#27

Thursday, February 13, 2020 7:58 AM

• Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution, $X_i \sim Exp(\theta)$.

a) Derive the generalized likelihood ratio test for $H_0: \theta = \theta_0$ against $H_A: \theta \neq \theta_0$. Determine an approximate critical value for size α using a large sample chi-square approximation.

S1. The PDF...

$$f(x, \theta) = \frac{1}{\theta} \exp\left\{ -\frac{x}{\theta} \right\} \cdot I\{x > 0\}$$

S2. Find the MLE

S2a) The JDF...

$$f(\underset{\sim}{x}; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x/\theta} \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

$$= \left(\frac{1}{\theta}\right)^n \left(e^{-\sum_{i=1}^{n} x_i / \theta}\right) \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

S2b) Likelihood...

$$L(f(\underset{\sim}{x}; \theta)) = \left(\frac{1}{\theta}\right)^n \left(e^{-\frac{\sum_{i=1}^{n} X_i}{\theta}}\right) \cdot \underbrace{\prod_{i=1}^{n} I\{X_i > 0\}}_{=1}$$

S2c) Taking the log likelihood

$$\ln[L(f(\underset{\sim}{x}; \theta))] = \ln\left[\left(\tfrac{1}{\theta}\right)^n \left(e^{-\sum_{i=1}^{n} X_i / \theta}\right)\right]$$

$$= n \ln\left[\tfrac{1}{\theta}\right] + \ln\left[e^{-\sum_{i=1}^{n} x_i / \theta}\right]$$

$$= n \ln\left[\tfrac{1}{\theta}\right] - \left(\sum_{i=1}^{n} x_i / \theta\right) \ln[e]$$

$$= n \ln[1/\theta] - \sum_{i=1}^{n} x_i / \theta$$

$$= n \ln[1] - n \ln[\theta] - \sum_{i=1}^{n} X_i / \theta$$

S2d) Taking the first derivative and setting equal to zero.

$$\frac{\partial \ln[L(\theta)]}{\partial \theta} = 0 - \frac{n}{\theta} + \sum_{i=1}^{n} X_i / \theta^2 = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\theta^2} = \frac{n}{\theta}$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \theta^2 / \theta$$

$$\frac{\;_{i=1}}{n} = \;/\theta$$

$$\hat{\theta} = \sum_{i=1}^{n} x_i / n$$

52 e) checking if this is a maximum by 2nd derivative test

$$\frac{\partial^2 \ln[L(\theta)]}{\partial\theta^2} = \quad \text{First derivative was } \frac{\sum x_i}{\theta} - n = 0$$

Then the second derivative is

$$-\frac{\sum x_i}{\theta^2} \quad \text{which is less than } 0 \ \forall \theta > 0 \quad \checkmark$$

$$\therefore MLE_{\exp} = \sum_{i=1}^{n} x_i / n = \bar{x}$$

53. Now we know the MLE is $\bar{x}$, we can compute the GLR using the $MLE_{\hat{\theta}}$.

$$\Lambda(\underset{\sim}{x}; \theta_0, \theta_1) = \frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}; \theta_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}; \hat{\theta})} = \frac{\left(\frac{1}{\theta_0}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\theta_0}\right)}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\bar{x}}\right)}$$

$$= \frac{\left(\frac{1}{\theta_0}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\theta_0}\right)}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-\sum_{i=1}^{n} \cancel{x_i} / \left(\sum_{i=1}^{n} \cancel{x_i} / n\right)}\right)}$$

$$= \frac{\left(\frac{1}{\theta_0}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\theta_0}\right)}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-n}\right)}$$

$$\Lambda(x) = \left(\frac{\bar{x}}{\theta}\right)^n \left(e^{n - \sum_{i=1}^{n} x_i / \theta_0}\right)$$

Now knowing that can reject H_0 if $-2\ln \Lambda(x) \geq \chi^2_{1-\alpha}(1)$ . . .

Now knowing that can reject H_0 if $-2\ln \lambda(x) \geq \chi^2_{1-\alpha}(1)$ . . .

$$-2\ln\left[\left(\frac{\bar{x}}{\theta}\right)^n \exp\{n - \sum_{i=1}^{n} x_i / \theta_0\}\right]$$

$$= -2n\ln[\bar{x}/\theta_0] - 2\ln[\exp\{n - \sum_{i=1}^{n} x_i/\theta_0\}]$$

$$= -2n\ln[\bar{x}/\theta_0] - 2\ln[\exp\{n - n\sum_{i=1}^{n} x_i / n\theta_0\}]$$

$$= -2n\ln[\bar{x}/\theta_0] - 2\ln[\exp\{n - \frac{n\bar{x}}{\theta_0}\}]$$

$$= -2n\ln[\bar{x}/\theta_0] - 2\ln[\exp\{n(1 - \frac{x}{\theta_0})\}]$$

$$= -2n\ln[\bar{x}/\theta_0] - 2(n(1 - x/\theta_0))$$

$$= -2n\ln[\bar{x}/\theta_0] - 2n(1 - x/\theta_0)$$

$$= -2n[\ln(\bar{x}/\theta_0) - \frac{\bar{x}}{\theta_0} + 1]$$

$\therefore$ Thus, we can reject H_0 if

$$-2n[\ln(\bar{x}/\theta_0) - (\bar{x}/\theta_0) + 1] \geq \chi^2_{1-\alpha}(1)$$

✓