8. Verify that $0 \le H(x) \le \log_2 m$
when m is the size of the alphabet of X.

Solution

(i) Lower bound

since $0 \le P(x_i) \le 1$

$$\frac{1}{P(x_i)} \ge 1 \quad \text{and} \quad \log_2 \frac{1}{P(x_i)} \ge 0$$

thus

$$P(x_i) \log_2 \frac{1}{P(x_i)} \ge 0$$

$$\sum_{j=1}^{m} P(x_i) \log_2 \frac{1}{P(x_i)} \ge 0$$

$$H(x) \ge 0$$

(ii) Upper bound.

consider two probability distribution $P(x_i) = P_i$ and $Q(x_i) = Q_i$ on the alphabet x_i $\{i = 1, 2, \ldots m\}$ such that

$$\sum_{i=1}^{m} P_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{m} Q_i = 1$$

Now,
$$\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} = \frac{1}{\log_e 2} \sum_{j=1}^{m} P_i \log_e \frac{Q_i}{P_i}$$

we know that $\log_e \alpha \le \alpha - 1$, $a \ge 0$
equality holds only if $\alpha = 1$

so
$$\sum_{j=1}^{m} P_i \log_e \frac{Q_i}{P_i} \le \sum_{j=1}^{m} P_i \left(\frac{Q_i}{P_i} - 1\right)$$

$$\le \sum_{j=1}^{m} Q_i - P_i$$

$$\le \sum_{j=1}^{m} Q_i - \sum_{i=1}^{m} P_i$$

thus
$$\sum_{j=1}^{m} P_i \log_e \frac{Q_i}{P_i} \le 0$$

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Solution

(i) Lower bound

since $0 \le P(x_i) \le 1$

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$$\le \sum_{j=1}^{m} Q_i - \sum_{j=1}^{m} P_i$$

thus $$\sum_{j=1}^{m} P_i \log_e \frac{Q_i}{P_i} \le 0$$

when the equality holds only if $Q_i = P_i$ for all i

$$Q_i = \frac{1}{m} \quad , \quad i = 1, 2, \ldots m$$

$$\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} = \sum_{i=1}^{m} P_i \log_2 \frac{1}{P_i m}$$

$$= -\sum_{i=1}^{m} P_i \log_2 P_i - \sum_{i=1}^{m} P_i \log_2 m$$

$$= H(x) - \log_2 m \sum_{i=1}^{m} P_i$$

$$= H(x) - \log_2 m \le 0$$

$$H(x) \le \log_2 m$$

Hence $\cancel{\log_2}$. $H(x) \le \log_2 m$