Statistical Pattern Recognition - Homework 1
Question 12

Vahid Maleki
Student ID: 40313004

October 11, 2025

Question 12

Consider two random variables x and y, with means μ_x, μ_y and standard deviations δ_x, δ_y, respectively. The covariance and correlation coefficient are defined as:

$$\operatorname{cov}(x, y) = E[(x - \mu_x)(y - \mu_y)] = E(xy) - \mu_x \mu_y,$$

$$\rho_{xy} = \frac{\operatorname{cov}(x, y)}{\delta_x \delta_y}.$$

(a) Prove that the absolute value of the correlation coefficient between two random variables is less than or equal to one, i.e., $|\rho_{xy}| \leq 1$. *Hint:* Use the Cauchy–Schwarz inequality: $[E(xy)]^2 \leq E(x^2)E(y^2)$.

(b) Under what conditions does $\rho_{xy} = 1$? Under what conditions does $\rho_{xy} = -1$?

Solution

Part (a): Prove $|\rho_{xy}| \leq 1$

To prove that the absolute value of the correlation coefficient $|\rho_{xy}| \leq 1$, we start with the definition of the correlation coefficient:

$$\rho_{xy} = \frac{\operatorname{cov}(x, y)}{\delta_x \delta_y},$$

where $\operatorname{cov}(x, y) = E[(x - \mu_x)(y - \mu_y)]$, and $\delta_x = \sqrt{E[(x - \mu_x)^2]}$, $\delta_y = \sqrt{E[(y - \mu_y)^2]}$ are the standard deviations of x and y.

Let's define centered random variables $u = x - \mu_x$ and $v = y - \mu_y$. Then:

$$\operatorname{cov}(x, y) = E[(x - \mu_x)(y - \mu_y)] = E[uv],$$

$$\delta_x = \sqrt{E[(x - \mu_x)^2]} = \sqrt{E[u^2]}, \quad \delta_y = \sqrt{E[(y - \mu_y)^2]} = \sqrt{E[v^2]},$$

$$\rho_{xy} = \frac{E[uv]}{\sqrt{E[u^2]}\sqrt{E[v^2]}}.$$

We need to show $|\rho_{xy}| \leq 1$, which implies:

$$\left|\frac{E[uv]}{\sqrt{E[u^2]}\sqrt{E[v^2]}}\right| \leq 1 \quad \text{or} \quad |E[uv]| \leq \sqrt{E[u^2]}\sqrt{E[v^2]}.$$

Apply the Cauchy-Schwarz inequality, which states for any two random variables a and b:

$$(E[ab])^2 \leq E[a^2]E[b^2].$$

Set $a = u = x - \mu_x$, $b = v = y - \mu_y$. Then:

$$(E[uv])^2 \leq E[u^2]E[v^2].$$

Taking the square root of both sides (noting that the expectations are non-negative):

$$|E[uv]| \leq \sqrt{E[u^2]E[v^2]} = \sqrt{E[u^2]}\sqrt{E[v^2]}.$$

Since $E[u^2] = E[(x - \mu_x)^2] = \delta_x^2$, and $E[v^2] = E[(y - \mu_y)^2] = \delta_y^2$, we have:

$$|E[uv]| \leq \delta_x \delta_y.$$

Thus:

$$\left|\frac{E[uv]}{\delta_x \delta_y}\right| = \frac{|E[uv]|}{\delta_x \delta_y} \leq \frac{\delta_x \delta_y}{\delta_x \delta_y} = 1.$$

Therefore:

$$|\rho_{xy}| = \left|\frac{\text{cov}(x, y)}{\delta_x \delta_y}\right| \leq 1.$$

This completes the proof for part (a).

Part (b): Conditions for $\rho_{xy} = 1$ and $\rho_{xy} = -1$

The correlation coefficient $\rho_{xy} = \pm 1$ when equality holds in the Cauchy-Schwarz inequality. For random variables u and v, equality in $(E[uv])^2 \leq E[u^2]E[v^2]$ holds if and only if u and v are linearly dependent, i.e., there exists a constant k such that $v = ku$ almost surely.

Rewrite in terms of x and y:

$$y - \mu_y = k(x - \mu_x).$$

Thus:

$$y = kx + (\mu_y - k\mu_x).$$

This is a linear relationship between x and y.

- **Condition for $\rho_{xy} = 1$:** Equality holds, and the covariance is positive, i.e., $\text{cov}(x, y) = \delta_x \delta_y$. This occurs when $k > 0$, meaning y increases linearly with x. Thus, $\rho_{xy} = 1$ when $y = ax + b$ with $a > 0$, where a and b are constants, ensuring a positive linear relationship.

- **Condition for** $\rho_{xy} = -1$: Equality holds, and the covariance is negative, i.e., $\text{cov}(x, y) = -\delta_x \delta_y$. This occurs when $k < 0$, meaning y decreases linearly with x. Thus, $\rho_{xy} = -1$ when $y = ax + b$ with $a < 0$.

In summary: - $\rho_{xy} = 1$ when $y = ax + b$ with $a > 0$. - $\rho_{xy} = -1$ when $y = ax + b$ with $a < 0$.