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Wednesday, February 12, 2020 5:00 PM

Q: Suppose $x_1, \dots, x_n$ is a random sample from a normal distribution $X_i \sim N(0, \sigma^2)$

a) Derive the UMP test of size α for $H_0: \sigma = \sigma_0$ and $H_A: \sigma > \sigma_0$

S1. The pdf...

$$f(x;\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x}{\sigma}\right)^2\left(\frac{1}{2}\right)}$$

S2. The JDF...

$$f_{\underline{x}}(\underline{x};\sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x_i}{\sigma}\right)^2\left(\frac{1}{2}\right)} \cdot \prod_{i=1}^{n} I\{-\infty \le x_i \le \infty\}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot \prod_{i=1}^{n} e^{\left(-\frac{x_i}{\sigma}\right)^2\left(\frac{1}{2}\right)} \cdot 1$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{\sum x_i^2}{2\sigma^2}}$$

S3. Then by the Neyman Pearson Lemma...

$$\frac{f_{\underline{x}}(\underline{x};\sigma_1)}{f_{\underline{x}}(x;\sigma_0)} = \frac{\left(\frac{1}{\sigma_1}\right)^n \cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n}\left(e^{-\frac{\sum x_i^2}{2\sigma_1^2}}\right)}{\left(\frac{1}{\sigma_0}\right)^n \cancel{\left(\frac{1}{\sqrt{2\pi}}\right)^n}\left(e^{-\frac{\sum x_i^2}{2\sigma_0^2}}\right)}$$

$$= \frac{\left(\frac{1}{\sigma_1}\right)^n \left(e^{-\frac{\sum x_i^2}{2\sigma_1^2}}\right)}{(\sigma_0)^{-n}\left(e^{-\frac{\sum x_i^2}{2\sigma_1^2}}\right)}$$

exponent signs flip when brought into numerator

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^n \left(e^{-\frac{\sum x_i^2}{2\sigma_1^2} + \frac{\sum x_i^2}{2\sigma_1^2}}\right)$$

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^n \left(e^{\left(-\frac{\sum x_i^2}{2}\right)\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)}\right)$$

... which is a monotone likelihood ratio of $f(\underline{x}) = \sum x_i^2$

$\therefore$ Thus by theorem 12.7.1, we can reject the null hypothesis if $\sum x_i^2 \ge k$ where $P[\sum x_i^2 \ge k \mid \sigma = \sigma_0] = \alpha$

✓

Then, since $X_i \sim N(0, \sigma^2)$ for some σ^2, $\frac{x}{\sigma}$ is a standard normal random variable.

Thus, under the null $\left(\sum_{i=1}^{n} x_i^2 / \sigma_0^2\right) \sim \chi^2(n)$

∴ Thus we can reject H_0 at size α if:

$$\frac{\sum x_i^2}{\sigma_0^2} \geq \chi^2_{1-\alpha}(n)$$