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Q: Let $X_1, \dots, X_n$ be random sample from a normal distribution, $X_i \sim N(\mu, 1)$.

a) Find a UMP test of $H_0: \mu = \mu_0$ against $H_A: \mu \leq \mu_0$

S1. The PDF...

$$f_X(x_i; \mu) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right) e^{-\left(\frac{x-\mu}{\sigma}\right)^2 \left(\frac{1}{2}\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) e^{-(x-\mu)^2\left(\frac{1}{2}\right)}$$

$(x_i - \mu)(x_i - \mu)$

$x_i^2 - 2\mu x_i + \mu^2$

S2. The JDF

$$f_{\underline{X}}(\underline{x}; \mu) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-(x-\mu)^2\left(\frac{1}{2}\right)} \cdot \prod_{i=1}^{n} I\{-\infty \leq x \leq \infty\}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{-(x_i-\mu)^2\left(\frac{1}{2}\right)} \cdot 1$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{-(x_i^2 - 2x_i\mu + \mu^2)\left(\frac{1}{2}\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{(-x_i^2)\left(\frac{1}{2}\right)} e^{(x_i\mu)} e^{(\mu^2)\left(\frac{1}{2}\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(e^{\frac{n\mu^2}{2}}\right) \prod_{i=1}^{n} e^{-x_i^2/2} \prod_{i=1}^{n} e^{x_i\mu}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(e^{\frac{n\mu^2}{2}}\right) \left(e^{-\frac{\sum x_i^2}{2}}\right) \left(e^{\mu\sum x_i}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(e^{\frac{n\mu^2}{2}}\right) \left(e^{-\frac{\sum x_i^2}{2}}\right) \left(e^{n\mu\bar{x}}\right)$$

$g(s;\mu)$ by factorization theorem

Clearly, this is of the exponential family where the parameter μ only depends on X through $t(\underline{x}) = \bar{x}$

$\therefore$ Thus, we can reject the null hypothesis for $\bar{x} \leq K$ where K is a value such that:

$$P[\bar{x} \leq K \mid \mu = \mu_0] = \alpha$$

Since we know X is normally distributed, we can standardize the above equation to get a z test:

$$P[\bar{x} \leq K \mid \mu = \mu_0] = P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \leq \frac{K - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

$$= P\left[Z \leq \frac{K - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

Finally, solving for K gives...

$$(z_\alpha)(\sigma/\sqrt{n}) = K - \mu_0$$

$$(Z_\alpha)(\sigma/\sqrt{n}) = K - \mu_0$$

$$\Rightarrow \quad (Z_\alpha)(\sigma/\sqrt{n}) + \mu_0 = K$$

$\therefore$ We reject the null hypothesis if

$$\bar{X} \le K$$

$$\Rightarrow \quad \bar{X} \le (Z_\alpha \cdot \sigma/\sqrt{n}) + \mu_0$$

Z_α because

$H_A: \mu < \mu_0$

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