Lecture 9

Tuesday, February 4, 2020 2:03 PM

- Agenda
- Examples $\begin{cases} 12.8.2 \\ 12.8.3 \end{cases}$
- Two sample problems using GLR
- Section 12.9 Conditional Tests

- Recall Ex 12.8.2
  - Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ with σ^2 unknown
  - Construct a size α test based on GLR for:

$H_0: \mu = \mu_0$

$H_A: \mu \neq \mu_0$

Then, $$\lambda(x) = \frac{\max_{(\mu,\sigma^2)} f_{\underline{x}}(x; \Omega_0)}{\max_{(\mu,\sigma^2)} f_{\underline{x}}(\underline{x}; \Omega)}$$

*Recall
- if $\bar{x} \le \mu_0$, then $\hat{\mu}_{MLE} = \hat{\mu}_0$
  - similarly, $\hat{\sigma}^2_{MLE} = \hat{\sigma}^2_0$

$H_0: \mu \le \mu_0$

$H_A: \mu > \mu_0$

$$\lambda(x) = \frac{\max_{\{(\mu,\sigma^2): \mu \le \mu_0, \sigma^2 > 0\}} f_{\underline{x}}(\underline{x}; \mu, \sigma^2)}{\max_{\{(\mu,\sigma^2): \mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}\}} f_{\underline{x}}(\underline{x}; \mu, \sigma^2)}$$

$$\lambda(x) = \begin{cases} 1 & \text{if } \bar{x} \le \mu_0 \\ \dfrac{f_{\underline{x}}(\underline{x}; \mu_0, \hat{\sigma}_0^2)}{f_{\underline{x}}(\underline{x}; \hat{\mu}_{MLE}, \hat{\sigma}^2_{MLE})} & \text{if } \bar{x} > \mu_0 \end{cases}$$

... where $\sigma_0^2 = \dfrac{\sum (x_i - \mu_0)^2}{n}$

- Unrestricted MLE for μ is $\bar{x} = \hat{\mu}_{MLE}$
- unrestricted MLE for σ^2 is $\dfrac{\sum (x_i - \bar{x})^2}{n} = \hat{\sigma}^2_{MLE}$

$$\lambda(\underline{x}) = \frac{(2\pi \hat{\sigma}_0^2)^{-n/2} \exp\left\{-\frac{\sum (x_i - \mu_0)^2}{2\hat{\sigma}_0^2}\right\}}{(2\pi \hat{\sigma}^2_{MLE})^{-n/2} \exp\left\{-\frac{\sum (x_i - \bar{x})^2}{2\hat{\sigma}^2_{MLE}}\right\}}$$

$$= \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2_{MLE}}\right)^{-n/2} \cdot \exp\left\{-\frac{1}{2}\left[\underbrace{\sum \left(\frac{x_i - \mu_0}{\hat{\sigma}_0}\right)^2}_{a} - \underbrace{\sum \left(\frac{x_i - \bar{x}}{\hat{\sigma}_{MLE}}\right)^2}_{b}\right]\right\}$$

$$a = \frac{\sum (x_i - \mu_0)^2}{\frac{\sum (x_i - \mu_0)^2}{n}}$$

Similarly:

$b = n$

$$\lambda(x) = \left(\frac{\hat{\sigma}_0^2}{\sigma^2_{MLE}}\right)^{-n/2} \exp\{0\}$$

$$= \left[\frac{\sum (x_i - \mu_0)^2}{\sum (x_i - \bar{x})^2}\right]^{-n/2}$$

Then, $[\lambda(x)]^{-\frac{2}{n}} = \dfrac{\sum (x_i - \mu_0)^2}{\quad}$

$\left[\sum (x_i - \bar{x})^2 \right]$

Then, $\left[\lambda(x)\right]^{-\frac{2}{n}} = \dfrac{\sum (x_i - \mu_0)^2}{\sum (x_i - \bar{x})^2}$

$$= \frac{\sum (x_i - \bar{x} + \bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum (x_i - \bar{x})^2 + \sum (\bar{x} - \mu_0)^2 - 2 \overbrace{\sum (x_i - \bar{x})(\bar{x} - \mu_0)}^{=0}}{\sum (x_i - \bar{x})^2}$$

$$1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}$$

Define $t(\underline{x}) = \dfrac{(\sqrt{n})(\bar{x} - \mu_0)}{\sqrt{\sum (x_i - \bar{x})^2 / (n-1)}}$

$$= \frac{(\sqrt{n})(\bar{x} - \mu_0)}{S} \sim t(n-1)$$

- we know that $t^2(\underline{x}) \sim F_{(1, n-1)}$, where in our case:

$$t^2(\underline{x}) = \frac{n(\bar{x} - \mu_0)^2}{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

- this implies:

$$\frac{t^2(\underline{x})}{n-1} = \frac{n(\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}$$

- this implies:

$$\left[\lambda(x)\right]^{-2/n} = 1 + \frac{t^2(\underline{x})}{n-1}$$

Under GLR based test:

$C = \{ \underline{x} \mid \lambda(x) \le k \} \Rightarrow$ we reject for large values of $\left[\lambda(\underline{x})\right]^{-2/n}$

- This also implies that we reject for large values of $t^2(\underline{x})$.
- For size α critical region, we reject for

$$t^2(x) \ge F_{1-\alpha}(1, n-1)$$

- We need to show that; for $H_A: \mu \ne \mu_0$, we reject H_0 if:

$$t(\underline{x}) \le -t_{1-\frac{\alpha}{2}}(n-1) \quad \text{OR} \quad t(\underline{x}) \ge t_{1-\alpha/2}(n-1)$$

- The test based on $t(\underline{x})$ for $H_A: \mu \ne \mu_0$ is UMPU (unbiased)

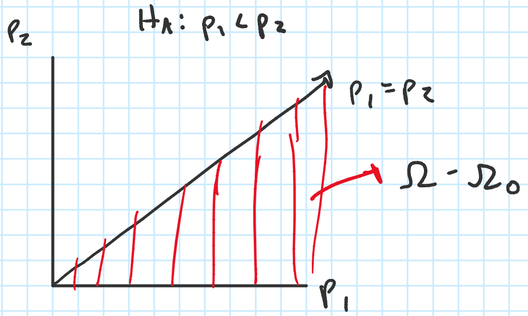
Ex 12.8-4 (2 sample problem where GLR may be used)

- The test based on $t(\underline{x})$ for $H_A: \mu \neq \mu_0$ is UMPU (unbiased)

Ex 12.8.4 (2 sample problem where GLR may be used)

- Suppose $X \sim Bin(n_1, p_1)$ and $Y \sim Bin(n_2, p_2)$ where X, Y are independent.
- Construct a test based on GLR to test:

$$H_0: p_1 = p_2 = p_0$$
$$H_A: p_1 < p_2$$



- Unrestricted MLEs:

$$\hat{p}_{1,MLE} = \frac{x}{n_1} \quad \text{and} \quad \hat{p}_{2,MLE} = \frac{y}{n_2}$$

- Restricted MLE under H_0: $\hat{p}_{0,MLE} = \frac{x+y}{n_1+n_2}$

- The GLR stat:

$$\Lambda(x,y) = \frac{\max_{p_0} f_X(x; p_0, n_1) \cdot f_Y(y; p_0, n_2)}{\max_{p_1,p_2} f_X(x; p_1, n_1) \cdot f_Y(y; p_2, n_2)}$$

$$= \frac{\cancel{\binom{n_1}{x}}\left(\frac{x+y}{n_1+n_2}\right)^x\left(1-\frac{x+y}{n_1+n_2}\right)^{n_1-x}\cancel{\binom{n_2}{y}}\left(\frac{x+y}{n_1+n_2}\right)^y\left(1-\frac{x+y}{n_1+n_2}\right)^{n_2-y}}{\cancel{\binom{n_1}{x}}\left(\frac{x}{n_1}\right)^x\left(1-\frac{x}{n_1}\right)^{n_1-x}\cancel{\binom{n_2}{y}}\left(\frac{y}{n_2}\right)^y\left(1-\frac{y}{n_2}\right)^{n_2-y}}$$

$$\left[\frac{\left(\frac{x+y}{n_1+n_2}\right)}{\left(\frac{x}{n_1}\right)}\right]^x \cdot \left(\frac{1-\frac{(x+y)}{n_1+n_2}}{1-\left(\frac{x}{n_1}\right)}\right)^{n_1-x}\left[\frac{\left(\frac{x+y}{n_1+n_2}\right)}{\left(\frac{y}{n_2}\right)}\right]^y\left(\frac{1-\frac{(x+y)}{n_1+n_2}}{1-\left(\frac{y}{n_2}\right)}\right)^{n_2-y}$$

- We reject values of x's y such that $-2\ln(\Lambda(x,y)) \geq \chi^2_{1-\alpha}(r)$, where '$r$' pertains to the number of linearly independent restrictions under H_0 (in this case, $r=1$)

$$H_0: p_1 = p_2 = p_3 = p_0$$

$$\begin{cases} p_1 - p_2 = 0 \\ p_2 - p_3 = 0 \end{cases} \Rightarrow 2 \text{ restrictions}$$

Section 12.9 (Conditional tests)

- Suppose our distribution has unknown parameters that are not involved in the hypothesis that we are interested in.
- Suppose the unknown parameter (nuissance parameter) has a sufficient statistic $S(\underline{X})$.
- Then, we can find a test free of the unknown parameter by conditioning on $S(\underline{x})$.

Example

- Let $X \sim Bin(n_1, p_1)$, $Y \sim Bin(n_2, p_2)$ } X & Y independent

Q: Find a size α test of $\begin{cases} H_0: p_1 = p_2 = p_0 \\ H_A: p_1 < p_2 \end{cases}$

- Here, p_0 is the "nuissance parameter".

$$f_{X,Y}(x,y) = \binom{n_1}{x}\binom{n_2}{y} p^{x+y} (1-p)^{n_1+n_2-(x+y)}$$

$$= \binom{n_1}{x}\binom{n_2}{y}\left(\frac{p}{1-p}\right)^{x+y}(1-p)^{n_1+n_2}$$

$$= \underbrace{\binom{n_1}{x}\binom{n_2}{y}}_{h(x,y)} \exp\left\{\underbrace{(x+y)}_{S(x,y)} \underbrace{\ln\left(\frac{p}{1-p}\right)}_{\eta(p)}\right\} \cdot \underbrace{(1-p)}_{c(p)}$$

$S(X,Y) = X+Y$ is sufficient for p