Lecture 15

Tuesday, March 3, 2020 2:03 PM

Section 13.8

- Recall the Cramer-Rao-Von Mises-Test
  - Empirical distribution function

$$\hat{F}_x(t) = \frac{1}{n} \sum I(x_i \le t)$$

...and more specifically for different values t...

$$\hat{F}_x(t) = \begin{cases} 0 & t < x_{1:N} \\ i/n & x_i \le t \le x_{i+1} \\ 1 & t > x_{N:N} \end{cases}$$

There are two types of tests:

#1) Completely specified H_0

#2) Incompletely specified H_0

Recall:

$$E\{\hat{F}_x(t)\} = E\left\{\frac{1}{n}\sum_{i=1}^{n} I(x_i \le t)\right\}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \{\underbrace{I(x_i \le t)}_{=P\{X_i \le t\}}\}$$

$\therefore E\{\hat{F}_x(t)\} = F_x(t)$

↳ This is an unbiased estimator of the CDF under the null.

Then since $CM = \left(\frac{1}{12n}\right) + \sum_{i=1}^{n}\left(F_x(x_i) - \left(\frac{i-0.5}{n}\right)\right)^2$,

we reject for large values of CM_{OB} if:

$CM_{OB} > CM_{1-\alpha}$ (Table a, Appendix C, pg 613)

Example 13.8.1

H_0: $X \sim$ Discrete Uniform ($K = 100$)

↳ $F_X(x) = \frac{x}{100}$ for $x = 1, 2, \ldots, 100$

- Assume r = # of system failures = n

  thus, $r/n = 1$

Then, $F_X(t) = \frac{i}{25}$ for $x_{(i)} \le t < x_{(i+1)}$

Then, $CM = \frac{1}{12(25)} + \sum_{i=1}^{r} \left(\frac{x_i}{100} - \left(\frac{i - 0.5}{25} \right) \right)^2 \approx 0.182$

- if $\alpha = 0.1$, then $CM_{0.90} \; (r/n = 1) = 0.347$

$\therefore$ $0.182 \le 0.347$, thus we fail to reject the null.

Now suppose instead that we have a total of 50 parts ($n = 50$)

Then, $r/n = 0.5$, $T = 100 \Rightarrow H_0: X \sim \text{Discrete Uniform}(K = 100)$

$CM_{OBS} = \frac{1}{12(50)} + \sum_{i=1}^{r=25} \left(\frac{x_{(i)}}{100} - \left(\frac{i - 0.5}{50} \right) \right)^2 \approx 3.032$

and $CM_{0.9} \; (r/n = 0.5) = 0.189$

$\therefore$ $3.032 \ge 0.189$, thus we reject the null.

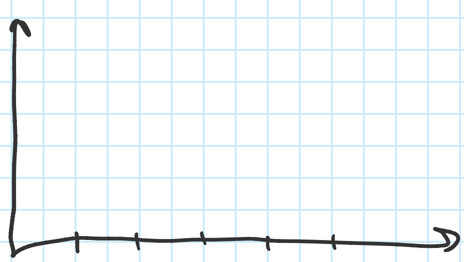
Now suppose we observe a Poisson Process,

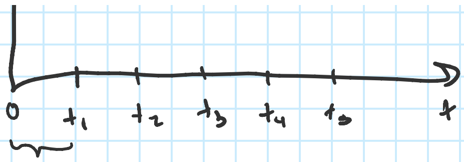
- X_t = count of event of interest up to some time t.
  - $X_t \sim \text{Poisson}(\lambda t)$

$\Rightarrow X_{[t, t+\delta]} \perp X_{[s, s+\delta']}$

$\Leftrightarrow$

waiting time for next occurrence of event is an iid process.





Y_i = waiting time for next failure

$H_0: Y_i \overset{iid}{\sim} Exp(\theta)$

#1 Fully specified $H_0: Y_i \underset{iid}{\sim} Exp(\theta = 5)$

$n = r = 25$

Failure	Time	Y_i (waiting time in-between failures)
1	5.2	5.2
2	13.6	8.4
3	14.5	0.9

$H_0: F_Y(y_i) = 1 - e^{-y_i/5}$

Thus, CM_{OB}:

$$CM_{OB} = \frac{1}{12(25)} + \sum_{i=1}^{25}\Big(\underbrace{F_Y(y_{(i)})}_{\text{CDF evaluated at ith ordered statistic}} - \Big(\frac{i-0.5}{25}\Big)\Big)^2 \approx 0.173$$

Then if $\alpha = 0.01$, $CM_{0.99}(\%n=1) = 0.743$

$\therefore$ we fail to reject the null

#2 Incompletely Specified H_0

- suppose that θ is unknown
- then we would estimate θ with $\hat{\theta}_{MLE} = \bar{y} = 3.7$
- then, $CM_{OB} \approx 0.7$ (text book says $CM_{OB} = 0.5$)

difference is due to rounding

$\therefore$ if $\alpha = 0.01$, we would now fail to reject the null.

Kolmogorov-Smirnov Test:

- this test is based on the largest observed difference between $F_X(x_{(i)})$ and $\hat{F}_X(x_{(i)})$.
- Suppose $H_0: F_X(x) \rightarrow$
  - General $F_X(x)$
  - $Exp(\theta)$

- Suppose $H_0: F_x(x) \rightarrow \begin{cases} \cdot \text{ General } F_x(x) \\ \cdot \text{ Exp}(\theta) \\ \cdot \text{ } N(\mu, \sigma^2) \\ \cdot \text{ weibull}(\theta, \beta) \end{cases}$

$$\begin{cases} D^+ = \max\limits_{\text{over } i} \left(\hat{F}_x(X_{(i)}) - F_x(X_{(i)}) \right) \\ \quad = \max\limits_{\text{over all } i} \left(i/n - F_x(X_{(i)}) \right) \\ D^- = \max\limits_{\text{over all } i} \left(F_x(X_{(i)}) - \frac{(i-1)}{n} \right) \end{cases}$$

$$D = \max(D^+, D^-)$$

- Now recall the example: $x \sim \text{Exp}(\theta)$
  - ↳ then the K.S. statistic (for $H_0: x \sim \exp(\theta)$)

$$= \left(\sqrt{n} + \underline{0.26} + \underline{0.5}/\sqrt{n}\right)\left(\hat{D} - \frac{\overline{0.2}}{n}\right)$$

* these values are constants which depend on the distribution from the null hypothesis

* pg: 613, table 11

if $\hat{\theta}_{MLE} = 3.7$, then

software says $D = 0.14$ (observed at $x_{(7)}$)

$\Rightarrow KS_{OB} \approx 0.584$

- then the critical value:

$KS_{0.99}(\text{Exp}) = 0.995$