Lecture 6

Thursday, January 23, 2020 1:58 PM

- Agenda
- Uniformly Most Powerful test (UMP tests)
- Monotone likelihood ratio

- Recall: Ex 12.6.3
  - Find the MP test based on:

$S \sim Bin(n, p)$

for $H_0: p = p_0$

$H_a: p = p_1 \quad (p_0 < p_1)$

- From the N-P-lemma:

⋮

$\lambda(s, p_0, p_1) = .$

⋮

$$= \left(\frac{p_0}{p_1}\right)^s \left(\frac{1-p_0}{1-p_1}\right)^{n-s}$$

$$C^* = \left\{ s \mid \left(\frac{p_0}{p_1}\right)^s \left(\frac{1-p_0}{1-p_1}\right)^{n-s} \le k \right\} \quad \text{where,}$$

$$s \ge \frac{\ln\left[k\left(\frac{1-p_0}{1-p_1}\right)^n\right]}{\ln\left[\underbrace{\frac{p_0(1-p_1)}{p_1(1-p_0)}}_{<1}\right]}$$

- we need to find k so we have a size α test:

$P[S \ge i \mid p = p_0] = 1 - B(i-1; n, p_0)$ B here is the binomial CDF, not beta.

* The N-P Lemma only guarantees that this test is most powerful amongst size α_i tests.

Extension of the N-P Lemma (illustrated in ex. 12.6.4)

ex 12.6.4

- Let $\underset{\sim}{X} \overset{\sim}{iid} \begin{cases} H_0: X_i \sim Unif(0,1) \ \forall i \\ H_A: X_i \sim exp(1) \ \forall i \end{cases}$

Q: Find the MP test:

$$\lambda(\underset{\sim}{x}; H_0, H_A) = \frac{1 \cdot I\{x_i \in (0,1) \ \forall i\}}{\exp\{-\Sigma x_i\} \cdot I\{X_i > 0 \ \forall i\}}$$

$$= \exp\{\Sigma x_i\} \cdot I\{x_i \in (0,1) \ \forall i\}$$

Recall

$$\lambda(\underset{\sim}{x}; H_0, H_A) = \frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}; \theta_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}; \theta_1)}$$

- The MP critical region:

$C^* = \{ x \mid \exp\{\Sigma x_i\} \le k \}$ if $0 < X_i < 1 \ \forall i$

• The MP critical region:

$$C^* = \{ \underline{x} \mid \exp\{\sum x_i\} \le K \} \text{ if } 0 \le X_i \le 1 \ \forall i$$

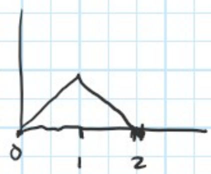
↳ Consider C^*: this implies:

$$\sum x_i \le \ln(K)$$

↳ Then, under the null H_0:

$X_i \sim \text{Unif}(0,1)$, thus:

$X_1 + X_2 \sim$ Triangle Distribution



$\sum_{i=1}^{3} X_i$

. . . .

$\sum_{i=1}^{n} X_i$

$\Rightarrow \sum X_i \sim N$ by CLT for "large" n.

$$\sum X_i \sim N \Rightarrow \bar{x} \sim N\left(\tfrac{1}{2}, \text{Var}(\bar{x}) = \frac{\sigma^2}{n} = \frac{1}{12n}\right)$$

• The approximate MP-CR is:

$$C^* = \left\{ \underline{x} \;\middle|\; \bar{x} \le \frac{\ln(K)}{n} \right\}$$

$$\bar{x} = \frac{\sum x_i}{n}$$

∴ Rejection region is then of form

$$\left[\frac{\bar{x} - \frac{1}{2}}{\sqrt{\frac{1}{12n}}}\right] \le \left[\frac{\frac{\ln(K)}{n} - \frac{1}{2}}{\sqrt{1/12n}}\right]$$

$$\Rightarrow \frac{\frac{\ln(K)}{n} - \frac{1}{2}}{\sqrt{1/12n}} = z_\alpha = z_{1-\alpha}$$

Recall the MP critical region:

$$C^* = \{ \underline{x} \mid \exp\{\sum x_i\} \le K \}$$

Recall $E(X_i) = \frac{1}{2}$ under H_0

$E(X_i) = 1$ under H_A

Section 12.7 Uniformly Most Powerful Tests:

• An extension of the MP test to compare with composite alternative hypothesis.

Section 12.7 Uniformly Most Powerful Tests.

- An extension of the MP test to compare with composite alternative hypothesis.

- If a test is most powerful against every possible value in a composite alternative, then we say it is the Uniformly Most Powerful test.

- Def 12.7.1

  - Let $X_1, \dots, X_n$ be jointly distributed $f_{\underline{X}}(\underline{x};\theta)$ for $\theta \in \Omega$ and consider hypotheses of form:

$$H_0: \theta \in \Omega_0$$

$$H_A: \theta \in \Omega - \Omega_0 \quad \text{where } \Omega_0 \subset \Omega$$

  - A critical region C^* and its associated test is said to be UMP of size α if:

    - $\max\limits_{\theta \in \Omega_0} \pi_{C^*}(\theta) = \alpha$

    - $\pi_{C^*}(\theta) \geq \pi_C(\theta) \ \forall \theta \in \Omega - \Omega_0$, for any other size α critical region C.

Ex.

$X_1, \dots, X_n \underset{iid}{\sim} \exp(\theta)$

$$\begin{cases} H_0: \theta \leq \theta_0 \\ H_A: \theta \geq \theta_0 \end{cases}$$

Q: Find the UMP test.

Recall: $C^* = \left\{ \underline{x} \,\middle|\, \frac{2\sum x_i}{\theta_0} \geq \chi^2_{1-\alpha}(2n) \right\}$

- The $\pi(\theta) = P(\text{reject } H_0 \mid \theta)$

$$= P\left(\frac{2\sum X_i}{\theta_0} \geq \chi^2_{1-\alpha}(2n) \,\middle|\, \theta\right)$$

$$= P\left(\frac{2\sum X_i}{\theta_0} \cdot \frac{\theta}{\theta} \geq \chi^2_{1-\alpha}(2n)\right)$$

$$= P\left(\frac{2\sum X_i}{\theta} \geq \frac{\theta_0}{\theta}\chi^2_{1-\alpha}(2n)\right)$$

$$\therefore \pi(\theta) = 1 - H\left[\frac{\theta_0}{\theta} \cdot \chi^2_{1-\alpha}(2n);\ 2n\right] \quad \checkmark$$

... where H is the CDF of $\chi^2(2n)$

- Similarity for $\begin{cases} H_0: \theta \geq \theta_0 \\ H_A: \theta < \theta_0 \end{cases}$, we get:

$$\pi(\theta) = H\left[\frac{\theta_0}{\theta} \cdot \chi^2_{\alpha}(2n);\ 2n\right] \quad \checkmark$$

Discrete Example

X	1	2	3	4	5	6	7
$f_X(x; H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f_X(x; H_A)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79
$\lambda(x, H_0, H_A)$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	1.2

$\mathcal{A}$ they sum to α under the null.

Q: Use N-P Lemma to find the MP test for $\alpha = 0.04$

$\hookrightarrow C^* = \{\lambda(x; H_0, H_A) \leq k\}$

... where k is such that:

$P(\text{Reject } H_0 \mid H_0) = 0.04$

$\Rightarrow C^* = \{X \leq 4\}$

Then $\beta = P(\text{Type II error}) = P(\text{fail to reject } H_0 \mid H_A \text{ is true})$

$= P(X \geq 5 \mid H_A) = 0.82$