Painlevé Systems and Eigenvalue Distributions

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Outline

- Painlevé equations
- Hamiltonian formulation
  - Auxiliary Hamiltonians
  - σ-form of the equations
- Relationship to eigenvalue distributions
  - Numerical results
- Discussion

Painlevé Equations (Historical Context)

- Late 19th Century: Fuchs/Poincaré classify first order differential equations:

$$P(y', y, t) = 0$$

- Reduced equations to Weierstrauss or Ricatti equations:

$$\left(\frac{dy}{dt}\right)^2 = 4y^3 - g_2 - g_3 \qquad \frac{dy}{dt} = a(t)y^2 + b(t)y + c(t)$$

- Painlevé wanted to do something similar with 2nd order ODEs

Painlevé Equations (Historical Context)

- Painlevé attempted to classify equations for y’’ rational in y, y’, and t:

$$y'' = R(y, y', t)$$

- Reduced to either first order equations, linear equations, or 6 equations:

PI $y'' = 6y^2 + t$

PII $y'' = 2y^3 + ty + \alpha,$

PIII $y'' = \frac{1}{y}\left(y'\right)^2 - \frac{1}{t}y' + \gamma y^3 + \frac{1}{t}\left(\alpha y^2 + \beta\right) + \frac{\delta}{y},$

PIV $y'' = \frac{1}{2y}\left(y'\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2\left(t^2 - \alpha\right)y + \frac{\beta}{y},$

PV $y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(y'\right)^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1},$

PVI $y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(y'\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y'$

$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right).$

σ-form of the equations

- (Auxiliary) Hamiltonians satisfy highly nonlinear differential equations

$$\sigma\mathrm{PII} \quad \left(\sigma''_{II}\right)^2 + 4\sigma'_{II}\left(\left(\sigma'_{II}\right)^2 - t\sigma'_{II} + \sigma_{II}\right) - a^2 = 0,$$

$$\sigma\mathrm{PIII}' \quad \left(t\sigma''_{III'}\right)^2 - v_1 v_2\left(\sigma'_{III'}\right)^2 + \sigma'_{III'}\left(4\sigma'_{III'} - 1\right)\left(\sigma_{III'} - t\sigma'_{III'}\right) - \frac{1}{4^3}\left(v_1 - v_2\right)^2 = 0,$$

$$\sigma\mathrm{PIV} \quad \left(\sigma''_{IV}\right)^2 - 4\left(t\sigma'_{IV} - \sigma_{IV}\right)^2 + 4\sigma'_{IV}\left(\sigma'_{IV} + 2\alpha_1\right)\left(\sigma'_{IV} - 2\alpha_2\right) = 0,$$

$$\sigma\mathrm{PV} \quad \left(t\sigma''_{V}\right)^2 - \left(\sigma_V - t\sigma'_V + 2\left(\sigma'_V\right)^2 + \left(\nu_0 + \nu_1 + \nu_2 + \nu_3\right)\sigma'_V\right)^2$$
$$+ 4\left(\nu_0 + \sigma'_V\right)\left(\nu_1 + \sigma'_V\right)\left(\nu_2 + \sigma'_V\right)\left(\nu_3 + \sigma'_V\right) = 0,$$

$$\sigma\mathrm{PVI} \quad \sigma'_{VI}\left(t(1-t)\sigma''_{VI}\right)^2 + \left(\sigma'_{VI}\left(2\sigma_{VI} - (2t-1)\sigma'_{VI}\right) + v_1 v_2 v_3 v_4\right)^2 = \prod_{k=1}^{4}\left(\sigma'_{VI} + v_k^2\right)$$

Hamiltonian Formulation

- Present a Hamiltonian H such that eliminating the momentum from the Hamilton equations of motion gives the desired Painlevé equation

$$H = H(p, q, t; \vec{v})$$

$$q' = \frac{\partial H}{\partial p} \qquad p' = -\frac{\partial H}{\partial q}$$

$$
\begin{aligned}
H_{I} =&\frac{1}{2}p^2-2q^3-tq,\\
H_{II} =&-\frac{1}{2}\left(2q^2-p+t\right)p-\frac{v_1-v_2}{2}q,\\
tH_{III'} =&q^2p^2-\left(q^2+v_1q-t\right)p+\frac{1}{2}\left(v_1+v_2\right)q,\\
H_{IV} =&(2p-q-2t)pq-2\left(v_1-v_2\right)p+\left(v_3-v_2\right)q,\\
tH_{V} =&q(q-1)^2p^2-\left\{\left(v_1-v_2\right)(q-1)^2-2\left(v_1+v_2\right)q(q-1)+tq\right\}p\\
&+\left(v_3-v_2\right)\left(v_4-v_2\right)(q-1),\\
t(t-1)H_{VI} =&q(q-1)(q-t)p^2-\left(\left(v_3+v_4\right)(q-1)(q-t)+\left(v_3-v_4\right)q(q-t)\right.\\
&\left.-\left(v_1+v_2\right)q(q-1)\right)p+\left(v_3-v_1\right)\left(v_3-v_2\right)(q-t),
\end{aligned}
$$

Relationships between Painlevé equation parameters and Hamiltonian parameters

PII $\quad v_1 + v_2 = 0, \quad \alpha = v_1 - \dfrac{1}{2},$

PIII$'$ $\quad \alpha = -4v_2, \quad \beta = 4(v_1 + 1), \quad \gamma = 4, \quad \delta = -4,$

PIV $\quad v_1 + v_2 + v_3 = 0, \quad \alpha = 1 + 2v_3 - v_1 - v_2, \quad \beta = -2\alpha_1^2,$

PV $\quad v_1 + v_2 + v_3 + v_4 = 0,\ \alpha = \dfrac{1}{2}(v_3 - v_4)^2,\ \beta = -\dfrac{1}{2}(v_1 - v_2)^2,\ \gamma = 2v_1 + 2v_2 - 1,\ \delta = -\dfrac{1}{2},$

PVI $\quad \alpha = \dfrac{1}{2}(v_1 - v_2)^2,\ \beta = -\dfrac{1}{2}(v_3 + v_4)^2,\ \gamma = \dfrac{1}{2}(v_3 - v_4)^2,\ \delta = \dfrac{1}{2}(1 - (1 - v_1 - v_2)^2).$

Relations with Auxiliary Hamiltonians and σ functions

$$h_{II}(t) = H_{II},$$

$$h_{III'}(t) = tH_{III'} + \frac{1}{4}v_1^2 - \frac{1}{2}t,$$

$$h_{IV}(t) = H_{IV} - 2v_2 t,$$

$$h_V(t) = tH_V + (v_3 - v_2)(v_4 - v_2) - v_2 t - 2v_2^2,$$

$$h_{VI}(t) = t(t-1)H_{VI} + e_2[-v_1, -v_2, v_3]\, t, -\frac{1}{2}e_2[-v_1, -v_2, v_3, v_4], \qquad e_p[a_1, \ldots, a_s] := \textstyle\sum_{1<j_1<\cdots<j_p<s} a_{j_1} a_{j_2} \cdots a_{j_p}$$

$$\sigma_{II}(t) = -\, 2^{1/3} h_{II}\left(-2^{1/3}t\right)\Big|_{(v_1,v_2)=(a,-a)},$$

$$\sigma_{III'}(t) = -h_{III'}(t/4) + \frac{t}{8} + \frac{v_1 v_2}{4},$$

$$\sigma_{IV}(t) = \left(h_{IV}(t) + 2v_2 t\right)\big|_{(1+v_3-v_1, v_1-v_2, v_2-v_3)=(\alpha_0,\alpha_1,\alpha_2)},$$

$$\sigma_V(t) = h_V(t) + v_2 t + 2v_2^2, \quad \nu_{j-1} = v_j - v_2 \quad (j = 1, \ldots, 4),$$

$$\sigma_{VI}(t) = h_{VI}(t)$$

Relationships to Eigenvalue Distributions

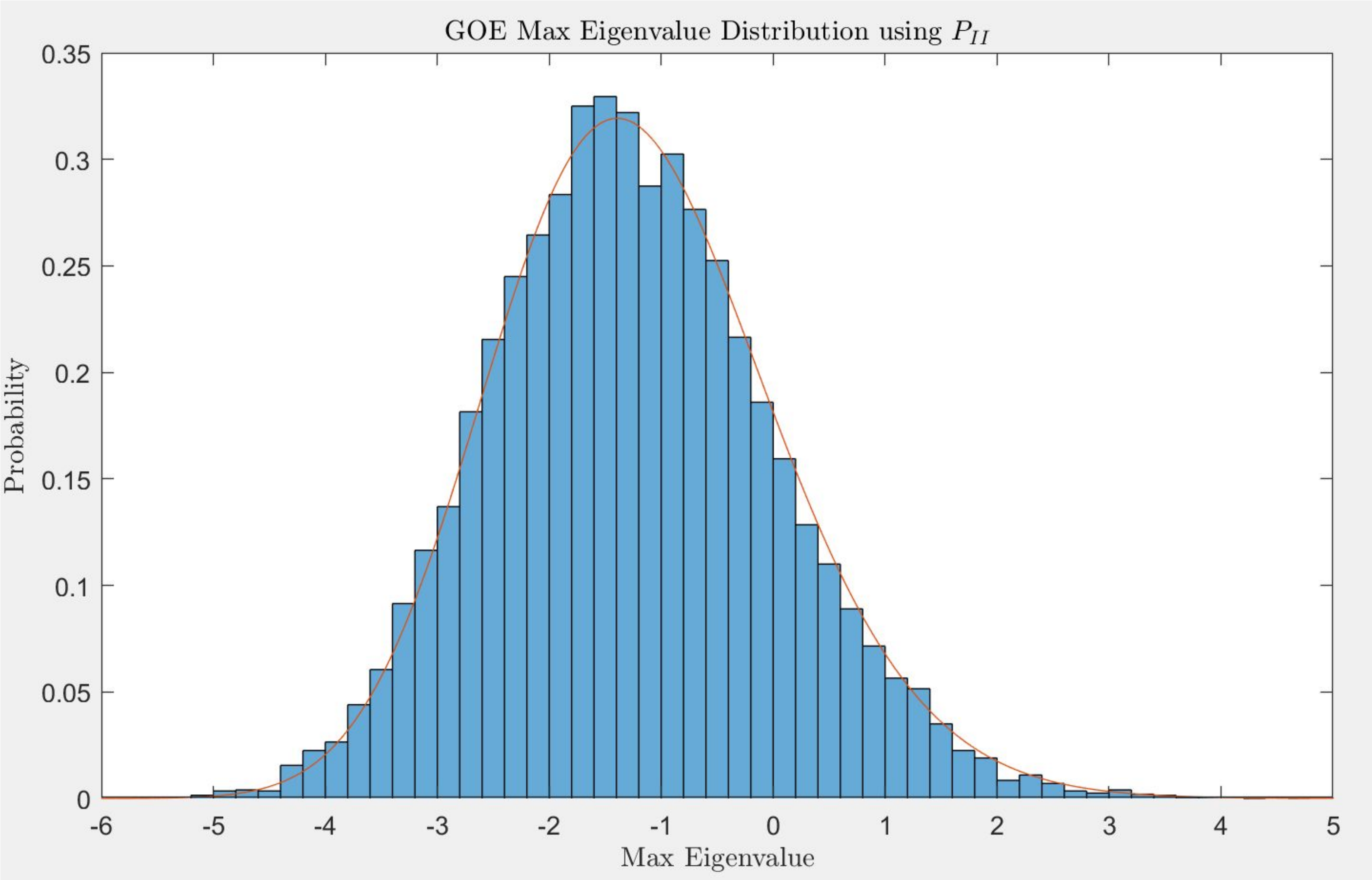
Painlevé II: $q(x)$

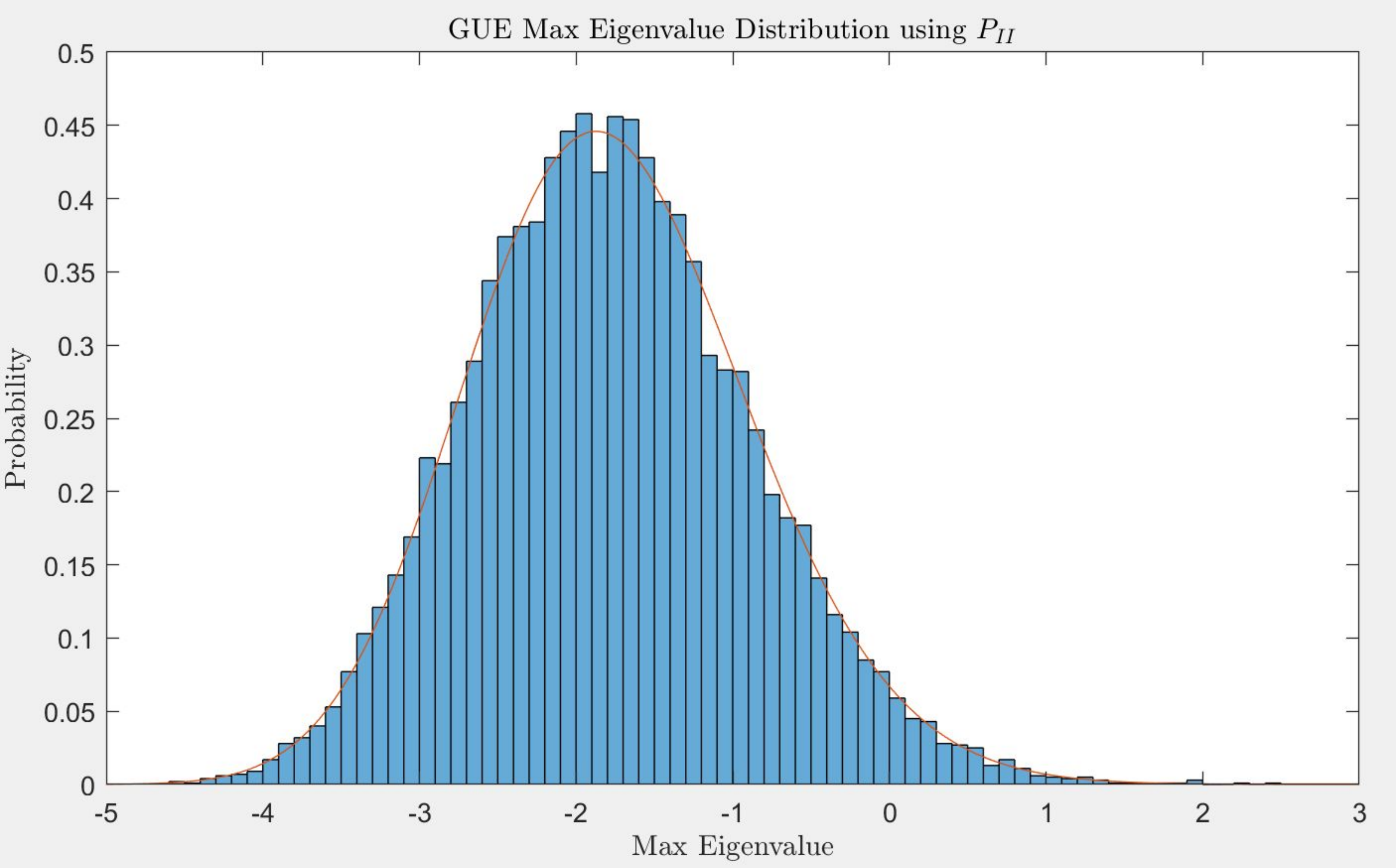
Using boundary condition that $q(x) \sim Ai(x)$ as $x \to \infty$

- Key part of finding Tracy-Widom PDF (for GUEs)
- Can also be used to find limiting distribution of the maximum GOE and GSE eigenvalues

Painlevé III

- Hard-edge scaling limit for Laguerre and Jacobi ensembles


GOE Max Eigenvalue Distribution using P_{II}
Probability
Max Eigenvalue
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5


GUE Max Eigenvalue Distribution using P_{II}
Probability
Max Eigenvalue
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
-5
-4
-3
-2
-1
0
1
2
3

Relationships to Eigenvalue Distributions

Painlevé IV (σ-form)

$\sigma \propto x^{2N-2} \exp(-x^2)$

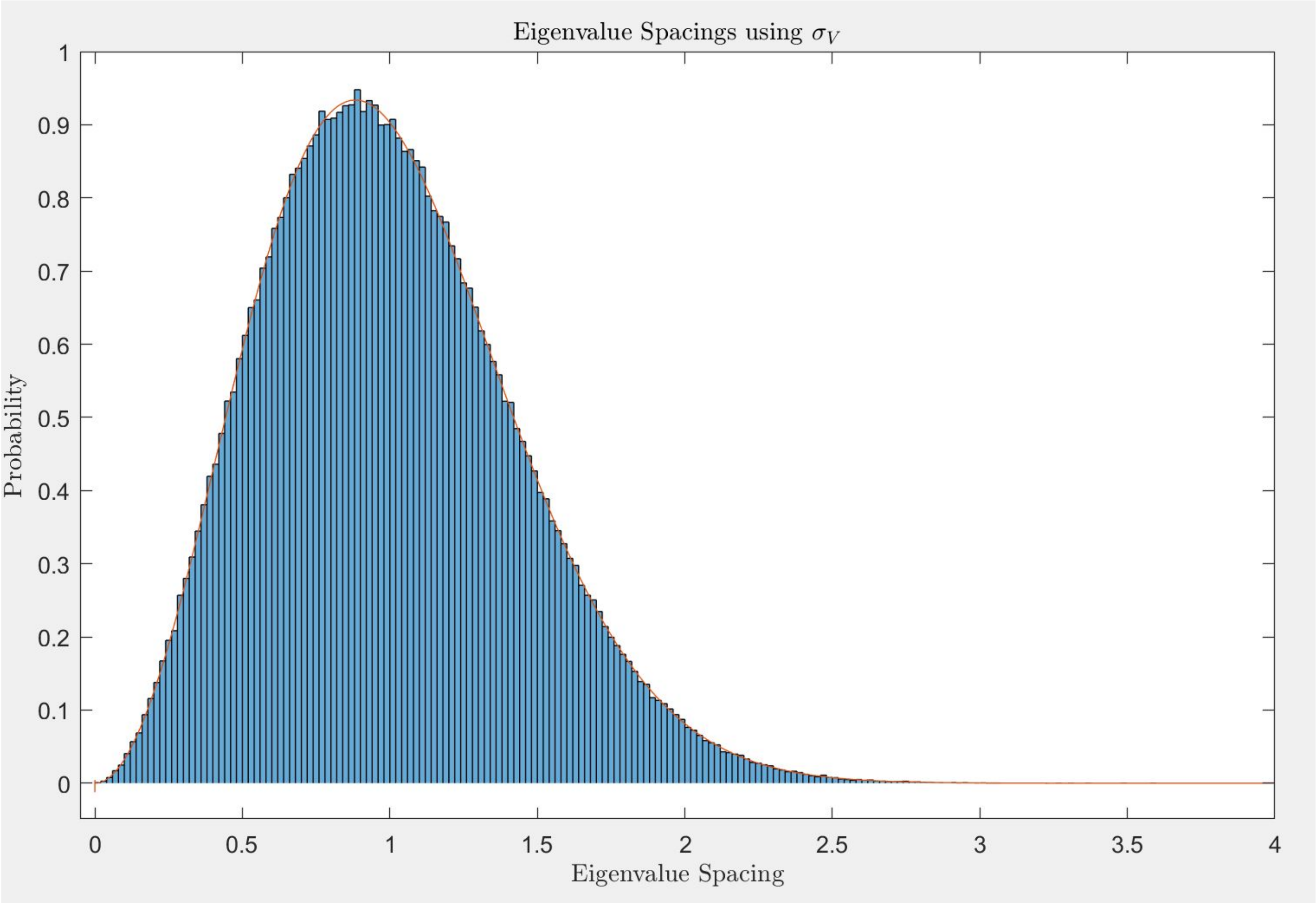
- Largest eigenvalue for GUE (finite N)

Painlevé V (σ-form)

- Bulk scaling limit for GUE
- Smallest eigenvalue for Laguerre ensemble

Painlevé VI (σ-form)

- Density for eigenvalues in Jacobi ensemble



Eigenvalue Spacings using σ_V
Probability
Eigenvalue Spacing
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3
3.5
4

Relationships to Eigenvalue Distributions

Painlevé VI (σ-form)

- Density for eigenvalues in Jacobi-weighted random matrices

Discussion

We now more fully understand issues with stability

- Asymptotic boundary conditions that vanish cause sensitivity to different choices for the boundary
- Highly nonlinear equations (especially the σ-forms) can be very sensitive to these