CS754 : Advanced Image Processing

Estimation of sample covariance matrix from compressive measurements

Pulkit Adil - 200070062
Vaibhav Kumar -20D070087

May 3, 2022

1 Introduction

Consider a collection of data samples $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n} \in \mathbb{R}^p$ and let $\mathbf{X} = [\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}] \in \mathbb{R}^{p \times n}$. Then, the Sample Covariance Matrix $\mathbf{C}_n$ of the data set $\mathbf{X}$ is defined as $\mathbf{C}_n := \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{x_i} \mathbf{x_i}^T$. We have demonstrated the reconstruction of $\mathbf{C}_n$ from compressed measurements $\{\mathbf{R}_i^T \mathbf{x}_i\}_{i=1}^n \in \mathbf{R}^m$, $m < p$ based on the research paper we have implemented. Here, $\{\mathbf{R}_i\}_{i=1}^n \in \mathbb{R}^{p \times m}$ have their entries from the set $\{-1, 0, +1\}$ with respective probabilities $\{1/2s, 1 - 1/s, 1/2s\}$. $\{\mathbf{R}_i^T \mathbf{x}_i\}$ thus represent **sparse random projections** of the data onto $\mathbb{R}^p$.

We shall compare the results of reconstruction with the following two estimators:

- **Biased Estimator:** This is taken as a reference from some other paper

$$\widehat{\mathbf{C}}_n = \frac{1}{(m^2 + m)\mu_2^2} \cdot \frac{1}{n} \sum_{i=1}^{n} \mathbf{R_i} \mathbf{R_i}^T \mathbf{x_i} \mathbf{x_i}^T \mathbf{R_i} \mathbf{R_i}^T$$

- **Unbiased Estimator:** This has been proposed in the paper

$$\widehat{\mathbf{\Sigma}}_n = \widehat{\mathbf{C}}_n - \alpha_1 \text{ diag } (\widehat{\mathbf{C}}_n) - \alpha_2 \text{ tr}(\widehat{\mathbf{C}}_n) \ \mathbf{I}_{p \times p}$$

  where α_1 and α_2 have been defined in the paper such that $\mathbb{E}[\widehat{\mathbf{\Sigma}}_n] = \mathbf{C}_n$.

We shall use the following real world data sets for comparison:

- **MNIST Data set**: Images of handwritten letters, numbers. We have specifically used the data for digit zero here.
- **Gen4 Data set**: Collection of real world sparse matrices from linear programming problems.
- **Traffic Data set**: Frames from stationary video surveillance camera.

We shall use the values of the following metric as a **Validation Criterion:**

$$\text{Normalised Estimation Error} = \frac{\left\| \widehat{\mathbf{\Sigma}}_n - \mathbf{C}_n \right\|_2}{\left\| \mathbf{C}_n \right\|_2}$$

The lower the values of the above metric, the better is the reconstruction performed. We have plotted the Normalised Estimation Error vs γ where $\gamma := \frac{m}{s} < 1$ is the **compression factor**. We have performed the experiment with $m = 0.4 * p$.

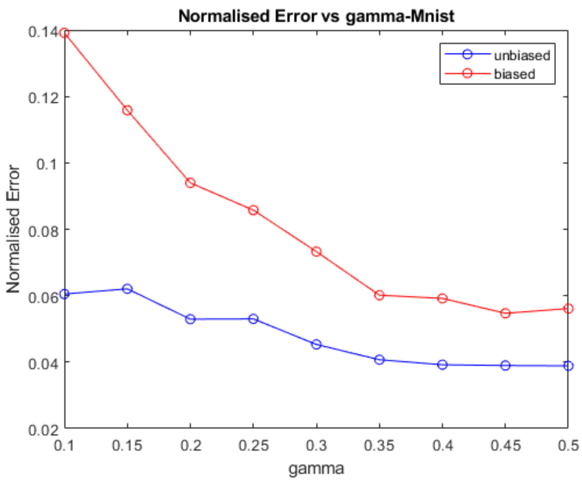
2 Experimental Results

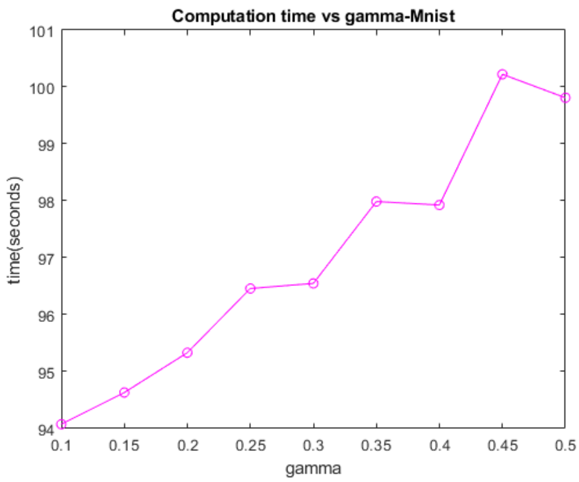
MNIST Dataset:

Some instances of MNIST Dataset are as follows:



Plots:





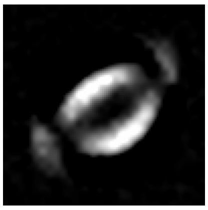
In this experiment , all images are 28×28 so p = 784 and there are 6903 such images so n = 6903.

Reconstruction using first Eigen Vector of C_n

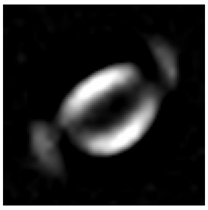
Here, we have calculated $\mathbf{C}_n$ with mean subtracted $\mathbf{X}$

$$\mathbf{C}_n := \frac{1}{n} \cdot \sum_{i=1}^{n} (\mathbf{x_i} - \mu)(\mathbf{x_i} - \mu)^T$$

where $\mu = \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{x}_i$

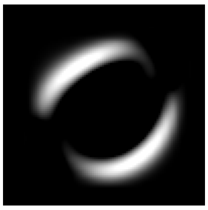


(a) $\gamma = 0.1$



(b) $\gamma = 0.3$

(a) $\gamma = 0.5$

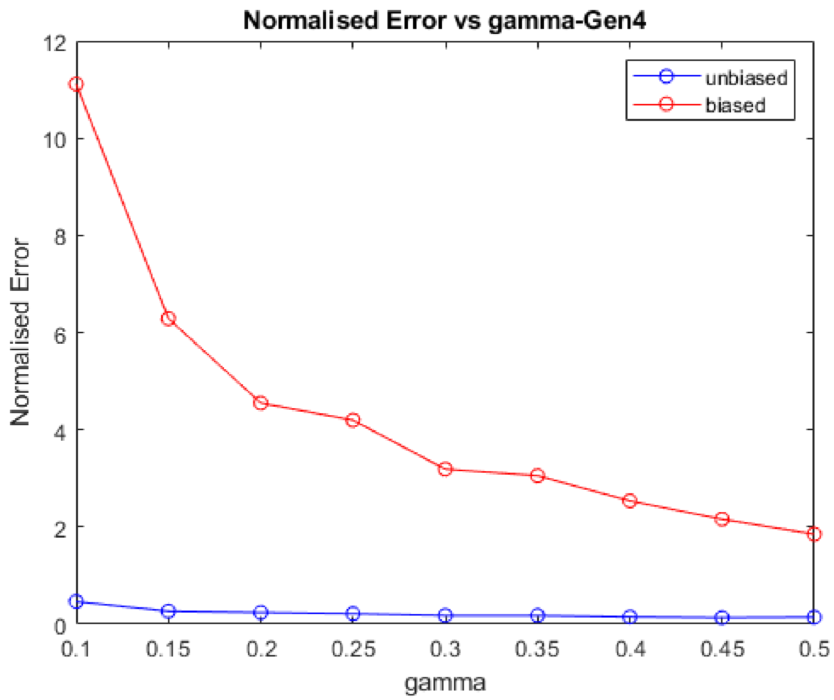


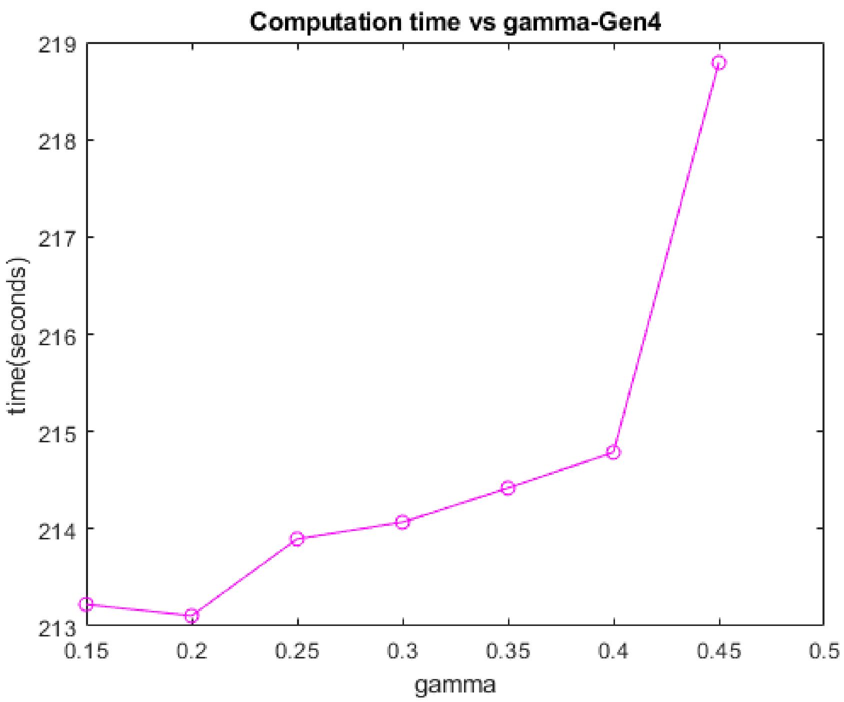
(b) full data

We can see that first eigen vector reveals the underlying structure and pattern in the dataset consisting of handwritten digits "0".

Gen4 Dataset:

Plots:



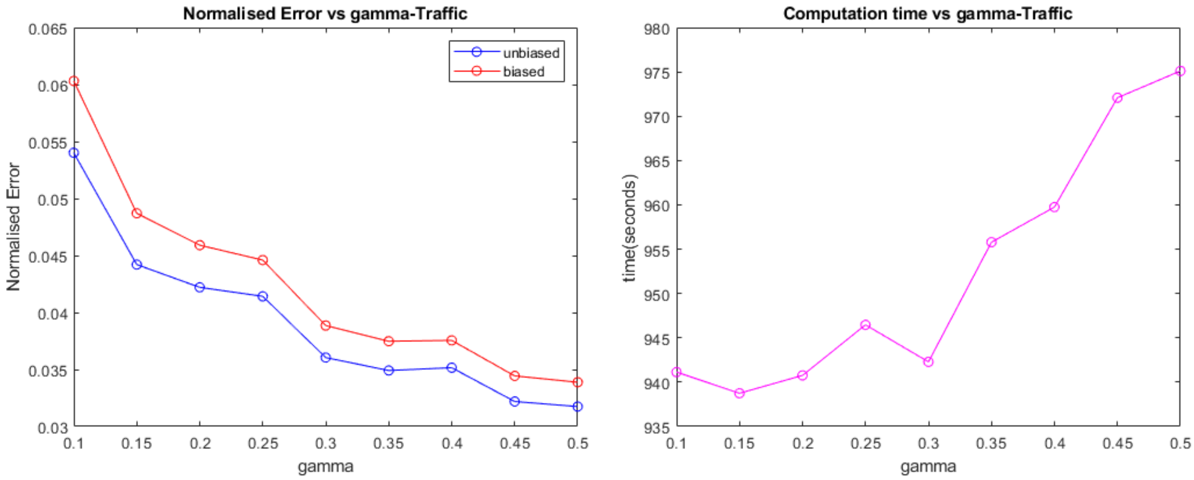


In this experiment , p =1537 and n= 4298

Traffic Data set:

Some instances of Traffic Dataset are as follows:

Plots:



In this experiment, all frames are 48 × 48 so p = 2304 and there are 5139 such images so n = 6903.

Reconstruction using first Eigen Vector of $\mathbf{C}_n$

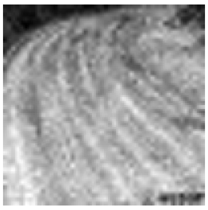
In this, we have calculated $\mathbf{C}_n$ with mean subtracted $\mathbf{X}$

$$\mathbf{C}_n := \frac{1}{n} \cdot \sum_{i=1}^{n} (\mathbf{x_i} - \mu)(\mathbf{x_i} - \mu)^T$$

where $\mu = \dfrac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{x}_i$

(a) $\gamma = 0.1$

(b) $\gamma = 0.3$



(a) $\gamma = 0.5$

(b) full data

We can see that first eigen vector reveals the underlying structure and pattern in the dataset consisting frames from stationary video surveillance camera.

A note on computation time of the two estimators

Computation time for the Unbiased estimator will be negligibly higher than Biased estimator because Unbiased is calculated from Biased by performing few additional multiplications and additions. The plot of computation time in all experiment for the biased and unbiased estimators were found to be similar and overlapping, as expected.

3 Observations and Conclusions

- Unbaised Covariance Estimator is better than Biased Covariance Estimator because it provides lower Loss for all γ in all three experiments.
- As γ increases, the reconstruction using first eigen vector becomes better because we have more information since sparcity is less.
- $\widehat{\mathbf{\Sigma}}_n$ leads to more accurate estimates of $\mathbf{C}_n$ when the eigenvalues of $\mathbf{C}_n$ decay faster.

Note that analysis holds for any random projection matrix consisting of i.i.d. zero-mean entries with finite first four moments and further that the proposed method does not require restrictive assumptions on the sample covariance matrix.

4 Acknowledgements

We would like to thank Prof. Ajit Rajwade to give us the opportunity to work on this interesting project.

The datasets with their respective links are listed below:

1. **MNIST dataset**: www.kaggle.com
2. **Gen4 dataset**: https://sparse.tamu.edu/
3. **Traffic dataset**: https://www.kaggle.com/datasets/aryashah2k/highway-traffic-videos-dataset

5 References

[1] F. Pourkamali-Anaraki, "Estimation of the sample covariance matrix from compressive measurements".

[2] F. Pourkamali-Anaraki and S. Hughes, "Memory and computation efficient PCA via very sparse random projections," in *Proceedings of the 31st International Conference on Machine Learning (ICML)*, pp. 1341–1349, 2014.