Assignment 9

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Abstract—**This document solves whether two system of linear equations are linear equivalent or not.**

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1 PROBLEM

Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

2 SOLUTION

The given system of linear equations can be written as,

$$\mathbf{Ax} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.2}$$

$$\mathbf{Bx} = 0 \tag{2.0.3}$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.4}$$

Now we can obtain **B** from matrix **A** by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{CA} \tag{2.0.5}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \tag{2.0.6}$$

where **C** is product of elementary matrices given as,

$$\mathbf{C} = (\mathbf{E_7 E_6 E_5 E_4 E_3 E_2 E_1})$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \tag{2.0.7}$$

Now, performing elementary operations on the right side of **A** we obtain matrix **B** given as,

$$\mathbf{B} = \mathbf{AC}' \tag{2.0.8}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{C}' \tag{2.0.9}$$

where, $\mathbf{C}'$ is product of elementary matrices given by,

$$\mathbf{C}' = (\mathbf{E_1 E_2 E_3 E_4 E_5})$$
$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \tag{2.0.10}$$

Thus (2.0.4) can be obtained from (2.0.2) by multiplying it with matrix **C** or $\mathbf{C}'$, where each one is product of elementary matrices and hence invertible. Thus the two given homogeneous systems are row equivalent. Now writing equations in matrix-vector form as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.11}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{4}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.12}$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.13}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{2}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.14}$$

Thus, each equation in second system can be expressed as linear combination of equations in first system. This can be verified from the entries of **C** as they take the linear combination of each rows of

matrix $\mathbf{A}$. Similarly,

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} \qquad (2.0.15)$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = (1)\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + (-2)\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \qquad (2.0.16)$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \qquad (2.0.17)$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2}\begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \qquad (2.0.18)$$