Properties of Laplase transformation:

1) Change scale property:

$$L[f(t)] = \phi(s)$$

$$L[F(at)] = \frac{1}{a}\,\phi\left(\frac{s}{a}\right)$$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[\sin 3t] = \cancel{\frac{1}{3\left(\left(\frac{s}{3}\right)^2+(3/3)^2\right)}} = \frac{3}{s^2+9}$$

IMP First shifting THM:

$$L[f(t)] = \phi(s)$$

$$L[e^{-at} f(t)] = \phi(s+a)$$

$$L[e^{at} f(t)] = \phi(s-a)$$

Q.1]

$$L[\sin t] = \frac{1}{s^2+1} = \phi(s)$$

$$L[e^{-3t} \sin 3t] = \phi(s+3)$$

$$= \frac{1}{(s+3)^2+1}$$

Q.2]

$$L[\cos 3t] = \frac{s}{s^2+9} = \phi(s)$$

$$L[e^{-2t} \cos 3t] = \phi(s+2)$$

$$= \frac{(s+2)}{(s+2)^2+9}$$

Q.2)

$$L[e^{2t} t^2] \Rightarrow$$

$$L[t^2] = \frac{2}{s^3} = \phi(s)$$

$$L[e^{2t} t^2] = \frac{2}{(s-2)^3} = \phi(s-2)$$

IMP

Q.1] $L[\cosh 2t \cos 2t]$

$\rightarrow$

$= L\left[\left(\frac{e^{2t}+e^{-2t}}{2}\right)\cos 2t\right]$

$= \frac{1}{2} L\left[e^{2t}\cos 2t + e^{-2t}\cos 2t\right]$

$f(t) = L[\cos 2t] = \frac{2}{s^2+4} = \phi(s)$

$\therefore L[e^{2t}\cos 2t] = \frac{2}{(s-2)^2+4} = \phi$

$\therefore L[e^{-2t}\cos 2t] = \frac{2}{(s+2)^2+4}$

$\therefore L[\cosh 2t \cos 2t] = \frac{1}{2}\left[\frac{2}{(s-2)^2+4} + \frac{2}{(s+2)^2+4}\right]$

Q.2] $L\left[\frac{\cos 2t \sin t}{e^t}\right] = L\left[e^{-t}(\cos 2t \sin t)\right]$

$L[\cos 2t \sin t] = \frac{1}{2} L[\sin 3t - \sin t]$

$= \frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$

$L[e^{-t}\cos 2t \sin t] = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$

Q.3] Find $L[e^{-3t}\cosh 5t \sin 4t]$

$\rightarrow$ $L[e^{-3t}\cosh 5t \sin 4t] = L\left[e^{-3t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\sin 4t\right]$

$= \frac{1}{2} L\left[(e^{5t}\sin 4t + e^{-5t}\sin 4t)e^{-3t}\right]$

$= \frac{1}{2} L\left[e^{2t}\sin 4t + e^{-8t}\sin 4t\right]$

$L[e^{-3t}\cosh 5t \sin 4t] = \frac{1}{2}\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+8)^2+16}\right]$

$\text{---} \left[\because L[\sin 4t] = \frac{4}{s^2+16}\right]$

Q.4] Find $L[\sin 2t \cos t \cosh 2t]$

$\rightarrow$ $= \frac{1}{4} L\left[(\sin 3t + \sin t)(e^{2t} + e^{-2t})\right]$

$= \frac{1}{4}\left[e^{2t}\sin 3t + e^{-2t}\sin 3t + e^{2t}\sin t + e^{-2t}\sin t\right]$

$= \frac{1}{4}\left[\frac{3}{(s-2)^2+9} + \frac{3}{(s+2)^2+9} + \frac{1}{(s-2)^2+1} + \frac{1}{(s+1)^2+1}\right]$

Q.6] Evaluate $\int_0^{\infty} e^{-t} \sin\frac{t}{2} \sinh\frac{\sqrt{3}t}{2}\, dt$

$\rightarrow$ $= \int_0^{\infty} e^{-t} \sin\frac{t}{2}\left(\frac{e^{\frac{\sqrt{3}}{2}t} - e^{-\frac{\sqrt{3}}{2}t}}{2}\right) dt$

$= \int_0^{\infty} \sin\frac{t}{2}\left(\frac{e^{\frac{\sqrt{3}}{2}t - t} - e^{-\frac{\sqrt{3}}{2}t - t}}{2}\right) dt$

$= \frac{1}{2}\int_0^{\infty} \sin\frac{t}{2}\left(e^{(\frac{\sqrt{3}}{2}-1)t} - e^{-(\frac{\sqrt{3}}{2}+1)t}\right) dt$

$\frac{1/_4}{s^2+1/4}$

$\frac{1}{2s^2+1/4}$

$\frac{4}{4s^2+1}$

$= \frac{1}{2}\int_0^{\infty}\left[e^{(\frac{\sqrt{3}}{2}-1)t}\sin\frac{t}{2} - e^{-(\frac{\sqrt{3}}{2}+1)t}\sin\frac{t}{2}\right] dt$

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$$L\left[\sin\frac{t}{2}\right] = \frac{1/2}{s^2+(1/4)} = \frac{2}{(4s^2+1)}$$

$$\text{ANS} = \frac{1}{2}\left[\frac{1/2}{(\sqrt{3}/2+1)^2+1/4} - \frac{1/2}{(-\sqrt{3}/2-1)^2+1/4}\right]$$

$$= \frac{1(1/2)}{2}\left[\frac{1}{\frac{(\sqrt{3}+2)^2}{4}+1/4} - \frac{1}{\frac{(-\sqrt{3}-2)^2}{4}+\frac{1}{4}}\right]$$

$\frac{2}{s^2}$

$\frac{-1+1}{s^0}$

Q. 6] $\int_0^\infty e^{-t}\left(t^2-3t+5+e^{2t}t^2\right)dt$ Property 2)

$\rightarrow L\left[t^2-3t+5+e^{2t}t^2\right] = \frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s} + \frac{2}{(s-2)^3}$

$\underline{\underline{s=1}}$

$(-1)^3$

$\therefore = \frac{2}{1} - \frac{3}{1} + \frac{5}{1} + \frac{2}{(-1)}$

$= \cancel{2} - 3 + 5 - \cancel{2}$

$= 2$

$\frac{1}{x}$

x^{-1}

$\sinh at = \frac{e^{at} - e^{-at}}{2}$ $\frac{1}{s-a} - \frac{1}{s+a}$ $\frac{s+a - s + a}{s^2 - a^2}$ $\frac{2a}{s^2 - a^2}$

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* $L[e^{at}] = \frac{1}{s-a}$ OR $L[e^{-at}] = \frac{1}{s+a}$
* $L[\sin at] = \frac{a}{s^2+a^2}$
* $L[\cos at] = \frac{s}{s^2+a^2}$
* $L[\sinh at] = \frac{a}{s^2-a^2}$ OR $L\left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{a}{s^2-a^2}$
* $L[\cosh at] = \frac{s}{s^2-a^2}$ $L\left[\frac{e^{at} + e^{-at}}{2}\right] = \frac{s}{s^2-a^2}$
* $L[t^n] = \frac{n!}{s^{n+1}}$ OR $= \frac{\Gamma(n+1)}{s^{n+1}}$
* $L[c] = \frac{c}{s}$ OR $L[ce^{0t}] = c\,L[e^{0t}] = \frac{c}{s}$

Property 2) Multiplication property:

$L[f(t)] = \phi(s)$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \phi(s)$

$\therefore L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} \phi(s)$

Ex. $L[te^{3t}] = (-1)^1 \frac{d}{ds} \phi(s) = -\frac{d}{ds}\left(\frac{1}{s-3}\right)$

$\therefore L[f(t)] = L[e^{3t}] = \frac{1}{s-3} = \phi(s)$

$L[te^{3t}] = -\left[\frac{-1}{(s-3)^2}\right]$

$L[te^{3t}] = \frac{1}{(s-3)^2}$

(2) $L[t^5 e^{3t}]$

$L[f(t)] = L[t^5] = \frac{5!}{s^6} = \phi(s)$

$\therefore L[e^{3t} f(t)] = L[e^{3t} t^5] = \phi(s-a) = \frac{5!}{(s-3)^6}$

Property 3] **Division 't' property:**

$L[f(t)] = \phi(s)$

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s)\, ds$

$L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_s^\infty \phi(s)\, ds\, ds$

Ex (1) $L\left[\frac{e^{5t} - e^{2t}}{t}\right]$

$L[f(t)] = L[e^{5t} - e^{2t}] = \frac{1}{s-5} - \frac{1}{s-2} = \phi(s)$

$\therefore L\left[\frac{e^{5t}}{t}\right] = \int_s^\infty \frac{1}{(s-5)}\, ds - \int_s^\infty \frac{1}{s-2}\, ds$

$= [\log(s-5)]_s^\infty - [\log(s-2)]_s^\infty$

$= -\log(s-5) + \log(s-2)$

$= \log\left(\frac{s-2}{s-5}\right)$

$\frac{1}{t^2+(2)^2}$

(2) $L\left[\frac{\sin 2t}{t}\right] = \frac{1}{2} \cancel{\tan^{-1}}\left(\frac{s}{2}\right)$

$L[f(t)] = L[\sin 2t] = \frac{2}{s^2+4}$

$L\left[\frac{f(t)}{t}\right] = L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \frac{2}{s^2+2^2}\, ds$

$= 2\left[\frac{1}{2} \tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

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$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2}\right)$

$= \left[\pi/2 - \tan^{-1}(s/2)\right]$

$= \cot^{-1}(s/2)$

(3) $L\left[\frac{\cos 5t}{t}\right]$

$L[F(t)] = L[\cos 5t] = \frac{s}{s^2+25}$

$\therefore L\left[\frac{\cos 5t}{t}\right] = \int_s^\infty \frac{s}{s^2+5^2}\, ds$

$= \cancel{\left[\frac{1}{5}\tan^{-1}\left(\frac{s}{5}\right)\right]_s^\infty}$

$= \cancel{\tan^{-1}(\infty) - \tan^{-1}(s/5)}$

$= \cancel{\pi/2}$

let $s^2+25 = t$ $\therefore s = \sqrt{t-25}$ when $s = s$, $t =$

$2s\, ds = dt$ $\therefore s\,ds = dt/2$

$\therefore L\left[\frac{\cos 5t}{t}\right] = \frac{1}{2}\int_{s^2+25}^\infty \frac{dt}{t}$

$= \frac{1}{2}\left[\log t\right]_{s^2+25}^\infty$

$= \frac{1}{2}\left[\log\infty - \log(s^2+25)\right]$

$=$

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Ex① Find laplace transformation:

$\frac{1}{t} e^{-t} \sin t$

→ $L\left[\frac{e^{-t}\sin t}{t}\right]$

$L[e^{-t}\sin t] \doteq L[\sin t] = \frac{1}{s^2+1} = \phi(s)$

$f(t)$

$\therefore L[e^{-t}\sin t] = \phi(s+a) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+1+1}$

$= \frac{1}{s^2+2s+2}$

$\therefore L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$

$= \left[\tan^{-1}\left(\frac{s+1}{1}\right)\right]_s^\infty$

$= \left[\pi/2 - \tan^{-1}(s+1)\right]$

$L\left[\frac{e^{-t}\sin t}{t}\right] = \cot^{-1}(s+1)$

③

→

② $L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right]$

→ here, $\cosh t = \frac{e^t + e^{-t}}{2}$

$\therefore L\left[\frac{e^{-(2t)}\sin 2t\,(e^t+e^{-t})}{2t}\right]$

$= \frac{1}{2}\left[\frac{e^{-2t}\, e^t \sin 2t}{t} + \frac{e^{-2t}\sin 2t\, e^{-t}}{t}\right]$

$= \frac{1}{2} L\left[\frac{e^{-t}\sin 2t}{t} + \frac{e^{-3t}\sin 2t}{t}\right]$

$= \frac{1}{2}\left[\int_s^\infty \frac{2}{(s+1)^2+4}\,ds + \int_s^\infty \frac{2}{(s+3)^2+4}\,ds\right]$

$= \frac{1}{2}\left\{\left[\tan^{-1}\left(\frac{s+1}{2}\right)\right]_s^\infty + \left[\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^\infty\right\}$

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$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s+1}{2}\right) + \frac{\pi}{2} - \tan^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$= \frac{1}{2}\left[\pi - \tan^{-1}\left(\frac{s+1}{2}\right) - \tan^{-1}\left(\frac{s+3}{2}\right)\right]$$

(3) $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

$\rightarrow$ $L[f(t)] = L[\sin t] = \frac{1}{s^2+1} = \phi(s)$

$L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1} = \phi(s+1)$

OR $L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{(s^2+1)}\,ds = \left[\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$

$$= \left[\frac{\pi}{2} - \tan^{-1}(s)\right]$$

$L\left[\frac{\sin t}{t}\right] = \cot^{-1}(s)$

$\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt \Rightarrow s = 1 \Rightarrow = \cot^{-1}(1)$

$= \frac{\pi}{4}$

$$\boxed{\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \frac{\pi}{4}}$$

Method 2: $L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}$

$L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$

$= \left[\tan^{-1}\left(\frac{s+1}{1}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$

$= \cot^{-1}(s+1)$

$\int_0^\infty e^{0t}\left(\frac{e^{-t}\sin t}{t}\right)dt = \cot^{-1}(0+1) = \cot^{-1}(1) = \frac{\pi}{4}$

$\because$ as $(s=0)$

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$\star\star\star$ 4 Evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt$

$\rightarrow$ $\therefore \Rightarrow \int_0^\infty \frac{\cos at}{t}\,dt - \int_0^\infty \frac{\cos bt}{t}\,dt$

$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$

$= -\frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$

$= -\frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

$= \frac{1}{2}\left[\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]$

To find $\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt$ put $s = 0$

$= \frac{1}{2}\left[\log\left(\frac{b^2}{a^2}\right)\right] = \log\left[\left(\frac{b}{a}\right)^2\right]^{1/2}$

$$\boxed{\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt = \log\frac{b}{a}}$$

$\frac{1-2}{1-3}$

$\frac{1}{2}$ 5 $\int_0^\infty e^{-t}\left(\frac{\cos 3t - \cos 2t}{t}\right) dt$

(6) Prove that $\int_0^{\infty} e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt = \frac{\pi}{8}$

$\rightarrow$ $\int_0^{\infty} e^{-\sqrt{2}t} \frac{\sin t}{t}\left(\frac{e^{t}-e^{-t}}{2}\right) = \int_0^{\infty} \frac{e^{-\sqrt{2}t}}{(2} \frac{e^{t}\sin t - e^{-t}e^{-\sqrt{2}t}\sin t}{t)}$

$= \int_0^{\infty} \frac{e^{-\sqrt{2}t}}{2}\left[\frac{e^{t}\sin t - e^{-t}\sin t}{t}\right] ds$

$L\left[e^{t}\sin t - e^{-t}\sin t\right] = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$

$\therefore L\left[\frac{e^{t}\sin t - e^{-t}\sin t}{t}\right] = \frac{1}{2}\left\{\int_s^{\infty}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]\right\} ds$

$= \frac{1}{2}\left\{\tan^{-1}\left(\frac{s-1}{1}\right) - \tan^{-1}\left(\frac{s+1}{1}\right)\right\}_s^{\infty}$

$= \frac{1}{2}\left\{\cancel{\tan^{-1}}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) - \left[\tan^{-1}(s-1) - \tan^{-1}(s+1)\right]\right\}$

$= \frac{1}{2}\left[0 - \tan^{-1}(s-1) + \tan^{-1}(s+1)\right]$

$= \frac{1}{2}\left[\tan^{-1}(\underset{a}{s+1}) - \tan^{-1}(\underset{b}{s-1})\right]$

$= \frac{1}{2}\left\{\tan^{-1}\left[\frac{s+1-s+1}{1+(s+1)(s-1)}\right]\right\}$

$= \frac{1}{2}\left\{\tan^{-1}\left(\frac{2}{1+(s^2-1)}\right)\right\}$

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$$L\left[\frac{e^{t}\sin t - e^{-t}\sin t}{t}\right] = \frac{1}{2}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2}{s^2}\right)$$

To find $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t \sinh t}{t}\,dt \Rightarrow \quad s = \sqrt{2}$

$$\therefore \quad = \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right)$$

$$= \tfrac{1}{2}\tan^{-1}(1)$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t \sinh t}{t}\,dt = \frac{\pi}{8}$$

Hence proved!

(7) P.T. $\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right) = \frac{3\pi}{4}$

$\rightarrow \quad \int_0^\infty e^{-t}\left(\frac{\sin 2t + \sin 3t}{t}\right)$

$$L[\sin 2t + \sin 3t] = L[\sin 2t] + L[\sin 3t]$$

$$= \frac{2}{s^2+4} + \frac{3}{s^2+9} \qquad \left[L[\sin at] = \frac{s}{s^2+a^2}\right]$$

$$L\left[\frac{\sin 2t + \sin 3t}{t}\right] = \int_s^\infty \left(\frac{2}{s^2+4} + \frac{3}{s^2+9}\right)ds$$

$$= \left[\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left[\left(\pi/2 + \pi/2\right) - \tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{3}\right)\right]$$

$$= \pi - \left[\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right]$$

$$= \pi - \left[\tan^{-1}\left(\frac{\frac{s}{2}+\frac{s}{3}}{1-\frac{s}{2}\cdot\frac{s}{3}}\right)\right]$$

$\frac{3s+2s}{6}$ $\frac{6-s^2}{6}$

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$$L\left[\frac{\sin 2t+\sin 3t}{t}\right] = \pi - \left[\tan^{-1}\frac{5s}{6\,\frac{(6-s^2)}{6}}\right]$$

$$= \pi - \left[\tan^{-1}\left(\frac{5s}{6-s^2}\right)\right]$$

$$\int_0^\infty e^{-t}\left[\frac{\sin 2t+\sin 3t}{t}\right] \Rightarrow \text{put } s = 1$$

$$= \pi - \tan^{-1}\left(\frac{5}{6-1}\right) = \pi - \tan^{-1}\left(\frac{5}{5}\right)$$

$$= \pi - \tan^{-1}(1) = \pi - \pi/4$$

$$\boxed{\int_0^\infty e^{-t}\left[\frac{\sin 2t+\sin 3t}{t}\right] = \frac{3\pi}{4}}$$

Property 4 Derivative property :

~~find~~ $L[f'(t)] = -f(0) + sL[f(t)]$

$f(t) = \sin 2t$

$L[f(t)] = \phi(s)$

$L[f'(t)] = -f(0) + sL[f(t)]$

$L[\sin 2t] = \frac{2}{s^2+4}$

& $L[\cos 2t] =$

* $L[f(t)] = \phi(s)$
* $L[f'(t)] = -f(0) + sL[f(t)]$
* $L[f''(t)] = -f'(0) - sf(0) + s^2 L[f(t)]$
* $L[f'''(t)] = -f''(0) - sf'(0) - s^2 f(0) + s^3 L[f(t)]$

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Q.1] $f(t) = \frac{\sin t}{t}$ find $L[f'(t)]$

$\rightarrow$

Formula : $L[f'(t)] = -f(0) + sL[f(t)]$ — (i)

$\frac{1}{0}$ $f(0) = 1$ — (ii) — $\left[t \to 0 \ \frac{\sin 0}{0} = 1 \text{ By L'HOSPITAL Rule} \right]$

$\frac{0}{0}$

$L[f(t)] = L\left[\frac{\sin t}{t}\right]$ -- $\left[L[\sin t] = \frac{1}{s^2+1} \right]$

$= \int_s^\infty \frac{1}{(s^2+1^2)} ds$ -- $\left[L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi s \, ds \right]$

$= \left[\tan^{-1}(s)\right]_s^\infty$

$= \pi/2 - \tan^{-1}(s)$

$L[f(t)] = \cot^{-1}(s)$ — (iii)

From (i), (ii) & (iii)

$\therefore$ $\boxed{L[f'(t)] = -1 + s\cot^{-1}(s)}$

Q.2] If $L[t\sin\omega t] = \frac{2\omega s}{s^2+\omega^2}$, find $L[\omega t\cos\omega t + \sin\omega t]$

$\rightarrow$ $L[\omega t\cos\omega t + \sin\omega t]$

$= \omega L[t\cos\omega t] + L[\sin\omega t]$

$= \omega\left[-\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)\right] + \frac{\omega}{s^2+\omega^2}$

$= -\omega\left[\frac{(s^2+\omega^2)(1) - s(2s)}{(s^2+\omega^2)^2}\right] + \frac{\omega}{s^2+\omega^2}$

$= \frac{2s^2\omega - \omega(s^2+\omega^2)}{(s^2+\omega^2)^2} + \frac{\omega(\omega^2+s^2)}{(s^2+\omega^2)^2} = \frac{2s^2\omega}{(s^2+\omega^2)^2}$

$f(t) = t\sin t$ $\therefore f(0) = 0$

$f'(t) = \omega t\cos\omega t + \sin\omega t$

$\therefore L[f'(t)] = L[\omega t\cos\omega t + \sin\omega t]$

$= -f(0) + sL[f(t)]$

$= -0 + s\left(\frac{2s\omega}{s^2+\omega^2}\right) = \frac{2s^2\omega}{s^2+\omega^2}$

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Property 5:

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\,\phi(s)$$

$$L\left[\int_0^t \int_0^t f(t)\,dt\right] = \frac{1}{s^2}\,\phi(s)$$

e.g. $L\left[\int_0^t \sin u\, du\right]$

$$L[\sin u] = 1/s^2+1$$

$$L\left[\int_0^t \sin u\, du\right] = \frac{1}{s}\left(\frac{1}{s^2+1}\right)$$

Q] $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u}\, du$

$\rightarrow$ $L\left[\int_0^t \frac{\sin u}{u}\right] du \Rightarrow L\left[\sin u\right] = \frac{1}{s^2+1}$

$\cancel{L\left[\int_0^t \frac{\sin u}{du}\right]}$ $L\left[\frac{\sin u}{u}\right] = \int_s^\infty \frac{1}{s^2+1}\, ds = \left[\tan^{-1}(s)\right]_s^\infty$

$$L\left[\frac{\sin u}{u}\right] = \cot^{-1}(s)$$

$$L\left[\int_0^t \frac{\sin u}{u}\, du\right] = \frac{1}{s}\cot^{-1}(s)$$

$\because \int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u}\, du$, $s = 1$

$\therefore$ $\Rightarrow \frac{1}{(1)} \cot^{-1}(1) = \pi/4$

$$\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u}\, du = \frac{\pi}{4}$$

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Q1] $\int_{t=0}^{\infty} e^{-t} \left(\int_{u=0}^{t} u^2 \sinh u \cosh u \, du \right) dt$

$\rightarrow$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sinh u \cosh u = \frac{1}{2} \sin 2hu$

$\therefore L[\sinh u \cosh u] = \frac{1}{2} L[\sin 2hu]$

$= \frac{1}{2}\left[\frac{2}{(s^2-4)}\right]$

$= \frac{1}{s^2-4}$

$L[u^2 \sinh u \cosh u] = \frac{d^2}{ds^2}\left[\frac{1}{s^2-4}\right]$

$= \frac{d}{ds}\left[-\frac{1(2s)}{(s^2-4)^2}\right]$

$= -2\frac{d}{ds}\left[\frac{s}{(s^2-4)^2}\right]$

$= -2\left[\frac{(s^2-4)^2(1) - s(2)(s^2-4)(2s)}{(s^2-4)^4}\right]$

$= -2\left[\frac{(s^2-4)^2 - 4s^2(s^2-4)}{(s^2-4)^4}\right]$

$-3s^2$ $= \frac{(s^2-4-4s^2)(-2)}{(s^2-4)^3}$

s^2-4s^2 $-3s^2$ $6s^2+8$ $= \frac{8+6s^2}{(s^2-4)^3}$

$L\left[\int_0^t u^2 \sinh u \cosh u \, du\right] = \frac{1}{s}\left(\frac{8+6s^2}{(s^2-4)^3}\right)$

Now, In $\int_0^{\infty} e^{-t}\left(\int_{u=0}^{t} u^2 \sinh u \cosh u \, du\right) dt$,

Put $s=1$.

$(-3)^3$

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$$\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u \, du\right)dt = \frac{1}{(1)}\left[\frac{6+8}{(1-4)^3}\right] = \frac{14}{-27}$$

Q2) $$\int_0^\infty e^{-4t}\left(\cosh t \int_0^t e^u \cosh u \, du\right)dt$$

$\longrightarrow$

$$L[\cosh u] \Rightarrow \frac{s}{s^2 - a^2} = \frac{1}{s^2 - 1}$$

$$L[e^u \cosh u] = \frac{1}{(s-1)^2 - 1}$$

$$L\left[\int_0^t e^u \cosh u \, du\right] = \frac{1}{s}\left(\frac{1}{(s-1)^2 - 1}\right)$$

$$L\left[\cosh t \int_0^t e^u \cosh u \, du\right] = L\left[\left(\frac{e^t + e^{-t}}{2}\right)\right]$$

$$\therefore \Rightarrow \int_0^\infty \frac{e^{-4t} e^t}{2}\left(\frac{1}{s}\frac{1}{(s-1)^2 - 1}\right) + \int_0^\infty \frac{e^{-4t} e^{-t}}{2}\left(\frac{1}{s}\frac{1}{(s-1)^2 - 1}\right)$$

$$\Rightarrow \int_0^\infty \frac{e^{-3t}}{2}\left[\frac{1}{s(s-1)^2 - s}\right] + \int_0^\infty \frac{e^{-5t}}{2}\left[\frac{1}{s(s-1)^2 - s}\right]$$

$$\Rightarrow \frac{1}{2[3(2)^2 - 3]} + \frac{1}{2[5(4)^2 - 5]}$$

$$\Rightarrow \frac{1}{2(12)} + \frac{1}{2(75)}$$

$15 - 3$

$$\Rightarrow \frac{1}{2}\left(\frac{75 + 12}{12 \times 75}\right)$$

$$\Rightarrow \frac{87}{2(900)}$$

INVERSE LAPLASE TRASFORMATION

\# $L[f(t)] = \phi(s)$

$L^{-1}[\phi(s)] = f(t)$

\# STANDARD FORMULA'S :

$L[c] = \frac{c}{s} \Rightarrow L^{-1}\left[\frac{c}{s}\right] = c$

$L[e^{at}] = \frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$L[e^{-at}] = \frac{1}{s+a} \Rightarrow L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$L[\sin at] = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$L[\cos at] = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$L[\sinh at] = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

$\frac{3}{2}-1$

$L[\cosh at] = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$\frac{1}{2}$

$L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

$\frac{1}{2}$

OR $= \frac{t^{n-1}}{\Gamma n}$

\# QUESTIONS:

1) $L^{-1}\left[\frac{1}{s^5}\right] = \frac{t^4}{4!}$

$\frac{s}{s^2-36}$

$\frac{1}{s-4}$

(2) $L^{-1}\left[\frac{1}{s^{1/2}}\right] = \frac{t^{-1/2}}{\Gamma 1/2}$

$\frac{2s}{s^2+4}$

(3) $L\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin 3t$

$2\left[\frac{s}{s^2+(2)^2}\right]$

$L\left[\frac{2s+3}{s^2+9}\right]$

$2\cos 3t + \sin 3t$

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Partial fraction

① $\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$

② $\frac{1}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$

③ $\frac{1}{(x+a)^2(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}$

Ex ① $\frac{1}{(S+2)(S+3)} = \frac{A}{(S+2)} + \frac{B}{(S+3)}$

② $\frac{1}{S^2(S+2)(S+3)} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{(S+2)} + \frac{D}{(S+3)}$

③ $\frac{1}{(S^2+2)(S)(S+3)} = \frac{AS+B}{(S^2+2)} + \frac{C}{S} + \frac{D}{(S+3)}$

④ $\frac{1}{(S^2+4S+4)(S+2)} = \frac{1}{(S+2)^2(S+3)} = \frac{A}{(S+2)} + \frac{B}{(S+2)^2} + \frac{C}{(S+}$

Q. 1] find $L^{-1}\left[\frac{2}{S} + \frac{1}{S^3} + \frac{1}{S+4}\right]$

→ $= 2(1) + \frac{t^2}{2!} + e^{-4t}$

Q. 2] $L^{-1}\left[\frac{S+3}{S^2+4}\right]$

→ $L^{-1}\left[\frac{S}{S^2+4}\right] + L^{-1}\left[\frac{3}{S^2+4}\right]$

$= \cos 2t + \frac{3}{2}\sin 2t$

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Q.3] $L^{-1}\left[\frac{s}{(s-2)^6}\right]$

$\rightarrow$ $= L^{-1}\left[\frac{s-2+2}{(s-2)^6}\right]$

$= L^{-1}\left[\frac{(s-2)}{(s-2)^6} + \frac{2}{(s-2)^6}\right]$

FST inverse $L^{-1}[\phi(s-a)] = e^{at} L^{-1}[\phi(s)]$

$L^{-1}[\phi(s+a)] = e^{-at} L^{-1}[\phi(s)]$

$\therefore L^{-1}\left[\frac{s}{(s-2)^6}\right] = e^{2t} L^{-1}\left[\frac{s}{s^6}\right] + 2L^{-1}\left[\frac{1}{(s-2)^6}\right]$

$= e^{2t} L^{-1}\left[\frac{1}{s^5}\right] + 2e^{2t} L^{-1}\left[\frac{1}{s^6}\right]$

$= e^{2t}\left[\frac{t^4}{4!} + 2\frac{t^5}{5!}\right]$

Q.4] $L^{-1}\left[\frac{1}{(s-2)^{3/2}}\right]$

$\rightarrow$ $= e^{2t} L^{-1}\left[\frac{1}{s^{3/2}}\right] = e^{2t}\left[t^{1/2}\right.$

Q.5] $L^{-1}\left[\frac{1}{(s+3)^2}\right]$

$\rightarrow$

Q.6] $L^{-1}\left[\frac{s-2}{(s-2)^2+2^2}\right]$

$\rightarrow$ L^{-1}

$e^{2t} L^{-1}\left[\frac{1}{s^2+2^2}\right] = \frac{e^{2t}}{2}\sin 2t$

Q.7
$\rightarrow$
Q.8]
Q.9
Q.10
=

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$$L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$$

$$= e^{3t}\, L^{-1}\left[\frac{s}{(s-3)^2+4} - \frac{3}{(s-3)^2+4}\right]$$

$$= e^{3t}\cos 2t - 3e^{3t}$$

Q.7] $L^{-1}\left[\frac{1}{(s-2)^2-5^2}\right]$

$\rightarrow$

$$= e^{2t} L^{-1}\left[\frac{1}{s^2-5^2}\right]$$

$$= e^{2t}\frac{1}{5}\sinh 5t$$

Q.8] $L^{-1}\left[\frac{s-4}{(s-4)^2-5^2}\right]$

e^{2t}

Q.9] $L^{-1}\left[\frac{s}{(s+4)^2+(2)^2}\right]$

$e^{-4t}\cos 2t$

Q.10] $L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right]$

$$= L^{-1}\left[\frac{1}{\sqrt{2}\sqrt{s+\frac{1}{2}}}\right] = \frac{1}{\sqrt{2}} L^{-1}\left[\frac{1}{(s+1/2)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{t}{2}} L^{-1}\left[\frac{1}{s^{1/2}}\right] \quad \text{— [FST inverse]}$$

$$= \frac{1}{\sqrt{2}} e^{-t/2} \frac{t^{-1/2}}{\Gamma 1/2}$$

$$L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right] = \frac{1}{\sqrt{2\pi}} e^{-t/2} e^{-1/2}$$

Q] $L^{-1}\left[\dfrac{s+2}{s^2-4s+13}\right]$

→

$= L^{-1}\left[\dfrac{s+2}{(s-2)^2+9}\right]$

$= L^{-1}\left[\dfrac{(s-2)+4}{(s-2)^2+9}\right]$

$= L^{-1}\left[\dfrac{(s-2)}{(s-2)^2+9}\right] + 4L^{-1}\left[\dfrac{1}{(s-2)^2+9}\right]$

$=$

Q] $L^{-1}\left[\dfrac{4s+12}{s^2+8s+12}\right]$

→ $= L^{-1}\left[\dfrac{4s+12}{s^2+8s+16-4}\right]$

$(s+4)^2$
$s^2+8s+16$

$= L^{-1}\left[\dfrac{4s+12}{(s+4)^2-4}\right]$

$= 4L^{-1}\left[\dfrac{s}{(s+4)^2-4}\right] + 12L^{-1}\left[\dfrac{1}{(s+4)^2-4}\right]$

$= 4L^{-1}\left[\dfrac{s}{(s+4)^2-(2)^2}\right]$

$= 4e^{-4t}L^{-1}\left[\dfrac{s}{s^2-2^2}\right] + 12e^{-4t}L^{-1}\left[\dfrac{1}{s^2-2^2}\right]$

$= 4e^{-4t}\cosh 2t + \overset{3}{\cancel{12}}e^{-4t}\dfrac{1}{2}\sinh 2t$

$= 4e^{-4t}\cosh 2t -$

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Q.] $L^{-1}\left[\dfrac{4s+12}{s^2+8s+12}\right]$

$\rightarrow$ $L^{-1}\left[\dfrac{4(s+4)-4}{(s+4)^2-4}\right]$

$= 4L^{-1}\left[\quad (s+4)-1\right.$

Q] $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]_{+4-4} = L^{-1}\left[\dfrac{3s+7}{s^2-2s-3+4-4}\right]$

$\rightarrow$ $= L^{-1}\left[\dfrac{3s+7}{(s-1)^2-4}\right]$

$(s^2-1)^2$ $= L^{-1}\left[\dfrac{3(s-1)+10}{(s-1)^2-(2)^2}\right]$

$= 3L^{-1}\left[\dfrac{(s-1)}{(s-1)^2-(2)^2}\right] + 10L^{-1}\left[\dfrac{1}{(s-1)^2-(2)^2}\right]$

$= 3e^t L^{-1}\left[\dfrac{s}{s^2-4}\right] + 10e^t L^{-1}\left[\dfrac{1}{s^2-2^2}\right]$

$= 3e^t \cosh 2t + \dfrac{10e^t}{2}\sinh 2t$

$= e^t\left[3\cosh 2t + 5\sinh 2t\right]$

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Q]
$$L^{-1}\left[\frac{s+2}{s^2-\cancel{2s}+17}\right]$$

$\rightarrow$
$$= L^{-1}\left[\frac{s+2}{(s-1)^2-18}\right]$$
$$= L^{-1}\left[\frac{(s-1)+3}{(s-1)^2-18}\right]$$
$$= e^{t}L^{-1}\left[\frac{s+3}{s^2-18}\right]$$
$$= e^{t}\left\{L^{-1}\left[\frac{s}{s^2-18}\right] + L^{-1}\left[\frac{3}{s^2-18}\right]\right.$$

Q]
$$\frac{s+29}{(s+4)(s^2+29)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+29}$$
$$\frac{(s+29)}{(s+4)(s^2+29)} = \frac{A(s^2+29)+(Bs+C)(s+4)}{(s+4)(s^2+29)}$$

$$0s^2+s+29 = As^2+A29+Bs^2+4Bs+Cs+4C$$
$$0s^2+s+29 = (A+B)s^2+(4B+C)s+(A29+4C)$$

$A+B=0$ — (i)

$0A+4B+C=1$ — (ii)

$29A+0B+4C=29$ — (iii)

#

Q.]

$\rightarrow$

Convolution Theorem:

$L^{-1}[\phi_1(s)\ \phi_2(s)]$

$L^{-1}[\phi_1(s)] = f_1(u)$ (say)

$L^{-1}[\phi_2(s)] = \cancel{f_2(t-u)}\ f_2(t-u)$ (say)

$$\boxed{L^{-1}[\phi_1(s)\,\phi_2(s)] = \int_0^t f_1(u)\,f_2(t-u)\,du}$$

Convolution theorem

Ex. $L^{-1}\left[\dfrac{s}{(s-1)(s^2-1)}\right]$

$\rightarrow$ $\phi_1(s) = \dfrac{1}{s-1} \Rightarrow L^{-1}[\phi_1(s)] = e^u = f_1(u)$

$\phi_2(s) = \dfrac{s}{(s^2-1)} \Rightarrow L^{-1}[\phi_2(s)] = \cos(t-u) = f_2(t-u)$

By Theorem, $\displaystyle\int_0^t e^u . \cos(t-u)\,du$

$= \left[e^u.\cos(t-u) - \int e^u(-\sin(t-u))\,du\right]_0^t$

$\displaystyle\int_0^t e^u \cos(t-u)\,du = \left[e^u \cos(t-u) + e^u \sin(t-u) - \int e^u \cos(t-u)\,du\right]_0^t$

$\displaystyle\int_0^t e^u \cos(t-u)\,du = \left[\dfrac{e^u\left[\cos(t-u) + \sin(t-u)\right]}{2}\right]_0^t$

Q.1] Find $L^{-1}\left[\dfrac{1}{s(s+1)}\right]$ by convolution theorem.

$\rightarrow$

$L^{-1}[\phi_1(s)\,\phi_2(s)]$; $\phi_1(s) = 1/s$; $\phi_2(s) = 1/s+1$

$L^{-1}[\phi_1(s)] = L^{-1}\left[\dfrac{1}{s}\right] \Rightarrow 1 = f_1(u) = 1$

$L^{-1}[\phi_2(s)] = L^{-1}\left[\dfrac{1}{s+1}\right] = e^{-t} \Rightarrow f_2(t) = e^{-u}$

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$$L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\,du$$

$$= \int_0^t e^{-u}(1)\,du$$

$$= \left[\frac{e^{-u}}{-1}\right]_0^t = -e^{-t} + e^0$$

$$= -e^{-t} + 1$$

$$\boxed{L^{-1}\left[\frac{1}{s(s+1)}\right] = 1 - e^{-t}}$$

Corrolary : $\left(\frac{1}{s}\right) \Rightarrow \int_0^t L^{-1}[\phi_1(s)]\,du$

$L[1/s] = 1$

Q. 2] Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$

$\rightarrow$ $L^{-1}[\phi_1(s)\,\phi_2(s)]$; $\phi_1(s) = \frac{1}{s}$; $\phi_2(s) = \frac{1}{s^2-a^2}$

$L^{-1}[\phi_1(s)] = 1$ $\qquad$ $L^{-1}[\phi_2(s)] = \frac{1}{a}\sinh at$

$f_1(u) = 1$ $\qquad$ $f_2(t-u) = \frac{1}{a}\sinh at$

$$L^{-1}[\phi(s)] = \int_0^t f_1(u)\, f_2(t-u)\,du$$

$$= \int_0^t (1)\cdot\frac{1}{a}\sinh a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \sinh a(t-u)\,du$$

$$= \frac{1}{a}\left[\frac{\cosh a(t-u)}{(-a)}\right]_0^t$$

$$= \frac{1}{a^2}\left[-\cosh a(0) + \cosh at\right]$$

$$L^{-1}[\phi(s)] = \frac{1}{a^2}\left[\cosh at - 1\right]$$

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(s^2-4)

Q.3] find $L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right]$

$\rightarrow$ $\phi_1(s) = \frac{1}{s-2}$ $L^{-1}\left[\frac{1}{(s-2)}\right] = e^{2t}$; $f_1(u) = e^{2u}$

$\phi_2(s) = \frac{1}{(s+2)^2} \Rightarrow L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}\, t$

$f_2(t-u) = e^{-2(t-u)}(t-u)$

$L^{-1}[\phi(s)] = \int_0^t f_1(u) f_2(t-u)\, du$

$= \int_0^t e^{2u} e^{-2(t-u)} (t-u)\, du$

$(s-2)$
$(s+2)$
s^2-2

$= e^{-2t} \int_0^t e^{2u} . e^{2u} (t-u)\, du$

$= e^{-2t} \int_0^t e^{4u} (t-u)\, du$

$\frac{1}{(s-2)(s+2)}$

$= e^{-2t} \left\{ \int_0^t e^{4u} t\, du - \int_0^t e^{4u} u\, du \right\}$

$= e^{-2t} \left\{ t \left(\frac{e^{4u}}{4}\right)_0^t - \left[u \left(\frac{e^{4u}}{4}\right) - \frac{e^{4u}}{4^2} \right]_0^t \right\}$

$= e^{-2t} \left\{ t \left(\frac{e^{4t}}{4} - \frac{1}{4} \right) - \left[\frac{t\, e^{4t}}{4} - \frac{e^{4t}}{4^2} + \frac{1}{16} \right] \right.$

$= e^{-2t} \left[\frac{(e^{4t}-1)t}{4} - \frac{t e^{4t}}{4} + \frac{e^{4t}}{16} - \frac{1}{16} \right]$

$= e^{-2t} \left[\frac{-t}{4} + \frac{(e^{4t}-1)}{16} \right]$

$= \frac{-e^{-2t}t}{4} + \frac{(e^{-2t}e^{4t} - e^{-2t})}{16}$

$= \frac{-e^{-2t}t}{4} + \frac{e^{2t} - e^{-2t}}{16}$

$= e^{-2t} \left[\frac{e^{4t} - 4t - 1}{16} \right]$

Q.4] $L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right]$

$\rightarrow$ $\phi_1(s) = \frac{s}{s^2+2^2} \Rightarrow L^{-1}[\phi_1(s)] = \cos 2t = f_1(u)$
$= \cos 2u$

$\phi_2(s) = \frac{s}{s^2+2^2} \Rightarrow L^{-1}[\phi_2(s)] = \cos 2t = f_2(t-u)$
$= \cos 2(t-u)$

By convolution theorem.

$$L^{-1}[\phi(s)] = \int_0^t \cos 2u \cdot \cos 2(t-u)\, du$$

$$= \int_0^t \frac{1}{2}\left[\cos(2u+2(t-u)) + \cos(2u-2t+2u)\right] du$$

$$= \frac{1}{2}\int_0^t \left(\cos 2t + \cos(4u-2t)\right) du$$

$$= \frac{1}{2}\left\{\cos 2t \cdot u + \frac{\sin(4u-2t)}{4}\right\}_0^t$$

$$= \frac{1}{2}\left[\cos 2t \cdot t + \frac{\sin(4t-2t)}{4} - \cos 2t(0) - \frac{\sin(-2t)}{4}\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4}\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right] = \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{2}\right]$$

Q.5] $L^{-1}\left[\frac{1}{(s^2+2^2)^2}\right]$

$\rightarrow$ $\phi_1(s) = \frac{1}{s^2+2^2}$ $L^{-1}[\phi_1(s)] = \frac{1}{2}\sin 2t = \frac{1}{2}\sin 2u$

$\phi_2(s) = \frac{1}{s^2+s^2}$ $L^{-1}[\phi_2(s)] = \frac{1}{2}\sin 2t = \frac{1}{2}\sin 2(t-u)$

$\therefore$ By convolution,

$$L^{-1}[\phi(s)] = \int_0^t \frac{1}{4}\sin 2u \sin 2(t-u)\, du$$

Q.6]

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$$= \frac{1}{4} \int_0^t \cdot \frac{1}{2}\left[\cos(2u-2t+2u) - \cos(2u+2t-2u)\right]$$

$$= -\frac{1}{8} \int_0^t \left[\cos(4u-2t) - \cos(2t)\right] du$$

$$= -\frac{1}{8}\left[\frac{\sin(4u-2t)}{4} - u\cos 2t\right]_0^t$$

$$= -\frac{1}{8}\left[\frac{\sin(2t)}{4} - t\cos 2t + \frac{\sin(2t)}{4}\right]$$

$$= -\frac{1}{8}\left[\frac{1}{2}\sin 2t - t\cos 2t\right]$$

$$L^{-1}\left[\frac{1}{(s^2+2^2)^2}\right] = \frac{1}{8}\left[t\cos 2t - \frac{\sin 2t}{2}\right]$$

Q.6] $L^{-1}\left[\dfrac{(s+2)^2}{(s^2+4s+8)^2}\right]$

$\rightarrow$ By first shifting theorem, of inverse laplace transformation

$L^{-1}[\phi(s+a)] = e^{at} L^{-1}[\phi(s)]$

$$L^{-1}\left[\frac{(s+2)^2}{((s+2)^2+4)^2}\right] = L^{-1}[\phi(s+2)]$$

2S

$$\therefore \Rightarrow L^{-1}[\phi(s)] = e^{-2t} L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right]$$

$$\therefore \phi_1(s) = \frac{s}{s^2+2^2} = \cos 2t = \cos 2u$$

$$\phi_2(s) = \frac{s}{(s^2+2^2)} = \cos 2t = \cos 2(t-u)$$

$$\therefore L^{-1}[\phi(s)] = \int_0^t e^{-2t}\cos 2(t-u)\cos 2u \, du$$

$$= e^{-2t} \int_0^t \cos 2(t-u)\cos 2u \, du$$

$$= \frac{e^{-2t}}{2} \int_0^t \left[\cos 2t + \cos(4u-2t)\right] du$$

$$= \frac{e^{-2t}}{2}\left[u\cos 2t + \frac{\sin(4u-2t)}{4}\right]_0^t$$

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$$= \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{\sin(4t-2t) - \cos 2t\,(0) + \sin ...}{4}\right]$$

$$L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{\sin 2t}{2}\right]$$

Q.8]

→

Q.7] $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$

→ $= L^{-1}\left[\frac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}\right]$

$= e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+1)(s^2+4)}\right]$ --- [By first shifting theorem]

$= e^{-t}\left\{L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] + 2L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right]\right\}$

$= e^{-t}L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] + 2e^{-t}L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right]$

$L^{-1}[\phi(s)]_1$ $\qquad$ $L^{-1}[\phi(s)]_2$

$L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s}{s^2+1^2}\right] = \cos t = \cos u$

$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t = \cos 2(t-u)$

$L^{-1}[\phi_3(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = \sin u$

$L^{-1}[\phi_4(s)] = L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t = \frac{1}{2}\sin 2(t-u)$

$\therefore L^{-1}[\phi(s)] = \int_0^t e^{-t}\left\{\cos u\cos 2(t-u) + \frac{1}{2}\sin u\sin 2(t-u)\right\}du$

$= e^{-t}\int_0^t \cos u\cos 2(t-u)\,du + \frac{e^{-t}}{2}\int_0^t \frac{\sin u}{\sin 2t + u}\,du$

$= \frac{e^{-t}}{2}\left[t\cos 2t + \frac{\sin 2t}{2}\right] + \frac{e^{-t}}{16}\left[t\cos t - \frac{\sin 2t}{2}\right]$

$\frac{1}{s^4(s+3)}$

$s^5 + 3s^4$

Q.8] $L^{-1}\left[\frac{1}{(s-2)^4 (s+3)}\right]$

$\rightarrow$ $\phi_1(s) = \frac{1}{(s-2)^4}$ $\phi_2(s) = \frac{1}{s+3}$

$L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t} L^{-1}\left[\frac{1}{s^4}\right] = e^{2t}\frac{t^3}{3!} = \frac{e^{2u}u^3}{3!}$

--[By FST]

$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t} = e^{-3(t-u)}$

$L^{-1}[\phi(s)] = \int_0^t \frac{e^{2u}u^3}{3!} \cdot e^{-3(t-u)}\, du$

$3\times2\times1$

$= \frac{1}{3!}\int_0^t e^{2u}u^3 \cdot e^{-3t} \cdot e^{3u}\, du$

$= \frac{e^{-3t}}{6}\int_0^t e^{5u} \cdot u^3\, du$

$= \frac{e^{-3t}}{6}\left[\frac{u^3 e^{5u}}{5} - \int \frac{e^{5u}3u^2}{5}\right]_0^t du$

$= \frac{e^{-3t}}{6}\left[\frac{u^3e^{5u}}{5} - \frac{e^{5u}3u^2}{5^2} + \frac{6ue^{5u}}{5^3} - \frac{6e^{5u}}{5^4}\right]_0^t$

$= \frac{e^{-3t}}{6}\left[\frac{t^3e^{5t}}{5} - \frac{e^{5t}3t^2}{5^2} + \frac{6te^{5t}}{5^3} - \frac{6e^{5t}}{5^4} + \frac{6}{5^4}\right]$

$= \frac{e^{-3t}}{6}\Big[$

H.W $L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right]$

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Multiplication theorem of laplace theorem :

e.g. $L^{-1}\left[\log\left(\frac{s+1}{s+2}\right)\right]$

$\rightarrow$ $\phi(s) = \log\left(\frac{s+1}{s+2}\right) = \log(s+1) - \log(s+2)$ — (i)

We know that,

$$L^{-1}[\phi(s)] = -\frac{1}{t} L^{-1}[\phi'(s)]$$

$\therefore$ Differentiating eqn. (i),

$$\phi'(s) = \frac{1}{s+1} - \frac{1}{(s+2)}$$

$$L^{-1}[\phi'(s)] = e^{-t} - e^{-2t}$$

$$\therefore L^{-1}[\phi(s)] = \frac{-1}{t}[e^{-t} - e^{-2t}]$$

$\frac{1+1}{2}$

Q. 2] $L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right] = L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2}\right)\right]$

$\rightarrow$ $\therefore L^{-1}[\log(s^2+a^2) - \log s^2]$

$$L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2}\right]$$

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Q.3] $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$

$\rightarrow$ $L^{-1}\left[\log(s^2+a^2) - \log\sqrt{s+b}\right]$

$L^{-1}\left[\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right]$

$L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$

$\therefore$ $= \frac{-1}{t} L^{-1}\left[\frac{2s}{(s^2+a^2)} - \frac{1}{2}\frac{1}{(s+b)}\right]$

$= \frac{-1}{t}\left\{2L^{-1}\left[\frac{s}{(s^2+a^2)}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+b}\right]\right\}$

$= \frac{-1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right]$

Q.4] $L^{-1}\left[\frac{s^2+a^2}{s^2+b^2}\right]$

$\rightarrow$ $= L^{-1}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{}\right.$

Q5] $L^{-1}\left[2\tan^{-1}s\right]$ $\qquad \tan^{-1}s = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$

→ $= L^{-1}\left[2\cdot\left(\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right)\right]$

$= L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right]$

$= L^{-1}\left[\log(1+s) - \log(1-s)\right]$

$= L^{-1}\left[\log(s+1) - \log(1-s)\right]$

$= \frac{-1}{t}\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{-1}{1-s}\right)\right]$

$= \frac{-1}{t}\left[e^{-t} - e^{t}\right]$

$= \frac{-1}{t}\,2\sinh t = \frac{-2\sinh t}{t}$

Q.6] $L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right]$

→ $L^{-1}\left[s\log(s+1) - s\log(s-1)\right]$

$= \frac{-1}{t}L^{-1}\left[\phi'(s)\right] = \frac{-1}{t}L^{-1}\left[\frac{s}{(s+1)} + \log(s+1) - \frac{s}{(s-1)} - \log(s-1)\right]$

$= \frac{-1}{t}\left\{L^{-1}\left[\frac{s}{s+1}\right] - L^{-1}\left(\frac{s}{s-1}\right) - L^{-1}\left[\log(s-1)\right] + L^{-1}\left[\log(s+1)\right]\right\}$

$= -\frac{1}{t}$

$= \frac{-1}{t}\left\{L^{-1}\left[\frac{s+1-1}{s+1}\right] - L^{-1}\left[\frac{s+1-1}{s-1}\right] + \frac{1}{t}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{t}L^{-1}\left[\frac{1}{s+1}\right]\right\}$

$= \frac{-1}{t}\left\{L^{-1}\left[1-\frac{1}{s+1}\right] - L^{-1}\left[1+\frac{1}{s-1}\right]\right\}$

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$= \frac{-1}{t}\left\{ L^{-1}\left[\frac{s}{s+1} - \frac{s}{s-1}\right] - \frac{1}{t}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{t}L^{-1}\left[\frac{1}{s-1}\right]\right\}$

$= \frac{-1}{t}\left\{ L^{-1}\left[\frac{s^2-s-s^2-s}{s^2-1}\right] - \frac{1}{t}e^{-t} + \frac{1}{t}e^{t}\right\}$

$= \frac{-1}{t}\left\{ -1L^{-1}\left[\frac{s}{s^2-1^2}\right] - \frac{1}{t}(e^{-t}+e^{t})\right\}$

$= \frac{1}{t}\left\{ \cosh t + \frac{2\cosh t}{t}\right]$

$= \frac{\cosh t}{t}\left[1+\frac{2}{t}\right] = \frac{\cosh t\,(t+2)}{t^2}$

Q.7] $L^{-1}\left[\log\frac{(s^2+1)}{s(s+1)}\right]$

→ $L^{-1}\left[\log(s^2+1) - \log s(s+1)\right]$

(s^2+s) $= \frac{-1}{t}\left\{ \cancel{L^{-1}\left[\frac{1}{s^2+1}\right]} - L^{-1}\cancel{\left[\frac{1}{s(s+1)}(2s+1)\right]}\right\}$

$= \frac{-1}{t}\left\{ L^{-1}\left[\frac{1}{s^2+1}\right] - L^{-1}\left[\log s - \log(s+1)\right]\right\}$

$= \frac{-1}{t}\left\{ \sin t - L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]\right\}$

$= \frac{-1}{t}\left\{ \sin t - 1 - e^{-t}\right]$

Q.8] $L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right]$

→ $L^{-1}[\phi(s)] = \frac{-1}{t}L^{-1}[\phi'(s)]$

$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+(a/s)^2}\left(\frac{-a}{s^2}\right)\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{-as^2}{s^2(s^2+a^2)}\right]$

$= \frac{a}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$

Q.9] →

Q.10] →

Q.11] →

$$L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{a}{t} \sin at$$

Q.9] $L^{-1}[\cot^{-1}(s+1)]$

→

$$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+(s+1)^2}\right]$$

$$= \frac{-1}{t} e^{-t} L^{-1}\left[\frac{-1}{s^2+1}\right]$$

$$= \frac{e^{-t}}{t} \sin t$$

Q.10] $L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]$

→

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+\left(\frac{s+a}{b}\right)^2}\left(\frac{1}{b}\right)\right]$$

$$= \frac{-1}{bt} L^{-1}\left[\frac{b^2}{b^2+(s+a)^2}\right]$$

$$= \frac{-1}{t} e^{at} L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= \frac{-1}{t} e^{at} \sin bt$$

Q.11] $L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]$

→

$$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+\left(\frac{s+3}{2}\right)^2}\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{-4}{4+(s+3)^2}\right]$$

$$= \frac{1}{t} e^{-3t} L^{-1}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{1}{t} e^{-3t} \sin 4t$$

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$(s^2)^2-(2)^2$

$(s^2-2)(s^2+2)$ $s \cdot 2$

Q3) $L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$

$\rightarrow$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{(-4)}{s^3}\right)\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{4s^4}{s^3(s^4+4)}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{4s}{(s^4+4)}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{s^4+4}\right]$$

$\frac{1}{\cancel{(s+4)(s+4)}}$ s^2

$$= \frac{4}{t} L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{(s^2+2s+2)(s^2-2s+2)}\right]$$

$(s^2-2)(s^2+2)$

$s^4+2s-2s^2-4$

s^2-4

$$\frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2-2s+2)} = \frac{s}{(s^2+2s+2)(s^2-2s}$$

$\therefore (As+B)(s^2-2s+2)+(Cs+D)(s^2+2s+2) = s$

$As^3-2As^2+2As+Bs^2-2Bs+2B+2D$

$+Cs^3+2Cs^2+2Cs+Ds^2+2Ds = s$

$\therefore (A+C)s^3+s^2(-2A+B+2C+D)+s(2A-2B+2C+2D)$

$+(2B+2D) = s$

$(A+C)s^3 = 0s^3$ $\quad A+C=0$ — (i)

$\quad C=-A$

$-2A+B+2C+D = 0$ — (ii)

$2(A+B+C+D) = 1$ — (iii)

$(2B+2D) = 0$ — (iv)

$\therefore$ ~~A=~~ $\cancel{2A+2(-A)}=1$

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$$L^{-1}[\phi(s)] = \frac{\cancel{*}}{t} \; L^{-1}\left[\frac{-1/4}{s^2+2s+2} + \frac{1/4}{s^2-2s+2}\right]$$

$$= \frac{4}{(4)t} \; L^{-1}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]$$

$$= \frac{1}{t}\left\{ L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]\right\}$$

$$= \frac{1}{t}\left\{ e^{t} L^{-1}\left[\frac{1}{s^2+1}\right] - e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]\right\}$$

$$L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{1}{t}\left\{ e^{t}\sin t - e^{-t}\sin t\right\}$$

$*$ $$L\left[\int_0^t f(t)\,dt\right] = \frac{\phi(s)}{s}$$

$$L^{-1}\left[\frac{\phi(s)}{s}\right] = \int_0^t f(t)\,dt$$

Q.1] $$L^{-1}\left[\frac{1}{s(s^2+4)}\right]$$

$\rightarrow$ $$\phi(s) = \frac{1}{s^2+4}$$

$$L^{-1}[\phi(s)] = \frac{1}{2}\sin 2t$$

$$\therefore L^{-1}\left[\frac{\phi(s)}{s}\right] = \int_0^t f(t)\,dt$$

$$= \frac{1}{2}\int_0^t \sin 2t\,dt$$

$$= \frac{1}{2}\left[\frac{-\cos 2t}{2}\right]_0^t = \frac{1}{2}\left[-\frac{\cos 2t}{2} + \frac{\cos 0}{2}\right]$$

$$= \frac{1}{4}\left[1-\cos 2t\right]$$

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Q. 2] $L^{-1}\left[\frac{1}{s^2(s+2)}\right]$

$\rightarrow$ $\phi_1(s) = \frac{1}{s+2}$

$L^{-1}[\phi(s)] = e^{-2t}$

$L^{-1}\left[\frac{\phi(s)}{s^2}\right] = \int_0^t \int_0^t f(t)\, dt\, dt$

$= \int_0^t \int_0^t e^{-2t}\, dt\, dt$

$= \int_0^t \left[\frac{e^{-2t}}{(-2)}\right]_0^t dt$

$= -\frac{1}{2}\int_0^t (e^{-2t} - 1)\, dt$

$= -\frac{1}{2}\left[\frac{e^{-2t}}{(-2)} - t\right]_0^t = \frac{1}{2}\left[\frac{e^{-2t}}{2} + t\right]_0^t$

$= \frac{1}{2}\left[\frac{e^{-2t}}{2} + t - \frac{e^0}{2}\right]$

$= \frac{1}{4}(e^{-2t} + 2t - 1)$

$L^{-1}\left[\frac{1}{\sqrt{s}}\right]$