Assignment 5

Gaydhane Vaibhav Digraj
Roll No. AI20MTECH11002

***Abstract*—This document explains the concept of finding the unknown value in an equation such that it is represents two straight lines.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_5

1 Problem

Find the value of k so that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0$ may represent two straight lines.

2 Explanation

Given equation,

$$12x^2 - 10xy + 2y^2 + 11x - 5y + k = 0 \tag{2.0.1}$$

is a second order equation. The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

$$\implies \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.5}$$

Equation (2.0.3) represents a pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.6}$$

Comparing equation (2.0.1) with (2.0.2), we can write in the form of (2.0.4) and (2.0.5) as,

$$\mathbf{V} = \begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{u} = \begin{pmatrix} \frac{11}{2} \\ \frac{-5}{2} \end{pmatrix} \tag{2.0.8}$$

$$f = k \tag{2.0.9}$$

The given equation represent two straight lines, substituting (2.0.7), (2.0.8), (2.0.9) in (2.0.6) to satisfy the equation.

$$\begin{vmatrix} 12 & -5 & \frac{11}{2} \\ -5 & 2 & \frac{-5}{2} \\ \frac{11}{2} & \frac{-5}{2} & k \end{vmatrix} = 0 \tag{2.0.10}$$

Expanding the above determinant along row 3,

$$\implies \frac{11}{2}\times(\frac{25}{2}-11)+\frac{5}{2}(-30+\frac{55}{2})+k\times(24-25) = 0$$

$$\implies \frac{33}{4} + (\frac{-25}{4}) - k = 0 \tag{2.0.11}$$

$$\implies \boxed{k = 2} \tag{2.0.12}$$

3 Solution

For $k = 2$ the given equation will represent two straight lines.

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \tag{3.0.1}$$

The pair of straight lines in vector form is given by,

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{3.0.2}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{3.0.3}$$

Equating their product with (2.0.3),

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{3.0.4}$$

Putting the values of $\mathbf{V}$, $\mathbf{u}$, f we get,

$$\mathbf{x}^T\begin{pmatrix} 12 & -5 \\ -5 & 2 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} \frac{11}{2} & \frac{-5}{2} \end{pmatrix}\mathbf{x} + 2 = 0 \tag{3.0.5}$$

Hence, from (3.0.5) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{3.0.6}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\mathbf{u} = -2\begin{pmatrix} \frac{11}{2} \\ \frac{-5}{2} \end{pmatrix} \tag{3.0.7}$$

$$c_1c_2 = f = 2 \tag{3.0.8}$$

The slopes of the lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \tag{3.0.9}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{3.0.10}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{3.0.11}$$

Substituting the given data in above equations (3.0.9) we get,

$$2m^2 - 10m + 12 = 0 \tag{3.0.12}$$

$$\implies m_i = \frac{-(-5) \pm \sqrt{-(-1)}}{2} \tag{3.0.13}$$

Solving equation (3.0.13) we get ,

$$m_1 = 3 \tag{3.0.14}$$

$$m_2 = 2 \tag{3.0.15}$$

$$\mathbf{n_1} = k_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{3.0.16}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.17}$$

Putting values of $\mathbf{n_1}$ and $\mathbf{n_2}$ in (3.0.6),

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -3k_1 & 0 \\ k_1 & -3k_1 \\ 0 & k_1 \end{pmatrix}\begin{pmatrix} -2k_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{3.0.18}$$

$$\implies \begin{pmatrix} 6k_1k_2 \\ -5k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 12 \\ -10 \\ 2 \end{pmatrix} \tag{3.0.19}$$

Thus from (3.0.19), $k_1k_2 = 2$. The possible combinations of (k_1, k_2) are (1,2), (2,1), (-1,-2), (-2,-1). Let's assume $k_1 = 2$, $k_2 = 1$, we get

$$\mathbf{n_1} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} \tag{3.0.20}$$

$$\mathbf{n_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.21}$$

From equation (3.0.7) we get,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\begin{pmatrix} \frac{11}{2} \\ \frac{-5}{2} \end{pmatrix} \tag{3.0.22}$$

$$\begin{pmatrix} -6 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix} \tag{3.0.23}$$

Converting (3.0.23) to augmented matrix and solving using row reduction,

$$\begin{pmatrix} -6 & -2 & -11 \\ 2 & 1 & 5 \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{R1}{-6}} \begin{pmatrix} 1 & \frac{1}{3} & \frac{11}{6} \\ 2 & 1 & 5 \end{pmatrix} \tag{3.0.24}$$

$$\begin{pmatrix} 1 & \frac{1}{3} & \frac{11}{6} \\ 2 & 1 & 5 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R2 - 2R1} \begin{pmatrix} 1 & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{pmatrix} \tag{3.0.25}$$

$$\begin{pmatrix} 1 & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1 - R2} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{pmatrix} \tag{3.0.26}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{pmatrix} \xleftrightarrow{R2 \leftarrow 3R2} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 4 \end{pmatrix} \tag{3.0.27}$$

Thus we get,

$$c_1 = 4 \tag{3.0.28}$$

$$c_2 = \frac{1}{2} \tag{3.0.29}$$

Equations (3.0.2), (3.0.3) can be modified as,

$$\begin{pmatrix} -6 & 2 \end{pmatrix}\mathbf{x} = 4 \tag{3.0.30}$$

$$\begin{pmatrix} -2 & 1 \end{pmatrix}\mathbf{x} = \frac{1}{2} \tag{3.0.31}$$

The figure below corresponds to the pair of straight lines represented by (3.0.30), (3.0.31).

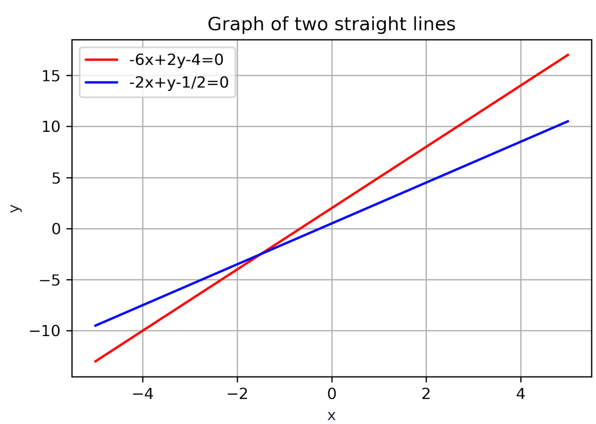

Fig. 1: Plot of Straight lines