Assignment 15

Gaydhane Vaibhav Digraj
Roll No. AI20MTECH11002

***Abstract*—This document solves a problem involving linear transformations.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_15

1 Problem

Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$ and $\mathbf{T}$ is a linear operator on $\mathbf{V}$. If $\mathbf{T}^2 = 0$, what can you say about the relation of the range of $\mathbf{T}$ to the null space of $\mathbf{T}$? Give an example of linear operator $\mathbf{T}$ on $\mathbf{R}^2$ such that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.

2 Solution

Let for some vector $\mathbf{y} \in$ Range($\mathbf{T}$) then there exists $\mathbf{x} \in \mathbf{V}$ such that,

$$\mathbf{T} : \mathbf{V} \to \mathbf{V} \tag{2.0.1}$$
$$\mathbf{T}(\mathbf{x}) = \mathbf{y} \tag{2.0.2}$$
$$\mathbf{T}(\mathbf{T}(\mathbf{x})) = \mathbf{T}(\mathbf{y}) \tag{2.0.3}$$
$$\implies \mathbf{T}^2(\mathbf{x}) = \mathbf{T}(\mathbf{y}) \tag{2.0.4}$$
$$\mathbf{0} = \mathbf{T}(\mathbf{y}) \tag{2.0.5}$$

$\therefore$ $\mathbf{y}$ lies in the Null space of $\mathbf{T}$. Hence $\mathbf{T}$ is singular. Thus, the range of $\mathbf{T}$ must be contained in Null space of $\mathbf{T}$ i.e., Range($\mathbf{T}$) $\subseteq$ NullSpace($\mathbf{T}$)

3 Example

Let a vector,

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \tag{3.0.1}$$
$$\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2 \tag{3.0.2}$$
$$\tag{3.0.3}$$

Consider,

$$\mathbf{T}(\mathbf{X}) = \mathbf{M}\mathbf{X} \tag{3.0.4}$$
$$\implies \mathbf{T}(\mathbf{X}) = \begin{pmatrix} 0 \\ x \end{pmatrix} \tag{3.0.5}$$

such that,

$$\mathbf{M} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.0.6}$$
$$\implies \mathbf{T} \neq 0 \tag{3.0.7}$$

Now, since $\mathbf{T}$ is a linear operator on $\mathbf{R}^2$,

$$\mathbf{T}^2(c\alpha) = \mathbf{T}(\mathbf{T}(c\alpha)) = \mathbf{T}(c\mathbf{T}(\alpha)) \tag{3.0.8}$$
$$= c\mathbf{T}(\mathbf{T}(\alpha)) = c\mathbf{T}^2(\alpha) \tag{3.0.9}$$

Thus $\mathbf{T}^2$ is also a linear operator on $\mathbf{R}^2$ such that,

$$\mathbf{T}^2 : \mathbf{R}^2 \to \mathbf{R}^2 \tag{3.0.10}$$

$$\mathbf{T}^2(\mathbf{X}) = \mathbf{T}(\mathbf{T}(\mathbf{X})) = \mathbf{T}(\mathbf{M}\mathbf{X}) = \mathbf{M}^2\mathbf{X} \tag{3.0.11}$$
$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\mathbf{X} = \mathbf{0} \tag{3.0.12}$$

Thus $\mathbf{T}^2$ is a zero transformation,

$$\implies \mathbf{T}^2 = \mathbf{0} \tag{3.0.13}$$

Thus from (3.0.7), (3.0.13) it is clear that $\mathbf{T}^2 = 0$ but $\mathbf{T} \neq 0$.