Assignment 18

Gaydhane Vaibhav Digraj
Roll No. AI20MTECH11002

***Abstract*—This document solves a problem based on polynomial vector spaces.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_18

1 Problem

Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers $a_0, a_1, \cdots, a_k$. For $p \in V$

$$max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\}$$

defines a norm on V

1) only if $k < n$
2) only if $k \geq n$
3) if $k + 1 \leq n$
4) if $k \geq n + 1$

2 Solution

Options 2 and 4 are correct as verified in the table 2

3 Example

The scalar multiplication and triangle inequality properties holds true for all k.

$$max\left\{\left|\alpha p(a_j)\right|\right\} = |\alpha|\, max\left\{\left|p(a_j)\right|\right\} \tag{3.0.1}$$

$$max\left\{\left|p(a_i) + p(a_j)\right|\right\} \leq max\left\{|p(a_i)|\right\} + max\left\{\left|p(a_j)\right|\right\} \tag{3.0.2}$$

The positivity property holds true only if $k \geq n$ as more than n roots are possible when,

$$p(x) = 0 \implies \left|p(a_j)\right|_{0 \leq j \leq k} = 0 \tag{3.0.3}$$

$$\implies max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} = 0 \tag{3.0.4}$$

Properties	**Norm** $\forall x \in V$
Positivity	$\|\|x\|\| \geq 0, \|\|x\|\| = 0 \iff x = 0$
Scalar Multiplication	$\|\|\alpha x\|\| = \|\alpha\| \|\|x\|\|, \alpha \in F$
Triangle Inequality	$\|\|x + y\|\| \leq \|\|x\|\| + \|\|y\|\|$

TABLE 1: Properties of Norm

<table>
<tr><td colspan="2">For $p \in V$ then the norm, $max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} = 0 \iff \left|p(a_j)\right|_{0\leq j\leq k} = 0$</td></tr>
<tr><th>Conditions</th><th>Explanation</th></tr>
<tr><td>only if $k < n$

Example:</td><td>A polynomial doesn't necessarily have k distinct real roots,
i.e., it may have repeated, complex roots.
let p be polynomial of degree $n = 2$ and $k = 1$ given by:-
$$p(x) = x^2 + 4x + 4 \quad (2.0.1)$$
$$\left|p(a_j)\right|_{0\leq j\leq 1} = 0 \implies a_0 = -2, a_1 = -2 \quad (2.0.2)$$
but $a_0, a_1, \cdots, a_k$ should be distinct real numbers.
This contradicts the property of Norm. Thus condition fails.</td></tr>
<tr><td>only if $k \geq n$</td><td>p is a polynomial of degree ≤n,
it can't have more than n roots and is only possible when,
$p(x) = 0 \implies \left|p(a_j)\right|_{0\leq j\leq k} = 0$
hence p is identically zero. Thus condition satisfies.</td></tr>
<tr><td>if $k + 1 \leq n$</td><td>Not a norm for $k < n$. Hence incorrect.</td></tr>
<tr><td>if $k \geq n + 1$</td><td>Norm for $k \geq n$. Hence correct.</td></tr>
</table>

TABLE 2: Verifying Positivity Property of Norm