Shannon - Hartley theorem (Channel Capacity of a Gaussian Channel)

Channel with Gaussian noise characteristics are known as Gaussian channel. The result obtained for a Gaussian channel often provide a lower bound on the performance of a system with the non-gaussian channel.

For a Gaussian channel, probability density function $P(x)$

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$$

Hence $$H(x) = -\int_{-\infty}^{\infty} P(x) \log P(x)\, dx$$

$$= -\int P(x) \log P(x)\, dx$$

But $$-\log P(x) = -\log\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\right)$$

$$= -\log \frac{1}{\sqrt{2\pi\sigma^2}} - \log e^{-x^2/2\sigma^2}$$

$$= \log\sqrt{2\pi\sigma^2} + \frac{x^2}{2\sigma^2}\log e.$$

$$H(x) = \int_{-\infty}^{\infty}\left(\log\sqrt{2\pi\sigma^2} + \frac{x^2}{2\sigma^2}\log e\right) P(x)\, dx$$

$$= \int_{-\infty}^{\infty} \log\sqrt{2\pi\sigma^2}\, P(x)\, dx + \int_{-\infty}^{\infty} \frac{x^2}{2\sigma^2}\log e\, P(x)\, dx$$

$$= \log\sqrt{2\pi\sigma^2} + \frac{\log e}{2\sigma^2}\int_{-\infty}^{\infty} x^2 P(x)\, dx$$

$$= \log\sqrt{2\pi\sigma^2} + \frac{1}{2}\log e = \log\sqrt{2\pi\sigma^2} + \log\sqrt{e}$$

$$= \log\sqrt{2\pi e\sigma^2}$$

Hartley theorem (Channel Capacity of a Gaussian C

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The Rate of information.

$$R(x) = 2B\, H(x)$$
$$= 2B \log \sqrt{2\pi e \sigma^2}$$
$$= B \log \left(\sqrt{2\pi e \sigma^2}\right)^2$$
$$= B \log(2\pi e \sigma^2)$$

If $P(x)$ is a bandlimited Gaussian noise with an average noise power N, then

$$R(n) = B \log(2\pi e N).$$

If the Received signal is composed of a transmitted signal x and noise n, then the joint entropy of the source and noise is given by.

$$R(x, n) = R(x) + R(n/x)$$

Assume that signal and noise are independent.

$$R(x, n) = R(x) + R(n)$$

Since the received signal is the sum of the transmitted signal x and the noise n, we may equate

$$H(x, y) = H(x, n)$$
$$H(y) + H(x/y) = H(x) + H(n).$$
$$R(y) + R(x/y) = R(x) + R(n)$$
$$R(x) - R(x/y) = R(y) - R(n)$$

Rate of Information from channel

$$R = R(x) - R(x/y)$$
$$= R(y) - R(n)$$

The Rate of information.

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Rate of Information from channel

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The Rate of information.

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$$R(x) - R(x/y) = R(y) - R(n)$$

Rate of Information from channel

$$R = R(x) - R(x/y)$$
$$= R(y) - R(n)$$

Channel Capacity $C = \text{Max } R$

$$= \text{Max}\left[R(y) - R(n)\right]$$

Maximizing R, require Maximizing R(y).

$$R(y) = B \log\left[2\pi e(S+N)\right]$$

$$R(n) = B \log\left[2\pi e N\right]$$

$$C = B \log\left[2\pi e(S+N)\right] - B \log\left[2\pi e N\right]$$

$$= B \log \frac{S+N}{N}$$

$$\boxed{C = B \log\left(1 + \frac{S}{N}\right)}$$

where B = Band width of channel.

S = Average signal power

N = average noise power $= \cancel{\text{Aut}}\ \frac{\eta}{2} \times 2B$

$$= \eta B$$

$$\boxed{C = B \log_2\left(1 + \frac{S}{\eta B}\right)}$$

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SNR Bandwidth Trade OFF

$$C = B\log_2\left(1+\frac{S}{N}\right)$$

$\frac{S}{N} = 15$ and $B = 5\,KHz$

$C = B\log\left(1+\frac{S}{N}\right)$

$= 5k\log(1+15)$

$= 5k\log 16$

$= 5k\log_2 2^4$

$= 5\times 4$

$= 20$ Kbps.

$\frac{S}{N} = 31$, $B = ?$ $C = 20$

$C = B\log\left(1+\frac{S}{N}\right)$

$20 = B\log(1+31)$

$20 = B\log 32$

$20 = B\log_2 2^5$

$20 = B\times 5\log_2 2$

$\frac{20}{5} = B\cancel{\log_2 2}$

$B = 4$ ~~kbps~~ KHz.

$\frac{S}{N} = 15$ $\qquad$ $\frac{S}{N} = 31$

$B = 5\,KHz$ $\qquad$ $B = 4\,KHz$.

Case I: 20% reduction in the B.W. (5KHz to 4KHz) require a 100% in the signal power to Noise Ratio.

Case II: 50% reduction in the SNR (from 31 to 15)

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$B = 4\,\cancel{Kbp}\,KHz.$

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Case I: 20% reduction in the B.W. (5 KHz to 4 KHz) require a 100% in the signal power to Noise Ratio.

Case II: 50% reduction in the SNR (from 31 to 15)

Infinite Bandwidth channel.

Channel ~~Bandwidth~~ Capacity of Infinite Bandwidth channel

$$C = B\log_2\left(1+\frac{S}{N}\right)$$

$N =$ Noise power

$$= \frac{n}{2} \times 2B = nB$$

$$C = B\log_2\left(1+\frac{S}{nB}\right)$$

$$= \frac{B}{\log_e 2}\log_e\left(1+\frac{S}{nB}\right).$$

If Bandwidth Infinite

$$C_\infty = \lim_{B\to\infty} \frac{B}{\log_e 2}\log_e\left(1+\frac{S}{nB}\right).$$

$$= \lim_{B\to\infty} \frac{B}{\log_e 2}\left(\frac{S}{nB} - \frac{1}{2}\left(\frac{S}{nB}\right)^2 + \frac{1}{3}\left(\frac{S}{nB}\right)^3 + \cdots\right)$$

$$= \lim_{B\to\infty} \frac{\cancel{B}}{\log_e 2} \times \frac{S}{n\cancel{B}}\left(1 - \frac{1}{2}\left(\frac{S}{nB}\right) + \frac{1}{3}\left(\frac{S}{nB}\right)^2 + \cdots\right)$$

$$= \frac{S}{n\log_e 2}\left(1 - 0 + 0 + \cdots\right)$$

$$\boxed{C_\infty = 1.44\frac{S}{n}}$$

Channel ~~Bandwidth~~ Capacity of Infinite Bandwidth channel.

$$C = B\log_2\left(1+\frac{S}{N}\right)$$

$N =$ Noise power

$$= \frac{\eta}{2} \times 2B = \eta B$$

$$C = B\log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{B}{\log_e 2}\log_e\left(1+\frac{S}{\eta B}\right).$$

If Bandwidth Infinite

$$C_\infty = \lim_{B\to\infty}\frac{B}{\log_e 2}\log_e\left(1+\frac{S}{\eta B}\right).$$

$$= \lim_{B\to\infty}\frac{B}{\log_e 2}\left(\frac{S}{\eta B} - \frac{1}{2}\left(\frac{S}{\eta B}\right)^2 + \frac{1}{3}\left(\frac{S}{\eta B}\right)^3 + \cdots\right)$$

$$= \lim_{B\to\infty}\frac{\cancel{B}}{\log_e 2} \times \frac{S}{\eta \cancel{B}}\left(1 - \frac{1}{2}\left(\frac{S}{\eta B}\right) + \frac{1}{3}\left(\frac{S}{\eta B}\right)^2 + \cdots\right)$$

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