Assignment 6

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***Abstract*—This document finds the equation of lines which are tangent to the curve.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_6

1 Problem

Find the equation of all lines having slope -2 which are tangents to the curve $\frac{1}{x-3}$, $x \neq 3$.

2 Solution

Given the curve,

$$y = \frac{1}{x-3} \tag{2.0.1}$$

$$\implies xy - 3y - 1 = 0 \tag{2.0.2}$$

From (2.0.2) we get,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u} = \frac{-3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, f = -1 \tag{2.0.3}$$

Now,

$$\because \left|V\right| = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = \frac{-1}{2} < 0 \tag{2.0.4}$$

(2.0.1) is equation of hyperbola. Now,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda & \frac{-1}{2} \\ \frac{-1}{2} & \lambda \end{vmatrix} = 0 \tag{2.0.5}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{2.0.6}$$

Thus the eigen values are,

$$\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{-1}{2} \tag{2.0.7}$$

The eigen vector p is given by,

$$(\lambda\mathbf{I} - \mathbf{V})\mathbf{p} = 0 \tag{2.0.8}$$

For $\lambda_1 = \frac{1}{2}$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.9}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.10}$$

Similarly for λ_2,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} \end{pmatrix} \xleftrightarrow[R_1 \leftarrow 2R_1]{R_2 \leftarrow R\_R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.11}$$

$$\implies \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.12}$$

Now,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{2.0.13}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 \tag{2.0.15}$$

$\because \text{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0$, there is no need to swap the axes. The hyperbola parameters are,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{2} \tag{2.0.17}$$

$$\sqrt{\frac{f - \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_1}} = \sqrt{2} \tag{2.0.18}$$

with the standard hyperbola becoming,

$$\frac{x^2}{2} - \frac{y^2}{2} = 1 \tag{2.0.19}$$

The direction and normal vectors of the tangent with slope -2 are given as,

$$\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.0.20}$$

Now considering the equations to find the point of contact,

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{n} - \mathbf{u}) \tag{2.0.21}$$

$$k = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.22}$$

Thus,

$$\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n} = 8 \tag{2.0.23}$$

$$k = \pm\frac{1}{2\sqrt{2}} \tag{2.0.24}$$

$$\mathbf{q_1} = \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix} \tag{2.0.25}$$

$$\mathbf{q_2} = \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix} \tag{2.0.26}$$

The desired tangents are,

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{1+3\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} \end{pmatrix}\right\} = 0 \tag{2.0.27}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 + 2\sqrt{2} \tag{2.0.28}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix}\left\{\mathbf{x} - \begin{pmatrix} \frac{-1+3\sqrt{2}}{\sqrt{2}} \\ -\sqrt{2} \end{pmatrix}\right\} = 0 \tag{2.0.29}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x} = 6 - 2\sqrt{2} \tag{2.0.30}$$

Below figure corresponds to the tangents on the hyperbola, represented by (2.0.28) and (2.0.30) each having slope of -2.

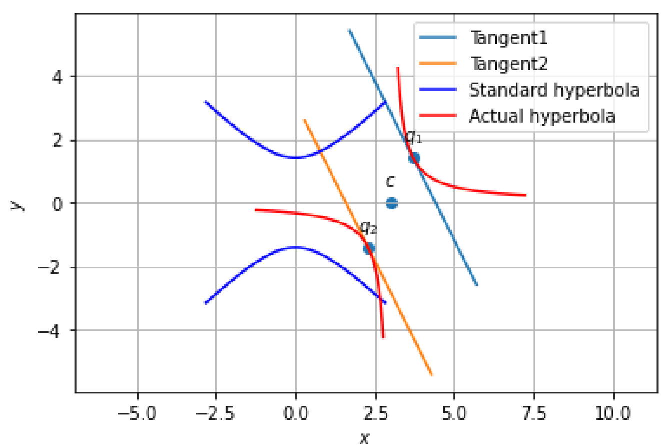

Fig. 1: Tangents to the hyperbola