MIT 18.338 Final Project Report
Determinantal Point Processes and Growth Models

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Abstract

A determinantal point process (DPP) is a probability distribution on a power set of a ground set, characterized by a kernel matrix K. A property of DPPs is that it allows for the sampling of diverse subsets of the ground set, due to the correlation structure it induces [1]. This had led to DPPs being used in describing models where repulsion is a driving factor [2] [3] [4] [5] [6]. Random growth models are growths in a plane based on a probabilistic mechanism. Famous examples of random growth models, such as Plancherel growth and Aztec diamond, have an intrinsic repulsive characteristic, which can be quantified using DPPs [7] [8]. The purpose of this project is to investigate the properties of DPPs that lead to the given repulsive factor, finding a generative form for DPPs, and using the DPPs to construct novel growth models. We start with discrete growth models that can be characterized by finite DPPs, and construct a flexible framework for generation.

1 Introduction

1.1 Background

A Point Process is a probability distribution defined over the power set of a ground set. A Determinantal Point Process is a Point Process, where the probability of a subset J being in the sample is equal to the J-minor of a given marginal kernel matrix K.

Thus, a Determinantal Point Process can be characterized by the marginal kernel matrix K. An alternate definition involves L ensembles. We can make the probability of selecting a given subset J as being proportional to the J-minor of a positive semidefinite matrix L. The formulations of K and L would be analogous to the CDF and PDF of a distribution, respectively.

A Random Growth Model are growths in a plane based on a probabilistic mechanism.

We can see that different structures of the matrices lead to unique DPPs. For example, if we were to choose a diagonal L matrix, this induces an independent sampling scheme on the elements of the ground set. Additionally, if we choose a Rank 1 L matrix, we induce singleton sampling, since every minor larger than size 1 would be 0. In general, low rank induces smaller subsets in the data.

1.2 Motivation

A key characteristic of a DPP is the diversity it induces in the point process. That is, sampling from a DPP causes nearby points in the process to have a lesser probability of appearing together. This repulsion can be quantified [1].

The intuition behind this can be seen if we assume a symmetric continuous kernel function, say ψ, as the generator for a DPP. The probability of two points (x, y) being in the random subset would be of the form $\psi(x, x)\psi(y, y) - \psi(x, y)^2$ which is much smaller for x close to y.

The main way that the connection between DPPs and Growth models have been studied in literature is by the quantification of repulsion. For example, when considering a random matrix, where elements are drawn independently from a complex normal distribution, the eigenvalues, which are a subset of the complex plane, are distributed as a DPP. In this case, the DPP could be used as a vessel to investigate the repulsion of eigenvalues, or as a quantification of said repulsion. The topic I was interested in exploring is using DPPs to induce repulsion, and then generalizing the framework for DPPs.

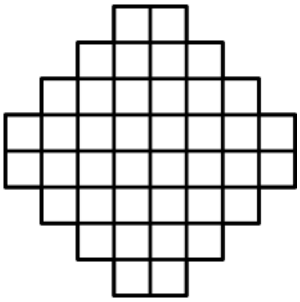

Figure 1: Setup of Aztec Diamond 1

2 Characterization of Repulsion using DPPs

In literature, the main way that DPPs are used to quantify repulsion in growth models is via non-intersecting paths. The core idea is that for a weighted graph $\mathcal{G}$, the process of randomly sampling a non-intersecting path could be characterized by a DPP on the adjacency matrix. Thus, if we were to find non-intersecting paths in the growth models, we could directly connect them with a DPP.

2.1 Aztec Diamond

Consider the Aztec Diamond setup:

- Consider a square lattice where all the centres satisfy $|x| + |y| \leq n$
- Consider a checkerboard pattern on this lattice
- Now, tile this structure with dominoes
- Dominoes are characterized by orientation, and position of the black squares. Thus, there are 4 types of dominoes
- We label the 4 types as follows:
  - N domino: horizontal domino with left white
  - S domino: horizontal domino with right white
  - W domino: vertical domino with upper white
  - E domino: vertical domino with lower white
- We construct a non-intersecting path within each domino as follows:
  - N domino: No path
  - W domino: From (0,0.5) to (1,1.5)
  - E domino: From (0,1.5) to (1,0.5)
  - S domino: From (0,0.5) to (2,0.5)
- Consider a weighting system on the different dominoes. Using this weighting system, there is a kernel matrix for the DPP related to the Aztec Diamond [9].

Figures 1-4 show the construction of the non-intersecting paths in the Aztec Diamond setup.

2.2 Corner Growth Model

Now for a Corner Growth Model, we consider a Geometric setup for every possible square. We can construct right/down paths, that start from $(0, L)$ and go to $(K, 0)$. Using these paths, we can construct a partition, that allows us to make a bijection between these paths and a weighted graph $\mathcal{G}$. While the specifics are not necessary, it does show that we can connect the Corner Growth Model to DPPs. This motivates the next part of this project.

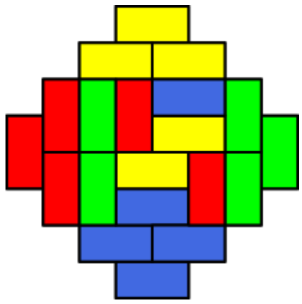

Figure 2: Setup of Aztec Diamond 2

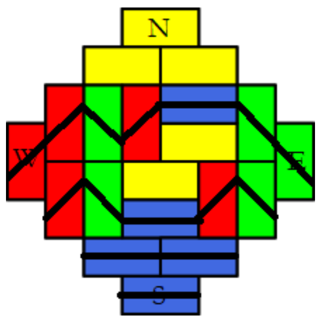

Figure 3: Construction of Non-intersecting Paths

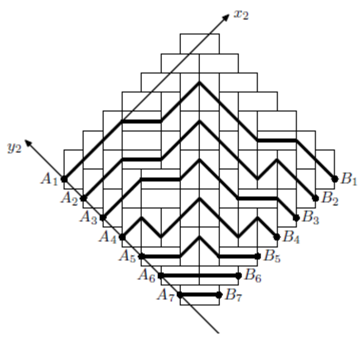

Figure 4: Reorientation

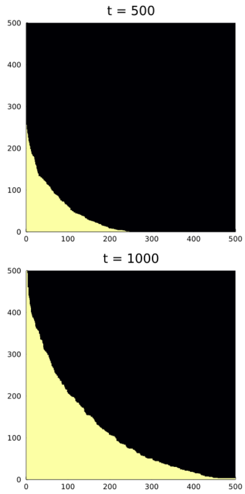

Figure 5: $p = 0.5$ growth

3 Generation of Growth Models

For example, consider a centre growth process on the integer lattice. Say at a time point T, the set $K_T \subset \mathbb{Z} \times \mathbb{Z}$ has points that could become squares in the next time point. If we were to treat this as a DPP, that is, assign a probability distribution to $\mathcal{P}(K_T)$, we can generate the squares for the next step. Iterating this will give us a growth model. Now, if we can replicate the generation model given by, say, the geometric interpretation, then we can assign a general DPP for the same model, and observe its growth. We can experiment with different kernel matrices for the same, and possibly extend to continuous or projection DPPs.

Now, if we choose $K = pI_{|C_T|}$, this is exactly the same as the Bernoulli Corner Growth with probability p. This is promising, as it shows that the original corner growth is a special case of the "DPP Growth".

Before we generalize further, we must find a way of preserving the structure of the DPP, while shifting in the dimensions. There are two ways this can be achieved:

- Use an embedding/trimming algorithm that preserves rank
- Construct a DPP on the whole lattice, and subset the given DPP

From a theoretical standpoint, both methods would work. The key to such a procedure would be to maintain the necessary properties for the kernel matrix (low rank, higher emphasis on initial elements of the ground set)

For embedding and trimming, there are theoretical guarantees in the low rank case [10]. Since the whole lattice method requires trimming, the same guarantees apply.

From a computational standpoint, both are quite inefficient, but there may be a dynamic programming approach for the second.

One structure I experimented with (under the independence regime) was one which made older squares more likely to be selected. There are two ways to order the possible squares: left to right (as is given in the code), distance from origin (ℓ_1 is an intuitive choice). Using the second ordering, we can construct L matrix that has larger elements along the diagonal.

Figures 5-10 show the results of simulations on specific types of DPPs.

Now, we can construct a random DPP, use the first ordering of the squares, and run a simulation. From the previous simulations, we expect a similar growth as the Bernoulli, since the repulsion does not manifest itself in a direct way.

4 Further Research

One thing that would be interesting to implement is to superimpose a limiting curve, to see if we still get a Plancherel-esque growth randomly. We can experiment with more complex kernel functions, particularly ones of value that deal with some implicit ordering of the squares. This would allow the repulsive characteristics of the DPP to manifest itself. Once this has been achieved, we can look at more complicated growth processes, and the extensions that would be necessary.

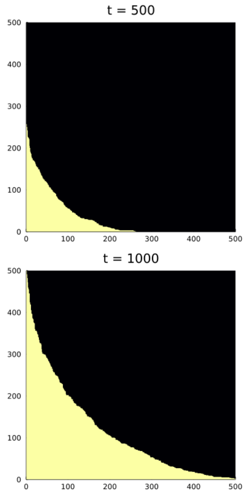

Figure 6: $L = I_{C_T}, K = 0.5 * I_{C_T}$ growth

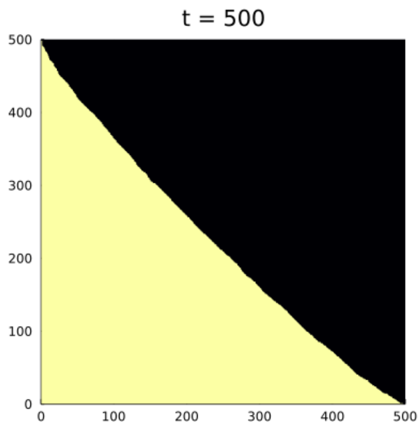

Figure 7: $L = \alpha I_{C_T}, \alpha = 1000$ growth

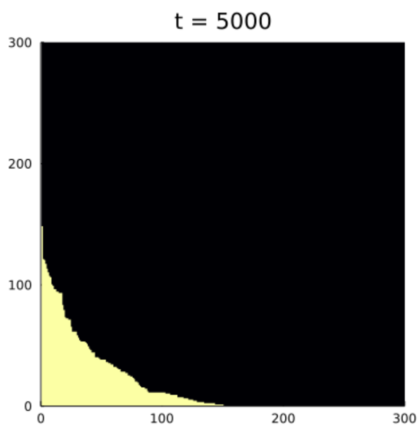

Figure 8: $L = kJ$ growth. Note that the value of k does not matter

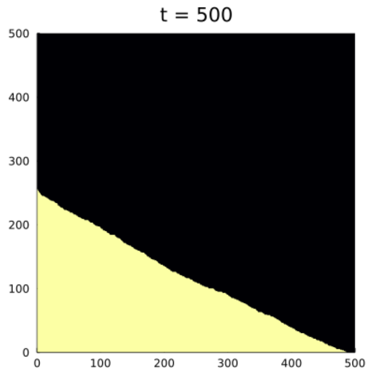

Figure 9: After using the second ordering of squares, the matrix has the probability of not choosing inversely proportional to the squares ranking ($L = \text{diag}(\frac{i}{i+1})$. We can see that the curve grows in this interesting fashion. This is due to the way the squares were initially sorted.

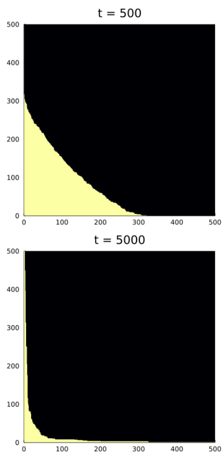

Figure 10: Consider the penalization linear in terms of the ℓ_1 norm. Construct large matrix L, and subset based on ℓ_1 norm of the possible squares. I tried the growths in both directions

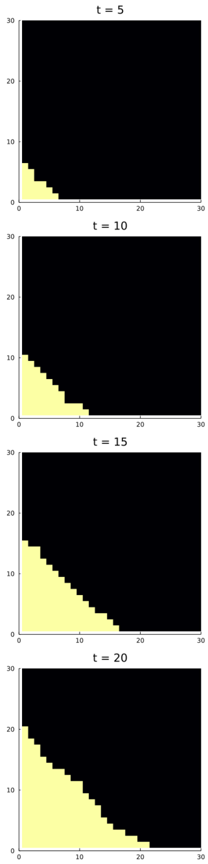

Figure 11: A random positive semidefinite matrix used for sampling

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