

Assignment 17

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Abstract—This document solves a problem involving linear functional.

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_17

1 PROBLEM

Similar matrices have the same trace. Thus we can define the trace of a linear operator on a finite-dimensional space to the trace of any matrix which represents the operator in a ordered basis. This is well-defined since all such representing matrices for one operator are similar.

Now let V be the space of all 2×2 matrices over the field F and let P be a fixed 2×2 matrix. Let T be the linear operator on V defined by $T(A) = PA$. Prove that $\text{trace}(T) = 2\text{trace}(P)$.

2 SOLUTION

Given V is the space of all 2×2 matrices over field F . P is a 2×2 matrix,

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad (2.0.1)$$

$$\text{trace}(P) = p_{11} + p_{22} \quad (2.0.2)$$

Let $\mathcal{B} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ be the ordered basis of V where,

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.3)$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.4)$$

Given, $T(A) = PA$,

$$T(e_{11}) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & 0 \end{pmatrix} \quad (2.0.5)$$

$$= p_{11}e_{11} + p_{21}e_{21} \quad (2.0.6)$$

$$T(e_{12}) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_{11} \\ 0 & p_{21} \end{pmatrix} \quad (2.0.7)$$

$$= p_{11}e_{12} + p_{21}e_{22} \quad (2.0.8)$$

$$T(e_{21}) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{12} & 0 \\ p_{22} & 0 \end{pmatrix} \quad (2.0.9)$$

$$= p_{12}e_{11} + p_{22}e_{21} \quad (2.0.10)$$

$$T(e_{22}) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & p_{12} \\ 0 & p_{22} \end{pmatrix} \quad (2.0.11)$$

$$= p_{12}e_{12} + p_{22}e_{22} \quad (2.0.12)$$

The matrix representation of linear functional T in the ordered basis \mathcal{B} is given as,

$$T = \begin{pmatrix} [T(e_{11})]_{\mathcal{B}} & [T(e_{12})]_{\mathcal{B}} & [T(e_{21})]_{\mathcal{B}} & [T(e_{22})]_{\mathcal{B}} \end{pmatrix} \quad (2.0.13)$$

$$T = \begin{pmatrix} p_{11} & 0 & p_{12} & 0 \\ 0 & p_{11} & 0 & p_{12} \\ p_{21} & 0 & p_{22} & 0 \\ 0 & p_{21} & 0 & p_{22} \end{pmatrix} \quad (2.0.14)$$

Thus,

$$\text{trace}(T) = 2(p_{11} + p_{22}) \quad (2.0.15)$$

$$\therefore \text{trace}(T) = 2\text{trace}(P) \quad (2.0.16)$$