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Q: Consider a random sample of size 'n' from a distribution with pdf $f(x;\theta) = (3x^2/\theta)e^{\frac{-x^3}{\theta}}$ if $0 < x$ and zero otherwise. Derive the form of the critical region for a uniformly most powerful test (UMP) test of size α for $H_0: \theta = \theta_0$ against $H_A: \theta > \theta_0$

S:

S1. The pdf:

$$f(x;\theta) = \left(\frac{3x^2}{\theta}\right) e^{\frac{-x^3}{\theta}}$$

S2. The jdf:

$$f(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \frac{3x^2}{\theta} \cdot e^{\frac{-x^3}{\theta}} \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

$$= \left(\frac{1}{\theta}\right)^n (3)^n \left(\prod_{i=1}^{n} x_i^2 \cdot e^{-\frac{x_i^3}{\theta}}\right)$$

$$= \left(\frac{3}{\theta}\right)^n \left(\prod_{i=1}^{n} x_i^2\right)\left(e^{\frac{-\Sigma x_i^3}{\theta}}\right)$$

S3. Drawing upon the Neymen-Pearson Lemma:

$$\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta_1)}{f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta_0)} = \frac{\left(\frac{3}{\theta_1}\right)^n \cancel{\left(\prod_{i=1}^{n} x_i^2\right)} \left(e^{-\frac{\Sigma x_i^3}{\theta_1}}\right)}{\left(\frac{3}{\theta_0}\right)^n \cancel{\left(\prod_{i=1}^{n} x_i^2\right)} \left(e^{-\frac{\Sigma x_i^3}{\theta_1}}\right)}$$

$$= \frac{\cancel{(3^n)}\left(\frac{1}{\theta_1}\right)^n \left(e^{-\frac{\Sigma x_i^3}{\theta_1}}\right)}{\cancel{(3^n)}\left(\frac{1}{\theta_0}\right)^n \left(e^{-\frac{\Sigma x_i^3}{\theta_0}}\right)}$$

$$= \frac{\left(\frac{1}{\theta_1}\right)^n \left(e^{-\frac{\Sigma x_i^3}{\theta_1}}\right)}{(\theta_0)^{-n}\left(e^{-\frac{\Sigma x_i^3}{\theta_0}}\right)}$$

Bringing into numerator flips the +/- of the denominator exponents

$$= \left(\frac{\theta_0}{\theta_1}\right)^n \left(e^{\frac{\Sigma x_i^3}{\theta_0} - \frac{\Sigma x_i^3}{\theta_1}}\right)$$

$$= \left(\frac{\theta_0}{\theta_1}\right)^n \left(e^{(-\Sigma x_i^3)\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}\right)$$

... which is a monotone likelihood ratio of $f(\underset{\sim}{x}) = \Sigma x_i^3$.

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Thm 12.7.1 :

- if a joint pdf $f_X(x, \theta)$ has an MLR in statistic $t(\underline{x})$, the UMP test of size α can be found for:

$$\begin{cases} H_0: \theta \leq \theta_0 \\ H_A: \theta > \theta_0 \end{cases}$$

- if we reject H_0 for $t(x) \geq k$ where k satisfies:

$$P[t(\underline{x}) \geq k \mid H_0] = \alpha$$

$\therefore$ Thus, we can reject the null hypothesis if

$$t(\underline{x}) = \sum x_i^3 \geq k \quad \text{where} \quad P[\sum x_i^3 \geq k \mid H_0] = \alpha$$

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