Possible Class Projects, 2022 FALL 18.338

Blue texts are hyperlinks

Computational

1. Implement in Julia, numerical computations of Jack Polynomial, Hypergeometric functions of a matrix argument and/or M.O.P.s. See *The Efficient Evaluation of the Hypergeometric Function of a Matrix Argument* and *MOPS: Multivariate Orthogonal Polynomials (symbolically)*. One could numerically demonstrate the orthogonality of the multivariate Hermite, Laguerre and Jacobi polynomials. Also one could compare Hypergeometric representations of eigenvalue statistics (both scaling limits, or finite RMTs) with the corresponding Fredholm deteriminant representations. See *On the Numerical Evaluation of Fredholm Determinants* for implementing Fredholm determinants. (A few of these may exist in MATLAB, C++, and maybe even Julia already.)

2. There are many many cool processes that are described by the Airy process. The coolest might be the Aztec diamond, but there are also, Dyson Brownian Motion (implementing DBM would need a very careful literature search), Large passage percolation, non-intersecting Brownian bridges, buses in Cuernavaca, parked cars. (Look up for various papers describing the KPZ class.) Implement any of these, and compare its limiting curve with the Airy process and see if they match up. (Sungwoo has a notebook with one approach in Julia)

3. Numerically solve Painlevé differential equations, using DifferentialEquations.jl. Check out if it is possible to numerically solve especially in Jimbo-Miwa-Okamoto σ-form of the Painlevé equations. See Forrester's book, Chapter 8.1 for definitions. See if the numerical solutions match Finite RMT laws: For example Painlevé IV should match the CDF of the largest eigenvalue of $n \times n$ GUE.

4. Write a Julia code to demonstrate Yau's universality spacing laws, see *Universality of Local Spectral Statistics of Random Matrices*.

5. Look up the presentation *Numerical calculation of random matrix distributions and orthogonal polynomials* given by Sheehan Olver (specifically the pictures on page 26, 27 and 47). Use his Mathematica code to reproduce his pictures and possibly explore a RMT experiment, but anyway explain what these pictures represent.

6. Modernize the *Cy's Beta Estimator* for the spacing data previously done by Cy Chan in 2006, and maybe relate it to Machine learning (see Ben Taska's papers on *Geometry of Diversity and Determinantal Point Processes*.)

7. On Haar measure, look up the talk *On Powers of a Random Orthogonal Matrix* and the paper *The "north pole problem" and random orthogonal matrices* by Muirhead, and do a numerical experiment to verify their results, and there may be a Jack polynomial proof. Also consider other Haar measure experiments, ask us for ideas.

8. Rewrite Odlyzko's Riemann Zeta Root finder in Julia, see *On the Distribution of Spacings between Zeros of the Zeta Function*. You will REALLY understand Riemann Zeta function if you do! (Mike Rubinstein at Waterloo may have some codes to look at)

9. Read up to page 4 of *The distribution of zeros of the derivative of a random polynomial* by Pemantle and Rivin. Redo more carefully the experiments in r Julia. The paper mentions experiments but not so much what they saw.

10. Carefully implement RSK algorithm (and its inverse) in Julia with an appropriate data structure. Demonstrate Plancherel growth and see its limiting curves. Ask us for the existing code. (with somewhat incomplete data structure)

11. Take as many codes in the course notes as you can, and turn them into elegant Julia code but first let's inventory what's already been done.

Theoretical

1. Give a presentation and write a summary about Brownian Carousels. See *Continuum limits of random matrices and the Brownian carousel*.

2. Extend the known table for $p(n,k)$ the probability that `a=randn(n)` has k real eigenvalues. (Read the paper on *How many eigenvalues of a random matrix are real* by Edelman and the table in page 9 of the thesis *On the computation of probabilities and eigenvalues for random and non-random matrices* by Sundaresh. Notice that there are three conjectures. This could also be a computational project. Also check *Statistics of Real Eigenvalues in Ginibre's Ensemble of Random Real Matrices*.

3. Hermite kernel could be generalized with a choice of non-integer n. See *generalized Hermite functions*. With DPP, one could even sample the "eigenvalues." It might not be a probability measure, but the concept of signed probability is out there. Be the first to explore.

4. Do the following

   (a) Explain the Weingarten formula for Haar measure (See page 380 of *Lectures on the Combinatorics of Free Probability* by Nica & Speicher) and find out if there is a real version ($\beta = 1$). (Maybe hard: analyze the (computational) complexity of this formula.)

(b) What do Schur Polynomials tell us? Compare and contrast.

5. Consider computing an exact formula for $\mathbb{E}[\mathrm{Tr}(A^k)]$ where A is an instance of β-Hermite ensemble. There is a method implemented in *M.O.P.s*. Alternatively, one can also try to use the Tridiagonal ensembles which has the best computational complexity.

6. A Monte-Carlo simulation on k-th largest eigenvalues of β-Hermite, with large n, for example $n > 10^6$, does not need a full $n \times n$ matrix. However, its $m \times m$ upper left corner (of the Hermite tridiagonal model) captures large eigenvalue behaviors. For example, $m = 10n^{1/3}$ is enough for the largest eigenvalue ($k = 1$). This is connected to the eigenvectors and the location of Airy roots. Approximate and justify the choice of m, for a given k.

7. Generalizing Tracy and Widom's derivation. Read Chapter 9.4 of Forrester's book. Tracy and Widom's original derivation of Painlevé equation from Airy, Bessel, Sine kernels rely on certain set of conditions defining these kernels. See (9.41) of Forrester. One could work on a generalization of the condition imposed on (9.56) and see if it leads to any differential equations.

8. Give a presentation and write a summary about Random Matrix Theories in Quantum Physics. One reference is *Random Matrix Theories in Quantum Physics: Common Concepts*.

9. Rigorously define β-ghost and shadows. See *The Random Matrix Technique of Ghosts and Shadows*.

10. Give a presentation and write a summary about RMT applications in Wireless communication. One reference might be *Random matrix theory and wireless communications* by Tulino & Verdus.

11. Find the eigenvectors of the Jacobian matrix for the change of variables from a symmetric Tridiagonal matrix T to its eigenvalues and the first component of the eigenvectors of T.

12. Is it true that Hermite, Laguerre and Jacobi roots can be computed to high relative accuracy (the small ones have nearly all exact digits) but probably one must use a good:

    (a) Tridiogonal eigensolver

    (b) Bidiagonal SVD solver

    (c) Perhaps CS decomposition that has not been invented yet or maybe in Brian Sutton's work.

13. Find the tridiagonal(?) Krawtchouck model for all β analogous to Hermite, Laguerre and Jacobi. Use the formula 18.22.2.

14. There should be a stochastic operator model for free probability. Specifically the eigenvalues of $A+QBQ'$ limit (for any β simultaneously) should be a stochastic operator itself.