RANDOM REFLECTIONS IN $\mathbb{R}^2$

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1. Summary

Much is known about reflections in $\mathbb{R}^2$, but iterated sequences of reflections are less well-studied. In this project, we consider reflections across lines that are tangent to circles centered at the origin, with the angle of each line uniformly distributed modulo 2π. The most obvious property to study is the expected norm of reflected points, but it turns out that the squared norms have a particular geometric significance which makes them a more natural object of study. We use classical geometry to find the expected squared norm of a point reflected across n random lines tangent to a circle of radius r. Then, we perform a more involved calculation to find the variance of the squared norm of a point reflected across n lines (Theorem 1, Corollary 1). Finally, we let r be a random variable. This allows us to provide answers to the otherwise ill-posed question: what is the expected squared norm of a point reflected across n random lines in $\mathbb{R}^2$? **All main results presented in Sections 2 and 3 have been computationally verified by running Monte Carlo simulations in Julia.**

Note to Prof. Edelman: when I came up with this project idea early in the class, I thought it would be a neat application of classical random matrix theory to classical geometry. I have since realized that the reflection matrices in $\mathbb{R}^2$ are too small for random matrix theory to offer an advantage over classical techniques! Nevertheless, I have been very happy with the math I have discovered during this project, so I'm glad I picked it as my final project. Perhaps a continuation of this project to consider reflections across hyperplanes in higher dimensional spaces would be a place where some random matrix theory results could be used.

2. Main Results

Theorem 1 (Reflection of z about n lines tangent to circle with radius r)**.**

$$\mathbb{E}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^2) = 4r^2n + \|z\|^2$$
$$\mathbb{E}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^4) = \|z\|^4 + 16\|z\|^2r^2n + 16r^4(2n^2 - n).$$

Corollary 1.

$$\mathrm{Var}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^2) = 8\|z\|^2r^2n + 16r^4(n^2 - n)$$

Proof. The first part of Thm 1 follows directly by Apollonius's Theorem, which relates the squared length of the median of a triangle to the squared lengths of the triangle's sides.

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Reflecting a point z with respect to a particular line, we can use a geometric construction to find the average squared norm of the reflected point (Fig. 1).

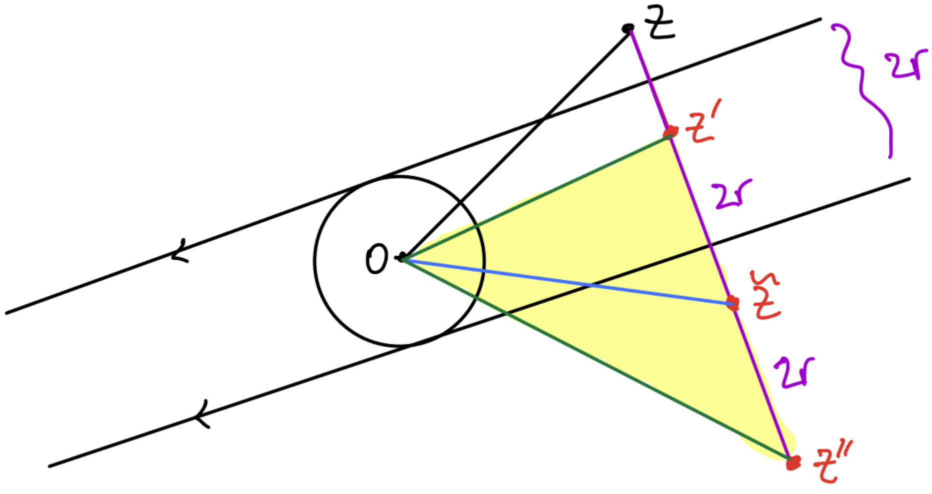


Figure 1. Illustration of the geometric construction.

Apollonius's Theorem states

$$\|z'\|^2 + \|z''\|^2 = 2(4r^2 + \|\tilde{z}\|^2). \tag{1}$$

Since the reflections are across parallel lines, we have that $\|\tilde{z}\|^2 = \|z\|^2$. Therefore,

$$\mathbb{E}(\|R_\theta z\|^2) = \frac{1}{2}(\|z'\|^2 + \|z''\|^2) = 4r^2 + \|z\|^2. \tag{2}$$

Since this process is additive with respect to $\|z\|^2$, taking n reflections affords:

$$\mathbb{E}(\|R_{\theta_n} R_{\theta_{n-1}} \cdots R_{\theta_1} z\|^2) = 4r^2 n + \|z\|^2. \tag{3}$$

The second part of Thm 1 is more involved, as I am not aware of a geometric interpretation of the fourth power of the norm of the reflected point. As a result, we will obtain it through direct calculation.

For a single reflection, we observe that $R_\theta z$ can be simplified by converting it to complex form:

$$R_\theta z = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta & 2r\cos\theta \\ -\sin 2\theta & \cos 2\theta & 2r\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos(2\theta)x - \sin(2\theta)y + 2r\cos(\theta) \\ -\sin(2\theta)x + \cos(2\theta)y + 2r\sin(\theta) \\ 1 \end{bmatrix} \tag{4}$$

$$\to -\overline{z}e^{2i\theta} + 2re^{i\theta}. \tag{5}$$

Using 5, we iteratively compute $R_{\theta_2}R_{\theta_1}z$, $R_{\theta_3}R_{\theta_2}R_{\theta_1}z$, etc. Since we only care about the norm of the reflected values, we can **assume** $z \in \mathbb{R}_{\geq 0}$, so $z = \overline{z}$. Inductively, it is straightforward to show that we obtain the following formula for n reflections:

$$R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z = (-1)^n\left[z\exp(iX_0) + 2r\sum_{k=1}^{n}(-1)^k\exp(iX_k)\right], \tag{6}$$

$$X_k := -(-1)^{n+k}\theta_k + \sum_{j=k}^{n}(-1)^{n+j}\,2\theta_j, \quad \text{where } \theta_0 = 0,\ \theta_j \sim \mathrm{Unif}(0, 2\pi) \text{ if } j \geq 1.$$

For ease of notation, define the following:

$$\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^4 = Q^2 \tag{7}$$

$$Q := R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z \cdot \overline{R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z} \tag{8}$$

In order to calculate Q^2, we must first simplify our expression for Q.

$$Q = \left[z\exp(iX_0) + 2r\sum_{k=1}^{n}(-1)^k\exp(iX_k)\right]\left[z\exp(-iX_0) + 2r\sum_{k=1}^{n}(-1)^k\exp(-iX_k)\right] \tag{9}$$

$$\begin{aligned} &= z^2 + 2rz\left(\sum_{k=1}^{n}(-1)^k(\exp(i(X_0 - X_k)) + \exp(-i(X_0 - X_k)))\right) \\ &+ 4r^2\left(\sum_{j=1}^{n}(-1)^j\exp(iX_j)\right)\left(\sum_{k=1}^{n}(-1)^k\exp(-iX_k)\right) \end{aligned} \tag{10}$$

The second term in 10 is just a sum of cosine terms. Considering the product term, we get 1 for all the diagonal terms ($j = k$), and the off-diagonal terms with index (j, k) and (k, j) pair to give cosine terms. Hence, we obtain

$$Q = z^2 + 4rz\sum_{k=1}^{n}(-1)^k\cos(X_0 - X_k) + 4r^2\left(n + 2\sum_{1\leq j<k\leq n}(-1)^{j+k}\cos(X_j - X_k)\right) \tag{11}$$

Now we observe an **important property** of the difference $X_j - X_k$, $j < k$: it always contains a factor of θ_k (see equation 12). Note that $\theta_k \sim \mathrm{Unif}(0, 2\pi)$ because $0 \leq j < k$.

This will be important for when we integrate to calculate the expected value of Q^2.

$$X_j - X_k = (-1)^{n+k}\theta_k - (-1)^{n+j}\theta_j + 2\sum_{m=j}^{k-1}(-1)^{n+m}\theta_m. \tag{12}$$

At this point, we can directly calculate the expected value of Q by integrating with respect to θ_k and noting that all the cosine terms in the sums will integrate to 0 (by the property above). This is a fine way to calculate the expected value, but it is much less elegant than the geometric approach we showed earlier.

Returning to the task at hand, we square both sides of (11) and simplify to afford

$$\begin{aligned} Q^2 &= z^4 + 8z^2r^2n + 16r^4n \\ &\quad + 8r(z^2+4r^2n)\left(z\sum_{k=1}^{n}(-1)^k\cos(X_0 - X_k) + 2r\sum_{1\le j<k\le n}(-1)^{j+k}\cos(X_j - X_k)\right) + 16r^2V, \end{aligned} \tag{13}$$

$$\begin{aligned} V &:= z^2\left(\sum_{k=1}^{n}(-1)^k\cos(X_0 - X_k)\right)^2 \\ &\quad + 4rz\left(\sum_{k=1}^{n}(-1)^k\cos(X_0 - X_k)\right)\left(\sum_{1\le j<k\le n}(-1)^{j+k}\cos(X_j - X_k)\right) \\ &\quad + 4r^2\left(\sum_{1\le j<k\le n}(-1)^{j+k}\cos(X_j - X_k)\right)^2. \end{aligned} \tag{14}$$

Expanding out the terms of V, we observe that every term (up to multiplicative constants) is of the form $\cos(X_j - X_k)\cos(X_\ell - X_m)$. Without loss of generality, this reduces to the following:

$$\cos(X_j - X_k)\cos(X_\ell - X_m) = \begin{cases} \cos(X_j - X_k)^2 \text{ if } (\ell,m)=(j,k) \text{ or } (\ell,m)=(k,j) \\ \frac{1}{2}(\cos(2X_j - X_k - X_m) + \cos(-X_k + X_m)) \text{ if } \ell = j \text{ but } m \neq k \\ \frac{1}{2}(\cos(X_j - X_k + X_\ell - X_m) + \cos(X_j - X_k - X_\ell + X_m)) \text{ otherwise.} \end{cases} \tag{15}$$

Now, integrating Q^2 term-by-term, we can choose the order of integration. We choose to integrate first with respect to θ_k. Making use of the important property of $X_j - X_k$ that we noted earlier, namely that $X_j - X_k$ always contains a term of θ_k, it is clear that every integral will be one of the following two forms:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(A+B\theta_k)d\theta_k=0, \tag{16}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2(A+B\theta_k)d\theta_k=\frac{1}{2}, \tag{17}$$

where A and B are arbitrary constants with respect to θ_k.

Applying (15), (16), and (17) to (13), we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}Q^2d\theta_k &= z^4+8z^2r^2n+16r^4n+16r^2\left(z^2\sum_{k=1}^{n}\frac{1}{2}+4r^2\sum_{1\le j<k\le n}\frac{1}{2}\right)\\ &= z^4+16z^2r^2n+16r^4(2n^2-n).\end{aligned} \tag{18}$$

Since integrating over θ_k has eliminated dependence on $\theta_1,\theta_2,\ldots\theta_n$, equation 18 implies that for any complex number z,

$$\mathbb{E}(Q^2)=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}Q^2d\theta_1\cdots d\theta_n=\|z\|^4+16\|z\|^2r^2n+16r^4(2n^2-n). \tag{19}$$

Note that Corollary 1 is a direct application of the fact that $\mathrm{Var}(Y)=\mathbb{E}(Y^2)-\mathbb{E}(Y)^2$. □

3. Generalizations and Examples

At first glance, the utility of Theorem 1 toward obtaining more general results appears limited by the constraint that the reflection lines are tangent to circles of radius r centered at the origin. However, every line in $\mathbb{R}^2$ is uniquely defined by an angle and a tangent circle centered at the origin. As such, we can vary r in order to parameterize every line in $\mathbb{R}^2$.

Letting r be a random variable, we can choose different distributions for r in order to obtain answers to the question: what is the expected squared norm of a point reflected about n random lines in $\mathbb{R}^2$?

Most naturally, we can choose $r\sim\mathrm{Unif}(0,N)$ for some nonnegative real number N. This means that all the lines pass through the disk of radius N centered at the origin. Intuitively, as we let $N\to\infty$, the expected squared norm of a reflected point will diverge. But how quickly? Making use of the 1 result and corollary,

$$\mathbb{E}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^2)=\frac{1}{N}\int_0^N(4r^2n+\|z\|^2)dr=\|z\|^2+\frac{4}{3}N^2n, \tag{20}$$

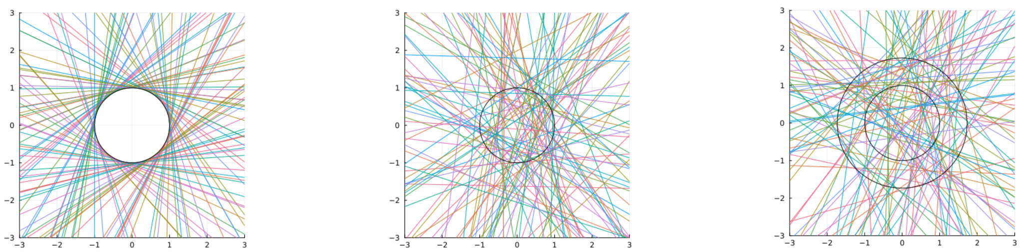
$$\mathrm{Var}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^2)=-\frac{16N^4n}{5}+\frac{208N^4n^2}{45}+\frac{8N^2n\|z\|^2}{3}. \tag{21}$$

Another natural choice is to consider lines where r is distributed as a truncated standard normal distribution, i.e. it has PDF $f(x) = \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}$. In this case,

$$\mathbb{E}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^2) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{1}{2}r^2}(4r^2n + \|z\|^2)dr = \|z\|^2 + 4n \tag{22}$$

$$\mathrm{Var}(\|R_{\theta_n}R_{\theta_{n-1}}\cdots R_{\theta_1}z\|^2) = 8n\|z\|^2 + 16(5n^2 - 3n) \tag{23}$$

Here, we get the same expected value as we did when all n lines were tangent to the unit circle! It turns out that we also get the same expected value in the case where $r \sim \mathrm{Unif}(0, \sqrt{3})$. At this point, I wanted to visualize these three cases so I could compare them side-by-side.



(A) $r = 1$ (B) $r \sim |N(0,1)|$ (C) $r \sim \mathrm{Unif}(0, \sqrt{3})$

FIGURE 2. Julia plots of the different distributions of lines ($n = 100$). In all cases, a unit circle is plotted for scale. In plot (C), we also plot a circle of radius $\sqrt{3}$ for reference.

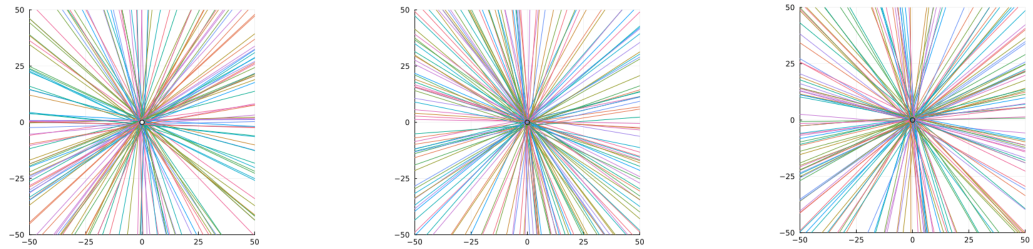
We observe that the three distributions of lines look very different, which makes it particularly striking that they all afford the same expected squared norm of a point after reflecting across n of the lines.

A natural next question to ask is how the variance differs in these three cases. The term with z dependence is the same in the three cases, so for all z and all $n \geq 1$, we obtain

$$80n^2 - 48n > \frac{16}{5}(13n^2 - 9n) > 16n^2 - 16n, \tag{24}$$

which implies that the variance of the squared norm is lowest for $r = 1$, second lowest for $r \sim \mathrm{Unif}(0, \sqrt{3})$, and highest for $r \sim |N(0,1)|$.

As expected, the three distributions look nearly the same when viewed at a larger scale:



(A) $r = 1$ (B) $r \sim |N(0,1)|$ (C) $r \sim \mathrm{Unif}(0, \sqrt{3})$

FIGURE 3. The same plots as Fig. 2 viewed with a larger scale.