Gauss Quadrature - Numerical Techniques and Computer Programming

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Consider the numerical evaluation of the integral

$$\int_a^b f(x)dx$$

Carl Friedrich Gauss derived a formula that gives better accuracy. His formula also called is expressed as:

$$\begin{aligned}\int_{-1}^{1} F(u)du &= W_1F(u_1) + W_2F(u_2) + W_3F(u_3) + \ldots + W_nF(u_n) \\ &= \sum_{i=1}^{n} W_iF(u_i) \end{aligned} \tag{1}$$

Where, W_i are the weights and the u_i are called the abscissa respectively.
The formula has an advantage that the abscissae and weights are symmetrical with respect to the middle point of the interval.
In Eq.(1) there are $2n$ arbitrary parameters. We can determine the weights and abscissae so that the formula is exact when $F(u)$ is a polynomial of degree not exceeding $2n-1$.

Therefore, lets see such a polynomial:

$$F(u) = C_0 + C_1u + C_2u^2 + C_3u^3 + \ldots + C_{2n-1}u^{2n-1} \tag{2}$$

Now, from Eq.(1),

$$\begin{aligned}\int_{-1}^{1} F(u)du &= \int_{-1}^{1} \left(C_0 + C_1u + C_2u^2 + C_3u^3 + \ldots + C_{2n-1}U^{2n-1}\right) du \qquad (3) \\ &= C_0u\Big|_{-1}^{1} + C_1\frac{u^2}{2}\Big|_{-1}^{1} + C_2\frac{u_3}{3}\Big|_{-1}^{1} + \ldots \\ &= 2C_0 + \frac{2}{3}C_2 + \frac{2}{5}C_4 + \ldots \end{aligned}$$

Now, lets put $u = u_i$ in Eq.(2), we get:

$$F(u_i) = C_0 + C_1u_i + C_2u_i^2 + C_3u_i^3 + \ldots + C_{2n-1}u_i^{2n-1}$$

Now, Eq.(1) becomes:

$$\begin{aligned}\int_{-1}^{1} F(u)du = W_1 &\left(C_0 + C_1u_1 + C_2u_1^2 + C_3u_1^3 + \ldots + C_{2n-1}u_1^{2n-1}\right)\\ &+ W_2\left(C_0 + C_1u_2 + C_2u_2^2 + C_3u_2^3 + \ldots + C_{2n-1}u_2^{2n-1}\right)\\ &+ W_3\left(C_0 + C_1u_3 + C_2u_3^2 + C_3u_3^3 + \ldots + C_{2n-1}u_3^{2n-1}\right) + \ldots\\ &+ W_n\left(C_0 + C_1u_n + C_2u_n^2 + C_3u_n^3 + \ldots + C_{2n-1}u_n^{2n-1}\right)\end{aligned}$$

This can be written as:

$$\begin{aligned}\int_{-1}^{1} F(u)du = C_0 &\left(W_1 + W_2 + \ldots + W_n\right)\\ &+ C_1\left(W_1u_1 + W_2u_2 + \ldots + W_nu_n\right)\\ &+ C_2\left(W_1u_1^2 + W_2u_2^2 + \ldots + W_nu_n^2\right) + \ldots\\ &+ C_{2n-1}\left(W_1u_1^{2n-1} + W_2u_2^{2n-1} + \ldots + W_nu_n^{2n-1}\right)\end{aligned} \tag{4}$$

Now, Since Eq.(3) and Eq.(4) are identical for all values of C_i, lets compare the coefficients of C_i and we get $2n$ equations:

$$\begin{aligned}2 &= W_1 + W_2 + W_3 + \ldots + W_n\\ 0 &= W_1u_1 + W_2u_2 + W_3u_3 + \ldots + W_nu_n\\ \frac{2}{3} &= W_1u_1^2 + W_2u_2^2 + W_3u_3^2 + \ldots + W_nu_n^2\\ \vdots &\quad \vdots \quad \vdots \quad \vdots \quad \vdots\\ 0 &= W_1u_1^{2n-1} + W_2u_2^{2n-1} + W_3u_3^{2n-1} + \ldots + W_nu_n^{2n-1}\end{aligned} \tag{5}$$

Now, for different values of n, we can find the corresponding Gauss-Quadrature formula.

Lets find for $n = 1$ first:

$$\int_{-1}^{1} F(x)dx = W_1F(x_1)$$

is exact for polynomials of degrees up to $2 \times 1 - 1 = 1$ i.e for x^0 and x^1

For $x^0 = 1$:

$$\int_{-1}^{1} 1dx = x\Big|_{-1}^{1} = 2 = W_1 \times 1 \qquad \Rightarrow \boxed{W_1 = 2}$$

For x^1:

$$\int_{-1}^{1} x dx = \frac{x^2}{2}\bigg|_{-1}^{1} = 0 = W_1 x_1 \qquad \Rightarrow 2x_1 = 0 \Rightarrow \boxed{x_1 = 0}$$

The gauss quadrature formula for $n = 1$ is therefore

$$\boxed{\int_{-1}^{1} F(x)dx = 2f(0)}$$

.

Now, let's find the Gauss quadrature formula for $n = 2$

$$\int_{-1}^{1} F(x)dx = W_1 F(x_1) + W_2 F(x_2)$$

is exact for polynomials of degrees up to $2 \times 2 - 1 = 3$ i.e for x^0, x^1, x^2 and x^3

For $x^0 = 1$:

$$\int_{-1}^{1} 1dx = x\bigg|_{-1}^{1} = 2 = W_1 + W_2 \qquad \Rightarrow \boxed{W_1 + W_2 = 2}$$

For x^1:

$$\int_{-1}^{1} x dx = \frac{x^2}{2}\bigg|_{-1}^{1} = 0 = W_1 x_1 \qquad \Rightarrow \boxed{W_1 x_1 + W_2 x_2 = 0}$$

For x^2:

$$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\bigg|_{-1}^{1} = \frac{2}{3} = W_1 x_1^2 + W_2 x_2^2 \qquad \Rightarrow \boxed{W_1 x_1^2 + W_2 x_2^2 = \frac{2}{3}}$$

For x^3:

$$\int_{-1}^{1} x^3 dx = \frac{x^4}{4}\bigg|_{-1}^{1} = 0 = W_1 x_1^3 + W_2 x_2^3 \qquad \Rightarrow \boxed{W_1 x_1^3 + W_2 x_2^3 = 0}$$

Upon solving those 4 equations, we get

$$W_1 = W_2 = 1, \text{ and } x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$$

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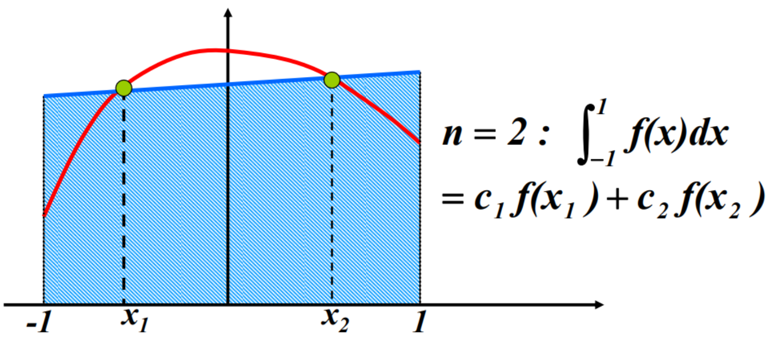
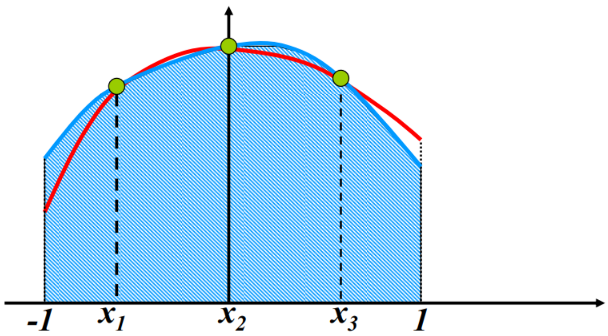
The Gauss quadrature formula for $n = 2$ is therefore

$$\boxed{\int_{-1}^{1} F(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$



$$\boxed{n = 3: \int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)}$$

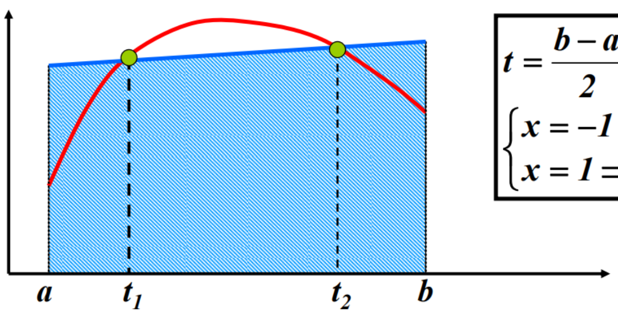


The table for up to $n = 5$ is as follows:

Number of points, n	Points, x_i		Weights, w_i	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm\sqrt{\frac{3}{5}}$	±0.774597...	$\frac{5}{9}$	0.555556...
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.339981...	$\frac{18+\sqrt{30}}{36}$	0.652145...
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.861136...	$\frac{18-\sqrt{30}}{36}$	0.347855...
5	0		$\frac{128}{225}$	0.568889...
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	±0.538469...	$\frac{322+13\sqrt{70}}{900}$	0.478629...
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	±0.90618...	$\frac{322-13\sqrt{70}}{900}$	0.236927...

The integral we discussed here is in the interval $[-1, 1]$. What to do if it is in the interval $[a, b]$?

Coordinate transformation from [a,b] to [-1,1]



$$\boxed{t = \frac{b-a}{2}x + \frac{b+a}{2} \qquad \begin{cases} x = -1 \Rightarrow t = a \\ x = 1 \Rightarrow \quad t = b \end{cases}}$$

$$\boxed{\int_a^b f(t)dt = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\left(\frac{b-a}{2}\right)dx = \int_{-1}^{1} g(x)dx}$$

Try this!

Approximate the integral value using gaussian quadrature.

$$\int_{-1}^{1} e^{-x^2}$$

Solution:
for n=1, $\int_{-1}^{1} f(x)dx = 2f(0)$

$$\int_{-1}^{1} e^{-x^2}dx = 2f(0) = 2 \times e^{0^2} = 2$$

for n=2, $\int_{-1}^{1} f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$

$$\int_{-1}^{1} e^{-x^2}dx = f(-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}) = e^{-\frac{1}{3}} + e^{\frac{1}{3}} = 2e^{\frac{1}{3}} \approx 2 \times 0.7165 \approx 1.433$$

Now, for n=3, $\int_{-1}^{1} f(x)dx = \frac{5}{9} \cdot f(-\sqrt{\frac{3}{5}}) + \frac{8}{9} \cdot f(0) + \frac{5}{9} \cdot f(\sqrt{\frac{3}{5}})$

$$\int_{-1}^{1} e^{-x^2}dx \approx 1.49868$$

References:

- M. Goyal. Computer-Based Numerical & Statistical Techniques. ISBN: 978-0-9778582-5-5
- Wikipedia
- CMSC460 Lecture notes of Ramani Duraiswami