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Thursday, February 13, 2020 9:42 AM

Q: Consider independent random samples of size n_1, n_2 from respective exponential distributions, $X_i \sim Exp(\theta_1)$, $Y_j \sim Exp(\theta_2)$.

Derive the GLR test of $H_0: \theta = \theta_0$ versus $H_A: \theta \neq \theta_0$

S1. The PDFs...

$$f_x(x; \theta_1) = \frac{1}{\theta_1} e^{-x_1/\theta_1}$$

$$f_y(y; \theta_2) = \frac{1}{\theta_2} e^{-y/\theta_2}$$

S2. The JDFs...

$$f_{\underline{x}}(\underline{x}; \theta_1) = \prod_{i=1}^{n} \frac{1}{\theta_1} e^{-x_i/\theta_1} \cdot \prod_{i=1}^{n} I\{x_i > 0\}$$

$$= \left(\frac{1}{\theta_1}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\theta_1}\right)$$

similarly...

$$f_y(y; \theta_2) = \left(\frac{1}{\theta_2}\right)^n \left(e^{-\sum_{i=1}^{n} x_i/\theta_2}\right)$$

S3. Then plugging these in to the GLR...

$$\Lambda(x, y) = \frac{f_{\underline{x}}(\underline{x}; \theta_1) \cdot f_{\underline{y}}(\underline{y}; \theta_2)}{f_{\underline{x}}(\underline{x}; \hat{\theta}) \cdot f_{\underline{y}}(\underline{y}; \hat{\theta})}$$

$$= \frac{\left[\left(\frac{1}{\theta_1}\right)^n \left(e^{-\sum x_i/\theta_1}\right)\right] \cdot \left[\left(\frac{1}{\theta_2}\right)^n \left(e^{-\sum y_i/\theta_2}\right)\right]}{\left(\frac{1}{\bar{x}}\right)^n \left(e^{-\sum \cancel{x_i}/(\sum \cancel{x_i}/n)}\right) \cdot \left(\frac{1}{\bar{y}}\right)^n \left(e^{-\sum \cancel{y_i}/(\sum \cancel{y_i}/n)}\right)}$$

Recall that the MLE for θ in an exponential distribution is $\overline{X}$

$$= \frac{\left(\frac{1}{\theta_1}\right)^n \left(\frac{1}{\theta_2}\right)^n \exp\left\{-\frac{\sum x_i}{\theta_1} - \frac{\sum y_i}{\theta_2}\right\}}{\ldots}$$

$$\overline{\left(\frac{1}{\bar{x}}\right)^n\left(\frac{1}{\bar{y}}\right)^n \exp\{-2n\}}$$

$$= \left(\frac{\bar{x}\cdot\bar{y}}{\theta_1\cdot\theta_2}\right)^n \cdot \exp\left\{-\frac{\sum x_i}{\theta_1} - \frac{\sum y_i}{\theta_2} + 2n\right\}$$

- We know we can reject the null hypothesis H_0 if

$$-2\ln[\lambda(x,y)] \geq \chi^2_{1-\alpha}(1)$$

- Thus we can reject the null hypothesis if

$$-2\ln\left[\left(\frac{\bar{x}\bar{y}}{\theta_1\theta_2}\right)^n \cdot \exp\left\{2n - \frac{n\bar{x}}{\theta_1} - \frac{n\bar{y}}{\theta_2}\right\}\right] \geq \chi^2_{1-\alpha}(1)$$

$$\Rightarrow (-2)\left[\ln\left[\left(\frac{\bar{x}\bar{y}}{\theta_1\theta_2}\right)^n\right] + \ln\left[\exp\left\{2n - \frac{n\bar{x}}{\theta_1} - \frac{n\bar{y}}{\theta_2}\right\}\right]\right]$$

$$\Rightarrow (-2)\left[n\ln\left(\frac{\bar{x}\bar{y}}{\theta_1\theta_2}\right) + 2n - \frac{n\bar{x}}{\theta_1} - \frac{n\bar{y}}{\theta_2}\right]$$

$$\Rightarrow (-2n)\left[\ln\left(\frac{\bar{x}\bar{y}}{\theta_1\theta_2}\right) + 2 - \frac{\bar{x}}{\theta_1} - \frac{\bar{y}}{\theta_2}\right] \geq \chi^2_{1-\alpha}(1)$$

$\therefore$ Thus, we can reject the null hypothesis if

$$(-2n)\left[\ln\left(\frac{\bar{x}\bar{y}}{\theta_1\theta_2}\right) + 2 - \frac{\bar{x}}{\theta_1} - \frac{\bar{y}}{\theta_2}\right] \geq \chi^2_{1-\alpha}(1)$$

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