Assignment 13

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***Abstract*—This document solves a problem involving basis and dimensions.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_13

1 Problem

Let V be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} \tag{1.0.1}$$

and let W_2 be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix} \tag{1.0.2}$$

1) Prove that W_1 and W_2 are subspaces of V.
2) Find the dimension of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

2 Theorem

A non-empty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c $\in$ F, the vector $c\alpha + \beta \in$ W.

3 Solution

1) Let $A_1, A_2 \in W_1$ where,

$$A_1 = \begin{pmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{pmatrix}, A_2 = \begin{pmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{pmatrix} \tag{3.0.1}$$

Let $c \in F$ then,

$$cA_1 + A_2 = \begin{pmatrix} cx_1 + x_2 & -cx_1 - x_2 \\ cy_1 + y_2 & cz_1 + z_2 \end{pmatrix} = \begin{pmatrix} u & -u \\ v & w \end{pmatrix} \tag{3.0.2}$$

Thus $cA_1 + A_2 \in W_1$. **Hence W_1 is a subspace.**
Similarly, let $A_1, A_2 \in W_2$ where,

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{pmatrix} \tag{3.0.3}$$

Let $c \in F$ then,

$$cA_1 + A_2 = \begin{pmatrix} ca_1 + a_2 & cb_1 + b_2 \\ -ca_1 - a_2 & cc_1 + c_2 \end{pmatrix} = \begin{pmatrix} u & v \\ -u & w \end{pmatrix} \tag{3.0.4}$$

Thus $cA_1 + A_2 \in W_2$. **Hence W_2 is a subspace.**

2) The subspace W_1 can be given as,

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = x\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.5}$$

$$= xA_1 + yA_2 + zA_2 \tag{3.0.6}$$

Now,

$$x\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.7}$$

$$\implies x = y = z = 0 \tag{3.0.8}$$

A_1, A_2, A_3 are linearly independent and spans W_1. Thus $\{A_1, A_2, A_3\}$ forms basis for W_1.
$\therefore$ **dimension of W_1 is 3.**
The subspace W_2 can be given as,

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.9}$$

$$= aA_1 + bA_2 + cA_2 \tag{3.0.10}$$

Now,

$$a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.11}$$

$$\implies a = b = c = 0 \tag{3.0.12}$$

A_1, A_2, A_3 are linearly independent and spans W_2. Thus $\{A_1, A_2, A_3\}$ forms basis for W_2.
$\therefore$ **dimension of W_2 is 3.**
Subspace $W_1 + W_2$ is given by,

$$\begin{pmatrix} x + a & -x + b \\ y - a & z + c \end{pmatrix} \tag{3.0.13}$$

For $x + a \neq -x + b \neq y - a \neq z + c$,

$$\begin{pmatrix} x+a & -x+b \\ y-a & z+c \end{pmatrix} = \begin{pmatrix} j & k \\ l & m \end{pmatrix} \tag{3.0.14}$$

$$= j\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + l\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + m\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.15}$$

$$= jA_1 + kA_2 + lA_3 + mA_4 \tag{3.0.16}$$

Now,

$$jA_1 + kA_2 + lA_3 + mA_4 = 0 \tag{3.0.17}$$

$$\implies j = k = l = m = 0 \tag{3.0.18}$$

A_1, A_2, A_3, A_4 are linearly independent and spans $W_1 + W_2$. Thus $\{A_1, A_2, A_3, A_4\}$ forms a basis.

$\therefore$ **dimension of $W_1 + W_2$ is 4.**

The subspace $W_1 \cap W_2$ is given as,

$$\begin{pmatrix} x & -x \\ -x & y \end{pmatrix} = x\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.19}$$

$$= xA_1 + yA_2 \tag{3.0.20}$$

Now,

$$x\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.21}$$

$$\implies x = y = 0 \tag{3.0.22}$$

A_1, A_2 are linearly independent and spans $W_1 \cap W_2$. Thus, $\{A_1, A_2\}$ forms a basis.

$\therefore$ **dimension of $W_1 \cap W_2$ is 2.**