Recurrence Relation

It is an equation that recursively defines a sequence where the next term is a function of ~~per~~ previous ~~term~~ term.

Recurrence relations have applications in many areas of mathematics:

→ The ~~p~~ fibonacci series

→ Euler's method

→ distribution of objects into bins.

Sometimes, a recurrence relation can be solved by defining the terms of a sequence in terms of its index rather than previous terms in the sequence.

This gives a closed form expression for each term in the sequence and eliminates the need for an iterative process to solve for terms in the sequence.

How to solve recurrence relation

Step 1: Define a base class.

Step 2: Develop more complicated cases and analyzes it.

Step 3: Write the recurrence relation of given problem

Q Find a recurrence relation and initial conditions for 1, 5, 17, 53, 161, 6485, ...

Solⁿ Step 1 : Initial condition is 1

Step 2 : Then look it 5, 17, 53, 161, 485, ...
Consider the series factor of 3. from previous one.

$1 \cdot 3 = 3$
$5 \cdot 3 = 15$
$17 \cdot 3 = 51$
$\vdots$

It appears that it always end up with 2 less than next term.

Step 3 : So $a_n = 3a_{n-1} + 2$ is our reccurrence relation & initial condition is $a_0 = 1$

Ex Check that $a_n = 2^n + 1$ is a solution to recurrence relation $a_n = 2a_{n-1} - 1$ with $a_1 = 3$

Solⁿ Firstly check initial condition.

$$a_1 = 2^1 + 1 = 3$$

Then, To check that our proposed solution satisfies the recurrence relation.

$$\begin{aligned} 2a_{n-1} - 1 &= 2(2^{n-1} + 1) - 1 \\ &= 2^n + 2 - 1 \\ &= 2^n + 1 \\ &= a_n \end{aligned}$$

Yes, it have a solution of our relation relation

Ex Solve the recurrence relation, write $a_n = a_{n-1} + n$ with initial term $a_0 = 4$.

Sol^n Write first few terms of the sequence from recurrence relation

$$a_0 = 4$$
$$a_1 = a_0 + 1 = 4 + 1 = 5$$
$$a_2 = a_1 + 2 = 5 + 2 = 7$$
$$a_3 = a_2 + 3 = 7 + 3 = 10$$
$$a_4 = a_3 + 4 = 10 + 4 = 14$$
$$a_5 = a_4 + 5 = 14 + 5 = 19$$

$\therefore$ Sequence are 4, 5, 7, 10, 14, 19, ...

Consider $a_n = a_{n-1} + n$

$$\boxed{n = a_n - a_{n-1}}$$

This gives difference between terms i.e. 'n'

$$a_1 - a_0 = 1$$
$$a_2 - a_1 = 2$$
$$a_3 - a_2 = 3$$
$$a_4 - a_3 = 4$$
$$\vdots$$
$$a_n - a_{n-1} = n$$

Adding all these equation

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) = 1 + 2 + 3 + \ldots + n$$

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) = \frac{n(n+1)}{2}$$

$$-a_0 + a_n = \frac{n(n+1)}{2}$$

$$\therefore a_n = a_0 + \frac{n(n+1)}{2}$$

Where $a_0 = 4$, then $\boxed{a_n = 4 + \frac{n(n+1)}{2}}$

Characteristic Roots

This is another method for solving recurrence relations.

Given a recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$

The characteristic polynomial is

$$x^2 + \alpha x + \beta$$

giving the characteristic equation:

$$~~x^2~~ x^2 + \alpha x + \beta = 0$$

If r_1 & r_2 are two distinct roots of characteristic polynomial, then the solution to the recurrence relation is

$$a_n = a r_1^n + b r_2^n ,$$

where a & b are constants determined by initial conditions.

Ex Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ & $a_1 = 3$

Sol^n Rewrite the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Now form characteristic equation:

$$x^2 - 7x + 10 = 0$$

and solve for x

$$(x-2)(x-5) = 0$$

$$\boxed{x = 2, 5}$$

$r_1 = 2$ & $r_2 = 5$ are the characteristic roots

Therefore, the solution to recurrence relation will have the form

$$a_n = a2^n + b5^n$$

To find a & b, put $n=0$ & $n=1$ to get a system of two equations with two unknowns:

$$a_0 = 2 = a2^0 + b5^0 = a+b$$

$$a_1 = 3 = a2^1 + b5^1 = 2a+5b$$

Solve these two equation to find a & b.

$$a+b = 2 \text{ — } ① \times 2$$

$$2a+5b = 3 \text{ — } ②$$

$$\cancel{2a}+2b = 4 \text{ — } ③$$

$$\pm \cancel{2a} \pm 5b = \pm 3 \text{ — } ②$$

$$-3b = 1$$

$$\boxed{b = -1/3}$$

but $b = -1/3$ in eqn ①.

$$a - \frac{1}{3} = 2$$

$$a = \frac{2}{1} + \frac{1}{3}$$

$$a = \frac{6+1}{3} = \frac{7}{3}$$

$$\boxed{a = 7/3}$$

∴ The solution of recurrence relation is

$$\boxed{a_n = \frac{7}{3}2^n - \frac{1}{3}3^n}$$