Logarithms and Their Properties Exercise 1 :
Single Option Correct Type Questions

- This section contains **20 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $\log_{10} 2 = 0.3010\ldots$, the number of digits in the number 2000^{2000} is

(a) 6601 (b) 6602 (c) 6603 (d) 6604

2. There exist a positive number λ, such that $\log_2 x + \log_4 x + \log_8 x = \log_\lambda x$, for all positive real numbers x.
If $\lambda = \sqrt[b]{a}$, where $a, b \in N$, the smallest possible value of $(a + b)$ is equal to

(a) 12 (b) 63 (c) 65 (d) 75

3. If a, b and c are the three real solutions of the equation

$$x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \frac{2}{\dfrac{1}{\sqrt{x+1} - 1} - \dfrac{1}{\sqrt{x+1} + 1}}$$

where, $a > b > c$, then a, b, c are in

(a) AP (b) GP
(c) HP (d) $a^{-1} + b^{-1} = c^{-1}$

4. If $f(n) = \prod_{i=2}^{n-1} \log_i (i+1)$, the value of $\sum_{k=1}^{100} f(2^k)$ equals

(a) 5010 (b) 5050 (c) 5100 (d) 5049

5. If $\log_3 27 \cdot \log_x 7 = \log_{27} x \cdot \log_7 3$, the least value of x, is

(a) 7^{-3} (b) 3^{-7} (c) 7^3 (d) 3^7

6. If $x = \log_5 (1000)$ and $y = \log_7 (2058)$, then

(a) $x > y$ (b) $x < y$
(c) $x = y$ (d) None of these

7. If $\log_5 120 + (x - 3) - 2\log_5 (1 - 5^{x-3})$
$= -\log_5 (0.2 - 5^{x-4})$, then x is

(a) 1 (b) 2 (c) 3 (d) 4

8. If $x_n > x_{n-1} > \ldots > x_2 > x_1 > 1$, the value of

$$\log_{x_1} \log_{x_2} \log_{x_3} \ldots \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{x_1}}}} \text{ is}$$

(a) 0 (b) 1
(c) 2 (d) undefined

9. If $\dfrac{x(y + z - x)}{\log x} = \dfrac{y(z + x - y)}{\log y} = \dfrac{z(x + y - z)}{\log z}$,
then $x^y y^x = z^y y^z$ is equal to

(a) $z^x x^z$ (b) $x^z y^x$ (c) $x^y y^z$ (d) $x^x y^y$

10. If $y = a^{\frac{1}{1 - \log_a x}}$ and $z = a^{\frac{1}{1 - \log_a y}}$, then x is equal to

(a) $a^{\frac{1}{1 + \log_a z}}$ (b) $a^{\frac{1}{2 + \log_a z}}$ (c) $a^{\frac{1}{1 - \log_a z}}$ (d) $a^{\frac{1}{2 - \log_a z}}$

11. If $\log_{0.3} (x - 1) < \log_{0.09} (x - 1)$, then x lies in the interval

(a) $(-\infty, 1)$
(b) $(1, 2)$
(c) $(2, \infty)$
(d) None of the above

12. The value of $a^x - b^y$ is (where $x = \sqrt{\log_a b}$ and $y = \sqrt{\log_b a}$, $a > 0$, $b > 0$ and $a, b \neq 1$)

(a) 1 (b) 2
(c) 0 (d) -1

13. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$, then $\dfrac{xyz}{xy + yz + zx}$ is equal to

(a) 0 (b) 1
(c) -1 (d) 2

14. The value of $a^{\frac{\log_b (\log_b N)}{\log_b a}}$ is

(a) $\log_a N$ (b) $\log_b N$
(c) $\log_N a$ (d) $\log_N b$

15. The value of $49^A + 5^B$, where $A = 1 - \log_7 2$ and $B = -\log_5 4$ is

(a) 10.5 (b) 11.5
(c) 12.5 (d) 13.5

16. The number of real values of the parameter λ for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} \lambda = 0$ with real coefficients will have exactly one solution is

(a) 1 (b) 2
(c) 3 (d) 4

17. The number of roots of the equation $x^{\log_x (x+3)^2} = 16$ is

(a) 1 (b) 0
(c) 2 (d) 4

18. The point on the graph $y = \log_2 \log_6 \{2^{\sqrt{(2x+1)}} + 4\}$, whose y-coordinate is 1 is

(a) (1, 1) (b) (6,1)
(c) (8, 1) (d) (12, 1)

19. Given, $\log 2 = 0.301$ and $\log 3 = 0.477$, then the number of digits before decimal in $3^{12} \times 2^8$ is

(a) 7 (b) 8
(c) 9 (d) 11

20. The number of solution(s) for the equation $2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$, is

(a) one (b) two
(c) three (d) four

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Logarithms and Their Properties Exercise 2 :
More than One Correct Option Type Questions

- This section contains **9 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

21. If $x^{(\log_2 x)^2 - 6\log_2 x + 11} = 64$, then x is equal to

(a) 2 (b) 4 (c) 6 (d) 8

22. If $\log_\lambda x \cdot \log_5 \lambda = \log_x 5, \lambda \neq 1, \lambda > 0$, then x is equal to

(a) λ (b) 5 (c) $\frac{1}{5}$ (d) None of these

23. If $S = \{x : \sqrt{\log_x \sqrt{3x}}$, where $\log_3 x > -1\}$, then

(a) S is a finite set (b) $S \in \phi$

(c) $S \subset (0, \infty)$ (d) S properly contains $\left(\frac{1}{3}, \infty\right)$

24. If x satisfies $\log_2(9^{x-1} + 7) = 2 + \log_2(3^{x-1} + 1)$, then

(a) $x \in Q$

(b) $x \in N$

(c) $x \in \{x \in Q : x < 0\}$

(d) $x \in N_e$ (set of even natural numbers)

25. $\log_p \log_p \underbrace{\sqrt[p]{\sqrt[p]{\sqrt[p]{\ldots \sqrt[p]{p}}}}}_{n \text{ times}}$, $p > 0$ and $p \neq 1$ is equal to

(a) n (b) $-n$

(c) $\frac{1}{n}$ (d) $\log_{1/p}(p^n)$

26. If $\log_a x = \alpha, \log_b x = \beta, \log_c x = \gamma$ and $\log_d x = \delta, x \neq 1$ and $a, b, c, d \neq 0, > 1$, then $\log_{abcd} x$ equals

(a) $\leq \frac{\alpha + \beta + \gamma + \delta}{16}$ (b) $\geq \frac{\alpha + \beta + \gamma + \delta}{16}$

(c) $\frac{1}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$ (d) $\frac{1}{\alpha\beta\gamma\delta}$

27. If $\log_{10} 5 = a$ and $\log_{10} 3 = b$, then

(a) $\log_{10} 8 = 3(1 - a)$ (b) $\log_{40} 15 = \frac{(a + b)}{(3 - 2a)}$

(c) $\log_{243} 32 = \left(\frac{1 - a}{b}\right)$ (d) All of these

28. If x is a positive number different from 1, such that $\log_a x, \log_b x$ and $\log_c x$ are in AP, then

(a) $\log b = \frac{2(\log a)(\log c)}{(\log a + \log c)}$ (b) $b = \frac{a + c}{2}$

(c) $b = \sqrt{ac}$ (d) $c^2 = (ac)^{\log_a b}$

29. If $|a| < |b|$, $b - a < 1$ and a, b are the real roots of the equation $x^2 - |\alpha| x - |\beta| = 0$, the equation $\log_{|b|} \left|\frac{x}{a}\right| - 1 = 0$ has

(a) one root lying in interval $(-\infty, a)$

(b) one root lying in interval (b, ∞)

(c) one positive root

(d) one negative root

Logarithms and Their Properties Exercise 3 :
Passage Based Questions

- This section contains **4 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Q. Nos. 30 to 32)

Let $\log_2 N = a_1 + b_1, \log_3 N = a_2 + b_2$ *and* $\log_5 N = a_3 + b_3$, *where* $a_1, a_2, a_3 \in I$ *and* $b_1, b_2, b_3 \in [0, 1)$.

30. If $a_1 = 5$ and $a_2 = 3$, the number of integral values of N is

(a) 16 (b) 32 (c) 48 (d) 64

31. If $a_1 = 6, a_2 = 4$ and $a_3 = 3$, the largest integral value of N is

(a) 124 (b) 63

(c) 624 (d) 127

32. If $a_1 = 6, a_2 = 4$ and $a_3 = 3$, the difference of largest and smallest integral values of N, is

(a) 2 (b) 8

(c) 14 (d) 20

Passage II
(Q. Nos. 33 to 35)

Let 'S' denotes the antilog of 0.5 to the base 256 and 'K' denotes the number of digits in 6^{10} *(given* $\log_{10} 2 = 0.301, \log_{10} 3 = 0.477$*) and G denotes the number of positive integers, which have the characteristic 2, when the base of logarithm is 3.*

33. The value of G is

(a) 18 (b) 24 (c) 30 (d) 36

34. The value of KG is

(a) 72 (b) 144 (c) 216 (d) 288

35. The value of SKG is

(a) 1440 (b) 17280
(c) 2016 (d) 2304

Passage III

(Q. Nos. 36 to 38)

Suppose 'U' denotes the number of digits in the number $(60)^{100}$ and 'M' denotes the number of cyphers after decimal, before a significant figure comes in $(8)^{-296}$. If the fraction U/M is expressed as rational number in the lowest term as p/q (given $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$).

36. The value of p is

(a) 1 (b) 2 (c) 3 (d) 4

37. The value of q is

(a) 5 (b) 2
(c) 3 (d) 4

38. The equation whose roots are p and q, is

(a) $x^2 - 3x + 2 = 0$ (b) $x^2 - 5x + 6 = 0$
(c) $x^2 - 7x + 12 = 0$ (d) $x^2 - 9x + 20 = 0$

Passage IV (Q. Nos. 39 to 41)

Let G, O, E and L be positive real numbers such that $\log(G \cdot L) + \log(G \cdot E) = 3$, $\log(E \cdot L) + \log(E \cdot O) = 4$, $\log(O \cdot G) + \log(O \cdot L) = 5$ (base of the log is 10).

39. If the value of the product $(GOEL)$ is λ, the value of $\sqrt{\log \lambda \sqrt{\log \lambda \sqrt{\log \lambda \ldots}}}$ is

(a) 3 (b) 4
(c) 5 (d) 7

40. If the minimum value of $3G + 2L + 2O + E$ is $2^{\lambda} 3^{\mu} 5^{\nu}$, where λ, μ and ν are whole numbers, the value of $\sum (\lambda^{\mu} + \mu^{\lambda})$ is

(a) 7 (b) 13
(c) 19 (d) None of these

41. If $\log\left(\frac{G}{O}\right)$ and $\log\left(\frac{O}{E}\right)$ are the roots of the equation

(a) $x^2 + x = 0$ (b) $x^2 - x = 0$
(c) $x^2 - 2x + 3 = 0$ (d) $x^2 - 1 = 0$

Logarithms and Their Properties Exercise 4 :

Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

42. If $x, y \in R^+$ and $\log_{10}(2x) + \log_{10} y = 2$ and $\log_{10} x^2 - \log_{10}(2y) = 4$ and $x + y = \frac{m}{n}$, where m and n are relative prime, the value of $m - 3n^6$ is

43. A line $x = \lambda$ intersects the graph of $y = \log_5 x$ and $y = \log_5(x + 4)$. The distance between the points of intersection is 0.5. Given $\lambda = a + \sqrt{b}$, where a and b are integers, the value of $(a + b)$ is

44. If the left hand side of the equation $a(b - c)x^2 + b(c - a)xy + c(a - b)y^2 = 0$ is a perfect square, the value of

$$\left\{\frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)}\right\}^2, (a, b, c \in R^+, a > c) \text{ is}$$

45. Number of integers satisfying the inequality $\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9$ is

46. If $x > 2$ is a solution of the equation $|\log_{\sqrt{3}} x - 2| + |\log_3 x - 2| = 2$, then the value of x is

47. Number of integers satisfying the inequality

$$\log_2 \sqrt{x} - 2\log^2_{1/4} x + 1 > 0, \text{ is}$$

48. The value of $b (> 0)$ for which the equation $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is

49. The value of $\dfrac{2^{\log_{2^{1/4}} 2} - 3^{\log_{27} 125} - 4}{7^{4\log_{49} 2} - 3}$ is

50. If x_1 and x_2 $(x_2 > x_1)$ are the integral solutions of the equation $(\log_5 x)^2 + \log_{5x}\left(\frac{5}{x}\right) = 1$, the value of $|x_2 - 4x_1|$ is

51. If $x = \log_{\lambda} a = \log_a b = \frac{1}{2}\log_b c$ and $\log_{\lambda} c = nx^{n+1}$, the value of n is

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Logarithms and Their Properties Exercise 5 :
Matching Type Questions

- This section contains **3 questions**. Questions 52 to 54 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

52.

	Column I		Column II
(A)	$\dfrac{\log_3 243}{\log_2 \sqrt{32}}$	(p)	positive integer
(B)	$\dfrac{2\log 6}{(\log 12 + \log 3)}$	(q)	negative integer
(C)	$\log_{1/3}\left(\dfrac{1}{9}\right)^{-2}$	(r)	rational but not integer
(D)	$\dfrac{\log_5 16 - \log_5 4}{\log_5 128}$	(s)	prime

53.

	Column I		Column II
(A)	The expression $\sqrt{\log_{0.5}^2 8}$ has the value equal to	(p)	1
(B)	The value of the expression $(\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3 + 3$, is	(q)	2
(C)	Let $N = \log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 \left(\dfrac{1}{6}\right)$. The value of $[N]$ is (where $[\cdot]$ denotes the greatest integer function)	(r)	3
(D)	If $(52.6)^a = (0.00526)^b = 100$, the value of $\dfrac{1}{a} - \dfrac{1}{b}$ is	(s)	4

54.

	Column I		Column II
(A)	If $\log_{1/x}\left\{\dfrac{2(x-2)}{(x+1)(x-5)}\right\} \geq 1$, then x can belongs to	(p)	$\left(0, \dfrac{1}{3}\right]$
(B)	If $\log_3 x - \log_3^2 x \leq \dfrac{3}{2}\log_{(1/2\sqrt{2})} 4$, then x can belongs to	(q)	$(1, 2]$
(C)	If $\log_{1/2}(4-x) \geq \log_{1/2} 2 - \log_{1/2}(x-1)$, then x belongs to	(r)	$[3, 4)$
(D)	Let α and β are the roots of the quadratic equation $(\lambda^2 - 3\lambda + 4)x^2 - 4(2\lambda - 1)x + 16 = 0$, if α and β satisfy the condition $\beta > 1 > \alpha$, then p can lie in	(s)	$(3, 8)$

Logarithms and Their Properties Exercise 6 :
Statement I and II Type Questions

- **Directions** Question numbers 55 to 60 are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
  (a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
  (b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
  (c) Statement-1 is true, Statement-2 is false
  (d) Statement-1 is false, Statement-2 is true

55. **Statement-1** $\log_{10} x < \log_3 x < \log_e x < \log_2 x$ $(x > 0, x \neq 1)$.

Statement-2 If $0 < x < 1$, then $\log_x a > \log_x b \Rightarrow 0 < a < b$.

56. **Statement-1** The equation $7^{\log_7 (x^3+1)} - x^2 = 1$ has two distinct real roots.

Statement-2 $a^{\log_a N} = N$, where $a > 0, a \neq 1$ and $N > 0$.

57. **Statement-1** $\left(\dfrac{1}{3}\right)^7 < \left(\dfrac{1}{3}\right)^4$

$$\Rightarrow \quad 7\log\left(\frac{1}{3}\right) < 4\log\left(\frac{1}{3}\right) \quad \Rightarrow \quad 7 < 4$$

Statement-2 If $ax < ay$, where $a < 0$, $x, y > 0$, then $x > y$.

58. **Statement-1** The equation $x^{\log_x (1-x)^2} = 9$ has two distinct real solutions.

Statement-2 $a^{\log_a b} = b$, when $a > 0, a \neq 1, b > 0$.

59. **Statement-1** The equation $(\log x)^2 + \log x^2 - 3 = 0$ has two distinct solutions.

Statement-2 $\log x^2 = 2\log x$.

60. **Statement-1** $\log_x 3 \cdot \log_{x/9} 3 = \log_{81}(3)$ has a solution.
Statement-2 Change of base in logarithms is possible.

Logarithms and Their Properties Exercise 7 :
Subjective Type Questions

- In this section, there are **27 subjective** questions.

61. (i) If $\log_7 12 = a$, $\log_{12} 24 = b$, then find value of $\log_{54} 168$ in terms of a and b.

(ii) If $\log_3 4 = a$, $\log_5 3 = b$, then find the value of $\log_3 10$ in terms of a and b.

62. If $\dfrac{\ln a}{b-c} = \dfrac{\ln b}{c-a} = \dfrac{\ln c}{a-b}$, prove the following.

(i) $abc = 1$

(ii) $a^a \cdot b^b \cdot c^c = 1$

(iii) $a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2} = 1$

(iv) $a + b + c \geq 3$

(v) $a^a + b^b + c^c \geq 3$

(vi) $a^{b^2+bc+c^2} + b^{c^2+ca+a^2} + c^{a^2+ab+b^2} \geq 3$

63. Prove that $\log_{10} 2$ lies between $\dfrac{1}{3}$ and $\dfrac{1}{4}$.

64. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the number of integers in

(i) 5^{200} (ii) 6^{20}

(iii) the number of zeroes after the decimal is 3^{-500}.

65. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find the value of $\log(3.375)$.

66. Find the least value of $\log_2 x - \log_x (0.125)$ for $x > 1$.

67. Without using the tables, prove that

$$\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > 2.$$

68. Solve the following equations.

(i) $x^{1+\log_{10} x} = 10x$

(ii) $\log_2(9 + 2^x) = 3$

(iii) $2 \cdot x^{\log_4 3} + 3^{\log_4 x} = 27$

(iv) $\log_4 \log_3 \log_2 x = 0$

(v) $x^{\frac{\log_{10} x + 5}{3}} = 10^{5+\log_{10} x}$

(vi) $\log_3\left(\log_9 x + \dfrac{1}{2} + 9^x\right) = 2x$

(vii) $4^{\log_{10} x+1} - 6^{\log_{10} x} - 2 \cdot 3^{\log_{10} x^2+2} = 0$

(viii) $\dfrac{\log_{10}(x-3)}{\log_{10}(x^2-21)} = \dfrac{1}{2}$

(ix) $x^{\log_2 x + 4} = 32$

(x) $\log_a x = x$, where $a = x^{\log_4 x}$

(xi) $\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$

69. Find a rational number, which is 50 times its own logarithm to the base 10.

70. Find the value of the expression

$$\frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6}.$$

71. Find the value of x satisfying
$\log_a\{1 + \log_b\{1 + \log_c(1 + \log_p x)\}\} = 0.$

72. Find the value of $4^{5\log_{4\sqrt{2}}(3-\sqrt{6}) - 6\log_8(\sqrt{3}-\sqrt{2})}$.

73. Solve the following inequations.

(i) $\log_{(2x+3)} x^2 < 1$

(ii) $\log_{2x}(x^2 - 5x + 6) < 1$

(iii) $\log_2(2 - x) < \log_{1/2}(x + 1)$

(iv) $\log_{x^2}(x + 2) < 1$

(v) $3^{\log_3 \sqrt{(x-1)}} < 3^{\log_3(x-6)} + 3$

(vi) $\log_{1/2}(3x - 1)^2 < \log_{1/2}(x + 5)^2$

(vii) $\log_{10} x + 2 \leq \log_{10}^2 x$

(viii) $\log_{10}(x^2 - 2x - 2) \leq 0$

(ix) $\log_x\left(2x - \dfrac{3}{4}\right) > 2$

(x) $\log_{1/3} x < \log_{1/2} x$

(xi) $\log_{2x+3} x^2 < \log_{2x+3}(2x + 3)$

(xii) $\log_2^2 x + 3\log_2 x \geq \dfrac{5}{2}\log_{4\sqrt{2}} 16$

(xiii) $(x^2 + x + 1)^x < 1$

(xiv) $\log_{(3x^2+1)} 2 < \dfrac{1}{2}$

(xv) $x^{(\log_{10} x)^2 - 3\log_{10} x + 1} > 1000$

(xvi) $\log_4\{14 + \log_6(x^2 - 64)\} \leq 2$

(xvii) $\log_2(9 - 2^x) \leq 10^{\log_{10}(3-x)}$

(xviii) $\log_a\left(\dfrac{2x+3}{x}\right) \geq 0$ for

(a) $a > 1$, (b) $0 < a < 1$

(xix) $1 + \log_2(x - 1) \leq \log_{x-1} 4$

(xx) $\log_{5x+4}(x^2) \leq \log_{5x+4}(2x + 3)$

74. Solve $\sqrt{\log_x (ax)^{1/5} + \log_a (ax)^{1/5}}$

$$+ \sqrt{\log_a\left(\frac{x}{a}\right)^{1/5} + \log_x\left(\frac{a}{x}\right)^{1/5}} = a.$$

75. It is known that $x = 9$ is root of the equation,

$$\log_\pi(x^2 + 15a^2) - \log_\pi(a - 2) = \log_\pi \frac{8ax}{a-2}$$

find the other roots of this equation.

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76. Solve $\log_4(\log_3 x) + \log_{1/4}(\log_{1/3} y) = 0$ and

$$x^2 + y^2 = \frac{17}{4}.$$

77. Find the real value(s) of x satisfying the equation $\log_{2x}(4x) + \log_{4x}(16x) = 4.$

78. Find the sum and product of all possible values of x which makes the following statement true

$$\log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36}\left(\frac{4}{9}\right).$$

79. Solve the equation

$$\frac{3}{2}\log_4(x+2)^3 + 3 = \log_4(4-x)^3 + \log_4(x+6)^3.$$

80. Solve $\log_2(4^{x+1} + 4)\cdot\log_2(4^x + 1) = \log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{8}}\right).$

81. Solve the system of equations $2^{\sqrt{x}+\sqrt{y}} = 256$ and

$$\log_{10}\sqrt{xy} - \log_{10}\left(\frac{3}{2}\right) = 1.$$

82. Solve the system of equations $\log_2 y = \log_4(xy - 2)$, $\log_9 x^2 + \log_3(x - y) = 1.$

83. Find the solution set of the inequality

$$2\log_{1/4}(x+5) > \frac{9}{4}\log_{\frac{1}{3\sqrt{3}}}(9) + \log_{\sqrt{(x+5)}}(2).$$

84. Solve $\log_3(\sqrt{x} + |\sqrt{x} - 1|) = \log_9(4\sqrt{x} - 3 + 4|\sqrt{x} - 1|).$

85. In the inequality

$$(\log_2 x)^4 - \left(\log_{1/2}\frac{x^5}{4}\right)^2 - 20\log_2 x + 148 < 0$$

holds true in (a, b), where $a, b \in N$. Find the value of $ab(a+b)$.

86. Find the value of x satisfying the equation

$$\sqrt{(\log_3 \sqrt[3]{3x} + \log_x \sqrt[3]{3x})\cdot\log_3 x^3} + \sqrt{\left(\log_3 \sqrt[3]{\left(\frac{x}{3}\right)} + \log_x \sqrt[3]{\left(\frac{3}{x}\right)}\right)\log_3 x^3} = 2.$$

87. If P is the number of natural numbers whose logarithm to the base 10 have the characteristic P and Q is the number of natural numbers reciprocals of whose 3 logarithms to the base 10 have the characteristic $-q$, show that $\log_{10} P - \log_{10} Q = p - q + 1$.

Logarithms and Their Properties Exercise 8 :

Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

88. Let $a = \log_3 \log_3 2$ and an integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, then k equals to **[IIT-JEE 2008, 1.5M]**

(a) 0 (b) 1
(c) 2 (d) 3

89. Let (x_0, y_0) be solution of the following equations $(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$, then x_0 is **[IIT-JEE 2011, 3M]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 6

90. The value of

$$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots}}}\right) \text{ is}$$

[IIT-JEE 2012, 4M]

91. If $3^x = 4^{x-1}$, then x equals **[JEE Advanced 2013, 3M]**

(a) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (b) $\frac{2}{2 - \log_2 3}$

(c) $\frac{1}{1 - \log_4 3}$ (d) $\frac{2\log_2 3}{2\log_2 3 - 1}$

Answers

Chapter Exercises

1. (c) 2. (d) 3. (b) 4. (b) 5. (a) 6. (a)
7. (a) 8. (b) 9. (a) 10. (c) 11. (c) 12. (c)
13. (b) 14. (b) 15. (c) 16. (b) 17. (b) 18. (d)
19. (c) 20. (b)
21. (a, b, d) 22. (b, c) 23. (c, d) 24. (a, b) 25. (b, d)
26. (a, c) 27. (a, b, c, d) 28. (a, d) 29. (c, d)
30. (b) 31. (d) 32. (a) 33. (a) 34. (b) 35. (d)
36. (b) 37. (c) 38. (b) 39. (b) 40. (a) 41. (d)
42. (9) 43. (6) 44. (4) 45. (3) 46. (9) 47. (3)
48. (4) 49. (7) 50. (1) 51. (2)
52. (A) $\rightarrow$ (p, s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)
53. (A) $\rightarrow$ (r), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (q)
54. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q, r), (D) $\rightarrow$ (s)
55. (d) 56. (d) 57. (d) 58. (d) 59. (c) 60. (d)
61. (i) $\dfrac{ab+1}{a\,(8-5b)}$ (ii) $\dfrac{ab+2}{2b}$ 64. (i) 140 (ii)16 (iii) 238 65. 0.528
66. $2\sqrt{3}$ 68. (i) 10, $\dfrac{1}{10}$ (ii) $x \in \phi$
(iii) $x = 16$ (iv) $x = 8$ (v) $\{10^{-5}, 10^{3}\}$
(vi) $x = \dfrac{1}{3}$ (vii) $x = \dfrac{1}{100}$ (viii) $x = 5$ (ix) $x = 2$ or $\dfrac{1}{32}$
(x) $x = 2$ (xi) $x = \dfrac{\pi}{3}$

69. 100 70. $\dfrac{1}{6}$ 71. 1 72. 9
73. (i) $x \in \left(-\dfrac{3}{2}, 3\right) \cup \{-1, 0\}$ (ii) $x \in \left(0, \dfrac{1}{2}\right) \cup (1, 2) \cup (3, 6)$
(iii) $x \in \left(-1, \dfrac{1-\sqrt{5}}{2}\right) \cup \left(\dfrac{1+\sqrt{5}}{2}, 2\right)$
(iv) $x \in (-2, 1) \cup (2, \infty) \sim \{-1, 0\}$ (v) $x > 6$
(vi) $x \in (-\infty, -5) \cup (-5, -1) \cup (3, \infty)$
(vii) $x \in (0, 10^{-1}] \cup [10^{2}, \infty)$
(viii) $x \in [-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$
(ix) $x \in \left(\dfrac{3}{8}, \dfrac{1}{2}\right) \cup \left(1, \dfrac{3}{2}\right)$ (x) $x \in (0, 1)$
(xi) $x \in \left(-\dfrac{3}{2}, -1\right) \cup (-1, 3)$ (xii) $x \in \left(0, \dfrac{1}{16}\right] \cup [2, \infty)$
(xiii) $x \in (-\infty, -1)$ (xiv) $x \in (-\infty, -1) \cup (1, \infty)$
(xv) $x \in (1000, \infty)$ (xvi) $x \in [-10, -8) \cup (8, 10]$
(xvii) $x \in (-\infty, 0]$
(xviii) (a) $x \in (-\infty, -3] \cup (0, \infty)$ (b) $x \in \left[-3, -\dfrac{3}{2}\right)$
(xix) $x \in (2, 3]$ (xx) $x \in \left(-\dfrac{3}{5}, -\dfrac{3}{2}\right) \cup [-1, 0) \cup (0, 3]$
74. $x = a^{4/5_{a2}}$ 75. $x = 15$ for $a = 3$
76. $x = 2$ or $\dfrac{1}{2}$, $y = \dfrac{1}{2}$ or 2
77. $x = 1, 2^{-3/2}$ 78. Sum $= \dfrac{9}{2}$, Product $= 2$
79. $x = 2$ 80. $x = 0$ 81. (9, 25) and (25, 9)
82. $x = 3,\ y = 2$ 83. $x \in (-5, -4) \cup (-3, -1)$
84. $x = \dfrac{25}{64}$ 85. 3456 86. $x \in (1, 3]$ 88. (b)
89. (c) 90. (4) 91. (a, b, c)

Solutions

1. $\log_{10} 2 = 0.3010$

Let $\quad y = 2000^{2000}$

$$\log_{10} y = 2000 \log_{10} 2000 = 2000 \times (\log_{10} 2 + 3)$$
$$= 2000 \times 3.3010 = 6602$$

So, the number of digits in $2000^{2000} = 6602 + 1 = 6603$.

2. $\because \lambda > 0$ and $\lambda \neq 1$ and $x > 0$

$$\log_2 x + \log_4 x + \log_8 x = \log_\lambda x$$
$$\Rightarrow \log_2 x + \frac{1}{2}\log_2 x + \frac{1}{3}\log_2 x = \log_\lambda x$$
$$\Rightarrow \frac{11}{6}\log_2 x = \log_\lambda x$$
$$\Rightarrow \frac{11}{6\log_x 2} = \frac{11}{\log_x \lambda}$$
$$\Rightarrow 11\log_x \lambda - 6\log_x 2 = 0$$
$$\Rightarrow \log_x\left(\frac{\lambda^{11}}{2^6}\right) = 0 \Rightarrow \frac{\lambda^{11}}{2^6} = 1$$
$$\Rightarrow \lambda^{11} = 2^6 \Rightarrow \lambda = 2^{6/11}$$
$$\Rightarrow \lambda = (2^6)^{1/11} \qquad \text{...(i)}$$

Given that, $\lambda = \sqrt[b]{a}$ and $a, b \in N$

$$\Rightarrow \lambda = a^{\frac{1}{b}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a = 2^6 \quad \text{and} \quad b = 11$$
$$\Rightarrow a + b = 64 + 11 = 75$$

3. $$x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \frac{2}{\dfrac{1}{\sqrt{x+1}-1} - \dfrac{1}{\sqrt{x+1}+1}}$$

Given, a, b and c are real solution Eq. (i) and $a > b > c$ and for Eq. (i) to be defined $x > 0$, $x > -1 \Rightarrow x > 0$ from Eq. (i),

$$x^{\log_{10}^2 x + 3\log_{10} x + 3} = \frac{2x}{2}$$

On taking logarithm both sides on base 10, then

$$(\log_{10}^2 x + 3\log_{10} x + 3)\log_{10} x = \log_{10} x$$
$$\Rightarrow (\log_{10}^2 x + 3\log_{10} x + 2)\log_{10} x = 0$$
$$\Rightarrow (\log_{10} x + 1)(\log_{10} x + 2)\log_{10} x = 0$$
$$\therefore \log_{10} x = -2, -1, 0$$
$$\therefore x = 10^{-2}, 10^{-1}, 10^0$$
$$x = \frac{1}{100}, \frac{1}{10}, 1$$

So, a, b, c can take values $a = 1, b = \frac{1}{10}, c = \frac{1}{100}$ $\quad (\because a > b > c)$

$$\therefore \quad a, b, c \in \text{GP}$$

4. $$f(n) = \prod_{i=2}^{n-1} \frac{\log(i+1)}{\log(i)} = \frac{\log(n)}{\log(2)} = \log_2 n$$

$$\therefore f(2^k) = k$$

Then, $$\sum_{k=1}^{100} f(2^k) = \sum_{k=1}^{100} k = \frac{100 \cdot (100+1)}{2} = 5050$$

5. $\log_3 27 \cdot \log_x 7 = \log_{27} x \cdot \log_7 3 \qquad$ ...(i)

Eq. (i) valid for $x > 0, x \neq 1$

On solving Eq. (i),

$$\log_3(3^3) \cdot \log_x 7 = \frac{1}{3}\log_3 x \cdot \log_7 3$$
$$\Rightarrow 9 \cdot \log_x 7 = \log_7 x$$
$$\Rightarrow 9 = (\log_7 x)^2$$
$$\Rightarrow \log_7 x = \pm 3$$
$$\Rightarrow x = 7^3 \quad \text{or} \quad x = 7^{-3}$$

Then, the least value of x is $\frac{1}{7^3}$ i.e., 7^{-3}.

6. $\because \quad x = \log_5(5^3 \times 8) = 3 + \log_5 8$

$$\Rightarrow x - 3 = \log_5 8 \qquad \text{...(i)}$$

and $\quad y = \log_7(7^3 \times 6) = 3 + \log_7 6$

$$\Rightarrow y - 3 = \log_7 6 \qquad \text{...(ii)}$$
$$\because \quad 8 > 6 \quad \text{and} \quad 7 > 5$$
$$\Rightarrow \log 8 > \log 6 \text{ and } \log 7 > \log 5$$

or $\quad (\log 8)(\log 7) > (\log 6)(\log 5)$

$$\Rightarrow \log_5 8 > \log_7 6$$
$$\Rightarrow x - 3 > y - 3 \qquad \text{[from Eqs. (i) and (ii)]}$$
$$\therefore x > y$$

7. $\because \log_5 120 + (x-3) - 2\log_5(1 - 5^{x-3}) = -\log_5(0.2 - 5^{x-4})$

$$\Rightarrow \log_5(5 \times 24) + (x-3)$$
$$= \log_5(1 - 5^{x-3})^2 - \log_5\left(\frac{1 - 5^{x-3}}{5}\right)$$
$$\Rightarrow 1 + \log_5 24 + (x-3) = \log_5\{5 \cdot (1 - 5^{x-3})\}$$
$$\Rightarrow 1 + \log_5(24 \cdot 5^{x-3}) = 1 + \log_5(1 - 5^{x-3})$$
$$\Rightarrow 24 \cdot 5^{x-3} = 1 - 5^{x-3}$$
$$\Rightarrow 25 \cdot 5^{x-3} = 1$$
$$\Rightarrow 5^{x-1} = 5^0$$
$$\therefore x - 1 = 0 \Rightarrow x = 1$$

8. Given, $x_n > x_{n-1} > \cdots > x_2 > x_1 > 1$

$$\therefore \log_{x_1} \log_{x_2} \log_{x_3} \cdots \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{\cdot^{x_1}}}}}$$
$$= \log_{x_1} \log_{x_2} \log_{x_3} \cdots \log x_{n-1}^{x_{n-2}^{\cdot^{\cdot^{\cdot^{x_1}}}}}$$
$$= \log_{x_1} x_1 = 1 \qquad (\because \log_a a = 1)$$

9. Let $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z} = \dfrac{1}{n}$

Then, $\quad \log x = nx(y+z-x) \qquad$ ...(i)

$\quad \log y = ny(z+x-y) \qquad$ ...(ii)

and $\quad \log z = nz(x+y-z) \qquad$ ...(iii)

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$\therefore \quad y\log x + x\log y = y\log z + z\log y$

$\qquad\qquad = z\log x + x\log z$

$\Rightarrow \quad \log(x^y \cdot y^x) = \log(y^z \cdot z^y) = \log(x^z \cdot z^x)$

$\Rightarrow \quad x^y \cdot y^x = y^z \cdot z^y = z^x \cdot x^z$

10. $\because \quad y = a^{\frac{1}{1-\log_a x}}$

$\Rightarrow \quad \log_a y = \dfrac{1}{1-\log_a x}$ ...(i)

and $\quad z = a^{\frac{1}{1-\log_a y}}$

or $\quad \log_a z = \dfrac{1}{1-\log_a y}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\log_a z = \frac{1}{1-\left(\dfrac{1}{1-\log_a x}\right)} = 1 - \frac{1}{\log_a x}$$

$\Rightarrow \quad \dfrac{1}{\log_a x} = (1-\log_a z) \Rightarrow \log_a x = \dfrac{1}{(1-\log_a z)}$

$\therefore \quad x = a^{\frac{1}{1-\log_a z}}$

11. $\log_{0.3}(x-1) < \log_{0.09}(x-1)$ ...(i)

Eq. (i) defined for $x > 1$, ...(ii)

$\Rightarrow \quad \log_{0.3}(x-1) - \log_{(0.3)^2}(x-1) < 0$

$\Rightarrow \quad \log_{0.3}(x-1) - \dfrac{1}{2}\log_{0.3}(x-1) < 0$

$\Rightarrow \quad \dfrac{1}{2}\log_{0.3}(x-1) < 0$

$\Rightarrow \quad \log_{0.3}(x-1) < 0$

$\Rightarrow \quad (x-1) > (0.3)^0$

[$\because$ base of log is lie in (0, 1)]

$\Rightarrow \quad x > 2$ ...(iii)

From Eqs. (ii) and (iii), we get

$x > 2 \quad \Rightarrow \quad x \in (2, \infty)$

12. $\because \quad a^x = a^{\sqrt{\log_a b}}$

$\quad = a^{\sqrt{\log_a b} \cdot \sqrt{\log_a b} \sqrt{\log_b a}} = a^{\log_a b \sqrt{\log_b a}} = b^{\sqrt{\log_b a}} = b^y$

$\therefore \quad a^x - b^y = 0$

13. $\because \quad x = 1 + \log_a bc = \log_a a + \log_a bc = \log_a(abc)$

$\therefore \quad \dfrac{1}{x} = \log_{abc} a$ ...(i)

Similarly, $\quad \dfrac{1}{y} = \log_{abc} b$ ...(ii)

and $\quad \dfrac{1}{z} = \log_{abc} c$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \log_{abc} abc = 1$$

$\Rightarrow \quad \dfrac{xy + yz + zx}{xyz} = 1$ or $\dfrac{xyz}{xy + yz + zx} = 1$

14. $\quad a^{\frac{\log_b(\log_b N)}{\log_b a}} = a^{\log_a(\log_b N)} = \log_b N$

15. $49^A + 5^B = ?$

$A = 1 - \log_7 2$

$A = \log_7 7 - \log_7 2$

$A = \log_7 \dfrac{7}{2} \quad \Rightarrow \quad 7^A = \dfrac{7}{2} \quad \Rightarrow \quad 49^A = \dfrac{49}{4}$

and $\quad B = -\log_5 4 = \log_5\left(\dfrac{1}{4}\right) \Rightarrow 5^B = \dfrac{1}{4}$

$\therefore \quad 49^A + 5^B = \dfrac{49}{4} + \dfrac{1}{4} = \dfrac{50}{4} = 12.5$

16. $(\log_{16} x)^2 - \log_{16} x + \log_{16} \lambda = 0$ ...(i)

Eq. (i) defined for $x > 0, \lambda > 0 \left(\log_{16} x - \dfrac{1}{2}\right)^2 - \dfrac{1}{4} + \log_{16}\lambda = 0$

For exactly one solution,

$\log_{16} x - \dfrac{1}{2} = 0$

$\therefore \quad -\dfrac{1}{4} + \log_{16}\lambda = 0 \quad \Rightarrow \quad \log_{16}\lambda = \dfrac{1}{4}$

or $\quad \lambda = (16)^{1/4} = 2$

17. $\quad x^{\log_x (x+3)^2} = 16$ ...(i)

From Eq. (i), $\quad x > 0$ and $x \neq 1$ ... (ii)

By Eq. (i), $\quad (x+3)^2 = 16$

$\Rightarrow \quad x + 3 = \pm 4$

$\Rightarrow \quad x = 1 \quad$ or $\quad x = -7$

From Eq. (ii), no values of x satisfy Eq. (i).

$\therefore$ Number of values of x satisfy Eq. (i)

$\therefore$ Number of roots $= 0$

18. Given, $y = \log_2 \log_6(2^{\sqrt{2x+1}} + 4)$ ...(i)

From Eq. (i) to be defined,

$2x + 1 > 0 \quad \Rightarrow \quad x > -\dfrac{1}{2}$ ...(ii)

We find value of x for which $y = 1$

$\therefore \quad 1 = \log_2 \log_6(2^{\sqrt{2x+1}} + 4)$

$\Rightarrow \quad \log_6(2^{\sqrt{2x+1}} + 4) = 2$

$\Rightarrow \quad 2^{\sqrt{2x+1}} + 4 = 36$

$\Rightarrow \quad 2^{\sqrt{2x+1}} = 32 = 2^5 \quad \Rightarrow \sqrt{2x+1} = 5$

$\Rightarrow \quad 2x + 1 = 25 \quad \Rightarrow \quad x = 12$

So, required point is (12, 1).

19. Given that, $\log 2 = 0.301$

$\log 3 = 0.477$

Let $\quad y = 3^{12} \times 2^8$

$\log y = 12\log 3 + 8\log 2$

$\quad = 12 \times (0.477) + 8(0.301) = 8.132$

So, number of digits before decimal in $3^{12} \times 2^8 = 8 + 1 = 9$

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20. Given, equation $2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$ ...(i)

$$\Rightarrow \quad \frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0 \quad \text{...(ii)}$$

Let $\log_a x = t$

Then, Eq. (ii),

$$\frac{2}{t} + \frac{1}{1+t} + \frac{3}{2+t} = 0 \Rightarrow 6t^2 + 11t + 4 = 0$$

$$\Rightarrow \quad t = -\frac{4}{3} \text{ or } -\frac{1}{2}$$

So, $x = a^{-4/3}$ or $x = a^{-1/2}$

Two value of x possible for which Eq. (i) is defined and satisfy.

21. Decimal on $x > 0$ and $x \neq 1$.

Taking logarithm on both sides on base 2, we get

$$\{(\log_2 x)^2 - 6\log_2 x + 11\}\log_2 x = 6$$

Let $\log_2 x = t$

$$\therefore \quad t^3 - 6t^2 + 11t - 6 = 0$$

$$\Rightarrow \quad (t-1)(t-2)(t-3) = 0 \Rightarrow t = 1, 2, 3$$

$$\Rightarrow \quad \log_2 x = 1, 2, 3$$

$$\Rightarrow \quad x = 2, 2^2, 2^3$$

22. $\log_\lambda x \cdot \log_5 \lambda = \log_x 5$ ...(i)

$$\lambda \neq 1, \lambda > 0 \text{ and } x > 0, \; x \neq 1$$

$$\Rightarrow \quad \log_5 x = \log_x 5 \Rightarrow (\log_5 x)^2 = 1$$

$$\Rightarrow \quad \log_5 x = \pm 1 \Rightarrow x = 5^1 \text{ and } 5^{-1}$$

$$\therefore \quad x = 5 \text{ and } \frac{1}{5}$$

23. $S = \{x : \sqrt{\log_x \sqrt{3x}} : \log_3 x > -1\}$

$$\log_3 x > -1$$

$$\Rightarrow \quad x > \frac{1}{3} \quad \text{...(i)}$$

Let $y = \sqrt{\log_x \sqrt{3x}}, x \neq 1$

To be defined y, $3x > 0 \Rightarrow x > 0$ ...(ii)

and $\log_x \sqrt{3x} \geq 0$ ...(iii)

From Eqs. (i) and (iii),

for $x \in \left(\frac{1}{3}, 1\right) \Rightarrow \sqrt{3x} \leq 1$

$$\Rightarrow \quad 3x \leq 1 \Rightarrow x \leq \frac{1}{3}$$

No solution for this case.

Now, for $x > 1$, from Eq. (iii), $\sqrt{3x} \geq 1 \Rightarrow x \geq \frac{1}{3}$

$$\therefore \quad x > 1$$

24. Given equation,

$$\log_2(9^{x-1} + 7) = 2 + \log_2(3^{x-1} + 1)$$

$$\Rightarrow \quad \log_2 \frac{\{3^{2(x-1)} + 7\}}{3^{(x-1)} + 1} = 2$$

$$\Rightarrow \quad 3^{2(x-1)} + 7 = 4 \cdot \{3^{(x-1)} + 1\}$$

$$\Rightarrow \quad \{3^{(x-1)}\}^2 - 4 \cdot 3^{(x-1)} + 3 = 0$$

$$\Rightarrow \quad (3^{x-1} - 3)(3^{x-1} - 1) = 0$$

$$\Rightarrow \quad x - 1 = 1 \text{ or } x - 1 = 0$$

$$\Rightarrow \quad x = 2 \text{ or } x = 1$$

25. $y = \log_p \log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots \sqrt[p]{p}}}}_{n \text{ times}})$ $[p > 0, p \neq 1]$

$$= \log_p \left\{\log_p (\underbrace{\sqrt[p]{\cdots \sqrt[p]{p}}}_{(n-1) \text{ times}})^{\frac{1}{p}}\right\} = \log_p \left\{\frac{1}{p}\log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots \sqrt[p]{p}}}}_{(n-1) \text{ times}})\right\}$$

$$= \log_p \left\{\frac{1}{p} \cdot \frac{1}{p}\log_p (\underbrace{\sqrt[p]{\sqrt[p]{\cdots \sqrt[p]{p}}}}_{(n-2) \text{ times}})\right\}$$

$$= \log_p \left(\frac{1}{p^n}\right) = -n, \; \log_{1/p} p^n = -n$$

26. $\log_a x = \alpha, \log_b x = \beta, \log_c x = \gamma, \log_d x = \delta$

$$\Rightarrow \quad \log_x a = \alpha^{-1} \quad \text{...(i)}$$

$$\Rightarrow \quad \log_x b = \beta^{-1} \quad \text{...(ii)}$$

$$\Rightarrow \quad \log_x c = \gamma^{-1} \quad \text{...(iii)}$$

$$\Rightarrow \quad \log_x d = \delta^{-1} \quad \text{...(iv)}$$

On adding Eqs. (i), (ii), (iii) and (iv) , we get

$$\log_x (abcd) = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} \quad \text{...(v)}$$

$$\therefore \quad \log_{abcd} x = \frac{1}{a^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$$

For $\alpha, \beta, \gamma, \delta$

$$\text{AM} \geq \text{HM} \Rightarrow \frac{\alpha + \beta + \gamma + \delta}{4} \geq \frac{4}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}}$$

or $$\frac{1}{\alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1}} \leq \frac{\alpha + \beta + \gamma + \delta}{16}$$

or $$\log_{abcd} x \leq \frac{\alpha + \beta + \gamma + \delta}{16} \quad \text{[from Eq. (v)]}$$

27. $\because$ $\log_{10} 5 = a$ and $\log_{10} 3 = b$ ...(i)

$$\therefore \quad \log_{10} 2 = \log_{10}\left(\frac{10}{5}\right) = 1 - a \quad \text{...(ii)}$$

Option (a)

$$\therefore \quad \log_{10} 8 = 3\log_{10} 2 = 3(1-a) \quad \text{[from Eq. (ii)]}$$

Option (b) $\log_{40} 15 = \dfrac{\log_{10} 15}{\log_{10} 40} = \dfrac{\log_{10}(5 \times 3)}{\log_{10}(2^3 \times 5)}$

$$= \frac{\log_{10} 5 + \log_{10} 3}{\log_{10} 2^3 + \log_{10} 5}$$

$$= \frac{a+b}{3(1-a)+a} = \frac{(a+b)}{(3-2a)}$$

Option (c) $\log_{243} 32 = \log_{3^5} 2^5 = \dfrac{5}{5}\log_5 2 = \dfrac{\log_{10} 2}{\log_{10} 2}$

$$= \frac{1-a}{a} \quad \text{[from Eqs. (i) and (ii)]}$$

Hence, all options are correct.

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28. $\because x > 0$ and $x \neq 1$

Given, $\log_a x$, $\log_b x$ and $\log_c x$ are in AP.

$\Rightarrow \quad 2\log_b x = \log_a x + \log_c x$

$\Rightarrow \quad \dfrac{2\log x}{\log b} = \dfrac{\log x}{\log a} + \dfrac{\log x}{\log c}$

$\Rightarrow \quad \dfrac{2}{\log b} = \dfrac{1}{\log a} + \dfrac{1}{\log c} \qquad \begin{bmatrix} \log x \neq 0 \\ \therefore x \neq 1 \end{bmatrix}$

$\Rightarrow \quad \log b = \dfrac{2(\log a)(\log c)}{(\log a + \log c)}$

Also, $\quad \dfrac{\log b}{\log a} = \dfrac{2\log c}{\log a + \log c}$

$\Rightarrow \quad \log_a b = \dfrac{\log c^2}{\log(ac)} = \log_{(ac)} c^2$

$\therefore \quad c^2 = (ac)^{\log_a b}$

29. $|a| < |b|$, $b - a < 1$

$a, b \in x^2 - |a|\, x - |\beta| = 0 \qquad$ ...(i)

So, $\quad \left.\begin{aligned} a + b &= |\alpha| \\ ab &= -|\beta| \end{aligned}\right\} \qquad$ ...(ii)

Given equation, $\log_{|b|}\left|\dfrac{x}{a}\right| - 1 = 0$, $\log_{|b|}\left|\dfrac{x}{a}\right| = 1$

$\Rightarrow \quad \left|\dfrac{x}{a}\right| = |b|^1$

$\Rightarrow \quad |x| = |ab|$

$\Rightarrow \quad |x| = |\beta| \qquad$ [from Eq. (ii)]

$\therefore \quad x = \pm\beta$

Sol. (Q. Nos. 30 to 32)

$\because \quad \log_2 N = a_1 + b_1$

$\Rightarrow \quad b_1 = \log_2 N - a_1$

Given, $\quad 0 \leq b_1 < 1 \Rightarrow 0 \leq \log_2 N - a_1 < 1$

$\Rightarrow \quad a_1 \leq \log_2 N < 1 + a_1$

$\Rightarrow \quad 2^{a_1} \leq N < 2^{1+a_1} \qquad$ ...(i)

Similarly, $\quad 3^{a_2} \leq N \;\; 3^{1+a_2} \qquad$ ...(ii)

and $\quad 5^{a_3} \leq N < 5^{1+a_3} \qquad$ ...(iii)

30. Here, $a_1 = 5$ and $a_2 = 3$, then from Eqs. (i) and (ii),

$$2^5 \leq N < 2^6 \quad \text{and} \quad 3^3 \leq N < 3^4$$

$\therefore$ Common values of N are 32, 33, 34, ..., 63

Number of integral values of N are 32.

31. Here, $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$, then from Eqs. (i), (ii) and (iii),

$$2^6 \leq N < 2^7, 3^4 \leq N < 3^5 \quad \text{and} \quad 5^3 \leq N < 5^4$$

$\Rightarrow$ 64, 65, 66, ..., 127, 81, 82, 83,..., 242 and 125, 126, ..., 624

$\therefore$ Largest common value = 127

32. Here, $a_1 = 6$, $a_2 = 4$ and $a_3 = 3$

From question number 31, we get

64, 65, 66,..., 127; 81, 82, 83, ..., 242 and 125, 126, ...,624

$\therefore$ Largest common value = 127

and smallest common value = 125

$\therefore$ Difference = 127 − 125 = 2

Sol. (Q. Nos. 33 to 35)

S = Antilog of (0.5) to the base 256

$\Rightarrow \quad \log_{256} S = 0.5$

$S = (256)^{0.5} = (2^8)^{1/2}$

$S = 2^4$

$S = 16 \qquad$ ...(i)

K = Number of digits in 6^{10}

[$\because \log_{10} 2 = 0.301$, $\log_{10} 3 = 0.477$]

Let $\quad \alpha = 6^{10}$

$\log\alpha = 10\log_{10} 6 = 10(0.301 + 0.477)$

$= 10(0.778)$

$\log(6^{10}) = 7.78$

So, $\quad x = 7 + 1$, $x = 8$

Number of positive integers which have characteristic 2, when the base of logarithm is 3

$= 3^{2+1} - 3^2 = 18$

$\therefore \quad G = 18$

33. The value of $G = 18$

34. The value of $KG = 8 \times 18 = 144$

35. The value of $SKG = 16 \times 8 \times 18 = 16 \times 144 = 2304$

Sol. (Q. Nos. 36 to 38)

U = Number of digits in $(60)^{100}$

Let $\quad \alpha = (60)^{100}$

$\log_{10}\alpha = 100\log_{10} 60 = 100(1 + \log_{10} 2 + \log_{10} 3)$

$= 100\,(1.778)$

$\log_{10}\alpha = 177.8$

So, $\quad U = 177 + 1 \Rightarrow U = 178 \qquad$ ...(i)

M = Number of cyphers after decimal, before a significant figure comes in $(8)^{-296}$

Let $\quad \beta = (8)^{-296}$

$\log_{10}\beta = (-296)\log_{10} 8 = (-296) \times 3\log_{10} 2$

$\log_{10}\beta = (-296) \times 3 \times (0.301)$

$= -267.288 = -267 - 0.288$

$= -267 - 1 + (1 - 0.288) = -268 + 0.712$

$\log_{10}\beta = \overline{268}.712$

$\therefore \quad M = 268 - 1 = 267$

Now, $\quad \dfrac{U}{M} = \dfrac{178}{267}$

According to the question,

$$\frac{U}{M} = \frac{2}{3}$$

$\Rightarrow \quad \dfrac{U}{M} = \dfrac{p}{q}$

So, $\quad p = 2$

and $\quad q = 3$

36. The value of $p = 2$.

37. The value of $q = 3$.

38. The equation whose roots are p and q is $x^2 - 5x + 6 = 0$.

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Sol. (Q. Nos. 39 to 41)

According to question, $G, O, E, L > 0$ and are real numbers.

Such that,

$\log_{10}(G\cdot L) + \log_{10}(G\cdot E) = 3 \Rightarrow \log_{10} G^2LE = 3$

$$\Rightarrow \quad G^2LE = 10^3 \quad \text{...(i)}$$

and $\quad \log_{10} E\cdot L + \log_{10} E\cdot O = 4$

$$\Rightarrow \quad \log_{10} E^2\cdot L\cdot O = 4$$

$$\Rightarrow \quad E^2\cdot L\cdot O = 10^4 \quad \text{...(ii)}$$

and $\log_{10}(O\cdot G) + \log_{10}(O\cdot L) = 5$

$$\Rightarrow \quad \log_{10} O^2GL = 5 \Rightarrow O^2GL = 10^5 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$G^3O^3E^3L^3 = 10^{12}$$

$$GOEL = 10^4 \quad \text{...(iv)}$$

$$\Rightarrow \quad \lambda = 10^4$$

39. Now, let

$$y = \sqrt{\log\lambda\sqrt{\log\lambda\sqrt{\log\lambda\cdots}}} = (\log\lambda)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}$$

$$= (\log\lambda)^{\frac{1/2}{1-1/2}} = (\log\lambda)$$

$$= \log 10^4 = 4\log 10 = 4$$

40. Minimum of $3G + 2L + 2O + E = 2^\lambda 3^\mu 5^\upsilon$

where $\lambda, \mu, \upsilon \in W$

Apply AM $\geq$ GM for $3G, 2L, 2O, E$

$$\frac{3G + 2L + 2O + E}{8} \geq \sqrt[8]{G^3 \times L^2 \times O^2 \times E}$$

So, $$8 \times \sqrt[8]{G^3L^2O^2E} = 2^\lambda 3^\mu 5^\upsilon \quad \text{...(v)}$$

(equality hold, if $G = L = O = E$)

From Eqs. (i) and (iii) of Q. 10, we get

$$G^3L^2O^2E = 10^8$$

From Eq. (v), $$8 \times (10^8)^{1/8} = 2^\lambda 3^\mu 5^\upsilon$$

$$8 \times 10 = 2^\lambda 3^\mu 5^\upsilon$$

$$2^4 \times 5^1 = 2^\lambda 3^\mu 5^\upsilon$$

$$\therefore \quad \lambda = 4, \upsilon = 1, \mu = 0$$

$$\Sigma(\lambda^\mu + \mu^\lambda) = (4^0 + 0^4) + (0^1 + 1^0) + (1^4 + 4^1)$$

$$= (1 + 0) + (0 + 1) + 1 + 4 = 7$$

41. $$\log_{10}\left(\frac{G}{O}\right) + \log_{10}\left(\frac{O}{E}\right) = \log_{10}\left(\frac{G}{E}\right) = \log_{10} 1 = 0$$

[divide Eq. (iv) and Eq. (ii) of Q. 39]

$$P = \log_{10}\frac{G}{O}\cdot\log_{10}\frac{O}{E} = \log\left(\frac{1}{10}\right)\log(10) = -1$$

[by dividing Eq. (i) by Eq. (ii) and dividing Eq. (iii) by Eq. (iv) in Q. 10]

$$= x^2 - 0\cdot x + (-1) = 0 = x^2 - 1$$

42. $\log_{10}(2x) + \log_{10} y = 2 \Rightarrow 2xy = 10^2$ ...(i)

and $\quad \log_{10} x^2 - \log_{10} 2y = 4$

$$\Rightarrow \quad \frac{x^2}{2y} = 10^4 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $x^3 = 10^6 \Rightarrow x = 100$

From Eq. (i), $\quad y = \frac{1}{2}$

$$\therefore \quad x + y = 100 + \frac{1}{2} = \frac{201}{2} = \frac{m}{n} \quad \text{(given)}$$

$$\therefore \quad m = 201 \text{ and } n = 2$$

$$\Rightarrow \quad m - 3n^6 = 201 - 3(2)^6 = 201 - 192 = 9$$

43. Solving, $x = \lambda$ and $y = \log_5 x$, we get

$$A \equiv (\lambda, \log_5\lambda), \lambda > 0$$

and solving $x = \lambda$ and $y = \log_5(x + 4)$, we get

$$B \equiv \{\lambda, \log_5(\lambda + 4)\}, \lambda > -4$$

Given, $\quad AB = 0.5$

$$\Rightarrow \quad \log_5(\lambda + 4) - \log_5\lambda = 0.5$$

$$\Rightarrow \quad \frac{\lambda + 4}{\lambda} = (5)^{1/2} = \sqrt{5}$$

$$\Rightarrow \quad \lambda = \frac{4}{\sqrt{5} - 1} = 4\frac{(\sqrt{5} + 1)}{4}$$

$$= 1 + \sqrt{5} = a + \sqrt{b} \quad \text{[given]}$$

$$\therefore \quad a = 1 \text{ and } b = 5$$

Then, $\quad a + b = 1 + 5 = 6$

44. $\therefore \quad a(b - c)x^2 + b(c - a)xy + c(a - b)y^2 = b, y \neq 0$ ...(i)

$$a(b - c)\left(\frac{x}{y}\right)^2 + b(c - a)\left(\frac{x}{y}\right) + c(a - b) = 0$$

Let $\quad \frac{x}{y} = X$

$$\Rightarrow \quad a(b - c)X^2 + b(c - a)X + c(a - b) = 0$$

$$\because \quad a(b - c) + b(c - a) + c(a - b) = 0$$

$$\therefore \quad X = 1$$

$\because$ Eq. (i) is perfect square.

$\therefore$ Roots are equal.

$$\therefore \quad 1 \times 1 = \frac{c(a - b)}{a(b - c)}$$

$$\Rightarrow \quad b = \frac{2ac}{a + c} \quad \text{...(ii)}$$

Now, $\log(a + c) + \log(a - 2b + c)$

$$= \log\{(a + c)^2 - 2b(a + c)\}$$

$$= \log\{(a + c)^2 - 4ac\} \quad \text{[from Eq. (ii)]}$$

$$= \log(a - c)^2 = 2\log(a - c)$$

$$\Rightarrow \quad \frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)} = 2$$

$$\therefore \quad \left\{\frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)}\right\}^2 = 4$$

45. According to the question, $x \in I$

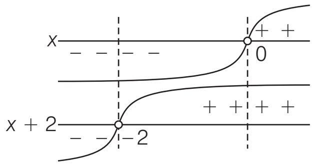
Given equation, $$\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9 \quad [x \neq \pm 2] \quad \text{...(i)}$$

$$\Rightarrow \quad 3^{\frac{|x+2|}{2-|x|}} > 3^2 \Rightarrow -\frac{|x + 2|}{2 - |x|} > 2$$

$$\Rightarrow \quad \frac{|x+2|}{|x|-2} - 2 > 0$$

$$\Rightarrow \quad \frac{|x+2| - 2|x| + 4}{|x|-2} > 0 \qquad \text{...(ii)}$$

Case **I** If $x < -2$, $-\dfrac{x - 2 + 2x + 4}{-x-2} > 0$



$$\frac{x+2}{-(x+2)} > 0 \Rightarrow -1 > 0$$

which is not possible.

***Case* II** $-2 < x < 0$, then Eq. (ii)

$$\Rightarrow \quad \frac{x + 2 + 2x + 4}{-x-2} > 0 \Rightarrow \frac{3x+6}{-(x+2)} > 0$$

$$\frac{-3(x+2)}{(x+2)} > 0 - 3 > 0$$

which is not possible.

***Case* III** when $x > 0$

From Eq. (ii),

$$\frac{x + 2 - 2x + 4}{x-2} > 0 \Rightarrow \frac{-x+6}{x-2} > 0$$

$$\frac{x-6}{x-2} < 0$$

$$2 < x < 6$$

So, the integer values of $x = 3, 4, 5$

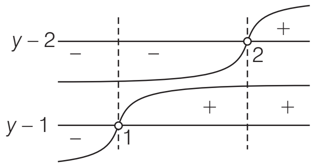
So, the number of integer values of x is 3.

46. $x > 2$

$$|\log_{\sqrt{3}} x - 2| + |\log_3 x - 2| = 2$$
$$|2\log_3 x - 2| + |\log_3 x - 2| = 2$$
$$2|\log_3 x - 1| + |\log_3 x - 2| = 2$$

Let $\qquad \log_3 x = y \qquad$ ...(i)

Then, Eq. (i) $\Rightarrow 2|y-1| + |y-2| = 2 \qquad$ ...(ii)



***Case* I** $\qquad y < 1$, then $x < 3$

Eq. (ii) becomes $-2y + 2 - y + 2 = 2$

$$-3y = -2, y = \frac{2}{3}$$

$$\log_3 x = \frac{2}{3} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x = 3^{2/3}$$

which is less than 2, so not acceptable.

***Case* II** $\qquad 1 < y < 2$, then $3 < x < 9$

From Eq. (ii), $\quad 2(y-1) - (y-2) = 2$

$\Rightarrow \qquad y = 2$

$\Rightarrow \qquad \log_3 x = 2$

$\therefore \qquad x = 3^2 = 9 \qquad$ [impossible]

***Case* III** $\qquad y \geq 2$, then $x \geq 9$

From Eq. (ii), $\quad 2(y-1) + (y-2) = 2$

$\therefore \qquad y = 2, \log_3 x = 2$

$\therefore \qquad x = 9 \qquad$ [acceptable]

47. Given equation is

$$\log_2 \sqrt{x} - 2\log^2_{1/4} x + 1 > 0 \qquad \text{...(i)}$$

From Eq. (i), $\qquad x > 0$

Eq. (i) $\Rightarrow \dfrac{1}{2}\log_2 x - \dfrac{2}{(-2)^2}\log_2^2 x + 1 > 0$

$$\Rightarrow \quad \frac{1}{2}\log_2 x - \frac{1}{2}\log_2^2 x + 1 > 0$$

$$\Rightarrow \quad (\log_2 x)^2 - (\log_2 x) - 2 < 0$$

$$\Rightarrow \quad (\log_2 x - 2)(\log_2 x + 1) < 0$$

$$\Rightarrow \quad -1 < \log_2 x < 2$$

$$\Rightarrow \quad 2^{-1} < x < 2^2$$

$$\Rightarrow \quad \frac{1}{2} < x < 4$$

$$\Rightarrow \quad x \in I, \text{ so } x = 1, 2, 3$$

So, number of integer value of x is 3.

48. Given that, $b > 0$

$$2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2) \qquad \text{...(i)}$$

$$\frac{2}{(-2)}\log_5(bx+28) = -\log_5(12 - 4x - x^2)$$

$\Rightarrow \qquad bx + 28 = 12 - 4x - x^2$

and $\qquad bx + 28 > 0$

and $\qquad 12 - 4x - x^2 > 0$

$\Rightarrow \qquad x^2 + (4 + b)x + 16 = 0 \qquad$ ...(ii)

and $\qquad x > \dfrac{-28}{b}$ and $-6 < x < 2$

Since, Eq. (i) has coincident roots, so discriminant Eq. (ii) is zero.

$$(4 + b)^2 - 64 = 0$$

$$b + 4 = \pm 8$$

$$b = 4 \text{ or } b = -12$$

Since, $\qquad b > 0$ so $b = 4$

for this value $x > -7$ and $-6 < x < 2$

49. $\dfrac{2^{\log_{1/4} 2} - 3^{\log_{27} 125} - 4}{7^{4\log_{49} 2} - 3} = \dfrac{2^{4\log_2 2} - 3^{\log_{3^3} 5^3} - 4}{7^{4\log_{7^2} 2^1}}$

$$= \frac{2^4 - 5 - 4}{7^{2\log_7 2} - 3} = \frac{16 - 9}{2^2 - 3} = 7$$

50. $(\log_5 x)^2 + \log_{5x}\left(\dfrac{5}{x}\right) = 1, x > 0, x \neq \dfrac{1}{5}$

$$\Rightarrow \quad (\log_5 x)^2 + \frac{\log_5\left(\frac{5}{x}\right)}{\log_5(5x)} = 1 \Rightarrow (\log_5 x)^2 + \frac{1 - \log_5 x}{1 + \log_5 x} = 1$$

Let $\log_5 x = t$, then

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$$t^2 + \frac{1-t}{1+t} = 1$$

$\Rightarrow \quad t^3 + t^2 - 2t = 0$

$\Rightarrow \quad t(t+2)(t-1) = 0 \quad \Rightarrow \quad t = -2, 0, 1$

$\Rightarrow \quad x = 5^{-2}, 5^0, 5^1$

$\Rightarrow \quad x = \frac{1}{25}, 1, 5$

$x_1, x_2 \in I$

$\therefore \quad x_1 = 1, x_2 = 5$

$\therefore \quad |x_2 - 4x_1| = |5 - 4| = 1$

51. Given, $x = \log_\lambda a = \log_a b = \frac{1}{2} \log_b c$ and $\log_\lambda c = nx^{n+1}$

$x = \log_\lambda a = \log_a b = \log_b \sqrt{c}$ and $\log_\lambda c = nx^{n+1}$ ...(i)

From Eq. (i), $\log_\lambda a \times \log_a b + \log_b \sqrt{c} = x^3$

$\log_\lambda \sqrt{c} = x^3, \frac{1}{2} \log_\lambda c = x^3$

$\log_\lambda c = 2x^3$

Compare with $\quad \log_\lambda c = nx^{n+1}$

$\Rightarrow \quad n = 2$

52. (A) $\dfrac{\log_3 243}{\log_2 \sqrt{32}} = \dfrac{\log_3 3^5}{-\frac{1}{2} \log_2 2^5} = \dfrac{5 \times 2}{5} = 2$ (p,s)

(B) $\dfrac{2 \log 6}{\log 12 + \log 3} = \dfrac{2 \log 6}{\log 36} = \dfrac{2 \log 6}{2 \log 6} = 1$ (p)

(C) $\log_{1/3} \left(\frac{1}{9}\right)^{-2} = -\log_3 3^4 = -4$ (q)

(D) $\dfrac{\log_5 16 - \log_5 4}{\log_5 128} = \dfrac{\log_5 \left(\frac{16}{4}\right)}{\log_5 (2)^7} = \dfrac{\log_5 (2)^2}{\log_5 (2)^7} = \dfrac{2}{7}$ (r)

53. (A) $\sqrt{\log^2_{(0.5)^2} 8} = \sqrt{\log^2_{1/2} 8} = \sqrt{(\log_{2^{-1}} 2^3)^2}$

$= \sqrt{\left(\frac{3}{-1} \log_2 2\right)^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$ (r)

(B) $(\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3$

$= (\log_{10} 2)^3 + 3 \log_{10} 2 \log_{10} 5 + (\log_{10} 5)^3$

$= (\log_{10} 2)^3 + 3 \cdot \log_{10} 2 \cdot \log_{10} 5 \cdot (\log_{10} 2 + \log_{10} 5)$
$+ (\log_{10} 5)^3$

$[\because \log_{10} 2 + \log_{10} 5 = \log_{10} 10 = 1]$

$= (\log_{10} 2 + \log_{10} 5)^3 = (\log_{10} 10)^3 = (1)^3 = 1$

$3 + (\log_{10} 2)^3 + \log_{10} 8 \cdot \log_{10} 5 + (\log_{10} 5)^3$

$= 3 + 1 = 4$ (s)

(C) $N = \log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 \frac{1}{6}$

$= \log_2 15 (-\log_6 2)(-\log_3 6)$

$= \dfrac{\log 15}{\log 2} \times \dfrac{\log 2}{\log 6} \times \dfrac{\log 6}{\log 3} = \log_3 15$

$9 < 15 < 27$

$2 < \log_3 15 < 3$

So, $[N] = 2$ (q)

(D) $(52.6)^a = (0.00526)^b = 100$

$(52.6)^a = 100$ and $(0.00526)^b = 100$

$52.6 = 10^{\frac{2}{a}}$ ...(i)

$(52.6)^b \times 10^{-4b} = 10^2$

$(52.6)^b = 10^{2+4b}$

$\Rightarrow \quad 52.6 = 10^{\left(\frac{2+4b}{b}\right)}$ ...(ii)

From Eqs. (i) and (ii), we get

$10^{\frac{2}{a}} = 10^{\frac{2+4b}{b}}$

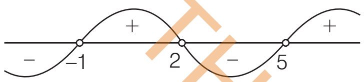
$\Rightarrow \quad \frac{2}{a} = \frac{2}{b} + 4$

$\Rightarrow \quad \frac{1}{a} - \frac{1}{b} = 2$ (q)

54. (A) Given that, $\log_{1/x} \dfrac{2(x-2)}{(x+1)(x-5)} \geq 1$ ...(i)

for log to be defined $\dfrac{(x-2)}{(x+1)(x-5)} > 0$,

then $\quad x \in (-1, 2) \cup (5, \infty)$



Let $\quad x > 0$ and $x \neq 1$

So, $\quad x \in (0, 1) \cup (1, 2) \cup (5, \infty)$

***Case* I** $\quad x \in (0, 1)$ ...(ii)

$\frac{1}{x} > 1$

$\therefore$ By Eq. (i), $\log_{\frac{1}{x}} \dfrac{2(x-2)}{(x+1)(x-5)} \geq 1$

$\Rightarrow \quad \dfrac{2(x-2)}{(x+1)(x-5)} \geq \dfrac{1}{x}$

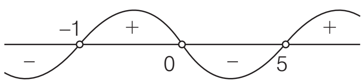
$\Rightarrow \quad \dfrac{2(x-2)}{(x+1)(x-5)} - \dfrac{1}{x} \geq 0$

$\Rightarrow \quad \dfrac{2x(x-2) - (x+1)(x+5)}{x(x+1)(x-5)} \geq 0$

$\Rightarrow \quad \dfrac{2x^2 - 4x - x^2 + 4x + 5}{x(x+1)(x-5)} \geq 0$

$\Rightarrow \quad \dfrac{x^2 + 5}{x(x+1)(x-5)} \geq 0$

$\Rightarrow \quad x(x+1)(x-5) > 0$



$\Rightarrow \quad x \in (-1, 0) \cup (5, \infty)$

But by Eq. (ii), $x \in (0, 1)$

So, no solution for this case.

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***Case* II** Let $x \in (1, 2) \cup (5, \infty)$ ...(iii)

$$\frac{1}{x} < 1$$

Eq. (i) $\Rightarrow$ $\log_{\frac{1}{x}} \frac{2(x-2)}{(x+1)(x-5)} \ge 1$

$$\frac{2(x-2)}{(x+1)(x-5)} \le \frac{1}{x}$$

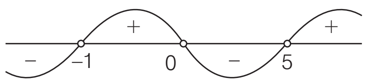
$\Rightarrow$ $\frac{2(x-2)}{(x+1)(x-5)} - \frac{1}{x} \le 0$

$\Rightarrow$ $\frac{x^2+5}{x(x+1)(x-5)} \le 0$ [by case I]

$\Rightarrow$ $x(x+1)(x-5) < 0$

$\Rightarrow$ $x \in (-\infty, -1) \cup (0, 5)$ ...(iv)

Eq. (iii), $x \in (1, 2) \cup (5, \infty)$



From Eqs. (iii) and (iv), $x \in (1, 2]$ (q)

(B) $\log_3 x - \log_3^2 x \le \frac{3}{2} \log_{\frac{1}{2\sqrt{2}}} 4$ ...(i)

defined, when $x > 0$

$$\log_3 x - \log_3^2 x \ge \frac{3}{2} \times \left(\frac{-2}{3}\right) \times 2 \times 1$$

$\Rightarrow$ $\log_3 x - \log_3^2 x + 2 \le 0$

$\Rightarrow$ $\log_3^2 x - \log_3 x - 2 \ge 0$

$\Rightarrow$ $(\log_3 x - 2)(\log_3 x + 1) \ge 0$

$\Rightarrow$ $\log_3 x \le -1$

or $\log_3 x \ge 2$

$\Rightarrow$ $x \le \frac{1}{3}$ or $x \ge 9$

From Eq. (i), $x > 0$

So, $x \in \left(0, \frac{1}{3}\right] \cup [9, \infty)$ (p)

(C) $\log_{\frac{1}{2}} (4-x) \ge \log_{\frac{1}{2}} 2 - \log_{\frac{1}{2}} (x-1)$ ...(i)

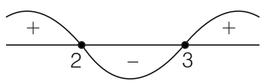
$\Rightarrow$ $\log_{\frac{1}{2}} \frac{(4-x)(x-1)}{2} \ge 0$

$\Rightarrow$ $-\frac{(x-4)(x-1)}{2} \le 1$

$\Rightarrow$ $(x-4)(x-1) \ge -2$

$\Rightarrow$ $x^2 - 5x + 4 + 2 \ge 0$

$\Rightarrow$ $x^2 - 5x + 6 \ge 0$



$(x-3)(x-2) \ge 0$

$x \le 2$ or $x \ge 3$ ...(ii)

From Eq. (i) to be defined, $4 - x > 0$ and $x - 1 > 0$

$x < 4$ and $x > 1$ ...(iii)

From Eqs. (ii) and (iii),

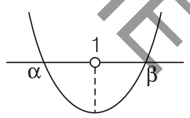
$x \in (1, 2] \cup [3, 4)$ (q, r)

(D) Given equation is

$(\lambda^2 - 3\lambda + 4)x^2 - 4(2\lambda - 1)x + 16 = 0$ ...(i)

$$\lambda^2 - 3\lambda + 4 = \lambda^2 - 3\lambda + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4 = \left(\lambda - \frac{3}{2}\right)^2 + \frac{7}{4}$$

So, $\lambda^2 - 3\lambda + 4 > 0, \forall \lambda \in R$



and $D > 0$

$\Rightarrow$ We get $\lambda > \frac{15}{8}$ ...(ii)

Let $f(x) = (\lambda^2 - 3\lambda + 4)x^2 = 4(2\lambda - 1)x + 16$

$\therefore$ $f(1) < 0$ by graph of $f(x)$

$\lambda^2 - 11\lambda + 24 < 0$

$(\lambda - 3)(\lambda - 8) < 0$

$3 < \lambda < 8$ ...(iii)

From Eqs. (ii) and (iii), we get

$3 < \lambda < 8 \Rightarrow \lambda \in (3, 8)$ (s)

55. If $0 < a < b$

Statement-1 If $x > 1$

$\Rightarrow$ $\log_x a < \log_x b$

$\therefore$ **Statement-2** If $0 < x < 1$

$\Rightarrow$ $\log_x a > \log_x b$

$\therefore$ Statement-2 is true, also

$10 > 3 > e > 2$

If $x > 1$,

then $\log_x 10 > \log_x 3 > \log_x e > \log_x 2$

$\Rightarrow$ $\frac{1}{\log_x 10} < \frac{1}{\log_x 3} < \frac{1}{\log_x e} < \frac{1}{\log_x 2}$

$\Rightarrow$ $\log_{10} x < \log_3 x < \log_e x < \log_2 x$

and for $0 < x < 1$

We get, $\log_{10} x > \log_3 x > \log_e x > \log_2 x$

It is clear that for $x > 0, x \ne 1$

Statement-1 is false.

56. **Statement-1** $7^{\log_7 (x^3+1)} - x^2 = 1$ ...(i)

$$\left.\begin{aligned} x^3 + 1 - x^2 &= 1 \\ x^3 - x^2 &= 0 \\ x^2(x-1) &= 0 \end{aligned}\right\} \begin{aligned} &\text{for this } x^3 + 1 > 0 \\ &\Rightarrow x^3 > -1 \\ &\Rightarrow x > -1 \end{aligned}$$

$x = 0$ (repeated) or $x = 1$

Thus, Eq. (i) has 2 repeated roots.

$\therefore$ Statement-1 is false.

Statement-2 $a^{\log_a N} = N, a > 0, a \ne 1$ and $N > 0$

which is true.

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57. Statement-1 $\left(\frac{1}{3}\right)^7 < \left(\frac{1}{3}\right)^4$. Taking log on both sides,

$$\log_e\left(\frac{1}{3}\right)^7 < \log_e\left(\frac{1}{3}\right)^4$$

$$7\log_e\frac{1}{3} < 4\log_e\frac{1}{3}$$

Now, $\log_e\frac{1}{3} < 0$ $[\because 2 < e < 3]$

So, $7 > 4$

Statement-1 is false. ...(i)

Statement-2 $ax < ay$

and $a < 0,\ x > 0,\ y > 0$

Eq. (i) divide by a, we get $x > y$

Statement-2 is true.

58. Statement-1 $x^{\log_x (1-x)^2} = 9$

$(1-x)^2 = 9$ $\left\{\begin{array}{l}\text{Eq. (i) is defined, if}\\ x \neq 1,\ x > 0\end{array}\right.$

$1 - x = \pm 3$

$\therefore$ $x = -2$ or 4

$x = 4$ [acceptable]

$\therefore$ Eq. (i) has only one solution.

Statement-1 is false.

Statement-2 $a^{\log_a b} = b$, where $a > 0,\ a \neq 1,\ b > 0$

which is true.

59. Statement-1 $(\log x)^2 + \log x^2 - 3 = 0$ ...(i)

$\Rightarrow$ $(\log x)^2 + 2\log x - 3 = 0$

$\Rightarrow$ $(\log x + 3)(\log x - 1) = 0$

$\Rightarrow$ $\log x = -3$ or $\log x = 1$

$\Rightarrow$ $x = 10^{-3}$ or $x = 10$

Eq. (i) is defined for $x > 0$.

So, Eq. (i) has 2 distinct solutions.

Statement-2 $\log x^2 \neq 2\log x$

$\because$ LHS has domain $x \in R$ and RHS has domain $x \in (0, \infty)$

$\therefore$ Statement-2 is false.

60. Statement-1

$$\log_x 3 \cdot \log_{x/9} 3 = \log_{81} 3 \quad \text{...(i)}$$

Eq. (i) holds, if $x > 0,\ x \neq 1,\ x \neq 9$

By Eq. (i), $\frac{1}{\log_3 x}\cdot\frac{1}{(\log_3 x + 2)} = \frac{1}{4}$

$(\log_3 x)^2 + 2\log_3 x - 4 = 0$

$(\log_3 x)^2 + 2\log_3 x + 4 = 8$

$(\log_3 x + 2)^2 = 8$

$\log_3 x + 2 = \pm 2\sqrt{2}$

$\log_3 x = 2(-1 \pm \sqrt{2})$

$\therefore$ $x = 3^{2(-1 \pm \sqrt{2})}$

Two values of x satisfying Eq. (i)

So, Statement-1 is false.

Statement-2 Change of bases in logarithm is possible.

$\therefore$ Statement-2 is true.

61. (i) $\because a = \log_7 12 = \frac{\log 12}{\log 7} = \frac{2\log 2 + \log 3}{\log 7}$

$$a = \frac{2 + \log_2 3}{\log_2 7} \quad \text{...(i)}$$

and $b = \log_{12} 24 = \frac{\log 24}{\log 12} = \frac{3\log 2 + \log 3}{2\log 2 + \log 3}$

$$= \frac{3 + \log_2 3}{2 + \log_2 3} \quad \text{...(ii)}$$

Let $\log_2 3 = \lambda$ and $\log_2 7 = \mu$

From Eq. (i), $a = \frac{2 + \lambda}{\mu}$

and from Eq. (ii), $b = \frac{3 + \lambda}{2 + \lambda}$, we get

$$\lambda = \frac{3 - 2b}{b - 1} \text{ and } \mu = \frac{1}{a(b-1)}$$

$\therefore$ $\log_{54} 168 = \frac{\log 168}{\log 54} = \frac{\log(2^3 \times 3 \times 7)}{\log(3^3 \times 2)}$

$$= \frac{3\log 2 + \log 3 + \log 7}{3\log 3 + \log 2}$$

$$= \frac{3 + \log_2 3 + \log_2 7}{3\log_2 3 + 1} = \frac{3 + \lambda + \mu}{3\lambda + 1}$$

$$= \frac{3 + \frac{3-2b}{b-1} + \frac{1}{a(b-1)}}{\frac{3(3-2b)}{b-1} + 1}$$

$$= \frac{(ab+1)}{a(8-5b)}$$

(ii) $\because$ $a = \log_3 4$ and $b = \log_5 3$

$\therefore$ $ab = \log_5 4$ ...(i)

Now, $\log_3 10 = \frac{\log_5 10}{\log_5 3} = \frac{2\log_5 10}{2\log_5 3}$

$$= \frac{\log_5(100)}{2b} = \frac{\log_5(4 \times 25)}{2b}$$

$$= \frac{\log_5 4 + 2}{2b} = \frac{ab + 2}{2b} \quad \text{[from Eq. (i)]}$$

62. $\because \frac{\ln a}{b - c} = \frac{\ln b}{c - a} = \frac{\ln c}{a - b}$ [by using law of proportion]

(i) $\because \frac{\ln a}{b - c} = \frac{\ln b}{c - a} = \frac{\ln c}{a - b}$

$$= \frac{\ln a + \ln b + \ln c}{b - c + c - a + a - b} = \frac{\ln(abc)}{0}$$

$\Rightarrow$ $\ln(abc) = 0$ $\Rightarrow$ $abc = 1$

(ii) $\frac{\ln a}{b - c} + \frac{\ln b}{c - a} + \frac{\ln c}{a - b} = \frac{a\ln a + b\ln b + c\ln c}{a(b-c) + b(c-a) + c(a-b)}$

$$= \frac{\ln a^a + \ln b^b + \ln c^c}{0} = \frac{\ln(a^a \cdot b^b \cdot c^c)}{0}$$

$\Rightarrow$ $\ln(a^a b^b c^c) = 0$

$\Rightarrow$ $a^a b^b c^c = 1$

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(iii) $\dfrac{\ln a}{b-c} = \dfrac{\ln b}{c-a} = \dfrac{\ln c}{a-b}$

$$= \frac{[(b^2+bc+c^2)\ln a + (c^2+ca+a^2)\ln b + (a^2+ab+b^2)\ln c]}{[(b^2+bc+c^2)(b-c) + (c^2+ca+a^2)(c-a) + (a^2+ab+b^2)(a-b)]}$$

$$= \frac{\ln a^{b^2+bc+c^2} + \ln b^{c^2+ca+a^2} + \ln c^{a^2+ab+b^2}}{(b^3-c^3)+(c^3-a^3)+(a^3-b^3)}$$

$$= \frac{\ln (a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2})}{0}$$

$$\Rightarrow \ln (a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2}) = 0$$

$$\therefore \quad a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2} = 1$$

(iv) $\because$ AM $\geq$ GM

$$\therefore \quad \frac{a+b+c}{3} \geq (abc)^{1/3} = (1)^{1/3} = 1 \qquad \text{[from Eq. (i)]}$$

$$\therefore \quad \frac{a+b+c}{3} \geq 1 \Rightarrow a+b+c \geq 3$$

(v) $\because$ AM $\geq$ GM

$$\Rightarrow \quad \frac{a^a+b^b+c^c}{3} \geq (a^a \cdot b^b \cdot c^c)^{1/3}$$

$$= (1)^{1/3} = 1 \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad \frac{a^a+b^b+c^c}{3} \geq 1 \Rightarrow a^a+b^b+c^c \geq 3$$

(vi) $\because$ AM $\geq$ GM $\dfrac{a^{b^2+bc+c^2} + b^{c^2+ca+a^2} + c^{a^2+ab+b^2}}{3}$

$$\geq (a^{b^2+bc+c^2} \cdot b^{c^2+ca+a^2} \cdot c^{a^2+ab+b^2})^{1/3}$$

$$= (1)^{1/3} \qquad \text{[from Eq. (iii)]}$$

$$= 1$$

$$\Rightarrow \quad \frac{a^{b^2+bc+c^2} + b^{c^2+ca+a^2} + c^{a^2+ab+b^2}}{3} \geq 1$$

$$\Rightarrow \quad a^{b^2+bc+c^2} + b^{c^2+ca+a^2} + c^{a^2+ab+b^2} \geq 3$$

63. To prove $\log_{10} 2$ lies between $\frac{1}{3}$ and $\frac{1}{4}$

$$2^{12} = 4096$$

$$1000 < 4096 < 10000$$

$$10^3 < 2^{12} < 10^4$$

Taking logarithm to the base 10,

$$\log_{10} 10^3 < \log_{10} 2^{12} < \log_{10} 10^4$$

$$3 < 12 \log_{10} 2 < 4 \Rightarrow \frac{1}{4} < \log_{10} 2 < \frac{1}{3}$$

64. $\log 2 = 0.301$

$\log 3 = 0.477$

(i) Let $\alpha = 5^{200}$

$$\log \alpha = 200 \log 5 = 200 (\log 10 - \log 2) = 200 (1 - 0.301)$$
$$= 200 \times 0.699 = 139.8$$

So, number of integers in $5^{200} = 139 + 1 = 140$.

(ii) $\alpha = 6^{20}$

$$\therefore \log \alpha = 20 \log 6 = 20 (\log 2 + \log 3)$$
$$= 20 (0.310 + 0.477)$$
$$= 20 \times 0.778 = 15.560$$

So, number of integers in $6^{20} = 15 + 1 = 16$

(iii) Let $\alpha = 3^{-500}$

$$\log \alpha = -500 \log 3 = -500 \times (0.477) = -238.5$$
$$= -239 + 0.5 = \overline{239}.5$$

So, number of zeroes after the decimal in

$$3^{-500} = 239 - 1 = 238$$

65. Given that, $\log_{10} 2 = 0.301$

and $\log_{10} 3 = 0.477$

$$\log 3.375 = \log(3375) - \log 10^3 = \log 5^3 \times 3^3 - 3 \log 5 \times 2$$
$$= 3 \log 5 + 3 \log 3 - 3 \log 5 - 3 \log 2$$
$$= 3 (0.477) - 3 (0.301) = 3 (0.176)$$
$$= 0.528$$

66. Let $P = \log_2 x - \log_x (0.125) = \log_2 x - \log_x \left(\frac{1}{8}\right)$

$$= \log_2 x + 3 \log_x 2$$

$$\therefore \quad \text{AM} \geq \text{GM}$$

$$\Rightarrow \quad \frac{\log_2 x + 3 \log_x 2}{2} \geq \sqrt{(\log_2 x)(3 \log_x 2)} = \sqrt{3}$$

$$\therefore \quad \frac{P}{2} \geq \sqrt{3}$$

$$\Rightarrow \quad P \geq 2\sqrt{3}$$

$\therefore$ Least value of $\log_2 x - \log_x (0.125)$ is $2\sqrt{3}$.

67. Let $y = \dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4$

$$= \log_\pi 12$$

Now, $\quad 12 > \pi^2$

$$\log_\pi 12 > \log_\pi \pi^2 \quad \therefore \quad y > 2$$

68. (i) $\therefore \quad x^{1 + \log_{10} x} = 10x \qquad \text{...(i)}$

$$\Rightarrow \quad x \cdot x^{\log_{10} x} = 10x$$

$$\Rightarrow \quad x [x^{\log_{10} x} - 10] = 0$$

$$x \neq 0, \text{ so } x^{\log_{10} x} - 10 = 0$$

$$\Rightarrow \quad x^{\log_{10} x} = 10$$

$$\Rightarrow \quad \log_{10} x = \log_x 10$$

$$\Rightarrow \quad (\log_{10} x)^2 = 1$$

$$\Rightarrow \quad \log_{10} x = \pm 1$$

$$\Rightarrow \quad x = 10^{\pm 1}$$

$$\Rightarrow \quad x = 10 \text{ or } \frac{1}{10} \qquad [\because x > 0]$$

(ii) $\quad \log_2 (9 + 2^x) = 3$

$$\Rightarrow \quad 9 + 2^x = 8$$

$$\Rightarrow \quad 2^x = -1$$

which is not possible, so $x \in \phi$.

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(iii) $$2\cdot x^{\log_4 3} + 3^{\log_4 x} = 27$$

$$2\cdot 3^{\log_4 x} + 3^{\log_4 x} = 27 \qquad \left[\because a^{\log_b c} = c^{\log_b a}\right]$$

$$3^{\log_4 (x+1)} = 3^3$$

$$\log_4 (x+1) = 3$$

$$\log_4 x = 2$$

$$x = 16$$

(iv) $\log_4 \log_3 \log_2 x = 0$ ...(i)

Defined for $x > 0$, $\log_2 x > 0$ and $\log_3 \log_2 x > 0$

$\Rightarrow$ $x > 0, x > 1, x > 3$

$\therefore$ $x > 3$

$$\log_3 \log_2 x = 1$$

$$\log_2 x = 3, x = 8$$

which satisfy Eq. (i).

(v) $$x^{\frac{\log_{10} x + 5}{3}} = 10^{5+\log_{10} x} \qquad \text{...(i)}$$

Defined for $x > 0$

Let $\log_{10} x = y$

$\Rightarrow$ $x = 10^y$

By Eq. (i), $10^{y\left(\frac{y+5}{3}\right)} = 10^{5+y}$

$\Rightarrow$ $y^2 + 5y = 15 + 3y$

$\Rightarrow$ $y^2 + 2y - 15 = 0$

$\Rightarrow$ $(y+5)(y-3) = 0$

$\Rightarrow$ $y = -5$ or $y = 3$

$\Rightarrow$ $x = \frac{1}{10^5}$ or $x = 10^3$

$\therefore$ $x = \{10^{-5}, 10^3\}$

(vi) $\log_3\left(\log_9 x + \frac{1}{2} + 9^x\right) = 2x$

Defined for $x > 0$,

$$\log_9 x + \frac{1}{2} + 9^x = 9^x$$

$\Rightarrow$ $\log_9 x = -\frac{1}{2} \Rightarrow x = 9^{-\frac{1}{2}}$

$\Rightarrow$ $x = 3^{-1}$

$\therefore$ $x = \frac{1}{3}$

(vii) $4^{\log_{10} x + 1} - 6^{\log_{10} x} - 2\cdot 3^{\log_{10} x^2 + 2} = 0$ ...(i)

$\Rightarrow$ $2^{2\log_{10} x + 2} - (2\times 3)^{\log_{10} x} - 2\cdot 3^{2\log_{10} x + 2} = 0$

Let $\log_{10} x = \lambda$, then

$$2^{2\lambda + 2} - (2\times 3)^{\lambda} - 2\cdot 3^{2\lambda + 2} = 0$$

$\Rightarrow$ $2^2 - \left(\frac{3}{2}\right)^{\lambda} - 2\cdot 3^2\cdot\left(\frac{3}{2}\right)^{2\lambda} = 0$

Let $\left(\frac{3}{2}\right)^{\lambda} = \mu$

$\therefore$ $18\mu^2 + \mu - 4 = 0$

$\Rightarrow$ $18\mu^2 + 9\mu - 8\mu - 4 = 0$

$\Rightarrow$ $9\mu(2\mu + 1) - 4(2\mu + 1) = 0$

$\therefore$ $\mu = -\frac{1}{2}, \mu = \frac{4}{9}$

$$\mu \neq -\frac{1}{2}$$

$\therefore$ $\mu = \frac{4}{9}$

$$\left(\frac{3}{2}\right)^{\lambda} = \left(\frac{3}{2}\right)^{-2} \Rightarrow \lambda = -2$$

Hence, $x = 10^{\lambda} = 10^{-2}\ \frac{1}{100}$

(viii) $\frac{\log_{10}(x-3)}{\log_{10}(x^2-21)} = \frac{1}{2}$

is defined for $x > 1$ and $x^2 > 21$.

$\therefore$ $x > \sqrt{21}$ ...(i)

$\Rightarrow$ $2\log_{10}(x-3) = \log_{10}(x^2 - 21)$

$\Rightarrow$ $\log_{10}(x-3)^2 = \log_{10}(x^2-21)$

$\Rightarrow$ $(x-3)^2 = x^2 - 21$

$$x^2 - 6x + 9 = x^2 - 21$$

$\therefore$ $x = 5$

satisfy Eq. (i), hence $x = 5$.

(ix) $x^{\log_2 x + 4} = 32$

Defined for $x > 0$,

$$\log_2 x + 4 = \log_x 2^5$$

$$\log_2 x + 4 = \frac{5}{\log_2 x}$$

$$(\log_2 x)^2 + 4\log_2 x - 5 = 0$$

$$(\log_2 x + 5)(\log_2 x - 1) = 0$$

$\Rightarrow$ $\log_2 x = -5$ or $\log_2 x = 1$

$\Rightarrow$ $x = 2^{-5}$ or $x = 2^1$

$\therefore$ $x = \frac{1}{32}$ or $x = 2$

which satisfy Eq. (i).

(x) $\log_a x = x$ ...(i)

and $a = x^{\log_4 x}$ ...(ii)

Defined for $x > 0$

From Eq. (i), $x = a^x$

$$a^x = x, a = x^{1/x}$$

From Eq. (ii), $x^{\frac{1}{x}} = x^{\log_4 x}$

$\Rightarrow$ $\frac{1}{x} = \log_4 x$

$\Rightarrow$ $x = \log_x 4 \Rightarrow x^x = 4$

$\therefore$ $x = 2$

(xi) $\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$ ...(i)

Defined for $1 + \cos x > 0$, $\sqrt{2}\sin x > 0$

and $\sqrt{2}\sin x \neq 1$, then

$$1 + \cos x = 2\sin^2 x$$

$$1 + \cos x = 2 - 2\cos^2 x$$

$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$
$$1 + \cos x \neq 0$$
So, $\cos x = \dfrac{1}{2}$

$x = \dfrac{\pi}{3}$, Eq. (i) is defined for that value of x.

69. Let rational number be x, then
$$x = 50\log_{10} x \Rightarrow 2x = 100 \cdot \log_{10} x$$
Taking logarithm to the base 10, then
$$\log_{10} 2 + \log_{10} x = 2 + \log_{10}(\log_{10} x)$$
Let $\log_{10} x = \lambda$

$\therefore$ $\log_{10} 2 + \lambda = 2 + \log_{10}(\lambda)$

$\Rightarrow$ $\log_{10}\left(\dfrac{\lambda}{2}\right) = \lambda - 2$

which is true for $\lambda = 2$.

$\therefore$ $\log_{10} x = 2$

$\Rightarrow$ $x = 10^2 = 100$

70. Let $y = \dfrac{2}{\log_4 (2000)^6} + \dfrac{3}{\log_5 (2000)^6}$
$$= 2\log_{(2000)^6} 4 + 3\log_{(2000)^6} 5$$
$$= \log_{(2000)^6} 4^2 + \log_{(2000)^6} 5^3$$
$$= \log_{(2000)^6} (4^2 \times 5^3)$$
$$= \frac{1}{6}\log_{2000} 2000 = \frac{1}{6}$$

71. $\log_a [1 + \log_b \{1 + \log_c (1 + \log_p x)\}] = 0$ ...(i)

$\Rightarrow$ $1 + \log_b \{1 + \log_c (1 + \log_p x)\} = 1$

$\Rightarrow$ $\log_b \{1 + \log_c (1 + \log_p x)\} = 0$

$\Rightarrow$ $1 + \log_c (1 + \log_p x) = 1$

$\Rightarrow$ $\log_c (1 + \log_p x) = 0$

$\Rightarrow$ $1 + \log_p x = 1$

$\Rightarrow$ $\log_p x = 0$

$\Rightarrow$ $x = p^0$

$\Rightarrow$ $x = 1$

Eq. (i) is satisfied for this value of x.

72. $\because 5\log_{4\sqrt{2}}(3 - \sqrt{6}) - 6\log_8(\sqrt{3} - \sqrt{2})$
$$= 5\log_{2^{5/2}}(3 - \sqrt{6}) - 6\log_{2^3}(\sqrt{3} - \sqrt{2})$$
$$= 5 \times \frac{1}{5/2}\log_2(3 - \sqrt{6}) - 6 \times \frac{1}{3}\log_2(\sqrt{3} - \sqrt{2})$$
$$= \log_2(3 - \sqrt{6})^2 - \log_2(\sqrt{3} - \sqrt{2})^2$$
$$= \log_2\left(\frac{3 - \sqrt{6}}{\sqrt{3} - \sqrt{2}}\right)^2 = \log_2\left(\frac{\sqrt{3}(\sqrt{3} - \sqrt{2})}{(\sqrt{3} - \sqrt{2})}\right)^2 = \log_2 3$$
$\therefore$ $= 4^{5\log_{4\sqrt{2}}(3 - \sqrt{6}) - 6\log_8(\sqrt{3} - \sqrt{2})}$
$$= 4^{\log_2 3} = 2^{2\log_2 3} = 2^{\log_2 9} = 9$$

73. (i) $\log_{2x+3} x^2 < 1$ ...(i)

***Case* I** $0 < 2x + 3 < 1$, i.e. $-\dfrac{3}{2} < x < -1$

Eq. (i), $x^2 > 2x + 3$
$$x^2 - 2x - 3 > 0$$
$$(x - 3)(x + 1) > 0$$
$$x < -1 \text{ or } x > 3$$
$\therefore$ $x \in \left(-\dfrac{3}{2}, -1\right)$ ...(ii)

***Case* II** $2x + 3 > 1 \Rightarrow x > -1$

Eq. (i), $x^2 < 2x + 3$
$$(x - 3)(x + 1) < 0 - 1 < x < 3$$
$\Rightarrow$ $x \in (-1, 3]$ ...(iii)

Eq. (i), $x \neq 0$ ...(iv)

Eqs. (ii), (iii) and (iv), $x \in \left(-\dfrac{3}{2}, 3\right) \cup \{-1, 0\}$

(ii) $\log_{2x}(x^2 - 5x + 6) < 1$ ...(i)

For Eq. (i) to be defined $2x > 0$ and $2x \neq 1$

So, $x > 0$ and $x \neq \dfrac{1}{2}$

and $x^2 - 5x + 6 > 0 \Rightarrow x < 2$ or $x > 3$ ...(ii)

***Case* I** $0 < 2x < 1 \Rightarrow 0 < x < \dfrac{1}{2}$ ...(iii)

From Eq. (i), $\log_{2x}(x^2 - 5x + 6) < 1$
$$x^2 - 5x + 6 < 2x$$
$$x^2 - 7x + 5 > 0$$
$$(x - 6)(x - 1) > 0$$
$$x < 1 \text{ or } x > 6$$ ...(iv)

From Eqs. (iii), (iv) and (ii)

$x \in \left(0, \dfrac{1}{2}\right)$ ...(A)

***Case* II** $2x > 1 \Rightarrow x > \dfrac{1}{2}$ ...(v)

From Eq. (i), $\log_{2x}(x^2 - 5x + 6) < 1$

$\Rightarrow$ $x^2 - 5x + 6 < 2x$

$\Rightarrow$ $x^2 - 7x + 6 < 0$

$\Rightarrow$ $1 < x < 6$ ...(vi)

From Eqs. (ii), (v) and (vi),

$x \in (1, 2) \cup (3, 6)$ ...(B)

From Eqs. (A) and (B), $x \in \left(0, \dfrac{1}{2}\right) \cup (1, 2) \cup (3, 6)$

(iii) $\log_2(2 - x) < \log_{1/2}(x + 1)$ ...(i)

From Eq. (i) to be defined $2 - x > 0 \Rightarrow x < 2$

and $x + 1 > 0 \Rightarrow x > -1$

So, $x \in (-1, 2)$ ...(ii)

Now, from Eq. (i), $\log_2(2 - x) + \log_2(x + 1) < 0$
$$(2 - x)(x + 1) < 1$$
$$(x - 2)(x + 1) + 1 > 0$$
$$x^2 - x - 2 + 1 > 0$$
$$x^2 - x - 1 > 0$$

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$$\Rightarrow \qquad x < \frac{1-\sqrt{5}}{2}$$

$$\text{or} \qquad x > \frac{1+\sqrt{5}}{2} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii),

$$x \in \left(-1, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, 2\right)$$

(iv) $\log_{x^2}(x+2) < 1$ ...(i)

From Eq. (i) to be defined, $x + 2 > 0 \Rightarrow x > -2$
and $x \in R,\ x \neq 0$ and $x \neq 1$ ...(A)

***Case* I** $x \in (-1, 1) \sim \{0\}$ ...(ii)

Eq. (i), $(x+2) > x^2$

$$x^2 - x - 2 < 0$$

$$(x-2)(x+1) < 0$$

$$x - 1 < x < 2 \qquad \text{...(iii)}$$

From Eqs. (ii), (iii) and (A),

$$x \in (-1, 0) \cup (0, 1) \qquad \text{...(B)}$$

***Case* II** $x \in (-\infty, -1) \cup (1, \infty)$ ...(iv)

Eq. (i), $x + 2 < x^2$

$$x^2 - x - 2 > 0$$

$$x < -1 \text{ or } x > 2 \qquad \text{...(v)}$$

From Eqs. (iv), (v) and (A),

$$x \in (-2, -1) \cup (2, \infty) \qquad \text{...(C)}$$

From Eqs. (B) and (C),

$$x \in (-2, 1) \cup (2, \infty) \sim \{-1, 0\}$$

(v) $3^{\log_3 \sqrt{x-1}} < 3^{\log_3 (x-6)} + 3$ ...(i)

From Eq. (i) to be defined

$$x - 1 > 0 \Rightarrow x > 1 \qquad \text{...(ii)}$$

and $x - 6 > 0 \Rightarrow x > 6$ ...(iii)

From Eqs. (ii) and (iii), $x > 6$ ...(iv)

Eq. (i), $\sqrt{x-1} - (x-6) - 3 < 0$

$$\sqrt{x-1} - x + 3 < 0$$

$$\sqrt{x-1} < (x-3)\ x - 1 < (x-3)^2$$

$$x^2 + 9 - 6x - x + 1 > 0$$

$$x^2 - 7x + 10 > 0$$

$$(x-5)(x-2) > 0$$

$$x < 2 \text{ or } x > 5 \qquad \text{...(v)}$$

From Eqs. (iv) and (v), $x > 6$

(vi) $\log_{1/2}(3x-1)^2 < \log_{1/2}(x+5)^2$ ...(i)

From Eq. (i) to be defined $x \neq \frac{1}{3},\ x \neq -5$ ...(ii)

Eq. (i), $(3x-1)^2 > (x+5)^2$

$$(3x - 1 - x - 5)(3x - 1 + x + 5) > 0$$

$$(2x-6)(4x+4) > 0$$

$$(x-3)(x+1) > 0$$

$$x < -1 \text{ or } x > 3 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii),

$$x \in (-\infty, -5) \cup (-5, -1) \cup (3, \infty)$$

(vii) $\log_{10} x + 2 \le \log_{10}^2 x$ ...(i)

From Eq. (i), $x > 0$ ...(ii)

$$\log_{10}^2 x + \log_{10} x - 2 \ge 0$$

$$(10_{10}\ x - 2)(\log_{10} x + 1) \ge 0$$

$$\log_{10} x \le -1 \text{ or } \log_{10} x \ge 2$$

$$x \le \frac{1}{10} \text{ or } x \ge 100 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii),

$$x \in \left(0, \frac{1}{10}\right] \cup [100, \infty)$$

$$\text{or} \qquad x \in (0, 10^{-1}] \cup [10^2, \infty)$$

(viii) $\log_{10}(x^2 - 2x - 2) \le 0$ ...(i)

From Eq. (i), $x^2 - 2x - 2 > 0$

$$x^2 - 2x + 1 - 3 > 0$$

$$(x-1)^2 - (\sqrt{3})^2 > 0$$

$$[x - (1+\sqrt{3})][x - (1-\sqrt{3})] > 0$$

$$\therefore \qquad x \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty) \qquad \text{...(ii)}$$

$$x^2 - 2x - 2 \le 1$$

$$x^2 - 2x - 3 \le 0$$

$$(x-3)(x+1) \le 0$$

$$-1 \le x \le 3 \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$x \in [-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$$

(ix) $\log_x \left(2x - \frac{3}{4}\right) > 2$ ...(i)

From Eq. (i) to be defined $x > 0,\ x \neq 1,\ 2x - \frac{3}{4} > 0$

$$x > 0,\ x \neq 1,\ x > \frac{3}{8} \qquad \text{...(ii)}$$

From Eq. (i), $\log_x \left(2x - \frac{3}{4}\right) > 2$

***Case* I** $0 < x < 1$ ...(iii)

$$2x - \frac{3}{4} < x^2$$

$$8x - 3 - 4x^2 < 0$$

$$4x^2 - 8x + 3 > 0$$

$$4x^2 - 6x - 2x + 3 > 0$$

$$(2x-1)(2x-3) > 0$$

$$x < \frac{1}{2} \text{ or } x > \frac{3}{2} \qquad \text{...(iv)}$$

From Eqs. (ii), (iii) and (iv),

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \qquad \text{...(v)}$$

***Case* II** $x > 1$ ...(vi)

Eq. (i) $\Rightarrow$ $2x - \frac{3}{4} > x^2$

$$8x - 3 > 4x^2$$

$$4x^2 - 8x + 3 < 0$$

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$$\frac{1}{2} < x < \frac{3}{2} \quad \text{...(vii)}$$

From Eqs. (ii), (vi) and (vii), we get

$$x \in \left(1, \frac{3}{2}\right) \quad \text{...(viii)}$$

From Eqs. (v) and (viii), we get

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$$

(x) $\log_{1/3} x < \log_{1/2} x \, (x > 0)$

$$\Rightarrow \quad \log_3 x > \log_2 x$$

$$\Rightarrow \quad \frac{\log x}{\log 3} - \frac{\log x}{\log 2} > 0$$

$$\log x \left(\frac{\log 3 - \log 2}{\log 2 \log 3}\right) < 0$$

$$\Rightarrow \quad \log x < 0 \Rightarrow x < 1$$

So, $\quad x \in (0, 1)$

(xi) $\log_{2x+3} x^2 < \log_{2x+3} (2x+3)$ ...(i)

From Eq. (i) to be defined,

$$2x + 3 > 0$$
$$x > -\frac{3}{2}$$
$$2x + 3 \neq 1$$
$$x \neq -1$$
$$x \in R - \{0\} \quad \text{...(A)}$$

From Eq. (i), $\log_{2x+3} x^2 < 1$ ...(ii)

***Case* I** $\quad 0 < 2x + 3 < 1 \Rightarrow -\frac{3}{2} < x < -1$ ...(iii)

From Eq. (ii), $\log_{2x+3} x^2 < 1$

$$\Rightarrow \quad x^2 > 2x + 3 \Rightarrow x^2 - 2x - 3 > 0$$
$$\Rightarrow \quad (x-3)(x+1) > 0$$
$$\Rightarrow \quad x < -1 \text{ or } x > 3 \quad \text{...(iv)}$$

From Eqs. (A), (iii) and (iv), $x \in \left(-\frac{3}{2}, -1\right)$ ...(B)

***Case* II** If $2x + 3 > 1 \Rightarrow x > -1$ ...(v)

$$\log_{2x+3} x^2 < 1$$
$$x^2 < 2x + 3$$
$$x^2 - 2x - 3 < 0$$
$$(x-3)(x+1) < 0$$
$$\Rightarrow \quad -1 < x < 3 \quad \text{...(vi)}$$

So, Eqs. (A), (v) and (vi), $x \in (-1, 3)$ ...(C)

From Eqs. (B) and (C),

$$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3)$$

(xii) $\log_2^2 x + 3 \log_2 x \geq \frac{5}{2} \log_{4\sqrt{2}} 16$ ...(i)

$$\log_2^2 x + 3 \log_2 x - \frac{5}{2} \times \frac{2}{5} \log_2 16 \geq 0$$
$$\log_2^2 x + 3 \log_2 x - 4 \geq 0$$
$$(\log_2 x + 4)(\log_2 x - 1) \geq 0$$

$$\log_2 x \leq -4 \text{ or } \log_2 x \geq 1 \Rightarrow x \leq \frac{1}{16}$$

or $\quad x \geq 2$ ...(ii)

From Eq. (i), $\quad x > 0$ ...(iii)

From Eqs. (ii) and (iii), $x \in \left(0, \frac{1}{16}\right] \in [2, \infty)$

(xiii) $\because \quad (x^2 + x + 1)^x < 1$

Taking logarithm on both sides, then

$$x \log (x^2 + x + 1) < 0$$
$$\because \quad x^2 + x + 1 > 0, \forall x \in R$$

***Case* I** If $\quad x > 0$ ...(i)

Then, $\quad \log (x^2 + x + 1) < 0$

$$\therefore \quad x^2 + x + 1 < 1$$
$$\Rightarrow \quad x(x+1) < 0$$
$$\Rightarrow \quad -1 < x < 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $x \in \phi$

***Case* II** If $x < 0$ ...(iii)

Then, $\quad \log (x^2 + x + 1) > 0$

$$\Rightarrow \quad x^2 + x + 1 > 1$$
$$\Rightarrow \quad x(x+1) > 0$$
$$\therefore \quad x \in (-\infty, -1) \cup (0, \infty) \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$x \in (-\infty, -1)$$

(xiv) $\log_{(3x^2+1)} 2 < \frac{1}{2}$

$$2 < (3x^2 + 1)^{1/2}$$
$$(3x^2 + 1 > 1, \forall x \in R)$$
$$4 < 3x^2 + 1$$
$$3x^2 > 3$$
$$x^2 > 1$$
$$x < -1 \text{ or } x > 1$$
$$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$$

(xv) $x^{(\log_{10} x)^2 - 3 \log_{10} x + 1} > 1000$ ...(i)

From Eq. (i) to be defined, $x > 0$ and $x \neq 1$

Let $\log_{10} x = y \Rightarrow x = 10^y$

From Eq. (i), $10^{y(y^2 - 3y + 1)} > 10^3$

$$\Rightarrow \quad y^3 - 3y^2 + y - 3 > 0$$
$$\Rightarrow \quad y^2(y-3) + 1(y-3) > 0$$
$$\Rightarrow \quad (y-3)(y^2+1) > 0$$
$$\Rightarrow \quad y > 3$$
$$\Rightarrow \quad \log_{10} x > 3$$
$$\Rightarrow \quad x > 1000$$
$$\Rightarrow \quad x \in (1000, \infty)$$

(xvi) $\quad \log_4 \{14 + \log_6 (x^2 - 64)\} \leq 2$ ...(i)

$$14 + \log_6 (x^2 - 64) \leq 16$$
$$\log_6 (x^2 - 64) \leq 2$$

$$x^2 - 64 \le 36$$
$$x^2 \le 100$$
$$-10 \le x \le 10 \quad \text{...(ii)}$$

From Eq. (i), $\quad x^2 - 64 > 0$

$\Rightarrow \quad x < -8$ or $x > 8 \quad$ ...(iii)

From Eqs. (ii) and (iii),

$$x \in [-10, -8) \cup (8, 10]$$

(xvii) $\quad \log_2 (9 - 2^x) \le 10^{\log_{10} (3 - x)} \quad$ ...(i)

From Eq. (i) to be defined,

$$9 - 2^x > 0 \;\Rightarrow\; 9 > 2^x$$

$\Rightarrow \quad 2^x < 9 \;\Rightarrow\; x < \log_2 9$

$$3 - x > 0 \;\Rightarrow\; x < 3$$

Then, $\quad x < 3 \quad$ ...(ii)

From Eq. (i), $\log_2 (9 - 2^x) \le 3 - x$

$\Rightarrow \quad 9 - 2^x \le 2^{3-x}$

$\Rightarrow \quad 9 - 2^x - 8 \cdot 2^{-x} \le 0$

$\Rightarrow \quad (2^x)^2 - 92^x + 8 \ge 0$

$\Rightarrow \quad (2^x - 8)(2^x - 1) \ge 0$

$\Rightarrow \quad 2^x \le 1$ or $2^x \ge 8$

$\Rightarrow \quad x \le 0$ or $x \ge 3 \quad$ ...(iii)

From Eqs. (ii) and (iii), $x \le 0 \Rightarrow x \in (-\infty, 0]$

(xviii) $\log_a \left(\dfrac{2x+3}{x} \right) \ge 0 \quad$ ...(i)

From inequation (a), $\quad a > 1$

By Eq. (i), $\quad \dfrac{2x+3}{x} > 0$

$\Rightarrow \quad \left[\dfrac{x - \left(-\frac{3}{2}\right)}{x - 0} \right] > 0$

$\Rightarrow \quad x < -\dfrac{3}{2}$ or $x > 0 \quad$ ...(ii)

From Eq. (i), $\quad \log_a \left(2 + \dfrac{3}{x} \right) \ge 0$

$$2 + \frac{3}{x} \ge 1$$
$$\frac{3+x}{x} \ge 0$$
$$\frac{x - (-3)}{x - 0} \ge 0$$

$x \le -3$ or $x \ge 0 \quad$ ...(iii)

From Eqs. (ii) and (iii),

$x \le -3$ or $x > 0 \quad$ ...(iv)

$\Rightarrow \quad x \in (-\infty, -3] \cup (0, \infty)$

From inequation in (b), $\; 0 < a < 1$

From Eq. (i), $\quad \dfrac{2x+3}{x} \le 1$

$\Rightarrow \quad \dfrac{x+3}{x} \le 0$

$\Rightarrow \quad -3 \le x \le 0$

$\Rightarrow \quad x \in [-3, 0] \quad$ ...(v)

From Eqs. (ii) and (v), we get $x \in \left[-3, -\dfrac{3}{2} \right)$

(xix) $\quad 1 + \log_2 (x - 1) \le \log_{(x-1)} 4 \quad$ ...(i)

From Eq. (i) to be defined, $x - 1 > 0 \Rightarrow x > 1$

and $\quad x - 1 \ne 1 \;\Rightarrow\; x \ne 2 \quad$ ...(ii)

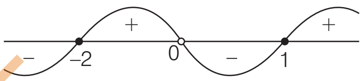
By Eq. (i), $1 + \log_2 (x-1) \le 2 \log_{(x-1)} 2$

Let $\log_2 (x - 1) = \lambda$, then

$$1 + \lambda \le \frac{2}{\lambda}$$

$\Rightarrow \quad \dfrac{\lambda^2 + \lambda - 2}{\lambda} \le 0$

$\Rightarrow \quad \dfrac{(\lambda + 2)(\lambda - 1)}{\lambda} \le 0$



$\Rightarrow \quad \lambda \le -2$ or $0 < \lambda \le 1$

$\Rightarrow \quad \log_2 (x - 1) \le -2$

or $\quad 0 < \log_2 (x-1) \le 1$

$\Rightarrow \quad x - 1 \le 2^{-2}$ or $2^0 < x - 1 \le 2^1$

$\Rightarrow \quad x \le \dfrac{5}{4}$ or $2 < x \le 3 \quad$ ...(iii)

From Eqs. (ii) and (iii), we get

$$x \in (2, 3]$$

(xx) $\log_{5x+4} x^2 \le \log_{5x+4} (2x + 3) \quad$ ...(i)

From Eq. (i) to be defined, $5x + 4 > 0 \Rightarrow x > -\dfrac{4}{5}$

$$5x + 4 \ne 1 \;\Rightarrow\; x \ne -\frac{3}{5}$$

$$2x + 3 > 0 \Rightarrow x > -\frac{3}{2}$$

and $\quad x \in (-\infty, \infty) - \{0\}$

$\Rightarrow \quad x \in \left(-\dfrac{4}{5}, -\dfrac{3}{5} \right) \cup \left(-\dfrac{3}{5}, 0 \right) \cup (0, \infty) \quad$ ...(ii)

From Eq. (i), $\; \log_{5x+4} x^2 \le \log_{5x+4} (2x+3)$

$\log_{5x+4} \dfrac{x^2}{2x+3} \le 0 \quad$ ...(iii)

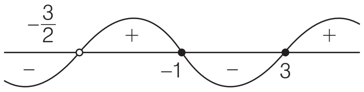
***Case* I** $\quad 0 < 5x + 4 < 1$

$\Rightarrow \quad -\dfrac{4}{5} < x < -\dfrac{3}{5} \quad$ ...(iv)

From Eq. (iii), $\dfrac{x^2}{2x+3} \ge 1$

$$\frac{x^2 - 2x - 3}{2x + 3} \ge 0$$

$$\frac{(x-3)(x+1)}{\left[x-\left(-\frac{3}{2}\right)\right]} \ge 0$$



$$x \in \left(-\frac{3}{2}, -1\right] \cup [3, \infty) \quad \text{...(v)}$$

From Eqs. (ii), (iv) and (v), $x \in \phi$ ...(vi)

***Case* II** $\quad 5x + 4 > 1 \quad \Rightarrow \quad x > -\frac{3}{5}$ ...(vii)

From Eq. (iii), $\frac{x^2}{2x+3} \le 1$

$$\left[\frac{(x-3)(x+1)}{\left\{x-\left(-\frac{3}{2}\right)\right\}}\right] \le 0$$

$\Rightarrow \quad x < -\frac{3}{2}$ or $x \in [-1, 3]$ ...(viii)

From Eqs. (ii), (vii) and (viii),

$$x \in \left(-\frac{3}{5}, -\frac{3}{2}\right) \cup [-1, 0) \cup (0, 3] \quad \text{...(ix)}$$

From Eqs. (vi) and (ix), we get

$$x \in \left(-\frac{3}{5}, -\frac{3}{2}\right) \cup [-1, 0) \cup (0, 3]$$

74. Given equation is

$$\sqrt{\log_x (ax)^{1/5} + \log_a (ax)^{1/5}} + \sqrt{\log_a \left(\frac{x}{a}\right)^{1/5} + \log_x \left(\frac{a}{x}\right)^{1/5}} = a \quad \text{...(i)}$$

$$\frac{1}{\sqrt{5}}\sqrt{1 + \log_x a + 1 + \log_a x} + \frac{1}{\sqrt{5}}\sqrt{\log_a x - 1 + \log_x a - 1} = a$$

$$\sqrt{\log_a x + \frac{1}{\log_a x} + 2} + \sqrt{\log_a x + \frac{1}{\log_a x} - 2} = \sqrt{5}a$$

$$\left|\sqrt{|\log_a x|} + \frac{1}{\sqrt{|\log_a x|}}\right| + \left|\sqrt{|\log_a x|} - \frac{1}{\sqrt{|\log_a x|}}\right| = \sqrt{5}a \quad \text{...(ii)}$$

Let $\quad \sqrt{|\log_a x|} = y \quad [y \ge 0]$

$$\left|y + \frac{1}{y}\right| + \left|y - \frac{1}{y}\right| = \sqrt{5}a \quad \text{...(iii)}$$

***Case* I** $x \ge a > 1$ Eq. (iii) $\Rightarrow y + \frac{1}{y} + y - \frac{1}{y} = \sqrt{5}a$

$\Rightarrow \quad 2y = \sqrt{5}a$

$$2\sqrt{|\log_a x|} = \sqrt{5}a$$

$$\sqrt{|\log_a x|} = \frac{\sqrt{5}}{2}a$$

$$\log_a x = \frac{5}{4}a^2$$

$$x = a^{\frac{5}{4}a^2}$$

***Case* II** $\quad 1 < x < a$ ...(v)

By Eq. (iii), $\quad y + \frac{1}{y} - y + \frac{1}{y} = \sqrt{5}a$

$$\frac{2}{y} = \sqrt{5}a$$

$$y = \frac{2}{\sqrt{5}a}$$

$$\sqrt{|\log_a x|} = \frac{2}{\sqrt{5}a}$$

$\Rightarrow \quad \log_a x = \frac{4}{5a^2}$

$$x = a^{4/5a^2}$$

75. Given equation,

$$\log_\pi (x^2 + 15a^2) - \log_\pi (a-2) = \log_\pi \frac{8ax}{a-2} \quad \text{...(i)}$$

Eq. (i) is defined, if $a - 2 > 0 \Rightarrow a > 2$ ...(ii)

$$\frac{8ax}{a-2} > 0$$

By Eq. (ii), $\quad a > 2$

So, $ax > 0$, then $x > 0$

Eq. (i) for $\quad x = 9, a > 0$

$$\log_\pi \frac{(x^2 + 15a^2)}{(a-2)} = \log_\pi \frac{8ax}{a-2}$$

$$x^2 + 15a^2 = 8ax \quad \text{...(iii)}$$

$$(x-3a)(x-5a) = 0$$

$\therefore \quad x = 3a$ and $x = 5a$

For $\quad a = 3, x = 9$ and $x = 15$

$\Rightarrow \quad x = 15$ for $a = 3$

76. Given that,

$$\log_4 (\log_3 x) + \log_{1/4} (\log_{1/3} y) = 0 \quad \text{...(i)}$$

$\Rightarrow \quad \frac{1}{2}\log_2 \log_3 x - \frac{1}{2}\log_2(-\log_3 y) = 0$

$\Rightarrow \quad \frac{1}{2}\left[\log_2 \left(\frac{\log_3 x}{-\log_3 y}\right)\right] = 0$

$\Rightarrow \quad -\frac{\log_3 x}{\log_3 y} = 1$

$\Rightarrow \quad \log_3 x = -\log_3 y$

$\Rightarrow \quad \log_3 x = \log_3 \left(\frac{1}{y}\right)$

$\Rightarrow \quad x = \frac{1}{y}$ ...(ii)

Also, given that, $\quad x^2 + y^2 = \frac{17}{4}$

$$x^2 + \frac{1}{x^2} = \frac{17}{4}$$

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$$\left(x+\frac{1}{x}\right)^2=\frac{17}{4}+2$$

$$x+\frac{1}{x}=\frac{5}{2} \quad \text{[by Eq. (i) } x>0, y>0]$$

$$x+\frac{1}{x}=2+\frac{1}{2}$$

$$\therefore \quad x=2 \text{ or } \frac{1}{2}$$

For these values of x, $y=\frac{1}{2}$ or 2 [by Eq. (ii)]

77. $\log_{2x} 4x+\log_{4x} 16x=4$ ...(i)

From Eq. (i) is defined for $x>0$, $x\neq\frac{1}{2}$, $x\neq\frac{1}{4}$ ...(ii)

$$\Rightarrow \quad \frac{\log 4x}{\log 2x}+\frac{\log 16x}{\log 4x}=4$$

$$\Rightarrow \quad \frac{2\log 2+\log x}{\log 2+\log x}+\frac{4\log 2+\log x}{2\log 2+\log x}=4$$

On dividing by log 2, then

$$\frac{2+\log_2 x}{1+\log_2 x}+\frac{4+\log_2 x}{2+\log_2 x}=4$$

Let $\log_2 x=\lambda$, then

$$(2+\lambda)^2+(1+\lambda)(4+\lambda)=4(1+\lambda)(2+\lambda)$$

$$\Rightarrow \quad 2\lambda^2+9\lambda+8=4\lambda^2+12\lambda+8$$

$$\Rightarrow \quad 2\lambda^2+3\lambda=0$$

$$\therefore \quad \lambda=0,\ \lambda=-\frac{3}{2}$$

$$\Rightarrow \quad \log_2 x=0,\ \log_2 x=-\frac{3}{2}$$

$$\therefore \quad x=2^0,\ x=2^{-3/2}$$

$$\text{or} \quad x=1,\ x=2^{-3/2}$$

78. Given equation,

$$\log_6 54+\log_x 16=\log_{\sqrt{2}} x-\log_{36}\frac{4}{9} \quad \text{...(i)}$$

Eq. (i) holds, if $x>0$, $x\neq 1$

From Eq. (i),

$$1+\log_6 9+4\log_x 2=2\log_2 x-\log_6\frac{2}{3}$$

$$\Rightarrow \quad 1+\log_6 9+\log_6\frac{2}{3}+4\log_x 2-2\log_2 x=0$$

$$\Rightarrow \quad 2+4\log_x 2-2\log_2 x=0$$

$$\Rightarrow \quad (\log_2 x)^2-\log_2 x-2=0$$

$$\Rightarrow \quad \log_2 x=2 \text{ or } \log_2 x=-1$$

$$\Rightarrow \quad x=4 \text{ or } x=\frac{1}{2}$$

Sum of the values of x satisfy Eq. (i) $=4+\frac{1}{2}=\frac{9}{2}$

Product of the values of x satisfy Eq. (i) $=4\times\frac{1}{2}=2$

79. Let $\frac{3}{2}\log_4(x+2)^3+3=\log_4(4-x)^3+\log_4(x+6)^3$ ...(i)

Eq. (i) holds, if $4-x>0$ and $x+0>0$, $x+2>0$

i.e., $-2<x<4$ ...(ii)

From Eq. (i),

$$\frac{3}{2}\times 2\times\frac{1}{2}\log_2|(x+2)|+3=\frac{1}{2}\times 3\log_2(4-x)+\frac{1}{2}\times 3\log_2(x+6)$$

$$\Rightarrow \quad \log_2(x+2)+2=\log_2(4-x)+\log_2(x+6)$$

$$\Rightarrow \quad \log_2\{4(x+2)\}=\log_2\{(4-x)(x+6)\}$$

$$\Rightarrow \quad 4(x+2)=(4-x)(x+6)$$

$$4x+8=-x^2-2x+24$$

$$x^2+6x-16=0$$

$$(x+8)(x-2)=0$$

$$\therefore \quad x=-8,\ x=2 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get $x=2$

80. $\log_2(4^{x+1}+4)\cdot\log_2(4^x+1)=\log_{1/\sqrt{2}}\left(\frac{1}{\sqrt{8}}\right)$ ...(i)

Eq. (i) defined, for $4^x+1>0$ which is true for all $x\in R$.

$$\log_2[4(4^x+1)]\cdot\log_2(4^x+1)=\log_{\sqrt{2}}\sqrt{8}=3$$

$$[2+\log_2(4^x+1)]\log_2(4^x+1)=3$$

$$\text{Let} \quad \log_2(4^x+1)=y$$

$$\therefore \quad (y+2)y=3$$

$$y^2+2y-3=0$$

$$(y+3)(y-1)=0$$

$$y=1 \text{ or } y=-3$$

$$\therefore \quad \log_2(4^x+1)=1 \quad \text{or} \quad \log_2(4^x+1)=-3$$

$$4^x+1=2 \quad \text{or} \quad 4^x+1=\frac{1}{8}$$

$$4^x=1 \quad \text{or} \quad 4^x=\frac{1}{8}-1$$

$$x=0 \quad \text{or} \quad 4^x=-\frac{7}{8} \text{ which is not possible.}$$

$$\therefore \quad x=0$$

81. $2^{\sqrt{x}+\sqrt{y}}=256$

$$\Rightarrow \quad 2^{\sqrt{x}+\sqrt{y}}=2^8$$

$$\Rightarrow \quad \sqrt{x}+\sqrt{y}=8 \quad \text{...(i)}$$

Also, given that, $\log_{10}\sqrt{xy}-\log_{10}\frac{3}{2}=1$ ...(ii)

which is defined, $xy>0$

So, Eq. (ii) $\Rightarrow$ $\log_{10}\sqrt{xy}=\log_{10}\left(10\times\frac{3}{2}\right)$

$$\Rightarrow \quad \sqrt{xy}=15$$

$$\Rightarrow \quad xy=225 \quad \text{...(iii)}$$

From Eq. (i), $x+y+2\sqrt{xy}=64$

$$x+y=64-30$$

$$x+y=34$$

From Eq. (iii), $xy=225$

After solving, we get $x=9$ or $x=25$, then $y=25$ or $y=9$

Hence, solutions are (9, 25) and (25, 9).

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82. Given, $\log_2 y = \log_4 (xy - 2)$ ...(i)

Eq. (i) defined for $y > 0$ and $xy - 2 > 0$ ...(ii)

$$xy > 0 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii) $\Rightarrow$ $y > 0,\ x > 0$

By Eq. (i), $y = \sqrt{xy - 2}$

$$y^2 - xy + 2 = 0 \quad \text{...(iv)}$$

$$y\,(x - y) = 2 \quad \text{...(v)}$$

Also given that,

$$\log_9 x^2 + \log_3 (x - y) = 1 \quad \text{...(vi)}$$

which is defined for $x \in R - \{0\}$ and $x - y > 0$

$$\Rightarrow \quad x > y$$

By Eq. (vi), $x(x - y) = 3 \Rightarrow x^2 - xy = 3$ ...(vii)

and $\quad x(x - y) = 3$ ...(viii)

Form Eqs. (iv) and (vii), $y^3 + 2 = x^2 - 3$

$$x^2 - y^2 = 5 \quad \text{...(ix)}$$

On dividing Eq. (v) by Eq. (viii),

$$\frac{y}{x} = \frac{2}{3} \Rightarrow y = \frac{2x}{3} \quad \text{...(x)}$$

From Eqs. (ix) and (x),

$$x = 3 \text{ and } y = 2$$

83. Given that,

$$2 \log_{1/4} (x + 5) > \frac{9}{4} \log_{1/3\sqrt{3}} 9 + \log_{\sqrt{x + 5}} 2 \quad \text{...(i)}$$

By Eq. (i), $x + 5 > 0 \Rightarrow x > -5$

$$x + 5 \neq 1 \Rightarrow x \neq -4$$

So, $\quad x \in (-5, -4) \cup (-4, \infty)$ ...(ii)

Now, by Eq. (i)

$$\frac{2}{-2} \log_2 (x + 5) - \frac{9}{4} \times \left(\frac{-2}{3}\right) \log_3 9 - 2 \log_{x+5} 2 > 0$$

$$- \log_2 (x + 5) + 3 - 2 \log_{x+5} 2 > 0$$

$$- \log_2 (x + 5) - \frac{2}{\log_2 (x + 5)} + 3 > 0 \quad \text{...(iii)}$$

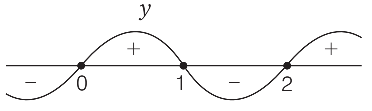
Now, let $\log_2 (x + 5) = y$, then Eq. (iii) becomes

$$- y - \frac{2}{y} + 3 > 0$$

$$\Rightarrow \quad \frac{- y^2 + 3y - 2}{y} > 0$$

$$\Rightarrow \quad \frac{y^2 - 3y + 2}{y} < 0$$

$$\Rightarrow \quad \frac{(y - 2)(y - 1)}{y} < 0$$



$$\Rightarrow \quad y < 0 \text{ or } 1 < y < 2$$

$$\Rightarrow \quad \log_2 (x + 5) < 0 \text{ or } 1 < \log_2 (x + 5) < 2$$

$$\Rightarrow \quad x + 5 < 1 \text{ or } 2 < x + 5 < 4$$

$$\Rightarrow \quad x < -4 \quad \text{...(iv)}$$

or $\quad -3 < x < -1$ ...(v)

From Eqs. (ii), (iv) and (v),

$$x \in (-5, -4) \cup (-3, -1)$$

84. $\log_3 (\sqrt{x} + |\sqrt{x} - 1|) = \log_9 (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$ ...(i)

From Eq. (i) is defined, if $x \geq 0$

then $\log_3 (\sqrt{x} + |\sqrt{x} - 1|) = \log_{3^2} (4\sqrt{x} - 3 + 4|\sqrt{x} - 1|)$

$$\Rightarrow \quad 2(\sqrt{x} + |\sqrt{x} - 1|) = 4\sqrt{x} - 3 + 4|\sqrt{x} - 1|$$

$$\Rightarrow \quad 3 - 2\sqrt{x} = 2|\sqrt{x} - 1|$$

On squaring both sides, then

$$9 + 4x - 12\sqrt{x} = 4x + 4 - 4\sqrt{x}$$

$$\Rightarrow \quad 8\sqrt{x} = 5$$

$$\therefore \quad x = \frac{25}{64}$$

85. $(\log_2 x)^4 - \left(\log_{1/2} \dfrac{x^5}{4}\right)^2 - 20 \log_2 x + 148 < 0$

From Eq. (i), $x > 0$

$\Rightarrow$ $(\log_2 x)^4 - (5 \log_2 x - 2)^2 - 20 \log_2 x + 148 < 0$ ...(i)

$$(\log_2 x)^4 - 25 \log_2^2 x - 4 + 20 \log_x x - 20 \log_2 x + 148 < 0$$

$$(\log_2 x)^4 - 25 \log_2^2 x + 144 < 0$$

$$\{(\log_2 x)^2 - 16\}\{(\log_2 x)^2 - 9\} < 0$$

$$9 < (\log_2 x)^2 < 16$$

$$3 < \log_2 x < 4 \text{ or } -4 < \log_2 x < -3$$

$$8 < x < 16 \quad \text{...(ii)}$$

or $\quad \dfrac{1}{16} < x < \dfrac{1}{8}$ ...(iii)

According to the question in Eq. (i) holds, for $x \in (a, b)$ where $a, b \in N$

So, from Eq. (ii), $a = 8,\ b = 16$

$$\therefore \quad ab\,(a + b) = 8 \times 16(8 + 16) = 144 \times 24 = 3456$$

86. $\sqrt{(\log_3 \sqrt[3]{3x}) + (\log_x \sqrt[3]{3x}) \log_3 x^3}$

$$+ \sqrt{\left(\log_3 \sqrt[3]{\frac{x}{3}} + \log_x \sqrt[3]{\frac{3}{x}}\right) \log_3 x^3} = 2 \quad \text{...(i)}$$

Eq. (i) is defined for $x > 0,\ x \neq 1$

From Eq. (i), $\sqrt{\dfrac{1}{3}(1 + \log_3 x + 1 + \log_x 3)\, 3 \log_3 x}$

$$+ \sqrt{\frac{1}{3}(\log_3 x - 1 + \log_x 3 - 1)\, 3 \log_3 x}$$

$$= 2\sqrt{\left(\log_3 x + \frac{1}{\log_3 x} + 2\right)} + \sqrt{\left(\log_3 x + \frac{1}{\log_3 x} - 2\right)}$$

$$= 2\sqrt{|\log_x 3|}$$

$$\Rightarrow \quad \sqrt{|\log_3 x|} + \frac{1}{\sqrt{|\log_3 x|}} + \left|\sqrt{|\log_3 x|} - \frac{1}{\sqrt{|\log_3 x|}}\right| = 2\sqrt{|\log_x 3|} \quad \text{...(ii)}$$

***Case* I** If $x \geq 3$, $\sqrt{\log_3 x} - \dfrac{1}{\sqrt{\log_3 x}} > 0$

$$2\sqrt{\log_3 x} = 2\sqrt{\log_x 3}$$

$$(\log_3 x)^2 = 1 \Rightarrow x = 3 \text{ or } \frac{1}{3}, \text{ so } x = 3$$

***Case* II** If $1 < x < 3$, $\dfrac{2}{\sqrt{\log_3 x}} = 2\sqrt{\log_x 3}$

$$\Rightarrow \quad \log_3 x \cdot \log_x 3 = 1$$

$$\Rightarrow \quad 1 = 1$$

which is true, for all $x \in (1, 3]$.

So, $x \in (1, 3]$

87. $P =$ Number of natural numbers, whose logarithms to the base 10 have characteristic p.

Let 'x' represent the natural number, i.e.

$$x = \lambda \times 10^p \; [\lambda = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \ldots]$$

So, $P =$ Number of natural numbers which have $(p+1)$ digits

$$= 9 \cdot 10^p - 1 + 1 = 9 \cdot 10^p$$

$Q =$ Number of natural numbers which have (q) digits.

$Q = 9 \cdot 10^{q-1} - 1 + 1 = 9 \cdot 10^{q-1}$

So, $\log_{10} P - \log_{10} Q = \log_{10}(9 \cdot 10^p) - \log_{10}(9 \cdot 10^{q-1})$

$$= (\log_{10} 9 + p) - (\log_{10} 9 + q - 1)$$

$$= p - q + 1$$

88. $\because \quad a = \log_3 \log_3 2$

$$\Rightarrow \quad 3^a = \log_3 2$$

$$\therefore \quad 3^{-a} = \log_2 3$$

Now, $\quad 1 < 2^{-k + 3^{-a}} < 2^1$

$$\Rightarrow \quad 2^0 < 2^{-k + 3^{-a}} < 2^1$$

$$\therefore \quad 0 < -k + 3^{-a} < 1$$

$$\Rightarrow \quad 0 < -k + \log_2 3 < 1$$

$$\Rightarrow \quad 0 > k - \log_2 3 > -1$$

$$\Rightarrow \quad \log_2 3 - 1 < k < \log_2 3$$

$$\therefore \quad k = 1$$

89. $\because (2x)^{\ln 2} = (3y)^{\ln 3}$

Taking log with base e on both sides, then

$$\ln 2 (\ln 2 + \ln x) = \ln 3 (\ln 3 + \ln y) \quad \ldots \text{(i)}$$

and $\quad 3^{\ln x} = 2^{\ln y}$

Taking log with base e on both sides, then

$$\ln x \cdot \ln 3 = \ln y \cdot \ln 2 \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\ln 2 (\ln 2 + \ln x) = \ln 3 \left(\ln 3 + \frac{\ln x \cdot \ln 3}{\ln 2} \right)$$

$$\Rightarrow \quad \ln x \left(\frac{(\ln 3)^2}{\ln 2} - \ln 2 \right) = -((\ln 3)^2 - (\ln 2)^2)$$

$$\therefore \quad \ln x = -\ln 2 = \ln \left(\frac{1}{2} \right)$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$\therefore \quad x_0 = \frac{1}{2}$$

90. Let $S = \dfrac{1}{3\sqrt{2}} \sqrt{4 - \dfrac{1}{3\sqrt{2}} \sqrt{4 - \dfrac{1}{3\sqrt{2}} \sqrt{4 - \dfrac{1}{3\sqrt{2}} \ldots \infty}}}$

$$\Rightarrow \quad S = \frac{1}{3\sqrt{2}} \sqrt{4 - S}$$

$$\text{or} \quad (3\sqrt{2}\, S)^2 = 4 - S$$

$$\Rightarrow \quad 18S^2 + S - 4 = 0$$

$$\Rightarrow \quad (9S - 4)(2S + 1) = 0$$

$$\therefore \quad 9S - 4 = 0 \quad [\because 2S + 1 \neq 0]$$

$$\text{or} \quad S = \frac{4}{9} = \left(\frac{3}{2} \right)^{-2}$$

$$\Rightarrow \quad \log_{3/2} S = -2 \Rightarrow 6 + \log_{3/2} S = 6 - 2 = 4$$

Hence,

$$6 + \log_{3/2} \left(\frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}} \ldots}}} \right) = 4$$

91. $(3/4)^x = 1/4$

Taking log with base 2

$$\Rightarrow \quad x(\log_2 3 - 2) = -2$$

$$\therefore \quad x = \frac{2}{2 - \log_2 3} = \frac{1}{1 - \log_4 3} \Rightarrow \text{(b, c)}$$

and taking log with base 3

$$\Rightarrow \quad x(1 - \log_3 4) = -2\log_3 2$$

$$\therefore \quad x = \frac{2\log_3 2}{2\log_3 2 - 1}$$

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