Assignment 9

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***Abstract*—This document solves whether two system of linear equations are linear equivalent or not.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_9

1 Problem

Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

2 Solution

The first system of linear equations can be written as,

$$\mathbf{A}\mathbf{x} = 0 \tag{2.0.1}$$
$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{2.0.2}$$

Forming augmented matrix and performing row reduction using Elementary Matrices,

$$\implies \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \tag{2.0.3}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \tag{2.0.4}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.5}$$
$$\leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.6}$$
$$\implies \mathbf{R}\mathbf{x} = 0 \tag{2.0.7}$$

where,

$$\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.8}$$

The second system of linear equations can be written as,

$$\mathbf{B}\mathbf{x} = 0 \tag{2.0.9}$$
$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{2.0.10}$$

Forming augmented matrix and performing row reduction using Elementary Matrices,

$$\implies \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{2.0.11}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix} \tag{2.0.12}$$
$$\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.13}$$
$$\leftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.14}$$
$$\leftrightarrow \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.15}$$
$$\implies \mathbf{R}^{'}\mathbf{x} = 0 \tag{2.0.16}$$

where,

$$\mathbf{R}^{'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.17}$$

From (2.0.8) and (2.0.17) we get,

$$\mathbf{R} = \mathbf{R}^{'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.18}$$

Thus, the two system of linear equations have same solution set as their row reduced echelon form are same. Hence the system of linear equations are equivalent. Now,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.19}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{4}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.20}$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.21}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{2}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.22}$$

Combining (2.0.20), (2.0.22) into a single matrix equation we get,

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.23}$$

Thus $\mathbf{Bx} = 0$ can be obtained by multiplying $\mathbf{Ax} = 0$ with matrix $\mathbf{C}$ where,

$$\mathbf{C} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \tag{2.0.24}$$

Similarly,

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} \tag{2.0.25}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = (1) \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + (-2) \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.26}$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.27}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.28}$$

Thus from (2.0.20), (2.0.22), (2.0.26) and (2.0.28) each equation is expressed as linear combination of equations in other system.