Homework 1 sample solution

Due 09/10/2024

September 2, 2024

Use the *formal definitions* of Big-Oh, Big-Omega, and Big-Theta to prove the following.

1. Prove that $\sqrt{(n+1)^3} = \Omega(n\sqrt{n})$. *Hint*: $\sqrt{ab} = \sqrt{a}\sqrt{b}$, for any nonnegative real numbers a and b.

   Proof.

   $$\begin{aligned}\sqrt{(n+1)^3} &= \sqrt{(n+1)^2(n+1)} \\ &= (n+1)\sqrt{n+1} \\ &\geq n\sqrt{n+1} \\ &\geq n\sqrt{n}\end{aligned}$$

   Since $\sqrt{(n+1)^3)} \geq n\sqrt{n}$, there exist positive constants c and n_0, namely $c = n_0 = 1$, such that $\sqrt{(n+1)^3} \geq cn\sqrt{n}$ for all $n \geq n_0$. Hence, $\sqrt{(n+1)^3} = \Omega(n\sqrt{n})$ by the formal definition of Big-Omega. □

2. Prove that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n)f_2(n) = O(g_1(n)g_2(n))$.

   Proof. If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, there exist positive constants c_1, c_2, n_1, and n_2 such that $f_1(n) \leq c_1g_1(n)$ for all $n \geq n_1$ and $f_2(n) \leq c_2g_2(n)$ for all $n \geq n_2$. Hence, $f_1(n) \leq c_1g_1(n)$ and $f_2(n) \leq c_2g_2(n)$ for all $n \geq \max\{n_1, n_2\}$, so $f_1(n)f_2(n) \leq (c_1g_1(n))(c_2g_2(n))$ for all $n \geq \max\{n_1, n_2\}$, so $f_1(n)f_2(n) \leq (c_1c_2)g_1(n)g_2(n)$ for all $n \geq \max\{n_1, n_2\}$. Thus, there exist positive constants c_3 and n_3, namely $c_3 = c_1c_2$ and $n_3 = \max\{n_1, n_1\}$, such that $f_1(n)f_2(n) \leq c_3g_1(n)g_2(n)$ for all $n \geq n_3$, so $f_1(n)f_2(n) = O(g_1(n)g_2(n))$ by the formal definition of Big-Oh. □

3. Prove that if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Proof. If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, there exist positive constants c_1, c_2, n_1, and n_2 such that $f(n) \leq c_1 g(n)$ for all $n \geq n_1$ and $g(n) \leq c_2 h(n)$ for all $n \geq n_2$. Since c_1 is positive, we can multiply both sides of the second inequality by this, yielding $c_1 g(n) \leq c_1 c_2 h(n)$ for all $n \geq n_2$. Thus, for $n \geq \max\{n_1, n_2\}$, $f(n) \leq c_1 g(n)$ and $c_1 g(n) \leq c_1 c_2 h(n)$, so $f(n) \leq c_1 c_2(h)$, transitively. Therefore, there exist positive constants c_3 and n_3, namely $c_3 = c_1 c_2$ and $n_3 = \max\{n_1, n_2\}$, such that $f(n) \leq c_3 h(n)$ for all $n \geq n_3$, so $f(n) = O(h(n))$ by the formal definition of Big-Oh. $\square$