HW 4.3-1

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• Apply classical Gram-Schmidt orthogonalization to find the full QR factorization of the following matrices:

a) $\begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$ | $q_1 = y_1 / \|y_1\|_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \frac{1}{(4^2+3^2)^{1/2}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \frac{1}{5} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$, $r_{11} = \|y_1\|_2 = 5$

• $y_2 = A_2 - q_1 q_1^T A_2$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} (3/5)$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 12/25 \\ 9/25 \end{bmatrix}$ $\quad q_2 = \dfrac{\begin{bmatrix} -12/25 \\ 16/25 \end{bmatrix}}{4/5} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = q_2$

$y_2 = \begin{bmatrix} -12/25 \\ 16/25 \end{bmatrix} \Rightarrow r_{22} = ((-12/25)^2 + (16/25)^2)^{1/2} = 4/5$

• then $r_{12} = q_1^T A_2 = \begin{bmatrix} 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3/5$

$r_{21} = q_2^T A_1 = \begin{bmatrix} -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 0$

$$A = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \underset{Q}{\begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}} \underset{R}{\begin{bmatrix} 5 & 3/5 \\ 0 & 4/5 \end{bmatrix}}$$

b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y_1$, $q_1 = y_1 / \|y_1\|_2 \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{(1+1)^{1/2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $r_{11} = \sqrt{2}$

$A_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $y_2 = A_2 - q_1 q_1^T A_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\dfrac{1}{1/\sqrt{2}} \quad \dfrac{1}{1} \Big/ 1/\sqrt{2}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} (3/\sqrt{2})$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

$y_2 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$, $q_2 = y_2 / \|y_2\|_2 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \cdot \frac{1}{(1/2)^{1/2}} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \cdot \sqrt{2} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = q_2$

$$A = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \underset{Q}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}}$$

c) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow A_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = y_1$, $q_1 = \frac{y_1}{\|y_1\|_2} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot 1/(2^2+1^2+2^2)^{1/2} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot 1/3 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = q_1$, $r_{11} = 3$

$A_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\quad y_2 = A_2 - q_1 q_1^T A_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} (1) = \begin{bmatrix} 1/3 \\ -4/3 \\ 1/3 \end{bmatrix} = y_2$, $q_2 = \begin{bmatrix} 1/3 \\ -4/3 \\ 1/3 \end{bmatrix} \cdot 1 \Big/ ((1/3)^2 + (-4/3)^2 + (1/3)^2)$

$= \begin{bmatrix} 1/3 \\ -4/3 \\ 1/3 \end{bmatrix} \cdot 1/\sqrt{2}$

$$A = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{bmatrix} 2/3 & 1/3\sqrt{2} \\ 1/3 & -4/3\sqrt{2} \\ 2/3 & 1/3\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$$

$q_2 = \begin{bmatrix} 1/3\sqrt{2} \\ -4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix}$

• since the input vectors

$\begin{bmatrix} r_{21} & r_{22} \end{bmatrix} \quad \begin{bmatrix} 2/3 & 1/3\sqrt{2} \end{bmatrix} \quad \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix}$

$$q_2 = \begin{bmatrix} 1/3\sqrt{2} \\ -4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix}$$

- since the input vectors are of length 3, we must enhance it with another vector.

$$A_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y_3 = A_3 - q_1 q_1^T A_3 - q_2 q_2^T A_3$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \underbrace{\begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{= 2/3} - \begin{bmatrix} 1/3\sqrt{2} \\ -4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} 1/3\sqrt{2} & -4/3\sqrt{2} & 1/3\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{= 1/3\sqrt{2}}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 4/9 \\ 2/9 \\ 4/9 \end{bmatrix} - \begin{bmatrix} 1/18 \\ -2/9 \\ 1/18 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \cdot 1 \Big/ \left((1/2)^2 + 0 + (-1/2)^2\right)^{1/2}$$

$$= \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \cdot \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

A full QR factorization is given by

$$\Rightarrow \underbrace{\begin{bmatrix} 2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ 1/3 & -4/3\sqrt{2} & 0 \\ 2/3 & 1/3\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 1 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}_{R}$$

D) $\begin{bmatrix} 4 & 8 & 1 \\ 0 & 2 & -2 \\ 3 & 6 & 7 \end{bmatrix}$, $A_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = y_1$, $r_{11} = (4^2 + 3^2)^{1/2} = 5$, $q_1 = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}$

$$y_2 = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix} \underbrace{\begin{bmatrix} 4/5 & 0 & 3/5 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix}}_{= 32/5 + 0 + 18/5}$$

$r_{12} = 10$

$$\Rightarrow y_2 = \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = y_2, \quad r_{22} = 2, \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} - \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix} \underbrace{\begin{bmatrix} 4/5 & 0 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}}_{= 4/5 + 21/5} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}}_{r_{23} = -2}$$

$r_{13} = 5$

$$\Rightarrow y_3 = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}, \quad r_{33} = (9 + 16)^{1/2} = 5, \quad q_3 = \begin{bmatrix} -3/5 \\ 0 \\ 4/5 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 4/5 & 0 & -3/5 \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 5 & 10 & 5 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{R}$$