OM

Singular Value Decomposition : (SVD)

- SVD is a matrix factorization technique commonly used in linear algebra
- SVD of a matrix $A(m \times n)$ is a factorisation of the form :

$$A = U \Sigma V$$

U: $m \times m$ unitary matrix

Σ: $m \times n$ rectangular diagonal matrix

V: $n \times n$ unitary matrix

- U & V are orthogonal matrices.
- The diagonal entries of Σ are known as singular value of A matrix
- The columns of U & V are called left singular and right singular vectors of matrix A, resp.
- SVD is generally used in PCA, once the mean of each variable has been removed since it is not advisable to remove the mean of a data attribute, especially when the data set is sparse (as in case of text data)
- SVD is a good choice for dimensionality reduction in these situations.

* SVD of a data matrix is expected to have following properties

(a) Patterns in the attribute are captured by right - singular vectors, i.e the columns of V.

(b) Patterns among the instances are captured by left- singular vectors, i.e the columns of U.

(c) Larger a singular value, larger is the part of matrix A, that it accounts for its associated vectors.

(d) New, data matrix with 'k' attributes is obtained using the equation

$$D' = D \times [V_1, V_2 \ldots\ldots V_k]$$

Thus, the Dimensionality gets reduced to K.

SVD - Algorithm steps :-

Step-1 : Compute the transpose (A^T) of given matrix 'A'. Also compute A^TA.

Step-2 : Determine the eigen values of A^TA and sort these in descending order, in the absolute sense.

↳ Singular values (σ) will be obtained as square root of these eigen values.

Step-3 : Construct diagonal matrix 'S' by placing singular value in descending order along its diagonal. Compute its inverse also as S^{-1}.

Step 4: Use the ordered eigen values from step-2 compute the eigen vectors of ATA. place these eigen vectors along the columns of V and compute its transpose, V^T.

Step 5: Compute U as $U = AVS^{-1}$

Compute full SVD using $A = USV^T$

Problem: 1 Find singular value Decomposition (SVD) of matrix $\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$

Sol

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

Eigen values of $A \cdot A^T$ will be :-

$$|A \cdot A^T - \lambda I| = 0$$

$$\begin{bmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{bmatrix} = 0$$

$$(16-\lambda)(34-\lambda)^2 - (12 \times 12) = 0$$

$$544 - 50\lambda + \lambda^2 - 144 = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

$$(\lambda - 10)(\lambda - 40) = 0$$

$$\lambda = 10, 40$$

Eigen vectors for $\lambda = 40$

$$(A \cdot A^T - \lambda I) U_1 = \left[\begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}\right]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 12 \\ 12 & 6 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$-24 U_1 + 12 U_2 = 0 \quad \text{—— (1)}$$

$$12 U_1 - 6 U_2 = 0 \quad \text{—— (2)}$$

from eq (1)

$$U_1 = 0.5 U_2$$

$$U_1 = \begin{bmatrix} 0.5 U_2 \\ U_2 \end{bmatrix} \quad \text{for} \quad U_2 = 1$$

$$U_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

we have to do normalisation :-

$$U_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{0.5}{\sqrt{0.5^2 + 1^2}} \\ \frac{1}{\sqrt{0.5^2 + 1^2}} \end{bmatrix}$$

$$\bar{U_1} = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix}$$

Similarly for $\lambda = 10$

$$[A \cdot A^T - \lambda I] U_2 = \left[\begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right] U_2$$

$$= \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$6 U_1 + 12 U_2 = 0 \quad \text{—— (1)}$$

$$12 U_1 + 24 U_2 = 0 \quad \text{—— (2)}$$

By eq - 1 $\quad U_1 = -2U_2 \quad\quad U_1 = -2U_2$

$$U_2 = \begin{bmatrix} -2U_2 \\ U_2 \end{bmatrix} \quad \text{for } U_2 = 1$$

$$U_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

After normalisation,

$$\overline{U_2} = \begin{bmatrix} \dfrac{-2}{\sqrt{-2^2 + 1^2}} \\ \dfrac{1}{\sqrt{-2^2 + 1^2}} \end{bmatrix}$$

$$\overline{U_2} = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix}$$

Similar calculations will be done for V

$$A^T A = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

For Eigen values :-

$$[A^T A - \lambda I] = 0$$

$$\begin{bmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{bmatrix} = 0$$

$$(25-\lambda)(25-\lambda) - (-15)(-15) = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

$$\lambda = 10, 40$$

Eigen vectors for $\lambda = 40$

$$(A^T A - \lambda I) V_1 = \left[\begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}\right] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$-15V_1 - 15V_2 = 0 \quad \text{— (1)}$$

$$-15V_1 - 15V_2 = 0 \quad \text{— (2)}$$

from eq (1)

$$V_1 = -V_2$$

$$V_1 = \begin{bmatrix} -V_2 \\ -V_2 \end{bmatrix}$$

for $V_2 = 1$, $\quad V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalisation}} \bar{V}_1' = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$

Eigen vector for $\lambda = 10$

$$(A^TA - \lambda I)V_2 = \left[\begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\right]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 0 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$15V_1 - 15V_2 = 0 \quad \text{— (1)}$$

$$-15V_1 + 15V_2 = 0 \quad \text{— (2)}$$

$$V_1 = V_2$$

$V_2 = \begin{bmatrix} V_2 \\ V_2 \end{bmatrix}$ for $V_2 = 1$,

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalisation}} \bar{V}_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

Now for Σ

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 6.3425 & 0 \\ 0 & 3.1622 \end{bmatrix}$$

$$\therefore\ U = [\bar{U}_1 \ \ \bar{U}_2] = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

[Alternatively
V is found as
$V_i = \frac{1}{\sigma_i} A^T . U_i$]

$$V = [\bar{V}_1 \ \ \bar{V}_2] = \begin{bmatrix} 0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}$$

[Alternatively
U is found as
$U_i = \frac{1}{\sigma_i} A_i . V_i$]

Verfication : $A = U \Sigma V^T$

$$U * \Sigma = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} \begin{bmatrix} 6.3246 & 0 \\ 0 & 3.1623 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8284 & -2.8284 \\ 5.6569 & 1.4142 \end{bmatrix}$$

$$U . \Sigma . V^T = \begin{bmatrix} 2.8284 & -2.8284 \\ 5.6569 & 1.4142 \end{bmatrix} \begin{bmatrix} 0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Hence
Verified