Statistical Pattern Recognition - Homework 1
Question 23

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Question 23

Which matrix has the eigenvector $\begin{bmatrix}1\\2\end{bmatrix}$ with eigenvalue 2 and the eigenvector $\begin{bmatrix}-2\\1\end{bmatrix}$ with eigenvalue 1? Note that the given eigenvectors are not normalized. Provide justification for your choice.

Options:

$$\text{a)} \begin{bmatrix}\frac{9}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{6}{5}\end{bmatrix}, \quad \text{b)} \begin{bmatrix}9 & -2\\ -2 & 6\end{bmatrix}, \quad \text{c)} \begin{bmatrix}6 & 2\\ 2 & 9\end{bmatrix}, \quad \text{d)} \begin{bmatrix}\frac{6}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{9}{5}\end{bmatrix}.$$

Solution

To find the correct matrix, we need to identify which 2x2 matrix A satisfies the eigenvalue-eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ for both given eigenvectors and their corresponding eigenvalues: - Eigenvector $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ with eigenvalue $\lambda_1 = 2$. - Eigenvector $\mathbf{v}_2 = \begin{bmatrix}-2\\1\end{bmatrix}$ with eigenvalue $\lambda_2 = 1$.

For each matrix, we will compute $A\mathbf{v}_1$ and check if it equals $2\mathbf{v}_1$, and compute $A\mathbf{v}_2$ and check if it equals $\mathbf{v}_2$. The matrix that satisfies both conditions is the correct choice.

Option (a): $\begin{bmatrix}\frac{9}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{6}{5}\end{bmatrix}$

Let $A = \begin{bmatrix}\frac{9}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{6}{5}\end{bmatrix}$.

For $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\lambda_1 = 2$:

$$A\mathbf{v}_1 = \begin{bmatrix}\frac{9}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{6}{5}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\frac{9}{5}\cdot 1 + \left(-\frac{2}{5}\right)\cdot 2\\ \left(-\frac{2}{5}\right)\cdot 1 + \frac{6}{5}\cdot 2\end{bmatrix} = \begin{bmatrix}\frac{9}{5} - \frac{4}{5}\\ -\frac{2}{5} + \frac{12}{5}\end{bmatrix} = \begin{bmatrix}\frac{5}{5}\\ \frac{10}{5}\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}.$$

Compare with $\lambda_1 \mathbf{v}_1 = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Since $A\mathbf{v}_1 \neq 2\mathbf{v}_1$, option (a) is incorrect without needing to check the second eigenvector.

Option (b): $\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

Let $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$.

For $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda_1 = 2$:

$$A\mathbf{v}_1 = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \cdot 1 + (-2) \cdot 2 \\ (-2) \cdot 1 + 6 \cdot 2 \end{bmatrix} = \begin{bmatrix} 9 - 4 \\ -2 + 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

Compare with $2\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$:

$$\begin{bmatrix} 5 \\ 10 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Since $A\mathbf{v}_1 \neq 2\mathbf{v}_1$, option (b) is incorrect.

Option (c): $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

Let $A = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$.

For $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda_1 = 2$:

$$A\mathbf{v}_1 = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 2 \cdot 2 \\ 2 \cdot 1 + 9 \cdot 2 \end{bmatrix} = \begin{bmatrix} 6 + 4 \\ 2 + 18 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}.$$

Compare with $2\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$:

$$\begin{bmatrix} 10 \\ 20 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Since $A\mathbf{v}_1 \neq 2\mathbf{v}_1$, option (c) is incorrect.

Option (d): $\begin{bmatrix} \frac{6}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{9}{5} \end{bmatrix}$

Let $A = \begin{bmatrix} \frac{6}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{9}{5} \end{bmatrix}$.

For $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\lambda_1 = 2$:

$$A\mathbf{v}_1 = \begin{bmatrix} \frac{6}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \cdot 1 + \frac{2}{5} \cdot 2 \\ \frac{2}{5} \cdot 1 + \frac{9}{5} \cdot 2 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} + \frac{4}{5} \\ \frac{2}{5} + \frac{18}{5} \end{bmatrix} = \begin{bmatrix} \frac{10}{5} \\ \frac{20}{5} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Compare with $2\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

This satisfies $A\mathbf{v}_1 = 2\mathbf{v}_1$.

For $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\lambda_2 = 1$:

$$A\mathbf{v}_2 = \begin{bmatrix} \frac{6}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \cdot (-2) + \frac{2}{5} \cdot 1 \\ \frac{2}{5} \cdot (-2) + \frac{9}{5} \cdot 1 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} + \frac{2}{5} \\ -\frac{4}{5} + \frac{9}{5} \end{bmatrix} = \begin{bmatrix} -\frac{10}{5} \\ \frac{5}{5} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Compare with $\lambda_2 \mathbf{v}_2 = 1 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

This satisfies $A\mathbf{v}_2 = \mathbf{v}_2$.

Since option (d) satisfies both eigenvalue-eigenvector pairs, it is a candidate for the correct answer.

To ensure correctness, note that the eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are linearly independent (their determinant is $1 \cdot 1 - 2 \cdot (-2) = 1 + 4 = 5 \neq 0$). A 2x2 matrix with two distinct eigenvalues and linearly independent eigenvectors is uniquely determined by its eigenvalue-eigenvector pairs. Since option (d) satisfies both conditions, it is the correct matrix.

The correct matrix is:

$$\text{d)} \begin{bmatrix} \frac{6}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{9}{5} \end{bmatrix}.$$

Justification: The matrix satisfies $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, as verified by direct computation. The other options fail to satisfy at least one of the eigenvalue-eigenvector conditions.