Shannon - limit (Rate/ B.W. & signal to Noise Ratio Trade off).

$$C = B\log_2(1 + S/N)$$

trammitted power $S = E_b C$ wher E_b is trammitted Energy per bit and C is channel capacity bits/sec

and Noise power $N = \frac{n}{2} \times 2B$

$$= nB$$

therefore above Equation becomes

$$C = B\log_2\left(1 + \frac{E_b C}{nB}\right)$$

$$\frac{C}{B} = \log_2\left(1 + \frac{E_b}{n} \times \frac{C}{B}\right)$$

$$\frac{C}{B} = \frac{1}{\log_e 2} \log_e\left(1 + \frac{E_b}{n} \times \frac{C}{B}\right)$$

$$\frac{C}{B}\log_e 2 = \log_e\left(1 + \frac{E_b}{n} \times \frac{C}{B}\right)$$

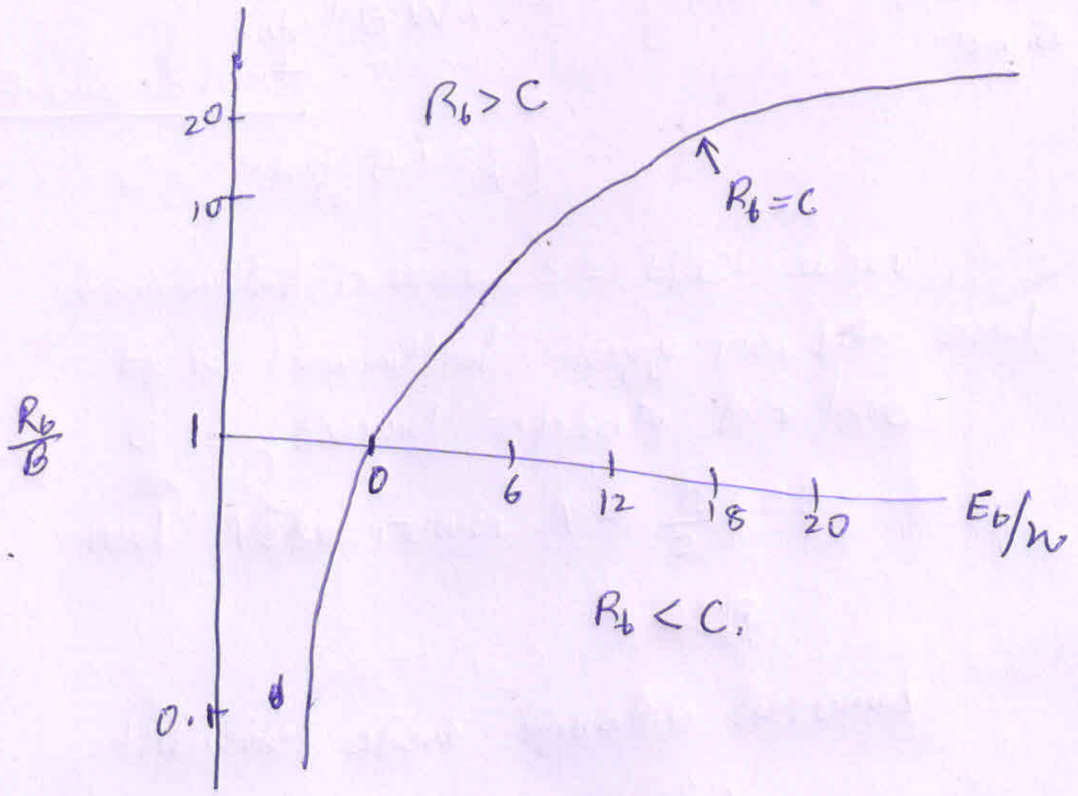
$$\log_e (2)^{C/B} = \log_e\left(1 + \frac{E_b}{n} \times \frac{C}{B}\right)$$

$$2^{C/B} = 1 + \frac{E_b}{n} \times \frac{C}{B}$$

$$\frac{E_b}{n} = \frac{2^{\frac{C}{B}} - 1}{\frac{C}{B}}$$

when $C = R_b$.

$$\frac{E_b}{n} = \frac{2^{R_b/B} - 1}{R_b/B}$$ $\frac{R_b}{B}$ = Rate B.W.



Diagram

We know that

$$C_\infty = 1.44 \frac{S}{\eta}$$

$$\because \quad S = E_b C = E_b C_\infty$$

$$C_\infty = 1.44 \frac{E_b C_\infty}{\eta}$$

$$\frac{E_b}{\eta} = \frac{1}{1.44} = 0.693$$

$$\left(\frac{E_b}{\eta}\right) dB = \boxed{-1.6\ dB}$$

Shannon's limit.

Q.1. Verify that $H(X,Y) = H(X|Y) + H(Y)$

Solution We know that

$$H(X|Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j) \log P(x_i|y_j) \quad \text{--- (1)}$$

$$H(Y) = -\sum_{j=1}^{n} P_{y_j} \log P_{y_j} \quad \text{--- (2)}$$

$$H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j) \log P(x_i\, y_j) \quad \text{--- (3)}$$

~~And~~

$$P(x_i\, y_j) = P(x_i|y_j)\, P(y_j) \quad \text{--- (4)}$$

and $\sum_{i=1}^{m} P(x_i\, y_j) = P(y_j)$ --- (5).

from eq (3)

so.

$$H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j) \log P(x_i\, y_j).$$

from eq (4)

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j)\left[\log P(x_i|y_j)\, P(y_j)\right]$$

$\because \log MN = \log M + \log N.$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j)\left[\log P(x_i|y_j) + \log P(y_j)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j) \log P(x_i|y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\, y_j) \log P(y_j)$$

from eq (1)

$$= H(X|Y) - \sum_{j=1}^{n}\left(\sum_{i=1}^{m} P(x_i\, y_j)\right) \log P(y_j)$$

from eq (5)

$$= H(X|Y) - \sum_{j=1}^{n} P(y_j) \log P(y_j)$$

$$\boxed{H(XY) = H(X|Y) + H(Y)}.$$

Q.2. Verify that $H(X,Y) = H(X) + H(Y/X)$.

<u>Solution</u> : Using definition.

$$H(X,Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log P(x_i\,y_j)$$

$$\because \; P(x_i y_j) = P(x_i)\,P(y_j|x_i)$$

$$H(X,Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log\{P(x_i)\,P(y_j|\,x_i)\}$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\left[\log P(x_i) + \log P(y_j|\,x_i)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i y_j)\log P(x_i) - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log P(y_j|\,x_i).$$

$$= -\sum_{i=1}^{m}\left(\sum_{j=1}^{n} P(x_i\,y_j)\right)\log P(x_i) - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i y_j)\log P(y_j|x_i)$$

$$= -\sum_{i=1}^{m} P(x_i)\log P(x_i) - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i\,y_j)\log P(y_j|x_i)$$

$$= H(X) + H(Y|X)$$

Mutual Information

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2 \frac{P(x_i/y_j)}{P(x_i)}.$$

$$I(Y:X) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2 \frac{P(y_j/x_i)}{P(y_j)}.$$

(A) $\because \quad P(x_i;y_j) = P(x_i)\,P(y_j/x_i) = P(x_i/y_j)\,P(y_j)$

$$\Rightarrow \quad \frac{P(y_j/x_i)}{P(y_j)} = \frac{P(x_i/y_j)}{P(x_i)}.$$

so $\quad I(X;Y) = I(Y;X)$

(B) <u>prove</u> $I(X;Y) = H(X) - H(X/Y)$.

$\because$ ~~H(X/Y)=~~

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2\frac{P(x_i|y_j)}{P(x_i)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\left[\log_2 P(x_i|y_j) - \log_2 P(x_i)\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2 P(x_i|y_j)$$

$$- \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2 P(x_i).$$

$$= -\sum_{i=1}^{m}\left(\sum_{j=1}^{n} P(x_i;y_j)\right)\log_2 P(x_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2 P(x_i|y_j)$$

$$= -\sum_{i=1}^{m} P(x_i)\log_2 P(x_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i;y_j)\log_2 P(x_i|y_j)$$

$$= H(X) - H(X|Y).$$