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Q: Let $X_1, \ldots, X_N$ have a joint pdf $f_{\underline{x}}(\underline{x};\theta)$ and let 'S' be the sufficient statistic for θ.

Show that a most powerful test $H_0: \theta = \theta_0$ versus $H_A: \theta = \theta_1$ can be expressed in terms of 'S'.

S: The given hypotheses are $H_0: \theta = \theta_0$, $H_A: \theta = \theta_1$.

Then by the Neyman-Pearson-Lemma we define a function $\lambda(\underline{x}; \theta_0, \theta_1)$ such that:

$$\lambda(\underline{x}; \theta_0, \theta_1) = \frac{f_{\underline{x}}(\underline{x}; \theta_1)}{f_{\underline{x}}(\underline{x}; \theta_0)}$$

Moreover, the Neyman-Pearson Lemma guarantees us that a test based on the critical region $C^* = \{ \underline{x} \mid \lambda(\underline{x}; \theta_0, \theta_1) \leq K \}$ gives us a "most powerful" test among size α tests.

Then, by some algebra (sorcery ☺)...

$$C^* = \{ \underline{x} \mid \lambda(\underline{x}; \theta_0, \theta_1) \leq K \}$$

$$\Rightarrow C^* = \{ \underline{x} \mid \frac{f_{\underline{x}}(\underline{x}; \theta_1)}{f_{\underline{x}}(\underline{x}, \theta_0)} \leq K \}$$

$$\Rightarrow C^* = \{ \underline{x} \mid f_{\underline{x}}(\underline{x}; \theta_1) \leq (K)(f_{\underline{x}}(\underline{x}; \theta_0)) \}$$

Now we can use the factorization theorem, $f_{\underline{x}}(\underline{x};\theta) = g(s;\theta) \cdot h(\underline{x})$, where $g(s;\theta)$ is a function of θ that only depends on s and $h(\underline{x})$ is a function of $\underline{x}$ which does not depend on θ.

Thus,

$$C^* = \{ \underline{x} \mid f_{\underline{x}}(\underline{x}; \theta_1) \leq (K)(f_{\underline{x}}(\underline{x}; \theta_0)) \}$$

$$\Rightarrow C^* = \{ \underline{x} \mid g(s;\theta_1) \cdot h(\cancel{\underline{x}}) \leq (K) \cdot g(s;\theta_0) \cdot h(\cancel{\underline{x}}) \}$$

$$\Rightarrow C^* = \{ \underline{x} \mid g(s;\theta_1) \leq (K) \cdot g(s;\theta_0) \}$$

∴ Thus, the most powerful test guaranteed by the NP Lemma is now expressed in terms of the sufficient statis 'S'.

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