Question #4

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- Let X be a vector st $|X| = m$
- Let Y be a vector st. $|Y| = n$
- Then let $Z = X \cup Y$
- Then $|Z| = m+n$
- Now let $W_{test} = \sum_{i=1}^{|Z|} rank(z_i) \cdot I\{z_i \in X\}$
- Thus, $E[W_{test}]$ gives...

$$E[W_x] = E\left[\sum^{m+n} rank(z_i) \cdot I\{z_i \in X\}\right]$$
$$= \sum^{m+n} rank(z_i) \cdot E[I\{z_i \in X\}]$$
$$(\star) \quad = \sum^{m+n} rank(z_i) \cdot P(z_i \in X)$$

- Then since $Z = X \cup Y$, $P(z_i \in X) = \frac{|X|}{|X|+|Y|} = \frac{m}{m+n}$

Continuing... $(\star)$

$$= \sum^{m+n} rank(z_i) \cdot \left(\frac{m}{m+n}\right)$$
$$= \left(\frac{m}{m+n}\right) \sum^{m+n} rank(z_i)$$

- let $N = m+n$ for ease of writing

$$= \frac{m}{N} \sum^{N} rank(z_i)$$
$$= \left(\frac{m}{N}\right)(1+2+3+4+\ldots+N-1+N)$$
$$= \left(\frac{m}{N}\right)\frac{2(1+2+\ldots+N)}{2} = \frac{\left(\frac{m}{N}\right)\left((1+2+\ldots+N-1+N)+(N+N-1+\ldots+2+1)\right)}{2}$$

- summing component wise makes each component $N+1$
- there are N components total

$$= \frac{\frac{m}{N}\left((N+1)+(N+1)+\ldots+(N+1)\right)}{2} = \frac{\frac{m}{N}(N(N+1))}{2}$$

$$= \frac{m\cancel{N}(N+1)}{2\cancel{N}} \qquad \boxed{\therefore E[W_x] = \frac{m(N+1)}{2}} \checkmark$$

- now to consider $Var[W_x]$, which is just a sum of the variances corresponding to the ranks of Z which are found in X, plus their co-variances.

ie, $var(W_x) = \sum_{i=1}^{m} Var(rank(z_i)) + \sum_{i=1}^{m}\sum_{j=1}^{m-1} Cov(rank(z_i), rank(z_j))$

$$= m\left[\frac{N^2-1}{12}\right] + m(m-1)\left[\frac{1-N}{12}\right]$$
$$= \frac{1}{12}m\left[N^2-1+(m-1)(1-N)\right]$$
$$= \frac{1}{12}m\left[(N^2-1)-(m-1)(1+N)\right]$$
$$= \frac{1}{12}m\left[(N+1)(N-1)-(m-1)(N+1)\right]$$
$$= \frac{(N+1)m}{12}\left[N-1-m+1\right]$$
$$= \frac{(N+1)m}{12}\left[N-m\right]$$

n^2-1

$(n+1)(n-1)$

$n^2+-n+n-1$

- Then since $N = n+m$, $N-m = n$

$$\boxed{\therefore Var[W_x] = \frac{mn(N+1)}{12}} \checkmark$$