TOWARDS A NONCOMMUTATIVE UNDERSTANDING OF EDELMAN'S GHOSTS AND SHADOWS

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Abstract. What follows is the preliminary investigation of some thoughts inspired by the isotropic ghost random variables as a final project for MIT 18.338.

Request

Please do not distribute this work, I intend to keep working on this in the coming months. :)

1. Background

1.1. **Edelman Ghosts and Shadows.** We reproduce some of the initial theory of isotropic ghost random variables and their related algebraic structures, as introduced by Alan Edelman [1]. These objects extend random matrix theory beyond the traditional dimensionalities of $\beta = 1, 2, 4$, corresponding to the real, complex, and quaternion cases, by generalizing to a continuous parameter $\beta \in \mathbb{R}_{>0}$. Ghost random variables should inspire an extension of understanding random matrix theory in arbitrary dimensions and provide new insights into matrix factorizations, polynomials, and stochastic processes.

We begin by giving the definition of an isotropic ghost random variable. For a given $\beta > 0$, an isotropic ghost is a β-dimensional random variable with spherically symmetric distribution. Formally, the probability density function of a ghost random variable x is defined as:

$$\int_0^\infty f_x(r) r^{\beta-1} S_{\beta-1}\, dr = 1,$$

where $S_\beta = \frac{2\pi^{\beta/2}}{\Gamma(\beta/2)}$ is the surface area of a β-dimensional sphere. This equation generalizes the concept of a normalized probability density in β dimensions.

Example 1.1 (Ghost Gaussian)**.** A standard β-dimensional Gaussian has the form:

$$f_x(r) = (2\pi)^{-\beta/2} e^{-r^2/2}.$$

The norm of a ghost random variable x is defined through the probability density given by the integrand of the previous equation:

$$\int_0^\infty f_x(r) r^{\beta-1}\, dr = 1.$$

In the case of a Gaussian ghost, the ghost norm corresponds to a random variable with a χ_β distribution.

Operations like addition and multiplication can be extended to independent ghosts. Given two independent ghosts x and y, their sum $z = x + y$ satisfies:

$$z^2 = x^2 + y^2 + 2cxy,$$

where c is an independent random variable distributed on the interval $[-1, 1]$ according to a beta distribution. The product of two ghosts $z = xy$ follows a simpler rule, defined by direct multiplication of the random variables.

Shadows are real or complex quantities derived from ghosts that allow the computation of meaningful results; which we won't have time to discuss much further. For now know that, the norm of a ghost is a shadow. Additionally, shadows play a crucial role in extending matrix factorizations like QR decomposition to ghost random matrices. The QR decomposition of a ghost matrix leads to orthogonal matrices of ghosts and provides a pathway to compute eigenvalues or singular values of the underlying structure.

1.2. **Free Probability:** Free probability theory is a non-commutative probability framework introduced by Dan Voiculescu in the 1980s. It provides a powerful tool for analyzing random matrices and operator algebras, offering an alternative to classical probability theory where independence is replaced by a notion of "freeness." This theory has found significant applications in random matrix theory, quantum mechanics, and large-scale statistical models.

In classical probability theory, two random variables x_1 and x_2 are independent if their joint distribution factorizes:

$$P(x_1 \leq y_1, x_2 \leq y_2) = P(x_1 \leq y_1)P(x_2 \leq y_2).$$

In contrast, in free probability, the concept of independence is replaced by "freeness," which applies to non-commutative random variables, typically operators on a Hilbert space. Two random variables x_1 and x_2 are said to be *free* if they satisfy certain algebraic relations involving their moments, which do not factorize as in the classical case. These algebraic relations depend on a "state" τ which is a linear functional acting on the random variables satisfying some (other) algebraic[1] properties that can be thought of as some abstraction of expectation.

Definition 1.2 ([2]). Elements (or random variables) $x_1, x_2, \ldots, x_n$ are called *free* or *freely independent* if for any m polynomials $p_k(x)$, $1 \leq k \leq m$, with $m \geq 2$, the expectation

$$\tau\left(p_1(x_{i_1})p_2(x_{i_2}) \cdots p_m(x_{i_m})\right) = 0,$$

provided that $\tau\left(p_k(x_{i_k})\right) = 0$ for all k, $1 \leq k \leq m$, and any two neighboring indices i_l and i_{l+1} are not equal, i.e., $1 \leq i_1, i_2, \ldots, i_m \leq n$ and $i_l \neq i_{l+1}$ for all l.

As an example, we see that from this condition, if two random variables x_1 and x_2 are free, then

$$\tau(x_1 x_2 x_1 x_2) = 0$$

whenever $\tau(x_1) = \tau(x_2) = 0$, where $m = 4$, $i_1 = 1$, $i_2 = 2$, $i_3 = 1$, $i_4 = 2$, and the polynomials $p_k(x) = x$ for $1 \leq k \leq 4$.

Comparing with the classical commutative case, it is immediately clear that real-valued random variables are not necessarily free. The terminology "free" originates in the motivating study of free groups, where there are no nontrivial relations between any generating elements.

Let $\mathcal{A}$ be a unital algebra with a trace $\tau : \mathcal{A} \to \mathbb{C}$. A collection of subalgebras $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \subseteq \mathcal{A}$ is said to be *free* if for any $a_i \in \mathcal{A}_{j_i}$ with $\tau(a_i) = 0$ and $j_1 \neq j_2 \neq \ldots \neq j_k$, we have:

$$\tau(a_1 a_2 \cdots a_k) = 0.$$

[1] To be defined later, see Section 2

Equivalently, these subalgbras are free when all of their elements are free as defined in Definition 1.2.

Example 1.3 ([3])**.** Taking $\mathcal{A} = M_{n\times n}(\mathbb{C})$, the space of $n \times n$ matrices with $\tau = \frac{1}{n}\operatorname{Tr}(T)$ for $T \in \mathcal{A}$ will be an example of a non-commutative probability space. Moreover, in the large n limit, elements of this probability space become free.

One of the central tools in free probability theory is the *R-transform*, which is analogous to the moment-generating function in classical probability. Given a non-commutative random variable X with its spectral distribution μ_X, the R-transform $R_X(z)$ is related to the moments of X and is defined through the relation:

$$R_X(G_X(z)) + \frac{1}{G_X(z)} = z,$$

where $G_X(z)$ is the Cauchy transform of μ_X:

$$G_X(z) = \int_{\mathbb{R}} \frac{d\mu_X(t)}{z-t}.$$

The R-transform linearizes the addition of free random variables, which is a key property that simplifies calculations in free probability theory.

The free convolution operation is central to free probability, analogous to the convolution of independent random variables in classical probability. For two free random variables X and Y, the spectral distribution of their sum is given by the free convolution of their individual distributions. This is computed through the R-transform:

$$R_{X+Y}(z) = R_X(z) + R_Y(z),$$

which makes the R-transform a particularly useful tool in analyzing sums of free random variables.

Free probability has important applications in random matrix theory, particularly in the study of large n limits of random matrices. For example, the empirical eigenvalue distribution of large random matrices converges to deterministic distributions (e.g. semicircular law), and free probability provides tools for calculating these limiting distributions. One famous result from Voiculescu states that the eigenvalue distributions of independent large random matrices become asymptotically free. This provides a powerful framework for understanding large-scale statistical behavior in matrix models. Moreover, free probability theory generalizes many classical concepts of probability into the non-commutative setting, allowing for the study of free random variables, which exhibit behaviors distinct from independent random variables. Its applications to random matrix theory and operator algebras make it a powerful framework for understanding large random systems, especially in the context of spectral analysis and large-scale models.

1.3. **Non-Commutative Geometry (NCG).** Non-commutative geometry, pioneered by Alain Connes, is a framework that extends the methods of differential geometry to spaces where the coordinates no longer commute. It allows the study of geometric and topological structures on spaces that may not have a classical pointwise description. This theory has deep connections with operator algebras, quantum physics, and number theory, and provides a new lens through which to view classical geometry in a broader context.

In classical geometry, a space is typically described by coordinates, or even a commutative algebra of functions. For example, a smooth manifold M is characterized by the algebra $C^\infty(M)$ of smooth functions on M, where pointwise multiplication is commutative:

$$f(x)g(x) = g(x)f(x) \quad \text{for all } x \in M.$$

It turns out that thinking of spaces by the associated algebra of functions is a rich topic, but we had better be more precise. Without providing every definition now[2], consider the following.

Definition 1.4. If X is a locally compact Hausdorff space, then the set $C_0(X)$ of all continuous complex-valued functions on X vanishing at infinity (i.e., for each $\epsilon > 0$, the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact), equipped with the supremum norm

$$\|f\| = \sup\{|f(x)| : x \in X\},$$

is a commutative C^*-algebra.

Moreover, we have the following theorem of Gelfand.

Theorem 1.5 (Gelfand)**.** *Taking X to be a locally compact Hausdorff space, then each commutative C^*-algebra A is of the form $C_0(X)$, where X is the maximal ideal space of A. Moreover, $C_0(X)$ is unital if and only if X is compact.*

In non-commutative geometry, the commutativity condition is relaxed, and spaces are described by non-commutative algebras. The guiding principle is to replace the algebra of functions on a space with a non-commutative algebra, while still retaining geometric and topological insights. This is seemingly best described by the following diagram, of which could probably be attributed to many, but was in this case taken from [4].

$$\begin{tikzcd}
\textbf{Manifold } \mathcal{M} \arrow[d] & \textbf{`Non-commutative manifold'?} \arrow[d, "\text{``?''}"] \\
\textbf{Commutative algebra } A \cong C_0(\mathcal{M}) \arrow[r] \arrow[d] & \textbf{A non-commutative algebra } \hat{A} \arrow[d] \\
\begin{array}{c}\textbf{All geometric objects} \\ \textbf{on } \mathcal{M} \textbf{ obtained from } A\end{array} \arrow[r] & \begin{array}{c}\textbf{Geometric objects obtained} \\ \textbf{in the same way from } \hat{A}\end{array}
\end{tikzcd}$$

Example 1.6. A central example of a non-commutative space is the *non-commutative torus* A_θ, which is defined as a C^*-subalgebra of $B(L^2(S^1))$, the algebra of bounded linear operators on the Hilbert space of square-integrable functions on the unit circle $S^1 \subset \mathbb{C}$. It is generated by two unitary operators U and V, defined as:

$$(Uf)(z) = zf(z), \quad (Vf)(z) = f(ze^{-2\pi i\theta}),$$

where $f \in L^2(S^1)$ and $z \in S^1$, and θ some irrational rotation parameter. A straightforward calculation verifies the commutation relation:

$$VU = e^{-2\pi i\theta}UV.$$

[2]See section 2

More generally, it turns out that the algebra A_θ can be defined up to isomorphism as the universal C^*-algebra generated by two unitary elements U and V satisfying the relation:

$$VU = e^{2\pi i\theta}UV,$$

which extends to the case when θ is rational. In particular, for $\theta = 0$, A_θ is isomorphic to the algebra of continuous functions on the 2-torus via the Gelfand transform.

Remark 1.7. One may find it peculiar that the noncommutative *torus* is defined with its base object as a circle! However, as discussed in [5], this algebra shows up in many places. Indeed, as above it is exactly the C^*-algebra generated by any pair of unitary operators U and V that satisfy the commutation relation $UV = \lambda VU$, where $\lambda = e^{-2\pi i\theta}$. However, it is also exactly the simple C^*-algebra on which the torus group T^2 has ergodic actions! There are many other ways where one finds this algebra, so we urge the interested reader to look at [5].

A fundamental tool in non-commutative geometry is the concept of a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, where:

- $\mathcal{A}$ is a non-commutative algebra,
- $\mathcal{H}$ is a Hilbert space on which $\mathcal{A}$ acts,
- D is a self-adjoint operator on $\mathcal{H}$, called the *Dirac operator*, that encodes geometric information.

The spectral triple generalizes the notion of a differential structure on a manifold, with the Dirac operator playing a central role in defining the geometry. The eigenvalues of D are related to the metric properties of the space, and its commutators with elements of $\mathcal{A}$ play the role of differential operators. The spectral triple is not a notion that will be particularly useful in what follows, however, in the spirit of generalizing differential structure, we will need integration.

In classical geometry, integration is defined using differential forms and the exterior derivative, which can be classified through the tools of de Rham cohomology. We recall the fact that de Rham cohomology classifies closed forms up to exact forms, or more concretely, the fact that the value of an integral on a submanifold depends only on the de Rham cohomology class of the form being integrated, not on the specific representative. In non-commutative geometry, integration is generalized through *cyclic cohomology*, which serves as a non-commutative analogue of de Rham cohomology. Indeed, at a higher level, K-theory gives a generalization of cohomology theories with respect to homotopy. Broadly, K-theory classifies vector bundles (or their analogues in a non-commutative setting) over a space or an algebra, capturing topological and geometric information. Moreover, K-theory gives us the tools to formally deal with classes of projectors in our algebra which is what we will focus need in the next sections. Without diving further, we note that the pairing of cyclic cohomology with K-theory allows for the definition of a trace on the algebra $\mathcal{A}$, generalizing the notion of integration or an expectation over a space:

$$\langle \varphi, [e] \rangle = \mathrm{Tr}_\varphi(e),$$

where φ is a cyclic cocycle and e is a projector in the algebra.

It is easy to find ourselves falling into the rabbit hole of abstraction. Consider the following example:

Example 1.8. Let $\mathcal{A} = C([0,1], M_N(\mathbb{C}))$, the algebra of continuous functions from the interval $[0,1]$ to the space of $N \times N$ complex matrices. Elements of $\mathcal{A}$ are continuous maps $a : [0,1] \to M_N(\mathbb{C})$, and the pointwise product and adjoint define the algebra structure.

Consider the normalized trace:

$$\varphi(a) = \frac{1}{N} \int_0^1 \mathrm{Tr}(a(t))\, dt,$$

where $\mathrm{Tr}(a(t))$ denotes the usual matrix trace at each point $t \in [0,1]$. This φ is a cyclic cocycle on $\mathcal{A}$ in the sense of non-commutative geometry. Indeed, if one was to take the unit operator defined as sending $1 : [0,1] \to I_N$ i.e. $1(t) = I_N$ for all t, then it is easy to see that

$$\varphi(1) = \frac{1}{N} \int_0^1 \mathrm{Tr}(I_N) dt = \frac{1}{N} \int_0^1 N dt = \frac{N}{N} = 1$$

Now suppose $e \in \mathcal{A}$ is a rank-k continuous projector, meaning $e(t)^2 = e(t)$ and $\mathrm{Tr}(e(t)) = k$ for all $t \in [0,1]$. The pairing of the cyclic cocycle φ with the K-theory class $[e]$ is given by:

$$\langle \varphi, [e] \rangle = \int_0^1 \varphi(e)\, dt = \frac{1}{N} \int_0^1 \mathrm{Tr}(e(t))\, dt.$$

Since $\mathrm{Tr}(e(t)) = k$ for all t, this simplifies to:

$$\langle \varphi, [e] \rangle = \frac{k}{N}.$$

This result has the following interpretation:

- The projector e continuously selects k dimensions at each point $t \in [0,1]$, effectively defining a subbundle of rank k over the interval $[0,1]$.
- The normalized trace φ integrates this selection over the interval, giving the fraction of the total space associated with e. This is analogous to integrating a density function over a classical space to measure the size of a subset.

The upshot is as follows. Non-commutative geometry provides a framework for extending classical geometric ideas to spaces described by non-commutative algebras. By generalizing tools such as integration, differentiation, and topology, it opens new avenues for understanding the structure of space in contexts where the classical notion of pointwise geometry breaks down.

1.4. **Path forward.** The idea hereforeward is to take tools from the last two and apply them to construct a free probability analogue of the first. Observe that in classical spaces there is no notion of sphere of β dimension for the lack of fields or division algebras of non-integers dimensions: and in that case we are “not bothered in the slightest!” by the fact that we cannot sample these objects by lacking a construction of a β-dimensional sphere [1]. Now, we throw out the idea that we need coordinates at all to deal with a theory of manifolds and can use the power of NCG and free probability to define these objects for arbitrary dimension.

In the next section we provide some more preliminary information and results that will be useful in our exploration of the area. Then in Section 3, we provide preliminary definitions of these objects and Section 4 discusses some ideas on wrapping everything back into the world of fractals.

2. Preliminaries

Definition 2.1 (C^*-Algebra)**.** A C^*-algebra $\mathcal{A}$ is an associative algebra that is normed and complete (Banach) with an involution $a \mapsto a^*$ and a norm satisfying the C^*-condition:

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in \mathcal{A}.$$

Definition 2.2 (Self-Adjoint Elements)**.** An element $a \in \mathcal{A}$ is called self-adjoint if $a = a^*$.

Definition 2.3 (Projections)**.** A projection $p \in \mathcal{A}$ is a self-adjoint element that satisfies $p^2 = p$.

In Noncommutative Geometry, the full algebra $\mathcal{A}$ defines the noncommutative space, while specific classes of elements like self-adjoint operators, projections, and positive elements encode important geometric features. As exposited in Theorem 1.5, C^* algebras correspond to locally compact Hausdorff spaces, in a similar vein, measurable spaces correspond to commutative von Neumann algebras, so in studying non-commutative von Neumann algebras, we can recover a non-commutative probability theory.

Definition 2.4 (State)**.** A *state*, τ, is a $*$-linear homomorphism that maps $1 \mapsto 1$ and satisfies for any associated unital algebra $\mathcal{A}$, that for every $x \in \mathcal{A}$,

$$\tau(xx^*) \geq 0.$$

Definition 2.5 (Trace Axiom)**.** We say that the state τ is *tracial* if for any two elements x and y, we have $\tau(xy) = \tau(yx)$.

Definition 2.6 (Faithfulness Axiom)**.** We say that the state τ is *faithful* when $\tau(x^*x) = 0$ if and only if $x = 0$.

Definition 2.7 ([6][7])**.** A non-commutative probability space $(\mathcal{A}, \tau)$ consists of a $*$-algebra $\mathcal{A}$ with identity 1, together with a $*$-linear functional $\tau : \mathcal{A} \to \mathbb{C}$, that maps 1 to 1 and obeys the non-negativity axiom. Additionally, if τ obeys the trace axiom, we say that the non-commutative probability space is *tracial*. If τ obeys the faithfulness axiom, we say that the non-commutative probability space is *faithful*.

Definition 2.8 ([3])**.** Let (A, τ) be a noncommutative probability space. We say $a \in A$ has a distribution μ, where μ is a probability measure on $\mathbb{R}$, if

$$\tau(a^k) = \int_{-\infty}^{\infty} t^k \, d\mu(t)$$

for all $k \in \mathbb{N}$.

Example 2.9 ([3])**.** Consider $(M_{n\times n}(\mathbb{C}), \tau_n)$, where $\tau_n(T) = \frac{1}{n}\operatorname{tr}(T)$. Let $T \in M_{n\times n}(\mathbb{C})$ be self-adjoint. Then there exists an orthonormal basis $e_1, \ldots, e_n$ in $\mathbb{C}^n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $Te_j = \lambda_j e_j$, $j = 1, \ldots, n$. We have

$$\tau_n(T^k) = \frac{1}{n}\sum_{j=1}^{n} \lambda_j^k,$$

where $\mu = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j}$. That is, T has distribution μ.

Example 2.10 (Construction of random matrix space [3])**.** We can construct the space of random matrices as follows. Given a probability space $(\Omega, \Sigma, \mathbb{P})$ and $n \in \mathbb{N}$, we define $\mathcal{A} = M_n(L^p(\Omega))$,

$1 \leq p < \infty$. For $X = [x_{i,j}] \in \mathcal{A}$, the entries $x_{i,j} : \Omega \to \mathbb{C}$ are random variables with $\mathbb{E}|x_{i,j}|^k < \infty$ for all $k \in \mathbb{N}$. Define $\varphi : \mathcal{A} \to \mathbb{C}$ by

$$\varphi(X) = \mathbb{E}(\tau_n(X)) = \frac{1}{n} \sum_{j=1}^{n} \int_{\Omega} x_{j,j} \, dP.$$

Then $(\mathcal{A}, \varphi)$ is a noncommutative probability space.

Suppose that $X(\omega) = X(\omega)^*$ for all $\omega \in \Omega$. Then

$$\varphi(X^k) = \mathbb{E}(\tau_n(X(\omega)^k)) = \mathbb{E}\left(\int_{-\infty}^{\infty} \lambda^k \, d\mu_\omega(\lambda) \right),$$

where μ_ω is the empirical eigenvalue distribution of $X(\omega)$. A distribution of X is then given by $\mathbb{E}\mu_\omega$.

2.1. **Von Neumann Algebras.** It turns out, when discussing von Neumann Algebras, one has some options on how to define them.

Definition 2.11. A von Neumann algebra is a C^* algebra $\mathcal{A}$ such that $\mathcal{A}$ is closed in the weak operator topology and has an identity element.

We recall the definition of the weak operator topology here as well:

Definition 2.12. The *weak operator topology* is the weakest topology on the set of bounded operators of a Hilbert space, H, such that the functional sending an operator $T \mapsto \langle Tx, y \rangle$ is continuous for any $x, y \in H$.

One can also describe von Neumann algebras in terms of the "bicommutant theorem" which we choose not to exposit. Most concretely, von Neumann algebras tend to be defined as weakly-closed C^* algebras of bounded operators acting on a Hilbert space containing the identity. The condition that the algebra be explicitly associated with a Hilbert space is not actually necessary. In [8], Sakai showed that von Neumann algebras, considered as a Banach space, can be defined more abstractly as being the dual of some other Banach space. So it is not necessary to specify a Hilbert space when working with the von Neumann algebras, although for concreteness in examples we often will.

2.2. **Factors.** Every von Neumann algebra can be constructed from so called "factors" via direct integration, a generalization of the direct sum. The definition of a factor is as follows:

Definition 2.13. A *factor* is a von Neumann algebra M whose center consists of only scalar multiples of the identity operator.

Factors are classified into three main types, and within each type, there are subclasses. Below is a description of these classifications:

2.2.1. *Type I.* A factor is said to be of type I if there is a minimal projection $E \neq 0$, i.e., a projection E such that there is no other projection F with $0 < F < E$. Any factor of type I is isomorphic to the von Neumann algebra of all bounded operators on some Hilbert space.

For separable Hilbert spaces, type I factors are further subdivided as follows:

- A factor of type I_n: The bounded operators on a Hilbert space of finite dimension n.
- A factor of type I_∞: The bounded operators on a separable infinite-dimensional Hilbert space.

Example 2.14 (Type I)**.** A classic example of a type I factor is the algebra of bounded operators on a Hilbert space H. For a finite-dimensional Hilbert space $H = \mathbb{C}^n$, the algebra $B(H)$ is isomorphic to $M_{n\times n}(\mathbb{C})$, the space of $n \times n$ complex-valued matrices, which is a type I_n factor.

2.2.2. *Type II.* A factor is said to be of type II if it contains no minimal projections but has non-zero finite projections. This implies that every projection E can be "halved," meaning there exist two projections F and G that are Murray–von Neumann equivalent and satisfy $E = F + G$.

Type II factors are further subdivided:

- **Type** II_1: The identity operator is finite, and there exists a unique finite tracial state. The set of traces of projections is $[0, 1]$.
- **Type** II_∞: The identity operator is infinite, but there exists a semifinite trace, unique up to rescaling. The set of traces of projections is $[0, \infty]$.

Type II_1 factors have a rich structure, with the hyperfinite type II_1 factor being the unique hyperfinite factor of this type which we construct below in Example 2.15. Similarly, the hyperfinite type II_∞ factor is the unique hyperfinite factor of type II_∞. Other type II factors, exists of course, but we will not be discussing them. The implication that projections can be "halved" gives us the notion of continuous geometry! Namely, the dimension of the subspaces given by the projections, measured by the tracial states[3], no longer are restricted to positive integer dimension.

In the following example we give a rough construction of R, the "hyperfinite type II_1 factor", equipped with a faithful, normal tracial state $\tau : R \to \mathbb{C}$. The hyperfinite type II_1 factor is the unique smallest infinite dimensional factor.

Example 2.15 (Hyperfinite Type II_1 Factor)**.** The algebra R is constructed as the *direct limit* of an inductive system of finite-dimensional matrix algebras:

$$R = \varinjlim M_{d_n}(\mathbb{C}),$$

where $M_{d_n}(\mathbb{C})$ denotes the algebra of $d_n \times d_n$ complex matrices, and the trace τ is normalized so that $\tau(I_{d_n}) = 1$ for the identity I_{d_n} in each $M_{d_n}(\mathbb{C})$.

More intuitively, R can be thought of as a "fancy union" or an infinite-dimensional limit of these matrix algebras, where the structure is preserved under the inclusion maps:

$$\phi_n : M_{d_n}(\mathbb{C}) \to M_{d_{n+1}}(\mathbb{C}),$$

defined as:

$$\phi_n(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

In this construction:

- Each $M_{d_n}(\mathbb{C})$ is embedded into $M_{d_{n+1}}(\mathbb{C})$ via ϕ_n,
- The direct limit R contains elements represented by sequences $(A_n)_{n\geq 1}$, where $A_n \in M_{d_n}(\mathbb{C})$ and the embeddings $\phi_n(A_n)$ are consistent across n.

This construction gives R its unique hyperfinite and type II_1 properties [9]. Namely, we have that R is the smallest infinite dimensional factor that is contained in any other infinite dimensional factor, and any infinite dimensional factor contained in R is isomorphic to R.

[3]Specifically, the Dixmier Trace is often used as the "correct" generalization of integration.

2.2.3. *Type III.* Type III factors are those that contain no non-zero finite projections. In these factors, the identity operator is infinite, and there is no finite trace. Instead, the only trace takes value ∞ on all non-zero positive elements.

Type III factors are further classified based on their Connes spectrum:

- **Type** III_0: The Connes spectrum is $\{1\}$.
- **Type** III_λ $(0 < \lambda < 1)$: The Connes spectrum is all integral powers of λ.
- **Type** III_1: The Connes spectrum is all positive reals.

Tomita–Takesaki theory provides a powerful framework for understanding type III factors, showing that any type III factor can be expressed as the crossed product of a type II_∞ factor and the real numbers. As a result the main study of factors boils down to the study of type I and II.

Example 2.16 (Type III)**.** With lack of time to give sufficient care to this example, but not wanting to omit it outright, we add that the crossed product of the hyperfinite type II_∞ factor with $\mathbb{R}$ will be of type III. Indeed if we let $\mathcal{R}$ be the hyperfinite type II_∞ factor, then this presents as:

$$M = \mathcal{R} \rtimes \mathbb{R}$$

Again without diving too deep into abstraction, a *crossed product* is a way to construct a new von Neumann algebra from an existing one that is acted on by a group. It can be thought of as a non-commutative analogue of forming a semidirect product in group theory. Roughly speaking, it combines the original algebra with the group action, encoding both the algebraic structure of the group and the dynamics of its action on the algebra.

Observing that the type II factors admit non-integer trace, it is this class of algebras that we will be interested in for the generalization of β-dimensional geometry as introduced in Edelman's ghosts.

2.3. **K-theory.** Perhaps unfortunately, it is most convenient to introduce a small amount of working K-theory in order to understand more properly the definitions and examples to come in Section 3.

Without discussing K-theory in full detail (or even motivating it!), we take K_0 to be the K-theory group corresponding to the equivalence classes of projections in a C^*-algebra or von Neumann algebra (technically, in algebraic K-theory, one cares about projective submodules in the ring, but these end up corresponding to projections in the C^* algebra!). Indeed, recall that projections are idempotent, self-adjoint operators ($p^2 = p$ and $p^* = p$) that generalize the notion of rank-k projectors in finite-dimensional settings.

Remark 2.17. In the context of von Neumann algebras, K_0 organizes the "sizes" or "dimensions" of subspaces represented by these projections. Two projections p and q are considered equivalent in K_0 if they are "isomorphic," meaning they represent the same underlying subspace up to re-labeling (technically, there exists a partial isometry connecting them).

Remark 2.18 (Utility). For a hyperfinite type II_1 factor R, the K_0-group reflects the structure of projections under the faithful, normal trace τ, which classifies projections up to equivalence. In R, the K_0-group is directly tied to the range of τ, which is $[0, 1]$. Every equivalence class of projections is uniquely identified by its trace value.

3. Free Random Ghosts: A Starter

Definition 3.1. A family of random variables $(f_t)_{t\in I} \subseteq A$ in a $*$-probability space (A,φ) is called *free* if for any $n \geq 1$, subalgebras $A_1, A_2, \ldots, A_n \subseteq A$, and elements $a_k \in A_{i_k}$ such that $\varphi(a_k) = 0$, the following holds:

$$\varphi(a_1 a_2 \cdots a_n) = 0 \quad \text{whenever } i_k \neq i_{k+1} \text{ for all } k.$$

Definition 3.2 ([6])**.** A family of random variables $(f_t)_{t\in I}$ in a $*$-probability space (A,φ) is called a *semicircular family* if:

(1) Each f_t is self-adjoint, i.e., $f_t = f_t^*$,
(2) The moments of f_t with respect to φ are given by the semicircle law:

$$\varphi(f_t^k) = \frac{2}{\pi}\int_{-1}^{1} x^k \sqrt{1-x^2}\, dx = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{1}{n+1}\binom{2n}{n}\sigma^{2n} & \text{if } k = 2n. \end{cases},$$

for all $k \geq 1$,

Definition 3.3. A family of random variables $(f_t)_{t\in I}$ in a $*$-probability space (A,ϕ) is called *isotropic* if the family $(f_t)_{t\in I}$ is invariant under a group of symmetries $G \subseteq A$ "analogous to the orthogonal group[4]", where for any $U \in G$, conjugation preserves the distributions:

$$\varphi((Uf_tU^*)^k) = \varphi(f_t^k) \quad \text{for all } t \in I \text{ and } k \geq 1.$$

Definition 3.4. Let (A,τ) be a Type II von Neumann algebra equipped with a faithful, normal tracial state τ. A projection $p \in A$ (i.e., $p^2 = p = p^*$) is said to define a *β-dimensional isotropic subspace* if:

$$\tau(p) = \beta, \quad \beta > 0.$$

The subspace $\mathrm{Im}(p) = \{v \in H \mid pv = v\}$ is invariant under the group of unitaries $\mathcal{U}_p = \{U \in A \mid Up = pU\}$. The dimension β is preserved under these symmetries, ensuring that $\mathrm{Im}(p)$ behaves isotropically with respect to rotations in the von Neumann algebraic sense.

Definition 3.5. A *free random ghost* is a family of random variables $(f_t^{(p)})_{t\in I}$, where $(f_t)_{t\in I}$ is a free family in (A,τ), and each $f_t^{(p)} = pf_tp$[5] is restricted to the isotropic subspace $\mathrm{Im}(p)$. The distribution of each $f_t^{(p)}$ is determined by the symmetry-preserving projection p, and the family *should* retain its freeness and rotational invariance under the action of $\mathcal{U}_p$.

Example 3.6. Let (A,τ) be a von Neumann algebra with a faithful, normal tracial state τ, and let (f_1, f_2) be a family of two freely independent random variables in A. Suppose each f_i is semicircular, with variance σ^2, meaning their distributions are given by the semicircle law:

$$\tau(f_i^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{1}{n+1}\binom{2n}{n}\sigma^{2n} & \text{if } k = 2n. \end{cases}$$

Now, let $p \in A$ be a projection with $\tau(p) = \beta > 0$. The subspace $\mathrm{Im}(p) = \{v \in H \mid pv = v\}$ is rotationally invariant because:

[4]What we mean here is not precise and will likely need refining. Indeed, we might want to make sure that G is "large enough" to capture a sufficient notion of rotational invariance. This might involve requiring that G acts transitively on the set of unit vectors in $\mathrm{Im}(p)$, or some other condition that ensures a suitable degree of symmetry.

[5]The cylic property of the trace implies: $\tau(pf_tp) = \tau(ppf_t) = \tau(pf_t)$ and $\tau(pf_tp) = \tau(ppf_t) = \tau(f_tpp) = \tau(f_tp)$

(1) The group of unitaries $\mathcal{U}_p = \{U \in A \mid Up = pU\}$ preserves $\mathrm{Im}(p)$ under conjugation.
(2) The restricted family $(f_1^{(p)}, f_2^{(p)})$, where $f_i^{(p)} = pf_ip$, retains the freeness of (f_1, f_2) and satisfies rotational invariance under $\mathcal{U}_p$.

The family $(f_1^{(p)}, f_2^{(p)})$ defines rotationally invariant free random ghosts confined to the isotropic subspace $\mathrm{Im}(p)$, preserving both their freeness and semicircular distributions.

Example 3.7 (Projection Operators in the Hyperfinite Type II_1 Factor)**.** Building on Examples 1.8 and 2.15, we now consider the hyperfinite type II_1 factor R. Just as $\mathcal{A} = C([0,1], M_N(\mathbb{C}))$ allows us to study matrix-valued functions over a continuous interval, the hyperfinite type II_1 factor R can be thought of as a 'wildly' infinite-dimensional generalization of such structures, where the normalized trace plays an analogous role to the functional φ in $\mathcal{A}$.

Let R be equipped with a faithful, normal trace $\tau : R \to \mathbb{C}$, satisfying:

$$\tau(p^2) = \tau(p), \quad \tau(p) \geq 0, \quad \forall p \in R.$$

Projections in R correspond to elements $p \in R$ such that $p^2 = p$ and $p^* = p$. These projections generalize the rank-k projectors from $\mathcal{A}$ and play a central role in the $K_0(R)$-group.

Just as the normalized trace φ in $\mathcal{A}$ integrates over the continuous interval $[0,1] \in \mathbb{R}$ (p is self adjoint), the trace τ in R induces a map on the K_0-group of projections:

$$\tau : K_0(R) \to \mathbb{R}, \quad [p] \mapsto \tau(p).$$

The range of τ on $K_0(R)$ is the interval $[0,1]$, capturing the possible "sizes" of projections in R. This includes projections of fractional trace, such as $\tau(p) = 0.4348$. To construct a projection $p \in R$ with such dimension, consider the following:

- Just as in Example 1.8, where a rank-k projector $e \in C([0,1], M_N(\mathbb{C}))$ satisfied $\varphi(e) = \frac{k}{N}$, in R, a projection p corresponds to a subspace whose "dimension" relative to τ is 0.4348.
- The trace $\tau(p)$ can be verified by pairing τ with the $K_0(R)$-class $[p]$, analogous to:

$$\langle \tau, [p] \rangle = \tau(p).$$

Here, the cocycle τ plays the same role as φ in $\mathcal{A}$, ensuring $\tau(p) = 0.4348$.

Now, consider a family of freely independent semicircular elements $(f_1, f_2, \ldots, f_n) \subset R$. The projection p restricts these elements to the 0.4348-dimensional subspace $\mathrm{Im}(p)$. Define the restricted elements as:

$$f_i^{(p)} = pf_ip, \quad \text{for all } i = 1, \ldots, n.$$

The restricted elements $(f_1^{(p)}, f_2^{(p)}, \ldots, f_n^{(p)})$ are self adjoint and retain their freeness under the trace τ, but now their action is confined to the subspace determined by p. One might hope that the trace of any moment of $f_i^{(p)}$ is given by

$$\tau\left((f_i^{(p)})^k\right) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \tau(p)\frac{1}{n+1}\binom{2n}{n}\sigma^{2n} & \text{if } k = 2n. \end{cases}$$

or, since $\tau(p) = 0.4348 \approx S_\beta = \frac{2\pi^{1/6}}{\Gamma(1/6)}$ for $\beta = \frac{1}{3}$, then we might have:

$$\tau\left((f_i^{(p)})^k\right) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ S_\beta\frac{1}{n+1}\binom{2n}{n}\sigma^{2n} & \text{if } k = 2n. \end{cases}$$

Remark 3.8. In the previous example we chose the trace to correspond to S_β where $\beta = 1/3$. Whether or not to choose projections such that we scale by the dimension or scale by the arbitrary dimensional spherical area is yet to be determined. Indeed, either of them may make sense, or Section 4 may inform us to do something entirely different.

Remark 3.9. As another remark to the previous example, we note that this is implicitly using the idea that

$$\tau(pxp) = \tau(px) = \tau(p)\tau(x)$$

While the first equality is true due to the cyclic property of the state, we have not found a source for the second equality, and so the final bit of the example should (for now) be taken with a grain of salt.

4. Analytic Ghosts

Everything that we just discussed above acts in operator space and is, of course, not defined pointwise. Moreover, most of what occurs in Section 3 comes from well established mathematics, and it is noted when it deviates. In the following, much less reading has gone into establishing these thoughts, and thus nothing is framed as being close to a 'definition' or 'result', but is instead quite speculative. With the warning in mind, it would be fun if we could somehow connect the original analytic description of isotropic ghosts to the generalization described above. In the following we outline how this might be realized, and it is likely this section where the most time will be spent sharpening in the coming months. Moreover, getting Section 3 to be defined such that the definitions line up with the following would make everything more concrete and frankly more exciting.

4.1. Fractal-Like Character Spaces in β-Ghosts.

In commutative algebras, the space of characters (algebra homomorphisms into $\mathbb{C}$) corresponds to the spectrum of the algebra, which often even surjects onto the complex plane. This allows for a complete analytic formulation of random variables, where we can evaluation at points in this spectrum to end up back in $\mathbb{C}$.

In the non-commutative setting, however, the space of characters is significantly constrained. Non-commutativity introduces relations that reduce the number of characters, leaving behind a space that is no longer dimensionally equivalent to $\mathbb{C}$. For $\beta \in (0, 1)$, this space might become reminiscent of a fractal or a Cantor-like set (or maybe we force the ghosts to obey such an analytic structure?) [10]. Such a structure would be characterized by:

- A fractional Hausdorff dimension corresponding to the trace $\tau(p) = \beta$ (or something related).
- A subset of $\mathbb{C}$ or another domain that is neither a line nor a plane, but something intermediate, analogous to a fractal?

This fractal-like character space, denoted $\mathcal{X}_\beta$, should naturally align with the structure of β-ghosts, where the projection p defines a subspace of dimension β. The following ideas show potential interactions between $\mathcal{X}_\beta$ and the analytic formulation of β-ghosts.

4.1.1. *Character Space $\mathcal{X}_\beta$.*

The space of characters $\mathcal{X}_\beta$ is defined as the subset of the spectrum of the non-commutative algebra determined by the projection p:

$$\mathcal{X}_\beta = \{\chi : \mathcal{A} \to \mathbb{C} \mid \chi \text{ is a character and respects } p\}.$$

For $\mathcal{A}$ representing some $*$-probability space $\mathcal{X}_\beta$ is a fractal-like subset of $\mathbb{C}$ or more specifically is the algebraic dual space of the projected subspace of the algebra. This subset reflects the dimensional reduction imposed by p and the constraints of non-commutativity.

Remark 4.1. An immediate issue stems from the fact that there is no reason why the character space of an arbitrary projection in a type II von Neumann algebra should have some sort of recursive fractal structure. That being said, there is reason to believe that in *can* be fractal in some cases, and it is likely narrowing down what those are that will help inform the path forward for this section.

4.1.2. *Fractal Integration.* Clearly sampling in the usual sense for β ghosts is not automatically well defined because of the lack of β dimensional commutative algebras. In the non-commutative setting, β-ghosts cannot be sampled in the conventional sense of evaluation at points in a spectrum, as would be possible in a commutative algebra. Thus, the random variables in this context might be better understood as an *integration process* over the fractal-like character space $\mathcal{X}_\beta$, which captures the non-commutative spectral behavior of the ghosts. Specifically:

- **Fractal Measure:** A measure μ_β defined on $\mathcal{X}_\beta$ respects the fractional dimensionality imposed by $\tau(p) = \beta$.
- **Analytic Formulation:** One might compute expectations or moments via integration against μ_β:
$$\mathbb{E}[f(Z_\beta)] \sim \int_{\mathcal{X}_\beta} f(x)\, d\mu_\beta(x),$$
where $f(Z_\beta)$ is a functional of the ghost random variable Z_β.

4.1.3. *Spectral Distributions on $\mathcal{X}_\beta$.* Operators acting on the ghost space are associated with spectral measures defined on $\mathcal{X}_\beta$. For example:

- A semicircular operator Z_β restricted to $\mathcal{X}_\beta$ has a spectral distribution defined by a measure μ_β similar to that given in Definition 3.2
- Spectral measures respect the fractional dimension imposed by the trace, ensuring $\tau(p) = \beta$.

4.1.4. *Fractal Dimension and Free Probability.* The fractal nature of $\mathcal{X}_\beta$ suggests connections to fractional-dimensional random processes. These include:

- **Fractional Spectral Measures:** Generalizing free probability distributions (e.g., semicircular or free convolution) to fractional-dimensional spaces [11].
- **Scaling Relations:** The trace $\tau(p)$ determines the scaling of moments and variance, consistent with fractal measures.

5. Concluding Remarks

I'll conclude with a few remarks on the project. This will continue to be worked on in the spring and so I request that it not be distributed or posted anywhere just yet. I have a few other thoughts about this material that I haven't included (brevity) but look forward to exploring. Any thoughts or comments on what has been written in this document so far are always welcome. I am also incredibly grateful to have taken the course, and it has easily been one of my favorite classes at MIT. I was unfamiliar with almost all of the material in this document even in October of this year, and getting the opportunity to speedrun the learning process for this material over the last two months has been remarkably exciting: so much so that I often would wake up in the middle of the night with ideas. So for this experience I am grateful. Lastly, I feel mostly confident in the material from sections 1, 2, and 3 and I expect, or at least hope, that the material from Section 4 will inform the choice of direction for tightening up the frayed ends in Section 3. I look forward to what is to come!

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