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Wednesday, February 12, 2020 9:17 PM

Q: Consider a random sample of size n from a bernoulli distribution, $X_i \sim Bin(1, p)$

a) Derive a UMP test of $H_0: p \le p_0$ versus $H_A: p > p_0$ using theorem 12.7.1

S1. The PDF...

$$f(x; p) = p^x (1-p)^{1-x}$$

S2. The JPDF...

$$f(\underline{x}; \theta) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} \cdot \prod_{i=1}^{n} I\{x =$$

$$= p^{\sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} (1-p)^{1-x_i}$$

$$= p^{\sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} (1-p) \cdot (1-p)^{-x_i}$$

$$= p^{\sum_{i=1}^{n} x_i} \cdot (1-p)^n \cdot (1-p)^{\sum -x_i}$$

$$= p^{\sum_{i=1}^{n} x_i} \cdot (1-p)^{n - \sum_{i=1}^{n} x_i}$$

S3. Then the monotone likelihood ratio

$$\frac{f_{\underline{x}}(x; p_0)}{f_{\underline{x}}(\underline{x}; p_1)} = \frac{p_0^{\sum_{i=1}^{n} x_i} \cdot (1-p_0)^{n - \sum_{i=1}^{n} x_i}}{p_1^{\sum_{i=1}^{n} x_i} \cdot (1-p_1)^{n - \sum_{i=1}^{n} x_i}}$$

and $H_A: p = p_1 > p_0$

$$= \left[\frac{(1-p_0)}{(1-p_1)}\right]^n \frac{p_0^{\sum_{i=1}^{n} x_i} (1-p_0)^{-\sum_{i=1}^{n} x_i}}{p_1^{\sum_{i=1}^{n}} (1-p_1)^{-\sum_{i=1}^{n} x_i}}$$

$$= \left(\frac{1-p_0}{1-p_1}\right)^n \frac{p_0^{\sum_{i=1}^{n} x_i} (1-p_1)^{\sum_{i=1}^{n} x_i}}{p_1^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n} x_i}}$$

$$= \left(\frac{1-p_0}{1-p_1}\right)^n \left(\frac{p_0 (1-p_1)}{p_1 (1-p_0)}\right)^{\sum_{i=1}^{n} x_i}$$

then since this is a monotone non-decreasing function on $\sum_{i=1}^{n} x_i$, $t(\underline{x}) = \sum_{i=1}^{n} x_i$ (ie, $S = \sum_{i=1}^{n} X_i$ is sufficient)

∴ Thus we can reject the null hypothesis if $\sum x_i \ge k$ where $P[\sum_{i=1}^{n} X_i \ge k \mid p = p_0] = \alpha$

✓

b) Recalling the JDF from S2 of part a, . .

JDF

$$f(x;p) = p^{\sum_{i=1}^{n} x_i} \cdot (1-p)^{n - \sum_{i=1}^{n} x_i}$$

Now, we want to use the factorization theorem by making this into some function of the form

$c(p) \cdot h(\underline{x}) \cdot \exp\{q(\theta) t(\underline{x})\}$ where $q(\theta)$ is an increasing function of θ.

S1. Take the natural log of the function.

S1. $$\ln[f(x;p)] = \ln\left[p^{\sum_{i=1}^{n} x_i} \cdot (1-p)^{n-\sum_{i=1}^{n} x_i}\right]$$

$$= \left(\sum_{i=1}^{n} x_i\right) \ln[p] + \underbrace{\left(n - \sum_{i=1}^{n} x_i\right) \ln[1-p]}_{n \ln[1-p] - \sum_{i=1}^{n} x_i \ln[1-p]}$$

$$= \left(\sum_{i=1}^{n} x_i\right) \ln[p] - \sum_{i=1}^{n} x_i \ln[1-p] + n \ln[1-p]$$

$$= \left(\sum_{i=1}^{n} x_i\right) \ln\left[\frac{p}{1-p}\right] + n \ln[1-p]$$

Then taking the exponential . . .

$$= \underbrace{\exp\left\{\left(\sum_{i=1}^{n} x_i\right) \ln\left[\frac{p}{1-p}\right]\right\}}_{\exp\{t(x) \cdot q(p)\}} \overset{h(x) = x^0 = 1}{\cdot} \underbrace{\exp\{n \ln[1-p]\}}_{c(p)}$$

$\therefore$ We can reject the null hypothesis if $\sum_{i=1}^{n} x_i \geq k$ for some k such that

$$P\left[\sum_{i=1}^{n} x_i \geq k \mid p < p_0\right] = \alpha$$

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