Pattern Recognition – Homework 3

Vahid Maleki Student ID: 40313004

Question 15

Problem Setup

We are dealing with a one-dimensional, two-class classification problem. The class-conditional densities are:

- For ω_1: $p(x|\omega_1) = \mathcal{N}(2,1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right)$
- For ω_2: $p(x|\omega_2) = \begin{cases} \frac{1}{3} & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

a) Maximum Likelihood (ML) Classifier

The ML classifier decides ω_1 if $p(x|\omega_1) > p(x|\omega_2)$, otherwise ω_2.

Outside $(0,3)$, $p(x|\omega_2) = 0$, and since $p(x|\omega_1) > 0$, decide ω_1.

Inside $(0,3)$, solve $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right) = \frac{1}{3}$.

This gives $\exp\left(-\frac{(x-2)^2}{2}\right) = \frac{\sqrt{2\pi}}{3} \approx 0.8356$.

Taking ln: $-\frac{(x-2)^2}{2} = \ln(0.8356) \approx -0.1795$.

So $(x-2)^2 = 0.359 \approx$, $x - 2 \approx \pm 0.599$, $x \approx 1.401$ or 2.599.

Since the Gaussian peaks at $0.399 > 0.333$, between 1.401 and 2.599, decide ω_1; elsewhere in $(0,3)$, decide ω_2.

Decision regions:

- R_1 (ω_1): $(-\infty, 0] \cup (1.401, 2.599) \cup [3, \infty)$
- R_2 (ω_2): $(0, 1.401] \cup [2.599, 3)$

b) Maximum a Posteriori (MAP) Classifier

Given $P(\omega_1) = 2P(\omega_2)$, so $P(\omega_1) = \frac{2}{3}$, $P(\omega_2) = \frac{1}{3}$.

Decide ω_1 if $p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2)$.

Outside $(0,3)$, decide ω_1.

Inside $(0,3)$: $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right) \cdot \frac{2}{3} = \frac{1}{3} \cdot \frac{1}{3}$.

So $\exp\left(-\frac{(x-2)^2}{2}\right) = \frac{\sqrt{2\pi}}{6} \approx 0.4178$.

Ln: $-\frac{(x-2)^2}{2} \approx -0.8727$, $(x-2)^2 \approx 1.7454$, $x - 2 \approx \pm 1.321$, $x \approx 0.679$ or 3.321.

Since $3.321 > 3$, the boundary in $(0,3)$ is at 0.679. The region for ω_1 is $(0.679, 3)$, for ω_2 is $(0, 0.679]$.

Overall:

- R_1 (ω_1): $(-\infty, 0] \cup (0.679, \infty)$
- R_2 (ω_2): $(0, 0.679]$

c) Minimum-Risk Bayes Classifier

Priors equal: $P(\omega_1) = P(\omega_2) = \frac{1}{2}$.

Loss matrix: $C = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ (costs for correct decisions are non-zero, unusual but proceeding).

Conditional risks:

- For deciding ω_1: $R(\alpha_1|x) = 3P(\omega_1|x)$
- For deciding ω_2: $R(\alpha_2|x) = 2P(\omega_2|x)$

Decide ω_1 if $3P(\omega_1|x) < 2P(\omega_2|x)$, i.e., $3p(x|\omega_1) < 2p(x|\omega_2)$ (since priors equal).

Outside $(0, 3)$, $2p(x|\omega_2) = 0$, condition false, decide ω_2.

Inside $(0, 3)$: $3 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right) = 2 \cdot \frac{1}{3}$.

$\exp\left(-\frac{(x-2)^2}{2}\right) = \frac{2\sqrt{2\pi}}{9} \approx 0.557$.

Ln: $-\frac{(x-2)^2}{2} \approx -0.585$, $(x-2)^2 \approx 1.17$, $x - 2 \approx \pm 1.082$, $x \approx 0.918$ or 3.082.

Since $3.082 > 3$, boundary at 0.918. Condition holds (decide ω_1) for $x < 0.918$ in $(0, 3)$.

Overall:

- R_1 (ω_1): $(0, 0.918]$
- R_2 (ω_2): $(-\infty, 0] \cup (0.918, \infty)$