Analyzing Higher Order Effects on Eigenvalues

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Motivation

- It is well known that the eigenvalues of a symmetric random matrix approach the Wigner semicircle, by the CLT.
- How do the spacings look? For an $n \times n$ matrix, the eigenvalues are $\lambda_1 \leq \lambda_1 \leq \cdots \leq \lambda_n$. We look at the distribution of $\lambda_{i+1} - \lambda_i$.

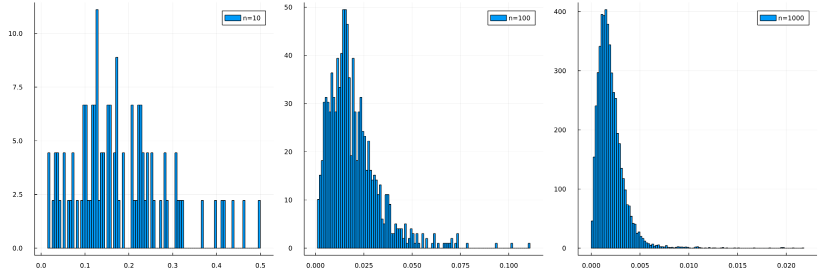

Figure: Raw spacings. As n increases, they're concentrated at 0.

Unfolded Spacings

- Want to "unfold" the eigenvalues so they're not bunched next to 0.
- Idea: Use cumulative density function.

$$p_w(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2} \implies f(\lambda) = (n+1)\int_{-1}^{\lambda} p_w(x)\,\mathrm{d}x$$

- The distribution of the CDF is uniform, so the kth smallest value has mean $\frac{k}{n+1} \implies$ the expected value of $n+1$ times a spacing is 1.

The Wigner Surmise

- The proposed distribution of these unfolded eigenvalues for a GOE is called the **Wigner surmise**.

$$\text{Wigner surmise} : p_s(x) = \frac{\pi}{2} x e^{-\pi x^2/4}$$

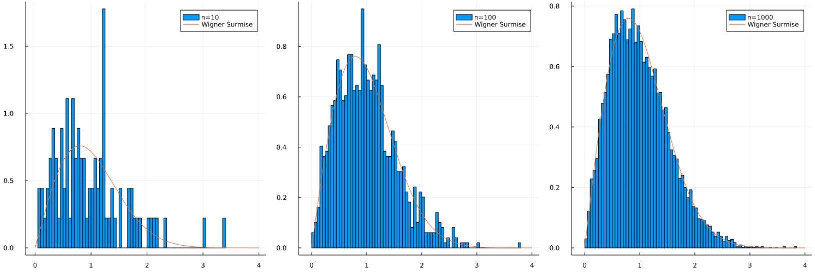

Figure: Unfolded spacings. These are centered at 1.

Evidence of Higher Order Effects

- If unfolding were the whole story, then the distribution would be the same as the spacings in $\mathcal{U}([0, n+1])$.

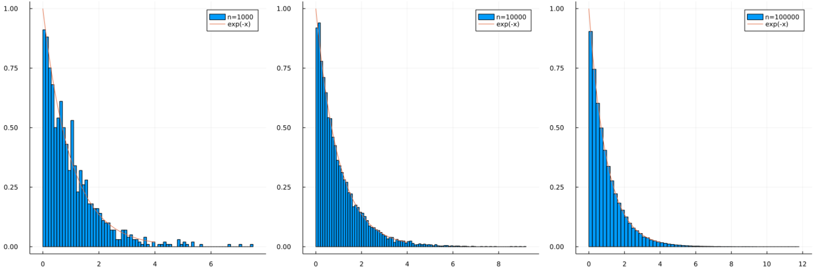

Figure: Uniform spacing. The limiting distribution is e^{-x}.

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Procedure

- Generate an $n \times n$ matrix A from distribution f.
- Calculate $\frac{A+A^T}{2\sqrt{2n}}$.
- Find the spacing distribution, as well as first order errors.

$$\text{Assumption} : p(x) = p_s(x)\left[1 + \frac{1}{n}e(x)\right] + O\left(\frac{1}{n^2}\right)$$

- Hopefully, $e(x)$ also depends on the normalized kurtosis γ.

Constraints

- Issues and their solutions
  - Outlier eigenvalues: Mean of f is 0
  - Normalization so range of semicircle is $[-1, 1]$: Variance of f is 1
- Potential distributions
  - $\mathcal{U}([-\sqrt{3}, \sqrt{3}])$
  - $\Gamma(1, 1)$, subtract 1 from samples
  - Bernoulli: -1 with probability $1/2$, 1 with probability $1/2$

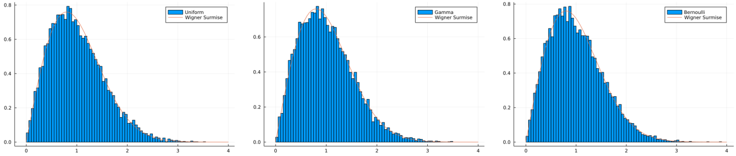

Figure: Comparison of various spacing distributions to surmise

Results

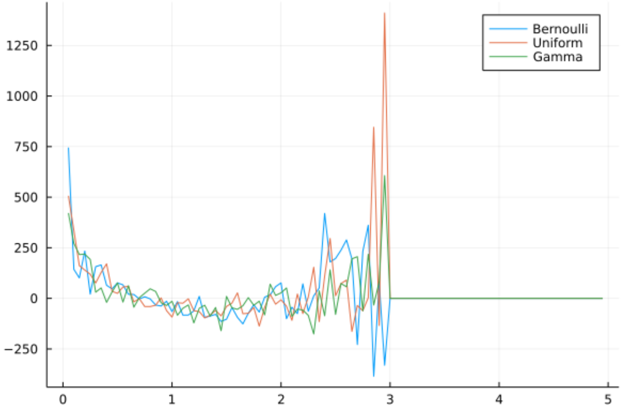

Figure: First order errors for each distribution

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Tracy-Widom

- Tracy Widom distribution is distribution of $(\lambda_{\max} - \sqrt{2n})\sqrt{2}n^{1/6}$
  - Used for GOE matrices ($\beta = 1$)

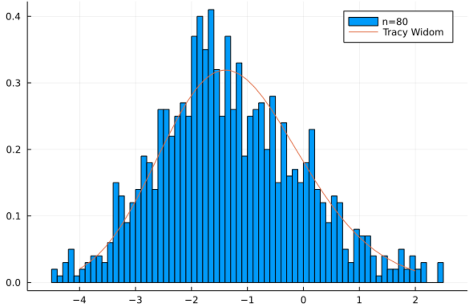

Figure: Visualization of basic Tracy-Widom

Other Distributions

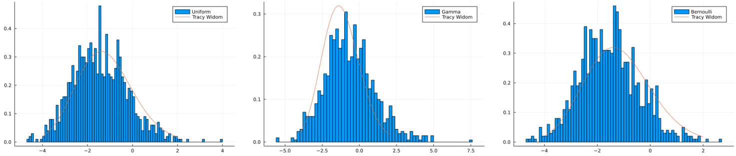

Figure: Comparison of various extreme distributions to Tracy-Widom

Results

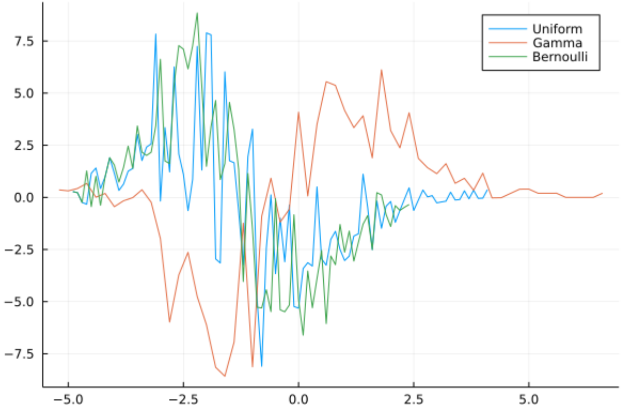

Figure: First order errors for each distribution

Future Steps

- More compute resources
- Obtain a quantitative measure of error
    - Perhaps this requires rethinking the format of the error

References

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