



27. Counting | Probability for
Gate data science and ai |...

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28. Permutation | Probability
for Gate data science and ai |...

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29. Combination | Probability
for Gate data science and ai |...

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30. Binomial Probability
| Probability for Gate data...

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explain me all this topics in really detailed as i am preparing for gate and i want to do top in gate exam i want u to explain me each topic with very easy example and then i want u to take upto gate level advancement

Please write in English language.

Awesome—let's nail these four pillars for GATE DA/CS: **Counting**, **Permutation**, **Combination**, and **Binomial Probability**. I'll go from super-simple intuition → standard formulas → tricky constraints → GATE-level tactics, with small solved examples and quick practice at the end of each section.

1) Counting (Basics → Advanced)

A. First principles

- **Rule of Sum (OR):** If task A can be done in m ways and task B in n **disjoint** ways → $m + n$.
- **Rule of Product (AND):** If you do step1 in m ways and step2 in n ways (independent choices) → $m \times n$.

Easy example: A password is 1 letter **or** 1 digit. Letters (26) + digits (10) = **36**.

Easy example (AND): A 4-digit PIN (0-9): $10^4 = 10\,000$.

Common constraints you'll see

- "Cannot start with 0"
- "No repetition" vs "repetition allowed"
- "At least one vowel/even digit..." → usually **complement counting** is easier.

Example (no leading zero, repetition allowed): 4-digit number

1st place: 9 (1-9), others: 10 each → $9 \cdot 10^3 = 9000$.

B. Inclusion-Exclusion (PIE)

Count “A or B (or C...)” when sets overlap.

$$|A \cup B| = |A| + |B| - |A \cap B|, \quad |A \cup B \cup C| = \sum |A_i| - \sum |A_i \cap A_j| + |A \cap B \cap C|$$

Example (classic GATE flavor): Integers 1..100 divisible by 2 or 5 or 7.

Divisible by 2: 50; by 5: 20; by 7: 14.

Pairs: by 10 \rightarrow 10; by 14 \rightarrow 7; by 35 \rightarrow 2.

Triple: by 70 \rightarrow 1.

Total = $50 + 20 + 14 - (10 + 7 + 2) + 1 = 66$.

C. Pigeonhole Principle (PHP)

If N objects in k boxes and $N > k \Rightarrow$ some box has ≥ 2 objects.

GATE-ish: Among 13 people, two share a birth-month (12 months $\Rightarrow 13 > 12$).

2) Permutations (Order matters)

A. Distinct objects, no repetition

- Arrange n distinct: $n!$
- Arrange r chosen from n : ${}^n P_r = \frac{n!}{(n-r)!}$

Easy example: Arrange A,B,C on a shelf: $3! = 6$.

B. With repetition *allowed*

- Length- r string from an alphabet of size k : k^r

Example: 5-char string from lowercase (26): 26^5 .

C. With repeated items (multiset)

If word has counts n_1, n_2, \dots, n_k (sum = n):

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

Example: “BALLOON” (B1,A1,L2,O2,N1): $\frac{7!}{2!2!} = 1260$.

D. Circular permutations

- Round table, rotations identical: $(n-1)!$.
- If reflections also identical (necklaces): $\frac{(n-1)!}{2}$ for $n > 2$.

Example: Seat 5 people around a round table: $4! = 24$.

E. "Together / Not together" tricks

- **Together:** treat the group as a block.
- **Not together:** total – together.

Example: 6 people; A & B together.

Make [AB] a block → now 5 units ⇒ $5! = 120$. Inside block AB/BA ⇒ $\times 2 \rightarrow 240$.

F. Derangements (no element in its own place)

$!n$ (advanced but useful):

$$!n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right)$$

or $!(n) = (n-1)(!(n-1) + !(n-2))$. Roughly $\approx \frac{n!}{e}$.

3) Combinations (Order doesn't matter)

A. Distinct objects, no repetition

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{nPr}{r!}$$

Example: Choose 3 books from 10: ${}^{10}C_3 = 120$.

B. With repetition (multicombinations; "stars & bars")

Number of solutions to $x_1 + x_2 + \cdots + x_k = n$ with $x_i \geq 0$:

$${}^{n+k-1}C_{k-1}$$

If $x_i \geq 1$ (at least one each): let $y_i = x_i - 1 \Rightarrow \text{total} = n - k$, so ${}^{(n-k)+k-1}C_{k-1} = {}^{n-1}C_{k-1}$.

Easy example: Distribute 7 identical balls into 4 boxes (allow empty):

$${}^{7+4-1}C_{4-1} = {}^{10}C_3 = 120.$$

At least one per box: ${}^{7-1}C_{4-1} = {}^6C_3 = 20$.

C. Mixed constraints (typical exam traps)

- "At least / at most" → convert via **shift** (≥ 1) or **complement** (total – forbidden).
- "No two adjacent" → use **gaps** method.

Example (gaps): Place 4 identical balls into slots so no two adjacent among 7 positions □.

Think of choosing 4 slots with at least 1 empty between ⇒ Equivalent to choose 4 from 7 with separation

→ use transformation $y_i \geq 0$ after fixing mandatory gaps.

4) Binomial Probability (Bernoulli trials)

A. Model

- n **independent** trials, each success with probability p (fail $q = 1 - p$).
- Random variable X = number of successes $\rightarrow X \sim \text{Bin}(n, p)$.
- **PMF**: $\Pr[X = k] = {}^nC_k p^k q^{n-k}$ for $k = 0, \dots, n$.
- **Mean**: $E[X] = np$; **Variance**: $\text{Var}(X) = npq$.

Easy example: Toss a fair coin 5 times, probability of exactly 2 heads:

$${}^5C_2 (0.5)^2 (0.5)^3 = 10/32 = 0.3125.$$

Quality-control example (numbers computed): 10 items, defect prob $p = 0.1$.

- Exactly 2 defectives: ${}^{10}C_2 (0.1)^2 (0.9)^8 = 45 \times 0.01 \times 0.43046721 \approx \mathbf{0.1937}$.
- At least 1 defective: $1 - (0.9)^{10} \approx \mathbf{0.6513}$.
- Mean defects = $np = 1$, variance = 0.9.

B. Cumulative & complement tricks

- "At most r " = sum $k = 0..r$ or use complementary if shorter.
- "At least one" = $1 - \Pr[0]$.
- Symmetry for $p = \frac{1}{2}$.

C. Approximations (edge GATE)

- For large n , small p , use **Poisson** with $\lambda = np$.
 - For large n , moderate p , use **Normal** with mean np , var npq (with continuity correction). These are rare but good to know.
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Common Pitfalls (high-yield)

- Mixing **order**: use permutations vs combinations correctly.
 - **Leading zero** in numbers.
 - **Mutual exclusivity vs independence** (they're different).
 - **Overcounting** when groups can be permuted internally (divide by $r!$ when order doesn't matter).
 - **Circular vs linear** arrangements.
 - For words with repeated letters: divide by repeat factorials.
-

Worked GATE-style mini set

Q1 (Counting + complement)

How many 6-digit numbers (no leading zero) have **at least one** repeated digit?

- Total 6-digit numbers: first 9, others 10 $\Rightarrow 9 \cdot 10^5 = 900\,000$.
 - With **all distinct** digits: first 9, then choose 5 from remaining 9 in order: $9 \cdot {}^9P_5 = 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 136,080$.
- Answer = $900,000 - 136,080 = \mathbf{763,920}$.

Q2 (PIE)

How many integers in 1..1000 are divisible by 3 **or** 5 **or** 7?

Compute with PIE (similar to earlier), careful with 1000.

- /3: 333, /5: 200, /7: 142
 - pairs: /15: 66, /21: 47, /35: 28
 - triple: /105: 9
- Answer = $(333 + 200 + 142) - (66 + 47 + 28) + 9 = \mathbf{543}$.

Q3 (Permutations with block)

Arrangements of the word **ASSESS** with the two S's at the ends.

Letters: A(1), S(4), E(1), S(we already count), total 6. Fix ends as S _ _ _ S \rightarrow remaining letters A,S,S,E to arrange with S(2),A(1),E(1): $\frac{4!}{2!} = 12$.

Q4 (Circular + restriction)

Seat 6 people around a table so that A and B are **not adjacent**.

Total circular: $5! = 120$. Together as a block: treat [AB] as one unit $\rightarrow 5! \times 2 = 240$.

Not adjacent = $120 - 240$ **✗** (trap!)

Correct: For circular with block: number of units = 5 $\Rightarrow 4! = 24$; internal AB/BA $\times 2 \Rightarrow 48$.

Answer = $120 - 48 = \mathbf{72}$.

Q5 (Stars & bars with caps)

Number of non-negative integer solutions to $x_1 + x_2 + x_3 = 10$ with $x_1 \leq 4$.

Total without cap: ${}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$.

Subtract $x_1 \geq 5$: let $x'_1 = x_1 - 5 \Rightarrow x'_1 + x_2 + x_3 = 5 \Rightarrow {}^{5+3-1}C_2 = {}^7C_2 = 21$.

Answer = $66 - 21 = \mathbf{45}$.

Q6 (Binomial)

A fair die rolled 8 times. Probability of **exactly two** "6"s.

${}^8C_2(1/6)^2(5/6)^6$.

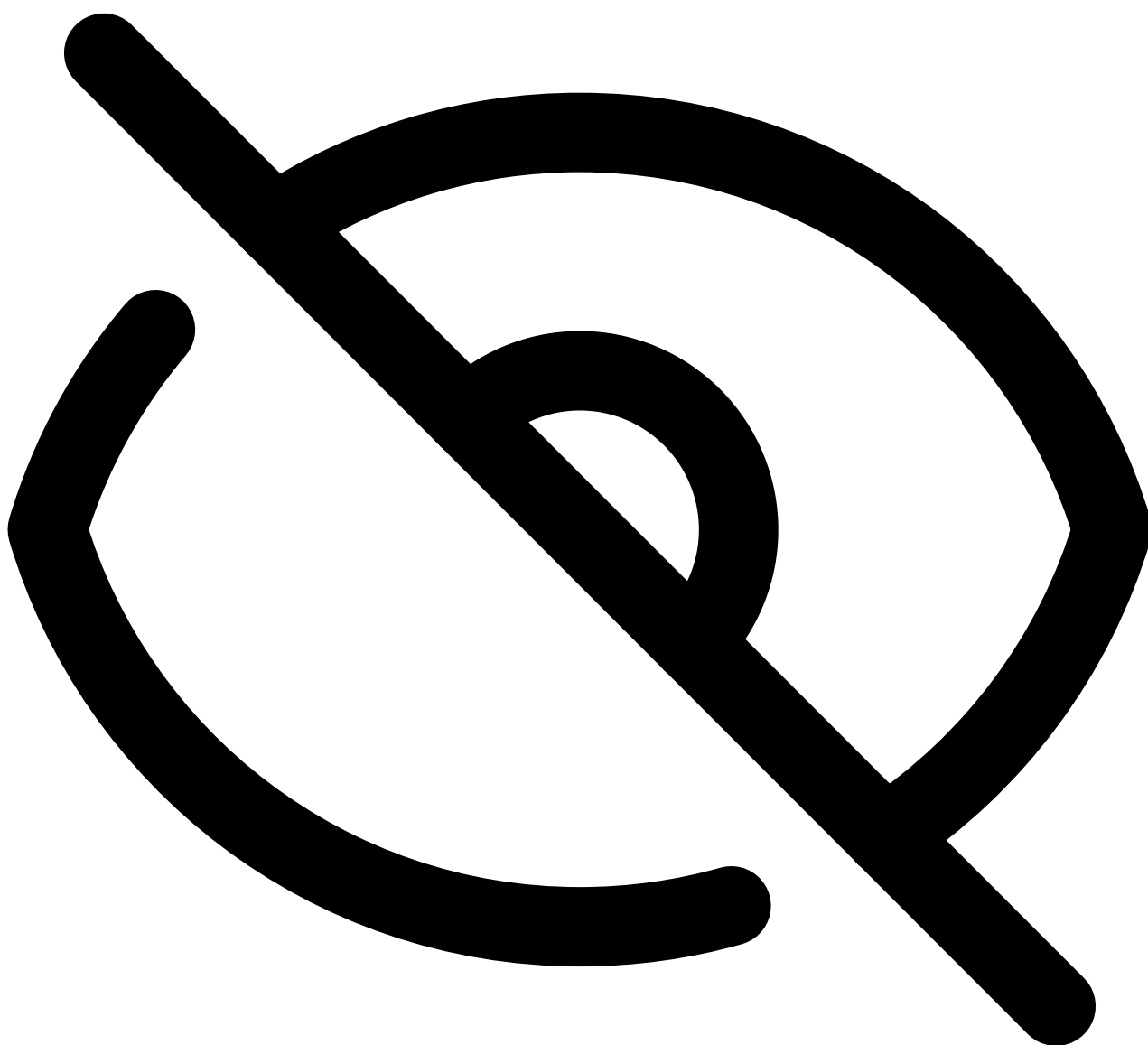
Quick Revision Sheet (memory hooks)

- **Sum/Product:** OR $\rightarrow +$, AND $\rightarrow \times$.
- **PIE 3-sets:** $\sum |A_i| - \sum |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|$.
- **Permute:** $n!$; nP_r ; circular $(n-1)!$; multiset $\frac{n!}{\prod n_i!}$.
- **Combine:** nC_r ; with repetition $n+r-1C_r$ (choose r from n types).
- **Stars & Bars:** $x_1 + \dots + x_k = n, x_i \geq 0 \Rightarrow {}^{n+k-1}C_{k-1}; x_i \geq 1 \Rightarrow {}^{n-1}C_{k-1}$.
- **Binomial:** $P(X = k) = {}^nC_k p^k (1-p)^{n-k}; E[X] = np, \text{Var} = np(1-p)$.
- **Derangements:** $!n \approx n!/e$.

Practice (try now)

1. How many 7-letter passwords using A-Z and digits 0-9 **without repeating** characters and **must start with a letter**?
2. Number of integers in 1..10,000 **not** divisible by 2, 3, or 5.
3. Arrangements of **MISSISSIPPI**.
4. Place 6 identical balls into 4 distinct boxes with each box holding **at most 3** balls.
5. A component works daily with probability 0.98. Over 50 days, probability it works on **all 50** days; on **at least 48** days (write the binomial sums).

If you want, I'll check your answers and then we can step up to **derangements proofs, advanced constraints (caps + adjacency), combinatorial identities, and binomial approximations (Poisson/Normal)** with tougher GATE PYQs.



get me more details in each topic and with easy examples i want more and more example as i want to clear my topic

Please write in English language.

Perfect — let's go *deep dive* into these four GATE-core topics:

Counting → **Permutation** → **Combination** → **Binomial Probability**, each explained **from school-level intuition** → **GATE-level reasoning**, with **many step-by-step examples** and **common traps**.

1. Counting Principles — the foundation of everything

◆ Rule of Product (AND rule)

If one event can occur in m ways **and** another in n ways \rightarrow total = $m \times n$.

Example 1:

A pizza shop offers 3 crusts and 5 toppings.

👉 Crust choice: 3 ways

👉 Topping choice: 5 ways

➡ Total = $3 \times 5 = 15$ pizzas

◆ Rule of Sum (OR rule)

If an event can occur in m ways **or** n ways (mutually exclusive) \rightarrow total = $m + n$.

Example 2:

You can go to college by **car (2 routes)** or **bus (3 routes)**.

Total = $2 + 3 = 5$ ways.

◆ Both rules together

Example 3:

License plates \rightarrow 2 letters + 2 digits

Letters: 26 choices each $\rightarrow 26^2$

Digits: 10 choices each $\rightarrow 10^2$

Total = $26^2 \times 10^2 = 67600$ plates

◆ Counting with restrictions

Example 4: 4-digit number **cannot start with 0**

\rightarrow first place: 9 choices (1–9)

\rightarrow rest: $10^3 = 1000$

Total = $9 \times 1000 = 9000$ numbers

Example 5: 4-digit number **no repetition**

\rightarrow first: 9 (1–9)

\rightarrow second: 9

\rightarrow third: 8

\rightarrow fourth: 7

Total = $9 \times 9 \times 8 \times 7 = 4536$

◆ Complement rule

Sometimes it's easier to count what you *don't* want and subtract.

Example 6:

How many 3-digit numbers have **at least one 7**?

\rightarrow Total 3-digit numbers: 900 (100–999)

→ Without 7: each digit 9 options (no 7) → $9 \times 9 \times 9 = 729$

→ Answer = $900 - 729 = 171$

◆ Inclusion-Exclusion (PIE)

Used when sets overlap (e.g., multiples of 2 or 3).

Example 7:

Numbers ≤ 30 divisible by 2 or 3

$|A| = 15$ (2s), $|B| = 10$ (3s), $|A \cap B| = 5$ (6s)

→ $|A \cup B| = 15 + 10 - 5 = 20$

◆ Pigeonhole Principle

If n items go into k boxes and $n > k \Rightarrow$ some box has ≥ 2 items.

Example 8: Among 13 people, two share a birth month (12 months).

★ 2. Permutations — order matters

◆ Formula

$${}_n P_r = \frac{n!}{(n-r)!}$$

Example 1: Arrange 4 books from 10 → $10P4 = 5040$.

◆ All n distinct → $n!$

Example 2: Arrange A, B, C → $3! = 6$ sequences.

◆ Repetition allowed

If each position can be filled by k choices: k^r

Example 3: 5-letter password (A-Z): 26^5 .

◆ Identical objects

If items repeat, divide by factorials of repeats.

$$\text{Arrangements} = \frac{n!}{n_1! n_2! n_3! \dots}$$

Example 4: "BALLOON" → 7 letters (L×2, O×2)

= $7! / (2! \cdot 2!) = 1260$

♦ Circular permutations

Rotations considered same $\rightarrow (n-1)!$

Reflections same too $\rightarrow (n-1)! / 2$

Example 5: 5 people around table $\rightarrow 4! = 24$.

♦ Restrictions — “Together / Not together”

Example 6: 6 people, A & B together.

Treat AB as block $\rightarrow 5$ units $\rightarrow 5! = 120 \times 2 = 240$

If NOT together $\rightarrow 6! - 240 = 480$.

♦ Word example

Example 7: “MISSISSIPPI”

Letters: 11; I \times 4, S \times 4, P \times 2

$= 11! / (4! \cdot 4! \cdot 2!) = 34650$

♦ Derangements (no one in own seat)

$$!n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \right)$$

Example 8: 3 letters A,B,C wrong envelopes $\rightarrow !3 = 2$.

★ 3. Combinations — order doesn't matter

♦ Formula

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Example 1: Choose 3 from 5 $\rightarrow {}^5C_3 = 10$.

♦ Relation with permutation

$${}^nP_r = {}^nC_r \times r!$$

♦ Repetition allowed (multicombination)

When you can choose same element multiple times:

$${}^{n+r-1}C_r$$

Example 2: Choose 3 fruits from {apple, mango, banana} (can repeat):

$= {}^{3+3-1}C_3 = {}^5C_3 = 10$.

◆ Stars and Bars (distribution)

Solutions of $x_1 + x_2 + \dots + x_k = n, x_i \geq 0$:

$${}^{n+k-1}C_{k-1}$$

Example 3: 5 identical balls into 3 boxes $\rightarrow 5+3-1 \text{ C } 3-1 = 7C2 = 21$

◆ If each box ≥ 1

Let $y_i = x_i - 1$, total = $n-k$

$${}^{n-1}C_{k-1}$$

Example 4: 5 balls, 3 boxes, each $\geq 1 \rightarrow 5-1 \text{ C } 3-1 = 4C2 = 6$.

◆ "At most / At least" constraints

Use complement or shift.

Example 5: How many ways to distribute 8 chocolates to 3 kids, each ≤ 5 .

Without restriction: $8+3-1 \text{ C } 3-1 = 10C2 = 45$

Subtract cases where someone gets $>5 \rightarrow$ treat separately.

◆ "No two adjacent"

Example 6: Place 3 stars among 7 boxes with gaps between \rightarrow use "gaps method."

Put 4 bars to separate \rightarrow count via transformed stars & bars.

◆ Combination identities

$$1. nCr = nC(n-r)$$

$$2. nCr + nC(r-1) = (n+1)Cr$$

$$3. \sum_r nCr = 2^n$$

\rightarrow GATE often tests identity simplification.

Example 7: $6C0 + 6C1 + \dots + 6C6 = 2^6 = 64$.

★ 4. Binomial Probability

◆ Model

n independent Bernoulli trials (success prob p , fail $q=1-p$).

X = number of successes.

$$P(X = k) = {}^nC_k p^k q^{n-k}$$

Example 1: 5 coin tosses → exactly 2 heads
 $= 5C2 (\frac{1}{2})^2 (\frac{1}{2})^3 = 10/32 = \mathbf{0.3125}$

◆ Properties

$E[X] = np$
 $\text{Var}[X] = npq$
 $\sigma = \sqrt{npq}$

Example 2: $n=10, p=0.3 \rightarrow E[X]=3, \text{Var}=2.1$.

◆ Cumulative (at most / at least)

Example 3: $P(X \geq 1) = 1 - P(X=0) = 1 - q^n$.
 For fair coin, 3 tosses: $1 - (\frac{1}{2})^3 = \mathbf{0.875}$

◆ "At most r"

Sum from 0 to r.

Example 4: $n=4, p=0.5, P(X \leq 2) = 4C0(0.5)^4 + 4C1(0.5)^4 + 4C2(0.5)^4$
 $= (1 + 4 + 6)/16 = \mathbf{11/16}$

◆ Mean verification via symmetry

Binomial is symmetric if $p=q=0.5$.

Example 5: For 5 tosses, $P(0)=P(5), P(1)=P(4)$, etc.

◆ Approximation (edge)

- Large n , small $p \Rightarrow \mathbf{Poisson}(\lambda=np)$
- Large n , moderate $p \Rightarrow \mathbf{Normal(np, npq)}$

Example 6: If $n=100, p=0.02 \rightarrow \lambda=2$
 $P(X=0) \approx e^{-2} = 0.135$.

◆ Application example

A bulb works with probability 0.9. In 5 bulbs, find:

- All work $\rightarrow (0.9)^5 = 0.59049$
 - Exactly 4 work $\rightarrow 5C4(0.9)^4(0.1)=0.32805$
 - At least 4 work $\rightarrow \text{sum} = 0.59049 + 0.32805 = \mathbf{0.91854}$
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GATE-Level Practice Examples

1. **(Counting)** 6-digit numbers from 1–9 that are divisible by 5 and have no repeated digits.
→ must end with 0 or 5, so handle both cases separately.
2. **(Permutation)** Arrange 8 people where 3 specific people never sit together.
→ Total – cases where they do sit together.
3. **(Combination)** Choose committee of 5 from 8 men and 6 women with ≥ 3 women.
 $= 3W+2M + 4W+1M + 5W = 6C3 \cdot 8C2 + 6C4 \cdot 8C1 + 6C5$.
4. **(Binomial)** A biased coin ($p=0.6$) tossed 10 times.
Find $P(\text{at least 8 heads})$. Sum $10C8(0.6)^8(0.4)^2 + 10C9(0.6)^9(0.4) + (0.6)^{10}$.

Quick Summary Sheet

Concept	Key Formula	Example Result
Rule of Sum	$m+n$	car/bus route
Rule of Product	$m \times n$	crust×topping
Permutation	$nPr = n!/(n-r)!$	$10P4=5040$
Repetition	n^r	26^5
Combination	$nCr = n!/[r!(n-r)!]$	$10C3=120$
With repetition	$n+r-1 \text{ C } r$	3 fruits repeat → 10
Circular	$(n-1)!$	$5 \rightarrow 24$
Binomial	$nCk p^k(1-p)^{n-k}$	coin toss
Expectation	np	$10, 0.3 \rightarrow 3$
Variance	npq	2.1

Would you like me to continue this series with **Part 2: Advanced GATE Examples + Problem-Solving Techniques** (covering tricky restrictions, inclusion–exclusion proofs, and PYQs from GATE 2022–2024)? I can explain every question step-by-step until mastery.

