Probability Distributions

Covering…

- What is Random Variable?
- Types of Random Variables
- What is Probability Distribution?
- Expected Value of Random Variable

Probability Distribution

- Random variable is a numerical quantity with uncertain values.
- The pattern of randomness of the random variable is the probability distribution of the random variable.

Types of Random Variables

- Discrete: Random Variables that take particular values are discrete random variables
- Continuous: Random Variables that can take any real value are continuous random variables

Properties of Probability Distributions

- For any probability distribution following things are always true:
  - Value of probability is between 0 and 1
  - Sum of all probabilities is 1
  - All the events are mutually exclusive
  - All the events are exhaustive

Examples of Probability Distributions

- Discrete:

Xi	Pi
4	0.1
6	0.2
8	0.5
10	0.2

$$\sum_{i=1}^{n} p_i = 0.1 + 0.2 + 0.5 + 0.2 = 1$$

- Continuous:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

$$\int_a^b \frac{1}{b-a} = 1$$

Probability Functions

- For Discrete Variables, functions are called Probability Mass Functions
- For Continuous Variables, functions are called Probability Density Function

Mathematical Expectation

- Expected Value of a variable for discrete distribution is given by

$$E(X) = \sum x_i p_i$$

- Expected Value of a variable for continuous distribution is given by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Expected Value Computation: Discrete

Xi	Pi
4	0.1
6	0.2
8	0.5
10	0.2

$$E(X) = \sum x_i p_i$$
$$= 4*0.1 + 6*0.2 + 8*0.5 + 10*0.2$$
$$= 7.6$$

Expected Value Computation: Continuous

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{2}(a+b)$$

Binomial Distribution

Binomial Distribution

- Considers experiment with two possible outcomes: success and failure.
- p = Probability of Success of single trial
- q = 1-p = Probability of Failure of single trial
- n = No. of Trials of the Experiment
- k = No. of successes out of n trials

$$f(k;n,p) = \Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boldsymbol{Mean = E(X) = np} \qquad \boldsymbol{Variance(X) = npq}$$

Python functions for Binomial Distribution

Syntax:

binom.pmf(k,n,p,…)

Applies for $P(X=k)$

binom.cdf(k,n,p,…)

Applies for $P(X\leq k)$

binom.sf(k,n,p,…)

Applies for $P(X>k)$

binom.stats(n,p,…)

Applies fore extracting mean, variance and other moments

Example

In a typical Month, an Insurance agent presents life insurance plans to 40 potential customers. Historically, one in four such customers chooses to buy Life Insurance from this agent. Based on the relevant binomial distribution , answer the following questions :

1. What is the probability that exactly 5 customers will buy life Insurance from this agent in the coming month ?
2. What is the probability that not more than 10 customers will buy life insurance from this agent in the coming month ?
3. What is the probability that at least 20 customers will buy life insurance from this agent in the coming month ?
4. Determine the mean and variance of the number of customers who will buy life insurance from this agent in the coming month.

Poisson Distribution

Poisson Distribution

- Considers experiment with two possible outcomes: success and failure.
- n is infinitely large (or very large)
- p is very small
- Characterized by a single parameter λ
- Let X : No. of successes

Siméon Denis Poisson

$$f(k;\lambda) = \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

where

- e is Euler's number ($e = 2.71828...$)
- $k!$ is the factorial of k.

$$\boldsymbol{Mean = E(X) = \lambda}$$

$$\boldsymbol{Variance(X) = \lambda}$$

Python Functions for Poisson Distribution

poisson.pmf(k,mu,…)

Applies for P(X=k)

poisson.cdf(k,mu,…)

Applies for $P(X \leq k)$

poisson.sf(k,mu,…)

Applies for P(X>k)

poisson.stats(mu,…)

Applies fore extracting mean, variance and other moments

Example

The annual number of industrial accidents occurring in a particular manufacturing plant is known to follow Poisson distribution with mean 12.

a) What is the probability of observing exactly 5 accidents at this plant during the coming year ?
b) What is the probability of observing not more than 12 accidents at this plant the coming year ?
c) What is the probability of observing at least 15 accidents at this plant during the coming year ?
d) What is the probability of observing between 10 and 15 accidents (inclusive) at this plant during the coming year ?

Normal Distribution

Probability Density Function

- PDF of Normal Distribution is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty,$$
$$-\infty < \mu < \infty, \sigma > 0$$

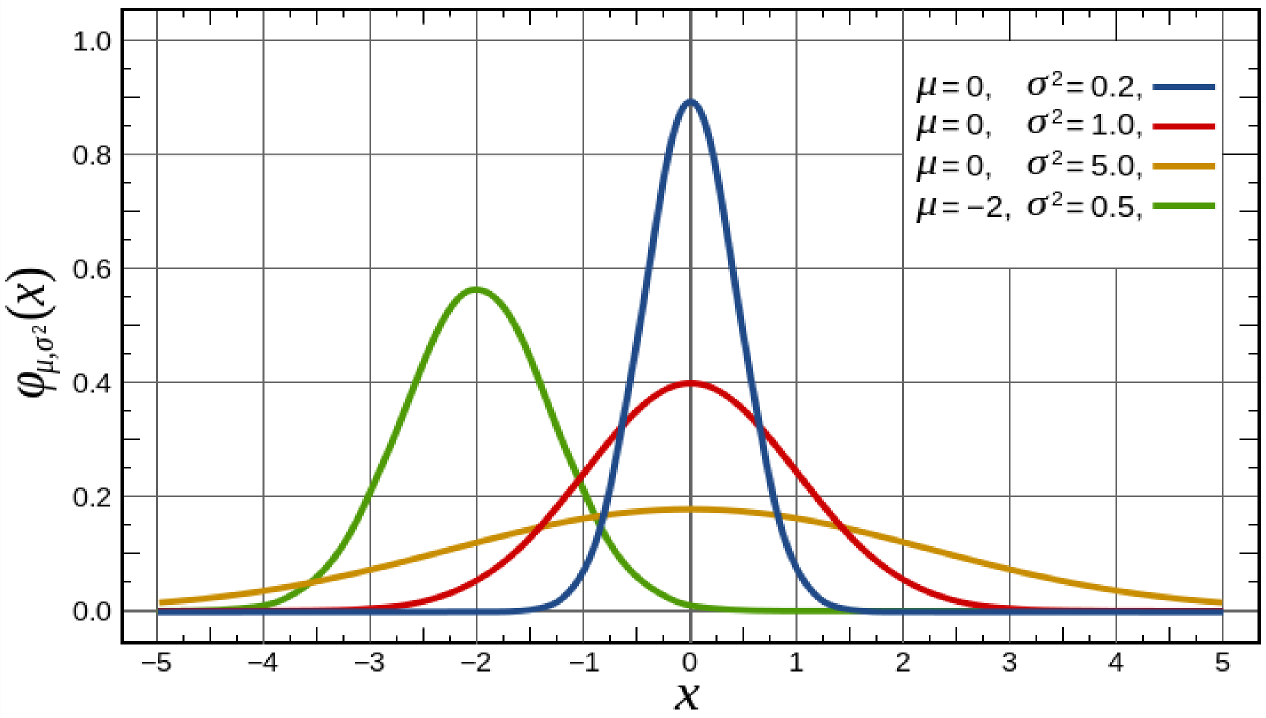

Image Courtesy: Wikipedia

Carl Friedrich Gauss

$$Mean = E(X) = \mu$$

$$Variance(X) = \sigma^2$$

Standard Normal Distribution

- Standard Normal Distribution is distribution with mean 0 and standard deviation 1.
- Standard Normal variable Z is formed by transforming X as:

$$Z = \frac{X-\mu}{\sigma}$$

$$\therefore\ f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

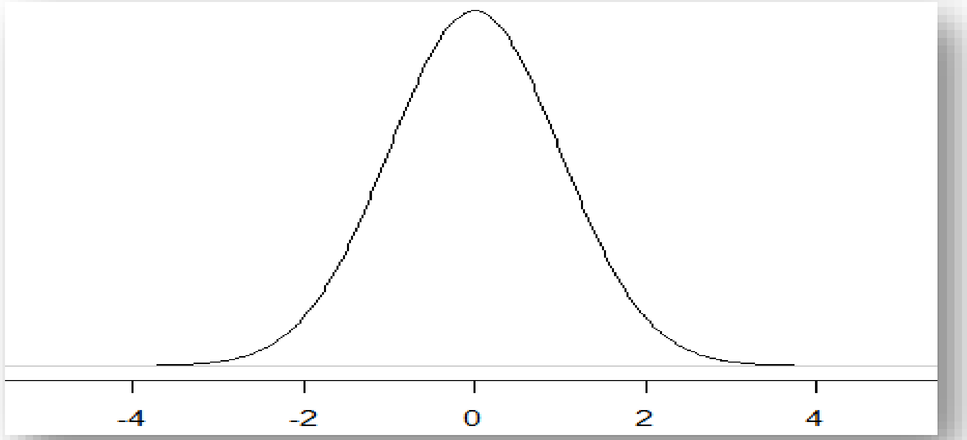

Image Courtesy: mathsisfun.com

Need for Standardization

- To measure variables with different means or standard deviations on a single scale.
- Easy to interpret Z-value
- Subtracting the mean from each value of the variable is called **centering**
- Dividing each value of the variable by standard deviation of the variable is called **scaling**
- We will often do **centering** and **scaling** for machine learning algorithms

Python Functions for Normal Distribution

norm.pdf(x,loc,scale)

Applies for $P(X=k)$

norm.cdf(x,loc,scale)

Applies for $P(X\leq k)$

norm.sf(x,loc,scale)

Applies for $P(X>k)$

norm.stats(loc,scale)

Applies for extracting mean, variance and other moments

norm.ppf(q, loc, scale)

Applies for inverse of cdf

By default if, loc and scale are not specified then standard normal distribution is assumed by all the functions

Examples:

Example 1:

Suppose that the height of a female in a geographical region is normally distributed with μ = 64 inches and σ = 4 inches.

- What is the probability of finding a woman who will be less than 58 inches tall ?

Example 2 :

Suppose the weight of a typical male in a geographical region follows a normal distribution with μ = 180lb and σ = 30lb.

What fraction of males weigh more than 200 pounds?

Example 3

A fast-food restaurant sells As and Bs. On a typical weekday the demand for As is normally distributed with mean 313 and standard deviation 57; the demand for Bs is normally distributed with mean 93 and standard deviation 22.

A) How many As must the restaurant stock to be 98% sure of not running out of stock on a given day ?

B) How many Bs must the restaurant stock to be 90% sure of not running out on a given day ?

C) If the restaurant stocks 450 As and 150 Bs for a given day, what is the probability that it will run out of As or Bs (or both) that day ? Assume that the demand for As and Bs are probabilistically independent.