18.338 Eigenvalues of Random Matrices

PS4 and Suggested Projects List

Proposal Due Date: Wed Nov. 11, 2020

Homework

Read chapter *4, 10, 11* of the class notes. notes. Please again give your feedback especially high-level style and where things did not make sense, in addition to spelling or technical errors.

1. Do Exercise 4.1 (Page 68), 10.5 (Page 202), and 11.1 (Page 213).
2. Submit a one-pager project proposal. You can choose from the **Suggested Project List** in the next section or come up with a topic of your own interest.

Suggested Project List

Pick any project from `https://github.com/mitmath/18338/issues`.

Computational

1. Write a Julia code to demonstrate Yau's universality spacing laws, see *Universality of Local Spectral Statistics of Random Matrices*.
2. Look up the presentation *Numerical calculation of random matrix distributions and orthogonal polynomials* given by Sheehan Olver (specifically the pictures on page 26, 27 and 47). Use his Mathematica code to reproduce his pictures and possibly explore a RMT experiment, but anyway explain what these pictures represent.
3. Modernize the *Cy's Beta Estimator* for the spacing data previously done by Cy Chan in 2006, and maybe relate it to Machine learning (see Ben Taska's papers on Geometry of Diversity and Determinantal Point Processes.)
4. On Haar measure, look up the talk *On Powers of a Random Orthogonal Matrix* and the paper *The "north pole problem" and random orthogonal matrices* by Muirhead, and do a numerical experiment to verify their results, and there may be a Jack polynomial proof. Also consider other Haar measure experiments, ask us for ideas.
5. Expand Bernie's Jack Polynomial code and demonstrate the orthogonality of the multivariate Hermite, Laguerre and Jacobi polynomials or do your own. Ask Bernie for his code.
6. Explore determinantal processes numerically.
7. Rewrite Odlyzko's Riemann Zeta Root finder in MATLAB, see *On the Distribution of Spacings between Zeros of the Zeta Function*. (You will REALLY understand Riemann Zeta function if you do!)
8. Read up to page 4 of *The distribution of zeros of the derivative of a random polynomial* by Pemantle and Rivin. Redo more carefully the experiments in MATLAB or Julia. The paper mentions experiments but not so much what they saw.

Theoretical

1. Give a presentation and write a summary about Brownian Carousels (Balint Virag and collaborators).

2. Read Zonal Spherical function on Wikipedia and tell us (with presentation and writeup) that story: Gelfand pairs, the Laplace-Beltrami Operator, and perhaps hypergeometric functions.

3. Extend the known table for $p(n, k)$ the probability that `a=randn(n)` has k real eigenvalues. (Read the paper on *How many eigenvalues of a random matrix are real* by Edelman and the table in page 9 of the thesis *On the computation of probabilities and eigenvalues for random and non-random matrices* by Sundaresh. Notice that there are three conjectures. This could also be a computational project. Also check .

4. Do the following

   (a) Explain the Weingarten formula for Haar measure (See p380 of *Lectures on the Combinatorics of Free Probability* by Nica & Speicher) and find out if there is a real version ($\beta = 1$). (Maybe hard: analyze the (computational) complexity of this formula.)

   (b) What do Schur Polynomials tell us? Compare and contrast.

5. Consider computing an exact formula for $\mathbb{E}[\mathbf{Tr}(A^k)]$ where A is an instance of β-Hermite ensemble. There is a method implemented in **MOPS**. Alternatively, one can also try to use the Tridiagonal ensembles which has the best computational complexity.

6. Get into the world of multivariate orthogonal polynomial theory.

7. Get into the world of q series.

8. Read the Tracy-Widom Law and explain it. The reference can be the last paper in Section 3.8 of *An Introduction to Random Matrices* by Anderson, Guionnet & Ofer Zeitouni.

9. Give a presentation and write a summary about Random Matrix Theories in Quantum Physics. One reference is *Random Matrix Theories in Quantum Physics: Common Concepts*.

10. Give a presentation and write a summary about RMT applications in Wireless communication. One reference might be *Random matrix theory and wireless communications* by Tulino & Verdus.

11. Find the eigenvectors of the Jacobian matrix for the change of variables from a symmetric Tridiagonal matrix T to its eigenvalues and the first component of the eigenvectors of T.

12. Is it true that Hermite, Laguerre and Jacobi roots can be computed to high relative accuracy (the small ones have nearly all exact digits) but probably one must use a good: 1) tridiogonal eigensolver; 2) bidiagonal SVD solver; 3) perhaps CS decomposition that has not been invented yet or maybe in Brian Sutton's work.

13. Find the tridiagonal (?) Krawtchouck model for all β analogous to Hermite, Laguerre and Jacobi. Use the formula 18.22.2.

14. Find a way to compute the level density for any beta-hermits. (may be known for even or 2/even betas)

15. Tabulate all known Painleve formulas for largest,smallest, bulk eigenvalues neatly or all known hypergeometric formulas

16. There should be a stochastic operator model for free probability. Specifically the eigenvalues of $A+QBQ'$ limit (for any beta simultaneously) should be a stochastic operator itself.