The Simplex Algorithm and Random Matrices

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Overview

- We review and implement the simplex method, an algorithm that solves Linear Programming (LP) problems.
- We also perform some experiments with random matrices to find interesting observations and conjectures.

The Simplex Method

- The LP problem in standard form is expressed as:

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where A is an $m \times n$ matrix, b is an $m \times 1$ vector and c is an $n \times 1$ vector.

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- Maximizing $\mathrm{c}'\mathrm{x}$ is equivalent to minimizing $-\mathrm{c}'\mathrm{x}$.

- Let $P = \{\mathrm{x} \in \mathbb{R}^n \,|\, \mathrm{Ax} = \mathrm{b}, \mathrm{x} \geq 0\}$ be a polyhedron corresponding to the feasible set of the LP problem in standard form.

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- Let $P = \{\mathsf{x} \in \mathbb{R}^n \,|\, \mathsf{A}\mathsf{x} = \mathsf{b}, \mathsf{x} \geq 0\}$ be a polyhedron corresponding to the feasible set of the LP problem in standard form.

- An optimal solution to the LP problem (if it exists) tends to occur at a "corner" of P (known as vertex, or basic feasible solution).

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- If an LP problem in standard form has an optimal solution, then there exists a basic feasible solution (a vertex of P) that is optimal.
- The simplex algorithm is based on this fact and searches for an optimal solution by moving from one basic feasible solution to another, along the edges of the feasible set P, always in a cost reducing direction.
- The algorithm terminates when we reach a basic feasible solution (that is optimal) at which none of the available edges leads to a cost reduction.

The Big-M Method

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- Note that we start the simplex algorithm by finding an initial basic feasible solution.
- However, this is not always straightforward, and in general, finding an initial basic feasible solution requires us to add some *artificial variables* and solve the problem in a different way.
- In this project, we adopt an approach known as the *big-M method*.

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- Note that we can assume, without loss of generality, that $b \geq 0$ (by multiplying some of the equality constraints by -1).

The Big-M Method

- We introduce a vector $\mathrm{y} \in \mathbb{R}^m$ of artifical variables and we modify the cost function to solve the following LP problem.

$$\begin{aligned} \text{minimize} \quad & \mathrm{c}'\mathrm{x} + M \sum_{i=1}^{m} y_i \\ \text{subject to} \quad & \mathrm{A}\mathrm{x} + \mathrm{y} = \mathrm{b} \\ & \mathrm{x} \geq 0 \\ & \mathrm{y} \geq 0 \end{aligned}$$

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- By following this approach, initialization of the simplex algorithm becomes easy: we let $\mathrm{x} = 0$ and $\mathrm{y} = \mathrm{b}$, and we get an initial basic feasible solution.

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& \mathrm{y} \geq 0
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- For large M, if the original problem is feasible and bounded, all of the artificial variables y_i will be 0 (i.e. $\mathrm{y} = 0$) and this will lead to the minimization of the original cost function $\mathrm{c}'\mathrm{x}$.

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- If the optimal solution of the problem above contains a nonzero artificial variable, we can conclude that the original problem is infeasible.

Time for some fun!

Acknowledgement

- I would like to thank Professor Edelman for his valuable discussions and comments.

Thank You!