Normal Sub-Groups

A sub-group $(H, *)$ of $(G, *)$ is called normal sub-group of G if for all $h \in H$, $g \in G$ and $ghg^{-1} \in H$

If H is normal in G then $H \trianglelefteq G$.

A sub-group $(H, *)$ of a group $(G, *)$ is for every $s \in G$ $ghg^{-1} \subseteq H$.

Q. Every sub-group of an abelian group is normal sub-group.

Solⁿ Let $(G, *)$ be an abelian group and $(H, *)$ be a sub-group of G

Now $g \in G$, $h \in H$

$$\Rightarrow g * h * g^{-1} = h * g * g^{-1}$$
$$= h * e$$
$$= h$$

$\therefore$ $g * h * g^{-1} = h \in H \ \forall g \in G, h \in H$

$(H, *)$ is normal in G.

Permutation Group

A permutation is a one-one mapping of a non-empty set onto itself.

① Equal permutation

Let S be a non-empty set. The permutation f and G defined on S, are said to be equal if $f(a) = g(a)$ for all $a \in S$.

eg:- let $S = \{1, 2, 3, 4\}$

and let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$, $g = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

we have $f(1) = g(1) = 3$

$f(2) = g(2) = 1$

$f(3) = g(3) = 2$

$f(4) = g(4) = 4$

i.e. $f(a) = g(a)$ $\forall a \in S$ therefore

$f = g$

Let $S = (a_1, a_2, a_3, \ldots, a_n)$ be a finite set. The number of permutations on S contains is $n!$, The set of all permutations on S is denoted by S_n. where $|S_n| = n!$

If $f \in S_n$ then f is of the form

$$f = \{(a_1, f(a_1)), (a_2, f(a_2)), \ldots (a_n, f(a_n))\}$$

It can also be written as

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ f(a_1) & f(a_2) & f(a_3) & \cdots & f(a_n) \end{pmatrix}$$

Ex: let $S = \{1, 2, 3\}$ and $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

we can write $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

Hence, there are $3! = 6$ ways of writing f.

② Identity Permutation

Let S be a finite non-empty set. An identity permutation on S denoted by I is defined $I(a) = a$ for all $a \in S$.

eg:- let $S = \{1, 2, 3, 4\}$, then

$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is identity permutation on S.

③ Product of Permutations (or Composition of Permutations)

Let $S = (a_1, a_2, \ldots, a_n)$ and let

$$f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix} \quad g = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g(a_1) & g(a_2) & \cdots & g(a_n) \end{pmatrix}$$

be two arbitrary on S.

Then composite of f and g as —

$$f \circ g = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix} \circ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g(a_1) & g(a_2) & \cdots & g(a_n) \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(g(a_1)) & f(g(a_2)) & \cdots & f(g(a_n)) \end{pmatrix}$$

Ex: Let $S = \{1, 2, 3\}$

and $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

be two permutations on S. Compute $f \circ g$

Sol^n

$$f \circ g = \overset{f}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}} \circ \overset{g}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ f(g(1)) & f(g(2)) & f(g(3)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ f(3) & f(2) & f(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

4) Inverse of Permutation (Symmetric Group)

If f is a permutation on $S = \{a_1, a_2, \dots a_n)$ such that $f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$

then there exists a permutation called inverse f, denoted f^{-1} such that $f \circ f^{-1} = f^{-1} \circ f = I$ (the identity permutation on S)

where, $f^{-1} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$

Cyclic Permutation

Let $S = \{a_1, a_2, \dots, a_n\}$ be a finite set of n symbols. A permutation f defined or S is said to be cyclic permutation if f is defined such that

$$f(a_1) = a_2,\ f(a_2) = a_3,\ f(a_3) = f(a_4), \dots f(a_{n-1}) = a_n \text{ and}$$
$$f(a_n) = f(1)$$

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Example let $S=\{1,2,3,4\}$

Then $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$ is cyclic permutation.

Q(1) If $A=\{1,2,3,4,5,6\}$

Compute $(5\ 6\ 3)\circ(4\ 1\ 3\ 5)$

Solⁿ

$$(4\ 1\ 3\ 5)=\begin{pmatrix}1&2&3&4&5&6\\3&2&5&1&4&6\end{pmatrix}$$

$$(5\ 6\ 3)=\begin{pmatrix}1&2&3&4&5&6\\1&2&5&4&6&3\end{pmatrix}$$

$$(5\ 6\ 3)\circ(4\ 1\ 3\ 5)=\begin{pmatrix}1&2&3&4&5&6\\1&2&5&4&6&3\end{pmatrix}\circ\begin{pmatrix}1&2&3&4&5&6\\3&2&5&1&4&6\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6\\5&2&6&1&4&3\end{pmatrix}$$

Q.(2) Let $A=\{1,2,3,4,5\}$

find $(1\ 3)\circ(2\ 4\ 5)\circ(2\ 3)$

Solⁿ

$$\begin{pmatrix}1&2&3&4&5\\3&2&1&4&5\end{pmatrix}\circ\begin{pmatrix}1&2&3&4&5\\1&4&3&5&2\end{pmatrix}\circ\begin{pmatrix}1&2&3&4&5\\1&3&2&4&5\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5\\3&4&1&5&2\end{pmatrix}\circ\begin{pmatrix}1&2&3&4&5\\1&3&2&4&5\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4&5\\3&1&4&5&2\end{pmatrix}$$

$$(1\ 3\ 4\ 5\ 2)$$

Q Show that $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 3 & 1 & 8 & 5 & 6 & 2 & 4 \end{pmatrix}$ is even

Solⁿ $f = \begin{pmatrix} 1 & 7 & 2 & 3 & 4 & 8 & 5 & 6 \\ 7 & 2 & 3 & 1 & 8 & 4 & 5 & 6 \end{pmatrix}$

$= (1\ 7\ 2\ 3) \circ (4\ 8) \circ \underline{(5) \circ (6)}$ → Ignore it because does not get pair

$= (1\ 7) \circ (1\ 2) \circ (1\ 3) \circ (4\ 8)$

f is expressed as product of 4 transpositions, therefore f is even.

Before Above Example.

$f = (a_1\ a_2\ \dots\ a_n)$

product of transpositions is

$f = (a_1, a_2) \circ (a_1\ a_3) \circ \dots \circ (a_1\ a_n)$

i.e. cycle of length can be expressed as product of $(n-1)$ transpositions.