Statistical Pattern Recognition - Homework 1
Question 24

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Suppose λ_i and $\mathbf{v}_i$ are the eigenvalues and eigenvectors of matrix A ($i = 1, \ldots, n$). The matrix AA^T represents the covariance matrix of a set of samples. Compute the eigenvalues and eigenvectors of AA^T in terms of λ_i, $\mathbf{v}_i$, and A.

Solution

We need to find the eigenvalues and eigenvectors of the matrix AA^T, given that A is an $n \times n$ matrix with eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ for $i = 1, \ldots, n$. Since AA^T is a covariance matrix, it is symmetric ($(AA^T)^T = AA^T$) and positive semi-definite, implying its eigenvalues are real and non-negative.

For the matrix A, the eigenvalue-eigenvector equation is:

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad i = 1, \ldots, n,$$

where $\mathbf{v}_i$ are the eigenvectors and λ_i are the corresponding eigenvalues. We assume the eigenvectors $\mathbf{v}_i$ are linearly independent, implying A is diagonalizable:

$$A = V\Lambda V^{-1},$$

where $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ is the matrix of eigenvectors, and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues.

The matrix AA^T is:

$$AA^T = (V\Lambda V^{-1})(V\Lambda V^{-1})^T = V\Lambda V^{-1}(V^{-1})^T\Lambda^T V^T.$$

Since Λ is diagonal, $\Lambda^T = \Lambda$. Also, for the eigenvector matrix V, if A is real and symmetric, V can be chosen orthogonal ($V^{-1} = V^T$), but we'll proceed generally for now. Simplify:

$$(V^{-1})^T = (V^T)^{-1},$$

so:

$$AA^T = V\Lambda(V^{-1}(V^T)^{-1})\Lambda V^T.$$

We need to compute the eigenvalues and eigenvectors of AA^T.

To find the eigenvalues of AA^T, consider its action on the eigenvectors of A. Let's test if $\mathbf{v}_i$ is an eigenvector of AA^T:

$$AA^T\mathbf{v}_i = A(A^T\mathbf{v}_i).$$

First, compute $A^T\mathbf{v}_i$. The eigenvalue equation for A^T is:

$$A^T(V^{-1})^T\mathbf{e}_i = \lambda_i(V^{-1})^T\mathbf{e}_i,$$

where $\mathbf{e}_i$ is the i-th standard basis vector, since:

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i \implies V^{-1}AV = \Lambda \implies A^T = (V\Lambda V^{-1})^T = (V^{-1})^T\Lambda V^T.$$

Thus, $(V^{-1})^T\mathbf{e}_i$ is an eigenvector of A^T with eigenvalue λ_i. However, we need $AA^T\mathbf{v}_i$:

$$A^T\mathbf{v}_i = A^TV\mathbf{e}_i = (V\Lambda V^{-1})^T\mathbf{v}_i = (V^{-1})^T\Lambda V^T\mathbf{v}_i.$$

Since $V^T\mathbf{v}_i = V^TV\mathbf{e}_i = \mathbf{e}_i$, we have:

$$A^T\mathbf{v}_i = (V^{-1})^T\Lambda\mathbf{e}_i = (V^{-1})^T\lambda_i\mathbf{e}_i.$$

Now apply A:

$$AA^T\mathbf{v}_i = A\left((V^{-1})^T\lambda_i\mathbf{e}_i\right) = \lambda_i V\Lambda V^{-1}(V^{-1})^T\mathbf{e}_i.$$

This is complex, so let's try the eigenvalues via the singular value decomposition (SVD) or properties of AA^T. Since AA^T is positive semi-definite, its eigenvalues are the squares of the singular values of A. If $A = V\Lambda V^{-1}$, the singular values of A are $|\lambda_i|$, and the eigenvalues of AA^T are:

$$\text{Eigenvalues of } AA^T = \lambda_i^2, \quad i = 1, \dots, n.$$

This follows because:

$$AA^T = V\Lambda V^{-1}(V\Lambda V^{-1})^T = V\Lambda(V^{-1}V^{-T})\Lambda V^T = V\Lambda^2V^T,$$

if V is orthogonal ($V^{-1} = V^T$). However, for a general A, compute the eigenvalues using the characteristic polynomial or test directly.

The eigenvectors of AA^T corresponding to λ_i^2 are typically the right singular vectors of A, but here we need them in terms of $\mathbf{v}_i$. If A is symmetric, $A = A^T$, then $AA^T = A^2$, and:

$$A^2\mathbf{v}_i = A(A\mathbf{v}_i) = A(\lambda_i\mathbf{v}_i) = \lambda_iA\mathbf{v}_i = \lambda_i(\lambda_i\mathbf{v}_i) = \lambda_i^2\mathbf{v}_i.$$

Thus, if A is symmetric, the eigenvectors of AA^T are $\mathbf{v}_i$, with eigenvalues λ_i^2.

Since AA^T is a covariance matrix, it is symmetric and positive semi-definite. If A is the data matrix (samples as columns), then AA^T relates to the covariance of the samples. Assume A is $m \times n$ (if samples are rows, adjust accordingly). The eigenvalues of AA^T are the squares of the singular values of A. For an $n \times n$ matrix A with eigenvalues λ_i, the eigenvalues of AA^T are λ_i^2, and the eigenvectors are $\mathbf{v}_i$ if A is symmetric. For a general A, we use SVD:

$$A = U\Sigma V^T,$$

$$AA^T = (U\Sigma V^T)(V\Sigma U^T) = U\Sigma^2U^T.$$

The eigenvalues of AA^T are σ_i^2, where $\sigma_i = |\lambda_i|$ if A is square and diagonalizable, and the eigenvectors are the columns of U, which may differ from $\mathbf{v}_i$.

Since AA^T is a covariance matrix, we assume A is real and possibly symmetric (common in statistical contexts). Thus: - **Eigenvalues of** AA^T: λ_i^2, for $i = 1, \ldots, n$. - **Eigenvectors of** AA^T: $\mathbf{v}_i$, the same as those of A, if A is symmetric. If A is not symmetric, the eigenvectors are the left singular vectors of A, which require computing U from the SVD of A.

For a covariance matrix, assuming symmetry of A:

$$AA^T\mathbf{v}_i = \lambda_i^2\mathbf{v}_i.$$

Thus, the eigenvalues are λ_i^2, and the eigenvectors are $\mathbf{v}_i$.

The eigenvalues of AA^T are λ_i^2, and the eigenvectors are $\mathbf{v}_i$, assuming A is symmetric, as is typical for a covariance matrix in statistical pattern recognition.