

Assignment 5

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Abstract—This document explains the concept of finding the unknown value in an equation such that it represents two straight lines.

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_5

1 PROBLEM

Find the value of k so that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0$ may represent two straight lines.

2 EXPLANATION

Given equation,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0 \quad (2.0.1)$$

is a second order equation. The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.2)$$

$$\Rightarrow \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.0.5)$$

Equation (2.0.3) represents a pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.6)$$

Comparing equation (2.0.1) with (2.0.2), we can write in the form of (2.0.4) and (2.0.5) as,

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \quad (2.0.7)$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ \frac{-5}{2} \end{pmatrix} \quad (2.0.8)$$

$$f = k \quad (2.0.9)$$

The given equation represent two straight lines, substituting (2.0.7), (2.0.8), (2.0.9) in (2.0.6) to satisfy the equation.

$$\begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & \frac{-5}{2} \\ \frac{11}{2} & \frac{-5}{2} & k \end{vmatrix} = 0 \quad (2.0.10)$$

Expanding the above determinant along row 3,

$$\begin{aligned} \Rightarrow \frac{11}{2} \times (\frac{25}{2} - 11) + \frac{5}{2} \times (-30 + \frac{55}{2}) + k \times (24 - 25) &= 0 \\ \Rightarrow \frac{33}{4} + (\frac{-25}{4}) - k &= 0 \quad (2.0.11) \end{aligned}$$

$$\Rightarrow \boxed{k = 2} \quad (2.0.12)$$

3 SOLUTION

For $k = 2$ the given equation will represent two straight lines.

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad (3.0.1)$$

From (3.0.1) we get,

$$f = 2 \quad (3.0.2)$$

$$\det(V) = \begin{vmatrix} 12 & -5 \\ -5 & 2 \end{vmatrix} = -1 < 0 \quad (3.0.3)$$

Since $\det(V) < 0$ we can say that two intersecting lines are obtained. The pair of straight lines in vector form is given by,

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (3.0.4)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (3.0.5)$$

Equating their product with (2.0.3),

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (3.0.6)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \{12, -10, 2\} \quad (3.0.7)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2\mathbf{u} = -2 \begin{pmatrix} \frac{11}{2} \\ \frac{-5}{2} \end{pmatrix} \quad (3.0.8)$$

$$c_1 c_2 = f = 2 \quad (3.0.9)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (3.0.10)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (3.0.11)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (3.0.12)$$

Substituting the given data in above equations (3.0.10) we get,

$$2m^2 - 10m + 12 = 0 \quad (3.0.13)$$

$$\Rightarrow m_i = \frac{-(-5) \pm \sqrt{-(-1)}}{2} \quad (3.0.14)$$

Solving equation (3.0.14) we get ,

$$m_1 = 3 \quad (3.0.15)$$

$$m_2 = 2 \quad (3.0.16)$$

$$\mathbf{n}_1 = k_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (3.0.17)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.18)$$

Substituting equations (3.0.17), (3.0.18) in equation (3.0.7) we get

$$k_1 k_2 = 2 \quad (3.0.19)$$

The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2), (-2,-1). Let's assume $k_1 = 2, k_2 = 1$, we get

$$\mathbf{n}_1 = \begin{pmatrix} -6 \\ 2 \end{pmatrix} \quad (3.0.20)$$

$$\mathbf{n}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.21)$$

From equation (3.0.8) we get

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} \frac{11}{2} \\ \frac{-5}{2} \end{pmatrix} \quad (3.0.22)$$

$$\begin{pmatrix} -6 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix} \quad (3.0.23)$$

$$-6c_2 - 2c_1 = -11 \quad (3.0.24)$$

$$2c_2 + 1c_1 = 5 \quad (3.0.25)$$

Solving equations (3.0.24) ,(3.0.25) we get

$$c_1 = 4 \quad (3.0.26)$$

$$c_2 = \frac{1}{2} \quad (3.0.27)$$

Equations (3.0.4), (3.0.5) can be modified as

$$\begin{pmatrix} -6 & 2 \end{pmatrix} \mathbf{x} = 4 \quad (3.0.28)$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \quad (3.0.29)$$

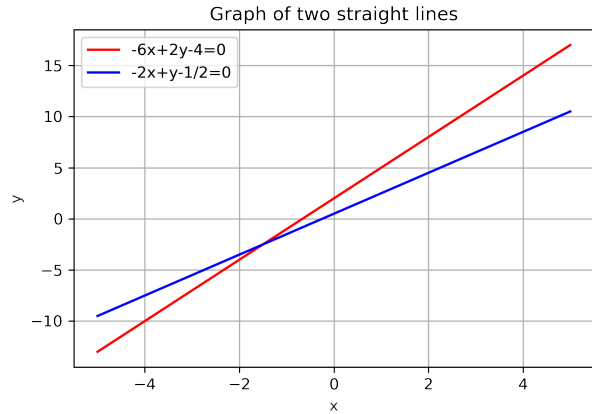


Fig. 1: Plot of Straight lines