Lecture 5

Tuesday, January 21, 2020 1:59 PM

Agenda:

- Section 12.6, (Most Powerful tools)
- NP Lemma

Recall Def 12.6.1:

- A test of $H_0: \theta = \theta_0$ vs $H_A: \theta = \theta_1$ based on a critical region C^* is said to be the most powerful test of size α if:

1) $\pi_{C^*}(\theta_0) = \alpha$

2) $\pi_{C^*}(\theta_1) \geq \pi_C(\theta_1)$ for all critical regions of size α

Thm 12.6.1 Neyman Pearson Lemma

- Suppose that $\underline{X} \sim f_{\underline{x}}(\underline{x}; \theta)$
  - Let $\lambda(\underline{x}; \theta_0, \theta_1) = \dfrac{f_x(\underline{x}; \theta_0)}{f_x(\underline{x}; \theta_1)}$
- Let $C^* = \{\underline{x} \mid \lambda(\underline{x}; \theta_0, \theta_1) \leq k\}$ where k is such that:

$$P(\underline{x} \in C^* | \theta_0) = \alpha$$

- Then we say C^* is the most powerful critical region of size α to $\begin{cases} H_0: \theta = \theta_0 \\ H_A: \theta = \theta_1 \end{cases}$

Proof:

Let A be an n dimensional event such that:

$$P(\underline{x} \in A | \theta) = \int_A \int f_{\underline{x}}(\underline{x}; \theta) \, dx_1 \, \dots \, dx_n$$

$\bar{C}^* = C^{*'}$

Since we know that C^* is a size α critical region.

Similarly, C is a size α critical region where $\bar{C} = C'$

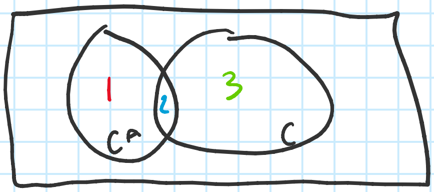
If $A \in C^*$, then:

$$P(\underline{x} \in A | \theta_0) \leq k \cdot P(\underline{x} \in A | \theta_1)$$

Else if $A \in \bar{C}^*$, then:

$$P(\underline{x} \in A | \theta_0) > k \cdot P(\underline{x} \in A | \theta_1)$$

Let C and C^* be the regions as below let C and C^* be the regions as below



$C^* = (C^* \cap \bar{C}) \cup (C^* \cap C)$

$C_1 = (C \cap \bar{C^*}) \cup (C \cap C^*)$

Then $\Pi_{C^*}(\theta) = P(\underline{x} \in C^* | \theta)$

$= P(\underline{x} \in [C^* \cap \bar{C}] | \theta) + P(\underline{x} \in [C^* \cap C] | \theta)$

$\Pi_C(\theta) = P(\underline{x} \in [C \cap \bar{C^*}] | \theta) + P(\underline{x} \in [C \cap C^*] | \theta)$

We want to evaluate:

$$\Pi_{C^*}(\theta_1) - \Pi_C(\theta_1) = \underbrace{P(\underline{x} \in [C^* \cap \bar{C}] | \theta_1)}_{a} - \underbrace{P(\underline{x} \in [C \cap \bar{C^*}] | \theta_1)}_{b}$$

$\hookrightarrow a \geq \left(\frac{1}{k}\right) P(\underline{x} \in [C^* \cap \bar{C}] | \theta_0)$

$\hookrightarrow b < \left(\frac{1}{k}\right) P(\underline{x} \in [C \cap \bar{C^*}] | \theta_0)$

$$\therefore \Pi_{C^*}(\theta_1) - \Pi_C(\theta_1) \leq \left[\left(\frac{1}{k}\right) \cdot P(\underline{x} \in [C^* \cap \bar{C}] | \theta_0)\right] - \left[\left(\frac{1}{k}\right) \cdot P(\underline{x} \in [C \cap \bar{C^*}] | \theta_0)\right]$$

$$= \left(\frac{1}{k}\right)\left[(\alpha - P(\underline{x} \in 2)) - (\alpha - P(\underline{x} \in 2))\right]$$

$= 0$ ✓

Ex 12.6.1

- Let $X_1, \ldots, X_n \underset{iid}{\sim} \exp(\theta)$
- Test $H_0 : \theta = \theta_0$
  $H_A : \theta = \theta_1 \quad (\theta_0 < \theta_1)$

Q: Find the most powerful critical region

S1. $f_X(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x \geq 0, \ \theta \geq 0 \\ 0 & \text{o.w} \end{cases}$

S2. $f_{\underline{X}}(\underline{x};\theta) = \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum x_i}{\theta}}$

S3. $\Lambda(\underline{x};\theta_0,\theta_1) = \left(\frac{1}{\theta_0}\right)^n e^{-\frac{\sum x_i}{\theta_0}}$

S3. $\lambda(\underline{x}; \theta_0, \theta_1) = \dfrac{\left(\frac{1}{\theta_0}\right)^n e^{-\frac{\sum x_i}{\theta_0}}}{\left(\frac{1}{\theta_1}\right)^n e^{-\frac{\sum x_i}{\theta_1}}} = \left(\dfrac{\theta_1}{\theta_0}\right)^n \cdot \exp\left\{\sum x_i \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right\}$

↳ Then since $\theta_0 < \theta_1$ in problem statement

$\Rightarrow \frac{1}{\theta_0} > \frac{1}{\theta_1}$

$C^* = \left\{ \underline{x} \mid \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left\{\sum x_i \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right\} \le k \right\}$

$\Rightarrow \sum x_i \underbrace{\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}_{<0} \le \ln\left[k\left(\frac{\theta_0}{\theta_1}\right)^n\right]$

$\Rightarrow \sum x_i \ge \ln\left[k\left(\frac{\theta_0}{\theta_1}\right)^n\right]\left\{\frac{1}{\theta_1} - \frac{1}{\theta_0}\right\}^{-1}$ ☆

Recall $\begin{cases} \sum x_i \sim \Gamma(n, \theta) \\ \chi^2_{(n)} = \Gamma\left(\frac{n}{2}, 2\right) \text{ ; } \chi^2_{(2n)} = \Gamma\left(n, \frac{2\theta}{\theta}\right) \end{cases}$

- Suppose $\begin{cases} Y = \sum x_i \\ Z = \frac{2Y}{\theta_0} = g(Y), \text{ then } g^{-1}(Z) = \frac{\theta_0 Z}{2} = Y, \quad \frac{\partial g^{-1}(Z)}{\partial Z} = \frac{\theta_0}{2} \\ Y = \frac{\theta_0 Z}{2} = \end{cases}$

then $f_Y(y; \theta) = \begin{cases} \frac{1}{\Gamma(n)\theta^n} \cdot y^{n-1} e^{-\frac{y}{\theta}} & y \ge 0, \ \theta \ge 0 \\ 0 \end{cases}$

then $f_Z(z) = \dfrac{1}{\Gamma(n)\cancel{\theta_0^n}} e^{-\frac{z\cancel{\theta_0}}{2\cancel{\theta_0}}} \left(\frac{\theta_0 z}{2}\right)^{n-1} \cdot \left|\frac{\cancel{\theta_0}}{2}\right|$

$= \dfrac{1}{\Gamma(n) 2^n} e^{-z/2} (z)^{n-1}$

$\therefore Z \sim \Gamma(n, 2)$

$\therefore Z \sim \chi^2_{(2n)}$

- The most powerful critical region of size α would be:

$C^* = \left\{ \underline{x} \mid \frac{2\sum x_i}{\theta_0} \ge \chi^2_{1-\alpha}(2n) \right\}$

- Now equate $\frac{2}{\theta_0}[☆] = \chi^2_{1-\alpha}(2n)$; solve for k

$\theta_0 [\ \ ] \quad \chi^2_{1-\alpha}(2n)$; solve for k

If: $\theta_0 > \theta_1$

Then $C_1^* = \left\{ \frac{2\sum x_i}{\theta_0} \leq \chi^2_{\alpha}(2n) \right\}$

for $\theta_1 > \theta_0$:

C_1^* is a most powerful critical region immaterial of value θ_1

↳ This leads to the notion of a uniformly most powerful test which is most powerful for all values in a composite alternative.

Ex 12.6.2

Suppose $X_1, \ldots, X_n \sim$ iid $N(0, \sigma^2)$

$H_0: \sigma^2 = \sigma_0^2$

$H_A: \theta^2 = \sigma_1^2$ where $(\theta_1^2 > \theta_0^2)$

* Find the most powerful critical region C_1^* and K,

S1. find

$$\Lambda(\underline{x}; \sigma_0^2, \sigma_1^2) = \frac{\left(\frac{1}{2\pi\sigma_0^2}\right)^{n/2} \cdot \exp\left\{-\frac{\sum x_i^2}{2\sigma_0^2}\right\}}{\left(\frac{1}{2\pi\sigma_1^2}\right)^{n/2} \cdot \exp\left\{-\frac{\sum x_i^2}{2\sigma_1^2}\right\}}$$

negative!!!

$$= \underbrace{\left(\frac{\sigma_1}{\sigma_0}\right)^n \cdot \exp\left\{\frac{\sum x_i^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right\}}$$

$C_1^* = \{ \underline{x} \mid \quad \leq K \}$

⋮

$$\Rightarrow \sum x_i^2 \geq 2\ln\left\{K\left(\frac{\sigma_0}{\sigma_1}\right)^n\right\}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)^{-1}$$

* we know that under H_0: $x_i \sim N(0, \sigma_0^2)$

↳ Thus $\frac{x_i}{\sigma_0} \sim N(0,1)$

and

$$\sum \left(\frac{x_i}{\sigma_0}\right)^2 \sim \chi^2_{(n)}$$

- Then for a size α test, we reject if we observe a statistic value such that:

$$\sum \left(\frac{x_i}{\sigma_0}\right)^2 \geq \chi^2_{1-\alpha}(n)$$

- We can then solve for K by setting equal:

$$\chi^2_{1-\alpha}(n) = \frac{2\ln\left\{K\left(\frac{\sigma_0}{\sigma_1}\right)^n\right\}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)^{-1}}{\sigma_0^2}$$

... and solve for K

Ex 12.6.3

Find the MP test for:

$H_0: p = p_0$

$H_A: p = p_1$ where $(p_1 > p_0)$ and based on $S \sim Bin(n,p)$

S1. $$\lambda(S; p_0, p_1) = \frac{\cancel{\binom{n}{s}} p_0^s (1-p_0)^{n-s}}{\cancel{\binom{n}{s}} p_1^s (1-p_1)^{n-s}} = \left(\frac{p_0}{p_1}\right)^S \left(\frac{1-p_0}{1-p_1}\right)^{n-S}$$

S2. Then $$C^* = \left\{ S \;\middle|\; \left(\frac{p_0}{p_1}\right)^S \left(\frac{1-p_0}{1-p_1}\right)^{n-S} \leq K \right\}$$

↳ since $p_1 > p_0$, $\left(\frac{p_0}{p_1}\right)^S < 1$

↳ Thus, $$\lambda(\underline{x}; \dots) = \left[\underbrace{\underbrace{\frac{p_0}{p_1}}_{<1} \underbrace{\frac{(1-p_1)}{(1-p_0)}}_{<1}}_{<1}\right]^S \left[\frac{1-p_0}{1-p_1}\right]^n$$

$\Rightarrow$

↳ Then your critical region is of form:

$$\left[\frac{p_0(1-p_1)}{p_1(1-p_0)}\right]^S \leq K\left(\frac{1-p_1}{1-p_0}\right)^n$$

$$\Rightarrow S \cdot \ln\left[\frac{p_0(1-p_1)}{p_1(1-p_0)}\right] \leq \ln\left[K\left(\frac{1-p_1}{1-p_0}\right)^n\right]$$

$$\Rightarrow S \geq \frac{\ln\left[K\left(\frac{1-p_1}{1-p_0}\right)^n\right]}{\ln\left[\frac{p_0(1-p_1)}{p_1(1-p_0)}\right]}$$