Hermite, Laguerre, Jacobi

Listen to Random Matrix Theory

It's trying to tell us something

Alan Edelman

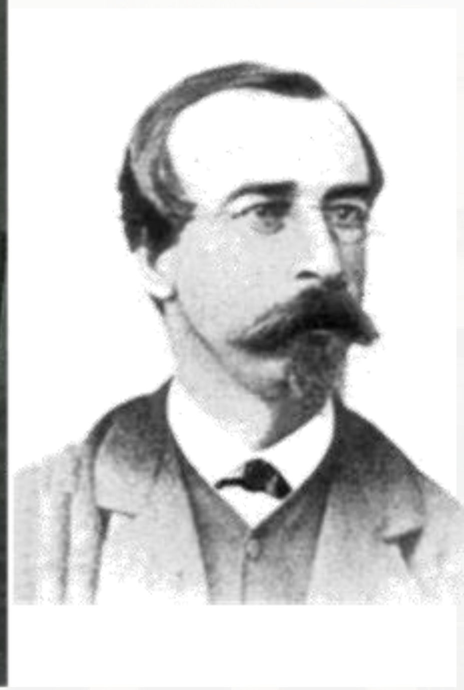
Mathematics

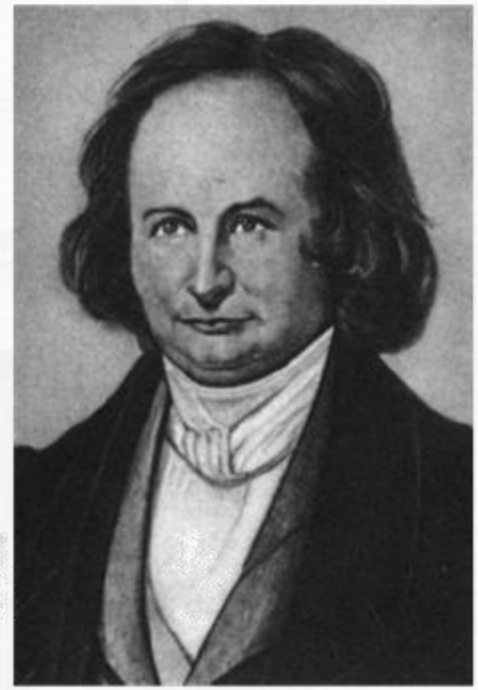
Presented originally 2014

Nice overview for 18.338 Lecture 1

MIT
MATHEMATICS

Hermite, Laguerre, and Jacobi





Hermite
1822-1901

Laguerre
1834-1886

Jacobi
1804-1851

An Intriguing Mathematical Tour

Sometimes out of my comfort zone

Opportunities Abound

Scalar Random Variables (n=1)

MATH	Julia	Probability Density	Remark
Standard Normal	Normal()	$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	
χ^2 Chi-Squared	Chisq(v)	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x}$	Using LinearAlgebra norm(randn(v))^2
Beta Distribution	Beta(a,b)	$\frac{x^{a-1}(1-x)^{b-1}}{\int_0^1 x^{a-1}(1-x)^{b-1}dx}$	x=rand(Chisq(2a)) y= rand(Chisq(2b)) x/(x+y)

If v = pos int

Julia Note: randn() in Base gives standard normal

```
julia> randn(5,3)
5×3 Matrix{Float64}:
 -0.617172   -2.17051   -1.25459
  0.0655657   0.968201   0.0666011
 -0.112315    0.750498  -1.1488
 -0.156859    0.879811  -1.00868
 -0.527015    0.836618   0.435882
```

```
julia> using Distributions

julia> rand( Chisq(2.4), 5, 3)
5×3 Matrix{Float64}:
 1.67742   2.36362  0.904712
 0.881535  4.70032  2.50146
 0.982194  1.9386   3.71603
 1.22735   1.68285  0.521132
 0.670757  4.86199  4.4683
```

Scalar Random Variables (n=1)

MATH	Julia	Probability Density	Remark	
Standard Normal	Normal()	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$		**Hermite**
χ^2 Chi-Squared	Chisq(v)	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\nu/2-1}e^{-x}$	Using LinearAlgebra norm(randn(v))^2	**Laguerre**
Beta Distribution	Beta(a,b)	$\frac{x^{a-1}(1-x)^{b-1}}{\int_0^1 x^{a-1}(1-x)^{b-1}dx}$	x=rand(Chisq(2a)) y= rand(Chisq(2b)) x/(x+y)	**Jacobi**

Random Matrices

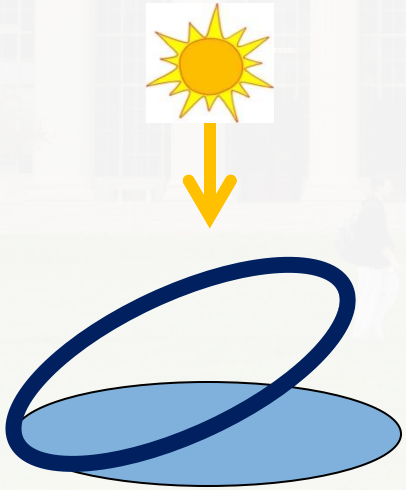
Ensembles	RMs	Julia		Joint Eigenvalue Density
Hermite	Gaussian Ensembles Wigner (1955)	G=randn(n,n) S=(G+G')/2	Symmetric	$c_H \prod_{i<j} \lvert\lambda_i - \lambda_j\rvert^\beta \times \prod_i e^{-\lambda_i^2/2}$
Laguerre	Wishart Matrices (1928)	G=randn(m,n) W=(G'G)/n	Positive Definite	$c_L \prod_{i<j} \lvert\lambda_i - \lambda_j\rvert^\beta \times \prod_i \lambda_i^{\beta(m-n+1)/2-1} e^{-\sum_i \lambda_i/2}$
Jacobi	MANOVA Matrices (1939)	W1=Wishart(m1,n) W2=Wishart(m2,n) J=W1/(W1+W2)	(Morally) Symmetric 0 < J < I	$c_J \prod_{i<j} \lvert\lambda_i - \lambda_j\rvert^\beta \times \prod_i \lambda_i^{\beta(m_1-n+1)/2-1} (1-\lambda_i)^{\beta(m_2-n+1)/2-1}$

Three biggies in numerical linear algebra: eig, svd, gsvd

	Random Matrix	Algorithm	JULIA
Hermite	Gaussian Ensembles	eig	G=randn(n,n) S=(G+G')/2 eig(S)
Laguerre	Wishart	svd	svd(randn(m,n))
Jacobi	MANOVA	gsvd	svd(randn(m1,n),randn(m2,n))

The Jacobi Ensemble: Geometric Interpretation

- Take reference n≤m dimensional subspace of R^m
- Take RANDOM n≤m dimensional subspace of R^m
- The shadow of the unit ball in the random subspace when projected onto the reference subspace is an ellipsoid
- The semi-axes lengths are the Jacobi ensemble cosines. (MANOVA Convention=Squared cosines)

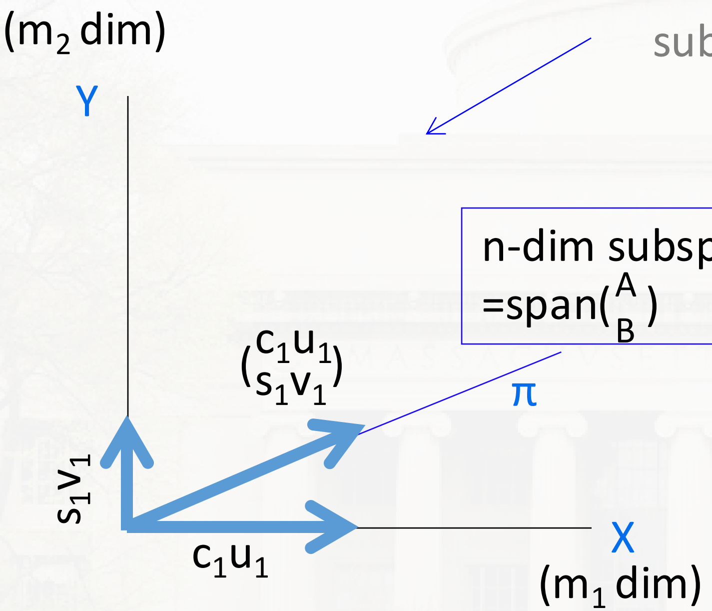


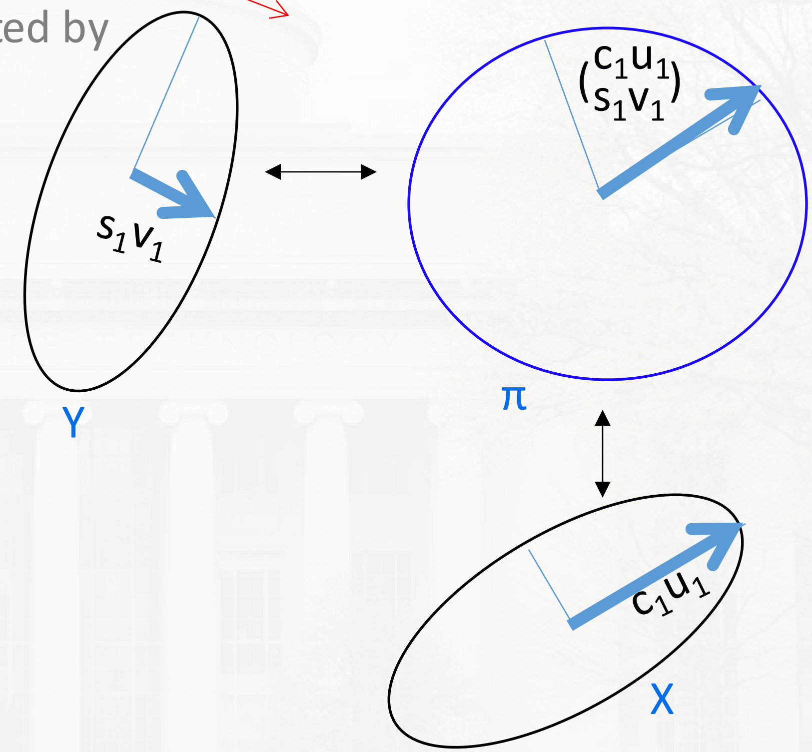
GSVD(A,B)

A,B have n columns

$m = m_1 + m_2$ dimensions
$n \le m$

Flattened View Expanded View
subspaces represented by
lines planes





Ex 1: Random line in R^2 through 0:
On the x axis: c
On the z axis: s

Ex 2: Random plane in R4 through 0:
On xy plane: c1,c2
On zw plane: s1,s2

GSVD(A,B)

A,B have n columns

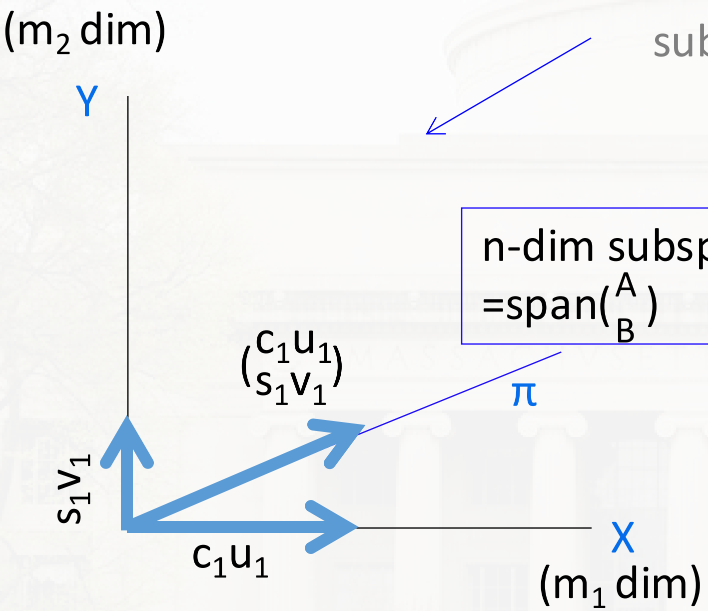
$m=m_1+m_2$ dimensions
$n \le m$

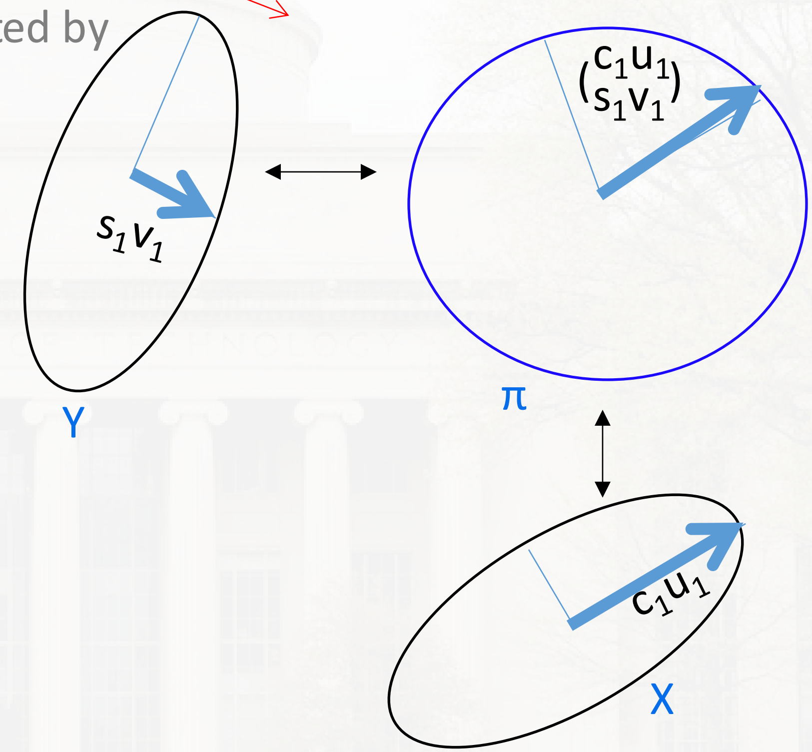
Flattened View Expanded View

subspaces represented by

lines planes

n-dim subspace
$=\mathrm{span}\begin{pmatrix} A \\ B \end{pmatrix}$



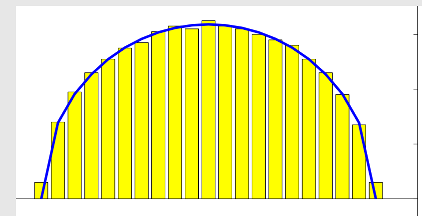
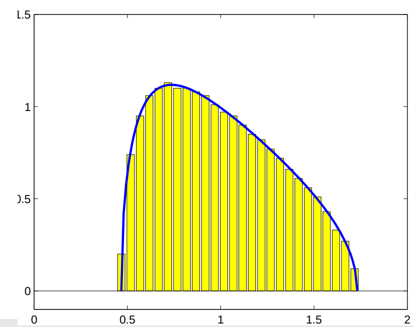
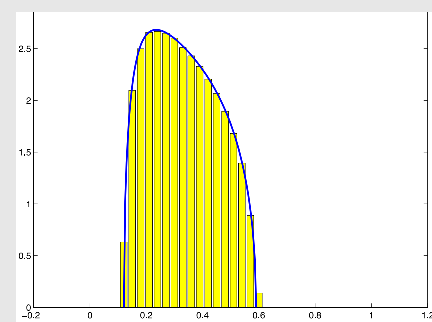


Ex 3: Random line in R3 through 0:
On the xy plane: c and 0
On the z axis: s

Ex 4:
Random plane in R3 through 0:
On the xy plane: c and 1 (one axis in the xy plane)
On the z axis: s

Infinite Random Matrix Theory & Gil Strang's favorite matrix

Limit Laws for Eigenvalue Histograms

	Law	Formula	
Hermite	Semicircle Law Wigner 1955 Free CLT	$$\frac{1}{2\pi}\sqrt{(2-x)(x+2)}$$	
Laguerre	Marcenko-Pastur Law 1967 $r = n/m$	$$\frac{1}{2\pi}\frac{\sqrt{(\lambda_+ - x)(x-\lambda_-)}}{xr}$$ $$\lambda_\pm = (1 \pm \sqrt{r})^2$$	
Jacobi	Wachter Law 1980 $a = m_1/n$ $b = m_2/n$	$$\frac{1}{2\pi}\frac{\sqrt{(\lambda_+ - x)(x-\lambda_-)}}{x(1-x)(a+b)^{-1}}$$ $$\lambda_\pm = \left[\left(\sqrt{\frac{a}{a+b}\left(1-\frac{1}{a+b}\right)} \pm \sqrt{\frac{1}{a+b}\left(1-\frac{a}{a+b}\right)}\right)^2\right]$$	

Three big laws: Toeplitz+boundary

$$\begin{pmatrix} x & y & & & & & \\ y & a & b & & & & \\ & b & a & b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & b & a & b & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

Anshelevich, Młotkowski (2010) (Free Meixner)
E, Dubbs (2014)

	Law	Equilibrium Measure
Hermite	Semicircle Law 1955 Free CLT	x=a y=b
Laguerre	Marcenko-Pastur Law 1967 Free Poisson	x=parameter y=b
Jacobi	Wachter Law 1980 Free Binomial	x=parameter y=parameter

That's pretty special!

Corresponds to 2nd order differences with boundary

Example Chebfun Lanczos Run

Verbatim from Pedro Gonnet’s November 2011 Run

```
% semicircle law
x = chebfun( 'x' , [-2,2] );
w = sqrt(4 - x.^2)/2/pi;
%
```

```
% % MP law
% r = 0.5;
% lmax = (1+sqrt(r))^2;
% lmin = (1-sqrt(r))^2;
%
% x = chebfun('x', [lmin, lmax]);
% w = 1/(2*pi) * sqrt((lmax-x).*(x-lmin)) ./ (x*r);
```

```
% % Wachter law
% a = 5; b = 10;
% c = sqrt(a/(a+b) * (1 - 1/(a+b)));
% d = sqrt(1/(a+b) * (1 - a/(a+b)));
%
% lmax = (c + d)^2;
% lmin = (c - d)^2;
% x = chebfun('x', [lmin, lmax]);
% w = (a+b) * sqrt((x-lmin).*(lmax-x))./(2*pi*x.*(1-x));
```

0	1.00	0	0	0
1.00	0.00	1.00	0	0
0	1.00	0	1.00	0
0	0	1.00	0.00	1.00
0	0	0	1.00	-0.00

1.00	0.71	0	0	0
0.71	1.50	0.71	0	0
0	0.71	1.50	0.71	0
0	0	0.71	1.50	0.71
0	0	0	0.71	1.50

0.33	0.12	0	0	0
0.12	0.36	0.12	0	0
0	0.12	0.36	0.12	0
0	0	0.12	0.36	0.12
0	0	0	0.12	0.36

Thanks to
Bernie Wang

Lanczos

```
P = chebfun( 1./sqrt(sum(w)) , domain(x) );
v = x.*P;
beta(1) = sum(w.*v.*P);
v = v - beta(1)*P;
gamma(1) = sqrt(sum( w.*v.^2 ));
P(:,2) = v / gamma(1);
```

```
for k=2:N
    v = x.*P(:,k) - gamma(k-1)*P(:,k-1);
    beta(k) = sum(w.*v.*P(:,k));
    v = v - beta(k)*P(:,k);
    gamma(k) = sqrt(sum( w.*v.^2 ));
    P(:,k+1) = v / gamma(k);
end
```

Why are

Hermite, Laguerre, Jacobi

all over mathematics?

I'm still wondering.

Varying Inconsistent Definitions of Classical Orthogonal Polynomials

- Hermite, Laguerre, and Jacobi Polynomials
- “There is no generally accepted definition of classical orthogonal polynomials, but …” (Walter Gautschi)
- Orthogonal polynomials whose derivatives are also orthogonal polynomials (Wikipedia: Honine, Hahn)
- Hermite, Laguerre*, Bessel,* and Jacobi … are called collectively the “classical orthogonal polynomials (L. Miranian)
- Orthogonal Polynomials that are eigenfunctions of a fixed 2^{nd}-order linear differential operator (Bochner, Grünbaum,Haine)
- All polynomials in the Askey scheme

Askey Scheme

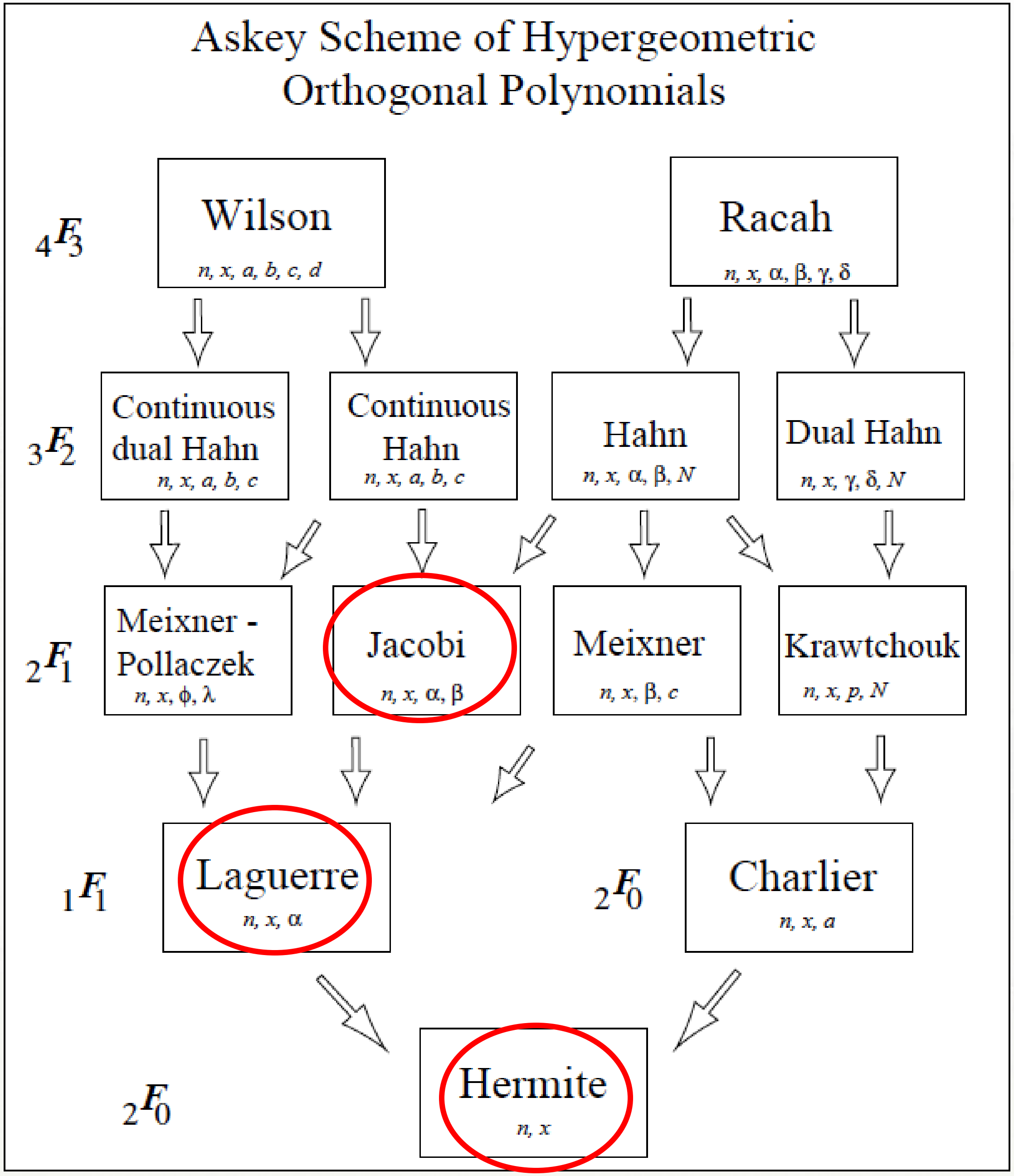


chart from Temme, et. al

Varying Inconsistent Definitions of Classical Orthogonal Polynomials

- The differential operator has the form

$$f(x) \mapsto Q(x)f''(x) + L(x)f'(x)$$

where Q(x) is (at most) quadratic and L(x) is linear

- Possess a Rodrigues formula (W(x)=weight)

$$p_n(x) = \frac{1}{e_n W(x)} \frac{d^n}{dx^n} \left(W(x)[Q(x)]^n \right)$$

- Pearson equation for the weight function itself:

$$\frac{d}{dx}(\sigma(x)w(x)) = \tau(x)w(x)$$

Yet more properties

- Sheffer sequence ($Qp_n = np_{n-1}$ for linear operator Q)
  - Hermite, Laguerre, and Jacobi are Sheffer
  - not sure what other orthogonal polynomials are Sheffer
- Appell Sequence $\frac{d}{dx}p_n = np_{n-1}$ must be Sheffer
  - Hermite (not any other orthogonal polynomial)

All the definitions are formulaic

- Formulas are concrete, useful, and departure points for many properties, but they don’t feel like they explain a mathematical core

Where else might we look?
Anything more structural?

A View towards Structure

- Hermite: Symmetric Eigenproblem:

  (Sym Matrices)/(Orthogonal matrices)

  $S = Q\Lambda Q'$ (Eigenvalues Λ)

- Laguerre: SVD

  (Orthogonals) \ (m x n matrices) / (Orthogonals)

  $A = U\Sigma V'$ (Singular Values Σ)

- Jacobi: GSVD

  (Grassmann Manifold)/(Stiefel m1 x Stiefel m2)

  $Y = \begin{bmatrix} U_1 C \\ U_2 S \end{bmatrix} V'$ (Cosine/Sine pairs C, S)

KAK Decompositions?

Homogeneous Spaces

- Take a Lie Group and quotient out a subgroup

Symmetric Space

- The subgroup is itself an open subgroup of the fixed points of an involution

class	noncompact type	compact type
A	$\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$	$\mathrm{U}(n)$
AI	$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$	$\mathrm{U}(n)/\mathrm{O}(n)$
AII	$\mathrm{U}^*(2n)/\mathrm{Sp}(n)$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$
AIII	$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$
BDI	$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$
CII	$\mathrm{Sp}(p,q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$
BD	$\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n)$	$\mathrm{SO}(n)$
C	$\mathrm{Sp}(n,\mathbb{C})/\mathrm{Sp}(n)$	$\mathrm{Sp}(n)$
CI	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
DIII	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$\mathrm{SO}(2n)/\mathrm{U}(n)$

Symmetric Space Charts

Hermite

What are these?

Circular Ensembles

Jacobi: m_1=n

Laguerre I'm told?

class	noncompact type	compact type
A	$\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$	$\mathrm{U}(n)$
$A\mathrm{I}$	$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$	$\mathrm{U}(n)/\mathrm{O}(n)$
$A\mathrm{II}$	$\mathrm{U}^*(2n)/\mathrm{Sp}(n)$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$
$A\mathrm{III}$	$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$
$BD\mathrm{I}$	$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$
$C\mathrm{II}$	$\mathrm{Sp}(p,q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$
BD	$\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n)$	$\mathrm{SO}(n)$
C	$\mathrm{Sp}(n,\mathbb{C})/\mathrm{Sp}(n)$	$\mathrm{Sp}(n)$
$C\mathrm{I}$	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
$D\mathrm{III}$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$\mathrm{SO}(2n)/\mathrm{U}(n)$

Haar also Jacobi

Jacobi: β=1,m_1=n+1,m_2=n+1

Jacobi: β=4, m_1=½,1½, m_2= ½

Random Matrix Story Clearly Lined up with Symmetric Spaces (Hermite, Circular)

class	noncompact type	compact type
A	$\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$	$\mathrm{U}(n)$
$A\mathrm{I}$	$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$	$\mathrm{U}(n)/\mathrm{O}(n)$
$A\mathrm{II}$	$\mathrm{U}^*(2n)/\mathrm{Sp}(n)$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$
$A\mathrm{III}$	$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$
$BD\mathrm{I}$	$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$
$C\mathrm{II}$	$\mathrm{Sp}(p,q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$
BD	$\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n)$	$\mathrm{SO}(n)$
C	$\mathrm{Sp}(n,\mathbb{C})/\mathrm{Sp}(n)$	$\mathrm{Sp}(n)$
$C\mathrm{I}$	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
$D\mathrm{III}$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$\mathrm{SO}(2n)/\mathrm{U}(n)$

also a Jacobi

What are these?
Some must be Laguerre

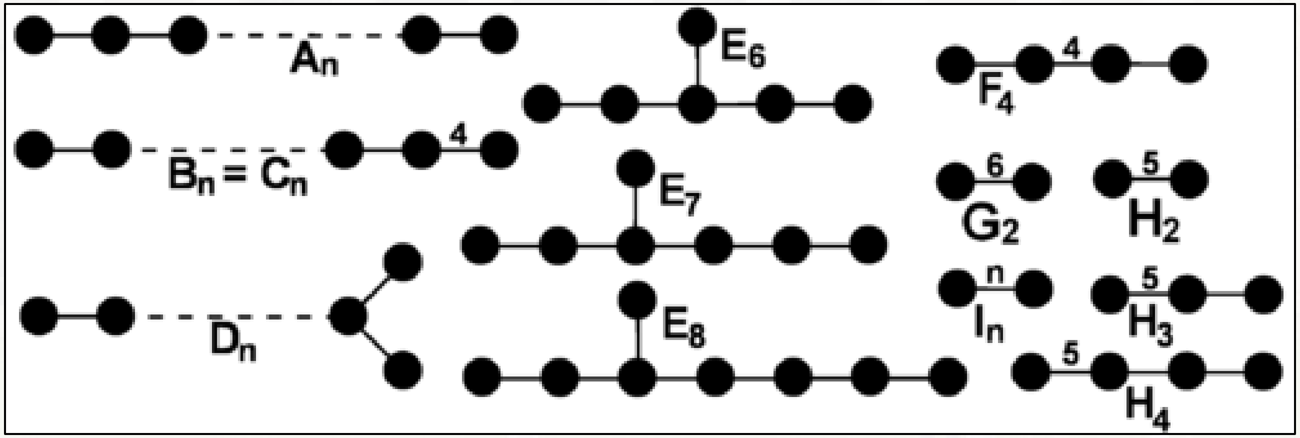
Symmetric Spaces fall a little short (where are the rest of the Jacobi's???)

(Zirnbauer, Dueñez)

Coxeter Groups?

- Symmetry group of regular polyhedra
- Weyl groups of simple Lie Algebras

Foundation for structure



Macdonald’s Integral form of Selberg’s Integral

Macdonald's integral

[edit]

Macdonald (1982) conjectured the following extension of Mehta's integral to all finite root systems, Mehta's original case corresponding to the A_{n-1} root system.

$$\frac{1}{(2\pi)^{n/2}}\int\cdots\int\left|\prod_r \frac{2(x,r)}{(r,r)}\right|^\gamma e^{-(x_1^2+\cdots+x_n^2)/2}\,dx_1\cdots dx_n=\prod_{j=1}^n\frac{\Gamma(1+d_j\gamma)}{\Gamma(1+\gamma)}$$

The product is over the roots r of the roots system and the numbers d_j are the degrees of the generators of the ring of invariants of the reflection group. Opdam (1989) gave a uniform proof for all crystallographic reflection groups. Several years later he proved it in full generality (Opdam (1993)), making use of computer-aided calculations by Garvan.

- Integrals can arise in random matrix theory
- A_n → Hermite B_n & D_n → Two special cases of Laguerre
- Connection to RMT, Very Structural, but ☹ does not line up

Graph Theory

- **Hermite**:
  - Random Complete Graph
  - Incidence Matrix is the semicircle law
- **Laguerre**:
  - Random Bipartite Graph
  - Incidence Matrix is Marcenko-Pastur law
- **Jacobi**:
  - Random d-regular graph (McKay)
  - Incidence Matrix is a special case of a Wachter law

Quantum Mechanics Analytically Solvable?

- **Hermite**: Harmonic Oscillator
- **Laguerre**: Radial Part of Hydrogen
  - Morse Oscillator
- **Jacobi**: Angular part of Hydrogen is Legendre
  - Hyperbolic Rosen-Morse Potential

Thanks to Jiahao Chen regarding Derizinsky,Wrochna

Representations of Lie Algebras

- **Hermite** Heisenberg group H_3
- **Laguerre** Third Order Triangular Matrices
- **Jacobi** Unimodular quasi-unitary group

In the orthogonal polynomial basis, the tridiagonal matrix and its pieces can be represented as simple differential operators

Wigner and Narayana

Narayana
Photo
Unavailable

ACKNOWLEDGMENT

I am much indebted to Dr. T. V. Narayana for a clarifying discussion of some of the papers in mathematical statistics which were referred to above.

[Wigner, 1957]

(Narayana was 27)

$$N_{k,j} = \frac{1}{k}\binom{k}{j}\binom{k}{j-1}, \quad N_k(r) = \sum_{j=1}^{k} N_{k,j} r^j .$$

$$m_k = E[\lambda^k] = \frac{1}{n} E\left[\mathbf{Tr}\left(\frac{1}{m}X^T X\right)^k\right] \to \frac{1}{r} N_k(r)$$

- Marcenko-Pastur = Limiting Density for Laguerre
- Moments are Narayana Polynomials!
- Narayana probably would not have known

Cool Pyramid

Narayana everywhere!

```
1
```

```
  3
1   1
```

```
    6
  6   4
1   3   1
```

```
      10
   20    10
10    20    5
1   6    6    1
```

```
        15
     50    20
   50   75   15
 15   60   45   6
1   10   20   10   1
```

```
21
105 35
175 210 35
105 315 189 21
21 140 210 84 7
1 15 50 50 15 1
```

```
28
196 56
490 490 70
490 1176 588 56
196 980 1176 392 28
28 280 700 560 140 8
1 21 105 175 105 21 1
```

```
36
336 84
1176 1008 126
1764 3528 1512 126
1176 4704 4704 1344 84
336 2520 5040 3360 720
36 504 1890 2520 1260 21
1 28 196 490 490 196 2
```

```
45
540 120
2520 1890 210
5292 9072 3402 252
5292 17 640 15 120 3780 210
2520 15 120 25 200 14 400 2700 120
540 5670 17 010 18 900 8100 1215 45
45 840 4410 8820 7350 2520 315 10
1 36 336 1176 1764 1176 336 36 1
```

```
55
825 165
4950 3300 330
13 860 20 790 6930 462
19 404 55 440 41 580 9240 462
13 860 69 300 99 000 49 500 8250 330
4950 41 580 103 950 99 000 37 125 4950 165
825 11 550 48 510 80 850 57 750 17 325 1925 55
55 1320 9240 25 872 32 340 18 480 4620 440 11
1 45 540 2520 5292 5292 2520 540 45 1
```

```
66
1210 220
9075 5445 495
32 670 43 560 13 068 792
60 984 152 460 101 640 20 328 924
60 984 261 360 326 700 145 200 21 780 792
32 670 228 690 490 050 408 375 136 125 16 335 495
9075 101 640 355 740 508 200 317 625 84 700 8470 220
1210 21 780 121 968 284 592 304 920 152 460 33 880 2904 66
66 1980 17 820 66 528 116 424 99 792 41 580 7920 594 12
1 55 825 4950 13 860 19 404 13 860 4950 825 55 1
```

Cool Pyramid

1

3
1 1

6
6 4
1 3 1

10
20 10
10 20 5
1 6 6 1

15
20 10
50 75 15
15 60 45 6
1 10 20 10 1

Narayana Triangle

The Narayana triangle is the number triangle

1
1 1
1 3 1
1 6 6 1
1 10 20 10 1
1 15 50 50 15 1

Cool Pyramid

1

3
1 1

6
6 4
1 3 1

10
20 10
10 20 5
1 6 6 1

15
20 10
50 75 15
15 60 45 6
1 10 20 10 1

Narayana Triangle

The Narayana triangle is the number triangle

1
1 1
1 3 1
1 6 6 1
1 10 20 10 1
1 15 50 50 15 1

Cool Pyramid

1

3
1 1

6
6 4
1 3 1

10
20 10
10 20 5
1 6 6 1

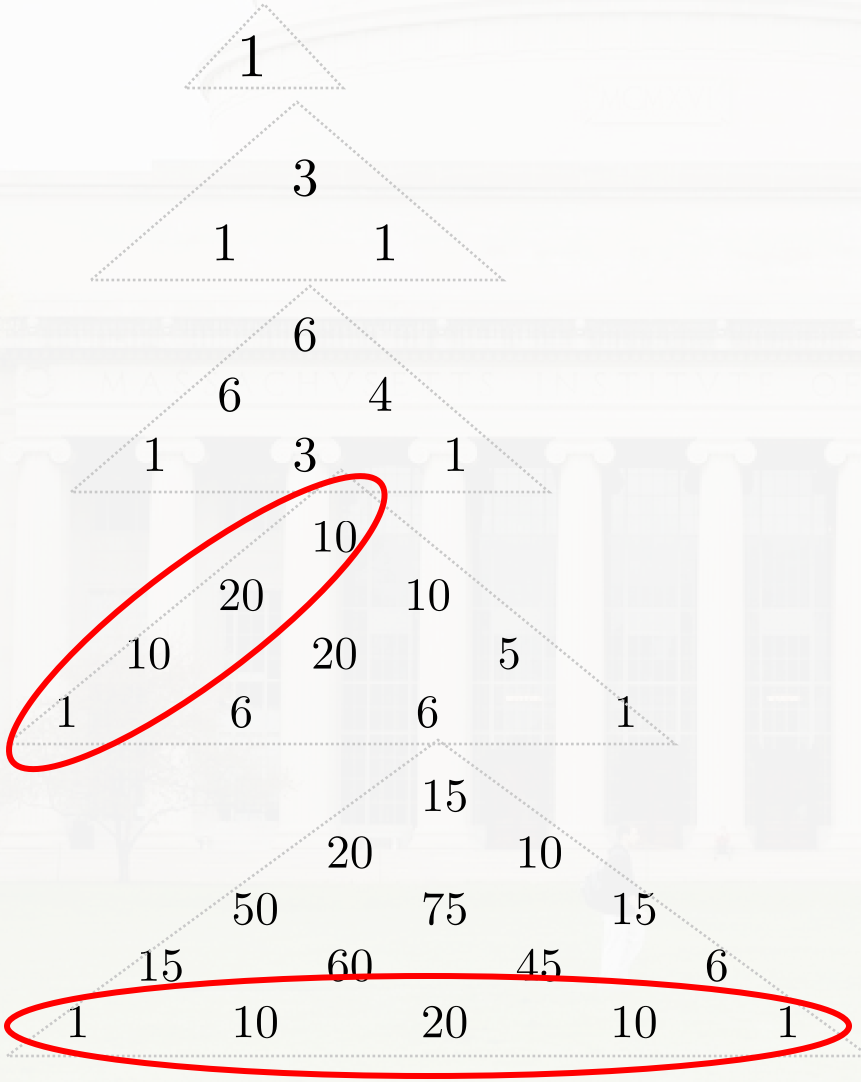
15
20 10
50 75 15
15 60 45 6
1 10 20 10 1

Narayana Triangle

The Narayana triangle is the number triangle

1
1 1
1 3 1
1 6 6 1
1 10 20 10 1
1 15 50 50 15 1

Cool Pyramid



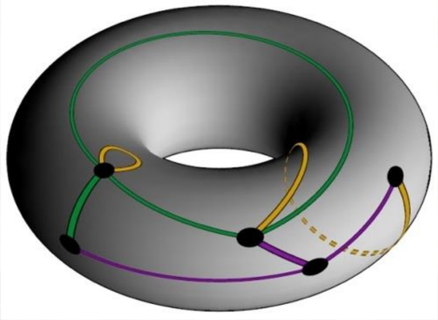
Narayana Triangle

The Narayana triangle is the number triangle

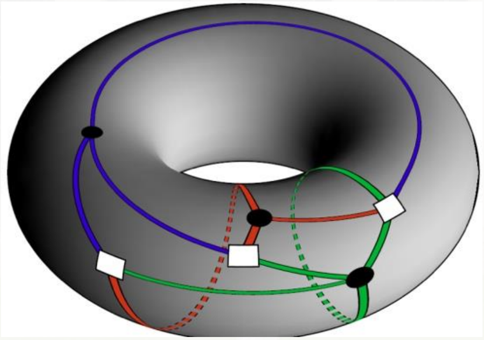
1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

Graphs on Surfaces???
Thanks to Mike LaCroix

- **Hermite**: Maps with one Vertex Coloring



- **Laguerre**: Bipartite Maps with multiple Vertex Colorings



- **Jacobi**: We know it's there, but don't have it quite yet.

The "How I met Mike" Slide

[*arm, conjugate, expH, expHjacks, expJ, expJjacks, expL, expLjacks, gbinomial, ghypergeom, gsfact, hermite, hermite2, issubpar, jack, jack2jack, jackidentity, jacobi, laguerre, leg, lhook, m2jack, m2m, m2p, p2m, par, ρ, sfact, subpar, uhook*] **(3)**

$$f := k \rightarrow series\left(\frac{a^{\frac{k}{2}}}{n^{\frac{k}{2}+1}} simplify(expH(a, m[k], n)), n, k+2\right)$$

$$k \rightarrow series\left(\frac{a^{\frac{1}{2}k}\, simplify(MOPS\text{:-}expH(a, m_k, n))}{n^{\frac{1}{2}k+1}}, n, k+2\right) \quad \textbf{(4)}$$

$f(2)$

$$\frac{a-1}{n} + 1 \quad \textbf{(5)}$$

$f(4)$

$$\frac{3a^2 - 5a + 3}{n^2} + \frac{5a - 5}{n} + 2 \quad \textbf{(6)}$$

$f(6)$

$$\frac{15a^3 - 32a^2 + 32a - 15}{n^3} + \frac{32a^2 - 54a + 32}{n^2} + \frac{22a - 22}{n} + 5 \quad \textbf{(7)}$$

Mops
Dumitriu, E, Shuman 2007
a=2/β

Multivariate Hermite and Laguerre Moments α=2/β=1+b

$$\langle \mathrm{Tr}(X^2)\rangle = bn + n^2$$

$$\langle \mathrm{Tr}(X^4)\rangle = (\alpha+3b^2)n + 5bn^2 + 2n^3$$

$$\langle \mathrm{Tr}(X^6)\rangle = (13\alpha b+15b^3)n + (10\alpha+32b^2)n^2 + 22bn^3 + 5n^4$$

$$\langle \mathrm{Tr}(X^8)\rangle = (21\alpha^2+160\alpha b^2+105b^4)n + (215\alpha b+260b^3)n^2 + (70\alpha+234b^2)n^3 + 93bn^4 + 14n^5$$

$$\langle \mathrm{Tr}(X^{10})\rangle = (753\alpha^2 b+2136\alpha b^3+945b^5)n + (483\alpha^2+3811\alpha b^2+2589b^4)n^2 + (2200\alpha b+2750b^3)n^3 + (420\alpha+1450b^2)n^4 + 386bn^5 + 42n^6$$

$$\langle \mathrm{Tr}(X^2)\rangle = (x+y)xy + bxy$$

$$\langle \mathrm{Tr}(X^3)\rangle = (x+y)^2xy + x^2y^2 + \alpha xy + 3b(x+y)xy + 2b^2xy$$

$$\langle \mathrm{Tr}(X^4)\rangle = (x+y)^3xy + 3(x+y)x^2y^2 + 5\alpha(x+y)xy + 6b(x+y)^2xy + 5bx^2y^2 + 7\alpha bxy + 11b^2(x+y)xy + 6b^3xy$$

$$\langle \mathrm{Tr}(X^5)\rangle = (x+y)^4xy + 6(x+y)^2x^2y^2 + 2x^3y^3 + 15\alpha(x+y)^2xy + 10\alpha x^2y^2 + 8\alpha^2xy + 10b(x+y)^3xy + 25b(x+y)x^2y^2 + 55\alpha b(x+y)xy + 35b^2(x+y)^2xy + 25b^2x^2y^2 + 46\alpha b^2xy + 50b^3(x+y)xy + 24b^4xy$$

$$\langle \mathrm{Tr}(X^6)\rangle = (x+y)^5xy + 10(x+y)^3x^2y^2 + 10(x+y)x^3y^3 + 35\alpha(x+y)^3xy + 70\alpha(x+y)x^2y^2 + 84\alpha^2(x+y)xy + 15b(x+y)^4xy + 75b(x+y)^2x^2y^2 + 22bx^3y^3 + 238\alpha b(x+y)^2xy + 142\alpha bx^2y^2 + 144\alpha^2bxy + 85b^2(x+y)^3xy + 182b^2(x+y)x^2y^2 + 505\alpha b^2(x+y)xy + 225b^3(x+y)^2xy + 141b^3x^2y^2 + 326\alpha b^3xy + 274b^4(x+y)xy + 120b^5xy$$

$$\langle \mathrm{Tr}(X^7)\rangle = (x+y)^6xy + 15(x+y)^4x^2y^2 + 30(x+y)^2x^3y^3 + 5x^4y^4 + 70\alpha(x+y)^4xy + 280\alpha(x+y)^2x^2y^2 + 70\alpha x^3y^3 + 469\alpha^2(x+y)^2xy + 245\alpha^2x^2y^2 + 180\alpha^3xy + 21b(x+y)^5xy + 175b(x+y)^3x^2y^2 + 154b(x+y)x^3y^3 + 756\alpha b(x+y)^3xy + 1351\alpha b(x+y)x^2y^2 + 1995\alpha^2b(x+y)xy + 175b^2(x+y)^4xy + 749b^2(x+y)^2x^2y^2$$

R-TRANSFORMS IN FREE PROBABILITY

ALEXANDRU NICA

Lectures in the special semester 'Free probability theory and operator spaces', IHP, Paris, 1999

14. The S-transform

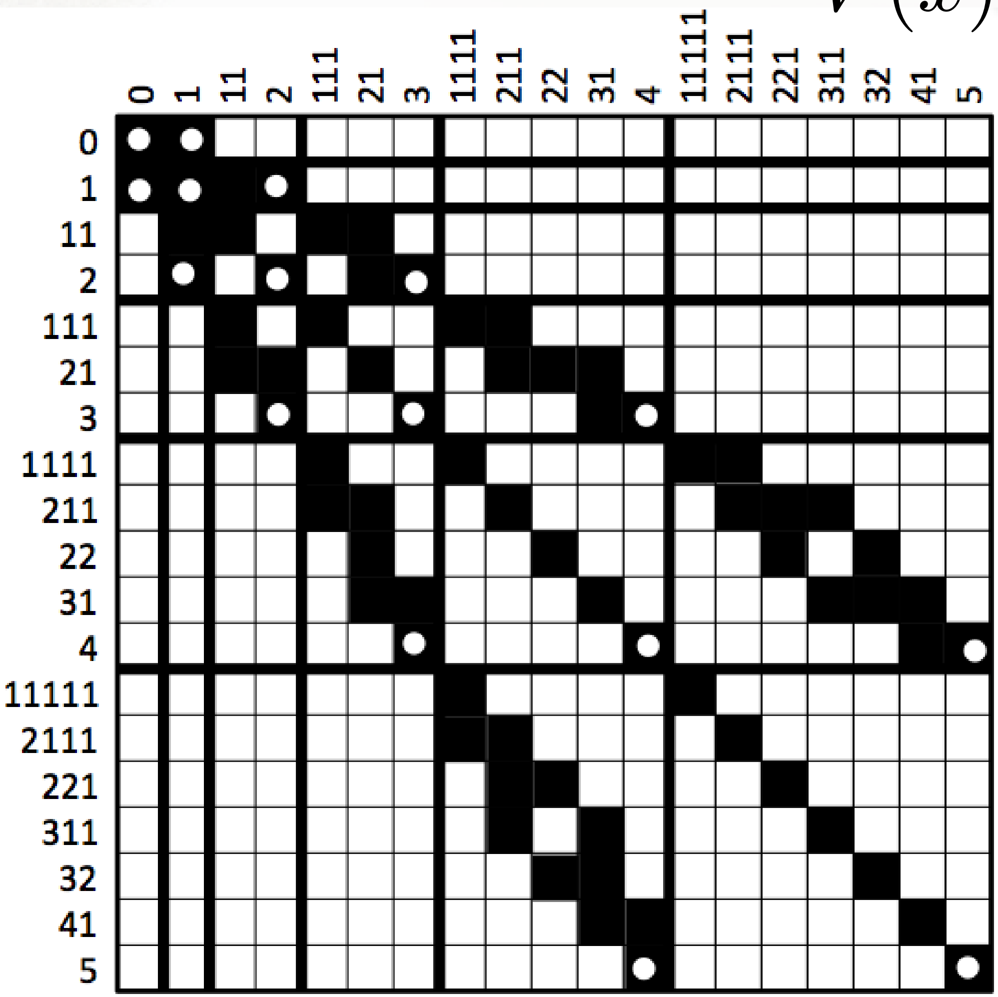
Recall that if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and if $a \in \mathcal{A}$ is such that $\varphi(a) \neq 0$, then the S-transform of a is defined to be the series:

$$S_a(z) := \frac{1}{z} R_a^{<-1>}(z) = \frac{1+z}{z} M_a^{<-1>}(z) \tag{1}$$

	Law	S-transform
Hermite	Semicircle Law	1
Laguerre	Marcenko-Pastur Law	$\dfrac{1}{z+\lambda}$
Jacobi	Wachter Law	$\dfrac{a+b+z}{a+z}$

Polynomials of matrix argument

$$P_\kappa(x) = \frac{\det\left[P_{\kappa_i+N-i}(x_j)\right]_{i,j=1}^{N}}{V(x)} \qquad \beta\text{=2}$$



Praveen and E (2014)

Young Lattice Generalizes
Sym tridiagonal

Always for β=2
Only HLJ for other β?

Schur :: Jack as
$P_\kappa(x)$ :: General β

Real, Complex, Quaternion is NOT Hermite, Laguerre,Jacobi

- We now understand that Dyson's fascination with the three division rings lead us astray
- There is a continuum that includes β=1,2,4
- Informal method, called ghosts and shadows for β- ensembles

E. (2010)

Finite Random Matrix Models

$$\begin{bmatrix} \sqrt{2}G & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \sqrt{2}G & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}G & \chi_{\beta} \\ & & & \chi_{\beta} & \sqrt{2}G \end{bmatrix}$$

Hermite

$$\begin{bmatrix} \chi_{n\beta} & & & \\ \chi_{(m-1)\beta} & \chi_{(n-1)\beta} & & \\ & \ddots & \ddots & \\ & & \chi_{2\beta} & \chi_{(n-m+2)\beta} & \\ & & & \chi_{\beta} & \chi_{(n-m+1)\beta} \end{bmatrix}$$

Laguerre

$$\left[\begin{array}{ccccc|ccccc} c_n & -s_n c'_{n-1} & & & & s_n s'_{n-1} & & & & \\ & c_{n-1}s'_{n-1} & -s_{n-1}c'_{n-2} & & & c_{n-1}c'_{n-1} & s_{n-1}s'_{n-2} & & & \\ & & c_{n-2}s'_{n-2} & \ddots & & & c_{n-2}c'_{n-2} & s_{n-2}s'_{n-3} & & \\ & & & \ddots & -s_2c'_1 & & & \ddots & \ddots & \\ & & & & c_1s'_1 & & & & c_1c'_1 & s_1 \\ \hline -s_n & -c_nc'_{n-1} & & & & c_ns'_{n-1} & & & & \\ & -s_{n-1}s'_{n-1} & -c_{n-1}c'_{n-2} & & & -s_{n-1}c'_{n-1} & c_{n-1}s'_{n-2} & & & \\ & & -s_{n-2}s'_{n-2} & \ddots & & & -s_{n-2}c'_{n-2} & c_{n-2}s'_{n-3} & & \\ & & & \ddots & -c_2c'_1 & & & \ddots & \ddots & \\ & & & & -s_1s'_1 & & & & -s_1c'_1 & c_1 \end{array}\right]$$

Jacobi

$\Theta = (\theta_n, \dots, \theta_1) \in [0, \frac{\pi}{2}]^n$	$\Phi = (\phi_{n-1}, \dots, \phi_1) \in [0, \frac{\pi}{2}]^{n-1}$
$c_k = \cos\theta_k$	$c'_k = \cos\phi_k$
$s_k = \sin\theta_k$	$s'_k = \sin\phi_k$
$c_k \sim \sqrt{\mathrm{Beta}\left(\frac{\beta}{2}(a+k), \frac{\beta}{2}(b+k)\right)}$	$c'_k \sim \sqrt{\mathrm{Beta}\left(\frac{\beta}{2}k, \frac{\beta}{2}(a+b+1+k)\right)}$

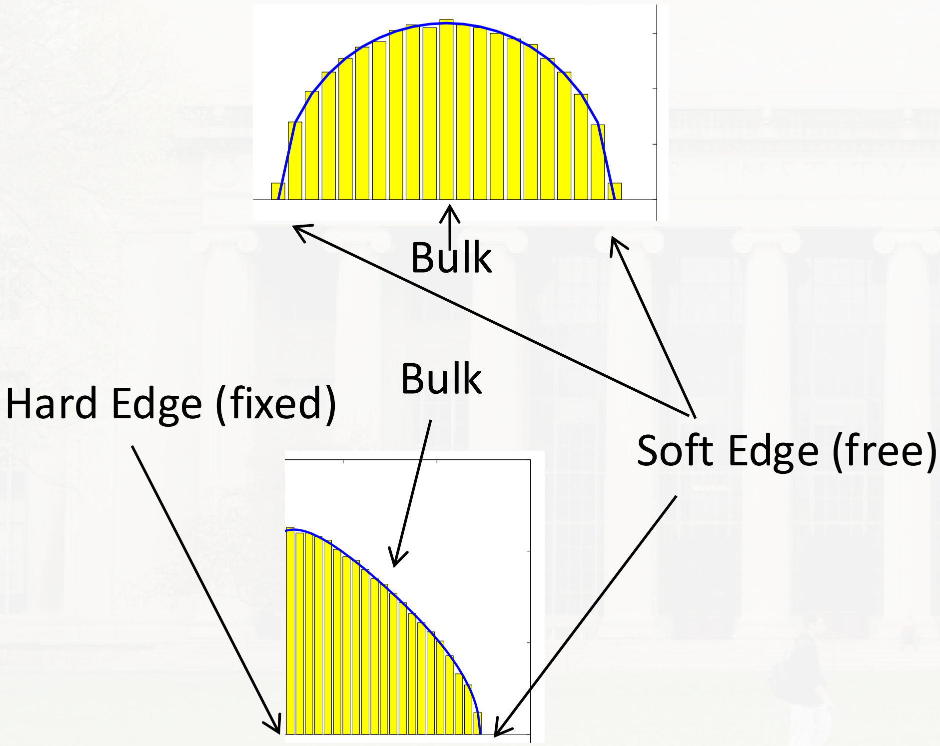
But ghosts lead to corner's process algorithms for H,L,J!

(see Borodin, Gorin 2013)

- Hermite: Symmetric Arrow Matrix Algorithm
- Laguerre: Broken Arrow Matrix Algorithm
- Jacobi: Two Broken Arrow Matrices Algorithm

Dubbs, E. (2013) and
Dubbs, E., Praveen (2013)
Also Forrester, etc.

Sine Kernel, Airy Kernel, Bessel Kernel
NOT
Hermite, Laguerre, Jacobi



Conclusion

Is there a theory???

- Whose answer is
    - 1) exactly Hermite, Laguerre, Jacobi
    - 2) or includes HLJ
- For Laguerre and Jacobi
    - includes all parameters
- Connects to a Matrix Framework?
- Can be connected to Random Matrix Theory
- Can Circle back to various differential,difference,hypergeometric,umbral definitions?
- Makes me happy!

?

Challenges for you

- In your own research: Find the hidden triad!!