HW 4.1-1

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Q: Solve the normal equations to find the least squares solution and 2-norm error for the following inconsistent systems.

a) $\begin{bmatrix}1&2\\0&1\\2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\1\\1\end{bmatrix}$

$a^Ta=\begin{bmatrix}1&0&2\\2&1&1\end{bmatrix}\begin{bmatrix}1&2\\0&1\\2&1\end{bmatrix}=\begin{bmatrix}(1)+(0)+(4)&(2)+(0)+(2)\\(2)+(0)+(2)&(4)+(1)+(1)\end{bmatrix}=\begin{bmatrix}5&4\\4&6\end{bmatrix}$

$a^Tb=\begin{bmatrix}1&0&2\\2&1&1\end{bmatrix}\begin{bmatrix}3\\1\\1\end{bmatrix}=\begin{bmatrix}3+0+2\\6+1+1\end{bmatrix}=\begin{bmatrix}5\\8\end{bmatrix}$

Then since $A^TA\bar{x}=A^Tb$

$6-4(4/5)=30/5-16/5=14/5=2.8$

$\left[\begin{array}{cc|c}5&4&5\\4&6&8\end{array}\right]\xrightarrow{R3-(4/5)R1}\begin{bmatrix}5&4&5\\0&14/5&4\end{bmatrix} \Rightarrow$

$14/5\,x_2=4 \Rightarrow 20/14=x_2 \Rightarrow 10/7=x_2$

$5x_1+4(10/7)=5 \Rightarrow\Rightarrow 5x_1=35/7-40/7 \Rightarrow x_1=-5/35 \Rightarrow -1/7=x_1$

$\boxed{\Rightarrow \bar{x}=\begin{bmatrix}-1/7\\10/7\end{bmatrix}}$

• Then the least squares approximation is given by

$A\bar{x}=\hat{b} \Rightarrow \begin{bmatrix}1&2\\0&1\\2&1\end{bmatrix}\begin{bmatrix}-1/7\\10/7\end{bmatrix}=\begin{bmatrix}-1/7+20/7\\0+10/7\\-2/7+10/7\end{bmatrix}=\begin{bmatrix}19/7\\10/7\\8/7\end{bmatrix}=\hat{b}$

• Then the residual vector is the difference between $b-\hat{b}$

$b=\begin{bmatrix}3\\1\\1\end{bmatrix}$, $\hat{b}=\begin{bmatrix}19/7\\10/7\\8/7\end{bmatrix}$, $b-\hat{b}=\begin{bmatrix}2/7\\-3/7\\-1/7\end{bmatrix}$, $\|b-\hat{b}\|_2=\sqrt{(2/7)^2+(-3/7)^2+(-1/7)^2}$

$=(4/49+9/49+1/49)^{1/2}$

$=(14/49)^{1/2}=\sqrt{14}/7$

$\boxed{\Rightarrow \|b-\hat{b}\|=\sqrt{14}/7}$

b) $\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix}$

$a^Ta=\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}=\begin{bmatrix}14&6\\6&3\end{bmatrix}$

$a^Tb=\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}1\\2\\0\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$

Then $a^Ta\bar{x}=a^Tb$ gives $\left[\begin{array}{cc|c}14&6&5\\6&3&3\end{array}\right]\xrightarrow{R2-6/14\,R1}\begin{bmatrix}14&6&5\\0&4/7&6/7\end{bmatrix}$

$4/7\,x_2=6/7 \Rightarrow x_2=3/2 \Rightarrow$

$14x_1+6(3/2)=5$
$14x_1+9=5$
$x_1=-2/7$

$\Rightarrow \boxed{\bar{x}=\begin{bmatrix}-2/7\\3/2\end{bmatrix}}$

• Then by least squares

$A\bar{x}=\hat{b} \Rightarrow \begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\begin{bmatrix}-2/7\\3/2\end{bmatrix}=\begin{bmatrix}-2/7+3/2\\-4/7+3/2\\-6/7+3/2\end{bmatrix}=\begin{bmatrix}-4/14+21/14\\-8/14+21/14\\-12/14+21/14\end{bmatrix}=\begin{bmatrix}17/14\\13/14\\9/14\end{bmatrix}=\hat{b}$

• $b-\hat{b}=\begin{bmatrix}1\\2\\0\end{bmatrix}-\begin{bmatrix}17/14\\13/14\\9/14\end{bmatrix}=\begin{bmatrix}-3/14\\15/14\\-9/14\end{bmatrix} \Rightarrow \sqrt{(-3/14)^2+(15/14)^2+(-9/14)^2}$

$=[(9/196)+(225/196)+(81/196)]^{1/2}$

$=(315/196)^{1/2}$

$\boxed{\|b-\hat{b}\|=3\sqrt{35}/14}$

c) $\begin{bmatrix}1&2\\1&1\\2&1\\2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\3\\3\\2\end{bmatrix}$

$a^Ta=\begin{bmatrix}1&1&2&2\\2&1&1&2\end{bmatrix}\begin{bmatrix}1&2\\1&1\\2&1\\2&2\end{bmatrix}=\begin{bmatrix}1+1+4+4&2+1+2+4\\2+1+2+4&4+1+1+4\end{bmatrix}=\begin{bmatrix}10&9\\9&10\end{bmatrix}$

$a^Tb=\begin{bmatrix}1&1&2&2\\2&1&1&2\end{bmatrix}\begin{bmatrix}3\\3\\3\\2\end{bmatrix}=\begin{bmatrix}3+3+6+4\\6+3+3+4\end{bmatrix}=\begin{bmatrix}16\\16\end{bmatrix}$

$a^Ta\bar{x}=a^Tb \Rightarrow \left[\begin{array}{cc|c}10&9&16\\9&10&16\end{array}\right]\xrightarrow{R3=R3-(9/10)R1}\begin{bmatrix}10&9&16\\0&19/10&16/10\end{bmatrix} \Rightarrow x_2=16/19$

$\Rightarrow 10x_1+9(16/19)=16$

$=10x_1+144/19=304/19$

$\Rightarrow x_1=160/190=16/19$

$\Rightarrow \boxed{\bar{x}=\begin{bmatrix}16/19\\16/19\end{bmatrix}}$

Then by least squares

$A\bar{x}=\hat{b} \Rightarrow \begin{bmatrix}1&2\\1&1\\2&1\\2&2\end{bmatrix}\begin{bmatrix}16/19\\16/19\end{bmatrix}=\begin{bmatrix}16/19+32/19\\16/19+16/19\\32/19+16/19\\32/19+32/19\end{bmatrix}=\begin{bmatrix}48/19\\32/19\\48/19\\64/19\end{bmatrix}=\hat{b}$

$\|b-\hat{b}\|=\begin{bmatrix}3\\3\\3\\2\end{bmatrix}-\begin{bmatrix}48/19\\32/19\\48/19\\64/19\end{bmatrix}=\begin{bmatrix}9/19\\25/19\\9/19\\-26/19\end{bmatrix} \Rightarrow \left((9/19)^2+(25/19)^2+(9/19)^2+(-26/19)^2\right)^{1/2}$

$=81/361+625/361+81/361+676/361$

$=(1463/361)^{1/2}$

$\boxed{\|b-\hat{b}\|=2.013}$