Generating Function

Generating function is a method to solve the recurrence relations.

Let us consider, the sequence $a_0, a_1, a_2, \dots a_r$ of real numbers. For some interval of real numbers containing zero values at t is given, the function $G(t)$ is defined by

$$G(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_r t^r \quad \text{——} \quad (1)$$

This function $G(t)$ is called generating function of Sequence a_r.

Now, for constant sequence $1, 1, 1, 1, \dots$ then generating function is

$$G(t) = \frac{1}{(1-t)}$$

It can be expressed as

$$G(t) = (1-t)^{-1} = 1 + t + t^2 + t^3 + \dots$$

[By Binomial Expansion]

For constant sequence $1, 2, 3, 4, 5, \dots$ the generating function is

$$G(t) = \frac{1}{(1-t)^2} = (1-t)^{-2} = 1 + 2t + 3t^2 + 4t^3 + \dots + (r+1)t^r$$

The generating function of z^r ($z \neq 0$ & z is a constant) is given by —

$$G(t) = 1 + zt + z^2 t^2 + z^3 t^3 + \dots + z^r t^r$$

$$G(t) = \frac{1}{1-zt} \quad [\text{Assume } |zt|<1]$$

Also, if $a_r^{(1)}$ has the generating function $G_1(t)$ and $a_r^{(2)}$ has the generating function $G_2(t)$ then $\lambda_1 a_r^{(1)} + \lambda_2 a_r^{(2)}$ has the generating function $\lambda_1 G_1(t) + \lambda_2 G_2(t)$. Here λ_1 & λ_2 are constants.

Application Areas

Generating functions can be used for the following purposes —

- For solving recurrence relations
- For proving some of combinatorial identities.
- For finding asymptotic formulae for terms of sequences.

Q Solve recurrence relation by generating function

$$a_r - 2a_{r-1} - 3a_{r-2} = 0 \quad \text{for } r \geq 2,\ a_0 = 3 \ \&\ a_1 = 1$$

Sol^n (i) Multiply both ~~sis~~ side by z^r

$$a_r z^r - 2a_{r-1} z^r - 3a_{r-2} z^r = 0$$

(ii) Since $r \geq 2$ by summing for all r we get

$$\underbrace{\sum_{r=2}^{\infty} a_r z^r}_{\text{I}^{st}\text{ term}} - 2\underbrace{\sum_{r=2}^{\infty} a_{r-1} z^r}_{\text{II}^{nd}\text{ term}} - 3\underbrace{\sum_{r=2}^{\infty} a_{r-2} z^r}_{\text{III}^{rd}\text{ term}} = 0 \quad \text{—— (1)}$$

$$\because \boxed{A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots} \quad \text{Standard equation}$$

where $a_1, a_2, a_3 \ldots$ are constants

I^st term $$\sum_{r=2}^{\infty} a_r z^r = a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots$$

$$= A(z) - a_0 - a_1 z \quad \text{—— (2)}$$

[On Comparing Standard eq^n]

$\underline{\text{II}^{nd}\ \text{term}}$

$$\sum_{r=2}^{\infty} a_{r-1} z^r = a_1 z^2 + a_2 z^3 + a_3 z^4$$
$$= (a_1 z^1 + a_2 z^2 + a_3 z^3) z$$
$$= (A(z) - a_0) z$$

$\underline{\text{III}^{rd}\ \text{term}}$

$$\sum_{r=2}^{\infty} a_{r-2} z^r = a_0 z^2 + a_1 z^3 + a_2 z^4 + a_3 z^5 + \dots$$
$$= [a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots] z^2$$
$$= A(z) z^2$$

This terms is ~~substitute~~ substitute in eq^n (1)

$$(A(z) - a_0 - a_1 z) - 2z(A(z) - a_0) - 3A(z)z^2 = 0 \quad \text{—— (3)}$$

Put initial value $a_0 = 3$, $a_1 = 1$ ~~the~~ in eq^n (3)

$$(A(z) - 3 - z) - 2z(A(z) - 3) - 3A(z)z^2 = 0$$

$$(1 - 2z - 3z^2) A(z) - 3 - z + 6z = 0$$

$$(1 - 2z - 3z^2) A(z) - 3 + 5z = 0$$

$$(1 - 2z - 3z^2) A(z) = 3 - 5z$$

$$A(z) = \frac{3 - 5z}{1 - 2z - 3z^2}$$

$$A(z) = \frac{3 - 5z}{(1 - 3z)(1 + z)} \quad \text{—— (4)}$$

$$= \frac{A}{1 - 3z} + \frac{B}{(1 + z)} \quad \text{—— (5) [By Partial function]}$$

$$= \frac{A(1 + z) + B(1 - 3z)}{(1 - 3z)(1 + z)}$$

$$= \frac{(A + B) + (A - 3B)z}{(1 - 3z)(1 + z)} \quad \text{—— (6)}$$

Compare ~~it~~ eq^n (4) & (5)

$$\cancel{A}+B = 3 \quad \text{---} (7)$$

$$\begin{array}{r} \pm \cancel{A} \mp 3B = \mp 5 \quad \text{---} (8) \\ \hline 4B = 8 \end{array}$$

$$\boxed{B=2}$$

Put $B=2$ in eq^n (7)

$$A+2=3$$

$$\boxed{A=1}$$

Put the value of A & B in eq (5)

$$A(z) = \frac{1}{1-3z} + \frac{2}{1+z}$$

∴ ~~Get~~

Thus, $G(t) = a_r = 1.(3)^r + 2.(-1)^r$

$$\boxed{a_r = 3^r + 2.(-1)^r}$$