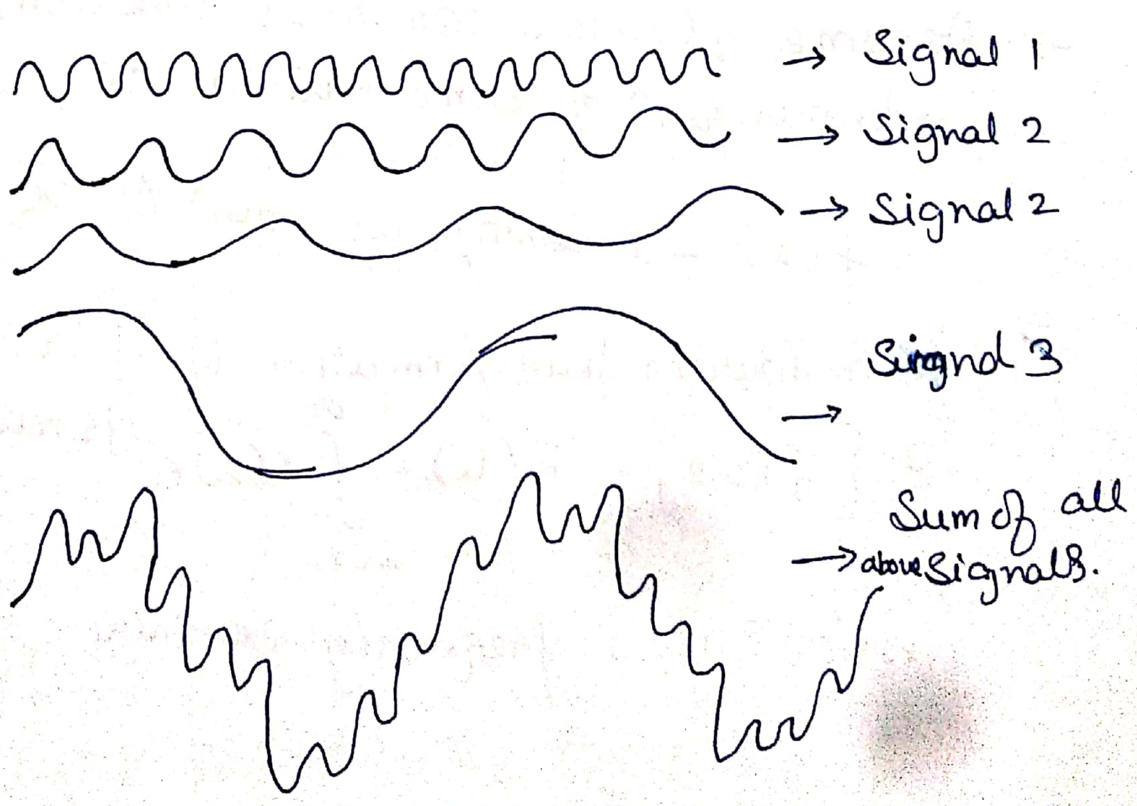
Fourier Transformation

→ Virtually everything in the world can be described by a waveform - a function of time, space some other variable.

For instance, Sound waves, Electronic field, Elevation of a hill v/s location, the price of your favorite stock v/s time, etc.

The fourier transform gives us a unique & powerful way of viewing these waveforms.

All the waveforms, no matter what you observe in the universe are actually just the sum of simple sinusoids of different frequencies.



→ In General waveforms are not made up of a discrete no. of frequencies, but rather a continous range of frequencies.

→ "The Fourier Transform is the mathematical tool that shows us how to deconstruct the waveform into its sinusoidal components.

→ Application :- TV signals, Cell phone Signals, Sound waves travel when you speak

→ Fourier Transform

$$\text{Fourier Transform} \begin{cases} \text{Continous Domain} \\ \text{Discrete Domain} \end{cases}$$

(I) Fourier Transform in Continous Domain

→ Assume $f(x)$ is a continous function, $f(x)$ is a continous fun^ of some variable x,

$f(x) \rightarrow$ Continous fun^ of x

then fourier transformation of $f(x)$ is

$$\mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

where u is frequency variable

→ Now for doing this continous Fourier transformation, $f(x)$ has to meet some requirement.

(i) $f(x)$ must be continous & integrable

→ Similary we have Inverse Fourier transformation if $F(u)$ must be Integrable

$$\mathcal{F}^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\, e^{j2\pi ux}\, du$$

Note :- From $f(x)$ using integral operation we can get Fourier Transformation which is $F(u)$. If $F(u)$ is integrable then using inverse fourier transformation, we can get back original continous funⁿ $f(x)$.

$F(u)$ & $f(x)$ are known as Fourier Transform pairs.

→ Because $f(x)\, e^{-j2\pi ux}$, $F(u)$ we get is in general complex quantity.

→ In general $F(u) \rightarrow$ Complex function., so we can write as.

$$F(u) = \underbrace{R(u)}_{\text{Real}} + j\, \underbrace{I(u)}_{\text{Imaginary}}$$

$$\Rightarrow |F(u)|\, e^{j\phi(u)}$$

$$|F(u)| = \sqrt{[R^2(u) + I^2(u)]}$$

Fourier Spectrum of $f(x)$.

$\phi = \tan^{-1} \dfrac{I(u)}{R(u)}$ // Phase Angle.

// Power Spectrum of $f(x)$

$$P(u) = |F(u)|^2$$
$$= R^2(u) + I^2(u)$$

→ Because we are doing image process & i/p is image i.e 2D representation.
So we'll see 2D fourier transformation.

2D Fourier Transform

$f(x,y)$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\,dy.$$

Forward Fourier Transformation

Inverse Fourier Transformation.

$$f(x,y) = \int_{-\infty}^{\infty}\int F(u,v)\, e^{j2\pi(ux+vy)}\, du\,dv$$

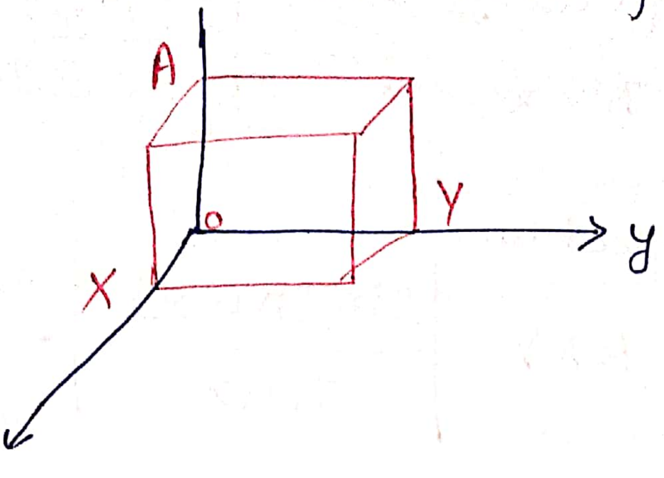
$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

// Phase $\phi(u,v) = \tan^{-1} \dfrac{I(u,v)}{R(u,v)}$

// Power Spectrum

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

→ Suppose we have a continous funⁿ $f(x,y) = A$



$$\text{for} \quad \left.\begin{array}{l} 0 \leq x \leq X \\ 0 \leq y \leq Y \end{array}\right\} \quad f(x,y) = A.$$

So we get rectangular function where
$f(x,y) = 0$ for $x > X$ & $y > Y$

→ Let use see how we can find Fourier Transformation of this 2 Dimensional Signal.

$$F(u,v) = \int_{-\infty}^{\infty}\int f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$\cancel{= A. \int_0^X \left[\frac{e^{-j2\pi ux}}{-j2\pi u}\right]_0^X}$$

$$= A. \int_0^X e^{-j2\pi ux}\, dx \;.\; \int_0^Y e^{-j2\pi vy}\, dy$$

$$= A \left[\frac{e^{-j2\pi ux}}{-j2\pi u}\right]_0^X . \left[\frac{e^{-j2\pi vy}}{-j2\pi v}\right]_0^Y$$

$$= A\,X\,Y \left[\frac{\sin(\pi u X) \cdot e^{-j\pi u X}}{\pi u X} \right] \left[\frac{\sin(\pi v Y) \cdot e^{-j\pi v Y}}{\pi v Y} \right]$$

Fourier Spectrum

$$|F(u,v)| = A\,X\,Y \left| \frac{\sin(\pi u X)}{\pi u X} \right| \cdot \left| \frac{\sin(\pi v Y)}{\pi v Y} \right|$$

Plot of fourier Spectrum.

(II) Fourier Transformation in Discrete Domain

→ All the Integration operations that we are doing in continous fun^ are replaced by the corresponding summation. operation.

→ 2-Dimensional Discrete Fourier Transform. (M X N)

Forward Discrete FT

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Because our images are discrete, the frequency variables are also going to be discrete.

So, $u = 0, 1, 2, \ldots M-1$

$v = 0, 1, 2, \ldots N-1$

Inverse Fourier Transformation.

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$x = 0, 1, 2, \ldots M-1$

$y = 0, 1, 2, \ldots N-1$

→ If the image is square array i.e $M=N$

$$F(u,v) = \frac{1}{N} \sum_{x,y=0}^{N-1}\sum f(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$f(x,y) = \frac{1}{N} \sum_{u,v=0}^{N-1}\sum F(u,v)\, e^{j\frac{2\pi}{N}(ux+vy)}$$

→ Fourier Spectrum

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

→ Phase

$$\phi(u,v) = \tan^{-1} \frac{I(u,v)}{R(u,v)}$$

→ Power Spectrum

$$P(u,v) = |F(u,v)|^2$$

$$= R^2(u,v) + I^2(u,v)$$

Real part of fourier co-efficient

Imaginary part of fourier co-efficient

Properties of Fourier Transformation.

1. Linear Property
2. Change of scale property
3. Shifting Property
4. Modulation Theorem

1. Linear Property

→ If $F(u)$ and $G(u)$ are fourier transform of $f(x)$ and $g(x)$ respectively

$$F\{a f(x) + b g(x)\} = a \cdot F(u) + b \cdot G(u)$$

where a, b are constants
$i = -j2\pi$

Proof : We know that

$$F(u) = \int_{-\infty}^{\infty} f(x) \cdot e^{iux} dx$$

$$G(u) = \int_{-\infty}^{\infty} g(x) \cdot e^{iux} dx$$

$$L.H.S = F\{a f(x) + b g(x)\}$$

$$= \int_{-\infty}^{\infty} \{a f(x) + b g(x)\} e^{iux} dx$$

$$= a \int_{-\infty}^{\infty} f(x) e^{iux} dx + b \int_{-\infty}^{\infty} g(x) e^{iux} dx$$

$$= a \cdot F(u) + b \cdot G(u) \quad R.H.S$$

Hence Proved.

② Change of Scale Property

→ If $F(u)$ is the complex fourier transformation of $f(x)$, then $i = -j2\pi$

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{u}{a}\right), \quad a \neq 0$$

<u>Proof:-</u> $F\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x) \cdot e^{iux} \, dx$

LHS

$\therefore F\{f(ax)\} = \int_{-\infty}^{\infty} f(ax) \, e^{iux} \, dx$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$dx = \frac{dt}{a}$

$$= \int_{-\infty}^{\infty} f(t) \cdot e^{iut/a} \, \frac{dt}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(t) \cdot e^{i(u/a)t} \, dt$$

$$= \frac{1}{a} F\left(\frac{u}{a}\right) \quad \text{R.H.S}$$

LHS = RHS

Hence Proved

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③ Shifting Property

→ If F(u) is the complex fourier transform of f(x) then $i = -j2\pi$

$$F\{f(x-a)\} = e^{iua} \cdot F(u)$$

Proof:- We Know that

$$F\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

L.H.S

$$\therefore F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)\, e^{iux}\, dx$$

Put $x - a = t$

$x = t + a$

$dx = dt$

$$= \int_{-\infty}^{\infty} f(t)\, e^{iu(t+a)}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{iut} \cdot e^{iua}\, dt$$

$$= e^{iua} \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt$$

$$= e^{iua} \cdot F(u) \quad \text{R.H.S}$$

So L.H.S = R.H.S Hence Proved.

④ Modulation Theorem

→ If $F(u)$ is the complex fourier transform of $f(x)$, then $F\{f(x).\cos ax\} = \frac{1}{2}[F(u+a)+F(u-a)]$

L.H.S ∴ $F\{f(x).\cos ax\} = \int_{-\infty}^{\infty} f(x) \cos ax . e^{iux} dx$

$$= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{iux} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x).e^{i(u+a)x} dx + \int_{-\infty}^{\infty} f(x).e^{i(u-a)x} dx\right]$$

$$= \frac{1}{2}\left[F(u+a) + F(u-a)\right] \text{ . R.H.S.}$$

Hence Proved L.H.S = R.H.S.

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