Q19 - HW2: Pattern Recognition

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October 18, 2025

1 Question 19

Suppose that using the **PCA (Principal Component Analysis)** algorithm, a random variable with the following **covariance matrix**

$$\Sigma = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

is projected onto a **two-dimensional space**.

1.1 (a)

On average, what **percentage of the total information (variance)** is **lost** after applying this transformation?

1.1.1 Solution

To determine the percentage of variance lost when projecting onto a two-dimensional space using PCA, we need to compute the eigenvalues of the covariance matrix Σ, calculate the total variance, and find the variance retained by the top two principal components.

Step 1: Find Eigenvalues of Σ

Given:

$$\Sigma = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

This is a symmetric matrix, so its eigenvalues are real. The middle row/column (index 2) is independent, so one eigenvalue is the diagonal entry:

$$\lambda_2 = 2$$

The remaining 2×2 block for rows/columns 1 and 3 is:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Compute the eigenvalues of this block:

$$\det\left(\begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}\right) = (4-\lambda)^2 - 1 = 0$$

$$(4-\lambda)^2 = 1 \Rightarrow 4-\lambda = \pm 1 \Rightarrow \lambda = 4 \pm 1$$

$$\lambda = 5, \quad \lambda = 3$$

Thus, the full set of eigenvalues is:

$$5, 3, 2$$

Step 2: Total Variance
The total variance is the sum of all eigenvalues:

$$5 + 3 + 2 = 10$$

Step 3: Variance Retained by Top 2 Components
Sort the eigenvalues in descending order: $5, 3, 2$. The top two eigenvalues are 5 and 3. The retained variance is:

$$5 + 3 = 8$$

Step 4: Percentage of Variance Lost
The variance lost is:

$$10 - 8 = 2$$

The percentage of variance lost is:

$$\frac{2}{10} \times 100\% = 20\%$$

Conclusion:
On average, **20% of the total variance** is lost when projecting onto the top two principal components.

$$\boxed{20\%}$$

1.2 (b)

If a sample

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

from this random variable is projected into the two-dimensional space using the above PCA transformation and then **reconstructed** back into the original feature space, what is the **reconstruction error** (i.e., the **Euclidean distance** between the original and reconstructed vectors)?

1.2.1 Solution

To compute the reconstruction error, we project the sample $\mathbf{x} = [1, 1, 1]^T$ onto the two-dimensional space spanned by the top two eigenvectors, reconstruct it back to the original space, and calculate the Euclidean distance between the original and reconstructed vectors.

Step 1: Find Eigenvectors
From part (a), the eigenvalues are $5, 3, 2$. We need the eigenvectors corresponding to the top two eigenvalues (5 and 3).
- For $\lambda = 2$ (middle dimension):

$$\Sigma - 2I = \begin{bmatrix} 4-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 4-2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The eigenvector satisfies:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_3 = 0, \quad x_1 + 2x_3 = 0$$

From $2x_1 + x_3 = 0$, we get $x_3 = -2x_1$. The second equation gives $x_1 + 2(-2x_1) = x_1 - 4x_1 = -3x_1 = 0$, so $x_1 = 0$, and thus $x_3 = 0$. The middle row gives $0 \cdot x_2 = 0$, so x_2 is free. Set $x_2 = 1$:

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- For the 2×2 block $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$:
For $\lambda = 5$:

$$\begin{bmatrix} 4-5 & 1 \\ 1 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$-x_1 + x_3 = 0 \Rightarrow x_1 = x_3$$

Eigenvector: $[1, 0, 1]^T$ (since middle component is zero in the 3×3 context). Normalize:

$$\sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3$:

$$\begin{bmatrix} 4-3 & 1 \\ 1 & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$$

Eigenvector: $[1, 0, -1]^T$. Normalize:

$$\sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The top two eigenvectors (for $\lambda = 5, 3$) are:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Step 2: Project x onto Top 2 Components
Project $\mathbf{x} = [1, 1, 1]^T$ onto the subspace spanned by $\mathbf{u}_1$ and $\mathbf{u}_3$:

$$z_1 = \mathbf{u}_1^T \mathbf{x} = \frac{1}{\sqrt{2}}(1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1) = \frac{1}{\sqrt{2}}(1 + 1) = \sqrt{2}$$

$$z_3 = \mathbf{u}_3^T \mathbf{x} = \frac{1}{\sqrt{2}}(1 \cdot 1 + 0 \cdot 1 + (-1) \cdot 1) = \frac{1}{\sqrt{2}}(1 - 1) = 0$$

Step 3: Reconstruct in Original Space
The reconstructed vector is:

$$\mathbf{x}' = z_1 \mathbf{u}_1 + z_3 \mathbf{u}_3 = \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Step 4: Compute Reconstruction Error
The reconstruction error is the Euclidean distance:

$$\mathbf{x} - \mathbf{x}' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\|\mathbf{x} - \mathbf{x}'\| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$$

Conclusion:
The reconstruction error for the sample $\mathbf{x} = [1, 1, 1]^T$ is:

$$\boxed{1}$$