Lower and Upper Reliability Bounds for Consecutive-k-Out-of-n:F Systems

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***Abstract*—After a comprehensive review of reliability bounds for consecutive-k-out-of-n:F systems with statistically independent components having the same failure probability q (i.i.d. components), we introduce new classes of lower and upper bounds. Our approach is different from previous ones, and relies on alternating summation as well as on the monotony of some particular sequences of real numbers. The starting point is represented by the original formula given by de Moivre; and, by considering partial sums approximating it, new lower and upper bounds on the reliability of a consecutive-k-out-of-n:F system have been established. Simulation results show that all the lower and upper bounds considered here present very similar behaviors, as all of them are exponentially closing in on the exact reliability of a consecutive-k-out-of-n:F system. Additionally, the accuracy of the different bounds depends not only on the particular values of k and n, but also on the particular range of q (some of the new bounds being the most accurate ones over certain ranges).**

***Index Terms*—Consecutive-k-out-of-n:F system, lower and upper bounds.**

NOTATIONS

n	total number of components of the system
k	minimum number of consecutive faulty components, which cause a system failure
p	probability that a component works correctly
q	probability that a component fails ($q = 1 - p$)
$R(k,n;q)$	reliability of a consecutive-k-out-of-n:F system with component failure probability q
$P_{n,k}$	probability of a run of length k in n trials ($P_{n,k} = 1 - R(k,n;q)$)
$\beta_{m,j}$	alternate summation, defined by (1)
C_m^j	combinations of m taken j times (without repetition)
a_j, b_j	sequences of decreasing positive numbers (defined in *Proposition* 2; see also (21))
S_r	alternate summation of a_j (subpart of $\beta_{n,k}$), defined by (27); $S_0 = 1$
S'_r	alternate summation of b_j (subpart of $\beta_{n-k,k}$), defined by (28); $S'_0 = 1$
L_r, U_r	lower, upper bounds of order r for $R(k,n;q)$ defined by (18), and (19)

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I. INTRODUCTION

A consecutive-k-out-of-n:F system is defined as a system having n components placed in a row (*i.e.*, sequentially), which fails iff at least k consecutive components fail. This concept was introduced by Kontoleon in 1980 [1] as an r-succesive-out-of-n:F system, r being used instead of k. Examples of this system structure include particular telecommunication systems, microwave broadcasting systems, oil pipeline systems, vacuum systems in accelerators, computer ring networks, supervision systems, pattern detection systems, and even rows of street lights. One year later, Chiang & Niu [2] discussed some of the possible applications of such systems, and also established the now well-known name of the consecutive-k-out-of-n:F system. Many generalizations have been introduced later [3]–[9], while the interested reader can consult a book on this topic [10], as well as three reviews [11]–[13].

In this paper, we shall consider consecutive-k-out-of-n:F systems having n identical and statistically independent components, with all components having a constant probability of failure q (i.i.d. components). Consecutive-k-out-of-n:F systems are of interest as they can achieve extremely high reliability at very low costs (*i.e.*, very low redundancy factors) [14]. Additionally, they are a good fit for novel nano-architectures (where reliability is of high interest) relying on beyond near-neighbor communications (where the reliability of transmissions would also need to be considered). Examples of technologies which allow for beyond near-neighbor communications, hence matching these types of consecutive systems, include magnetic and molecular technologies, as well as multiple gate field effect transistors.

This paper will start by reviewing previous results obtained for such systems in Section II, while the new upper and lower bounds are detailed in Section III. Simulations of consecutive-k-out-of-n:F systems (which are able to operate reliably regardless of how unreliable the elementary components are) are going to be presented and discussed in Section IV, and will be followed by conclusions.

II. State-of-the-Art

A. Exact Formulas

The reliability of a consecutive-k-out-of-n:F system has been calculated exactly. Still, as far as we know, this is the first time that this problem is linked to an equivalent nontrivial well-known problem which can be traced back to the beginning of the 18th century and the famous book *The Doctrine of Chances* [15]. It is in this book that de Moivre proposed and solved *Problem LXXIV: "To find the Probability of throwing a Chance assigned a given number of times without intermission, in any given number of Trials."* Assuming that an event E (*"a Chance"*) occurs with constant probability p (*"probability of throwing"*) in a single trial, if in a series of n independent trials (*"any given number of Trials"*) the event E occurs at least k times (*"a given number of times"*) in succession (*"without intermission"*), we say that we have a run of length k (obviously, $k < n$). Problem LXXIV as stated by de Moivre (on [15, page 254]) is nothing else but the probability of having k successes (*i.e.*, a run of length k) in n trials, $P_{n,k}$. The solution detailed by de Moivre in [15] was based on deriving the generating function of the complementary probability. This classical problem was rephrased in present day mathematical writing style in [16], where the probability of a run of length k in n trials ($P_{n,k}$) was determined following de Moivre's approach (see Problem 2 on page 77; in [16, Chapter V]).

Proposition 1: If we denote —

$$\beta_{m,k} = \sum_{j=0}^{\lfloor m/(k+1)\rfloor} (-1)^j C_{m-jk}^{j} \left(pq^k\right)^j, \tag{1}$$

the probability of a run of length k in n trials is

$$P_{n,k} = 1 - \beta_{n,k} + q^k \beta_{n-k,k}.$$

It is a straightforward consequence of *Proposition* 1 that the reliability $R\,(k,n;q)$ of a consecutive-k-out-of-n:F system is

$$R\,(k,n;q) = \beta_{n,k} - q^k \beta_{n-k,k}, \tag{2}$$

where $\beta_{n,k}$ and $\beta_{n-k,k}$ are given by (1).

It was not until the 1980s [17]–[19] that the Markov chain method was used for studying the reliability of consecutive-k-out-of-n:F systems. This work led to the compact matrix formula

$$R\,(k,n;q) = (1,0,\ldots,0) \times M^n \times (1,\ldots,1,0)^t, \tag{3}$$

where M is a square matrix of size $(k+1)\times(k+1)$ representing the transition probability matrix

$$M = \begin{pmatrix} p & q & 0 & \ldots & 0 \\ p & 0 & q & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & 0 & 0 & \cdots & q \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

while $(1,\ldots,1,0)^t$ is the transpose (column) of the row vector $(1,\ldots,1,0)$.

Both (2) and (3) compute $R\,(k,n;q)$ exactly, but a noteworthy difference between them is that while (2) relies on alternate summation (*i.e.*, a sum of real numbers of alternating signs $x_0 - x_1 + x_2 - \cdots$), (3) is a summation of positives ($x_0 + x_1 + x_2 + \cdots$, as multiplication of matrices of positive entries). Thus, (2) should lead to an alternating approach towards $R\,(k,n;q)$ (alternating higher and lower estimates as the summation is evaluated), while (3) will always approach $R\,(k,n;q)$ from below (the summation increasing without exceeding $R\,(k,n;q)$).

In the following two sub-sections, we shall go in chronological order over many of the lower and upper bounds determined over the last three decades (see also [20], [21]), reviewing the current state-of-the-art for establishing what we should improve upon.

B. Lower Bounds

In 1981, Chiang & Niu introduced the lower bound ((5) in [2])

$$\left(1-q^k\right)^{n-k+1} \le R(k,n;q). \tag{4}$$

This lower bound was rediscovered later by Fu in 1985 ((5) and (20) in [22]), and 1986 ([23, (3.4)]); by Papastavridis & Koutras in 1993 (Theorem 1 in [24]); as well as by Muselli in 1997 ((5) in [25]).

Salvia in 1982 ((3) in [26]) proposed a new lower bound

$$1-(n-k+1)\,q^k \le R(k,n;q). \tag{5}$$

It took a decade until Barbour *et al.* [27], using a different approach, established

$$e^{-(n-k+1)pq^k} - (2kp-1)\,q^k \le R(k,n;q).$$

This bound was later improved by Barbour *et al.* [28] as

$$\nu_1 - \varepsilon_1 \le R(k,n;q) \tag{6}$$
$$\nu_1 = e^{-(n-k+1)pq^k} - q^{k+1}e^{-(n-2k)pq^k} \tag{7}$$
$$\varepsilon_1 = \left\{1 - e^{-(n-k+1)pq^k} + q^{k+1}\left[1-e^{-(n-2k)pq^k}\right]\right\} \times (2k+1)\,pq^k. \tag{8}$$

Finally, in 2000, Muselli presented several lower bounds (some of which were detailed in technical reports one or two years earlier) as follows ((5) from [29] as well as (4) and (9) in [30]) .

$$\left(1-q^k\right)^{1+(n-k)p/\left(1-q^k\right)^k} \le R(k,n;q) \tag{9}$$

which holds for $\max(q/p,1) \le k$, and also ((13) and (14) in [29])

$$\left(1-q^k\right)^{n-k+1-l(q,k)[h(q,k)-1]} \le R(k,n;q),$$
$$l\,(q,k) = \left\lfloor \frac{n-k}{h\,(q,k)+1} \right\rfloor,$$
$$h\,(q,k) = \left\lfloor \frac{1-q^k}{p} \right\rfloor, \tag{10}$$

which hold for $k \le n - h\,(q,k)$.

Corollary 3 in [29] detailed another lower bound

$$\left(1-q^k\right)^{2l'(q,k)} \le R(k,n;q),$$
$$l'\,(q,k) = \left\lceil \frac{n-k+1}{h\,(q,k)+1} \right\rceil, \tag{11}$$

TABLE I
LOWER BOUNDS FOR THE RELIABILITY OF A CONSECUTIVE-k-OUT-OF-n:F SYSTEM.

Year	Author(s)	Lower Bound	Supplementary	Condition(s)	Color	Ref.	Eq(s).
1981	Chiang & Niu	$(1-q^k)^{n-k+1}$			Cyan	[2]	(5)
1982	Salvia	$1-(n-k+1)q^k$			Black	[26]	(3)
1992	Barbour *et al.*	$e^{-(n-k+1)pq^k}-(2kp-1)q^k$			Blue	[27]	
1995	Barbour *et al.*	$\nu_1-\epsilon_1$	$\nu_1=e^{-(n-k+1)pq^k}-q^{k+1}e^{-(n-2k)pq^k}$ $\epsilon_1=(2k+1)\{1-e^{-(n-k+1)pq^k}+q^{k+1}[1-e^{-(n-2k)pq^k}]\}pq^k$		Blue	[28]	
2000	Muselli	$(1-q^k)^{1+(n-k)p/(1-q^k)^k}$		$max(q/p,1)\le k$	Green	[29] [30]	(5) (4), (9)
2000	Muselli	$(1-q^k)^{n-k+1-l(q,k)[h(q,k)-1]}$	$l(q,k)=\lfloor(n-k)/[h(q,k)+1]\rfloor$ $h(q,k)=\lfloor(1-q^k)/p\rfloor$	$k\le n-h(q,k)$	Yellow	[29]	(13) (14)
2000	Muselli	$(1-q^k)^{2l'(q,k)}$	$l'(q,k)=\lceil(n-k+1)/[h(k,q)+1]\rceil$ $h(q,k)=\lfloor(1-q^k)/p\rfloor$		Red	[29]	Corollary 3
2000	Muselli	$(1-q^k)^{1+(n-k)/h_L(q,k)}$	$h_L(q,k)=(1-q^k)^{kp/(1-q^k)^k}/p$	$max(q/p,1)\le k$	Red	[30]	(8), (9) (15)

TABLE II
UPPER BOUNDS FOR THE RELIABILITY OF A CONSECUTIVE-k-OUT-OF-n:F SYSTEM.

Year	Author(s)	Upper Bound	Supplementary	Condition(s)	Color	Ref.	Eq(s).
1981	Chiang & Niu	$(1-q^k)^{\lfloor n/k\rfloor}$			Cyan	[2]	(6)
1982	Salvia	$1-(n-k+1)p^{n-k}q^k$			Black	[26]	(3)
1985	Fu	$(1-q^k+q^{k+1})^{n-k+1}$ or $(1-pq^k)^{n-k+1}$			Yellow	[22]	(5), (22)
1992	Barbour *et al.*	$e^{-(n-k+1)pq^k}+(2kp-1)q^k$			Blue	[27]	
1995	Barbour *et al.*	$\nu_1+\epsilon_1$	$\nu_1=e^{-(n-k+1)pq^k}-q^{k+1}e^{-(n-2k)pq^k}$ $\epsilon_1=(2k+1)\{1-e^{-(n-k+1)pq^k}+q^{k+1}[1-e^{-(n-2k)pq^k}]\}pq^k$		Blue	[28]	(2-3)
2000	Muselli	$(1-q^k)^{1+(n-k)p/(1-q^k)}$			Red	[29]	(4), (5)

while (8) and (15) in [30] revealed

$$\left(1-q^k\right)^{1+(n-k)/h_L(q,k)} \le R(k,n;q),$$
$$h_L(q,k)=\frac{\left(1-q^k\right)^{kp/\left(1-q^k\right)^k}}{p}, \quad (12)$$

which is valid for $\max(q/p,1)\le k$.

All these lower bounds are presented in a compact form in Table I.

C. Upper Bounds

Upper bounds were in most but not all cases introduced together with lower bounds. In 1981, Chiang & Niu proposed the upper bound ((6) in [2])

$$\left(1-q^k\right)^{\lfloor n/k\rfloor} \ge R(k,n;q). \quad (13)$$

This upper bound was rediscovered by Muselli in 1997 ((5) in [25]).

Salvia in 1982 ((3) in [26]) proved that

$$1-(n-k+1)\,p^{n-k}q^k \ge R(k,n;q). \quad (14)$$

Fu in 1985 ((5) and (22) in [22]; also in [31]), and Papastavridis & Koutras in 1993 (Theorem 1 in [24]) presented

$$\left(1-q^k+q^{k+1}\right)^{n-k+1}=\left(1-pq^k\right)^{n-k+1} \ge R(k,n;q). \quad (15)$$

In [32], it was mentioned that this upper bound is in fact equivalent to one described by Gončarov in 1944 [33].

In 1992, Barbour *et al.* [27] established that

$$e^{-(n-k+1)pq^k}+(2kp-1)\,q^k \ge R(k,n;q),$$

and later improved on it [28] as

$$\nu_1+\epsilon_1 \ge R(k,n;q) \quad (16)$$

with ν_1, and ε_1 given respectively by (7), and (8).

Finally, in 2000, Muselli showed that ((5) from [29])

$$\left(1-q^k\right)^{1+(n-k)p/\left(1-q^k\right)} \ge R(k,n;q). \quad (17)$$

All these upper bounds are presented in a compact form in Table II.

III. NEW LOWER AND UPPER BOUNDS

One way to classify the different bounds for $R(k,n;q)$ presented in Section II could be based on the methods they have relied upon, as follows.

- Product-type inequalities include those given by Fu [23] (probabilistic), Chao & Fu [34], Papastavridis & Koutras [24] (analyzing the roots of the denominator of the generating function), and Muselli [30].

- The Stein-Chen method was used by Chryssaphinou & Papastavridis [35], and Barbour *et al.* [27], [28] (which perform well for smaller q, and larger k).
- The use of Bonferroni's inequalities is due to Derman *et al.* [36].

The new lower and upper bounds to be presented here are obtained based on a completely different approach starting from (1) and (2). This new approach leads to an entire family of lower and upper bounds. The following two propositions prove the existence of the new families of lower and upper bounds for $R(k,n;q)$.

Proposition 2: Let $n,k \in \mathbf{N}^*$, $n > k$, and $p \in (0,1)$. We denote $a_j = C^j_{n-jk}\left(pq^k\right)^j$ for any $j = 0,1,\ldots,\lfloor n/(k+1)\rfloor$, and $b_j = C^j_{n-k-jk}\left(pq^k\right)^j$ for any $j = 0,1,\ldots,\lfloor (n-k)/(k+1)\rfloor$. If $1/(n-k) > pq^k$, then the following are true.

1. $a_j > a_{j+1}$ for any $j = 0,1,\ldots,\lfloor n/(k+1)\rfloor - 1$.
2. $b_j > b_{j+1}$ for any $j = 0,1,\ldots,\lfloor (n-k)/(k+1)\rfloor - 1$.

Proof: See the Appendix.

Proposition 3: If $1/(n-k) > pq^k$, we have

$$L_r \le R(k,n;q) \le U_r,$$

where

$$L_r = \sum_{j=0}^{2r+1} (-1)^j C^j_{n-jk}\left(pq^k\right)^j - q^k \sum_{j=0}^{2r} (-1)^j C^j_{n-k-jk}\left(pq^k\right)^j, \quad (18)$$

and

$$U_r = \sum_{j=0}^{2r} (-1)^j C^j_{n-jk}\left(pq^k\right)^j - q^k \sum_{j=0}^{2r+1} (-1)^j C^j_{n-k-jk}\left(pq^k\right)^j \quad (19)$$

for any $r = 0,1,\ldots,\lfloor (n-2k-1)/(2k+2)\rfloor$.

Proof: See the Appendix.

Obviously, by increasing the index r, one can get increasingly tighter lower and upper bounds, which are being traded for increasingly complex formulas. Still, as we shall see in the next section, these lower and upper bounds give very accurate approximations of $R(k,n;q)$, even for the smallest values of r (*i.e.*, $r = 0$ and $r = 1$).

IV. Simulation Results

Besides all the lower and upper bounds reviewed in Section II, we will also use two lower and two upper bounds which follow from *Proposition* 3. They are for the particular values $r = 0$ and $r = 1$. Based on (18) and (19), we have

$$L_0 = 1 - [(n-k)p + 1]q^k,$$
$$U_0 = 1 - \left[1 - (n-2k)pq^k\right]q^k;$$

and

$$L_1 = 1 - (n-k)pq^k + \frac{(n-2k-1)(n-2k)}{2}\left(pq^k\right)^2 - \frac{(n-3k-2)(n-3k-1)(n-3k)}{6}\left(pq^k\right)^3 - q^k\left[1 - (n-2k)pq^k + \frac{(n-3k-1)(n-3k)}{2}\left(pq^k\right)^2\right],$$

$$U_1 = 1 - (n-k)pq^k + \frac{(n-2k-1)(n-2k)}{2}\left(pq^k\right)^2 - q^k\left[1 - (n-2k)pq^k + \frac{(n-3k-1)(n-3k)}{2}\left(pq^k\right)^2 - \frac{(n-4k-2)(n-4k-1)(n-4k)}{6}\left(pq^k\right)^3\right].$$

All the lower bounds presented in Table I together with L_0 and L_1, as well as all the upper bounds presented in Table II together with U_0 and U_1, were coded in Matlab. Although we are aware of a few other articles which have presented different lower and upper bounds (*e.g.*, [37], [38]), we did not include them here as there seem to be errors in the formulas they have presented. For simplicity (and uniformity), the particular conditions of the different bounds were not implemented, leading to continuous simulation results over the whole range considered $q = 0.001\ldots 1$. This simplification made some bounds exhibit abnormal behaviors for $q > 0.5$ (if the particular conditions are not met, those bounds are in fact not defined). Hence, some of the simulation results are illustrative only in the range $q = 0.001\ldots 0.5$.

We have computed $R(k,n;q)$ exactly using both (2) and (3). It became apparent that (3) is numerically more stable than (2) at very high precision and very large n. In fact, (2) has a saturating behavior revealing oscillations at high precision (which are most certainly due to alternating summations). Fig. 1 presents $R(k,n;q) - Lower\ Bound$ as a function of q, while Fig. 2 presents $Upper\ Bound - R(k,n;q)$. The step used for all the simulations was $\Delta q = 0.001$, while we have used $n = 10^1, 10^2, 10^3, 10^4$, and have taken $k = 1 + \log_{10} n$. Both horizontal and vertical axes for all the plots are logarithmic (the base being 10).

Fig. 1 shows that, although the formulas for the lower bounds are different, some of the lower bounds behave quite similarly. In particular, the lower bounds given by (4), (5), (10), and (11) (introduced in [2], [26], and [29]) are a first group we could refer to as an overlapping set of lower bounds. Another overlapping set of lower bounds is represented by (6), (9), and (12) (introduced in [28]–[30]). L_0 falls between these two sets, while L_1 seems to be the best for small values of q, n, and k.

The upper bound plots reveal very similar behaviors, as can be easily understood by comparing Fig. 2 with Fig. 1. The upper bounds given by (13) and (14) (introduced in [2] and [26]) represent the first overlapping set. The second overlapping set of upper bounds is represented by (15), (16), and (17) (introduced in [22], [28], and respectively [29]). U_0 clearly belongs to the first overlapping set of upper bounds. On the other side, similarly to L_1, U_1 is again the best for small values of q, n, and k.

From Figs. 1 and 2, it becomes apparent that, for all the lower and upper bounds B analyzed here, there is a particular value q_0 (weakly depending on both k and n), starting from which the absolute errors $\log_{10}|R(k,n;q) - B|$ depend linearly on $\log_{10} q$, as all the plots look like straight lines for $q < q_0$ (q_0 could be taken as 0.1; see Figs. 1 and 2). Therefore,

$$\log_{10}|R(k,n;q) - B| = a \cdot \log_{10} q + b \quad (20)$$

where $a, b > 0$. Because there exists $c > 0$ such that $b = \log_{10} c$, (20) can be rewritten as

$$\log_{10}|R(k,n;q) - B| = \log_{10}(c \cdot q^a),$$

which gives

$$B = R(k,n;q) \pm c \cdot q^a.$$

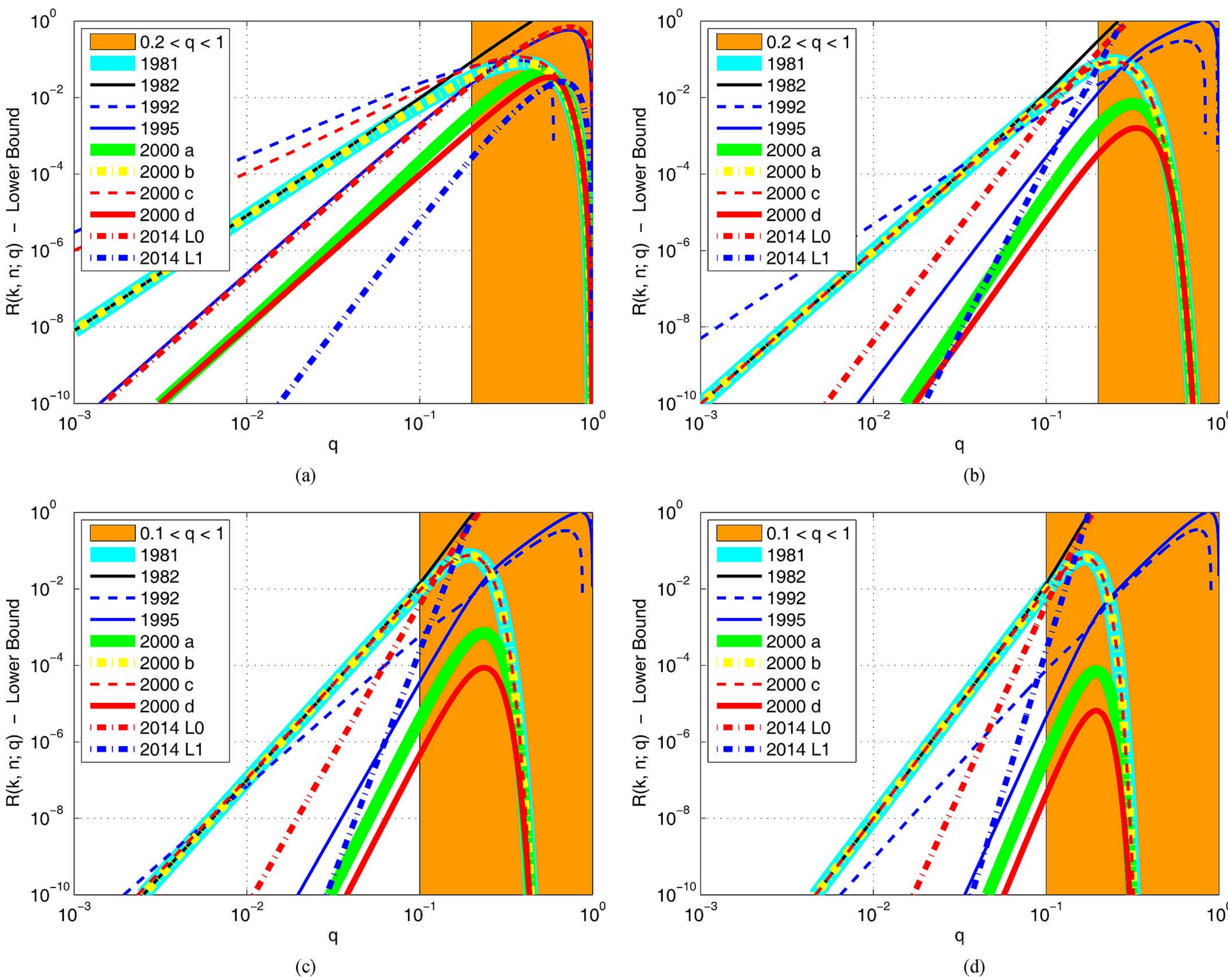

Fig. 1. Lower bounds absolute errors for consecutive systems: (a) 2-out-of-10, (b) 3-out-of-100, (c) 4-out-of-1000, and (d) 5-out-of-10000.

It follows that, for small enough q ($q < q_0$), the absolute approximation errors of all the lower and upper bounds presented here are off by an exponential ($c \cdot q^a$). Obviously, the larger a and the smaller c the better (*i.e.*, more accurate) the approximation. Of all the bounds presented here, L_1 and U_1 seem to exhibit the largest a values (as they are the steepest), while (12) and (17) exhibits the smallest c values (as they start lower than any other bound).

Lastly, the usefulness of such bounds will be only briefly suggested here through an example [39]. While algorithms for calculating $R(k,n;q)$ exactly even for large n and k are known [40]–[42], doing so takes time and requires memory (especially when this has to be repeated many times). Fig. 3(a) presents a 3D view of the reliability of consecutive-k-out-of-n:F systems for $q = 0.5$, *i.e.*, $R(k,n;0.5)$. In fact, such 3D plots should be evaluated for different q, ideally in steps of 0.001 (as done for Figs. 1 and 2), and for n as large as 10^6 [14]. These plots require repeated calculations for all (n,k) pairs. Various exact algorithms ranging from $O(k^3 \log(n/k))$ [40] to $O(k \log k \log(\log k) \log n + k^w)$ (with $2 < w < 3$) [42] arithmetic operations have been proposed. Obviously, using bounds (like the ones presented in Sections II and III) would require only a fixed number of arithmetic operations per (n,k) pair. A comparison of four different solutions with respect to the total number of arithmetic operations required for generating a 3D plot like the one from Fig. 3(a) is shown in Fig. 3(b) (*i.e.*, for the case when $n \times k = 5n \log n$). For this case, the exact algorithms presented in [41], [42] are about $10\times$ faster than the algorithm presented in [40], while using a bound (like the ones presented in this paper) is about $100\times$ faster than the algorithms from [41], [42], and $1000\times$ faster than [40].

V. Conclusions

A new class of lower and upper bounds for the reliability of consecutive-k-out-of-n:F systems was introduced in this paper. The approach used is different from previous ones, and starts from the original formula given by de Moivre.

All the lower and upper bounds analyzed here behave very similarly, exponentially approaching the exact reliability of a consecutive-k-out-of-n:F system. The new upper and lower bounds perform quite well when compared to other known lower and upper bounds. In particular, they seem to be the most

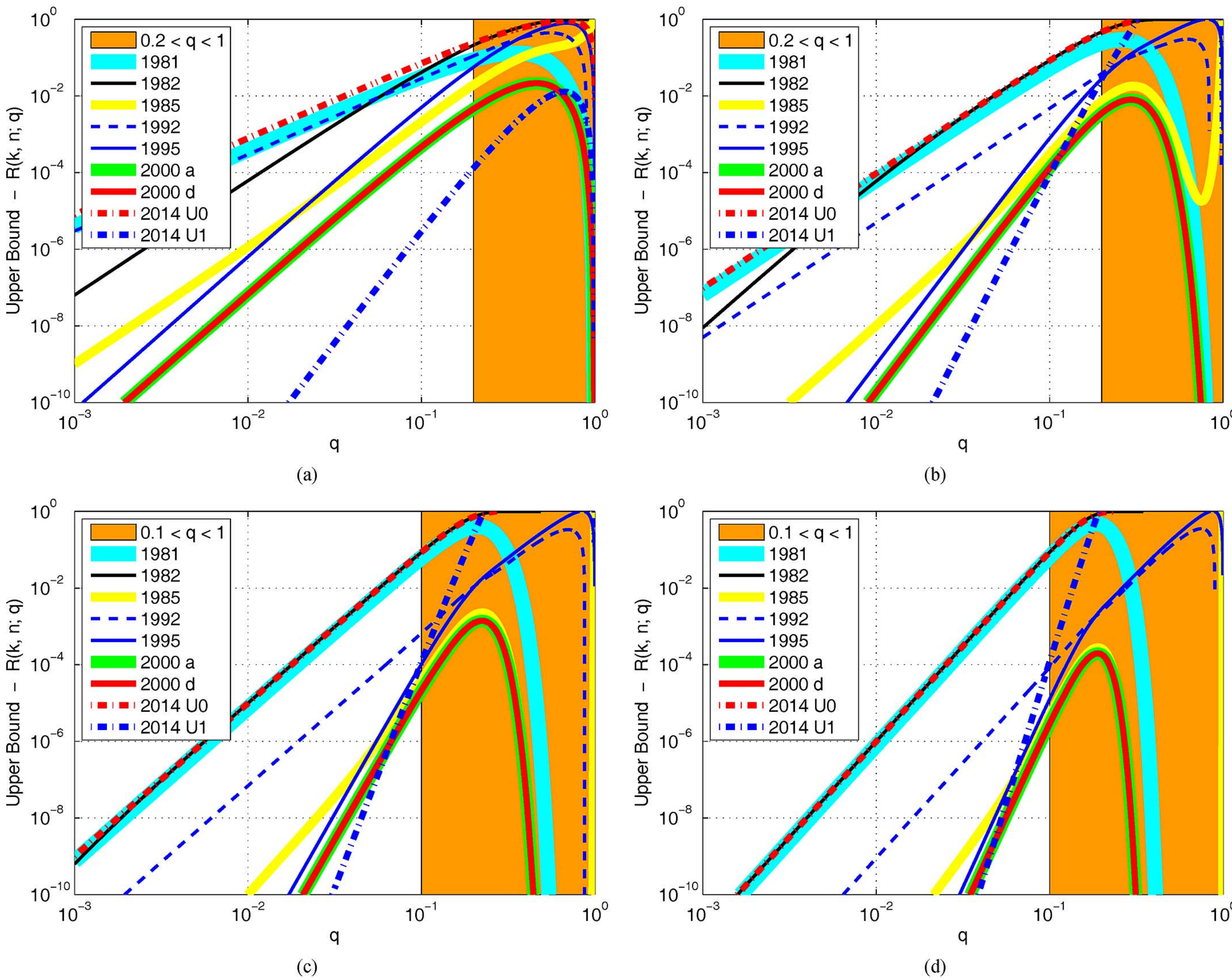

Fig. 2. Upper bound absolute errors for consecutive systems: (a) 2-out-of-10, (b) 3-out-of-100, (c) 4-out-of-1000, and (d) 5-out-of-10000.

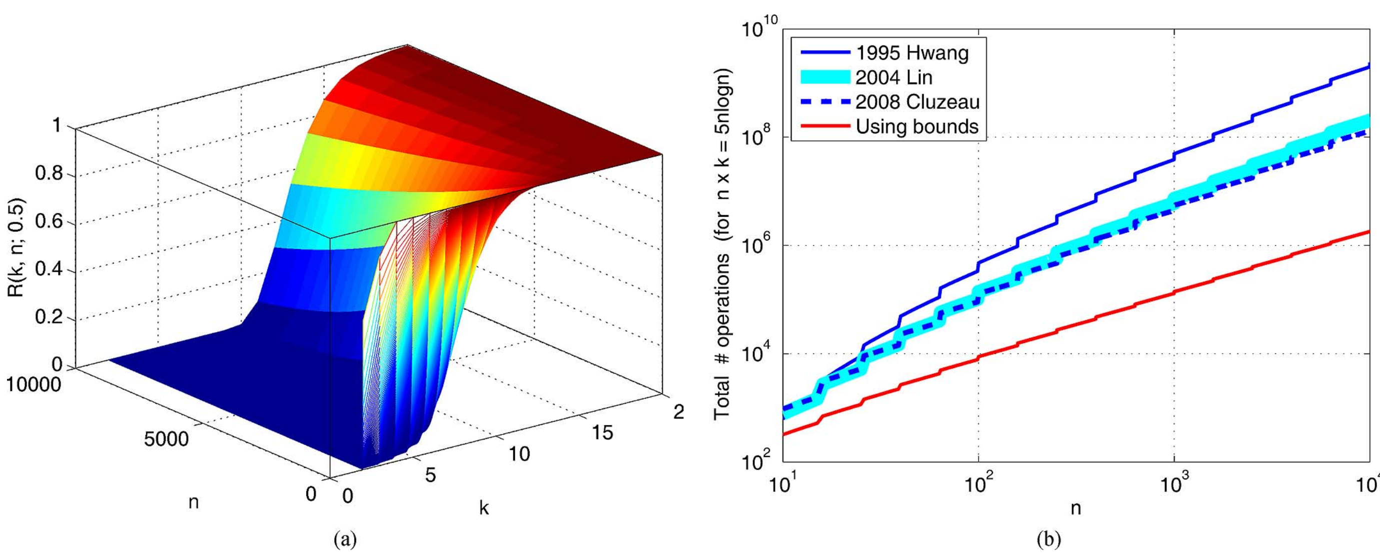

Fig. 3. Consecutive-k-out-of-n:F for n up to 10,000 and k up to $5 \log n$: (a) 3D view of $R(k, n; 0.5)$, (b) total number of arithmetic operations.

accurate for small values of q, n, and k; and two of them seem to have the steepest slope (in log-log scale) of the absolute error versus q.

Using bounds instead of exact calculations might be useful for preliminary investigations of the reliability of large consecutive-k-out-of-n:F systems (if bounds are accurate enough), as the execution time can be significantly reduced over that required by exact algorithms.

APPENDIX

Proof of Proposition 2: We start by expanding

$$a_j = C_{n-jk}^{j}\left(pq^k\right)^j = \frac{(n-jk)!}{j!\,(n-jk-j)!}\left(pq^k\right)^j = \frac{(n-jk-j+1)\,(n-jk-j+2)\cdots(n-jk)}{j!}\left(pq^k\right)^j. \tag{21}$$

Therefore, $a_j > a_{j+1}$ iff

$$\frac{(n-jk-j+1)(n-jk-j+2)\cdots(n-jk)}{j!}\left(pq^k\right)^j > \frac{[n-(j+1)k-(j+1)+1][n-(j+1)k-(j+1)+2]\cdots[n-(j+1)k]}{(j+1)!}\left(pq^k\right)^{j+1}, \tag{22}$$

or equivalently

$$\begin{aligned}(n-jk-j+1)\,(n-jk-j+2)\cdots(n-jk)\,(j+1) &> [n-(j+1)\,k-(j+1)+1] \\ &\quad\times[n-(j+1)\,k-(j+1)+2] \\ &\quad\cdots[n-(j+1)\,k]\,pq^k.\end{aligned} \tag{23}$$

Comparing term by term, we have

$$\begin{aligned} n-jk-j+1 &> n-(j+1)\,k-j+1 \\ &= n-(j+1)\,k-(j+1)+2 \\ n-jk-j+2 &> n-(j+1)\,k-j+2 \\ &= n-(j+1)\,k-(j+1)+3 \\ &\cdots \\ n-jk &> n-(j+1)\,k. \end{aligned} \tag{24}$$

For proving that $a_j > a_{j+1}$, we still need to show that

$$j+1 > [n-(j+1)\,k-(j+1)+1]\,pq^k. \tag{25}$$

As $j \in \mathbf{N}$ ($j = 0, 1, \ldots, \lfloor n/(k+1) \rfloor$), the left hand side of (25) is $j+1 \geq 1$, while the right hand side is < 1 as

$$\begin{aligned}&[n-(j+1)\,k-(j+1)+1]pq^k \\ &= [n-(j+1)\,k-j]\,pq^k \\ &= [n-(k+1)\,j-k]\,pq^k < (n-k)\,pq^k < 1.\end{aligned} \tag{26}$$

Taking (26) in conjunction with (23) and (24), we have (22), hence $a_j > a_{j+1}$ for any $j = 0, 1, \ldots, \lfloor n/(k+1)-1 \rfloor$.

The proof that $b_j > b_{j+1}$ is similar.

Proof of Proposition 3: We remark that $\beta_{n,k}$ and also $\beta_{n-k,k}$ are alternating sums. Using the notations from *Proposition* 1, we have

$$\beta_{n,k} = 1 - a_1 + a_2 - \cdots + (-1)^{\lfloor n/(k+1) \rfloor} a_{\lfloor n/(k+1) \rfloor},$$

and $\beta_{n-k,k} = 1 - b_1 + b_2$

$$- \cdots + (-1)^{\lfloor (n-k)/(k+1) \rfloor} b_{\lfloor (n-k)/(k+1) \rfloor}.$$

We define two partial sums

$$S_r = \sum_{j=0}^{r} (-1)^j a_j, \tag{27}$$

$$S'_r = \sum_{j=0}^{r} (-1)^j b_j. \tag{28}$$

Based on *Proposition* 1, if $1/(n-k) > pq^k$, the following two sequences $(a_j)_{1 \leq j \leq \lfloor n/(k+1) \rfloor}$ and $(b_j)_{1 \leq j \leq \lfloor (n-k)/(k+1) \rfloor}$ of positive numbers are both decreasing. Hence,

$$S_1 < S_3 < \cdots \leq \beta_{n,k} \leq \cdots < S_2 < S_0 = 1, \tag{29}$$

$$S'_1 < S'_3 < \cdots \leq \beta_{n-k,k} \leq \cdots < S'_2 < S'_0 = 1. \tag{30}$$

By multiplying (30) with $-q^k$, we get

$$\begin{aligned}-q^k S'_0 < -q^k S'_2 < \cdots &\leq -q^k \beta_{n-k,k} \\ &\leq \ldots < -q^k S'_3 < -q^k S'_1;\end{aligned}$$

and adding this to (29) we have

$$\begin{aligned}S_1 - q^k S'_0 < S_3 - q^k S'_2 < \cdots &\leq \beta_{n,k} - q^k \beta_{n-k,k} \\ &\leq \ldots < S_2 - q^k S'_3 < S_0 - q^k S'_1.\end{aligned}$$

Therefore,

$$\begin{aligned}S_1 - q^k S'_0 < S_3 - q^k S'_2 < \cdots &\leq R(k, n; q) \\ &\leq \ldots < S_2 - q^k S'_3 < S_0 - q^k S'_1.\end{aligned} \tag{31}$$

This result reveals the entire family of lower and upper bound for $R(k, n; q)$. Substituting (27) and (28) in (31) gives the lower and upper bounds L_r and U_r.

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