Assignment 9

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***Abstract*—This document solves whether two system of linear equations are linear equivalent or not.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_9

1 Problem

Let $\mathbb{F}$ be the field of complex numbers. Are the following two systems of linear equations equivalent? If so, express each equation in each system as a linear combination of the equations in the other system.

$$x_1 - x_2 = 0$$
$$2x_1 + x_2 = 0$$

and

$$3x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$

2 Solution

The given system of linear equations can be written as,

$$\mathbf{A}\mathbf{x} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.2}$$

$$\mathbf{B}\mathbf{x} = 0 \tag{2.0.3}$$

$$\implies \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.0.4}$$

Now we can obtain $\mathbf{B}$ from matrix $\mathbf{A}$ by performing elementary row operations given as,

$$\mathbf{B} = \mathbf{C}\mathbf{A} \tag{2.0.5}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{C} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \tag{2.0.6}$$

where $\mathbf{C}$ is product of elementary matrices given as,

$$\begin{aligned} \mathbf{C} &= (\mathbf{E_7}\mathbf{E_6}\mathbf{E_5}\mathbf{E_4}\mathbf{E_3}\mathbf{E_2}\mathbf{E_1}) \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \end{aligned} \tag{2.0.7}$$

Now, performing elementary operations on the right side of $\mathbf{A}$ we obtain matrix $\mathbf{B}$ given as,

$$\mathbf{B} = \mathbf{A}\mathbf{P} \tag{2.0.8}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{P} \tag{2.0.9}$$

where, $\mathbf{P}$ is product of elementary matrices given by,

$$\begin{aligned} \mathbf{P} &= (\mathbf{E_1}\mathbf{E_2}\mathbf{E_3}\mathbf{E_4}\mathbf{E_5}) \\ &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{-1}{3} \end{pmatrix} \end{aligned} \tag{2.0.10}$$

Similarly, $\mathbf{A}$ can be obtained from matrix $\mathbf{B}$ from (2.0.5) as,

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B} \tag{2.0.11}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \mathbf{C}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.0.12}$$

Matrix $\mathbf{C}$ is product of elementary matrices and hence invertible and is given as,

$$\begin{aligned} \mathbf{C}^{-1} &= \left(\mathbf{E_1}^{-1}\mathbf{E_2}^{-1}\mathbf{E_3}^{-1}\mathbf{E_4}^{-1}\mathbf{E_5}^{-1}\mathbf{E_6}^{-1}\mathbf{E_7}^{-1}\right) \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{3} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{aligned} \tag{2.0.13}$$

Matrix $\mathbf{A}$ can also be obtained from (2.0.8) given as,

$$\mathbf{A} = \mathbf{B}\mathbf{P}^{-1} \tag{2.0.14}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{P}^{-1} \tag{2.0.15}$$

where,

$$\begin{aligned} \mathbf{P}^{-1} &= \left(\mathbf{E_5}^{-1} \mathbf{E_4}^{-1} \mathbf{E_3}^{-1} \mathbf{E_2}^{-1} \mathbf{E_1}^{-1} \right) \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{5}{2} & 2 \end{pmatrix} \end{aligned} \tag{2.0.16}$$

Thus (2.0.4) can be obtained from (2.0.2) by multiplying it with matrix $\mathbf{C}$, and by inverse row operations (2.0.2) can be obtained back from (2.0.4) since $\mathbf{C}$ is product of elementary matrices and hence invertible.

Thus the two given homogeneous systems are row equivalent.

Now writing equations in matrix-vector form as,

$$3x_1 + x_2 = \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.17}$$

$$\implies \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{4}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.18}$$

$$x_1 + x_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.19}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} + \frac{2}{3} \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.20}$$

(2.0.18), (2.0.20) is same as multiplying $\mathbf{C}$ with $\mathbf{A}$ as it takes the linear combination of each rows of matrix $\mathbf{A}$ i.e, (2.0.6)

$$x_1 - x_2 = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} \tag{2.0.21}$$

$$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = (1) \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + (-2) \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.22}$$

$$2x_1 + x_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.23}$$

$$\implies \begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} 3 & 1 \end{pmatrix} \mathbf{x} + \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} \tag{2.0.24}$$

(2.0.22), (2.0.24) is same as multiplying $\mathbf{C}^{-1}$ with $\mathbf{B}$ as it takes the linear combination of each rows of matrix $\mathbf{B}$ i.e, (2.0.12)

Thus each equation in each system can be expressed as a linear combination of the equations in the other system when they are equivalent.