Lecture 4

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- **Recall**: Central T Distribution

$$T = \frac{Z}{\sqrt{V/\nu}} \quad \text{where } \begin{cases} Z \sim N(0,1) \\ V \sim \chi^2(\nu) \\ Z \perp V \end{cases}$$

- **Recall Also**:

$$\frac{s/\sqrt{n}}{\sigma/\sqrt{n}} = \sqrt{\frac{V}{n-1}} \quad \text{where } V = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

(II) Power Function:

$H_0: \mu \le \mu_0$

$H_A: \mu > \mu_0$

$$C = \left\{ x \,\middle|\, \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \ge t_{1-\alpha}(n-1) \right\}$$

$$\pi(\mu) = P\left[\frac{\bar{x}-\mu_0}{s/\sqrt{n}} \ge t_{1-\alpha}(\nu) \,\middle|\, \mu\right]$$

$$= P\left[\frac{\bar{x}-\mu}{s/\sqrt{n}} + \frac{\mu-\mu_0}{s/\sqrt{n}} \ge t_{1-\alpha}(n-1)\right]$$

$$= P\left[\frac{(\bar{x}-\mu)/(\sigma/\sqrt{n})}{(s/\sqrt{n})/(\sigma/\sqrt{n})} + \frac{(\mu-\mu_0)/(\sigma/\sqrt{n})}{(s/\sqrt{n})/(\sigma/\sqrt{n})} \ge t_{1-\alpha}(n-1)\right]$$

$$= P\left[\underbrace{\frac{Z}{\sqrt{\frac{V}{n-1}}} + \frac{\delta}{\sqrt{V/n-1}}}_{(A)} \ge t_{1-\alpha}(n-1)\right] \quad \text{where } \delta = \frac{\mu-\mu_0}{\sigma/\sqrt{n}}$$

(A) Distributed non central $T(n-1$, non-centrality parameter $= \delta)$

Ex. 12.3.1 Pg 400

$X_1, \dots, X_n \sim N(\mu, \sigma^2)$, σ^2 unknown.

$H_0: \mu = 10$

$H_A: \mu > 10$

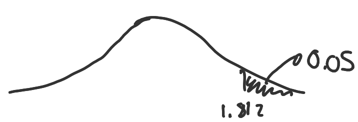
- Want power of $1-\beta = 0.99$ for $d = \frac{\mu-\mu_0}{\sigma} = 2$, at a significance level of $\alpha = 0.05$

- To detect $d=2$, $n=6$
- To detect $d=1$, $n=18$
  - ↳ to detect smaller differences, we need larger n

Extension of problem:

- Find c for $\alpha = 0.05$, $n=11$

(1) $C = \left\{ \bar{x} \,\middle|\, \frac{\bar{x}-\mu_0}{s/\sqrt{n}} \ge \underbrace{t_{1-0.05}(10)}_{1.812} \right\}$



- Find $\pi(\mu)$

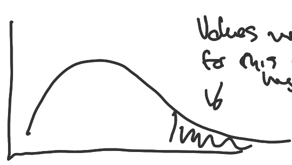
(2) $\pi(\mu) = P\left(\frac{Z}{\sqrt{V/\nu}} + \frac{\delta}{\sqrt{V/\nu}} \ge 1.812\right)$

(3) With a sample size of 11, if the true value $\mu = 12$, then what is the probability of rejecting H_0 if $\sigma = 2$??

$$\delta = \frac{\mu-\mu_0}{\sigma/\sqrt{n}} = \frac{12-10}{2/\sqrt{11}}$$

Test For Variance: σ^2

$\left.\begin{array}{l} H_0: \sigma^2 = \sigma_0^2 \\ H_A: \sigma^2 > \sigma_0^2 \end{array}\right\}$ (I)



Values we would reject for this particular hypothesis.

Thm 12.12.3:

- Let $x_1, \dots, x_n$ be the observed sample from $N(\mu, \sigma^2)$ and let $V_0 = \dfrac{(n-1)S^2}{\sigma_0^2}$

① A size α test Ⓘ rejects if $V_0 \geq \chi^2_{1-\alpha}(n-1)$

- The power function:

$\pi(\sigma^2) = 1 - H\left[(\sigma_0^2/\sigma^2) \cdot \chi^2_{1-\alpha}(n-1), n-1\right]$ where H is CDF of $\chi^2(n-1)$ evaluated at $\dfrac{\sigma_0^2}{\sigma^2} \cdot \chi^2_{1-\alpha}(n-1)$

$$\hookrightarrow \pi(\sigma^2) = P\left(V_0 \geq \chi^2_{1-\alpha}(n-1) \mid \sigma^2\right)$$

$$= P\left(\frac{(n-1)S^2}{\sigma_0^2} \geq \chi^2_{1-\alpha}(n-1) \mid \sigma^2\right)$$

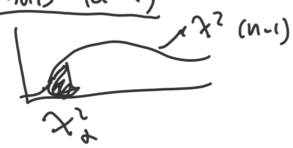
$$= P\Bigg(\underbrace{\frac{\sigma^2}{\sigma^2} \cdot \frac{(n-1)S^2}{\sigma_0^2}}_{\sim \chi^2} \geq \chi^2_{1-\alpha}(n-1)\Bigg)$$

$$= P\left(\frac{(n-1)S^2}{\sigma^2} \geq \frac{\sigma_0^2}{\sigma^2} \cdot \chi^2_{1-\alpha}(n-1)\right)$$

$$= 1 - H\left(\frac{\sigma_0^2}{\sigma^2} \cdot \chi^2_{1-\alpha}(n-1), n-1\right)$$

For hypotheses of this form,

$H_0: \sigma^2 \geq \sigma_0^2$ Ⓘⓘ
$H_A: \sigma^2 < \sigma_0^2$



... we reject H_0 if $V_0 \leq \chi^2(n-1)$

$\pi(\sigma^2) = H\left[\dfrac{\sigma_0^2}{\sigma^2} \cdot \chi_\alpha(n-1), n-1\right]$

Ⓘⓘⓘ $H_0: \sigma^2 = \sigma_0^2$
$H_A: \sigma^2 \neq \sigma_0^2$,

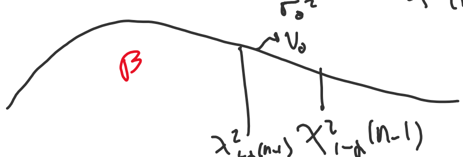
we reject if either

$V_0 \leq \chi^2_{\frac{\alpha}{2}}(n-1)$ or $V_0 \geq \chi^2_{1-\frac{\alpha}{2}}(n-1)$

Sample Size Determination:

Ⓘ $H_0: \sigma^2 \leq \sigma_0^2$
$H_A: \sigma^2 > \sigma_0^2$

- We reject if $V_0 \geq \chi^2_{1-\alpha}(n-1)$ in this case.
- The test statistic: $V_0 = \dfrac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$



$\pi(\sigma^2) = 1 - H\left[\sigma_0^2/\sigma^2 \cdot \chi^2_{1-\alpha}(n-1), (n-1)\right]$

$\chi^2_\beta(n-1) = \dfrac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha}(n-1)$

- Suppose we want to find the sample size needed to reject H_0 when when the true value $= \sigma_1^2$
- An alternative to the iterated method is the approximation:

$$n \approx 1 + 2\left[\frac{z_\beta - \left(\frac{\sigma_0^2}{\sigma_1^2} \cdot z_{1-\alpha}\right)}{1 - \frac{\sigma_0^2}{\sigma_1^2}}\right]^2$$

Definition: 12.6.1

- A test of the hypothesis $H_0: \theta = \theta_0$ vs $H_A: \theta = \theta_1$, based on a critical region C^* is said to be most powerful test of size α, if:
  1) $\pi_{C^*}(\theta_0) = \alpha$
  2) $\pi_{C^*}(\theta_1) \geq \pi_C(\theta_1)$ for any other critical region C of size α.

How do we find such a test??
Thm. 12.6.1 (N-P lemma)

- suppose that $X_1, \ldots, X_N$ is jointly $\sim f_X(\underline{x}; \theta)$.
- Let $\lambda(\underline{x}; \theta_0, \theta_1) = \dfrac{f_X(\underline{x}; \theta_0)}{f_X(\underline{x}, \theta_1)}$.
- Let $C^* = \{\underline{x} \mid \lambda(\underline{x}; \theta_0, \theta_1) \leq K\}$, where K is a constant that satisfies:

$$P[\underline{X} \in C^* \mid \theta_0] = \alpha$$

- If all of these conditions are satisfied, we say that C^* is the <u>most</u> <u>powerful</u> critical region of size α.