Assignment 17

Gaydhane Vaibhav Digraj
Roll No. AI20MTECH11002

***Abstract*—This document solves a problem involving linear functional.**

Download latex-tikz codes from

https://github.com/Vaibhav11002/EE5609/tree/master/Assignment_17

1 Problem

Similar matrices have the same trace. Thus we can define the trace of a linear operator on a finite-dimensional space to the trace of any matrix which represents the operator in a ordered basis. This is well-defined since all such representing matrices for one operator are similar.

Now let V be the space of all 2×2 matrices over the field F and let P be a fixed 2×2 matrix. Let T be the linear operator on V defined by $T(A) = PA$. Prove that $tr(T) = 2tr(P)$.

2 Solution

Given V is the space of all 2×2 matrices over field F. P is a 2×2 matrix,

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \tag{2.0.1}$$

Let $\mathcal{B} = \{e_{11}, e_{12}, e_{21}, e_{22}\}$ be the ordered basis of V where,

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.2}$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.3}$$

Given, $T(A) = PA$,

$$T(e_{11}) = Pe_{11} = \begin{pmatrix} p_{11} & 0 \\ p_{21} & 0 \end{pmatrix} \tag{2.0.4}$$

$$= p_{11}e_{11} + p_{21}e_{21} \tag{2.0.5}$$

$$T(e_{12}) = Pe_{12} = \begin{pmatrix} 0 & p_{11} \\ 0 & p_{21} \end{pmatrix} \tag{2.0.6}$$

$$= p_{11}e_{12} + p_{21}e_{22} \tag{2.0.7}$$

$$T(e_{21}) = Pe_{21} = \begin{pmatrix} p_{12} & 0 \\ p_{22} & 0 \end{pmatrix} \tag{2.0.8}$$

$$= p_{12}e_{11} + p_{22}e_{21} \tag{2.0.9}$$

$$T(e_{22}) = Pe_{22} = \begin{pmatrix} 0 & p_{12} \\ 0 & p_{22} \end{pmatrix} \tag{2.0.10}$$

$$= p_{12}e_{12} + p_{22}e_{22} \tag{2.0.11}$$

The matrix representation of linear functional T in the ordered basis $\mathcal{B}$ is given as,

$$T = \begin{pmatrix} [T(e_{11})]_{\mathcal{B}} & [T(e_{12})]_{\mathcal{B}} & [T(e_{21})]_{\mathcal{B}} & [T(e_{22})]_{\mathcal{B}} \end{pmatrix} \tag{2.0.12}$$

$$T = \begin{pmatrix} p_{11} & 0 & p_{12} & 0 \\ 0 & p_{11} & 0 & p_{12} \\ p_{21} & 0 & p_{22} & 0 \\ 0 & p_{21} & 0 & p_{22} \end{pmatrix} \tag{2.0.13}$$

$$= \begin{pmatrix} p_{11}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & p_{12}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ p_{21}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & p_{22}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{2.0.14}$$

$$= \begin{pmatrix} p_{11}I & p_{12}I \\ p_{21}I & p_{22}I \end{pmatrix} = P \otimes I \tag{2.0.15}$$

Now by using the property of kronecker product we get,

$$tr(T) = tr(P \otimes I) = tr(P)tr(I) \tag{2.0.16}$$

$$= 2tr(P) \tag{2.0.17}$$

$$\therefore tr(T) = 2tr(P) \tag{2.0.18}$$