ASSIGNMENT 2

Thursday, November 7, 2024 12:36 AM

Given:

$$E[u_c] = |u_c - g_c|^2_\Omega + \lambda |\nabla u_c|^2_{2,1}$$

where:

1) Data fidelity term:

$$|u_c - g_c|^2_\Omega = \sum_{i,j} \Omega[i,j] \left(u_c[i,j] - g_c[i,j] \right)^2$$

2) Total variation regularization term:

$$|\nabla u_c|^2_{2,1} = \sum_{i,j} |\nabla u_c[i,j]|^2_2$$

PART 1:

Using Forward differences, we approximate the gradient at each pixel (i,j) with respect to its neighbours.

Step-by-step Calculations:

1) Data Fidelity term Approximation:

- Measures difference b/w the inpainted image u_c and the original image g_c in regions specified by Ω. Since, this term is already in a summation form, no finite differencing is required. We have,

$$|u_c - g_c|^2_\Omega = \sum_{i,j} \Omega[i,j] \left(u_c[i,j] - g_c[i,j] \right)^2$$

2). Total Variance regularisation term (using Forward differences):

- We compute the gradient at each pixel (i,j) as follows:

$$|\nabla u_c|^2_{2,1} = \sum_{i,j} \left(\left(u_c[i+1,j] - u_c[i,j] \right)^2 + \left(u_c[i,j+1] - u_c[i,j] \right)^2 \right)$$

$u_c[i+1,j] - u_c[i,j]$ approximates the gradient in x-direction at pixel (i,j)

$u_c[i,j+1] - u_c[i,j]$ approximates the gradient in y-direction at pixel (i,j)

SUMMARY OF MAIN STEPS:

1) Data Fidelity term:

Computed as a sum of squared differences between u_c and g_c over domain Ω

2) Gradient Approximation:
Forward differences are applied in both x- and y- directions to approximate the gradient at each pixel

3) Combined Objective Function:
The finite difference approximation of $E[v_c]$ is formed by summing both the data fidelity and total variance terms, with regularization parameter balancing them.

★ The discrete approximation of $E[v_c]$ is now suitable for optimisation to minimize the energy and solve the inpainting problem

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PART 2:

(i) Data Fidelity term:

$$\sum_{i,j} \Omega[i,j]\left(v_c[i,j]-g_c[i,j]\right)^2$$

⟹ differentiating w.r.t. $v_c[i,j]$

$$\frac{\partial}{\partial v_c[i,j]} \sum_{i,j} \Omega[i,j]\left(v_c[i,j]-g_c[i,j]\right)^2$$

$$= \sum_{i,j} 2\,\Omega[i,j]\left(v_c[i,j]-g_c[i,j]\right)$$

- The gradient component encourages each $v_c[i,j]$ to match $g_c[i,j]$ where $\Omega[i,j]=1$

(ii) Total Variation regularisation term:

$$\lambda \sum_{i,j} \left(\left(v_c[i+1,j]-v_c[i,j]\right)^2 + \left(v_c[i,j+1]-v_c[i,j]\right)^2\right)$$

(a) <u>Horizontal Difference $(v_c[i+1,j]-v_c[i,j])$</u>

★ From the horizontal term $\left(v_c[i+1,j]-v_c[i,j]\right)^2$, $v_c[i,j]$ appears as:

1) The current pixel in $\left(v_c[i+1,j]-v_c[i,j]\right)^2$
2) The neighbouring pixel in $\left(v_c[i,j]-v_c[i-1,j]\right)^2$
(The pixel below it)

Hence,

1). $$\frac{\partial}{\partial v_c[i,j]}\left(v_c[i+1,j]-v_c[i,j]\right)^2 = -2\left(v_c[i+1,j]-v_c[i,j]\right)$$

2). $$\frac{\partial}{\partial v_c[i,j]}\left(v_c[i,j]-v_c[i-1,j]\right)^2 = 2\left(v_c[i,j]-v_c[i-1,j]\right)$$

(b) Vertical difference $\left(u_c[i,j+1] - u_c[i,j]\right)$

* From the vertical term $(u_c[i,j+1] - u_c[i,j])^2$, $u_c[i,j]$ appears as:
  1). The current pixel in $(u_c[i,j+1] - u_c[i,j])^2$
  2) The Neighbouring pixel in $(u_c[i,j] - u_c[i,j-1])^2$ (The pixel to the right of it)

Hence,

1). $$\frac{\partial}{\partial u_c[i,j]}\left(u_c[i,j+1] - u_c[i,j]\right)^2 = -2\left(u_c[i,j+1] - u_c[i,j]\right)$$

2). $$\frac{\partial}{\partial u_c[i,j]}\left(u_c[i,j] - u_c[i,j-1]\right)^2 = 2\left(u_c[i,j] - u_c[i,j-1]\right)$$

Combining these, the gradient of the total Variation term with respect to $u_c[i,j]$ becomes:

$$\lambda\Big[-2\left(u_c[i,j] - u_c[i-1,j]\right) + 2\left(u_c[i+1,j] - u_c[i,j]\right) - 2\left(u_c[i,j] - u_c[i,j-1]\right) + 2\left(u_c[i,j+1] - u_c[i,j]\right)\Big]$$

Handling Boundary Conditions:

Left and Top boundaries:

For $i=0$ and $j=0$, skip terms involving $u_c[i-1,j]$ and $u_c[i,j-1]$

Right and bottom boundaries:

For i = max row index and j = max column index, skip terms involving $u_c[i+1,j]$ and $u_c[i,j+1]$

Combined Gradient $\nabla_u E$:

Summing up gradient from data fidelity and regularization terms:

$$\nabla_u E\left[u_c[i,j]\right] = 2\,\Omega[i,j]\left(u_c[i,j] - g_c[i,j]\right) + \lambda\Big[-2\left(u_c[i,j] - u_c[i-1,j]\right) + 2\left(u_c[i+1,j] - u_c[i,j]\right) - 2\left(u_c[i,j] - u_c[i,j-1]\right) + 2\left(u_c[i,j+1] - u_c[i,j]\right)\Big]$$

* The terms involving **i-1** will not be involved in gradient calculation for points on the **top** border
* The terms involving **i+1** will not be involved in gradient calculation for points on the **bottom** border

* The terms involving j-1 will not be involved in gradient calculation for points on the **left** border
* The terms involving j+1 will not be involved in gradient calculation for points on the **right** border

* The corner points will contain the data term and values from 2 neighbours in the regularization term