Free Probability & The Free Central Limit Theorem

18.338 Final Presentation, 12/13/23

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Introduction

- **Free probability** is the study of non-commutative random variables
- Combines probability, random matrices, operator algebras, functional and complex analysis, and more
- Idea first came about when Voiculescu [10] equated free independence to freeness of subgroups $(G_i)_{i \in \mathcal{I}}$ of a group algebra $\mathbb{C}G$.
- Extends classical results in probability theory to this new setting: a **free law of large numbers**, **free central limit theorem**, etc.

Free Probability Background

To understand free probability, we first consider Kolmogorov's classical foundations:

1. We select a **sample space** Ω, with **states** $\omega \in \Omega$.
2. We then construct a σ-algebra $\mathcal{B} \subseteq 2^{\Omega}$ of **events**, such that $P(\Omega) = 1$.
3. We build a commutative algebra of random variables $X : \Omega \to \mathbb{C}$, with an expectation function $\mathbb{E}(X)$.

$(\Omega, \mathcal{B}, P)$ is our **probability space**. We "abstract away" both (1) and (2), to help us both compute limits and work with more general, non-commutative objects.

Free Probability Background

Definition

A (**non-commutative**) **C*-probability space** $(\mathcal{A}, \varphi)$ consists of a unital C*-algebra $\mathcal{A}$ over $\mathbb{C}$, and a unital, positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$.

- Recall that a C*-algebra is a vector space over $\mathbb{C}$ equipped with a bi-linear product, and has a norm under which it is complete.

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- We often also assume that $\varphi(ab) = \varphi(ba)$ (**tracial**) and $\varphi(a^*a) = 0 \implies a = 0$ (**faithful**).
- A good working example to keep in mind is $\mathcal{A} = M_d\left(L^{\infty-}(\Omega, P)\right)$, the space of $d \times d$ random matrices, and the linear functional

$$\varphi(a) = \frac{1}{d}\mathbb{E}[\mathrm{tr}(a)].$$

$*$-Distributions

- We now wish to understand the distribution of an arbitrary $a \in \mathcal{A}$.
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Definition

The $*$-**distribution** of $a \in \mathcal{A}$ is the linear functional $\mu : \mathbb{C}\langle X, X^* \rangle \to \mathbb{C}$ such that

$$\mu(X^{\varepsilon_1} \cdots X^{\varepsilon_k}) = \varphi(a^{\varepsilon_1} \cdots a^{\varepsilon_k}),$$

where each $\varepsilon_i \in \{1, *\}$.

- Having a $*$-distribution makes computing moments such as $\varphi(a^k) = \int t^k d\mu(t)$ much easier.

Spectrum

Definition

The **spectrum** of $a \in \mathcal{A}$ is given by

$$\operatorname{spec}(a) := \{z \in \mathbb{C} : z1_{\mathcal{A}} - a \text{ is not invertible}\}.$$

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Proposition

Let $a \in \mathcal{A}$ be normal. Then a has a $*$-distribution μ whose support is contained in $\operatorname{spec}(a)$. If φ is faithful, then the support is exactly $\operatorname{spec}(a)$.

Free Independence

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Definition

Let (A, φ) be a C*-probability space. Then $(A_i)_{i \in \mathcal{I}}$ are **freely independent** (**free**) if

$$\varphi(a_1 \cdots a_k) = 0$$

whenever $a_j \in \mathcal{A}_{i_j}$ with $\varphi(a_j) = 0$ for all $1 \leq j \leq k$, and no neighboring indices are from the same subalgebra: $i_1 \neq i_2$, $i_2 \neq i_3$, . . ., $i_{k-1} \neq i_k$.

• Random variables $(a_i)_{i \in \mathcal{I}}$ are free ($*$-free) when their generated subalgebras $\sigma(1, a_i)$ $(\sigma(1, a_i, a_i^*))$ are free.

Connection to Freeness

- Recall that in group theory, if G be a group, then subgroups $(G_i)_{i\in\mathcal{I}}$ are **free** if for every $k \geq 1$ and $i_1, \ldots, i_k \in \mathcal{I}$ all distinct, we have

$$g_1 \in G_{i_1} \setminus \{e\}, \ldots, g_k \in G_{i_k} \setminus \{e\} \implies g_1 \cdots g_k \neq e.$$

Proposition

Let G be a group. Then the subgroups $(G_i)_{i\in\mathcal{I}}$ are free in G if and only if the subalgebras $(\mathbb{C}G_i)_{i\in\mathcal{I}}$ are freely independent in $(\mathbb{C}G, \tau_G)$.

- The group algebra $\mathbb{C}G$ consists of all finite linear combinations of the form $\sum_{g\in G} \alpha_g g$ with coefficients in $\mathbb{C}$, and τ_G maps such a linear combination to its identity coefficient α_e
  - This is a proper non-commutative probability space!

- In the non-commutative case, we say that a_N converges in distribution to a if every moment converges: $\varphi(a_N^n) \to \varphi(a^n)$.

Theorem (Free CLT)

Let $(\mathcal{A}, \varphi)$ be a C*-probability space and $a_1, a_2, \ldots \in \mathcal{A}$ be free, identically distributed, and self-adjoint with common mean $\varphi(a_i) = 0$ and variance $\varphi(a_i^2) = \sigma^2$. Then

$$\frac{a_1 + \ldots + a_N}{\sqrt{N}} \xrightarrow{d} \mu_s,$$

where μ_s is the distribution of a semi-circular element of radius 2σ.

- In the lecture notes, we are given a proof using R-transforms. I will provide a short combinatorial proof on the board!

Free Probability for Concentration Inequalities

- Free probability can be applied toward concentration inequalities in random matrix theory.

Theorem (Non-commutative Khintchine Inequality)

Let $X = \sum_{i=1}^n g_i A_i$ for $g_i \sim \mathcal{N}(0,1)$ independent and A_i fixed coefficient matrices. Then

$$\sigma(X) \lesssim \mathbb{E}\|X\| \lesssim \sigma(X)\sqrt{\log(d)},$$

where $\sigma(X)^2 = \|E(X^2)\| = \|A_1^2 + \ldots + A_n^2\|$.

- Bounds expected spectral norm up to $\log(d)^{1/2}$ factor
- Can be sub-optimal in high dimensions ($d \gg n$)

- Let our original and "free" models be defined as

$$X := A_0 + \sum_{i=1}^{n} g_i A_i, \quad X_{\text{free}} := A_0 \otimes 1 + \sum_{i=1}^{n} A_i \otimes s_i.$$

Theorem 2.1 of [2]

For the above model with $A_0, \ldots, A_n$ all self-adjoint, we have that for every $t \geq 0$,

$$P\left(\text{spec}(X) \subseteq \text{spec}(X_{\text{free}}) \pm c_t\right) \geq 1 - e^{-t^2},$$

where $c_t := C\tilde{v}(X)\log(d)^{3/4} + C\sigma_*(X)t$ quantifies the non-commutativity of the matrices A_i (for some universal $C > 0$).

- They also provide bounds of the form

$$P\left(\|X\| > \|X_{\text{free}}\| + c_t\right) \leq e^{-t^2}, \quad \mathbb{E}\|X\| \leq \|X_{\text{free}}\| + C\tilde{v}(X)\log(d)^{3/4}.$$

Free Probability for Concentration Inequalities

- In order for these bounds to be useful, $\|X_{\text{free}}\|$ must be readily computable in practice, in which case the following lemma can help us:

Lemma

When the A_i are self-adjoint, we have

$$\|X_{\text{free}}\| = \max_{\varepsilon=\pm 1} \inf_{Z>0} \lambda_{\max}\left(Z^{-1} + \varepsilon A_0 + \sum_{i=1}^{n} A_i Z A_i\right),$$

where this infimum is over all positive-definite, self-adjoint $Z \in M_d(\mathbb{C})$, and $\lambda_{\max}(\cdot)$ is the largest eigenvalue.

- In the proof, authors show that even though the moments of the matrix X might depend on all pairings, the crossing pairings still come close to vanishing in many cases via the non-commutativity of the A_i.
    - Similar to how only non-crossing pairings survive in free CLT

Extensions of Free CLT

- Of course, the Free CLT relies on the crucial assumption of free independence, and only works for empirical averages
- Not very desirable for real-world applications as independence is often violated, such as the case of a single random matrix of growing dimension
- Many popular statistics, such as U-statistics $\frac{1}{n^2}\sum_{i,j=1}^{n} f(a_i, a_j)$ do not fit this form
- Not a problem for the classical CLT, in which independence, taking empirical averages, and even identical distribution assumptions can be relaxed to varying degrees of generality.

Extensions of Free CLT

- In [1], they study under what conditions a free CLT can be extended to both empirical averages and U-statistics of dynamical systems
  - stationary or quantum exchangeable sequences

Definition

For a sequence of classical random variables, the **strong-mixing coefficients** are defined as

$$\alpha_i := \sup_{\substack{A \in \sigma(X_{-\infty:0}) \\ B \in \sigma(X_{i:\infty})}} |P(A \cap B) - P(A)P(B)| \,.$$

Can do a similar method to define “free mixing” coefficients in the non-commutative case (but is much more elaborate), and these coefficients provide a bound on the normed difference between such a dynamical system and semi-circular elements.

Conclusion

- Many results in the non-commutative case mirror those in the classical setting, but with subtle changes
  - Partitions $\to$ non-crossing partitions
  - Normal distribution $\to$ semi-circular distribution
- Free probability has recently found applications in several fields, and is an active area of research (dependent free CLT, high-dimensional free CLT, etc.)
  - Another main goal is to construct new invariants of von Neumann algebras via free probability.

Thanks for a great course!!!

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