Characteristic root technique for repeated roots

Suppose the recurrence relation $a_n = \alpha a_{n-1} + \beta a_{n-2}$ has a characteristic polynomial with only one root r. Then the solution to recurrence relation is

$$a_n = ar^n + bnr^n$$

Where a and b are constants determined by the initial conditions.

Ex: Solve recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ & $a_1 = 4$.

Solⁿ: The characteristic polynomial equation of given recurrence relation is $x^2 - 6x + 9 = 0$ by factoring $(x-3)^2 = 0$

$$x = 3, 3.$$

This is only characteristic root.

The solution ~~of~~ to recurrence relation has the form $a_n = a3^n + bn3^n$

for some constant a & b. Now use initial condition

$$a_0 = 1 = a3^0 + b.0.3^0 = a$$

$$a_1 = 4 = a.3^1 + b.1.3^1 = 3a + 3b$$

Put $a = 1$ in $3a + 3b = 4$

$$3 + 3b = 4$$

$$3b = 4 - 3 = 1$$

$$\boxed{b = 1/3}$$

The solution of recurrence relation is —

$$a_n = 1.3^n + (1/3).n3^n.$$

$$\boxed{a_n = 3^n + \frac{1}{3} n 3^n}$$

Characteristic root technique for complex roots.

Suppose a ~~tot~~ recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$

the characteristic polynomial is

$$ax^2 + bx + c = 0$$

If r_1 & r_2 are two distinct roots of characteristic ploynomial. from quadratic equation i.e.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then solution to recurrence relation is

$$a_n = A(r_1)^n + B(r_2)^n$$

when r_1 & r_2 are same then

$$a_n = A(r_1)^n + B\,n(r_1)^n$$

$$\boxed{\begin{aligned} \text{use} \quad x + iy &= r(\cos\theta + i\sin\theta) \\ r = \sqrt{x^2+y^2}, &\quad \tan\theta = y/x \end{aligned}}$$

Q $a_n + 2a_{n-1} + 2a_{n-2} = 0$ and initial value $a_0 = 1, a_1 = 3$ for $n \geq 2$

Solⁿ ~~Ist~~ The characteristic polynomial equation is

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= -1 \pm i$$

$\therefore r_1 = -1+i \quad r_2 = -1-i$

(a) $r_1 = -1+i$

$x + iy = r(\cos\theta + i\sin\theta)$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = \frac{1}{-1} = -1 = -\pi/4$

$\boxed{\theta = -\pi/4}$

(b) $r_2 = -1-i$

$x + iy = r(\cos\theta + i\sin\theta)$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = \frac{-1}{-1} = 1 = \tan 45^\circ$

$\boxed{\theta = \pi/4}$

$$r_1 = \sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right) = \sqrt{2}\left(\cos\pi/4 - i\sin\pi/4\right)$$

$$r_2 = \sqrt{2}\left(\cos\pi/4 + i\sin\pi/4\right)$$

The solution of recurrence relation is

$$a_n = A\cdot(r_1)^n + B(r_2)^n$$

$$a_n = A(-1+i)^n + B(-1-i)^n$$

$$a_n = A\left((\sqrt{2})^n\cdot\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)\right) + B\left((\sqrt{2})^n\cdot\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right)$$

$\rightarrow$ constants

Let suppose $k_1 = A+B$ & $k_2 = (B-A)i$, we get

$$a_n = (\sqrt{2})^n\cdot k_1\left(\cos\left(\frac{n\pi}{4}\right)\right) + k_2\left(\sin\left(\frac{n\pi}{4}\right)\right)$$

$$a_0 = 1 = (\sqrt{2})^0\cdot\left[k_1\left[\cos 0\right] + k_2\left[\sin 0\right]\right]$$

$$= (\sqrt{2})^0\cdot(k_1\cdot 1 + k_2\cdot 0)$$

$$\boxed{1 = k_1}$$

$$a_1 = 3 = (\sqrt{2})^1\cdot\left(k_1(\cos\pi/4) + k_2(\sin\pi/4)\right)$$

$$= \sqrt{2}\left(k_1\cdot\frac{1}{\sqrt{2}} + k_2\cdot\frac{1}{\sqrt{2}}\right)$$

$$3 = k_1 + k_2$$

$$3 = 1 + k_2 \quad (\text{Put } k_1 = 1)$$

$$\boxed{k_2 = 3}$$

$$\therefore a_n = (\sqrt{2})^n\cdot\left[1\cdot\cos\frac{n\pi}{4} + 2\cdot\left(\sin\frac{n\pi}{4}\right)\right]$$

$$\boxed{a_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + 2\sin\frac{n\pi}{4}\right)}$$