Group Homomorphism

Let $(G, *)$ and $(\overline{G}, \Delta)$ be any two groups. A mapping $f: G \to \overline{G}$ is called homomorphism of G to $\overline{G}$ if.

$$f(a * b) = f(a) \, \Delta \, f(b) \quad \forall a, b \in G.$$

eg:- Let $G = R$, the set of real number and $\overline{G} = R - \{0\}$

$(G, +)$ and $(\overline{G}, \cdot)$ are groups. Define a mapping $f: G \to \overline{G}$ by $f(a) = 2^a \; \forall a \in G$.

Clearly $f(a+b) = 2^{a+b} = 2^a \cdot 2^b = f(a) \cdot f(b)$

$\therefore f$ is a homomorphism of G into $\overline{G}$.

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Ex: If for a group G, $f: G \to G$ is given by $f(x) = x^2$, $x \in G$ is homomorphism, prove that G is abelian.

Solⁿ $a, b \in G \Rightarrow ab \in G$

$f: G \to G$ is a homomorphism

$f(a) = a^2$, $f(b) = b^2$ and $f(ab) = (ab)^2$

Now $f(ab) = (ab)^2$

$f(a) \cdot f(b) = (ab)(ab)$

$a^2 \cdot b^2 = (ab)(ab)$

$(aa)(bb) = (ab)(ab)$

$a \cdot (ab) \cdot b = a(ba)b$

$(ab) \cdot b = (ba)b$ [Cancellation Law]

$ab = ba$ [Cancellation law]

$\therefore G$ is abelian because it holds commutative property.

Rings

An algebraic system $(R, +, \cdot)$ is called a ring if binary operations '+' and '.' R satisfy the following properties:

1. $(R, +)$ is an abelian group
2. $(R, \cdot)$ is a semi-group
3. The operation '.' is distributive over + i.e. for any $a, b, c \in R$

$a \cdot (b + c) = a \cdot b + a \cdot c$ and

$(b + c) \cdot a = b \cdot a + c \cdot a$

eg: The set of integers Z, with respect to operation + & × is a ring.

Special type of Rings

① **Commutative Ring or Abelian Ring**

A ring R is said to be commutative ring or an Abelian ring if it satisfies the commutative Law, $\forall a, b \in R$

$$a \cdot b = b \cdot a$$

② **Ring with Unity**

A ring R which contains multiplicative identity (Called unity) is called a ring with unity

Thus if $1 \in R$ such that $a \cdot 1 = a = 1 \cdot a \quad \forall a \in R$, then ring with unity.

③ **Ring without unity**

A ring R, which does not contain multiplicative identity is called ring without unity.

④ **Finite and infinite Ring**

If the number of elements in the ring R. Is finite then $\langle R, +, \cdot \rangle$ is called finite ring otherwise it is called infinite ring.

⑤ **Order of a Ring**

The number of elements in a finite ring R is called order of ring R. This is denoted by $|R|$.

Properties of Ring

① If R is ring then:

(i) $a \cdot 0 = 0 = 0 \cdot a \quad \forall a \in R$

we know that $a + 0 = a \;\forall a \in R$

$a \cdot (a + 0) = a \cdot a$

$a \cdot a + a \cdot 0 = a \cdot a$

$a \cdot 0 = 0$ (By left cancellation under addition)

Similarly; prove $\boxed{a.0 = 0}$.

(ii) $a(-b) = (-a)b = -(ab) \quad \forall a, b \in R$

$b \in R \Rightarrow -b \in R$ such that $b + (-b) = 0$

$$a.(b+(-b)) = a.0$$

$$a.b + a.(-b) = 0$$

$$a.(-b) = -(a.b)$$

Similarly, prove that

$$(-a).b = -(a.b)$$

(iii) $(-a).(-b) = ab \quad \forall a, b \in R$

$\because a.(-b) + (-a).(-b)$

$= (a + (-a)).(-b)$

$= 0.(-b)$

$= 0$

$= a.((-b)+b)$

$= a.(-b) + a.b$

$\therefore a.(-b) + (-a).(-b) = a.(-b) + a.b$

By left cancellation law,

$$(-a).(-b) = a.b$$

Let $(R, +, .)$ be a ring, then

$$a.(b-c) = a.b - a.c$$

$$(b-c).a = b.a - c.a \text{ for all } a, b, c \in R$$

$a.(b-c) = a.(b+(-c))$

$a.(b-c) = a.b + a.(-c)$

$a.(b-c) = a.b - a.c$

Similarly prove $(b-c).a = b.a - c.a$

Sub-Rings

Let $(R, +, \cdot)$ be a ring and S be a non-empty subset of R. If ~~$\mathcal{S}$~~ $(S, +, \cdot)$ is a ring then $(S, +, \cdot)$ is called a sub-ring of R.

Ex: let E denote the set of even integers. $(E, +, \cdot)$ is a Sub-ring of $(Z, +, \cdot)$ where Z denotes the set of integers.

Fields

A finite integral domain is called fields.

Ex: If R is a non-zero ring so that $a^2 = a \ \forall a \in R$. prove that the characterstic of R is 2.

Soln: Since $a^2 = a \ \forall a \in R$

we have $(a+a)^2 = (a+a)$

$$(a+a)(a+a) = (a+a)$$

$$a(a+a) + a(a+a) = a+a$$

$$aa + aa + aa + aa = a+a$$

$$(a^2 + a^2) + (a^2 + a^2) = a+a$$

$$(a+a) + (a+a) = (a+a) + 0$$

Apply left cancellation law

$$\Rightarrow a+a = 0$$

$$2a = 0$$

2 is the least positive integer so that

$$2a = 0 \quad \forall a \in R$$

Hence, the characterstic of R is 2.