Assignment 9

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***Abstract*—This document solves whether two system of linear equations are linear equivalent or not.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_9

1 Problem

Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

2 Solution

The given system of linear equations can be written as,

$$\mathbf{A}\mathbf{x} = 0 \tag{2.0.1}$$
$$\implies \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.0.2}$$
$$\mathbf{B}\mathbf{x} = 0 \tag{2.0.3}$$
$$\implies \begin{pmatrix}3 & 1\\1 & 1\end{pmatrix}\mathbf{x} = 0 \tag{2.0.4}$$

Consider $\mathbf{A}$, performing row operation using elementary matrices,

$$\mathbf{A} = \begin{pmatrix}1 & -1\\2 & 1\end{pmatrix} \tag{2.0.5}$$
$$\leftrightarrow \begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & -1\\0 & 3\end{pmatrix} \tag{2.0.6}$$
$$\leftrightarrow \begin{pmatrix}1 & 0\\0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 3\end{pmatrix} = \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} \tag{2.0.7}$$
$$\leftrightarrow \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.0.8}$$
$$\leftrightarrow \begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}3 & 0\\0 & 1\end{pmatrix} \tag{2.0.9}$$
$$\leftrightarrow \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}3 & 1\\0 & 1\end{pmatrix} \tag{2.0.10}$$
$$\leftrightarrow \begin{pmatrix}1 & 0\\0 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}3 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}3 & 1\\0 & \frac{2}{3}\end{pmatrix} \tag{2.0.11}$$
$$\leftrightarrow \begin{pmatrix}1 & 0\\\frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}3 & 1\\0 & \frac{2}{3}\end{pmatrix} = \begin{pmatrix}3 & 1\\1 & 0\end{pmatrix} = \mathbf{B} \tag{2.0.12}$$

Thus we get $\mathbf{B}$ from matrix $\mathbf{A}$ by multiplying it with matrix $\mathbf{C}$, where $\mathbf{C}$ is product of elementary matrices given as,

$$\mathbf{C} = \begin{pmatrix}1 & 0\\\frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}$$
$$= \begin{pmatrix}\frac{1}{3} & \frac{4}{3}\\\frac{-1}{3} & \frac{2}{3}\end{pmatrix} \tag{2.0.13}$$

Thus, since matrix $\mathbf{B}$ can be obtained from matrix $\mathbf{A}$ by a finite sequence of elementary row operations on $\mathbf{A}$, the given two system of linear equations are equivalent. Thus,

$$\mathbf{B} = \mathbf{C}\mathbf{A}$$
$$\implies \begin{pmatrix}3 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\\frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$
$$\begin{pmatrix}1 & 0\\0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\2 & 1\end{pmatrix} \tag{2.0.14}$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \tag{2.0.15}$$

From (2.0.15), it can be seen that each equation in system **B** is a linear combination of equations in system **A**.