Multivariate Orthogonal Polynomials Symbolically in Julia

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Abstract

Your abstract.

1 Introduction

In this project, a Julia package is implemented for both symbolically and numerically evaluating Jack Polynomials. In the following sections, we firstly introduce the basic concepts of orthogonal polynomials, describe the computation algorithm for Jack Polynomials, and present the Julia package, MOPS.jl.

2 Introduction to orthogonal polynomials

Scalar orthogonal polynomials come from a positive weight function $w(x)$ defined over a domain $I \in \mathbb{R}$.

The inner product is defined as

$$\langle f, g \rangle_w = \int_I f(x)g(x)w(x)dx \tag{1}$$

and the sequence of polynomials $p_0(x), p_1(x), \ldots$, such that $p_k(x)$ has degree k and

$$\langle p_i, p_j \rangle_w = C_i \delta_{ij} \tag{2}$$

In multi-dimensional case, an n-dimensional weight function is needed

$$W(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} |x_i - x_j|^{2/\alpha} \prod_{i=1}^{n} w(x_i) \tag{3}$$

Note that besides the product $\prod_{i=1}^{n} w(x_i)$, there is an additional term which depends on a "Boltzmann" constant β or a temperature factor $\alpha = 2/\beta$.

Multivariate orthogonal polynomials $p_\kappa^\alpha(x_1, \ldots, x_n)$ with respect to the weight $W(x_1, \ldots, x_n)$ is defined as follows. The polynomials are `symmetric` and satisfy

$$\int_{I^n} p_\kappa^\alpha(x_1, \ldots, x_n) p_\mu^\alpha(x_1, \ldots, x_n) \prod_{i<j} |x_i - x_j|^\beta \prod_{j=1}^{n} w(x_i) dx_1 \cdots dx_n = \delta_{\kappa\mu}$$

3 Computation of Jack polynomials

[DES04] is a Maple package for symbolically computing multivariate orthogonal polynomials, their integrals and eigenvalue statistics. In this class project, we solely focus on Jack polynomials.

There are mainly 2 definitions for Jack polynomials. One comes from inner product.

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{l(\lambda)} z_\lambda \delta_{\lambda\mu} \tag{4}$$

$$z_\lambda = \prod_{i=1}^{l(\lambda)} a_i! i^{a_i} \tag{5}$$

$$P_\lambda^\alpha = m_\lambda + \sum_{\mu \preceq \lambda} u_{\lambda,\mu}^\alpha m_\mu \tag{6}$$

In the other one, polynomials are eigenfunctions of the differential operator

$$D^* = \sum_{i=1}^{m} x_i^2 \frac{d^2}{dx_i^2} + \frac{2}{\alpha} \sum_{1 \le i \ne j \le m} \frac{x_i^2}{x_i - x_j} \frac{d}{dx_i} \tag{7}$$

with eigenvalue $\rho_\kappa^\alpha + k(m-1)$.

Furthermore, there are 3 normalizations

- "J": The coefficient of the smallest monomial, corresponding to the partition $[1, 1, \ldots, 1]$ is $|\kappa|!$.
- "P": The coefficient of the highest monomial, corresponding to κ is 1.
- "C": The sum of all Jack polynomials on the variables $x_1, x_2, \ldots, x_n$ coresponding to partitions of $|\kappa|$ is $\left(\sum_{i=1}^{n} x_i\right)^{|\kappa|}$.

For computing Jack polynomials, we recurrence relations. We need to find an expansion for the Jack polynomials of the type

$$C_\kappa^\alpha(x_1, x_2, \ldots, x_m) = \sum_{\lambda \le \kappa} c_{\kappa,\mu}^\alpha m_\lambda(x_1, x_2, \ldots, x_m) \tag{8}$$

where the monomial function m is

$$m_\lambda = \sum_{\sigma \in S_\lambda} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(m)}^{\lambda_m} \tag{9}$$

using the recurrence

$$c_{\kappa,\lambda}^\alpha = \frac{\frac{2}{\alpha}}{\rho_\kappa^\alpha - \rho_\lambda^\alpha} \sum_{\lambda \prec \mu \preceq \kappa} \left((l_i + t) - (l_j - t)\right) c_{\kappa,\mu}^\alpha \tag{10}$$

where $\lambda = (l_1, \ldots, l_i, \ldots, l_j, \ldots, l_m), \mu = (l_1, \ldots, l_i + t, \ldots, l_j - t, \ldots, l_m)$.

More details can be found in [Dum03].

4 MOPS.jl

The code is open-source and available at `https://github.com/bowenszhu/MOPS.jl`.

It is able to symbolically and numerically computing Jack polynomials. Only the "J" normalization due to issues of Symbolics.jl.

References

[DES04] Ioana Dumitriu, Alan Edelman, and Gene Shuman. MOPS: Multivariate Orthogonal Polynomials (symbolically), 2004.

[Dum03] Ioana Dumitriu. *Eigenvalue Statistics for Beta-Ensembles.* PhD thesis, Massachusetts Institute of Technology, 2003.