Lecture 7

Tuesday, January 28, 2020 1:59 PM

- Agenda
  - Monotone Likelihood Ratio
  - Unbiased tests
  - Tests for simple parameter exponential families
  - *Section 12.8
    - ↳ Generalized Likelihood Ratio Test

Clarification from lecture 6

↳ Ex 12.6.4

$X_1, \dots, X_n \overset{\sim}{iid} \begin{cases} H_0: \text{Unif}(0,1) \\ H_A: \text{Exp}(1) \end{cases}$ Dealing with different distributions

... after some work ...

$C^* = \{ \underline{x} \mid \exp\{\sum x_i\} \leq k \}$

$\Rightarrow \sum x_i \leq \ln(k)$

$\Rightarrow \bar{x} \leq \left(\frac{1}{n}\right) \cdot \ln(k)$ where $\underbrace{\bar{x} \sim N\left(E(\bar{x}) = 1/2,\ var(\bar{x}) = \frac{1}{12n}\right)}$

↳ Approximation when n is large

- Then, a size α test would reject if:

$$\frac{\bar{x} - \frac{1}{2}}{\sqrt{\frac{1}{12n}}} \leq -Z_{1-\alpha}$$

Suppose: $\begin{cases} H_0: \mu = \mu_0 \\ H_A: \mu \neq \mu_0 \end{cases}$

$\Rightarrow \pi_{C^*}(\mu) \geq \pi_C(\mu) \quad \forall \mu \neq \mu_0$ → condition for UMP test

- We usually cannot find a UMP size α test under H_A (if we have a 2-sided alternative).
  - ↳ So, we consider a more restricted class of of tests that are **unbiased**.
- This implies that UMP tests are "usually" available for one-sided alternatives.

Monotone Likelihood Ration (MLR):

Def 12.7.2

- A joint pdf $f_{\underline{x}}(x;\theta)$ is said to have an MLR in statistic $T = t(x)$ if for any two values $\theta_1 < \theta_2$, the ratio of:

$$\frac{f_{\underline{x}}(\underline{x};\theta_2)}{f_{\underline{x}}(\underline{x};\theta_1)}$$

... depends on $\underline{X}$ only through $t(\underline{x})$ } if if it is non-decreasing in $t(\underline{x})$.

Ex 12.7.2

- let $X_1, \ldots, X_n \sim Exp(\theta)$
- let $\theta_1 < \theta_2$
- then:

$$\frac{f_{\underline{x}}(\underline{x};\theta_2)}{f_{\underline{x}}(\underline{x};\theta_1)} = \frac{\left(\frac{1}{\theta_2}\right)^n \cdot \exp\left\{-\frac{\Sigma x_i}{\theta_2}\right\}}{\left(\frac{1}{\theta_1}\right)^n \cdot \exp\left\{-\frac{\Sigma x_i}{\theta_1}\right\}} \quad \text{(assuming } x_i > 0\text{)}$$

$$= \left(\frac{\theta_1}{\theta_2}\right)^n \cdot \exp\left\{\Sigma x_i \cdot \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\right\}$$

> 0 on $\theta_1 < \theta_2$

- Since $\theta_1 < \theta_2$, the above is a non-decreasing function of $t(\underline{X}) = \Sigma X_i$

$\therefore$ $Exp(\theta)$ has an MLR in ΣX_i

Connection with UMP tests???

Thm 12.7.1:

- if a joint pdf $f_{\underline{x}}(x,\theta)$ has an MLR in statistic $t(\underline{x})$, the UMP test of size α can be found for:

$$\begin{cases} H_0: \theta \leq \theta_0 \\ H_A: \theta > \theta_0 \end{cases}$$

- if we reject H_0 for $t(x) \geq k$ where k satisfies:

$$P[t(\underline{x}) \geq k \mid H_0] = \alpha$$

- Similarly, if instead we have:

$$\begin{cases} H_0: \theta \geq \theta_0 \\ H_A: \theta < \theta_0 \end{cases}$$

... then we reject H_0 if $t(\underline{x}) \le k$ (etc, etc...)

Example 12.7.3

- $X_1, \dots, X_n \overset{iid}{\sim} \text{Exp}(1, \eta)$ where η is the location parameter.

Q: Find the UMP tests (size α) for:

$$\begin{cases} H_0: \eta \le \eta_0 \\ H_A: \eta > \eta_0 \end{cases}$$

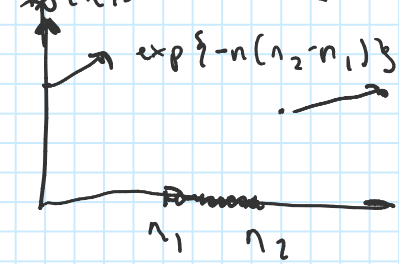
$X_{(1)} = X_{1:N}$

- The joint pdf of $\underline{x}$ is $f_x(\underline{x}; 1, \eta) = \exp\{-\Sigma(x_i - \eta)\} \cdot I\{x_{(1)} \ge \eta\}$

- For $\eta_1 < \eta_2$ we have:

$$\frac{f_x(\underline{x}; 1, \eta_2)}{f_x(\underline{x}; 1, \eta_1)} = \frac{\exp\{-\Sigma(x_i - \eta_2)\} \cdot I\{x_{(1)} > \eta_2\}}{\exp\{-\Sigma(x_i - \eta_1)\} \cdot I\{x_{(1)} > \eta_1\}}$$

$$\frac{f_x(\eta_2)}{f_x(\eta_1)} = \exp\{-n(\eta_2 - \eta_1)\} \cdot I\{x_{(1)} > \eta_2\}$$



- Thus, the ratio is a non-decreasing function of $t(\underline{x}) = X_{(1)}$. The distribution has an MLR in $X_{(1)}$.

↳ Then by theorem 12.7.1, a UMP test of size α is of the form:

$$C^* = \{\underline{x} \mid x_{(1)} \ge k\}, \text{ where } \alpha = P[X_{(1)} \ge k \mid \eta_0] \quad ✓$$

Unbiased Tests: (for 2-sided test)

Def 12.7.3: A test of $H_0: \theta \in \Omega_0$

$H_A: \theta \in \Omega - \Omega_0$

...is unbiased in $\min_{\theta \in \Omega - \Omega_0} \pi(\theta) > \max_{\theta \in \Omega_0} \pi(\theta)$.

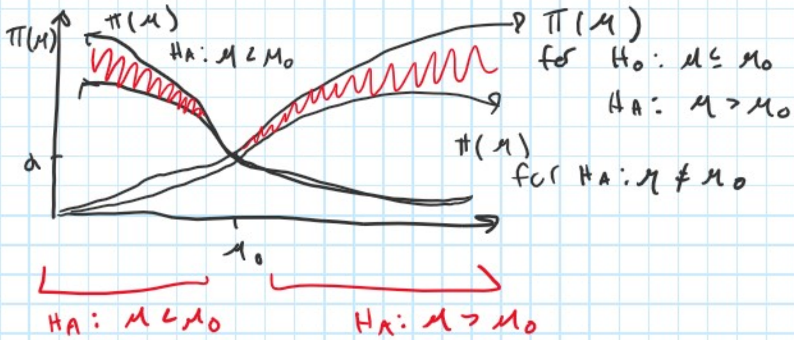
- Suppose $X_1, \dots, X_N \sim N(\mu, \sigma^2)$, σ^2 known.
- We want: $H_0: \mu = \mu_0$

  $H_A: \mu \ne \mu_0$

- The test that rejects for $\bar{x} \ge \mu_0 + z_{1-\alpha}\, \sigma/\sqrt{n}$ OR $\bar{x} \le \mu - z_{1-\alpha}\, \sigma/\sqrt{n}$

$H_A: \mu \neq \mu_0$

- The test that rejects for $\bar{x} \geq \mu_0 + z_{1-\alpha}\,\sigma/\sqrt{n}$ OR $\bar{x} \leq \mu - z_{1-\alpha}\,\sigma/\sqrt{n}$ is **not** a UMP test because:



- Result pertaining to single parameter exponential families.
  - ↳ Theorem 12.7.2
    - Suppose $X_1, \dots, X_n \overset{iid}{\sim} f_{\underline{x}}(\underline{x};\theta)$
    - Suppose $f_{\underline{x}}(\underline{x};\theta) = c(\theta)\,h(x)\exp\{q(\theta)\,t(\underline{x})\}$ where $q(\theta)$ is an increasing function of (θ).

1) A UMP test of $\begin{cases} H_0: \theta \leq \theta_0 \\ H_A\ \theta > \theta_0 \end{cases}$

... rejects H_0 if $t(\underline{x}) \geq k$ for $P[t(\underline{x}) \geq k \mid H_0] = \alpha$

2) A UMP test of $H_0: \theta \geq \theta_0$
$H_A: \theta < \theta_0$

... rejects H_0 if $t(\underline{x}) \leq k$ for $P[t(\underline{x}) \leq k \mid H_0] = \alpha$

Proof:

- If $\theta_1 < \theta_2$, then $q_1(\theta_1) < q(\theta_2)$ because $q(\cdot)$ is increasing.
- The ratio of the joint distributions is then:

$$\frac{f_{\underline{x}}(\underline{x};\theta_2)}{f_{\underline{x}}(\underline{x};\theta_1)} = \frac{c(\theta_2)\,h(\underline{x})\exp\{q(\theta_2)\,t(\underline{x})\}}{c(\theta_1)\,h(\underline{x})\exp\{q(\theta_1)\,t(\underline{x})\}}$$

$$= \frac{c(\theta_2)}{c(\theta_1)}\exp\{\underbrace{[q(\theta_2) - q(\theta_1)]}_{>0}\cdot t(\underline{x})\}$$

- This is an increasing function of $t(\underline{x})$.

$\therefore$ This has an MLR in $t(\underline{x})$ ✓

- By Theorem 12.7.1, a UMP test of size α can be found as stated above.

Ex 12.7.4

- let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\mu)$
- Find $T(\underline{x})$ such that a UMP test can be found based on it's observed value.

$$f_X(\underline{x};\mu) = \frac{e^{-n\mu} \cdot \mu^{\sum x_i}}{\left(\prod_{i=1}^{n} x_i\right)}$$

$$= \underbrace{e^{-n\mu}}_{c(\mu)} \cdot \underbrace{\frac{1}{\left(\prod_{i=1} x_i\right)}}_{h(\underline{x})} \cdot \exp\left\{ \underbrace{\sum x_i}_{t(\underline{x})} \cdot \underbrace{\ln \mu}_{q(\mu)} \right\}$$

- A UMP test can be conducted for

$H_0: \mu \leq \mu_0$
$H_A: \mu > \mu_0$

where we reject $\sum x_i > K$

and where K is such that $P[\sum x_i \geq K \mid H_0] = \alpha$