Algebraic Structures (Algebraic System)

Definition - If A is a set and $*$ is a binary operation on A, then $(A, *)$ is called an Algebraic Structure.

Ex - let R be the set of real numbers, then $(R, +)$ is an Algebraic structure.

Some of the most fundamental algebraic structures — groupoids, semi-groups, monoids, groups and rings.

① Groupoids

If $*$ is a binary operation on a non-empty set A, such that $a * b \in A$ for all $a, b \in A$ then we say that A is closed under the operation.

Ex - If $A = \{0, 1\}$, then A is closed under multiplication

Sol we have $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 0 = 0$, $1 \times 1 = 1$.

But A is not closed under Addition

we have $1 + 1 = 2 \notin A$.

A Groupoid is an algebraic structure consisting of non-empty set A and a binary operation $*$, which is such that A is closed under $*$.

Ex - The set of real numbers is closed under addition. therefore, $(R, +)$ is a groupoid.

② Semi-Group

Let S be a non-empty set and $*$ be a binary operation on S. The algebraic $(S, *)$ is called semi-group if operation $*$ is associative. In other words, the groupoid is ~~a~~ semi-group if $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

Thus, a semi-group requires the following:

(i) A set S

(ii) A binary operation $*$ defined on element of S.

(iii) Closure, $a*b$ whenever $a,b \in S$

(iv) Associativity $(a*b)*c = a*(b*c)$ for all $a,b,c \in S$

Ex let N be a set of Natural numbers. Then $(N,+)$ and $(N,*)$ are semi-groups.

Homomorphism of Semi-Groups

Let $(S,*)$ and ~~T~~ $(T,\circ)$ be any two semi-groups. A mapping $f: S \to T$ such that for any two element $a,b \in S$.

$$f(a*b) = f(a) \circ f(b)$$

is called a semi-group homomorphism.

A homomorphism of a semi-group into itself is called a semi-group endomorphism.

Isomorphism of semi-groups

Let $(S,*)$ and $(T,\circ)$ by any two semi-groups. A homomorphism $f: S \to T$ is called semi-group isomorphism if f is one-to-one and onto.

If $f: S \to T$ is an isomorphism then $(S,*)$ and $(T,\circ)$ are said to be isomorphic.

An isomorphism of a semi-group onto itself is called semi-group automorphism.

Theorem: Let $(S, *)$, $(T, \circ)$ and (V, Δ) be semi-groups $f: S \to T$ and $g: T \to V$ be semi-group homomorphism. Then $gof: S \to V$ is a semi-group homomorphism from $(S, *)$ to (V, Δ).

Proof – Let $a, b \in S$ then

$$\begin{aligned} gof(a * b) &= g[f(a * b)] \\ &= g[f(a) \circ f(b)] \\ &= g(f(a)) \,\Delta\, g(f(b)) \\ &= gof(a) \,\Delta\, gof(b) \end{aligned}$$

Hence, $gof: S \to V$ is a semi-group homomorphism