Lattices

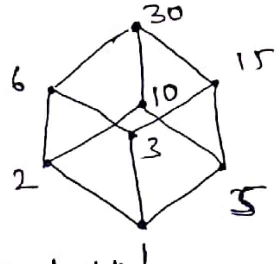
- A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.

Example

Let Z^+ denote the set of all positive integers and let R denote the relation "division" in Z^+ such that for any two elements $a, b \in Z^+$, aRb, if a divides b. Then (Z^+, R) is a lattice in which join of a and b is least common multiple of a and b i.e. $a \vee b = a \oplus b =$ LCM of a and b, and meet of a & b, i.e. $a * b = a \wedge b =$ GCD of a and b.

Example

Set of all divisors of 30. Then $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ let R denote the relation division. Then (S_{30}, R) is a lattice.



Different lattices can be represented by same Hasse Diagram. If $(L, \leq)$ is a lattice, then $(L, \geq)$ is also a lattice. The operations of meet and join on $(L, \leq)$ become the operations of join and meet on $(L, \geq)$. The statement involving the operations $*$ and $\oplus$ and $\leq$ hold if we replace $*$ by $\oplus$, $\oplus$ by $*$ and $\leq$ by $\geq$. The lattices $(L, \leq)$ and $(L, \geq)$ are duals each other.

Some properties of Lattices

Let $(L, \leq)$ be a lattice and '$\cdot$' and '$+$' denote two binary operation meet and join on $(L, \leq)$. Then for any $a, b, c \in L$ we have

1) ~~Inpo~~ Idempotent Laws

$$a \cdot a = a$$
$$a + a = a$$

2) Commutative Laws

$$a \cdot b = b \cdot a$$
$$a + b = b + a$$

3) Associative Laws.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$(a + b) + c = a + (b + c)$$

4) Absorption Laws

$$a \cdot (a + b) = a \Rightarrow a \cdot a = a$$
$$a + (a \cdot b) = a \Rightarrow a + a = a$$

Theorem: Let $(L, \leq)$ be a lattice in which '$\cdot$' and '$+$' denote the operations of meet and join respectively. Then $a \leq b \Leftrightarrow a \cdot b = a \Leftrightarrow a + b = b \quad \forall a, b, c \in L$

Proof: Let $a \leq b$

We know that $a \leq a$, therefore $a \leq a \cdot b$ but from definition we have $a \cdot b \leq a$.

$\therefore \quad a \leq b \Rightarrow a \cdot b = a$

Let us assume that $a \cdot b = a$

but this is possible only if $a \leq b$

i.e. $a \cdot b = a \Rightarrow a \leq b$

$\therefore \quad a \leq b \Rightarrow a \cdot b = a$ and $a \cdot b = a \Rightarrow a \leq b$

Combining these two, $a \leq b \Leftrightarrow a \cdot b = a$

Now let $a \cdot b = a$, then we have

$$b+(a\cdot b) = b+a+a+b$$

but $b+(a\cdot b) = b$

Hence $a+b=b$

Similarly by assuming $a+b=b$, we can show that $a\cdot b=a$

Hence $a \leq b \Leftrightarrow a\cdot b = a \Leftrightarrow a+b=b=a$

<u>Theorem</u>: let $(L, \leq)$ be a lattice. Then $b \leq c \Rightarrow \begin{cases} a*b \leq a*c \\ a\oplus b \leq a \oplus c \end{cases}$

<u>Proof</u>: By theorem: $a \leq b \Leftrightarrow a*b=a \Leftrightarrow a\oplus b = b$

$\therefore$ $\underline{a*b \leq a*c}$

$\downarrow$

This is to prove when $(a*b)*(a*c) = (a*b)$

$$(a*b)*(a*c) = a*(b*a)*c$$
(Associative Law)
$$= a*(a*b)*c$$
(Commutative Law)
$$= (a*a)*(b*c)$$
$$= a*(b*c)$$
(Absorption law)
$$= (a*b)$$
(Absorption law)

$\therefore$ If $b \leq c$ then $a*b \leq a*c$

let $b \leq c \Rightarrow b \oplus c = c$

To show that $a\oplus b \leq a \oplus c$. It is sufficient to show that $(a\oplus b)\oplus(a\oplus c) = (a\oplus c)$

Consider, $(a\oplus b)\oplus(a\oplus c) = a\oplus(b\oplus a)\oplus c$

$$= a\oplus$$

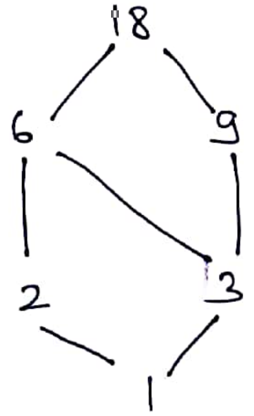
Well Ordered Set

Given a poset $(X, \leq)$, $\leq$ is a well order (well ordering) and that is well-ordered by $\leq$ iff every non-empty subset of X has a least element. When X is non-empty if we pick any two element subset, $\{a, b\}$ of X, Since the subset $\{a, b\}$ must have a least element, either $a \leq b$ or $b \leq a$ i.e., every well-order is a total order.

eg:- The set of natural number $(\mathbb{N})$ is a well-ordered.

Bounded Lattice

A lattice L is said to be bounded if it has greatest element 1 and a least element 0. eg:- $D_{18} = \{1, 2, 3, 6, 9, 18\}$ is a bounded lattice



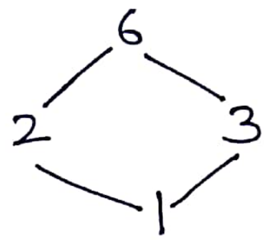
Hasse Diagram of D_{18}

[Every finite lattice is always bounded].

Complemented Lattice

A lattice L is said to be complemented if it is bounded and if it is bounded and if every element L has a complement. Here, each element should have atleast one complement.

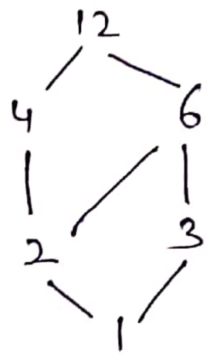
eg:- $D_6 \cdot \{1,2,3,6\}$ is a complemented lattice.



Hasse Diagram of D_6

Every element has a complement

Q What are the complements pair for the following lattices?



There are two complement pair here →

meet (4, 3) = 1
join (4, 3) = 12

There is another pair here → upper & lower bounds are also compliments of each other.

meet (1, 12) = 1
join (1, 12) = 12