Assignment 9

Gaydhane Vaibhav Digraj
Roll No. AI20MTECH11002

***Abstract*—This document solves whether two system of linear equations are linear equivalent or not.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_9

1 Problem

Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{aligned} \mathbf{x_1} - \mathbf{x_2} &= 0 \\ 2\mathbf{x_1} + \mathbf{x_2} &= 0 \end{aligned}$$

and

$$\begin{aligned} 3\mathbf{x_1} + \mathbf{x_2} &= 0 \\ \mathbf{x_1} + \mathbf{x_2} &= 0 \end{aligned}$$

2 Solution

The first system of linear equations can be written as,

$$\mathbf{Ax} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.2}$$

Forming augmented matrix and performing row reduction using Elementary Matrices,

$$\implies \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \tag{2.0.3}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \tag{2.0.4}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.5}$$

$$\leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.6}$$

$$\implies \mathbf{Rx} = 0 \tag{2.0.7}$$

where,

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.8}$$

The second system of linear equations can be written as,

$$\mathbf{Bx} = 0 \tag{2.0.9}$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.10}$$

Forming augmented matrix and performing row reduction using Elementary Matrices,

$$\implies \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.0.11}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix} \tag{2.0.12}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.14}$$

$$\leftrightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{R}^{'}\mathbf{x} = 0 \tag{2.0.16}$$

where,

$$\mathbf{R}^{'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.17}$$

From (2.0.8) and (2.0.17) we get,

$$\mathbf{R} = \mathbf{R}^{'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.18}$$

Thus, the two system of linear equations have same solution set as their row reduced echelon form are same. Hence the system of linear equations are equivalent. Combining (2.0.1), (2.0.9) into a single matrix equation and forming the augmented matrix,

$$\implies \begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{B} & 0 \end{pmatrix} \tag{2.0.19}$$

$$\leftrightarrow \begin{pmatrix} \mathbf{R} & 0 \\ \mathbf{R}' & 0 \end{pmatrix} \tag{2.0.20}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R} & 0 \\ \mathbf{R}' & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.21}$$

$$\implies \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.0.22}$$

Thus, (2.0.22) is only possible if each equation in each system is a linear combination of the equations in other system.