] arrived at an *arithmetic* formulation of the *concept of limit*, as it is essentially still common today, by abandoning the cumbersome *apagogic* procedure of the ancients and translating *Cavalieri's direct geometric* method into the *arithmetic* - in meaning and in today's expression approximately as follows:

A number a is considered as the *limit* of an unboundedly continuable sequence of numbers a_ν ($\nu = 0, 1, 2, \ldots$ in inf.), if the difference $a - a_\nu$ becomes *arbitrarily small*[78] with sufficiently increasing values of ν.

This *definition*, which completely fixes the *arithmetic* relation of that *limit* a to the numbers a_ν as soon as the number a is *known* or at least its *existence* is established from the outset, *does not* yet provide a *criterion* to possibly infer the *existence* of a limit from the nature of the numbers a_ν. In this respect, one repeatedly took refuge in *geometric* ideas and analogies, from which one then believed to be able to infer without further ado the *existence* of the limit in question[79]. Thus, for example, in the *quadrature* of curvilinearly bounded

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That conclusion agrees perfectly with our current view that the *rectification* of a curved line cannot be defined at all *without* the *general concept* of irrational number. Cf. No. 11, 12.

74) Nova stereometria doliorum. Linz 1615. (Cf. *M. Chasles*, Histoire de la Géométrie (2de éd. 1875), p. 56. *M. Cantor*, Gesch. der Math. 2, p. 750.)

75) Geometria indivisibilibus continuorum nova quadam ratione promota. Bologna 1635. (Details in *Klügel*, 1, Art. *"Cavalieri's Method des Untheilbaren"*. *M. Cantor* l.c. p. 759.)

76) *H. Hankel* l.c. p. 189. *Chasles* l.c. p. 57.

77) Arithmetica Infinitorum (1655), Prop. 43, Lemma. (In the complete edition of *W.*'s works - Opera omnia, Oxoniae, 1695. 1, p. 383.) Cf. *M. Cantor*, 2, p. 823.

78) "Quolibet assignabili minor." L.c. Prop. 40.

79) I deliberately omit here again all attempts to establish the concept of limit epistemologically and psychologically, as belonging to the *philosophy* of mathematics (thus according to VI A 2 a, 3 a).

plane pieces, in the *rectification* of curve arcs (with the help of the quadrature or rectification of a series of unboundedly approximated polygons), one regarded the *existence* of a definite *area or length number* as something *self-evident*, existing *a priori* on the basis of *geometric* intuition[80]. The *decisive turn* toward eliminating this inadequate conception is marked by *Cauchy's* definition and existence proof[81] for the *definite integral* of a continuous function; with this, indeed, not only is the *necessity* made clear for the first time to explicitly prove *arithmetically* the *existence* of an *area number*, but this proof is actually delivered *at least in the main part*, i.e., it is shown that for the definition of that *area number*, sequences of numbers are available which fulfill the *criterion* required for the *existence* of a definite *limit* (to be discussed more closely immediately)[82]. Although *Cauchy* lacks (and indeed not only at the relevant point, but generally in his works) the *proof* that this *criterion* is actually *sufficient* for the existence of a definite *limit*, one can nevertheless say that through *Cauchy's* mentioned achievement, the true *arithmetic* nature of the general limit problem has been sharply characterized for the first time and the way has been shown for its final resolution.

13. The Criterion for the Existence of a Limit Value. The mentioned *criterion* for the existence of a definite *limit*, in its *basic form*, i.e., for a simple, unboundedly continuable series of real numbers (simply-infinite *sequence of numbers*, simply-*countable*[83] set of numbers) and in connection

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80) In stereometry, the analogous difficulty already arises with the cubature of the *pyramid*; cf. *R. Baltzer*, Die Elemente der Mathematik 2 (1883), p. 229. *Stolz*, Math. Ann. 22 (1883), p. 517.

81) Both are already found in the "Résumé des leçons données à l'école polytechnique sur le calcul infinitésimal" (Paris 1823), p. 81 (not first, as is often assumed, in the "Leçons sur le calcul différentiel et intégral" 2, p. 2, published by *M. Moigno* 1840-44).

82) For the full rigor of the proof, the recognition of the *uniform* continuity of a simply continuous function would still be required - which, however, does not weigh essentially in the context at hand. Cf. II A 1.

83) Cf. I A 5, No. 2. In the present article, essentially only the *limit values of countable sets of numbers* are dealt with, since the limit values of *uncountable*, especially *continuous* sets of numbers (cf. I A 5, No. 2, 13, 16) belong to *analysis* (II A, B). Of course, this separation cannot be strictly maintained with regard to the historical development of the different limit value considerations cannot always be strictly maintained (as e.g. above

with the *definition of limit* given above, is as follows:

For the unbounded sequence of numbers (a_ν) to possess a definite *limit* (a definite *limit value* or *limes*) a, in symbols[84]:

$$a = \lim a_\nu \quad (\nu = \infty) \quad \text{or:} \quad \lim_{\nu=\infty} a_\nu = a$$

it is *necessary* and *sufficient* that $a_{n+\varrho} - a_n$ becomes *arbitrarily small* for *a sufficiently large* value of n and *every* value of ϱ[85].

The sequence of numbers (a_ν) is then called *convergent.*

That the above condition is a *necessary* one follows directly from the *definition* of the *limit* and may well have been known since one has dealt with such limit values at all. That it is also *sufficient* was regarded as *self-evident* until recent times, but *was never explicitly proven.* The merit of having first emphasized this necessity belongs to *Bolzano*[86], who at least *attempted* to provide the corresponding proof for the special case of series convergence[87]. This is, however, inadequate, as is also a proof given by *Herm. Hankel* (relating to the more general case of *arbitrary* sets of numbers)[88].

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regarding the citation about the definite integral or in the following remarks about the proof of the limit value criterion).

84) The *symbol* lim, which has become completely indispensable to us today, seems to me to have been first used by *Simon L'Huilier* (Exposition élément. des calculs supérieurs, Berlin 1786 - also under the title: Principiorum calc. diff. et integr. expositio, Tübingen 1795). It probably became generally common only since *Cauchy* (Anal. algébr. p. 13) (i.e., since 1821; in the great Traité de calc. diff. et integr. by *Lacroix*, 1810-1819, each individual limit transition is still laboriously designated with words). The above-mentioned work of *L'Huilier* (awarded a prize by the Berlin Academy as the solution to a prize question posed in 1784), which in the historical presentations known to me is by no means appreciated according to merit, contains the first rigorous presentation of the concept of limit based on the Euclidean theory of proportions and the method of exhaustion.

85) This theorem with its transfer to *arbitrary* (e.g., continuous) sets of numbers - designated by *Du Bois-Reymond* as the *"general convergence principle"* (Allg. Funct.-Theorie, pp. 6, 260) is the actual *fundamental theorem of the entire analysis* and should stand at the head of every rational textbook of analysis with sufficient emphasis on its fundamental character.

86) "Rein analytischer Beweis des Lehrsatzes, etc." Prague 1817. Cf. *Stolz*, Math. Ann. 18 (1881), p. 259.

87) Cf. also No. 21 of this article.

88) *Ersch* u. *Gruber* l.c. p. 193; Math. Ann. 20 (1882), p. 106. Cf. *Stolz* l.c. p. 260, footnote.

Since the correctness of the theorem in question depends essentially and exclusively on the *well-defined existence of irrational numbers*, the first rigorous proofs of it naturally coincide with the emergence of the *arithmetic* theories of irrational numbers and the associated revision and sharpening of the older *geometrizing* mode of explanation (cf. No. 7). In *Cantor*, the theorem appears as a quite direct *consequence* resulting from the *definition* of irrational numbers, as is sharply emphasized by himself [89)]. *Dedekind* has also, in connection with *his* theory of irrational numbers, provided a complete proof of it (for the more general case of *arbitrary* sets of numbers)[90)]. The latter has been somewhat simplified by *U. Dini*[91)] and then adapted by *Du Bois-Reymond* to the view he represents[92)]. Other modifications of that proof have been given by *Stolz*, *J. Tannery*, *C. Jordan*[93)] and *P. Mansion*[94)].

If the sequence of numbers (a_ν) is *monotonic*[95)], i.e., never *de*- or never *in*creasing, then for its *convergence* the condition suffices that the a_ν remain numerically below a fixed number (example: the systematic fractions). One can also take this *simplest* form of convergent sequences of numbers as the starting point for the theory of irrational numbers and limit values[96)]. But then, in order to be able to define subtraction and division, one always needs *two* such sequences (one never *de*- and one never *in*creasing)[97)].

14. The Infinitely Large and Infinitely Small. If the terms of an unbounded sequence of well-defined numbers (a_ν) have the property that, no matter how large a positive number G is prescribed, from a certain index ν onwards throughout: $a_\nu > G$ (or $a_\nu < -G$), then one says that the *limit value*

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89) Math. Ann. 21 (1883), p. 124.

90) L.c. p. 30.

91) Fondamenti per la teorica etc. p. 27.

92) Allg. Funct.-Theorie p. 260.

93) Cf. my remarks in the Münch. Sitzber. 27 (1897), pp. 357, 358. - *Stolz* has also proven the existence of the limit value through its representation in systematic form: Allg. Arithm. 1, p. 115 ff. (cf. No. 9 of this article).

94) Mathesis 5 (1885), p. 270.

95) This expression comes from *C. Neumann*: "Über die nach Kreis-, Kugel- und Cylinder-Funct. fortschr. Entw.", Leipzig 1881, p. 26.

96) Cf. *Mansion*, Mathesis 5, p. 193.

97) *Bachmann* l.c. pp. 12, 13. Cf. No. 4, Note 20. If one chooses even more special sequences of numbers for the definition of irrational numbers, e.g., the systematic fractions, a difficulty already arises in the definition of addition and multiplication

of the a_ν is positive (negative) *infinity*, in symbols:

$$\lim_{\nu=\infty} a_\nu = +\infty \quad (\text{or} \quad \lim_{\nu=\infty} a_\nu = -\infty).$$

The sequence of numbers (a_ν) is then called *properly divergent*.

This statement, according to today's conception, is to be regarded as a *definition of the infinite*[98], while older analysts used to view it as a provable *theorem*[99]; in truth, however, any such proof would have to amount to a mere *circular reasoning* as long as no other *mathematically tangible definition* of the *infinite* existed[100] (which has been the case only since very recent times - see somewhat further below).

Based on the definition given above, among the numbers a_ν, no matter how large ν may be assumed, *none is infinitely large*; nevertheless, one uses the *expression*: the numbers a_ν *become infinitely large* with *unboundedly* increasing values of ν. The *infinite*, which in this form of definition appears merely as a *variably-finite*, thus as a *becoming*, not a *become*, is designated as *potential*[101] or *improper*[102] *infinite*.

But also *independently* of any such *process of becoming*, the *infinite* can be strictly arithmetically defined as an *actual* or *proper infinite*. *B. Bolzano*[103] has emphasized as a peculiar characteristic of an *infinite* set of elements that the elements which form merely a certain *part* of that set can be *uniquely-invertibly* assigned to the elements of the *total* set (e.g., to the *total* set of numbers $0 \leq y \leq 12$ the *partial* set of numbers $0 \leq x \leq 5$ on the basis of the stipulation: $5y = 12x$). *G. Cantor* has formulated the same property to the effect that in an *infinite* set, and only in such a set, a *part* of the set can possess the *same cardinality* as itself[104].

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98) Approximately since *Cauchy*: Analyse algébr. pp. 4, 27.

99) See e.g. *Jac. Bernoulli*, Positiones arithmeticae de seriebus infinitis (1689), Prop. II (Opera, Genevae 1744, 1, p. 379).

100) Also what e.g. *DuBois-Reymond* says in his Allg. Functionen-Theorie p. 69 ff. about the distinction of the *"infinite"* from the *"unbounded"* appears untenable. Cf. my remarks Münch. Sitzber. 1897, p. 322, footnote 1.

101) The *Infinitum potentia* or *syncategorematic infinite* of the philosophers, in contrast to the *Infinitum actu* or *categorematic* (actual) *infinite* to be mentioned immediately.

102) According to *G. Cantor*, Math. Ann. 21 (1883), p. 546.

103) Paradoxien des Unendlichen. Leipzig 1851. § 20.

104) Journ. f. Math. 84 (1878), p. 242.

Independent of the two mentioned[105)], *Dedekind* has actually elevated this property to the *definition* of the *infinite*, i.e. (retaining the *Cantorian* terminology just used): A set is called *infinite* if it contains a *sub*set of *equal cardinality*; in the opposite case, it is called *finite*[106)]. *Dedekind* then *proves* the *existence of infinite* sets[107)], derives from this the concept of the *natural sequence of numbers* and finally that of the *number* of a *finite* set.

Conversely, *G. Cantor*, regarding the concept of *number* for *finite* sets in the usual way as something given *a priori*, has transferred this concept to *infinite* sets and has thereby been led to the establishment of a consistently developed *system of properly-infinite* (*"supra-finite"* or *"transfinite"*) numbers[108)].

In *arithmetic*, the *infinite* always appears only as *improperly-infinite*, thus as *variably-finite* whose absolute value is not bound to any upper limit. In *function theory*, especially for complex variables, it has nevertheless proven expedient to introduce, besides this *improperly-infinite*, also a *properly-infinite* in such a way that to all possible *finite* values of which a variable is capable, *the value* ∞ *is added like a single, definite one* (geometrically represented by a definite point)[109)].

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105) Cf. the preface to the 2nd edition of the work: Was sind und was sollen die Zahlen? Braunschweig 1895. (First edition 1887.)

106) L.c. No. 64. *D.* thereby designates two sets of *"equal cardinality"* (thus those whose elements can be uniquely-invertibly assigned to each other) as *"similar"* (or also more explicitly as such that can be similarly mapped into each other). Another, in a certain respect even simpler definition of the infinite is given by *D.* in the above-cited preface, p. XVII. Cf. also *Franz Meyer*, Zur Lehre vom Unendlichen. Antr.-Rede, Tübingen 1889; *C. Cram*, Wundts Philos. Studien 21 (1895), p. 1; *E. Schröder*, Nova acta Leop. 71 (1898), p. 303.

107) Similarly, as already *Bolzano* l.c. § 13.

108) Math. Ann. 21 (1883), p. 545 ff. The relevant treatise also contains a historical-critical discussion of the concept of infinity provided with numerous citations. More on *transfinite* numbers see I A 5, No. 3 ff.

109) This proper *infinite* of function theory can by no means always be replaced without further ado by the *improper infinite*; in other words: the behavior of a function $f(x)$ *for all possible arbitrarily large* values of x need by no means yet determine that for the value $x = \infty$. If one sets e.g.

$$f(x) = \lim_{n=\infty} f_n(x), \quad where: f_n(x) = \frac{n}{n+x}$$

The so-called *infinitely small* behaves somewhat differently. If $\lim a_\nu = 0$, one often uses the *expression*: the numbers a_ν *become infinitely small* with *unboundedly* increasing values of ν[110]. Wherever in *arithmetic*, *function theory*, *geometry* the so-called *infinitely small* appears, it is always only a *becoming infinitely small*, thus according to the terminology used above an *improperly-infinitely small*[111]. Although it has recently been possible to establish self-consistent systems of *properly-infinitely small "magnitudes"*[112], these are merely *systems of symbols* with *purely formally* defined laws, which partly differ from those valid for real numbers. Such fictitious *properly-infinitely small magnitudes* have no direct relation to *real numbers*; they find no place in proper *arithmetic* and *analysis* and cannot, like real numbers, serve to *describe geometric magnitude* relations without contradiction. In particular, from the possibility of such arithmetic constructions, the *existence of infinitely small geometric magnitudes* (e.g., line elements) *cannot* be inferred. *G. Cantor* has rather explicitly shown that from the assumption of numbers that are numerically *smaller* than *any* positive number, precisely the *non-existence of infinitely small line segments* can be inferred[113].

15. Upper and Lower Limits. From an *improperly divergent*, i.e.

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so one has for *every arbitrarily large* finite x without exception:

$$f(x) = 1,$$

whereas:

$$f_n(\infty) = 0, \quad \textit{thus also}: \quad f(\infty) = 0.$$

In general, the behavior of a function $f(x)$ for that *value* or *point* $x = \infty$ is defined by that of $f(\frac{1}{x})$ for $x = 0$. Cf. II B 1. Another type of the properly-infinite ("the infinitely distant line") has proven expedient in projective *geometry*.

110) *Cauchy*, Anal. algébr. p. 4, 26.

111) With the *properly-infinite* $x = \infty$ of function theory corresponds *not* a *properly-infinitely* small value x, but the value $x = 0$.

112) *O. Stolz* has, using *Du Bois-Reymond's* investigations, constructed two different systems of properly-infinitely small magnitudes: Ber. d. naturw.-medic. Vereins, Innsbruck 1884, p. 1 ff. 37 ff. Allg. Arithm. 1, p. 205 ff. Cf. I A 5, No. 17. On *P. Veronese's "Infiniti und Infinitesimi attuali"* cf. I A 5, footnote 103, 107. - A detailed historical-critical presentation of the theory of infinitely small magnitudes is given in *G. Vivanti's* work: Il concetto d'infinitesimo, Mantova 1894.

113) Z. f. Philos. 91, p. 112. Cf. also *O. Stolz*, Math. Ann. 31 (1888), p. 601. *G. Peano*, Rivista di Mat. 2 (1872), p. 58.

neither convergent nor properly divergent sequence of numbers (a_ν), *two convergent* or *properly divergent* sequences of numbers (a_{m_ν}), (a_{n_ν}) of the following nature can always be extracted:

If one sets
$$\lim_{\nu=\infty} a_{m_\nu} = A, \quad \lim_{\nu=\infty} a_{n_\nu} = a, \tag{1}$$
(where $A > a$ and A, a either represent definite numbers or can also be $A = +\infty$, $a = -\infty$), then from the sequence (a_ν) *no* sequence can be extracted which possesses a *greater limit* than A or a *smaller limit* than a. A is accordingly called the *greatest* or *upper*, a the *smallest* or *lower limit* of the a_ν, in symbols[114]:
$$\lim_{\nu=\infty} \sup a_\nu = A, \quad \lim_{\nu=\infty} \inf a_\nu = a, \tag{2}$$
or more briefly[115]:
$$\overline{\lim_{\nu=\infty}} a_\nu = A, \quad \underline{\lim_{\nu=\infty}} a_\nu = a. \tag{3}$$

By means of this generalization of the concept of limit, the *convergent* and *properly divergent* sequences of numbers appear as that limiting case in which *upper and lower limits coincide*, so that:
$$\overline{\lim_{\nu=\infty}} a_\nu = \underline{\lim_{\nu=\infty}} a_\nu = \lim_{\nu=\infty} a_\nu. \tag{4}$$

The concept of the *upper and lower limits* is already found by *Cauchy*[116], who has made an extremely important application of it specifically in the theory of series[117]. *Du Bois-Reymond* has introduced for the *upper and lower limits* the designation *upper and lower indeterminacy bounds*[118] and is therefore often falsely regarded as the *inventor* of the *concept* thus designated. Nevertheless, one can say that he was the first to explicitly emphasize the great and general significance of that concept for the theory of series and functions and to give occasion for its consistent application[119].

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114) According to *Pasch*, Math. Ann. 30 (1887), p. 134.

115) According to a notation recently introduced by me: Münch. Sitzber. 28 (1898), p. 62. The occasional use of the notation $\overline{\underline{\lim}}_{\nu=\infty} a_\nu$ is meant to indicate that in the relevant context *either the upper or the lower limit* may be taken.

116) Anal. algébr. p. 132, 151 etc. "la plus grande des limites". *C.* designates the upper limit as "la limite vers laquelle tend la plus grande valeur". "La plus petite des limites" in *N. H. Abel*: Oeuvres 2, p. 198.

117) Cf. No. 23.

118) Antritts-Progr. d. Univ. Freiburg (1871), p. 3. Münch. Abh. 12, I. Abth. (1876), p. 125. Allg. Funct.-Th. p. 266.

119) Cf. especially the above-cited essay by *Pasch*.

16. Upper and Lower Bounds. Related to the concept of the *upper (lower) limit*, yet to be sharply distinguished from it, is the concept of the *upper (lower) bound*[122] first noted by *Bolzano*[120], but especially emphasized by *Weierstrass* (in his lectures)[121]: Every sequence of numbers (a_ν) with terms remaining finite (i.e., contained between two definite numbers) possesses a definite *upper* and *lower bound* G, g, i.e., one has for *every* ν: $g \le a_\nu \le G$ and for at least one value each $\nu = m$, $\nu = n$: $G - \varepsilon < a_m \le G, \quad g \le a_n < g + \varepsilon$ for arbitrarily small prescribed positive ε. If there is a term $a_m = G$ (possibly also several or even infinitely many), then the *upper bound* of the a_ν is also called their*maximum*[123]. If there is *no* such term, then there must be infinitely many terms a_{m_ν} for which: $G - \varepsilon < a_{m_\nu} < G$, i.e., in this case the *upper bound* G is simultaneously the *upper limit* of the a_ν. This obviously also occurs when for *infinitely many* values of ν the relation $a_\nu = G$ holds.

The analogous applies regarding the lower bound g.

If the a_ν *do not* remain below a certain *positive* or above a certain *negative* number, then $G = +\infty$ or $g = -\infty$. Also in this case, the upper or lower *bound* appears simultaneously as the upper or lower *limit*[124].

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120) Beweis des Lehrs. etc. p. 41. Cf. *Stolz*, Math. Ann. 18 (1881), p. 257.

121) *Pincherle* l.c. p. 242 ff.

122) *Pasch* l.c. designates what is here (following *Weierstrass*) called upper (lower) *bound* as upper (lower) *barrier*, and uses the expression upper (lower) *bound* for the upper (lower) *limit*. French (and Italian) authors tend to use the expression *limite supérieure* (*limite superiore*) etc. sometimes in one sense, sometimes in the other, which can easily give rise to ambiguities.

123) *Darboux* (Ann. de l'école norm. (2) 4, p. 61) calls the upper (lower) bound: *"la limite maximum (minimum)"* a designation that should not be confused with *maximum (minimum)*. In the case $a_m = G$, I tend to designate the upper bound even more concisely as the *real maximum* of the a_ν, and call it their *ideal maximum* if *no* term reaches the upper bound G (an assumption that also encompasses the case $G = \infty$). Then one can say: The upper *limit* coincides with the upper *bound* if and only if the latter is an *ideal* or *infinitely often occurring real maximum*. Analogously for the lower bound.

124) *G. Peano* has pointed out that in certain cases (e.g., def. of the definite integral, of rectification, etc.) one operates more easily and precisely with the concept of the upper (lower) *bound* than with that of the *limit*: Ann. di Mat. (2), 23 (1895), p. 153.

17. Operations with Limit Values. The number $e = \lim(1+\frac{1}{\nu})^\nu$.

If (a_ν), (b_ν) are convergent sequences of numbers, then the elementary rules of calculation to be directly connected to the definition of irrational numbers yield the relations:

$$\left.\begin{array}{l} \lim a_\nu \pm \lim b_\nu = \lim(a_\nu \pm b_\nu), \quad \lim a_\nu \cdot \lim b_\nu = \lim(a_\nu b_\nu), \\ \dfrac{\lim a_\nu}{\lim b_\nu} = \lim(\dfrac{a_\nu}{b_\nu}) \ {}^{125)} \end{array}\right\} \tag{5}$$

(where in the last equation the case $\lim b_\nu = 0$ is to be excluded), and in general:

$$f(\lim a_\nu, \lim b_\nu, \lim c_\nu, \ldots) = \lim f(a_\nu, b_\nu, c_\nu, \ldots), \tag{6}$$

when f denotes any combination of the 4 operations (excluding division by 0).

If the calculation symbol f contains other requirements, e.g., extraction of roots, then eq. (6) is valid as a *definition* equation, provided the right side *converges*. With the help of this principle, in particular the theory of fractional and irrational powers and their inversions, the logarithms, can be consistently and rigorously founded[126].

The distinguished arithmetic properties which the *natural* logarithms (i.e., those with base e) have over all others are based on the relations:

$$\lim(1 + \frac{1}{\nu})^\nu = e, \quad \lim(1 + \frac{a}{\nu})^\nu = e^a \tag{7}$$

(a an arbitrary real number). While the latter appear in *Euler*[127] only in the context that the equality of the limit values on the left with the series serving to *define* e, e^a is derived (in a manner certainly inadequate by today's concepts), *Cauchy*[128] has directly proven the *existence* of those limit values and *based* on them the *definition* of exponential quantities and natural logarithms - a method that has since passed into most textbooks of analysis[129].

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125) From now on, I always write just lim instead of $\lim_{\nu=\infty}$, as far as a misunderstanding seems excluded.

126) Cf. *Stolz*, Allg. Arithm. p. 125-148.

127) Introductio in anal. inf. 1 § 115-122.

128) Résumé des leçons etc. (1823), p. 2.

129) Thereby the definition of the power with arbitrary (possibly irrational) exponents is usually presupposed as already known.

18. So-called Indeterminate Expressions. If the $\lim a_\nu$, $\lim b_\nu$, ... occurring in eq. (5), (6) become ∞ or 0, then on the *left* sides of those equations arise in part so-called *indeterminate expressions*[130)] (such as: $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, 0^0, ∞^0 etc.), whose *"true values"* one is accustomed to designate, not very aptly, as the *limit values* standing on the *right* (insofar as these have a definite meaning). Although the methods for determining such limit values gain their full generality only through the introduction of a *continuous* variable in place of the variable *whole* number ν and in this form belong to differential calculus[131)], they are nevertheless ultimately based (like the whole theory of functions of *continuous* variables) on certain simple theorems about limit values of ordinary sequences of numbers. Here belong the following relations due to *Cauchy*[132)].

One has

$$\lim \frac{a_\nu}{\nu} = \lim(a_{\nu+1} - a_\nu) \quad (\text{Example: } \lim \frac{\log \nu}{\nu} = 0, \lim \frac{e^\nu}{\nu} = \infty) \tag{8}$$

and for $a_\nu > 0$:

$$\lim a_\nu^{\frac{1}{\nu}} = \lim \frac{a_{\nu+1}}{a_\nu} \quad (\text{Example: } \lim \sqrt[\nu]{\nu} = 1, \quad \lim \sqrt[\nu]{\nu!} = \infty), \tag{9}$$

provided that the limit values standing on the *right* (in the broader sense) *exist*[133)] (but not conversely).

Stolz has generalized the first of these theorems as follows[134)]:

If (m_ν) is *monotonic* and: $\lim m_\nu = \pm\infty$ or: $\lim m_\nu = 0$, then:

$$\lim \frac{a_\nu}{m_\nu} = \lim \frac{a_{\nu+1} - a_\nu}{m_{\nu+1} - m_\nu}, \tag{10}$$

if the limit value standing on the *right* (in the broader sense) exists.

One can, however, also use the existence of the limit value $\lim(1 + \frac{1}{\nu})^\nu$ for the *definition* of the power with arbitrary real exponents: cf. *Th. Wulf*, Wiener Monatsh. 8, p. 43 ff. This method can, by the way, also be transferred to complex values of a; cf. *J. A. Serret*, Calcul diff. 1 (or *Serret-A. Harnack* 1), Art. 366.

130) In *Cauchy*: Valeurs singulières (Anal. algébr. p. 45).

131) Cf. II A 1.

132) Anal. algébr. p. 59. (The relevant theorems are there initially proven in the more general form, where $f(x)$ takes the place of a_ν, and derived through specialization $x = \nu$.)

133) I.e., are finite or infinite with definite sign.

134) Math. Ann. 14 (1879), p. 232. Allg. Arithm. 1, p. 173.

19. Gradation of Becoming Infinite and Zero. On the investigation of quotients of the form $\frac{\infty}{\infty}$ (i.e., of $\lim \frac{a_\nu}{b_\nu}$ where $\lim a_\nu = \infty, \lim b_\nu = \infty$) is based the *gradation* of becoming infinite of sequences of numbers (or of functions). If $\lim a_\nu = +\infty$, $\lim b_\nu = +\infty$, then, *if* $\lim \frac{a_\nu}{b_\nu}$ *exists at all*[135], the following three cases are to be distinguished:

$$\lim \frac{a_\nu}{b_\nu} = 0, \quad \lim \frac{a_\nu}{b_\nu} = g > 0, \quad \lim \frac{a_\nu}{b_\nu} = \infty, \tag{11}$$

for which *Du Bois-Reymond* has introduced the notations[136]

$$a_\nu \prec b_\nu, \quad a_\nu \sim b_\nu, \quad a_\nu \succ b_\nu, \tag{12}$$

in words:
a_ν becomes infinite of lower order (weaker, slower) than b_ν
a_ν becomes infinite of the same order (equally) as b_ν
a_ν becomes infinite of higher order (stronger, faster) than b_ν:
or more briefly:

a_ν is *infinitarily* smaller than b_ν
a_ν is *infinitarily* equal to b_ν
a_ν is *infinitarily* greater than b_ν

I am accustomed to apply the notation $a_\nu \sim b_\nu$ and the corresponding expression also when it is only established that $\underline{\lim} \frac{a_\nu}{b_\nu}$ and $\overline{\lim} \frac{a_\nu}{b_\nu}$ are *both* finite and different from zero (thus:

$$0 < g \leq \overline{\underline{\lim}} \frac{a_\nu}{b_\nu} \leq G < \infty)$$

and have added to the above notations also the following[137]:

$$a_\nu \cong g \cdot b_\nu, \quad if: \quad \lim \frac{a_\nu}{b_\nu} = g. \tag{13}$$

If (M_ν) is monotonically increasing, $\lim M_\nu = \infty$[138], then one has:

$$\cdots \prec (\log_2 M_\nu)^{p''} \prec (\log_2 M_\nu)^{p'} \prec M_\nu^p \prec (e^{M_\nu})^{p_1} \prec (e^{e^{M_\nu}})^{p_2} \prec \cdots, \tag{14}$$

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135) This need not even be the case when a_ν, b_ν are *both monotonic*, see e.g. *Stolz*, Math. Ann. 14, p. 232 and cf. No. 29, 30 of this article.

136) Ann. di Mat. Ser. II 4 (1870), p. 339. The development and utilization of the algorithm defined in (11) (12) forms the content of *Du Bois-Reymond's infinitary calculus*.

137) Math. Ann. 35 (1890), p. 302.

138) In the following, the symbol (M_ν) shall once and for all represent a sequence of numbers of this kind.

where $p^{(\chi)}$, p, p_χ denote entirely arbitrary (e.g., also *increasing*) positive numbers and $\log_\chi M_\nu$ represents the χ-fold iterated logarithm[139]. One can thus, starting from an arbitrarily chosen *"infinite"* $\lim M_\nu$, establish a scale unbounded on both sides of ever *weaker* or ever *stronger "infinites"*, so-called *order types* of the infinite. This scale can be thickened in infinitely many *ways*[140]. One is also not restricted to logarithms and exponential functions in its formation; but they are the *analytically simplest* functions of this kind. One can, however, also construct sequences of numbers or functions that become infinite more weakly (strongly) not only than any specific *individual*, but than *all possible* iterated logarithms[141] (exponential functions). The analogous also holds for *any arbitrary* scale of such order types[142].

In connection with No. 14, it may be noted that these *"different types"* of *infinite* are by no means *proper* infinites in the sense specified there. The so-called infinitary relations of the form (12) are merely compilations of an unbounded number of relations between *finite* numbers that are not bound to any upper limit[143].

The analogous considerations can be made regarding becoming zero or infinitely small. Only naturally in the case $\lim a_\nu = 0$, $\lim b_\nu = 0$, the relation $a_\nu \prec b_\nu$ has the meaning: a_ν becomes infinitely small of *higher* order (stronger, faster) than b_ν, and so on[144].

20. Limit Values of Doubly Infinite Sequences of Numbers. The *limit values of doubly infinite* sequences of numbers have, to my knowledge, not yet been explicitly treated in the literature; one has only investigated *special* forms of such limit values (double series) and limit values of *functions*

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139) One was led to the consideration of such iterated logarithms (and, as a natural complement, to that of iterated exponential quantities) through investigations on series convergence; cf. No. 26 of this article. *Abel* was, as far as I could determine, the first who made use of the iterated logarithms in this sense: Oeuvres compl. Ed. Sylow-Lie 1, p. 400; 2, p. 200. Scales of similar form as (14) are found first in *A. de Morgan*, Diff. and integr. calculus (London 1839) p. 323.

140) *Du Bois-Reymond* l.c. p. 341.

141) *Du Bois-Reymond*, J. f. Math. 76 (1873), p. 88.

142) *Du Bois-Reymond*, Math. Ann. 8, p. 365, footnote. *Pincherle*, Mem. Acad. Bologn. (4), 5 (1884), p. 739. *J. Hadamard*, Acta math. 18 (1894), p. 331.

143) Cf. my remarks in the Münch. Sitzber. 27 (1897), p. 307.

144) More on *"types of infinity"* see I A 5, No. 17.

of two variables, of which *at least one* (in the infinite series $\sum f_\nu(x)$) appears as *continuously* variable. Since the characteristic of the possibilities coming into question here emerges most simply in sequences of numbers of the form $a_{\mu\nu}$ ($\mu = 0, 1, 2, \ldots;\ \nu = 0, 1, 2, \ldots$)[145], I have recently briefly compiled the most important theorems about such limit values[146]. As a *criterion* for the existence of a finite or positively infinite $\lim_{\mu,\nu=\infty} a_{\mu\nu}$ appears thereby a condition of the form: $|a_{\mu+\varrho,\nu+\sigma} - a_{\mu\nu}| \leq \varepsilon$ or $> G$ for $\mu \geq m$, $\nu \geq n$. Hereby the existence of $\lim_{\nu=\infty} a_{\mu\nu}$ for any specific μ and $\lim_{\mu=\infty} a_{\mu\nu}$ for any specific ν is in no way prejudiced. On the other hand, there obviously exist under all circumstances: $\underline{\lim}_{\nu=\infty} a_{\mu\nu}$, $\overline{\lim}_{\nu=\infty} a_{\mu\nu}$ ($\mu = 0, 1, 2, \ldots$), $\underline{\lim}_{\mu=\infty} a_{\mu\nu}$, $\overline{\lim}_{\mu=\infty} a_{\mu\nu}$ ($\nu = 0, 1, 2, \ldots$), and the main theorem holds:

$$(15) \qquad \lim_{\mu,\nu=\infty} a_{\mu\nu} = \lim_{\mu=\infty}\big(\overline{\underline{\lim}}_{\nu=\infty} a_{\mu\nu}\big) = \lim_{\nu=\infty}\big(\overline{\underline{\lim}}_{\mu=\infty} a_{\mu\nu}\big), \qquad (15)$$

if the *first* of these limit values (in the broader sense) *exists*.

Part Two. Infinite Series, Products, Continued Fractions and Determinants.

III. Infinite Series.

21. Convergence and Divergence. The simplest type of lawfully defined sequences of numbers is formed by the *infinite series* (s_ν), in which each term s_ν is generated from the preceding one by a simple *addition*, so that:

$$s_\nu = s_{\nu-1} + a_\nu = a_0 + a_1 + \cdots + a_\nu.$$

One then says the *infinite series* $\sum_{\nu=0}^{\infty} a_\nu$ is *convergent*, properly or improperly *divergent*, according to whether the sequence of numbers (s_ν) *converges* or properly or improperly *diverges*. If $\lim s_\nu = s$ (where s is a definite number incl. 0), then s is called the *sum* of the series[148].

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145) This applies, e.g., also regarding the fundamental concept of uniform convergence. Cf. II A 1.

146) Münch. Ber. 27 (1897), p. 103 ff.

147) L.c. p. 105.

148) Some authors initially designate s only as the *limit value* of the series and use the expression *sum* only when $\lim s_\nu$ is *commutative*, thus the series converges *absolutely* (cf. No. 31). On the meaning of the symbol $\sum_{\nu=-\infty}^{\infty} a_\nu$ cf. No. 59, footnote 448.

One also frequently uses the expression that the *sum* of the series is *infinitely large* or *indeterminate* (it *oscillates*) when (s_ν) *properly* or *improperly* diverges. As a *necessary and sufficient* condition for the *convergence* of the series, it follows from No. 13:

The quantity $|s_{n+\varrho} - s_n| \equiv |a_{n+1} + a_{n+2} + \cdots + a_{n+\varrho}|$ must become *arbitrarily small* solely through the choice of n for *every* ϱ.

Although the introduction of infinite series dates back to the 17th century[149] and their treatment occupies an exceedingly broad space in the mathematical literature of the 18th, one will search in vain for such a *criterion* of convergence[150]. If one *asked* at all about the *convergence* of a series development obtained through any formal operations (which in itself was already an exception), one considered the determination that $\lim a_\nu = 0$ already sufficient, although *Jac. Bernoulli* had already demonstrated the *divergence* of the harmonic series $\sum \frac{1}{\nu}$ [151]. Even *J. L. Lagrange* in his treatise on the solution of literal equations by series still stands completely on this standpoint[152].

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149) On the earlier developmental history of the theory of infinite series cf. *Reiff* l.c.

150) *Reiff* (p. 119) seems to me to err when he interprets a passage in *Euler* (Comm. Petrop. 7, 1734, p. 150) to mean that the latter actually already knew the convergence condition in the (*Cauchy*ian) form: $\lim_{n=\infty}(s_{n+\varrho} - s_n) = 0$. The relevant passage in *Euler* only states that a series *diverges* when: $\lim_{n=\infty} |s_{kn} - s_n| > 0$.

151) Pos. arithm. de seriebus 1689. Prop. XVI (Opera omnia 1, p. 392). *B.* gives there *two* proofs, and designates his brother *Johann* as the originator of the *first* (based on the so-called *Bernoullian paradox* $\sum_{\nu=2}^{\infty} \frac{1}{\nu} = \sum_{\nu=1}^{\infty} \frac{1}{\nu}$). The *second* (with the help of the inequality

$$\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a^2} < \frac{1}{a} + \frac{a^2 - a}{a^2} = 1)$$

is in principle still the usual one today.

152) Berl. Mem. 24 (1770). Oeuvres 3, p. 61. "... pour qu'une série puisse être regardée comme représentant réellement la valeur d'une quantité cherchée, il faut qu'elle soit convergente à son extrémité, c'est à dire que ses derniers termes soient infiniment petits, de sorte que l'erreur puisse devenir moindre qu'aucune quantité donnée". The convergence investigation that follows is limited to showing that the individual series terms eventually converge to zero. After this, it can hardly seem surprising that, for example, in the 1st volume of *Klügel*'s W. B. printed in 1803, p. 555, one still finds the following definition: "A series is convergent if its terms in their sequence become ever smaller. The sum of the terms then approaches ever more closely the value of the quantity which is the sum of the

And the introduction of the *remainder term* of the *Taylor* series occurs in *Lagrange* by no means with the intention of proving its *convergence* (this is not touched upon with a single word as something self-evident), but merely to be able to *estimate* the error bound when truncating the series at a finite point[153)].

The first essentially rigorous formulation of the necessary and sufficient condition for the convergence of a series is usually attributed to *Cauchy*[154)]. *Herm. Hankel*[155)] and *O. Stolz*[156)] have, however, pointed out that the same can be found already some years before *Cauchy* in *Bolzano*[157)]. The latter's version, which (apart from the notation) agrees exactly with the one given above, appears even more precise than that given by *Cauchy*, which does not exclude the possibility of a misunderstanding[158)]. Since *Bolzano*'s writings received little attention until very recent times, it must nevertheless be said that *Cauchy* is to be regarded as the actual founder of an exact general theory of series[159)].

22. The Convergence Criteria of Gauss and Cauchy. The *true criterion* for the convergence and divergence of a series indicated above is usable for determining convergence or divergence only in a few cases (e.g., for the geometric progression, for series of the form $\sum(a_\nu - a_{\nu+1})$, for the harmonic series). This circumstance led to the establishment of more convenient

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series continued to infinity."

153) Théorie des fonctions (1797). Oeuvres 9, p. 85.

154) Anal. algébr. (also 1821), p. 125.

155) *Ersch u. Gruber*, Art. Grenze, p. 209.

156) Math. Ann. 18 (1881), p. 259.

157) Beweis des Lehrsatzes etc. 1817.

158) This applies to an even greater extent to a later formulation appearing in the Anc. exerc. 2 (1827), p. 221: $\lim_{n=\infty}(s_{n+\varrho} - s_n) = 0$, which has indeed been misunderstood and consequently contested. Cf. my note in the Münch. Sitzber. 27 (1897), p. 327. *N. H. Abel*, who expresses himself almost verbatim the same way in his treatise on the binomial series (J. f. Math. 1, 1826, p. 313), gives in a note dating from 1827 but only found in his estate (Oeuvres 2, p. 197) a formulation agreeing with ours, free from objection.

159) *K. F. Gauss* in his investigation of the hypergeometric series (1812), which admittedly provides the *first example* of exact convergence investigation, does not go into *general* convergence questions.

manageable *convergence* and *divergence* criteria, i.e., conditions which, while *not necessary* for convergence or divergence, prove to be *sufficient.* The first criteria of this kind come from *Gauss*[160] and relate to series with all positive terms a_ν, for which:

$$\frac{a_{\nu+1}}{a_\nu} = \frac{\nu^m + A\nu^{m-1} + B\nu^{m-2} + \cdots}{\nu^m + a\nu^{m-1} + b\nu^{m-2} + \cdots}.$$

The series *diverges* when $A - a \geq -1$, it *converges* when $A - a < -1$[161]. The *divergence* in the case $A - a > 0$ or $A - a = 0$ is directly concluded from the fact that the terms of the series grow to infinity or approach a finite limit different from zero. On the other hand, in the case $A - a < 0$, the *divergence* or *convergence* is obtained with the help of the principle of *series comparison* (i.e., the term-by-term comparison of the series to be investigated with another *series already recognized as divergent or convergent*, e.g., through direct summation), which is to be regarded as *fundamental* for all further convergence investigations.

23. Continuation. After *Cauchy* had established that the *convergence* of a series with *positive* and *negative* terms is assured if the series of *absolute values* converges[162], it was primarily a matter of developing convergence criteria for series with all *positive terms.* By comparison with the geometric progression, he first obtained the two *fundamental criteria of the first and second kind*[163], namely:

(I) $\sum a_\nu$ *diverges*, when $\overline{\lim}\sqrt[\nu]{a_\nu} > 1$; *converges*, when $\overline{\lim}\sqrt[\nu]{a_\nu} < 1$,
(II) $\sum a_\nu$ *diverges*, when $\lim \frac{a_{\nu+1}}{a_\nu} > 1$; *converges*, when $\lim \frac{a_{\nu+1}}{a_\nu} < 1$.

What is to be emphasized is the sharp distinction in the formulation of these two criteria; for (I), the nature of the *upper* limit of $\sqrt[\nu]{a_\nu}$ is already sufficient to decide, with the exclusion of the *single* case $\overline{\lim}\sqrt[\nu]{a_\nu} = 1$

160) See the just cited treatise: Opera 3, p. 139.

161) An extension of these criteria to the case of *complex* a_ν has been given by *Weierstrass*: J. f. Math. 51 (1856), p. 22 ff.

162) Anal. algébr. p. 142. The formulation of the proof is admittedly inadequate. More rigorous: Résumé analyt. p. 39.

163) L.c. p. 133, 134. We designate a criterion, following *Du Bois-Reymond* (J. f. Math. 76, p. 61), as one of the *first* or *second* kind, depending on whether it depends exclusively on a_ν or on $\frac{a_{\nu+1}}{a_\nu}$.

the divergence or convergence; for (II), only the case is explicitly considered where *a definite* $\lim \frac{a_{\nu+1}}{a_\nu}$ *exists*, i.e., besides the case $\lim \frac{a_{\nu+1}}{a_\nu} = 1$, all those remain unresolved where *no definite limit* exists[164]. This superiority of criterion (I) over (II) has been specially emphasized by *Cauchy*[165], and he has further shown how it can serve to *precisely fix the convergence interval*[166] (the *radius of convergence*[167]) of a power series $\sum a_\nu x^\nu$ in *every* case[168]. For the possible resolution of the case which the application of criterion (I) leaves undecided, *Cauchy* proves an auxiliary theorem about the *simultaneous* divergence and convergence of the series $\sum a_\nu$ and $\sum 2^\nu \cdot a_{2^\nu-1}$ (if $a_{\nu+1} \le a_\nu$), infers from it the divergence of the series $\sum \frac{1}{\nu^{1+\varrho}}$ for $\varrho \le 0$, the convergence for $\varrho > 0$, and derives from it a sharpened *criterion of the first kind*:

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164) Somewhat more completely, one can formulate (II) as follows: $\sum a_\nu$ *diverges* when $\underline{\lim} \frac{a_{\nu+1}}{a_\nu} > 1$, *converges* when $\overline{\lim} \frac{a_{\nu+1}}{a_\nu} < 1$. The question remains undecided when simultaneously:

$$\underline{\lim} \frac{a_{\nu+1}}{a_\nu} \le 1, \qquad \overline{\lim} \frac{a_{\nu+1}}{a_\nu} \ge 1.$$

165) L.c. p. 135 "le *premier* de ces théorèmes etc."

166) L.c. p. 151. Résumés analyt. (1833), p. 46.

167) L.c. p. 286. Rés. analyt. p. 113. Exerc. d'Anal. 3 (1844), p. 390.

168) It is peculiar that this result, extremely important for function theory (to which *Cauchy* himself evidently attached great value), seems to have been completely overlooked or fallen into oblivion in many cases. Only a few years ago it was rediscovered by *J. Hadamard* (J. de Math. (4) 8 [1892], p. 107) and has since then often been cited as the *"Hadamard theorem"*. – On the other hand, despite the flawlessly correct formulation of criterion (I) and the explicit emphasis on its more *special* character, the opinion has partly formed that the three assumptions $\lim \frac{a_{\nu+1}}{a_\nu} < 1, = 1, > 1$ exhaust *all* possibilities coming into consideration, or that at least the *convergence* of $\sum a_\nu$ in the case of the *non*-existence of a definite $\lim \frac{a_{\nu+1}}{a_\nu}$ appears as a *special curiosity* (cf. my remarks Math. Ann. 35 [1890], p. 308). And accordingly, in many textbooks (even belonging to the most recent times), the *whole theory of power series* is founded on the much too special assumption that $\lim |\frac{a_{\nu+1}}{a_\nu}|$ *exists*.

$$\lim \frac{\lg \frac{1}{a_\nu}}{\lg \nu} \quad \begin{cases} < 0: & \textit{Divergence}, \\ > 0: & \textit{Convergence}. \end{cases} \tag{16}$$

Elsewhere[169], *Cauchy* shows that the divergence or convergence of the series $\sum_{m=\nu}^{\infty} f(\nu)$ under certain conditions coincides with that of the integral $\int_m^\infty f(x)dx$, and from this obtains the *criterion pair*:

$$\begin{cases} \lim \nu \cdot a_\nu > 0: & \textit{Divergence}, \\ \lim \nu^{1+\varrho} \cdot a_\nu = 0: & \textit{Convergence}, \quad (\varrho > 0), \end{cases} \tag{17}$$

which, incidentally, is easily recognized as essentially *equivalent* to the *disjunctive double criterion* (16) and could have been derived more simply directly from the behavior of the series $\sum \frac{1}{\nu^{1+\varrho}}$ $(\varrho \geq 0)$. More important, it seems to me, is that *Cauchy* here for the first time proves the *divergence* of $\sum \frac{1}{\nu \lg \nu}$, the *convergence* of $\sum \frac{1}{\nu (\lg \nu)^{1+\varrho}}$ for $\varrho > 0$, whereby the path for the further sharpening of criteria (16) and (17) appears directly indicated.

24. Kummer's General Criteria. The criteria of *J. L. Raabe*, *J. M. C. Duhamel*, *de Morgan*, *Bertrand*, *P. O. Bonnet*, *M. G. v. Paucker* (whose publication falls in the period from 1832-1851 and which will be discussed later) provide merely such *sharpenings* of *Cauchy*'s criteria, to which they also essentially adhere in form and method of derivation.

While all the criteria mentioned so far have a *special* character, insofar as they are consistently based on the comparison of a_ν with one of the *special* sequences a^ν, ν^p, $\nu \cdot (\lg \nu)^p$ etc., *E. E. Kummer*[170] has derived the following *convergence* criterion of surprisingly *general* character: $\sum a_\nu$ *converges* if there exists any positive sequence of numbers (p_ν)[171] such that:

$$\lim \lambda_\nu \equiv \lim(P_\nu \cdot \frac{a_\nu}{a_{\nu+1}} - P_{\nu+1} > 0. \tag{18}$$

169) Anc. Exerc. 2 (1827), p. 221 ff. The theorem on the connection of the integral with the series is already found in geometric form in *Colin Mac Laurin* (Treatise of fluxions 1742, p. 289). On the transformation of this criterion by *B. Riemann*, cf. No. 36.

170) J. f. Math. 13 (1835), p. 171 ff.

171) *Kummer* adds the additional condition: $\lim p_\nu \cdot a_\nu = 0$, which is, however, in truth superfluous, as *Dini* first showed in a work to be mentioned immediately.

At the same time, K. shows that $\sum a_\nu$ *diverges* when:

$$\lim P_\nu \cdot a_\nu = 0,^{172)} \quad \lim \lambda_\nu = 0, \quad \lim \frac{P_\nu \cdot a_\nu}{\lambda_\nu} > 0, \tag{19}$$

and demonstrates that there always actually *exist* (infinitely many) sequences of numbers (P_ν) which satisfy one of the criteria (18) (19); but to be able to determine them in each *case*, one would have to be oriented in advance about the *convergence* and *divergence* of $\sum a_\nu$.

25. The Theories of Dini, du Bois-Reymond and Pringsheim. Significant generalizations of the whole theory of convergence criteria are then brought by *Dini*'s extensive treatise, initially directly connecting to *Kummer*'s investigation: *"Sulle serie a termini positivi"*[173)], which, however, does not seem to have found the deserved dissemination.

Du Bois-Reymond's "New Theory of Convergence and Divergence of Series with Positive Terms"[174)] seems to have originated quite independently of *Dini*'s work. Although his investigation methods and main results are not essentially different from those of *Dini*, he goes beyond *Dini* in *principle* through the expressed tendency "to give the theory of convergence and divergence, through more rigorous foundation and through appropriate connection of its theorems, the character of a mathematical theory which it has lacked until now." Since, however, *Du Bois-Reymond* does not seem to me to have achieved this goal by any means[175)], I have taken up the problem posed by him anew and resolved it as[176)]: Rules of the greatest possible generality are derived from the completely uniformly implemented, most obvious principle of series comparison, which not only include all previously known criteria as special cases[177)], but also

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172) Here this condition is *essential*.

173) Pisa 1867 (Tipogr. Nistri). Also: Ann. dell'Univ. Tosc. 9 (1867), p. 41-76.

174) J. f. Math. 76 (1873), p. 61-91.

175) Cf. my critical remarks Math. Ann. 35 (1890), p. 298.

176) Math. Ann. 35 (1890), p. 297-394. Addendum to this: Math. Ann. 39 (1891), p. 125. An extract of this theory can be found: Math. Pap. Congr. Chicago [1896] 1893, p. 305-329.

177) An exception is *Kummer*'s *divergence* criterion, because it depends not, like all other criteria, on *one*, but on *three* conditions. The same, however, is made completely dispensable by the more general divergence criteria of the 2nd kind. Cf. my treatise l.c. p. 365, footnote.

make their scope and their more or less hidden connection more clearly recognizable. In particular, *Kummer*'s convergence criterion of the *second kind*, which has so far stood completely apart in its generality, appears as a natural member of this theory and finds its complete analogue among the criteria of the *first kind*.

26. The Criteria of First and Second Kind. I denote by $d_\nu \equiv D_\nu^{-1}$ or $c_\nu \equiv C_\nu^{-1}$ the general term of a series recognized as *divergent* or *convergent*, and by a_ν that of a series to be judged. Then the *main form* of the criteria of the *first and second kind* emerges:

$$\begin{cases} \lim D_\nu \cdot a_\nu > 0: & \textit{Divergence}, \\ \lim C_\nu \cdot a_\nu < \infty: & \textit{Convergence}^{178)}. \end{cases} \tag{20}$$

$$\begin{cases} \lim(D_\nu \cdot \frac{a_\nu}{a_{\nu+1}} - D_{\nu+1}) < 0: & \textit{Divergence}, \\ \lim(C_\nu \cdot \frac{a_\nu}{a_{\nu+1}} - C_{\nu+1}) > 0: & \textit{Convergence}. \end{cases} \tag{21}$$

One can give these criteria manifold *other forms* if one compares not a_ν directly with d_ν, c_ν, but $F(a_\nu)$ with $F(d_\nu)$, $F(c_\nu)$, where F is understood to be a *monotonic* function. On this is based in particular the transformation of the *criterion pairs* (20) into *disjunctive double criteria*, in which a *single* expression decides on divergence and convergence.

If for any specific choice of D_ν, C_ν one of those criteria fails in such a way that the equality sign appears in place of the signs $<$ *or* $>$, then there arises the *possibility* of obtaining *more effective* criteria if one introduces instead of D_ν, C_ν such $\overline{D_\nu}$, $\overline{C_\nu}$ which satisfy the condition: $\overline{D_\nu} \prec D_\nu$ or $\overline{C_\nu} \succ C_\nu$, in which case the series $\sum \overline{D_\nu}^{-1}$ or $\sum \overline{C_\nu}^{-1}$ is said to be *more weakly* divergent or convergent than $\sum D_\nu{}^{-1}$ or $\sum C_\nu{}^{-1}$[179]. But such D_ν, C_ν can be produced not only in unlimited number, but *all possible ones* with the help of the following theorems:

If $0 < M_\nu < M_{\nu+1}$, $\lim M_\nu = \infty$, then each of the three expressions

178) The notation: $< \infty$ means: *not* ∞, thus *under a finite bound.* Furthermore, note that lim here stands in the sense of $\overline{\lim}$, i.e., there need not exist *a definite limit* of the nature in question.

179) The concept of *"weaker"* divergence and convergence can be formulated more generally. Cf. l.c. p. 319, 327.

$$(a) \quad M_{\nu+1} - M_\nu, \quad (b) \quad \frac{M_{\nu+1} - M_\nu}{M_\nu}, \quad (c) \quad \frac{M_{\nu+1} - M_\nu}{M_{\nu+1}} \tag{22}$$

represents a d_ν, and conversely, every d_ν can be represented in the form (a), (b), and in the case $d_\nu < 1$ also in the form (c)[180)].

Furthermore, the expression:

$$\frac{M_{\nu+1} - M_\nu}{M_{\nu+1} \cdot M_\nu} \tag{23}$$

represents a c_ν – *vice versa*.

The series in question diverge or converge *more weakly* the *more slowly* M_ν increases with ν[181)].

By introducing M_ν^ϱ $(0 < \varrho < 1)$ instead of M_ν, one recognizes with the help of the relation:

$$\frac{M_{\nu+1}^\varrho - M_\nu^\varrho}{M_{\nu+1}^\varrho \cdot M_\nu^\varrho} \sim \frac{M_{\nu+1} - M_\nu}{M_{\nu+1} \cdot M_\nu^\varrho} \preceq \frac{M_{\nu+1} - M_\nu}{M_{\nu+1}^{1+\varrho}} \tag{24}$$

each of these terms as the general term of a *convergent* series[182)].

Then the substitution of $\lg_\chi M_\nu$ $(\chi = 1, 2, 3, \ldots)$ and (24), if one sets:

$$x \cdot \lg_1 x \cdot \lg_2 x \cdots \lg_\chi x = L_\chi(x), \tag{25}$$

with the help of elementary infinitary relations, provides the two indefinitely continuable sequences:

$$(a)\, \frac{M_{\nu+1} - M_\nu}{L_\chi(M_\nu)}, \quad (b)\, \frac{M_{\nu+1} - M_\nu}{L_\chi(M_{\nu+1}) \cdot (\lg_\chi M_{\nu+1})^\varrho}\, (\varrho > 0,\ \chi = 1, 2, 3, \cdots) \tag{26}$$

as general terms of constantly *more weakly* diverging or converging series. These expressions contain for $\chi = 0$ the corresponding initial terms in

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180) The comparison of expressions (22) (b) and (c) with (a) shows directly that for *every divergent* series there exist *more weakly* diverging ones. If one sets $M_{\nu+1} - M_\nu = d_\nu$, $M_0 = 0$, thus: $M_{\nu+1} = d_0 + d_1 + \ldots + d_\nu = s_\nu$, it follows: With the series $\sum d_\nu$ also *diverges* $\sum \frac{d_\nu}{s_{\nu+1}}$ (theorem of *Abel*: J. f. Math. 3 [1828], p. 81) and $\sum \frac{d_\nu}{s_\nu}$ (*Dini* l.c. p. 8).

181) One can directly designate M_ν as the *measure* of divergence or convergence of $\sum(M_{\nu+1} - M_\nu)$ or $\sum \frac{M_{\nu+1} - M_\nu}{M_{\nu+1} \cdot M_\nu}$. Cf. *Du Bois-Reymond* l.c. p. 64.

182) From this follows, with application of the notation used immediately before, that $\sum \frac{d_\nu}{s_\nu^{1+\varrho}}$ *converges*. This theorem is also found already in *Abel* (in the posthumous note mentioned above: 2, p. 198), additionally in *Dini* (l.c. p. 8).

(22), (24), if one further sets $L_0(x) = \lg_0(x) = x$.

Also, one can in the denominator of expression (26b) replace $M_{\nu+1}$ without further ado by M_ν if one introduces the restriction $M_{\nu+1} \sim M_\nu$, which proves expedient for the formation of criteria.

27. Continuation. According to this, the *main form of all possible criteria of the first kind* is contained in the two relations:

$$\begin{cases} \lim \frac{M_\nu}{M_{\nu+1}-M_\nu} \cdot a_\nu > 0: \ \textit{Divergence}, \\ \lim \frac{M_{\nu+1}\cdot M_\nu}{M_{\nu+1}-M_\nu} \cdot a_\nu < \infty: \ \textit{Convergence}, \end{cases} \tag{27}$$

and the relations:

$$\begin{cases} \lim \frac{L_\chi(M_\nu)}{M_{\nu+1}-M_\nu} \cdot a_\nu > 0: \ \textit{Divergence}, \\ \lim \frac{L_\chi(M_\nu)\cdot \lg_\chi^\varrho M_\nu}{M_{\nu+1}-M_\nu} \cdot a_\nu < \infty: \ \textit{Convergence}, \end{cases} \qquad \begin{pmatrix} M_{\nu+1} \sim M_\nu \\ \varrho > 0 \end{pmatrix} \tag{28}$$

for $\chi = 0, 1, 2, \ldots$ represent a *scale* of *increasingly* effective criteria.

The special choice $M_\nu = \nu$ then provides for $\chi = 0$ the *Cauchy* criterion (17), for $\chi = 1, 2, \ldots$ that series which was first established by *de Morgan*[183)], later by *Bonnet*[184)].

The criteria (28) can also be replaced by the following scale of disjunctive criteria[185)]:

$$\begin{cases} (a) \quad \lim \dfrac{\lg \frac{M_{\nu+1}-M_\nu}{a_\nu}}{M_\nu} \begin{cases} < 0 & \textit{Divergence}, \\ > 0 & \textit{Convergence}, \end{cases} \\ (b) \quad \lim \dfrac{\lg \frac{M_{\nu+1}-M_\nu}{L_\chi(M_\nu)\cdot a_\nu}}{L_{\chi+1}(M_\nu)} \begin{cases} < 0 & \textit{Divergence}, \\ > 0 & \textit{Convergence}. \end{cases} \qquad (\chi = 0, 1, 2, \ldots). \end{cases} \tag{29}$$

If one again specializes $M_\nu = \nu$, then (a) provides the *Cauchy* fundamental criterion (I), (b) for $\chi = 0$ the *Cauchy* criterion (16), for $\chi = 1, 2, \ldots$ a series first derived by *Bertrand*[186)].

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183) Diff. and Integr. Calc. (1839), p. 326. *De Morgan* derives from this yet another seemingly more general criterion form, whose scope is, however, exactly the same, as *Bertrand* and *Bonnet* (J. de Math. 7, p. 48; 8, p. 86) have shown.

184) J. de Math. 8 (1843), p. 78.

185) Derived in somewhat different form by *Dini* l.c. p. 14.

186) J. de Math. 7 (1842), p. 37. – A more elementary derivation is given by *Paucker* (J. f. Math. 42 [1851], p. 139) and *Cauchy* (C. R. 1856, 2me sem., p. 638), who on this

Finally, the *convergence* criterion contained in (a) permits the following generalization:

$$\lim \frac{\lg P_\nu \cdot a_\nu}{S_\nu} < 0 : \ \textit{Convergence},^{187)} \tag{30}$$

where (P_ν) can mean *any arbitrary positive* sequence of numbers and

$$S_\nu = P_0 + P_1 + \cdots + P_\nu.$$

This *most general convergence criterion of the first kind* then forms the analogue to *Kummer's convergence criterion of the second kind.*

By substituting the general expression (23) for C_ν^{-1} in the *convergence* criterion of the *second* kind (21), the remarkable result emerges that the same can also be brought to the form:

$$\lim(D_\nu \cdot \frac{a_\nu}{a_{\nu+1}} - D_{\nu+1}) > 0 : \ \textit{Convergence.} \tag{31}$$

Since *every arbitrary* positive sequence of numbers (P_ν) must belong either to the type (D_ν) or to the type (C_ν), one finds by combination of (31) with the *convergence* criterion (21) directly *Kummer*'s convergence criterion (18), with the *divergence* criterion (21) the *disjunctive criterion of the second kind*:

$$\lim(D_\nu \cdot \frac{a_\nu}{a_{\nu+1}} - D_{\nu+1}) \begin{cases} < 0 : & \textit{Divergence}, \\ > 0 : & \textit{Convergence}, \end{cases} \tag{32}$$

into which one need only substitute from (22a), (26a):

$$D_\nu = \frac{1}{M_{\nu+1} - M_\nu} \quad \textit{or} \quad D_\nu = \frac{L_\chi(M_\nu)}{M_{\nu+1} - M_\nu} \quad (\chi = 0, 1, 2, \cdots) \tag{33}$$

to obtain *scales* of increasingly effective[188)] criteria. For $M_\nu = \nu$ there results from this in succession the *Cauchy fundamental criterion* (II), *Raabe*'s[189)]

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occasion rightfully claims the fundamental idea and the methods used for himself.

187) Written differently: $\lim(P_\nu \cdot a_\nu)^{\frac{1}{s_\nu}} < 1$.

188) On the character (not so immediately visible here as with the criteria of the first kind) of the successive *sharpenings* to be achieved, cf. my treatise l.c. p. 364.

189) Z. f. Phys. u. Math, von *Baumgartner* u. *Ettingshausen* 10 (1832), p. 63. Rediscovered by *Duhamel*, J. de Math. 4 (1839), p. 214. Cf. also 6 (1841), p. 85. The criterion in question can be brought to the form:

$$\lim \nu \cdot (\frac{a_\nu}{a_{\nu+1}} - 1) \begin{cases} < 1 : & \textit{Divergence}, \\ > 1 : & \textit{Convergence}. \end{cases}$$

and (apart from an un-essential difference in form) a criterion sequence likewise established by *Bertrand*[190].

Besides the *main form* (32) of the *disjunctive criterion of the second kind*, I have emphasized the following as particularly simple and of equal scope:

$$\lim D_{\nu+1} \lg \frac{D_\nu a_\nu}{D_{\nu+1} a_{\nu+1}} \begin{cases} < 0: & \textit{Divergence} \\ > 0: & \textit{Convergence} \end{cases} \tag{34}$$

Here too it proves admissible to replace the D_ν in the *convergence* criterion by the terms of a completely *arbitrary* positive sequence of numbers (P_ν), so that a criterion of the same generality as *Kummer*'s results.

28. Other Criterion Forms. The theory of *criteria of the first and second kind* may be considered completely closed. If nevertheless from time to time "new" such criteria keep appearing, these involve either the rediscovery of long-known criteria or special formations of subordinate importance.

On the other hand, the unlimited possibility of further general criterion formations emerges if one compares instead of a_ν or $\frac{a_\nu}{a_{\nu+1}}$ some other, suitably chosen combinations $F(a_\nu, a_{\nu+1}, \ldots)$ with the corresponding ones of the d_ν or c_ν. On this principle are based the criteria of the *third kind* (difference criteria)[191] established by me, as well as the *"extended criteria of the second kind"*, in which, instead of the quotients of two *consecutive* terms, those of two *arbitrarily distant* terms or also those of two *groups of terms* are taken into consideration. I arrive by the latter path to the following *extended main criterion of the second kind*:

$$\begin{cases} \lim_{x=\infty} \frac{(M_{x+h}-M_x)\cdot f(M_{x+h})}{(m_{x+h}-m_x)\cdot f(m_x)} > 1: & \textit{Divergence} \text{ of the series} \sum f(\nu), \\ \lim_{x=\infty} \frac{(M_{x+h}-M_x)\cdot f(M_x)}{(m_{x+h}-m_x)\cdot f(m_{x+h})} < 1: & \textit{Convergence} \text{ of the series} \sum f(\nu) \end{cases} \tag{35}$$

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190) J. de Math. 7, p. 43. Cf. also: *Bonnet*, J. de Math. 8, p. 89 and *Paucker*, J. f. Math. 42, p. 143. – The *Gauss* criteria can be derived with the help of *Raabe*'s and the *first Bertrand* criterion, as *B.* has shown l.c. p. 52; incidentally also with the help of *Kummer*'s criteria (*Kummer* l.c. p. 178). The analogous holds for the somewhat more general case: $\frac{a_{\nu+1}}{a_\nu} = 1 + \frac{c_1}{\nu} + \frac{c_2}{\nu^2} + \cdots$. *Schlömilch*, Z. f. Math. 10 (1865), p. 74.

191) L.c. p. 379.

when $M_x > m_x$ and M_x, m_x are monotonically increasing, $f(x)$ a monotonically decreasing function of the positive variable x. From the same result for $h = 1$ the criteria derived by *G. Kohn*[192], for $\lim h = 0$ the criteria of *Ermakoff*[193] distinguished by formal simplicity and great scope:

$$\lim_{x=\infty} \frac{M'_x \cdot f(M_x)}{m'_x \cdot f(m_x)} \begin{cases} > 1: & \textit{Divergence}, \\ < 1: & \textit{Convergence}, \end{cases} \tag{36}$$

The latter I have recently generalized in such a way that $f(x)$ no longer needs to be assumed *monotonic*[194].

29. Scope of the Criteria of First and Second Kind. The field of application of any criterion of the *second kind* is naturally a noticeably *narrower* one than that of the corresponding (i.e., formed with the same D_ν, C_ν) criterion of the *first kind*[195]. *Cauchy* has, on the basis of the limit theorem mentioned in No. 18, Eq. (9), more precisely established the connection between his fundamental criteria of the first and second kind. The relevant result can be generalized in the following way: If the disjunctive criterion of the *second kind* (32) for $D_\nu^{-1} = M_{\nu+1} - M_\nu$ provides a *decision* or *fails* by the appearance of the *limit value zero*, then the same holds for the criterion of the *first kind* (29a). On the other hand, the latter *can* still *provide* a decision when the *former* fails through the appearance of *indeterminate limits*[196].

The limits for the scope of the ordinary *criterion pairs of the first kind* (20) result from the observation that they fail not only when directly:

$$(A) \quad \lim D_\nu \cdot a_\nu = 0, \quad \lim C_\nu \cdot a_\nu = \infty,$$

but also when those limit values *do not exist* at all and *simultaneously*:

$$(B) \quad \underline{\lim}\, D_\nu \cdot a_\nu = 0, \quad \overline{\lim}\, C_\nu \cdot a_\nu = \infty.$$

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192) Archiv f. Math. 67 (1882), p. 82, 84.

193) *Darboux* Bulletin 2 (1871), p. 250; 18 (1883), p. 142. The criterion resulting for $M_x = e^x$, $m_x = 1$: $\lim \frac{e^x f(e^x)}{f(x)} >$ *or* < 1 possesses, for example, the same scope as the *entire scale* of logarithmic criteria.

194) Chicago Papers p. 328. There also a shorter proof based on the theory of definite integrals (improvement of that originally given by *W. Ermakoff*) and more precise establishment of the relationship between $\sum_{m=\nu}^{\infty} f(\nu)$ and $\int_m^\infty f(x)dx$.

195) Cf. l.c. p. 308.

196) *Pringsheim* l.c. p. 376.

If one chooses, as happens with the criteria exclusively applied in practice, the D_ν, C_ν *monotonically* increasing[197], their applicability obviously extends only to such a_ν which either directly decrease *monotonically* or at least *"essentially" monotonically*, i.e., so that any fluctuations remain within certain bounds.

30. The Boundary Regions of Divergence and Convergence. The first example of a *convergent* series for which the ordinary logarithmic (*Bonnet*) scale *fails* according to the mode of Eq. (A) was constructed by *Du Bois-Reymond*[198]. I have then given a somewhat more general series type with a more transparent formation law, which simultaneously also provides *divergent* series of the nature in question[199]. The method used here can, as *Hadamard* has shown[200], easily be transferred to any *arbitrary* criterion scale.

But there are also infinitely many *monotonic* a_ν for which an arbitrarily chosen criterion scale must completely fail in the sense of equations (B). The investigations I have undertaken in this direction[201] lead to the following general theorem: "However strongly $\sum C_\nu^{-1}$ may *converge*, there always exist monotonic *divergent* series $\sum a_\nu$ for which: $\underline{\lim} C_\nu a_\nu = 0$. However *slowly* m_ν may grow to infinity with ν, there always exist monotonic *convergent* series $\sum a_\nu$ for which: $\overline{\lim}\, \nu \cdot m_\nu \cdot a_\nu = \infty$; on the other hand, one always has: $\lim \nu \cdot a_\nu = 0$." There exists therefore no M_ν of *arbitrarily high infinity* such that $\lim M_\nu \cdot a_\nu > 0$ forms a *necessary* condition for the *divergence* of $\sum a_\nu$. On the other hand, the relation $\lim \nu \cdot a_\nu = 0$ does indeed form a *necessary*[202]

197) E.g.,

$$D_\nu = \nu, \quad \nu \lg \nu, \quad \ldots$$
$$C_\nu = \nu^{1+\varrho}, \quad \nu \cdot (\lg \nu)^{1+\varrho}, \quad \ldots$$

(*Bonnet* criteria: see No. 27).

198) J. f. Math. 76 (1873), p. 88.

199) L.c. p. 353 ff.

200) Acta Math. 18 (1894), p. 325.

201) L.c. p. 347, 356. Math. Ann. 37 (1890), p. 600. Münch. Ber. 26 (1896), p. 609 ff.

202) That the same always *suffices* for convergence was claimed by *Th. Olivier* (J. f. Math. 2, 1827, p. 34), refuted by *Abel* (l.c. 3, p. 79. Oeuvres 1, p. 399) by pointing to the series $\sum \frac{1}{\nu \lg \nu}$. *Kummer* has, on the other hand, shown that that condition always suffices

condition for the *convergence* of $\sum a_\nu$, but *no*[203] relation of the form: $\lim \nu \cdot m_\nu \cdot a_\nu = 0$ with *arbitrarily weak infinity* of $\lim m_\nu$. In other words: There exists, even when one restricts oneself to the consideration of *monotonic*[204] a_ν, *no boundary* of *divergence* at all, i.e., *no* sequence of numbers (c_ν) such that from some definite ν onward *constantly* $a_\nu > c_\nu$ would have to hold if $\sum a_\nu$ *diverges*. And while every sequence of numbers of the form $\left(\frac{\varepsilon}{\nu}\right)$, where $\varepsilon > 0$, forms a *boundary* of *convergence* (i.e., from some definite ν onward *constantly* $a_\nu < \frac{\varepsilon}{\nu}$ must hold if $\sum a_\nu$ is to *converge*), no sequence of numbers of the form $\left(\frac{\varepsilon_\nu}{\nu}\right)$ does, however *slowly* ε_ν with $\frac{1}{\nu}$ may approach *zero*.

According to this, the fiction of a *"boundary between convergence and divergence"* introduced by *Du Bois-Reymond*[205] rests from the outset on a false fundamental conception. But even if one understands the same in an essentially narrower sense, namely as a presumptive boundary between any two *definite* divergent and convergent scales, such as: $\frac{1}{L_\chi(\nu)}$ and $\frac{1}{L_\chi(\nu)\cdot(\lg_\chi^\nu)^\varrho}$ $\quad(\chi = 1, 2, 3, \cdots; \varrho < 0)$, it appears untenable, as I have attempted to demonstrate in detail[206].

31. Conditional and Unconditional Convergence. A series with *positive* and *negative* terms u_ν is called *absolutely* convergent if $\sum |u_\nu|$ *converges*; that under this assumption it actually always *converges itself* as well was, as already noted in No. 23, proved by *Cauchy*. That there also exist *convergent* series $\sum u_\nu$ for which $\sum |u_\nu|$ *diverges* was already shown by the

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for convergence when $\frac{a_\nu}{a_{\nu+1}}$ can be expanded in ascending powers of $\frac{1}{\nu}$ (J. f. Math. 13, 1835, p. 178).

203) One frequently finds the *false theorem* (cf. my Conv.-Theory l.c. p. 343) that *generally*: $\lim D_\nu \cdot a_\nu = 0$ forms a *necessary* condition for *convergence*, while for $D_\nu \succ \nu$ in truth only: $\underline{\lim} D_\nu \cdot a_\nu = 0$ needs to hold. This uniquely correct formulation is already given by *Abel* in the posthumous note cited above: Oeuvres 2, p. 198.

204) For *non-monotonic* a_ν the existence of such convergence and divergence boundaries appears excluded *a priori*; see my Conv.-Theory l.c. p. 344, 357.

205) Münch. Abh. 12 (1876), p. XV. Math. Ann. 11 (1877), p. 158 ff.

206) Münch. Ber. 27 (1897), p. 203 ff.

example of the *Leibniz* series[207]: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. Also *Leibniz* proved generally the *convergence* of every series of the form $\sum(-1)^\nu \cdot a_\nu$ (where $a_\nu \geq a_{\nu+1} > 0$, $\lim a_\nu = 0$)[208]. To such series, specifically to $\sum(-1)^\nu \cdot \frac{1}{\nu+1}$, *Cauchy* attached the important observation[209] that *their convergence depends essentially on the arrangement of the terms*, such that they become *divergent* under certain rearrangements. With this he uncovered that property which one is accustomed to designate today as *conditional* convergence. *Lej.-Dirichlet* added[210] that under certain rearrangements the *convergence* is indeed preserved, but the *sum* undergoes a change; and he has particularly sharply emphasized that an *absolutely* convergent series always converges *unconditionally*, i.e., independently of the arrangement of the terms, to the same sum[211]. Through *Cauchy* and *Dirichlet* it had at most been proved that *certain* non-absolutely converging series converge only *conditionally*; that this must in truth be the case for *every* non-absolutely converging series was first taught by a theorem proved by *Riemann*[212], according to which two *arbitrary divergent* series of the form $\sum a_\nu$, $\sum(-b_\nu)$ ($a_\nu > 0$, $b_\nu > 0$, $\lim a_\nu = \lim b_\nu = 0$) can be combined into a *convergent* series with an *arbitrarily prescribed sum*[213]. With this the complete equivalence of *absolute* and *unconditional*, *non-absolute*

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207) De vera proportione circuli ad quadratum circumscriptum. Acta erud. Lips. 1682. (Opera, Ed. Dutens 3, p. 140.) The series is already found in *James Gregory*. Cf. *Reiff* l.c. p. 45. *M. Cantor* 3, p. 72.

208) Letter to *Joh. Bernoulli*, Jan. 1, 1714. (Commerc. epist. 2, p. 329.)

209) Résumé anal. p. 57.

210) Berl. Abh. 1837, p. 48. (Ges. W. 1, p. 318.)

211) This was explicitly proved probably for the first time by *W. Scheibner*: Über unendliche Reihen und deren Konvergenz. Gratulationsschrift, Lpzg. 1860, p. 11. The expression *"unconditional"* convergence probably stems from *Weierstrass* (J. f. Math. 51 [1856], p. 41). Individual German and almost all French and English authors designate the *conditionally* convergent series as *semiconvergent*. This expression is in itself little fitting (for the addition *"semi"* designates not so much a special *mode* as rather the partial *negation* of convergence) and appears also for the reason little recommendable because it (or the synonymous *half-convergent* series, série *demi-convergente*) has already acquired a completely different meaning following the precedent of *Legendre* (Exerc. de calc. intégr. 1, p. 267). Cf. No. 38.

212) Gött. Abh. 13 (1867). (Ges. W. p. 221.)

213) *Dini* has noted that one can in analogous manner also produce proper or improper *divergence*; Ann. di Mat. (2) 2 (1868), p. 31.

and *conditional* series convergence definitively established.

32. Value Changes of Conditionally Convergent Series. For the *change* which the harmonic series

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}\cdot\frac{1}{\nu+1}=\lg 2$$

undergoes if one lets q negative terms follow every p positive terms, the value $\frac{1}{2}\lg\frac{p}{q}$ was found by *Mart. Ohm* (with the help of integral calculus)[214]. A direct generalization of this result is formed by the theorem proved by *Schlömilch*[215] that the series $\sum(-1)^{\nu}\cdot a_{\nu+1}$ undergoes the value change $(\lim\nu\cdot a_{\nu})\cdot\frac{1}{2}\lg\frac{p}{q}$ under analogous rearrangement. I have investigated in completely general manner[216] what value changes a convergent series composed of the two divergent components $\sum a_{\nu}$, $\sum(-b_{\nu})$ undergoes when the *relative frequency* of the a_{ν} and $(-b_{\nu})$ (while maintaining the original order within the two individual groups (a_{ν}) and (b_{ν})) is changed in an arbitrarily prescribed manner, and conversely, what such rearrangement is required to produce an arbitrarily prescribed value change. The investigation of *"singular series remainders"* of the form: $\lim_{n=\infty}\sum_{\nu=n+1}^{n+\varphi(n)}a_{\nu}$ required for this and completely feasible for the case $\lim\frac{a_{\nu+1}}{a_{\nu}}=1$ teaches that the value changes in question depend not on the special formation law of the a_{ν}, but solely on their behavior for $\lim\nu=\infty$: If $\lim\sum_{\nu=n+1}^{n+\varphi(n)}a_{\nu}=a$ (finite), then also $\lim\sum_{\nu=n+1}^{n+\varphi(n)}a'_{\nu}=a$, if $a'_{\nu}\cong a_{\nu}$; on the other hand $\lim\sum_{\nu=n+1}^{n+\varphi(n)}a'_{\nu}=0$ or $=\infty$, if $a'_{\nu}\prec a_{\nu}$ or $\succ a_{\nu}$ respectively. The $\varphi(n)$ required for producing a certain remainder value (incl. 0 and ∞) (i.e., ultimately the *rearrangement law* leading to a certain *value change*)

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214) De nonnullis seriebus summandis. Antr.-Programm, Berlin 1839. An elementary derivation in *H. Simon*, Die harm. Reihe. Dissert. Halle 1886.

215) Z. f. Math. 18 (1873), p. 520.

216) Math. Ann. 22 (1883), p. 455 ff.

depends then in precisely specifiable manner on the infinitary nature of the a_ν. If $a_\nu \succ \frac{1}{\nu}$, the series sum undergoes the change 0, a, ∞, according as $\lim \varphi(n) \cdot a_n = 0$, a, ∞. The analogous holds in the case: $a_\nu \cong \frac{g}{\nu}$, with the single difference that the change, if $\lim \varphi(n) \cdot a_n = a$, here takes the value: $\frac{1}{g} \lg(1 + ag)$. If finally $a_\nu \prec \frac{1}{\nu}$, the *two* assumptions $\lim \varphi(n) \cdot a_n = 0$ and $= a$ yield *no* value change; in the case: $\lim \varphi(n) \cdot a_n = \infty$ there then results a definite finite or infinitely large change, according to the particular manner of the becoming infinite of $\lim \varphi(n) \cdot a_n$.[217)]

A somewhat more general type of rearrangements which leave the sum of a conditionally converging series *unchanged* has been considered by *E. Borel*[218)].

33. Criteria for Possibly Only Conditional Convergence. For establishing the *simple*, i.e., possibly only *conditional* convergence of a series with positive and negative terms, one possesses no general criteria. The *measure of term decrease* is here completely irrelevant for judging convergence, as the *Leibniz* criterion for alternating series (No. 31) shows: $\sum(-1)^\nu \cdot a_\nu$ converges even when the a_ν approach zero monotonically *arbitrarily slowly.* A useful aid in many cases is given by the transformation originating from *Abel* ("partial summation")[219)]:

$$\sum_{\nu=0}^{n} u_\nu v_\nu = u_n V_n - \sum_{\nu=0}^{n-1} (u_\nu - u_{\nu+1}) \cdot V_\nu + u_n V_n \tag{37}$$

$$(where: V_\nu = v_0 + v_1 + \cdots + v_\nu),$$

which for $\lim n = \infty$ provides the following convergence theorem: "If $\sum(u_\nu - u_{\nu+1})$ is absolute and $\sum v_\nu$ is convergent at all, then $\sum u_\nu v_\nu$ converges at least in the prescribed arrangement. This also holds when $\sum v_\nu$ oscillates within finite bounds, provided that $\lim u_\nu = 0$." The application of *Abel*'s transformation for such convergence considerations originates from *Dirichlet*[220)], the above theorem in somewhat more special formulation from *Dedekind*[221)];

217) Details l.c. p. 496 ff.
218) Bull. d. Sc. (2) 14 (1890), p. 97.
219) J. f. Math. 1 (1826), p. 314. Oeuvres 1, p. 222.
220) Vorl. über Zahlentheorie, herausgeg. von *R. Dedekind*, 3. Aufl. (1879), § 101.
221) Ibid., Supplem. 9, § 143.

The here given is found along with some simple modifications in *Du Bois-Reymond*[222].

From this theorem follows, for example, immediately the convergence of $\sum a_\nu \cdot \cos \nu x$ (excl. $x = 0 \pm 2k\pi$) and $\sum a_\nu \cdot \sin \nu x$, first proved otherwise by *Malmsten*[223], when the a_ν approach zero *monotonically*[224], as well as that of some other trigonometric series[225]. Also the convergence proof for the *Fourier* series can be reduced to it under certain simplifying assumptions[226].

The *Abel* transformation in connection with the convergence of the series $\sum \frac{M_{\nu+1} - M_\nu}{M_{\nu+1} \cdot M_\nu^\varrho}$ emphasized in No. 26 has been used by me[227] to obtain a very general criterion for judging the so-called *Dirichlet* series: $\sum k_\nu \cdot M_\nu^{-\varrho}$ (k_ν arbitrary, $\varrho > 0$). Special cases of the same have been found earlier by *Dedekind*[228] and *O. Hölder*[229] by other methods.

A useful generalization of the ordinary convergence theorem for alternating series results from the *Weierstrass* convergence investigations[230]. According to this, $\sum (-1)^\nu \cdot a_\nu$ still converges *conditionally* when $\frac{a_\nu}{a_\nu + 1} = 1 + \frac{\chi}{\nu} + \frac{\lambda}{\nu^2} + \cdots$ and $0 < \chi \leq 1$.[231]

The most important category of series which generally need converge only *conditionally* are the *Fourier series*[232]. The general investigations concerning their convergence and divergence are based on the representation of s_n by a

222) Antr.-Programm, p. 10.

223) With the unnecessary restriction $\lim \frac{a_{\nu+1}}{a_\nu} = 1$. Nova acta Upsal. 12 (1844), p. 255. Without that restriction and simpler: *Hj. Holmgren*, J. de Math. 16 (1851), p. 186.

224) *G. Björling* (J. de Math. 17 (1852), p. 470) erroneously considers the conditions: $a_\nu > 0$, $\lim a_\nu = 0$ already sufficient.

225) *Du Bois-Reymond* l.c. p. 12, 17.

226) Ibid. p. 13.

227) Math. Ann. 37 (1886), p. 41.

228) Vorl. über Zahlentheorie, Suppl. 9, § 144.

229) Math. Ann. 20 (1882), p. 545.

230) J. f. Math. 51 (1856), p. 29; Werke 1, p. 185.

231) This also holds for *complex* a_ν, if the real part of χ satisfies the condition given in the text. For $\chi > 1$, $\sum a_\nu$ *converges absolutely* (most simply by *Raabe*'s criterion), for $\chi \leq 0$ it *diverges*. Cf. also *Stolz*, Allg. Arithm. 1, p. 268.

232) Cf. II A8.

definite integral and its discussion for $\lim n = \infty$.

34. Addition and Multiplication of Infinite Series. For the *addition* or *subtraction* of two convergent series there results immediately from the relation: $\lim a_\nu \pm \lim b_\nu = \lim(a_\nu \pm b_\nu)$ (No. 17, Eq. (15)) the rule:

$$\sum_{\nu=0}^{\infty} u_\nu \pm \sum_{\nu=0}^{\infty} v_\nu = \sum_{\nu=0}^{\infty}(u_\nu \pm v_\nu). \tag{38}$$

For *multiplication Cauchy* established the theorem:

$$(\sum_{\nu=0}^{\infty} u_\nu) \cdot (\sum_{\nu=0}^{\infty} v_\nu) = \sum_{\nu=0}^{\infty} w_\nu \quad (w_\nu = u_0 v_\nu + u_1 v_{\nu-1} + \cdots + u_\nu v_0), \tag{39}$$

under the assumption that $\sum u_\nu$, $\sum v_\nu$ *converge absolutely*[233)], with the explicit indication that the formula can *fail* for *non-absolutely* converging series[234)]. *Abel* has shown that the same is *valid* as soon as (besides the series $\sum u_\nu$, $\sum v_\nu$ naturally assumed *convergent*) the series $\sum w_\nu$ *converges* at all[235)]. Since this convergence proof (apart from the case of *absolute* convergence of $\sum u_\nu$, $\sum v_\nu$ settled by *Cauchy*) must be provided specially each time, it appears by no means superfluous that *F. Mertens* has extended the validity of the multiplication theorem (39) to the case that only *one* of the two series $\sum u_\nu$, $\sum v_\nu$ converges *absolutely*[236)]. The case that *both* series converge only *conditionally* has been considered by me in detail[237)]. If one of the two series, say $\sum u_\nu$, possesses the property that $\sum |u_\nu + u_{\nu+1}|$ converges, then the condition $\lim w_\nu = 0$ appears as *necessary and sufficient* for the validity of formula (39); from this result in particular simple criteria for the case

233) Anal. algébr. p. 147.

234) Ibid. p. 149, probably the first place where the *different* behavior of *absolutely* and *non-absolutely* converging series is emphasized.

235) J. f. Math. 1 (1826), p. 318. (Oeuvres 1, p. 226.) *Abel*'s proof is based on the consideration of the series $\sum u_\nu x^\nu$, $\sum v_\nu x^\nu$ for $\lim x = 1$, thus on a *continuous* limit transition. A proof *without* use of this aid belonging to function theory has been given by *E. Cesàro*: Bull. d. Sc. (2) 14 (1890), p. 114. Similarly *Jordan*, Cours d'Anal. 1, p. 282.

236) J. f. Math. 79 (1875), p. 182. Other proof by *W. V. Jensen*, Nouv. Corresp. math. 1879, p. 430.

237) Math. Ann. 21 (1883), p. 327.

of two *alternating* series with *monotonic* terms[238]. Under the more general assumption that $\sum u_\nu$ becomes *absolutely* convergent when one groups the u_ν in groups of p_ν terms (p_ν constant or variable, but remaining finite), I have given a *sufficient* condition which includes the *Cesàro* and *Mertens* theorem as a special case. For the case $p_\nu = 2$, *A. Voss*[239] then, for arbitrary *constant*[240] and *finitely-variable*[241] p_ν, *F. Cajori* established the *necessary and sufficient* conditions.

35. Double Series. The addition formula (38) is indeed directly transferable to an arbitrary *finite* number of series:

$$\sum_{\nu=0}^{\infty} u_\nu^{(\mu)} \quad (\mu = 0, 1, \dots m),$$

but not to the case $m = \infty$, i.e., for the validity of the relation:

$$\sum_{\mu=0}^{\infty}(\sum_{\nu=0}^{\infty} u_\nu^{(\mu)}) = \sum_{\nu=0}^{\infty}(\sum_{\mu=0}^{\infty} u_\nu^{(\mu)}) \tag{40}$$

it appears by no means *sufficient* that the *left* side has a definite meaning, thus *converges* absolutely. *Cauchy* has shown that Eq. (40) holds when also $\sum_{\mu=0}^{\infty}(\sum_{\nu=0}^{\infty} |u_\nu^{(\mu)}|)$ converges[242]; the multiplication theorem (39) for two absolutely convergent series proves to be a special case of this theorem[243]. At the same time *Cauchy* connected to the consideration of a *doubly-infinite scheme* of terms $u_\nu^{(\mu)}$ (where perhaps the index μ may characterize the *rows*, the index ν the *columns*) the concept of the *double series*. Setting $\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} u_\nu^{(\mu)} = s_n^{(m)}$, the *double series* $\sum_{\mu,\nu=0}^{\infty} u_\nu^{(\mu)}$ formed from the terms $u_\nu^{(\mu)}$ is called *convergent* and s its *sum*, when in the

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238) An application to the multiplication of two *trigonometric series* in Math. Ann. 26 (1886), p. 157.

239) Math. Ann. 24 (1884), p. 42.

240) Am. J. of Math. 15 (1893), p. 339.

241) N. Y. Bull. (2), 1 (1895), p. 180. *Cajori* gives a brief analysis of the results found by me and *Voss*: N. Y. Bull. 1 (1892), p. 184.

242) Anal. algébr. p. 541. A more general form of a sufficient condition valid for *power series*, originating from *Weierstrass* (Werke 2, p. 205), is essentially *function-theoretic* in nature. Cf. II B 1.

243) *Cauchy* l.c. p. 542.