

Lecture 7

Tuesday, January 28, 2020 1:59 PM

Agenda

- Monotone Likelihood Ratio
- Unbiased tests
- Tests for simple parameter exponential families
- Section 12.8
 - ↳ Generalized Likelihood Ratio Test

Clarification from Lecture 6

↳ Ex 12.6.4

$$X_1, \dots, X_n \stackrel{iid}{\sim} \begin{cases} H_0: \text{Unif}(0,1) \\ H_A: \text{Exp}(1) \end{cases} \quad \text{Dealing with different distributions}$$

... after some work ...

$$C^* = \{x \mid \exp\{\sum x_i\} \leq k\}$$

$$\Rightarrow \sum x_i \leq \ln(k)$$

$$\Rightarrow \bar{x} \leq \left(\frac{1}{n}\right) \cdot \ln(k) \quad \text{where } \bar{x} \sim N(E(\bar{x})=1/2, \text{var}(\bar{x})=\frac{1}{12n})$$

Then, a size α test would reject if:

$$\frac{\bar{x} - 1/2}{\sqrt{\frac{1}{12n}}} \leq -Z_{1-\alpha}$$

Approximation when n is large

$$\text{Suppose: } \begin{cases} H_0: \mu = \mu_0 \\ H_A: \mu \neq \mu_0 \end{cases}$$

$$\Rightarrow \pi_{C^*}(\mu) \geq \pi_C(\mu) \quad \forall \mu \neq \mu_0 \rightarrow \text{condition for UMP test}$$

• We usually cannot find a UMP size α test under H_A (if we have a 2-sided alternative).

↳ So, we consider a more restricted class of tests that are unbiased.

• This implies that UMP tests are "usually" available for one-sided alternatives.

Monotone Likelihood Ratio (MLR):

Def 12.7.2

• A joint pdf $f_{\underline{x}}(\underline{x}; \theta)$ is said to have an MLR in statistic $T = t(\underline{x})$ if for any two values $\theta_1 < \theta_2$, the ratio of:

$$\frac{f_{\underline{x}}(\underline{x}; \theta_2)}{f_{\underline{x}}(\underline{x}; \theta_1)}$$

• depends on \underline{x} only through $t(\underline{x})$ if it is non-decreasing in $t(\underline{x})$.

Ex 12.7.2

• let $x_1, \dots, x_n \sim \text{Exp}(\theta)$

• let $\theta_1 < \theta_2$

• then:

$$\begin{aligned} \frac{f_{\underline{x}}(\underline{x}; \theta_2)}{f_{\underline{x}}(\underline{x}; \theta_1)} &= \frac{\left(\frac{1}{\theta_2}\right)^n \cdot \exp\left\{-\frac{\sum x_i}{\theta_2}\right\}}{\left(\frac{1}{\theta_1}\right)^n \cdot \exp\left\{-\frac{\sum x_i}{\theta_1}\right\}} \quad (\text{assuming } x_i > 0) \\ &= \left(\frac{\theta_1}{\theta_2}\right)^n \cdot \exp\left\{\sum x_i \cdot \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\right\} \end{aligned}$$

• Since $\theta_1 < \theta_2$, the above is a non-decreasing function of $t(\underline{x}) = \sum x_i$ > 0 on $\theta_1 < \theta_2$

• $\therefore \text{Exp}(\theta)$ has an MLR in $\sum x_i$

Connection with UMP tests???

Thm 12.7.1:

• if a joint pdf $f_{\underline{x}}(\underline{x}; \theta)$ has an MLR in statistic $t(\underline{x})$, the UMP test of size α can be found for:

$$\begin{cases} H_0: \theta \leq \theta_0 \\ H_A: \theta > \theta_0 \end{cases}$$

• if we reject H_0 for $t(\underline{x}) \geq k$ where k satisfies:

$$P[t(\underline{x}) \geq k \mid H_0] = \alpha$$

• Similarly, if instead we have:

$$\begin{cases} H_0: \theta \geq \theta_0 \\ H_A: \theta < \theta_0 \end{cases}$$

... then we reject H_0 if $t(\underline{x}) \leq k$ (etc, etc...)

Example 12.7.3

$X_1, \dots, X_n \stackrel{iid}{\sim} \text{Exp}(1, \eta)$ where η is the location parameter.

Q: Find the UMP tests (size α) for:

$$\begin{cases} H_0: \eta \leq \eta_0 \\ H_A: \eta > \eta_0 \end{cases}$$

$$X_{(1)} = X_{1:n}$$

The joint pdf of \underline{x} is $f_{\underline{x}}(\underline{x}; 1, \eta) = \exp\{-\sum(x_i - \eta)\} \cdot I\{x_{(1)} \geq \eta\}$

For $\eta_1 < \eta_2$ we have:

$$\frac{f_{\underline{x}}(\underline{x}; 1, \eta_2)}{f_{\underline{x}}(\underline{x}; 1, \eta_1)} = \frac{\exp\{-\sum(x_i - \eta_2)\} \cdot I\{x_{(1)} \geq \eta_2\}}{\exp\{-\sum(x_i - \eta_1)\} \cdot I\{x_{(1)} \geq \eta_1\}}$$

$$\frac{f_{\underline{x}}(\eta_2)}{f_{\underline{x}}(\eta_1)} = \exp\{-n(\eta_2 - \eta_1)\} \cdot I\{x_{(1)} \geq \eta_2\}$$

Thus the ratio is a non-decreasing function of $t(\underline{x}) = X_{(1)}$.
The distribution has an MLR in $X_{(1)}$.

↳ Then by theorem 12.7.1, a UMP test of size α is of the form:

$$C^* = \{\underline{x} \mid x_{(1)} \geq k\}, \text{ where } \alpha = P[X_{(1)} \geq k \mid \eta_0] \quad \checkmark$$

Unbiased Tests: (for 2-sided test)

Def 12.7.3: A test of $H_0: \theta \in \Omega_0$

$$H_A: \theta \in \Omega - \Omega_0$$

... is unbiased in $\min_{\theta \in \Omega - \Omega_0} \pi(\theta) > \max_{\theta \in \Omega_0} \pi(\theta)$

Suppose $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, σ^2 known.

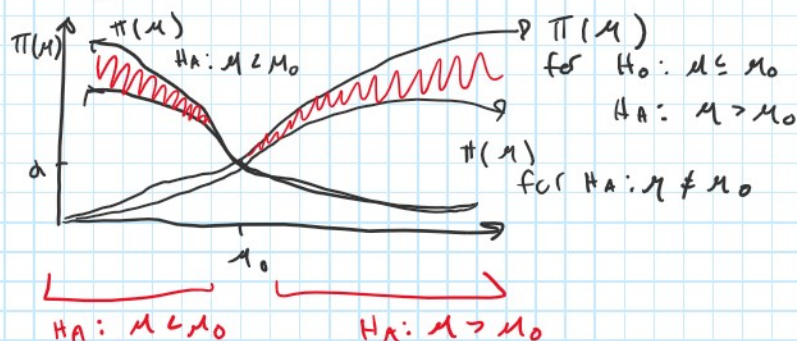
We want: $H_0: \mu = \mu_0$

$$H_A: \mu \neq \mu_0$$

The test that rejects for $\bar{X} \geq \mu_0 + z_{1-\alpha/2} \sigma/\sqrt{n}$ OR $\bar{X} \leq \mu_0 - z_{1-\alpha/2} \sigma/\sqrt{n}$

$$H_A: \mu \neq \mu_0$$

• The test that rejects for $\bar{X} \geq \mu_0 + z_{1-\alpha} \sigma/\sqrt{n}$ OR $\bar{X} \leq \mu_0 - z_{1-\alpha} \sigma/\sqrt{n}$ is not a UMP test because:



• Result pertaining to single parameter exponential families.

↳ Theorem 12.7.2

• Suppose $X_1, \dots, X_n \stackrel{iid}{\sim} f_X(x; \theta)$

• Suppose $f_X(x; \theta) = c(\theta) h(x) \exp \{q(\theta) + t(x)\}$ where $q(\theta)$ is an increasing function of θ .

1) A UMP test of $\begin{cases} H_0: \theta \leq \theta_0 \\ H_A: \theta > \theta_0 \end{cases}$

... rejects H_0 if $t(x) \geq k$ for $P[t(x) \geq k | H_0] = \alpha$

2) A UMP test of $\begin{cases} H_0: \theta \geq \theta_0 \\ H_A: \theta < \theta_0 \end{cases}$

... rejects H_0 if $t(x) \leq k$ for $P[t(x) \leq k | H_0] = \alpha$

Proof:

• If $\theta_1 < \theta_2$, then $q(\theta_1) < q(\theta_2)$ because $q(\cdot)$ is increasing.

• The ratio of the joint distributions is then:

$$\begin{aligned} \frac{f_X(x; \theta_2)}{f_X(x; \theta_1)} &= \frac{c(\theta_2) h(x) \exp \{q(\theta_2) + t(x)\}}{c(\theta_1) h(x) \exp \{q(\theta_1) + t(x)\}} \\ &= \frac{c(\theta_2)}{c(\theta_1)} \exp \{ \underbrace{[q(\theta_2) - q(\theta_1)]}_{>0} + t(x) \} \end{aligned}$$

• This is an increasing function of $t(x)$.

• This has an MLR in $t(x)$ ✓