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Thursday, February 13, 2020 7:58 AM

• Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution, $X_i \sim Exp(\theta)$.

a) Derive the generalized likelihood ratio test for $H_0: \theta = \theta_0$ against $H_A: \theta \neq \theta_0$. Determine an approximate critical value for size α using a large sample chi-square approximation.

S1. The PDF...

$$f(x,\theta) = \frac{1}{\theta}\exp\left\{-\frac{x}{\theta}\right\} \cdot I\{x>0\}$$

S2. Find the MLE

S2a) The JDF...

$$f(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x/\theta} \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

$$= \left(\frac{1}{\theta}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\theta}\right) \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

S2b) Likelihood...

$$L(f(\underset{\sim}{x};\theta)) = \left(\frac{1}{\theta}\right)^n \left(e^{-\frac{\sum_{i=1}^{n} X_i}{\theta}}\right) \cdot \underbrace{\prod_{i=1}^{n} I\{X_i > 0\}}_{=1}$$

S2c) Taking the log likelihood

$$\ln[L(f(\underset{\sim}{x};\theta))] = \ln\left[\left(\frac{1}{\theta}\right)^n \left(e^{-\sum_{i=1}^{n} X_i/\theta}\right)\right]$$

$$= n\ln\left[\frac{1}{\theta}\right] + \ln\left[e^{-\sum_{i=1}^{n} x_i/\theta}\right]$$

$$= n\ln\left[\frac{1}{\theta}\right] - \left(\sum_{i=1}^{n} x_i/\theta\right)\ln[e]$$

$$= n\ln[1/\theta] - \sum_{i=1}^{n} x_i/\theta$$

$$= n\ln[1] - n\ln[\theta] - \sum_{i=1}^{n} x_i/\theta$$

S2d) Taking the first derivative and setting equal to zero.

$$\frac{\partial \ln[L(\theta)]}{\partial\theta} = 0 - \frac{n}{\theta} + \sum_{i=1}^{n} X_i/\theta^2 = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\theta^2} = \frac{n}{\theta}$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \theta^2/\theta$$

$$\frac{\sum_{i=1}^{n} x_i}{n} = \hat{\theta}$$

$$\hat{\theta} = \sum_{i=1}^{n} x_i / n$$

S2 e) checking if this is a maximum by 2nd derivative test

$$\frac{\partial^2 \ln[L(\theta)]}{\partial \theta^2} =$$ First derivative was $\frac{\sum x_i}{\theta} - n = 0$

Then the second derivative is

$$-\frac{\sum x_i}{\theta^2}$$ which is less than $0 \ \forall \theta > 0$ ✓

$$\therefore MLE_{\exp} = \sum_{i=1}^{n} x_i / n = \bar{x}$$

S3. Now we know the MLE is $\bar{x}$, we can compute the GLR using the $MLE_{\hat{\theta}}$.

$$\lambda(\underset{\sim}{x}; \theta_0, \theta_1) = \frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}; \theta_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}; \hat{\theta})} = \frac{\left(\frac{1}{\theta_0}\right)^n \left(e^{-\sum_{i=1}^{n} x_i / \theta_0}\right)}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-\sum_{i=1}^{n} x_i / \bar{x}}\right)}$$

$$= \frac{\left(\frac{1}{\theta_0}\right)^n \left(e^{-\sum_{i=1}^{n} x_i / \theta_0}\right)}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-\sum_{i=1}^{n} \cancel{x_i} / \left(\sum_{i=1}^{n} \cancel{x_i} / n\right)}\right)}$$

$$= \frac{\left(\frac{1}{\theta_0}\right)^n \left(e^{-\sum_{i=1}^{n} x_i / \theta_0}\right)}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-n}\right)}$$

$$\lambda(x) = \left(\frac{\bar{x}}{\theta}\right)^n \left(e^{n - \sum_{i=1}^{n} x_i / \theta_0}\right)$$

Now knowing that can reject H_0 if $-2 \ln \lambda(x) \geq \chi^2_{1-\alpha}(1)$ . . .

Now knowing that can reject H_0 if $-2\ln \lambda(x) \geq \chi^2_{1-\alpha}(1)$ . . .

$$-2\ln\left[\left(\frac{\bar{x}}{\theta}\right)^n \exp\left\{n - \sum_{i=1}^{n} x_i / \theta_0\right\}\right]$$

$$= -2n\ln\left[\bar{x}/\theta_0\right] - 2\ln\left[\exp\left\{n - \sum_{i=1}^{n} x_i / \theta_0\right\}\right]$$

$$= -2n\ln\left[\bar{x}/\theta_0\right] - 2\ln\left[\exp\left\{n - n\sum_{i=1}^{n} x_i / n\theta_0\right\}\right]$$

$$= -2n\ln\left[\bar{x}/\theta_0\right] - 2\ln\left[\exp\left\{n - \frac{n\bar{x}}{\theta_0}\right\}\right]$$

$$= -2n\ln\left[\bar{x}/\theta_0\right] - 2\ln\left[\exp\left\{n\left(1 - \frac{x}{\theta_0}\right)\right\}\right]$$

$$= -2n\ln\left[\bar{x}/\theta_0\right] - 2\left(n\left(1 - x/\theta_0\right)\right)$$

$$= -2n\ln\left[\bar{x}/\theta_0\right] - 2n\left(1 - x/\theta_0\right)$$

$$= -2n\left[\ln\left(\bar{x}/\theta_0\right) + \frac{\bar{x}}{\theta_0} - 1\right]$$

$\therefore$ Thus, we can reject H_0 if

$$-2n\left[\ln\left(\bar{x}/\theta_0\right) + \left(\bar{x}/\theta_0\right) - 1\right] \geq \chi^2_{1-\alpha}(1)$$

✓