~~Unit 5~~

Graph Theory

A Graph $G=(V,E)$ is a mathematical structure consisting of two finite sets V & E. 'V' can be represents vertices of a graph and 'E' can be represents edges of a graph. Each edge is associated with a set of consisting of either one or two vertices. called its endpoints.

Directed Graph

A Graph which have directed edges are present is called Directed Graph.

Edges which have direction i.e. directed to one vertex to another is called directed edges.

Undirected Graph

A Graph which have no directed edges are present is called ~~Direct~~ undirected Graph.

Edges which have no direction is called undirected edges.

1736 - Euler's

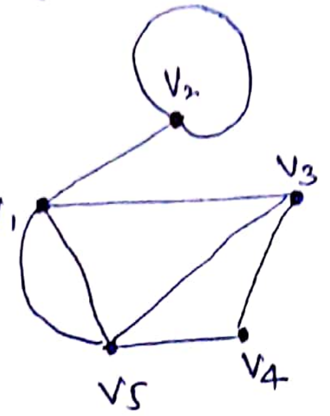
(Graph Theory with Application to Engg. & CS - Narsingh Deo

Matrix Representation of a Graph

(a) Ajacency Matrix

Let $G=(V,E)$ be a graph with n vertices where $V=\{v_1,\dots,v_n\}$. The adjacency matrix A_G is an $n\times n$ matrix such that $(i,j)^{th}$ entry a_{ij} of A_G is the number of edges from v_i to v_j. That is.

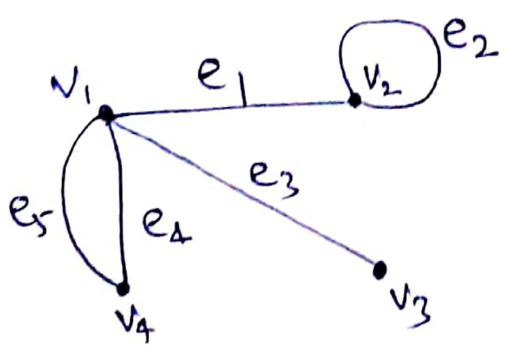
$$a_{ij} = \text{the no. of edges from } v_i \text{ to } v_j$$



$$\begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(b) Incidence Matrix

$$a_{ij} = \begin{cases} 0 & \text{if } v_i \text{ is not an end vertex of } e_j \\ 1 & \text{if } v_i \text{ is an end vertex of } e_j \text{ but is not a loop} \\ 2 \rightarrow & \text{if } v_i \text{ is a loop at } v_i \end{cases}$$



$$I_G = \begin{array}{c c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Sub-Graph

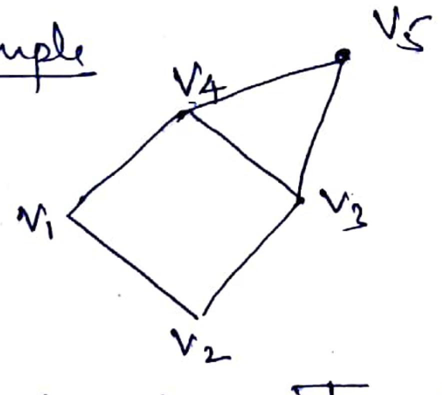
Let $G = (V, E)$ be a graph. A Graph $G_1 = (V_1, E_1)$ is called a sub-graph of G if V_1 is a non-empty subset of B & E_1 is a subset of E such that whenever $e = \{u, v\} \in E_1 \cup E$

Subgraphs of a graph can be obtained by deleting the vertex or the edge between that graph:

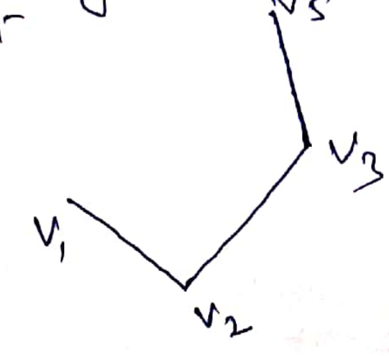
(i) Subgraph obtained from $G = (V, E)$ by deleting vertex $v \in V$ is the graph $G_1 = (V_1, E_1)$ where $V_1 = V - \{v\}$ and $E_1 = \{e \in E \mid v$ is not an end vertex of $e\}$.

(ii) Subgraph obtained from $G = (V, E)$ be deleting an edge $e \in E$ is the graph $G_1 = (V_1, E_1)$ where $V_1 \in V$ and $E_1 \in E - \{e\}$.

Example



Deleting vertex V_4. in above graph. we get



Null Graph

A Graph whose vertex & edge sets are empty or A Graph in which all ~~vertex~~ the vertices are isolated vertices is called Null Graph.

Initial & terminal Vertex

A vertex where edges are started ~~i.e. A vertex where tail of the edges are present which~~ is called Initial vertex.

A vertex where edge will be terminated is called terminated vertex.

Ajacent vertex

A pair of vertices which ~~have~~ lie on common edge is called Adjacent vertex.

Adjacent edges

A pair of edges which have one common vertex is called Adjacent edges.

parallel edges | Multi-edges

If two or more edges of a graph G have same vertices, then these edges are said to be parallel or multi-edges.

Loop

An edge with just one endpoints is called a loop or a self-loop. ~~An ed~~ end points of a loop is said to be adjacent to itself.

Isolated Vertex

A ~~k~~ vertex on which no edges ~~are~~ is incident is called isolated vertex.

Simple Graph

A Graph without multiple edges (parallel edges) and loops is called Simple Graph.

Multiple Graph

A Graph which have multiple edges ~~are~~ & loops are present is called Multiple Graph / Multi-Graph.

Notation of Graph

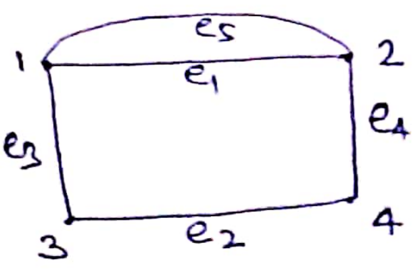
In pictorial representations of a graph, the vertices will be denoted by dots & edges by line segments.

$V = \{1, 2, 3, 4\}$ & $E = \{e_1, e_2, e_3, e_4, e_5\}$

Let γ be defined by

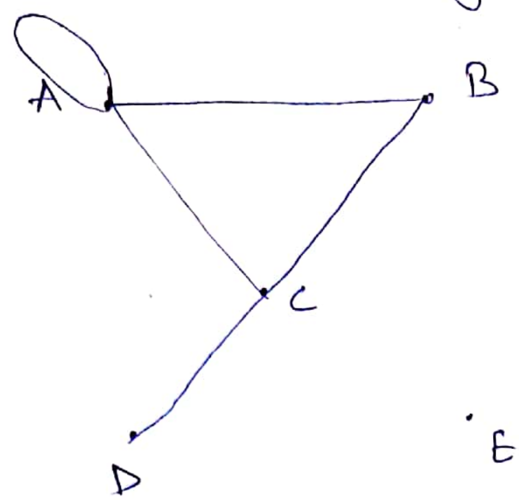
$\gamma(e_1) = \gamma(e_5) = \{1, 2\}$, $\gamma(e_2) = \{4, 3\}$, $\gamma(e_3) = \{1, 3\}$

$\gamma(e_4) = \{2, 4\}$.



Degree of a vertex

The number of edges in a graph G which are incident on a vertex is called degree of that vertex.



Number of edges is 5

Degree of vertex A is 4 (because self-loop has degree 2)

Degree of vertex B is 2

Degree of vertex C is 3

Degree of vertex D is 1

Degree of vertex E is 0

Sum of the degree of vertices = 4+2+3+1+0 = 10.

Thus, we observe that

$$\sum_{i=1}^{5} \deg(v_i) = 2e,$$

where $\deg(v_i)$ denoted the degree of vertex v_i & e denotes the number of edges.

Euler's Theorem

The sum of the degrees of vertices of a graph G is equal to twice the number of edges in G. Thus total degree of a graph is even.

Corollary

There can be only an even number of vertices of odd degree in a given graph G.

Discrete Graph

Let D_n denote the graph with n vertices & no edges for each integer $n \geq 1$. Then D_n is called discrete graph on n vertices.

For example

D_3

and

D_5

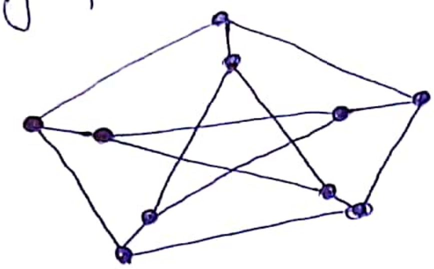
Regular Graph

If each vertex of a graph G has same degree as every other vertex, then G is called a regular graph.

A k-regular graph is a regular graph whose common degree is k.

Petersen Graph

The petersen graph is the 3-regular graph



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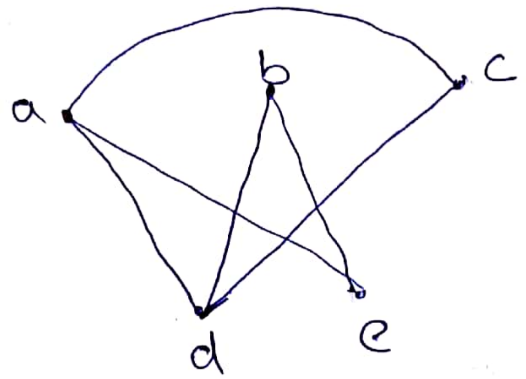
Q ~~Find~~ Draw a graph with five vertices a, b, c, d, e such that deg(a) = 3, deg(b) = 2, deg(c) = 2, deg(d) = 3, deg(e) = 2 and a & b are adjacent of e.

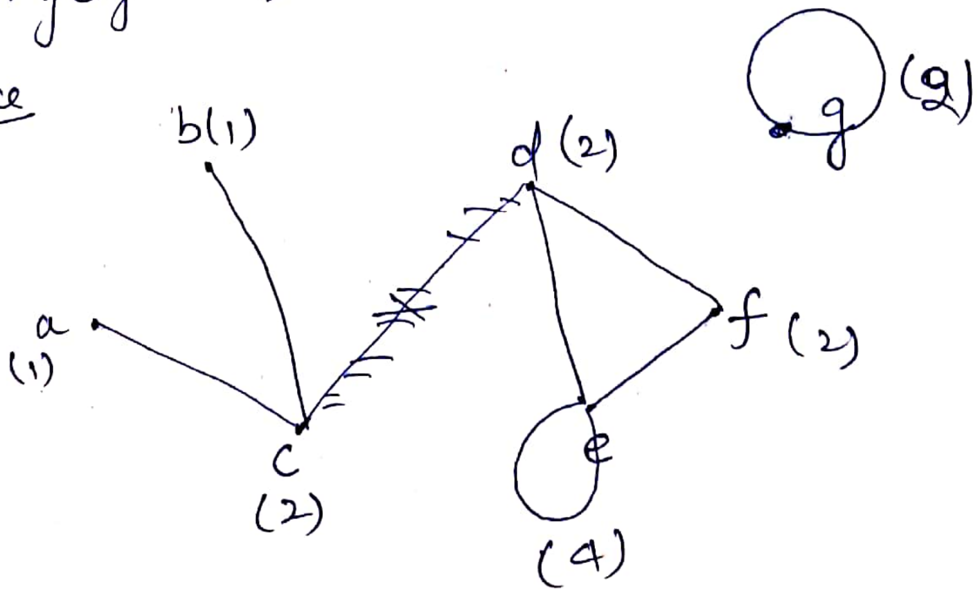
Solⁿ Total no. of edges = $\frac{1}{2}$ (sum of the degree of vertices)
$= \frac{1}{2}(3+2+2+3+2) = 6.$



Q A graph has degree sequence 1, 1, 2, 2, 2, 2, 4, find the number of edges of this graph & draw the graph.

Solⁿ no. of vertices is 7
No. of edges $= \frac{1}{2}(1+1+2+2+2+2+4) = 7$

Hence



Q How many edges are in each of the following graphs:

(a) K_3 (b) K_5 (c) $K_{2,3}$ (d) $K_{4,3}$

Sol^n A complete graph on n-vertices has $\frac{n(n-1)}{2} = {}^nC_2$ edges. Therefor.

(a) $K_3 = \frac{3(3-1)}{2} = 3$ edges

(b) $K_5 = \frac{5(5-1)}{2} = 10$ edges.

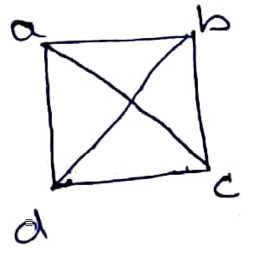
A complete bipartite group m&n vertices has m.n edges. Therefore,

(c) $K_{2,3} = 2 \times 3 = 6$ edges

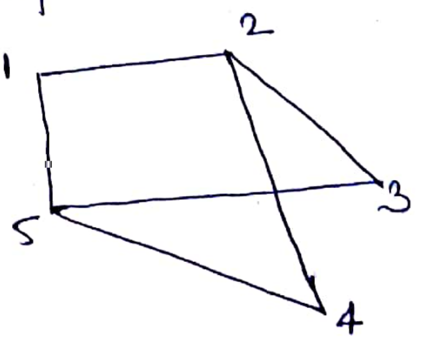
(d) $K_{4,3} = 4 \times 3 = 12$ edges.

Question Determine whether the following graphs are bipartite. If yes, give the bipartition sets.

(a)



(b)



NO.

Yes