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Wednesday, February 12, 2020 8:13 PM

Q: Let $X_1, \ldots, X_N$ be random variables from a discrete distribution:

$$f(x;\theta) = \frac{\left[\frac{\theta}{\theta+1}\right]^x}{\theta+1} \quad \text{if } x = 0, 1, \ldots$$

where $\theta = 0$.

Find a UMP test for $H_0: \theta = \theta_0$ against $H_A: \theta > \theta_0$

S1. The pdf:

$$f(x;\theta) = \frac{\left[\frac{\theta}{\theta+1}\right]^x}{\theta+1}$$

S2. The JDF...

$$f_{\underline{x}}(\underline{x};\theta) = \prod_{i=1} \left[\frac{\theta}{\theta+1}\right]^x \left[\frac{1}{\theta+1}\right] \cdot \prod_{i=1}^{n} I\{x = 0, 1, \ldots\}$$

$$= \left[\frac{1}{\theta+1}\right]^n \cdot \prod_{i=1} \left[\frac{\theta}{\theta+1}\right]^x$$

$$= \left[\frac{1}{\theta+1}\right]^n \left[\frac{\theta}{\theta+1}\right]^{\Sigma x_i}$$

S3. Using the NP Lemma to find a monotone likelihood ratio looks challenging, so let's try to find a sufficient statistic by the factorization theorem and then use S as a test statistic.

We can take the natural log...

$$\ln[f_{\underline{x}}(\underline{x};\theta)] = \ln\left(\left[\frac{1}{\theta+1}\right]^N \left[\frac{\theta}{\theta+1}\right]^{\Sigma x_i}\right)$$

$$= \ln\left(\left(\frac{1}{\theta+1}\right)^N\right) + \ln\left(\left(\frac{\theta}{\theta+1}\right)^{\Sigma x_i}\right)$$

$$= \ln\left(\left(\frac{1}{\theta+1}\right)^{n}\right) + \ln\left(\left(\frac{\theta}{\theta+1}\right)^{\sum x_i}\right)$$

$$= \ln\left(\left(\frac{1}{\theta+1}\right)^{N}\right) + \left(\sum x_i\right)\ln\left(\frac{\theta}{\theta+1}\right)$$

Then taking the exponential . . .

$$= \exp\left\{\ln\left(\frac{1}{\theta+1}\right)^{N} + \left(\sum x_i\right)\ln\left(\frac{\theta}{\theta+1}\right)\right\}$$

$$= \exp\left\{\ln\left(\frac{1}{\theta+1}\right)^{N}\right\} \cdot \exp\left\{\left(\sum x_i\right)\ln\left(\frac{\theta}{\theta+1}\right)\right\}$$

$$= \left(\frac{1}{\theta+1}\right)^{n} \cdot \underbrace{\exp\left\{\left(\sum x_i\right)\ln\left(\frac{\theta}{\theta+1}\right)\right\}}_{g(S;\theta)}$$

$h(\underline{x}) = x^0 = 1$

Thus $S = t(\underline{x}) = \sum x_i$ is sufficient by the factorization theorem.

$\therefore$ Thus we can reject the null hypothesis if

$\sum_{i=1}^{n} x_i \geq K$ where K is a number such that

$$P\left[\sum_{i=1}^{n} x_i \geq K \mid \theta = \theta_0\right] = \alpha$$

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