Statistical Pattern Recognition - Homework 1
Question 1

Vahid Maleki
Student ID: 40313004

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Given the prior probabilities $p(w_1) = p(w_2)$ and the class-conditional densities functions $p(x|w_1) = N_x(0, 1)$ and $p(x|w_2) = N_x(1, 2)$, calculate and draw the posterior probabilities $p(w_1|x)$ and $p(w_2|x)$.

Solution

Since $p(w_1) = p(w_2)$, and the priors must sum to 1, we have:

$$p(w_1) = p(w_2) = 0.5.$$

The class-conditional densities functions are:

$$p(x|w_1) = N_x(0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

$$p(x|w_2) = N_x(1, 2) = \frac{1}{\sqrt{2\pi \cdot 2}} \exp\left(-\frac{(x-1)^2}{2 \cdot 2}\right) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(x-1)^2}{4}\right).$$

To find the posterior probabilities $p(w_1|x)$ and $p(w_2|x)$, we use Bayes' theorem:

$$p(w_i|x) = \frac{p(x|w_i)p(w_i)}{p(x)}, \quad i = 1, 2,$$

where $p(x)$ is the evidence, given by:

$$p(x) = p(x|w_1)p(w_1) + p(x|w_2)p(w_2).$$

Substitute the known values:

$$p(x) = p(x|w_1) \cdot 0.5 + p(x|w_2) \cdot 0.5 = 0.5p(x|w_1) + 0.5p(x|w_2).$$

So:

$$p(w_1|x) = \frac{p(x|w_1) \cdot 0.5}{p(x)}, \quad p(w_2|x) = \frac{p(x|w_2) \cdot 0.5}{p(x)}.$$

Since the factor of 0.5 appears in both numerator and denominator, we can simplify:

$$p(w_1|x) = \frac{p(x|w_1)}{p(x|w_1) + p(x|w_2)}, \quad p(w_2|x) = \frac{p(x|w_2)}{p(x|w_1) + p(x|w_2)}.$$

Substitute the class-conditional densities functions:

$$p(x|w_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

$$p(x|w_2) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(x-1)^2}{4}\right).$$

Thus:

$$p(w_1|x) = \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(x-1)^2}{4}\right)}.$$

Simplify the denominator:

$$\sqrt{4\pi} = \sqrt{2 \cdot 2\pi} = \sqrt{2} \cdot \sqrt{2\pi},$$

so:

$$\frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{2} \cdot \sqrt{2\pi}}.$$

To simplify $p(w_1|x)$, multiply numerator and denominator by $\sqrt{2\pi}$:

$$p(w_1|x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2}} \exp\left(-\frac{(x-1)^2}{4}\right)}.$$

Similarly:

$$p(w_2|x) = \frac{\frac{1}{\sqrt{2}} \exp\left(-\frac{(x-1)^2}{4}\right)}{\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2}} \exp\left(-\frac{(x-1)^2}{4}\right)}.$$

Notice that the posterior probabilities can be expressed using the logistic function. Let:

$$a = \frac{p(x|w_1)}{p(x|w_2)} = \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}{\frac{1}{\sqrt{4\pi}} \exp\left(-\frac{(x-1)^2}{4}\right)} = \sqrt{2} \exp\left(-\frac{x^2}{2} + \frac{(x-1)^2}{4}\right).$$

Simplify the exponent:

$$-\frac{x^2}{2} + \frac{(x-1)^2}{4} = -\frac{x^2}{2} + \frac{x^2 - 2x + 1}{4} = -\frac{2x^2}{4} + \frac{x^2 - 2x + 1}{4} = \frac{-x^2 - 2x + 1}{4}.$$

Thus:

$$a = \sqrt{2} \exp\left(\frac{-x^2 - 2x + 1}{4}\right).$$

Then:

$$p(w_1|x) = \frac{a}{a+1}, \quad p(w_2|x) = \frac{1}{a+1}.$$

This form is suitable for numerical computation.

Plotting $p(w_1|x)$ and $p(w_2|x)$

To draw $p(w_1|x)$ and $p(w_2|x)$, plot the functions:

$$p(w_1|x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2}}\exp\left(-\frac{(x-1)^2}{4}\right)},$$

$$p(w_2|x) = \frac{\frac{1}{\sqrt{2}}\exp\left(-\frac{(x-1)^2}{4}\right)}{\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2}}\exp\left(-\frac{(x-1)^2}{4}\right)}.$$

These can be plotted using computational tools like Python (with matplotlib) or MATLAB over a range of x, e.g., $x \in [-5, 5]$. The plot will show: - $p(w_1|x)$ starting high for negative x, decreasing as x increases. - $p(w_2|x)$ starting low for negative x, increasing as x increases. - The curves intersect where $p(w_1|x) = p(w_2|x) = 0.5$, which occurs when $p(x|w_1) = p(x|w_2)$.

To find the intersection, solve:

$$\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = \frac{1}{\sqrt{4\pi}}\exp\left(-\frac{(x-1)^2}{4}\right).$$

This requires numerical methods, but the plot will visually confirm the behavior.

The final expressions for the posterior probabilities are provided above, and the plots can be generated using the given formulas in a suitable programming environment.