Q Prove that if ${}^nC_x = {}^nC_y$ then either $x = y$ or $x + y = n$

Sol^n ${}^nC_x = {}^nC_y$

from this, $\boxed{x = y}$ — (1)

${}^nC_x = {}^nC_y$

${}^nC_x = {}^nC_{n-y}$ [W.K.T. ${}^nC_r = {}^nC_{n-r}$]

$\therefore x = n - y$

$\boxed{x + y = n}$ — (2)

from eq^n (1) & (2)

either $x = y$ or $x + y = n$

The number of subsets of a set A of n elements, of which p are alike of one kind, q are alike of a second kind, and r are alike of third kind, is

$$(p+1)(q+1)(r+1)\,2^{n-p-q-r}$$

$$\boxed{n \geq p+q+r}$$

Ex: Find the number of subsets of

$$A = \{2, 2, 2, 3, 3, 5, 11\}$$

Solⁿ: There are three 2's two 3's in the set. The remaining two elements i.e., 5 and 11 are distinct, we have $p = 3$, $q = 2$.

The number of subsets of A is

$$= (3+1)(2+1)\, 2^{7-3-2}$$

$$= 4 \times 3 \times 2^2$$

$$= 48$$

Ex: How many selections any number at a time, may be made from three white balls, four green balls, one red ball and one black ball, if atleast one must be chosen.

Solⁿ: Total number of balls = 3 white + 4 green + 1 red + 1 black

$$= 9$$

we have $p = 3$, $q = 4$

Hence the number of selections that can be made is

$$= (3+1)(4+1)\, 2^{9-7}$$

$$= 4 \times 5 \times 2^2$$

$$= 80 \text{ (This include the null set)}$$

If atleast are ball is to be chosen then the number of selections $= 80 - 1 = 79$

Ex. Show that $C(n, r) = C(n-1, r-1) + C(n-1, r)$

Solⁿ: we know that $C(n, r) = {}^nC_r = \frac{n!}{r!\, n-r!}$

$\therefore$ RHS

$$C(n-1, r-1) + C(n-1, r).$$

$$= {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

$$= \frac{n-1!}{r-1!\; n\cancel{-1}-r\cancel{+1}!} + \frac{n-1!}{r!\; n-1-r!}$$

$$= \frac{n-1!}{r-1!\; n-r!} + \frac{n-1!}{r!\; n-r-1!}$$

$$= \frac{(n-1)!}{(n-r).(r-1)!\,(n-r-1)!} + \frac{(n-1)!}{r.(r-1)!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{\cancel{r} + n - \cancel{r}}{r.(n-r)}\right]$$

$$= \frac{n.(n-1)!}{r.(r-1)!\,(n-r).(n-r-1)!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

$$= {}^nC_r \text{ or } C(n, r) = \underline{\underline{\text{LHS}}}$$

Hence Proved.

Binomial Theorem

Let n be a positive integers then for all a and b

$$(a+b)^n = C(n,0)a^n + C(n,1)a^{n-1}b + C(n,2)a^{n-2}b + \dots$$
$$\dots + C(n,r)a^{n-r}b^r + \dots + C(n,n)b^n$$

$$= \sum_{r=0}^{n} C(n,r)\, a^{n-r} \cdot b^r$$

Proof: let r denote the set of all positive integers, we use the identities:

$$C(n,r) + C(n,r+1) = C(n+1,r+1)$$
$$\text{and } C(k,k) = C(k+1,k+1) = 1.$$

Let S be the set of positive integers for which:

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)\, a^{n-r} \cdot b^r$$

Taking $n=1$, we obtaining

$$\sum_{r=0}^{1} C(1,r)\, a^{1-r}\, b^r = C(1,0)a^1 b^0 + C(1,1)a^0 b^1$$
$$= (a+b)$$

Assume that the theorem is true for $n=k$, i.e. $k \in S$

hence $(a+b)^k = \sum_{r=0}^{k} C(k,r)\, a^{k-r} \cdot b^r$

$$= C(k,0)a^k + C(k,1)a^{k-1}b + \dots + C(k,r-1)a^{k-r+1}b$$
$$+ C(k,r)a^{k-r}b^r + \dots + C(k,k)b^k.$$

Then

$$(a+b)^{k+1} = (a+b)(a+b)^k$$
$$= a(a+b)^k + b(a+b)^k$$
$$= C(k,0)a^{k+1} + C(k,1)a^k b + C(k,2)a^{k-1}b^2 + \dots +$$
$$C(k,r-1)a^{k-r+1}b^{r-1} + C(k,r)a^{k-r+1}b^r + \dots + C(k,k)ab^k$$
$$+ C(k,0)a^k b + C(k,1)a^{k-1}b^2 + C(k,2)a^{k-2}b^2 + \dots +$$
$$C(k,r-1)a^{k-r+1}b^r + C(k,r)a^{k-r+1}b^{r+1} + \dots + C(k,k)b^{k+1}$$

$$= C(k,0)a^{k+1} + [C(k,0)+C(k,1)]\, a^k b + [C(k,1)+C(k,2)]\, a^{k-1}b^2 + \ldots\ldots +$$
$$[C(k,r-1)+C(k,r)]\, a^{k-r+1}b^r + \ldots + C(k,k)\, b^{k+1}.$$

$$= C(k+1,0)\, a^{k+1} + C(k+1,1)\, a^k b + C(k+1,2)\, a^{k-1}b^2 + \ldots + C(k+1,r)\, a^{k-r+1}b^r$$
$$+ \ldots\ldots + C(k+1,k+1)\, b^{k+1}.$$

$$= \sum_{r=0}^{k+1} C(k+1,r)\, a^{k+1-r}\, b^r$$

Therefore, the theorem is true for $x=k+1$ also i.e., $k+1 \in S$, if $k \in S$.

Hence $S = N$.

i.e., by principle of mathematical induction the theorem is true for all positive integers &

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)\, a^{n-r}.\, b^r \text{ for all } n \in N.$$