Singular Value Decomposition :- (SVD)

→ SVD is a matrix factorization technique commonly used in linear algebra.

→ SVD of a matrix $A(m \times n)$ is a factorisation of the form:

$$A = U \Sigma V$$

U: $m \times m$ unitary matrix

Σ: $m \times n$ rectangular diagonal matrix

V: $n \times n$ unitary matrix

[U & V are orthogonal matrices]

→ The diagonal entries of Σ are k/as singular values of A matrix.

→ The columns of U & V are c/d left-singular & right singular vectors of matrix A, respectively.

→ SVD is generally used in PCA, once the mean of each variable has been removed. Since it is not advisable to remove the mean of a data attribute, especially when the data set is a sparse (as in case of text data).

→ SVD is a good choice for dimensionality reduction in those situations.

⊛ SVD of a data matrix is expected to have following properties:-

a) Patterns in the attributes are captured by right-singular vectors, i.e. the columns of V.

b) Patterns among the instances are captured by left-sing vectors i.e. the columns of U.

c) Larger a singular value, larger is the part of matrix A that it accounts for & its associated vectors.

d) New, data matrix with 'k' attributes is obtained using the eqn, $D' = D \times [v_1, v_2 \; -- \; v_k]$

Thus, the dimensionality gets reduced to k.

SVD - Algorithm Steps :

Step 1 : Compute the transpose (A^T) of given matrix 'A'. Also compute A^TA.

Step 2 : Determine the eigenvalues of A^TA and sort these in descending order, in the absolute sense.

→ Singular values (σ) will be obtained as square roots of these eigen values.

Step 3 : Construct diagonal matrix 'S' by placing singular values in descending order along its diagonal. Compute its inverse also as S^{-1}.

Step 4 :- Use the ordered eigenvalues from step 2 and compute the eigenvectors of A^TA. Place these eigenvectors along the columns of V and compute its transpose, V^T.

Step 5 : Compute U as $U = AVS^{-1}$.

Compute full SVD using $A = USV^T$

SVD Problem

Problem:1 Find Singular value Decomposition (SVD) of Matrix $\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \quad \& \quad A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$A.A^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

Eigen values of $A.A^T$ will be :-

$$|A.A^T - \lambda I| = 0$$

$$\begin{bmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (16-\lambda)(34-\lambda) - 12 \times 12 = 0$$

$$\therefore 544 - 50\lambda + \lambda^2 - 144 = 0$$

$$\therefore \lambda^2 - 50\lambda + 400 = 0$$

$$\therefore (\lambda - 10)(\lambda - 40) = 0$$

$$\lambda = 10, 40$$

⊛ Eigen vectors for $\lambda = 40$

$$(A.A^T - \lambda I)u_1 = \left[\begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}\right] U_1 = \begin{bmatrix} -24 & 12 \\ 12 & -6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$-24u_1 + 12u_2 = 0 \quad -① \qquad \& \qquad 12u_1 - 6u_2 = 0 \quad -②$$

Solving it, $u_1 = 0.5u_2$

$$U_1 = \begin{bmatrix} 0.5u_2 \\ u_2 \end{bmatrix} \quad \text{for } u_2 = 1$$

$$U_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalisation}} \overline{U_1} = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix}$$

Similarly for $\lambda = 10$

$$[A \cdot A^T - \lambda I] U_2 = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} U_2 = \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$6u_1 + 12u_2 = 0$ — ① & $12u_1 + 24u_2 = 0$

Solving it, we get $u_1 = -2u_2$ $\qquad u_1 = -2u_2$

$$U_2 = \begin{bmatrix} -2u_2 \\ u_2 \end{bmatrix} \quad \text{for } u_2 = 1 \Rightarrow U_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

After normalisation, $\bar{U}_2 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix}$

Resultant $U = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$

Similar calculations will be done for V

$$A^T \cdot A = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

For eigen values:- $(A^T A - \lambda I) = 0$

$$\begin{bmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{bmatrix} = 0$$

$(25-\lambda)(25-\lambda) - (-15)(-15) = 0$

$\lambda^2 - 50\lambda + 400 = 0$

$\lambda = 10 \ \& \ 40$

Eigen vectors for $\lambda = 40$

$$(A^T \cdot A - \lambda I) V_1 = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} V_1 = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$-15v_1 - 15v_2 = 0$ — ① $\quad$ & $-15v_1 - 15v_2 = 0$ — ②

$-15v_1 = 15v_2$

$v_1 = -v_2$

$$V_1 = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix}$$

for $v_2 = 1$, $V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalisation}} \bar{V}_1 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$

Eigen vectors for $\lambda = 10$

$$(A^T.A - \lambda I) V_2 = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} V_2 = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$15v_1 - 15v_2 = 0$ — ① & $-15v_1 + 15v_2 = 0$ — ②

$v_1 = v_2$

$V_2 = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix}$ for $v_2 = 1$, $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalisation}} \bar{V}_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 6.3425 & 0 \\ 0 & 3.1622 \end{bmatrix}$$

$$\therefore U = [\bar{U}_1 \quad \bar{U}_2] = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

Alternatively, V is found as $V_i = \frac{1}{\sigma_i} A^T . u_i$

$$\therefore V = [\bar{V}_1 \quad \bar{V}_2] = \begin{bmatrix} +0.7071 & \mp 0.7071 \\ \mp 0.7071 & \mp 0.7071 \end{bmatrix}$$

Alternatively U is found as $U_i = \frac{1}{\sigma_i} A . V_i$

Verification :- $A = U \Sigma V^T$

$$U * \Sigma = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} . \begin{bmatrix} 6.3246 & 0 \\ 0 & 3.1623 \end{bmatrix} = \begin{bmatrix} 2.8284 & -2.8284 \\ 5.6569 & 1.4142 \end{bmatrix}$$

$$U.\Sigma.V^T = \begin{bmatrix} 2.8284 & -2.8284 \\ 5.6569 & 1.4142 \end{bmatrix} . \begin{bmatrix} 0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Hence ~~pro~~ verified