

Pattern Recognition – Homework 3

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Question 8

Problem Statement

We have a two-class problem with feature $x \in \{1, 2\}$. The class-conditional probabilities are:

x	$p(x \omega_1)$	$p(x \omega_2)$
1	1/3	2/3
2	2/3	1/3

The loss matrix is

$$L = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad \lambda_{ki} = \text{cost of deciding } \omega_i \text{ when true class is } \omega_k.$$

The conditional risk for class ω_k is

$$r_k = \sum_{i=1}^2 \lambda_{ki} \sum_{x \in R_i} p(x|\omega_k).$$

a) Conditional Risks for All Possible Partitions

Since x is discrete, we consider the four possible decision rules.

Rule 1: $R_1 = \{1, 2\}$, $R_2 = \emptyset$ (always choose ω_1) For ω_1 :

$$r_1 = 0 \cdot (1/3 + 2/3) + 2 \cdot 0 = 0.$$

For ω_2 :

$$r_2 = 1 \cdot (2/3 + 1/3) + 0 \cdot 0 = 1.$$

Rule 2: $R_1 = \emptyset$, $R_2 = \{1, 2\}$ (always choose ω_2) For ω_1 :

$$r_1 = 0 \cdot 0 + 2 \cdot (1/3 + 2/3) = 2.$$

For ω_2 :

$$r_2 = 1 \cdot 0 + 0 \cdot (2/3 + 1/3) = 0.$$

Rule 3: $R_1 = \{1\}$, $R_2 = \{2\}$ For ω_1 :

$$r_1 = 0 \cdot (1/3) + 2 \cdot (2/3) = \frac{4}{3}.$$

For ω_2 :

$$r_2 = 1 \cdot (2/3) + 0 \cdot (1/3) = \frac{2}{3}.$$

Rule 4: $R_1 = \{2\}$, $R_2 = \{1\}$ For ω_1 :

$$r_1 = 0 \cdot (2/3) + 2 \cdot (1/3) = \frac{2}{3}.$$

For ω_2 :

$$r_2 = 1 \cdot (1/3) + 0 \cdot (2/3) = \frac{1}{3}.$$

b) Average Risk with $P(\omega_1) = 1/4$, $P(\omega_2) = 3/4$

The average risk is $r = r_1P(\omega_1) + r_2P(\omega_2)$.

- Rule 1: $r = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} = \frac{3}{4}$.
- Rule 2: $r = 2 \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} = \frac{1}{2}$.
- Rule 3: $r = \frac{4}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{3}{4} = \frac{4}{12} + \frac{6}{12} = \frac{5}{6}$.
- Rule 4: $r = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$.

c) Optimal Bayes Rule

The risks are $\frac{3}{4}$, $\frac{1}{2}$, $\frac{5}{6}$, $\frac{5}{12}$. The smallest is $\frac{5}{12}$, achieved by Rule 4.

Thus the optimal decision is:

- If $x = 1$, decide ω_2 .
- If $x = 2$, decide ω_1 .