HW 1.4-3

Friday, June 26, 2020 11:52 AM

- Use theorem 1.11 or 1.12 to estimate the error e_{i+1} in terms of the previous error e_i as Newton's Method converges to the given roots. Is the convergence linear or quadratic?

a) $x^5 - 2x^4 + 2x^2 - x = 0$; $r = -1$, $r = 0$, $r = 1$

Theorem 1.11

- Let f be twice continuously differentiable and $f(r) = 0$. If $f'(r) \neq 0$, then Newton's method is locally and quadratically convergent to r. The error e_i at step i satisfies

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i^2} = M, \text{ where } M = \frac{f''(r)}{2f'(r)}$$

- $f'(x) = 5x^4 - 8x^3 + 4x - 1$
- $f''(x) = 20x^3 - 24x^2 + 4$
- $f'(r=-1)$: $5(-1)^4 - 8(-1)^3 + 4(-1) - 1$

$= -5 + 8 - 5$

$= 8 \neq 0$

- $e_{i+1} = \left|\frac{f''(r=-1)}{2f'(r=-1)}\right| e_i^2 = \left|\frac{20(-1)^3 - 24(-1)^2 + 4}{16}\right| e_i^2 = \left|\frac{-20 - 24 + 4}{16}\right| e_i^2 = \left|\frac{-40}{16}\right| e_i^2 = \frac{5}{2} e_i^2$

$\therefore$ when $r = -1$, $e_{i+1} = (5/2) e_i^2$ and Newton's Method convolges quadratically to the root $r = -1$ since $f'(-1) \neq 0$

- $f'(r = 0)$: -1
  - $e_{i+1} = \left|\frac{f''(r=0)}{2f'(r=0)}\right| e_i^2 = \left|\frac{4}{-2}\right| e_i^2 = 2e_i^2$

$\therefore$ when $r = 0$, $e_{i+1} = 2e_i^2$ and Newton's Method is quadratically convergent to the root $r = 0$ since $f'(0) \neq 0$

- $f'(r = 1)$: $5(1)^4 - 8(1)^3 + 4(1) - 1$

$= 5 - 8 + 4 - 1$

$= 0$

- thus we must find the multiplicity of the root $r = 1$
  - $f''(r) = f''(1) = 20(1)^3 - 24(1)^2 + 4$

  $= 20 - 24 + 4$

  $= 0$

  - $f'''(x) = 60x^2 - 48x$
  - $f'''(r=1) = 60(1)^2 - 48(1)$

  $= 12 \neq 0$

    - thus the root $r = 1$ has the multiplicity $m = 3$
    - $e_{i+1} = \frac{m-1}{m} e_i = \frac{3-1}{3} e_i = \frac{2}{3} e_i$

- thus the root $r=1$ has the multiplicity $m=3$
  - $e_{i+1} = \frac{m-1}{m} e_i = \frac{3-1}{3} e_i = \frac{2}{3} e_i$

  $\therefore$ thus when $r=1$, $e_{i+1} = \frac{2}{3} e_i$ and Newton's Method is linearly convergent to the root $r=1$ since $f'(r=1) = 0$

b) $f(x) = 2x^4 - 5x^3 + 3x^2 + x - 1 = 0$, $r = -\frac{1}{2}$, $r = 1$

- $f'(x) = 8x^3 - 15x^2 + 6x + 1$
- $f''(x) = 24x^2 - 30x + 6$
- Now checking $f'(r = -\frac{1}{2})$
  - $f'(r=-1/2) = 8(-1/2)^3 - 15(-1/2)^2 + 6(-1/2) + 1$

    $= 8(-1/8) - 15(1/4) + 3 + 1$

    $= -27/4 \neq 0$
  - then $e_{i+1} = \left|\frac{24(-1/2)^2 - 30(-1/2) + 6}{-54/4}\right| e_i^2 = \left|\frac{6 + 15 + 6}{-54/4}\right| e_i^2 = \left|\frac{-108}{54}\right| e_i^2 = 2e_i^2$

  $\therefore$ thus when $r=-1/2$, $e_{i+1} = 2e_i^2$ and Newton's Method converges quadratically since $f'(r=-1/2) \neq 0$
- Now checking $f'(r=1)$
  - $f'(r=1) = 8(1)^3 - 15(1)^2 + 6(1) + 1$

    $= 8 - 15 + 7$

    $= 0$
  - since $f'(r=1) = 0$, we must find the multiplicity of the root at $r=1$
    - $f''(r=1) = 24(1)^2 - 30(1) + 6$

      $= 0$
    - $f'''(x) = 48x - 30$
      - $f'''(r=1) = 18 \neq 0$
        - thus $m = 3$
    - $e_{i+1} = \frac{m-1}{m} e_i = \frac{2}{3} e_i$

    $\therefore$ thus when $r=1$, $e_{i+1} = (2/3) e_i$ and Newton's Method converges linearly to the root $r=1$ since $f'(r=1) = 0$