Lecture 12

Thursday, February 20, 2020 2:00 PM

Agenda:

- Problem #8 from review
- Section 13.4 (1 sample MN)
- Section 13.5 (r-sample MN)

Problem #8:

- $X_1, \dots, X_N \sim N(\mu_1, \sigma^2)$ and $Y_1, \dots, Y_N \sim N(\mu_2, \sigma^2)$, where σ^2 is unknown, $X \perp Y$

$$\begin{cases} H_0: \mu_1 = \mu_2 = \mu_0 \\ H_A: \mu_1 \neq \mu_2 \end{cases}$$

Q: Find a conditional test based on $\sum X_i + \sum Y_i$

$$\frac{f_{\underline{x}, s}}{f_s} = f_{x|s}$$

- suppose $S_i = X_i + Y_i$ and $S = \sum_{i=1}^{n} S_i$
- Then under $H_0 \dots$
  - $S_i \sim N(2\mu_0, 2\sigma^2)$
  - $S \sim N(2n\mu_0, 2n\sigma^2)$
- Suppose $n=1$ (thus $S_i = S$)
  - Then $S \sim N(2\mu_0, 2\sigma^2)$ under H_0 and

$$f_{x,y} = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu_0)^2\right\} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu_0)^2\right\}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left[(x-\mu_0)^2 + (\underbrace{(s-x)}_{s^2-2sx+x^2} - \mu_0)^2\right]\right\}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right) \exp\left\{\frac{-1}{2\sigma^2}\left[x^2 - 2s\mu_0 + \mu_0^2 + s^2 - 2sx + x^2 - 2(s-x)\mu_0 + \mu_0^2\right]\right\}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right) \exp\left\{-\frac{1}{2\sigma^2}\left[2x^2 + 2\mu_0^2 + s^2 - 2sx - 2s\mu_0\right]\right\}$$

Then $f_S(s) = \left(\frac{1}{2\pi(2\sigma^2)}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{s-2\mu_0}{\sqrt{2}}\right)^2\right\}$

- The constants of the ratio

$$= \left(\frac{1}{2\pi\sigma^2}\right) \cdot \left[2\pi(2\sigma^2)\right]^{1/2} = \frac{\sqrt{2}}{\sqrt{2\pi\sigma^2}}$$

- Then the ratio gives:

$$\exp\left\{-\frac{1}{2\sigma^2}\left[2x^2 + 2\mu_0^2 + s^2 - 2sx - 2s\mu_0 - \underbrace{\left(\frac{s-2\mu_0}{\sqrt{2}}\right)^2}\right]\right\}$$

$$\frac{s^2}{2} - \frac{2(2\mu_0)s}{\sqrt{2}\sqrt{2}} + \frac{4\mu_0^2}{2}$$

$$= \frac{s^2}{2} - 2\mu_0 s + 2\mu_0^2$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(2x^2 + 2\mu_0^2 + \frac{s^2}{2} - 2sx - 2s\mu_0 - \frac{s^2}{2} + 2\mu_0 s - 2\mu_0^2\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(2x^2 + \frac{s^2}{2} - \underbrace{2sx}_{2\, s/\sqrt{2}\,(\sqrt{2}x)}\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left(\sqrt{2}x - s/\sqrt{2}\right)^2\right\}$$

$$= \exp\left\{-\frac{2}{2\sigma^2}\left(x - s/2\right)^2\right\}$$

Thus
$$\frac{f_{x,s}(x,s)}{f_s(s)} = \frac{\sqrt{2}}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{2}{2\sigma^2}\left(x - \frac{s}{2}\right)^2\right\}$$

Density: $N(s/2, \sigma^2/n)$ ✓

$\therefore$ You can then use the above for tests

Recall Example

	F	NF	Total
D	16	28	44
C	24	32	56
B	33	17	50
Total	73	77	150

<u>1st Case</u>: (3 degree of freedom test)

$$H_0: \left.\begin{aligned} P_D &= 0.5 \\ P_C &= 0.5 \\ P_B &= 0.5 \end{aligned}\right\}$$ where P_i = Prob of FPO, $i = D, C, B$

<u>2nd Case</u>: (2 degrees of freedom)

$H_0: P_D = P_C = P_B = P_0$ (unknown)

$\hat{P}_0 = 0.49$

Ex 2:

	F	NF	Total
D	16	28	44
C	24	32	56
B	15	35	50
	10	40	
			150

Section B.4 : (1 sample multinomial)

- extension of binomial with more than strictly two outcomes.

	A_1	A_2	A_3	...	A_c
Observed Count	O_1	O_2	O_3	...	O_c
	$p_1 = \frac{O_1}{n}$	$p_2 = \frac{O_2}{n}$	p_3	...	p_c

(c different outcomes)

$$\sum_{i=1}^{c} O_i = n$$

$$\sum_{i=1}^{c} p_i = \underline{1}$$

- we want to test : $H_0 : p_i = p_{i0}$ for $i = 1 \ldots c$ where $\sum_{i=1}^{c} p_{i0} = 1$
- There are $c-1$ restrictions under H_0.
- $e_i = n \cdot p_{i0}$, $i = 1, \ldots, c$
- $$\chi^2_{OB} = \sum_{i=1}^{c} \frac{(O_i - e_i)^2}{e_i} \sim \chi^2(c-1)$$

Ex

	F	NF	Total	E(F
D	16	28	44	$73 \times 0.25 = 18.25$
C	24	32	56	18.25
B	33	17	50	36.5
Total	73	77	150	

H_0

$p_D = 0.25$

$p_C = 0.25$

$p_B = 0.5$

$$\chi^2_{OB} = \frac{(16 - 18.25)^2}{18.25} + \frac{(24 - 18.25)^2}{18.25} + \frac{(33 - 36.5)^2}{36.5}$$

$$= 2.49$$

$$\chi^2_{0.95}(2) = 5.99$$

$\therefore$ $2.49 < 5.99$, thus we fail to reject. ✓

Section B.5 (r-sample multinomial) :

- r samples of $c \geq 2$ category multi-nomials.
- As before, we can do two types of tests.

Completely specified Tests

$H_0: P_0 \ , \ i = 1, \ldots, r$

Unspecified Tests

$H_0: P_1 = P_2 = \ldots = P_r = P_0$

(P_0 is unknown)

$$P_0 = \frac{\sum_{i=1}^{r} O_i}{\sum_{i=1}^{r} n_i}$$

r-sample MN:

	A_1	A_2	$\ldots$	A_c
Probability of outcome j in population 1	$P_{1/1}$	$P_{2/1}$	$\ldots$	$P_{c/1}$
$\vdots$			$\ldots$	
probability of outcome j in population r	$P_{1/r}$	$P_{2/r}$		$P_{c/r}$

where $\begin{cases} P_{j|i} = \text{Probability of category } j \text{ in population } 1 \\ \sum_{j=1}^{c} P_{j|i} = 1 \text{ for } i = 1, \ldots, r \end{cases}$

Completely Specified Hypothesis:

$H_0: P_{j|i} = P^{\circ}_{j|i}$ for each jth category in population i $\left(\sum_{i=1} P^{\circ}_{j|i} = 1 \right)$

$i = 1, \ldots, r$

- Though we are setting $r \times c$ $P_{j|i}$'s, only $[r(c-1)]$ are allowed to vary freely.
  - e_{ij} = Expected count for jth category in ith population.
    - $e_{ij} = n_i \cdot P^{\circ}_{j|i}$
  - $$\chi^2_{OB,i} = \sum_{j=1} \frac{(O_{ij} - e_{ij})^2}{e_{ij}} \sim \chi^2_{(c-1)}$$
  - Thus, $$\chi^2_{OB} = \sum_{i=1}^{r} \chi^2_{OB,i} \sim \chi^2_{(r(c-1))}$$