Questions of the day

- How can we compare algorithms without ever implementing them?
- What does O(n^2) *really* mean?
- How can we use our understanding of Big-Oh to show that this:

Input: *data*: an array of integers to sort
Input: n: the number of values in *data*
Output: permutation of *data* such that $data[1] \leq \ldots \leq data[n]$

for $i = 1$ to $n - 1$ **do**
  Let m be the location of the min value in $data[i..n]$;
  Swap $data[i]$ and $data[n]$;
end
return *data*;

always takes $\Theta(n^2)$ time?

Algorithm analysis

William Hendrix

Lecture 1

Today

- Syllabus
- RAM model of computation
- Big-Oh notation
  - Motivation
  - Formal definitions
  - Properties
- Analyzing algorithms
- Summations

Syllabus

- Read at: sit.instructure.com
- Contact: whendrix@stevens.edu
- Office hours:
  - TR 12:30-1:30 pm
  - GS 251
- CAs: TBD
- Office hours: TBD

- Course objectives, grading scale, exams, etc.
- Class participation
- Feedback form
- Slides

What will we learn in this class?

- **How to determine if an algorithm is efficient**
  - RAM model
  - Big-Oh definition and properties
- **How to improve your algorithms by organizing data**
  - Stacks and queues
  - Binary search trees
    - Balanced BSTs
  - Priority queues and heaps
  - Hash tables
- **How to develop your own algorithms**
  - Greedy algorithms
  - Divide-and-conquer
  - Dynamic programming
  - Graph traversals
- **Classical sort, search, and graph algorithms**

Brainstorm: time complexity

- What are the factors that contribute to the running time of an algorithm?
  - Processor speed
  - Number of instructions executed
  - Cache coherency
  - Resource conflicts (network, hard disk, etc.)

- Which of these are important when comparing algorithms?
  - Processor speed affects fast and slow algorithms equally
    - Not an important factor

- What can we *most reliably* control when designing an algorithm?
  - Number of instructions executed

RAM model of computation

- Set of assumptions that make analysis more reasonable

Assumptions

1. All "basic" operations (assignment, arithmetic, branching, memory access, etc.) take 1 operation
   - Loops and functions do not qualify
2. We have "infinite" memory

Cons

- Different operations take different number of clock cycles
  - Cache locality has significant impact
- Virtual memory can slow performance

Pros

- Can actually analyze algorithms

RAM model example

Input: *data*: array of integers
Input: n: size of *data*
Output: index *min* such that $data[min] \leq data[j]$, for all j from 1 to n

Algorithm: FindMin

$min = 1$;
for $i = 2$ to n **do**
    if $data[i] < data[min]$ **then**
        $min = i$;
    end
end
return min;

RAM model example

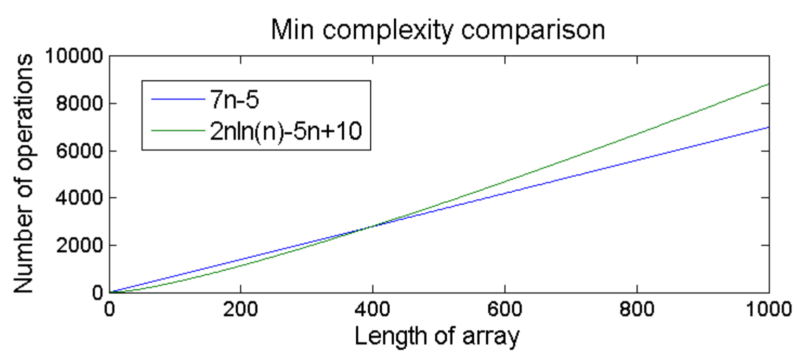
```
Input: data: array of integers
Input: n: size of data
Output: index min such that
        data[min] ≤ data[j], for all j from
        1 to n
Algorithm: FindMin
min = 1;
for i = 2 to n do
    if data[i] < data[min] then
        min = i;
    end
end
return min;
```

Ops per line	Times executed
1	1
2	n-1 (arguably n)
3	n-1
1	??? ($\leq$ n-1)
0	n/a
1	n-1
1	1

Total ops: $\leq 7(n-1) + 2 = 7n - 5$

Question: is this better or worse than an algorithm that takes at most $2n \ln n - 5n + 10$ ops?

Better unless n < 395



Big-Oh notation

- Technique for *abstracting away details* of complexity
  - Can be used for time complexity, space complexity, etc.

- **Main idea:** most important aspect of complexity is *how fast it grows* relative to input size
  - Focus on asymptotic (eventual) growth rate
  - "Fast" functions will eventually pass "slow" functions for large *n*
  - Coefficients only matter if growth rate is similar
  - Predicting behavior for small *n* is difficult and often pointless

- Big-Oh notation
  - Organizes growth rates into classes
  - Three main symbols: $O(f(n)), \Omega(f(n)), \Theta(f(n))$
    - Analogous to "at most", "at least", and "similar to" *f(n)*

Justification of Big-Oh

- Algorithm runtime with c=1, running at 1 GHz:

n=	lg(n)	n	$n$lg(n)	n^2	n^3	2^n	$n!$
10	3 ns	10 ns	33 ns	100 ns	1 µs	1 µs	3.6 ms
20	4 ns	20 ns	86 ns	400 ns	8 µs	1 ms	77 yrs
30	5 ns	30 ns	147 ns	900 ns	27 µs	1 s	
40	5 ns	40 ns	213 ns	1.6 µs	64 µs	18.3 min	
50	6 ns	50 ns	282 ns	2.5 µs	125 µs	13 days	
100	7 ns	100 ns	644 ns	10 µs	1 ms		
1,000	10 ns	1 µs	9.97 µs	1 ms	1 s		
1,000,000	20 ns	1 ms	19.9 ms	16.7 min	31.7 yrs		
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"Fails" at: Never! billions millions 10k 40ish 16ish

Lesson: on large data, coefficients not as important

Big-Oh

- Upper bound *("at most")*

$f(n) = O(g(n))$ if and only if there exist positive constants c and n_0 such that $f(n) \le cg(n)$ for all $n \ge n_0$.

Big-Oh in pictures

- Upper bound *("at most")*

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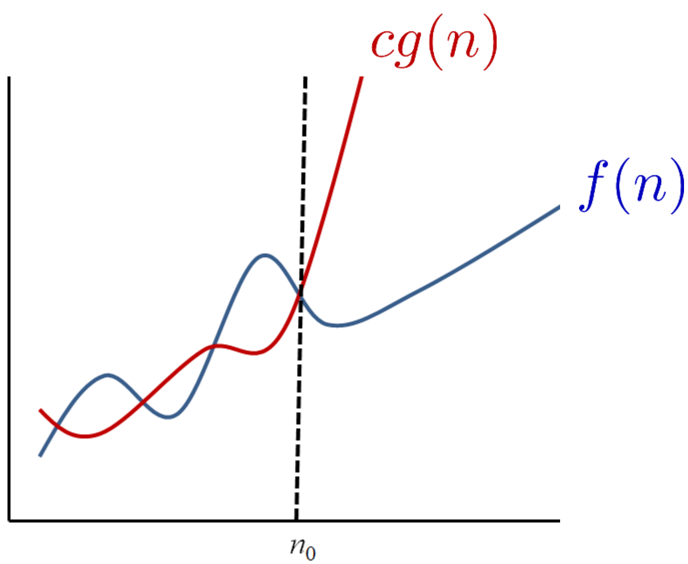

Translation: *f* is smaller than some multiple of *g* eventually (and stays smaller)

Small values of *n* don't matter

f isn't growing faster than *g*

Big-Oh

- Upper bound *("at most")*

$f(n) = O(g(n))$ if and only if there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.

- We say "*g(n)* dominates *f(n)*" when *f(n)* = *O(g(n))*
- Notation weirdness:
  - *O*, *Ω*, and *Θ* are classes (sets) of functions
  - BUT: we use = to assign class, not ∈
- **Example**
  - Prove that $7n^2 + 19n - 4444 = O(n^2)$.

Big-Oh

- Upper bound *("at most")*

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  - *O*, *Ω*, and *Θ* are classes (sets) of functions
  - BUT: we use = to assign class, not ∈
- **Example**
  - Prove that $7n^2 + 19n - 4444 = O(n^2)$.

Proof. If $n \geq 19$,

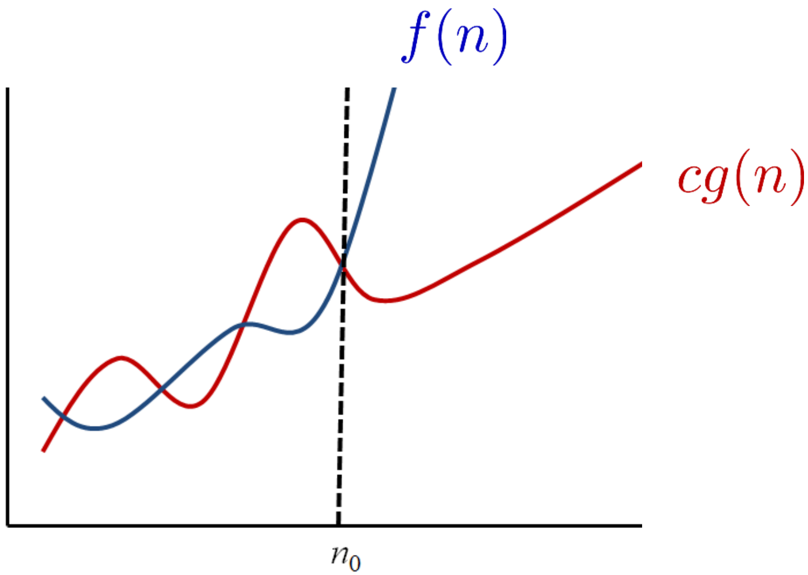
$$\begin{aligned} 7n^2 + 19n - 4444 &\leq 7n^2 + 19n \\ &\leq 7n^2 + n^2 \\ &= 8n^2 \end{aligned}$$

Therefore, there exist positive constants $c = 8$ and $n_0 = 19$ such that $7n^2 + 19n - 4444 \leq cn^2$ for all $n \geq n_0$. □

Big-Omega picture

- Lower bound *("at least")*

$f(n) = \Omega(g(n))$ if and only if there exist positive constants c and n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$.



Translation: f is bigger than some multiple of g eventually (and stays bigger)

Small values of n don't matter

g isn't growing faster than f

Big-Omega

- Lower bound *("at least")*

$f(n) = \Omega(g(n))$ if and only if there exist positive constants c and n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$.

- **Example**
  - Prove that $7n^2 + 19n - 4444 = \Omega(n^2)$.

Big-Omega

- Lower bound *("at least")*

$f(n) = \Omega(g(n))$ if and only if there exist positive constants c and n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$.

- **Example**
  - Prove that $7n^2 + 19n - 4444 = \Omega(n^2)$.

Proof. If $n \geq 4444$,

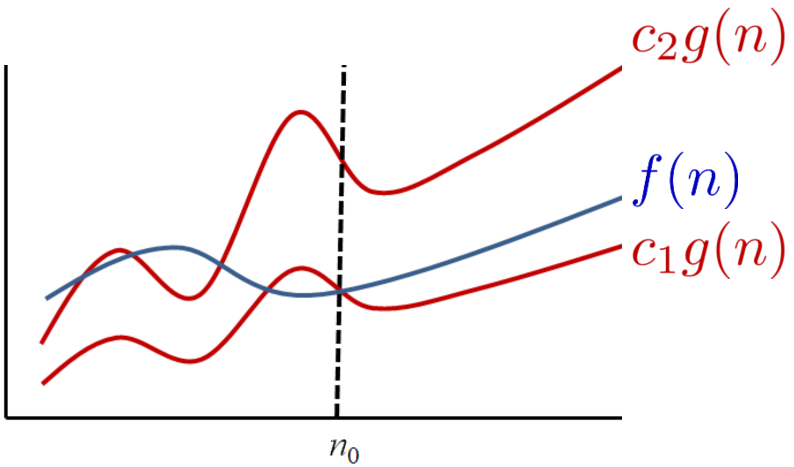
$$\begin{aligned} 7n^2 + 19n - 4444 &\geq 7n^2 - 4444 \\ &\geq 7n^2 - n^2 \\ &= 6n^2 \end{aligned}$$

Therefore, there exist positive constants $c = 6$ and $n_0 = 4444$ such that $7n^2 + 19n - 4444 \geq cn^2$ for all $n \geq n_0$. $\square$

Big-Theta picture

- Upper *and* lower bound *("same rate as")*

$f(n) = \Theta(g(n))$ if and only if there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.



Translation: *f* can be sandwiched between two multiples of *g* eventually (and stays between them)

f and *g* are growing at the same rate

Small values of *n* don't matter

Big-Theta

- Upper *and* lower bound *("same rate as")*

$f(n) = \Theta(g(n))$ if and only if there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$.

- **Example**
  - Prove that $7n^2 + 19n - 4444 = \Theta(n^2)$.

Big-Theta

- Upper *and* lower bound *("same rate as")*

> $f(n) = \Theta(g(n))$ if and only if there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$.

- **Example**
  - Prove that $7n^2 + 19n - 4444 = \Theta(n^2)$.

Proof. If $n \ge 4444$,

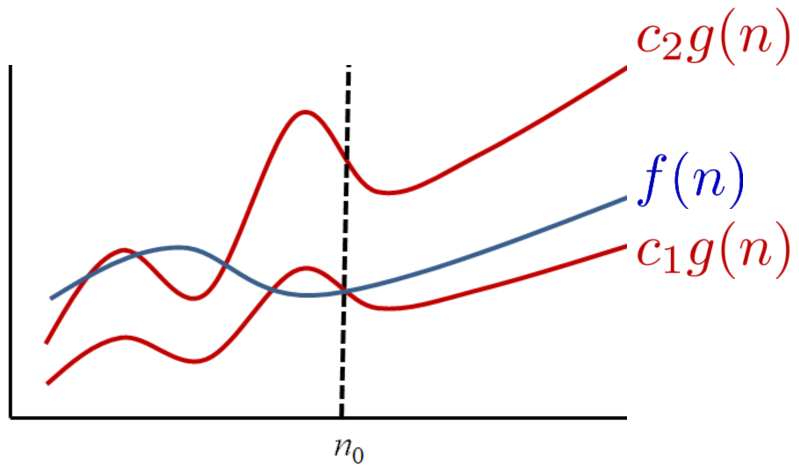
$$\begin{aligned} 7n^2 + 19n - 4444 &\ge 7n^2 + 19n - n \\ &= 7n^2 + 18n \\ &\ge 7n^2 \\ 7n^2 + 19n - 4444 &\le 7n^2 + 19n \\ &\le 7n^2 + n^2 \\ &= 8n^2 \end{aligned}$$

Therefore, there exist positive constants $c_1 = 7$, $c_2 = 8$, and $n_0 = 4444$ such that $c_1 n^2 \le 7n^2 + 19n - 4444 \le c_2 n^2$ for all $n \ge n_0$. $\square$

Big-Theta picture

- Upper *and* lower bound *("same rate as")*

$f(n) = \Theta(g(n))$ if and only if there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.



Translation: *f* can be sandwiched between two multiples of *g* <u>eventually</u> (and <u>stays between them</u>)

f and *g* are growing at the same rate

Small values of *n* don't matter

Big-Oh example

- Use the *formal definition* of Big-Oh to prove:

$$\sum_{i=1}^{n} i = O(n^2)$$

Big-Oh example

- Use the *formal definition* of Big-Oh to prove:

$$\sum_{i=1}^{n} i = O(n^2)$$

Proof. Consider the sum $\sum_{i=1}^{n} i$, which equals $1 + 2 + \ldots + n$. Note that every term in this sum is at most n. So, $\sum_{i=1}^{n} i \leq n + n + \ldots + n = n(n) = n^2$. Since $\sum_{i=1}^{n} i \leq n^2$, there exist positive constants $c = n_0 = 1$ such that $\sum_{i=1}^{n} i \leq cn^2$ for all $n \geq n_0$, so $\sum_{i=1}^{n} i = O(n^2)$. □

Big-Oh example (series)

- Prove that $\sum_{i=1}^{n} i = \Omega(n^2)$.

Big-Oh example (series)

- Prove that $\displaystyle\sum_{i=1}^{n} i = \Omega(n^2)$.

Proof. Consider $\sum_{i=1}^{n} i$:

<u>Good trick:</u>
drop lower half
of series

$$
\begin{aligned}
\sum_{i=1}^{n} i &= 1 + 2 + \ldots + (n-1) + n \\
&\geq \underbrace{\lceil n/2 \rceil + \ldots + (n-1) + n}_{\geq n/2 - 1} \\
&\geq n/2(n/2 - 1) \\
&\geq n/2(n/4) && \forall n \geq 4 \\
&\geq n^2/8 && \forall n \geq 4
\end{aligned}
$$

Thus, there exist positive constants c and n_0, namely $c = 1/8$ and $n_0 = 2$, such that $\sum_{i=1}^{n} i \geq cn^2$ for all $n \geq n_0$, so $\sum_{i=1}^{n} i = \Omega(n^2)$. $\square$

Analysis of Big-Oh

Pros

- Provides a useful summary of the growth rate of the complexity
- Compact
- Simple: eight classes cover most useful algorithms

$$O(1) \ll O(\lg n) \ll O(n) \ll O(n \lg n) \ll O(n^2) \ll O(n^3) \ll O(2^n) \ll O(n!)$$

Cons

- Ignores contributions from coefficients and lower-order terms
- Doesn't rank algorithms with same growth rate
- Doesn't rank algorithms on small inputs
- Some of the "best" algorithms have extremely large coefficients, making them impractical for many purposes

Connection to calculus

- You can also determine *O*, *Ω*, and *Θ* by limits:

g grows faster $\longrightarrow$ $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0 \quad \to f(n) = O(g(n))$$

$$\text{Actually } f(n) = o(g(n))$$

Same growth rate $\longrightarrow$ $$\lim_{n\to\infty}\frac{f(n)}{g(n)} \in (0,\infty) \to f(n) = \Theta(g(n))$$

g grows slower $\longrightarrow$ $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty \quad \to f(n) = \Omega(g(n))$$

$$\text{Actually } f(n) = \omega(g(n))$$

- Standard rules for taking limits apply
  - Including L'Hôpital's Rule

Formal definition extra practice

- Use the *formal definition* of Big-Theta to prove:

  For any $x > 0$, if $f(n) = \Theta(g(n))$, then $xf(n) = \Theta(g(n))$

Formal definition extra practice

- Use the *formal definition* of Big-Theta to prove:

  For any $x > 0$, if $f(n) = \Theta(g(n))$, then $xf(n) = \Theta(g(n))$

Proof. Since $f(n) = \Theta(g(n))$, there exist positive constants c and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$, for all $n \geq n_0$. Since $x > 0$, we may multiply all sides of this inequality by x, yielding $xc_1 g(n) \leq xf(n) \leq xc_2 g(n)$. Thus, there exist constants $c_3 = xc_1$, $c_4 = xc_2$, and n_0 such that $c_3 g(n) \leq xf(n) \leq c_4 g(n)$ for all $n \geq n_0$, so $xf(n) = \Theta(g(n))$. $\square$

Properties of Big-Oh notation

- Reflexivity

  $$f(n) = O(f(n)),\ f(n) = \Omega(f(n)), \text{ and } f(n) = \Theta(f(n))$$

- Antisymmetry

  $$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

  $$f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n)) \Leftrightarrow f(n) = \Theta(g(n))$$

- Symmetry (Θ only)

  $$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

- Transitivity

  $$f(n) = O(g(n)) \text{ and } g(n) = O(h(n)) \rightarrow f(n) = O(h(n))$$
  $$f(n) = \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \rightarrow f(n) = \Omega(h(n))$$
  $$f(n) = \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \rightarrow f(n) = \Theta(h(n))$$

- Alternately:

  $$O(O(h(n))) = O(h(n))$$
  $$\Omega(\Omega(h(n))) = \Omega(h(n))$$
  $$\Theta(\Theta(h(n))) = \Theta(h(n))$$

Combination properties

- Envelopment
  - Addition

$$O(f(n)) + O(g(n)) = O(f(n) + g(n))$$
$$\Omega(f(n)) + \Omega(g(n)) = \Omega(f(n) + g(n))$$
$$\Theta(f(n)) + \Theta(g(n)) = \Theta(f(n) + g(n))$$

  - Multiplication

$$O(f(n))O(g(n)) = O(f(n)g(n))$$
$$\Omega(f(n))\Omega(g(n)) = \Omega(f(n)g(n))$$
$$\Theta(f(n))\Theta(g(n)) = \Theta(f(n)g(n))$$

- All three ignore constant coefficients

$$\forall x > 0, \quad \begin{array}{l} f(n) = O(g(n)) \rightarrow xf(n) = O(g(n)) \\ f(n) = \Omega(g(n)) \rightarrow xf(n) = \Omega(g(n)) \\ f(n) = \Theta(g(n)) \rightarrow xf(n) = \Theta(g(n)) \end{array}$$

- Only the largest term matters

$$f(n) = O(g(n)) \rightarrow O(f(n) + g(n)) = O(g(n))$$
$$f(n) = O(g(n)) \rightarrow \Omega(f(n) + g(n)) = \Omega(g(n))$$
$$f(n) = O(g(n)) \rightarrow \Theta(f(n) + g(n)) = \Theta(g(n))$$

Big-Oh properties example

- Use Big-Oh properties to establish the following:

1. If $f(n) = 13n^2 + 1234n + 91.2n\sqrt{n}$, then $f(n) = \Theta(n^2)$. Use the facts that $n = O(n^2)$ and $\sqrt{n} = O(n)$.

Big-Oh properties example

- Use Big-Oh properties to establish the following:

1. If $f(n) = 13n^2 + 1234n + 91.2n\sqrt{n}$, then $f(n) = \Theta(n^2)$. Use the facts that $n = O(n^2)$ and $\sqrt{n} = O(n)$.

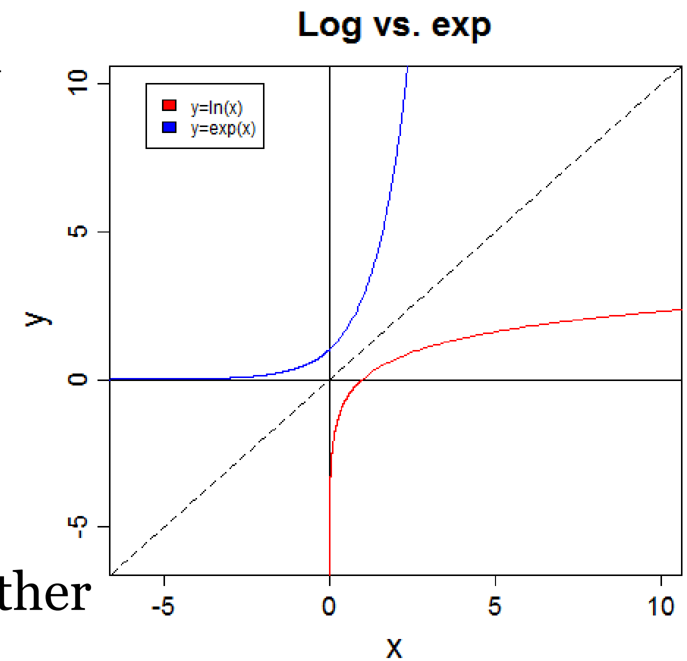
$$
\begin{aligned}
f(n) &= 13n^2 + 1234n + 91.2n\sqrt{n} \\
&= \Theta(13n^2) + \Theta(1234n) + \Theta(91.2n\sqrt{n}) && \text{Reflexive} \\
&= \Theta(n^2) + \Theta(n) + \Theta(n\sqrt{n}) && \text{Constant coefficients} \\
&= \Theta(n^2 + n) + \Theta(n\sqrt{n}) && \text{Envelopment } (+) \\
&= \Theta(n^2) + \Theta(n\sqrt{n}) && \text{Largest term } (n = O(n^2)) \\
&= \Theta(n^2 + n\sqrt{n}) && \text{Envelopment } (+) \\
&= \Theta(n(n + \sqrt{n})) \\
&= \Theta(n)\Theta(n + \sqrt{n}) && \text{Envelopment } (\cdot) \\
&= \Theta(n)\Theta(n) && \text{Largest term } (\sqrt{n} = O(n)) \\
&= \Theta(n^2) && \text{Envelopment } (\cdot)
\end{aligned}
$$

Revenge of the logarithms

- **Logarithm:** inverse exponential function

$$y = \ln x \Leftrightarrow x = e^y$$

- Natural log (ln): inverse of e^x
- Logarithms of other base: $\log_b(x)$
  - $\log_2(x)$ is very common in algorithms
- Computing logs of other bases
  - $\log_b(x) = \frac{\ln x}{\ln b}$
  - All logs are *scalar multiples* of one another



- **Log properties**

Base 2 → $\lg(ab) = \lg(a) + \lg(b)$

$$\lg(a^b) = b \lg(a)$$

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\lg n)$$

Because

$$2^A 2^B = 2^{A+B}$$

$$\left(2^A\right)^b = 2^{Ab}$$

$$\int_1^n \frac{1}{x} dx = \ln n$$

Logarithm property example

- Prove that $\lg(n!) = \Theta(n \lg(n))$.

Logarithm property example

- Prove that $\lg(n!) = \Theta(n\lg(n))$.

Proof. First, note that:

$$
\begin{aligned}
\lg(n!) &= \lg(n(n-1)(n-2)\cdots(2)(1)) && (1)\\
&= \lg(n) + \lg(n-1) + \ldots + \lg(2) + \lg(1) && (2)
\end{aligned}
$$

Since each term of this sum is less than or equal to $\lg(n)$,

$$
\begin{aligned}
\lg(n!) &\leq \lg(n) + \lg(n) + \ldots + \lg(n) + \lg(n) && (3)\\
&\leq n\lg(n) && (4)
\end{aligned}
$$

So, $\lg(n!) = O(n\lg(n))$. On the other hand, the sum in line 2 has at least $n/2$ elements that are $\lg(n/2)$ or larger, so:

$$
\begin{aligned}
\lg(n!) &\geq (n/2)\lg(n/2) && (5)\\
&\geq (n/2)(\lg n - \lg 2) && (6)\\
&\geq (n/2)(\lg n - 1) && (7)\\
&\geq (n/2)\lg n - n/2 && (8)
\end{aligned}
$$

Since $n/2\lg n = \Omega(n\lg n)$ and $n/2 = \Omega(n)$, $(n/2)\lg n - n/2 = \Omega(n\lg n)$. Since $\lg(n!)$ bounded below by $\Omega(n\lg n)$, $\lg(n!) = \Omega(n\lg n)$. Therefore, $\lg(n!) = \Theta(n\lg n)$, because $\lg(n!) = O(n\lg n)$ and $\lg(n!) = \Omega(n\lg n)$. $\square$

Coming up

- Algorithm analysis
- Recursive analysis
- Data structures

- **Recommended reading (today):** Sections 1.1 and 1.2
  - *Practice problems:* R-1.3, R-1.7, R-1.19, R-1.21, C-1.8
- **Recommended reading (next week):** Section 1.3
  - *Practice problems:* R-1.11, R-1.15, R-1.26, C-1.3, A-1.4