Algebraic Structures

Binary Operation

Let S be a non-empty set. If $f: S \times S \to S$ is a mapping Then f is called a binary operation or binary composition in S.

The Symbols $+, \cdot, *, \oplus$ etc are used to denote binary operations on a set.

→ for $a, b \in S \Rightarrow a+b \in S \Rightarrow +$ is a binary operation in S.

→ for $a, b \in S \Rightarrow a \cdot b \in S \Rightarrow \cdot$ is a binary operation in S.

→ for $a, b \in S \Rightarrow a * b \in S \Rightarrow *$ is a binary operation in S.

→ for $a, b \in S \Rightarrow a \circ b \in S \Rightarrow \circ$ is a binary operation in S.

This is said to the Closure property of binary operation and the set S is said to be closed with respect to binary operation.

Ex '$\circ$' is operation defined on Z such that $a \circ b = a+b-ab$ for $a, b \in Z$. Is the operation '$\circ$' a binary operation in Z? If so, is it associative and commutative in Z?

Soln: If $a, b \in Z$, we have $a+b \in Z$, $ab \in Z$ and $a+b-ab \in Z$

$$\Rightarrow a \circ b = a+b-ab \in Z$$

$\therefore$ '$\circ$' is a binary operation in Z

$$\Rightarrow a \circ b = b \circ a$$

$\therefore$ '$\circ$' is commutative in Z

Now,
$$\begin{aligned}(a \circ b) \circ c &= (a \circ b) + c - (a \circ b)c \\ &= a+b-ab+c-(a+b-ab)c \\ &= a+b-ab+c-ac-bc+ab\end{aligned}$$

and

$$a\circ(b\circ c) = a+(b\circ c) - a(b\circ c)$$
$$= a+b+c-bc-a(b+c-bc)$$
$$= a+b+c-bc-ab-ac+abc$$

$$\Rightarrow (a\circ b)\circ c = a\circ(b\circ c)$$

$\therefore$ 'o' is associative in Z.

Q Fill in the blanks in the following composition table so that 'o' is associative in $S=\{a,b,c,d\}$

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d				

Solⁿ (i) $d\circ a = (c\circ b)\circ a \quad [\because c\circ b = d]$
$$= c\circ(b\circ a) \; [\text{associative Law}]$$
$$= c\circ b$$
$$= d$$

(ii) $d\circ b = (c\circ b)\circ b = c\circ(b\circ b) = c\circ a = c$

(iii) $d\circ c = (c\circ b)\circ c = c\circ(b\circ c) = c\circ c = c$

(iv) $d\circ d = (c\circ b)\circ(c\circ b) = c\circ(b\circ c)\circ b$
$$= c\circ c\circ b$$
$$= c\circ(c\circ b)$$
$$= c\circ d$$
$$= d.$$

Hence, the required composition table is

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	c	c	d

Theorem – The identity elements of a binary operation $*$ in a set A, if it exists and is unique.

Proof – If possible, let e' and e'' be two identity element in A with respect to binary operation $*$ e' is an identity element in A

$$\Rightarrow e' * e'' = e'' * e' = e''$$

and e'' is an identity element in A

$$\Rightarrow e' * e'' = e''e' = e'$$

Which together show that $e' = e''$

Theorem If $*$ is an associative binary operation in A, Then the inverse of every invertible element is unique.

proof – Let $a \in A$, be an invertible element with respect to $*$, If possible let b and c be two distinct inverses of the element a in A.

Let e be identity elements in A with respect to $*$

Then we have

$$b * a = a * b = e$$

and, $$c * a = a * c = e$$

now $(b * a) * c = b * (a * c)$ ($\because$ $*$ is associative in A)

$$\Rightarrow e * c = b * e$$

$$c = b$$

This completes the proof of the theorem.

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