

Assignment 9

Gaydhane Vaibhav Digraj
Roll No. AI20MTECH11002

Abstract—This document solves whether two system of linear equations are linear equivalent or not.

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_9

1 PROBLEM

Let \mathbb{F} be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned}x_1 - x_2 &= 0 \\ 2x_1 + x_2 &= 0\end{aligned}$$

and

$$\begin{aligned}3x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0\end{aligned}$$

2 SOLUTION

The given system of linear equations can be written as,

$$\mathbf{A}\mathbf{x} = 0 \quad (2.0.1)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.0.2)$$

$$\mathbf{B}\mathbf{x} = 0 \quad (2.0.3)$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.0.4)$$

Now we can obtain \mathbf{B} from matrix \mathbf{A} by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \quad (2.0.5)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \quad (2.0.6)$$

where \mathbf{C} is product of elementary matrices given as,

$$\begin{aligned}\mathbf{C} &= (\mathbf{E}_7 \mathbf{E}_6 \mathbf{E}_5 \mathbf{E}_4 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1) \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \quad (2.0.7)\end{aligned}$$

Now, performing elementary operations on the right side of \mathbf{A} we obtain matrix \mathbf{B} given as,

$$\mathbf{B} = \mathbf{A}\mathbf{P} \quad (2.0.8)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{P} \quad (2.0.9)$$

where, \mathbf{P} is product of elementary matrices given by,

$$\begin{aligned}\mathbf{P} &= (\mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_4 \mathbf{E}_5) \\ &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \quad (2.0.10)\end{aligned}$$

Similarly, \mathbf{A} can be obtained from matrix \mathbf{B} from (2.0.5) as,

$$\mathbf{A} = \mathbf{C}^{-1} \mathbf{B} \quad (2.0.11)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad (2.0.12)$$

Matrix \mathbf{C} is product of elementary matrices and hence invertible and is given as,

$$\begin{aligned}\mathbf{C}^{-1} &= (\mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \mathbf{E}_3^{-1} \mathbf{E}_4^{-1} \mathbf{E}_5^{-1} \mathbf{E}_6^{-1} \mathbf{E}_7^{-1}) \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (2.0.13)\end{aligned}$$

Matrix **A** can also be obtained from (2.0.8) given as,

$$\mathbf{A} = \mathbf{B}\mathbf{P}^{-1} \quad (2.0.14)$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{P}^{-1} \quad (2.0.15)$$

where,

$$\begin{aligned} \mathbf{P}^{-1} &= (\mathbf{E}_5^{-1} \mathbf{E}_4^{-1} \mathbf{E}_3^{-1} \mathbf{E}_2^{-1} \mathbf{E}_1^{-1}) \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \quad (2.0.16) \end{aligned}$$

Thus (2.0.4) can be obtained from (2.0.2) by multiplying it with matrix **C**, and by inverse row operations (2.0.2) can be obtained back from (2.0.4) since **C** is product of elementary matrices and hence invertible.

Thus the two given homogeneous systems are row equivalent.

Now writing equations in matrix-vector form as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.17)$$

$$\Rightarrow \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{4}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.18)$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.19)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{2}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.20)$$

(2.0.18), (2.0.20) is same as multiplying **C** with **A** as it takes the linear combination of each rows of matrix **A** i.e, (2.0.6)

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} \quad (2.0.21)$$

$$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = (1) \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + (-2) \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.22)$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.23)$$

$$\Rightarrow \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \quad (2.0.24)$$

(2.0.22), (2.0.24) is same as multiplying **C**⁻¹ with **B** as it takes the linear combination of each rows of matrix **B** i.e, (2.0.12)

Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.