Lecture 3

Tuesday, January 14, 2020 1:55 PM

Agenda

- Power Function
  - simple vs. simple hypothesis

* Section 12.1 Composite Hypothesis

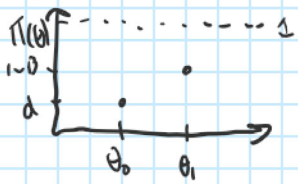
- A method to evaluate the performance of a test via the power funtion $\pi(\cdot)$

Def (12.1.4)

↳ The power function $\pi(\theta)$ for a test is the probability of rejecting H_0 when the true value of the parameter is θ.

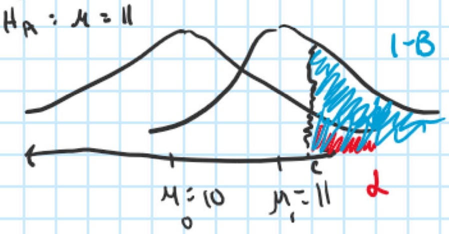
↳ $\pi : \Omega \rightarrow [0,1]$

- Suppose $\begin{cases} H_0: \theta = \theta_0 \\ H_A: \theta = \theta_1 \quad (\theta_1 > \theta_0) \end{cases}$



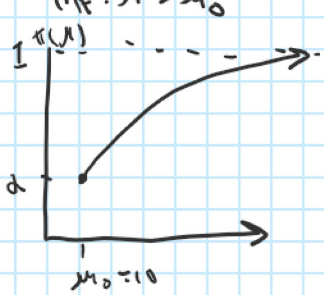
Recall Example

$H_0 : \mu = 10$
$H_A : \mu = 11$

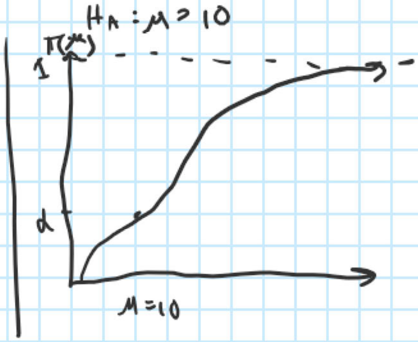


* Thus power grows as $\mu_1 - \mu_0$ gets bigger

$H_0: \mu_0 = 10$
$H_A: \mu > \mu_0$



$H_0 : \mu \leq 10$
$H_A : \mu > 10$



- $\max \pi(\mu) = \alpha$ when evaluated over all values of the parameter in the null space.

- max $\pi(\mu) = \alpha$ when evaluated over all values of the parameter in the null space.

Composite Vs Composite Hypothesis

- Suppose $X \sim N(\mu, \sigma^2)$ where σ^2 known
- we want to test $\begin{cases} H_0: \mu = \mu_0 \\ H_A: \mu > \mu_0 \end{cases}$ (A)

Step #1) Find c.

→ Recall $H_0: \mu = \mu_0$
$H_A: \mu = \mu_1$ where $(\mu_1 > \mu)$

$$C = \{ \underline{x} \mid \bar{x} \geq c \} \quad \text{where} \quad c = \mu_0 + Z_{1-\alpha} \cdot (\sigma/\sqrt{n})$$

C also holds for hypothesis of the form (A)

Step #2) Power:

$$\pi(\mu) = P(\text{Rejecting } H_0 \mid \mu)$$
$$= P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \geq Z_{1-\alpha} \,\middle|\, \mu \right]$$
$$= P\left[\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq Z_{1-\alpha} \right]$$
$$= P\left[\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \geq Z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \right]$$
$$= 1 - \underbrace{\Phi\left(Z_{1-\alpha} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \right)}_{\beta}$$

Φ → standard normal CDF

$$\therefore \pi(\mu) = 1 - \beta(\mu)$$

- At $\mu = \mu_0$:

$$\pi(\mu_0) = 1 - \Phi(Z_{1-\alpha})$$
$$= 1 - (1-\alpha)$$
$$= \alpha$$

- If $\mu > \mu_0$, then:
  $\pi(\mu) > \alpha$

  and similarly if
  $\mu < \mu_0$, then:
  $\pi(\mu) < \alpha$

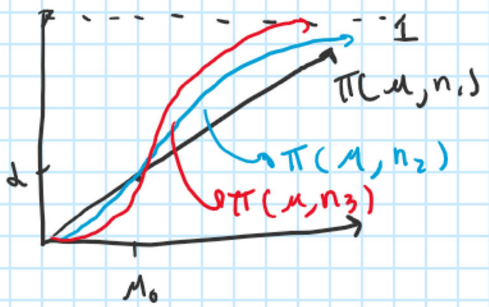
If hypothesis from (A)

↳ otherwise, these flip.

III) Sample Size and $\pi(\mu)$

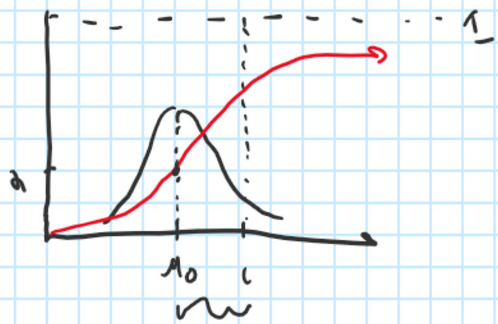
(III) Sample Size and $\pi(\mu)$

For (A): $\pi(\mu) = 1 - \Phi\left(z_{1-\alpha} + \frac{(\mu_0 - \mu)\sqrt{n}}{\sigma}\right)$



$n_3 > n_2 > n_1$

power function converges to α as n grows.



P Value:

$\alpha = 0.05$ need not be the approximate Type I error rate even within the same field.

p-value: This is the smallest size of α at which H_0 can be rejected; based on observed value of the test statistic. $T(\underline{X})$ is the stat and t is the observed value.

#1 ($\leq$):

$H_0: \mu = \mu_0$

$H_A: \mu > \mu_0$

p-value $= P(T(\underline{X}) \geq t \mid H_0 \text{ is true})$

#2 ($\geq$)

$H_0: \mu = \mu_0$

$H_A: \mu < \mu_0$

p-value $= P(T(\underline{X}) \leq t \mid H_0 \text{ is true})$

#3

$H_0: \mu = \mu_0$

$H_A: \mu \neq \mu_0$

p-value $= 2 \cdot P(T(\underline{X}) \geq |t|)$ when $t > \mu_0$

Quick Example (12.2.1)

$$p\text{-value} = 2 \cdot P(T(\underline{x}) \geq |t|) \text{ when } t > \mu_0$$

Quick Example (12.2.1)

- $X_1, \dots, X_{25} \underset{iid}{\sim} N(\mu, \sigma^2 = 16)$

  $H_0: \mu = 10$

  $H_A: \mu > 10$

- Suppose the observed value of $\bar{x} = 11.4$

  - Then p-value $= P(\bar{X} \geq 11.4 \mid \mu = 10)$

$$= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{11.4 - 10}{4/5}\right)$$

$$= P(Z \geq 1.75)$$

$$\approx 0.0401$$

→ we might be making a type I error at rate of about 0.0401 if we reject values bigger than $\bar{x}$.

The probability of observing a value of 11.4
Given that the actual mean is 10 is about 4%

Alpha is basically the cutoff for how unusual something
Needs to be for us to decide something is 'different'. Alpha
Is usually about 5% so this would flag.

- $H_0: \mu = 10$ (B)

  $H_A: \mu \neq 10$, $\bar{x} = 11.4$

  p-value $= 2 \cdot P(\bar{X} > 11.4)$

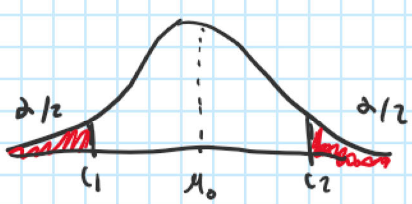
  ≈ 0.08

2-sided alternative:

For (A):

$$C = \{\underline{x} \mid \bar{x} \leq c_1 \text{ or } \bar{x} \geq c_2\}$$

where $\begin{cases} c_1 = \mu_0 - z_{1-\alpha/2}(\sigma/\sqrt{n}) \\ c_2 = \mu_0 + z_{1-\alpha/2}(\sigma/\sqrt{n}) \end{cases}$



- For (B):

$$\pi(\mu) = 1 - \beta = 1 - P(c_1 \leq \bar{X} \leq c_2 \mid \mu)$$

$$= 1 - \left[P\left(-z_{1-\alpha/2} \leq \frac{\bar{X} - \mu_0}{(\sigma/\sqrt{n})} \leq z_{1-\frac{\alpha}{2}} \,\middle|\, \mu\right)\right]$$

$$= 1 - \Phi\left(z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$$

$$+ \Phi\left(z_{1-\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right)$$

- A $(1-\alpha)100\%$ CI for μ:

$$\left[\bar{x} - z_{1-\frac{\alpha}{2}}\left(\sigma/\sqrt{n}\right),\ \bar{x} + z_{1-\alpha/2}\left(\sigma/\sqrt{n}\right)\right]$$

ie, it is the compliment of the Rejection Region.

(Read pg 398 in text)

Section 12.3 (σ^2 unknown)

- $X_i \sim N(\mu, \sigma^2)$ both unknown

I) Critical Region

- Theorem 12.3.2:

Let $x_1, \dots, x_n$ be an observed random sample from the $N(\mu, \sigma^2)$ distribution, where σ^2 is unknown. Let

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad \text{under the null.}$$

If (1) $H_0: \mu \le \mu_0$
$H_A: \mu > \mu_0$

- Then $C = \left\{ \underline{x} \mid t_0 = \frac{x - \mu_0}{s/\sqrt{n}} \ge t_{1-\alpha}(n-1) \right\}$

where $t_{1-\alpha}(n-1)$ is such that:

$$P(T(n-1) \ge t_{1-\alpha}(n-1)) = \alpha$$

- The statistic

$$T_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t(n-1)$$

(2) $H_0: \mu \ge \mu_0$
$H_A: \mu < \mu_0$

then $C = \left\{ \underline{x} \mid t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \le -t_{1-\alpha}(n-1) \right\}$

(3) $H_0: \mu = \mu_0$
$H_A: \mu \ne \mu_0$

Then $C = \{ x \mid t_0 = \frac{\bar{x} - \mu_0}{\ldots} \le -t \quad (n-1)$

③ $H_0: \mu = \mu_0$

$H_A: \mu \neq \mu_0$

Then $C = \left\{ \underline{x} \;\middle|\; t_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} \leq -t_{1-\alpha/2}(n-1) \right.$

or

$\left. t_0 \geq t_{1-\alpha/2}(n-1) \right\}$

Recall Central T-Distribution:

· student t-distribution is of form:

$$T = \frac{Z}{\sqrt{\frac{V}{\nu}}} \quad \text{where} \begin{cases} Z \sim N(0,1) \\ V \sim \chi^2(\nu) \\ Z \underset{iid}{\perp} V \end{cases}$$

$$T = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$= \frac{(x-\mu)/(\sigma/\sqrt{n})}{(s/\sqrt{n})/(\sigma/\sqrt{n})}$$

$$= \sqrt{\frac{s^2(n-1)}{\sigma^2(n-1)}}$$

·Recall: $\dfrac{s}{\sqrt{n}} = \sqrt{\dfrac{s^2}{n}}$

$$\frac{s/\sqrt{n}}{\sigma/\sqrt{n}} = \frac{\sqrt{s^2/n}}{\sqrt{\sigma^2/n}}$$

Recall: $\dfrac{s^2(n-1)}{\sigma^2} \sim \chi^2(n-1)$

$$T = \frac{Z}{\sqrt{\frac{V}{\nu}}} \quad \text{where} \begin{cases} Z \sim N(0,1) \\ V = \chi^2(n-1) \\ Z \perp V \end{cases}$$