Assignment 8

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***Abstract*—This document solves a problem using Singular Value Decomposition.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_8

1 Problem

Find the point on the plane closest to the point $\begin{pmatrix}6\\5\\9\end{pmatrix}$ and the plane is determined by the points

$$\mathbf{A} = \begin{pmatrix}3\\-1\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}5\\2\\4\end{pmatrix}, \mathbf{C} = \begin{pmatrix}-1\\-1\\6\end{pmatrix}$$

2 Solution

The equation of plane is given by,

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.1}$$

$$\mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = \mathbf{n}^T\mathbf{C} = c \tag{2.0.2}$$

$$\implies \begin{pmatrix}\mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C}\end{pmatrix}^T \mathbf{n} = 0 \tag{2.0.3}$$

Using row reduction on above matrix,

$$\begin{pmatrix}-2 & -3 & -2\\6 & 3 & -2\end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix}1 & \frac{3}{2} & 1\\6 & 3 & -2\end{pmatrix} \tag{2.0.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 6R_1} \begin{pmatrix}1 & \frac{3}{2} & 1\\0 & -6 & -8\end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{pmatrix}1 & \frac{3}{2} & 1\\0 & 3 & 4\end{pmatrix} \tag{2.0.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{2}} \begin{pmatrix}1 & 0 & -1\\0 & 3 & 4\end{pmatrix} \tag{2.0.6}$$

Thus,

$$\mathbf{n} = \begin{pmatrix}1\\\frac{-4}{3}\\1\end{pmatrix} = \begin{pmatrix}3\\-4\\3\end{pmatrix} \tag{2.0.7}$$

$$c = \mathbf{n}^T\mathbf{A} = 19 \tag{2.0.8}$$

Thus the equation of the plane is,

$$\begin{pmatrix}3 & -4 & 3\end{pmatrix}\mathbf{x} = 19 \tag{2.0.9}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be the two orthogonal vectors to the given normal. Let, $\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, then

$$\mathbf{m}^T\mathbf{n} = 0 \tag{2.0.10}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}3\\-4\\3\end{pmatrix} = 0 \tag{2.0.11}$$

$$\implies 3a - 4b + 3c = 0 \tag{2.0.12}$$

Let $a = 1$, $b = 0$ we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{2.0.13}$$

Let $a = 0$, $b = 1$ we get,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\\frac{4}{3}\end{pmatrix} \tag{2.0.14}$$

Solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{2.0.15}$$

Putting the values in (2.0.15),

$$\begin{pmatrix}1 & 0\\0 & 1\\-1 & \frac{4}{3}\end{pmatrix}\mathbf{x} = \begin{pmatrix}6\\5\\9\end{pmatrix} \tag{2.0.16}$$

To solve (2.0.16), we perform Singular Value Decomposition on $\mathbf{M}$,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.17}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{4}{3} \\ -1 & \frac{4}{3} & \frac{25}{9} \end{pmatrix} \tag{2.0.19}$$

Putting (2.0.17) in (2.0.15) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{2.0.20}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^{\mathbf{T}}\mathbf{b} \tag{2.0.21}$$

Where $\mathbf{S}_{+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$. Now, calculating eigen values of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.22}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & \frac{4}{3} \\ -1 & \frac{4}{3} & \frac{25}{9}-\lambda \end{pmatrix} = 0 \tag{2.0.23}$$

$$\implies \lambda^3 - \frac{43}{9}\lambda^2 + \frac{34}{9}\lambda = 0 \tag{2.0.24}$$

Thus the eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{34}{9} \tag{2.0.25}$$

$$\lambda_2 = 1 \tag{2.0.26}$$

$$\lambda_3 = 0 \tag{2.0.27}$$

The eigen vectors comes out to be,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{25} \\ \frac{12}{25} \\ 1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} 1 \\ \frac{-4}{3} \\ 1 \end{pmatrix} \tag{2.0.28}$$

Normalising the eigen vectors,

$$\mathbf{u_1} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} \\ \frac{12}{5\sqrt{34}} \\ \frac{5}{\sqrt{34}} \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} \frac{3}{\sqrt{34}} \\ \frac{-4}{\sqrt{34}} \\ \frac{3}{\sqrt{34}} \end{pmatrix} \tag{2.0.29}$$

Hence we obtain $\mathbf{U}$ matrix as,

$$\mathbf{U} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \tag{2.0.30}$$

Now,

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.31}$$

Calculating the eigen values of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.32}$$

$$\implies \begin{pmatrix} 2-\lambda & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9}-\lambda \end{pmatrix} = 0 \tag{2.0.33}$$

$$\implies \lambda^2 - \frac{43}{9}\lambda + \frac{34}{9} = 0 \tag{2.0.34}$$

The eigen values are,

$$\lambda_1 = \frac{34}{9} \tag{2.0.35}$$

$$\lambda_2 = 1 \tag{2.0.36}$$

The eigen vectors are,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \tag{2.0.37}$$

Normalising the eigen vectors,

$$\mathbf{v_1} = \begin{pmatrix} \frac{-3}{5} \\ \frac{4}{5} \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{2.0.38}$$

Hence we obtain $\mathbf{V}$ matrix as,

$$\mathbf{V} = \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \tag{2.0.39}$$

Thus we get the Singular Value Decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} \frac{-9}{5\sqrt{34}} & \frac{4}{5} & \frac{3}{\sqrt{34}} \\ \frac{12}{5\sqrt{34}} & \frac{3}{5} & \frac{-4}{\sqrt{34}} \\ \frac{5}{\sqrt{34}} & 0 & \frac{3}{\sqrt{34}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{34}}{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}^T \tag{2.0.40}$$

The Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_{+} = \begin{pmatrix} \frac{3}{\sqrt{34}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.41}$$

From (2.0.21) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{231}{5\sqrt{34}} \\ \frac{39}{5} \\ \frac{25}{\sqrt{34}} \end{pmatrix} \tag{2.0.42}$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{693}{170} \\ \frac{39}{5} \end{pmatrix} \tag{2.0.43}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \tag{2.0.44}$$

Verifying the solution of (2.0.44) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.45}$$

Evaluating the R.H.S in (2.0.45) we get,

$$\mathbf{M}^T\mathbf{b} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \tag{2.0.46}$$

$$\implies \begin{pmatrix} 2 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{25}{9} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -3 \\ 17 \end{pmatrix} \tag{2.0.47}$$

Solving the augmented matrix of (2.0.47) we get,

$$\begin{pmatrix} 2 & \frac{-4}{3} & -3 \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ \frac{-4}{3} & \frac{25}{9} & 17 \end{pmatrix} \tag{2.0.48}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R-1} \begin{pmatrix} 1 & \frac{-2}{3} & \frac{-3}{2} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \tag{2.0.49}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{6}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & \frac{17}{9} & 15 \end{pmatrix} \tag{2.0.50}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{9}{17}R_2} \begin{pmatrix} 1 & 0 & \frac{129}{34} \\ 0 & 1 & \frac{135}{17} \end{pmatrix} \tag{2.0.51}$$

Hence, solution of (2.0.45) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{129}{34} \\ \frac{135}{17} \end{pmatrix} \tag{2.0.52}$$

Comparing results of $\mathbf{x}$ from (2.0.44) and (2.0.52) we conclude that the solution is verified.