For integers n, r, i, let $c_{n,r,i} = \frac{n!(n-1)!}{(n-r-1)!(i-1)!(r-i)!i!(n-i)!(r+1)r}$ if $1 \le i \le r \le n-1$ and 0 otherwise. Also, recall that for positive integer n we have the Narayana polynomials $N_n(x) = \sum_{i=1}^{n} \binom{1}{n}\binom{n}{i}\binom{n}{i-1}x^i$.

Theorem. For any positive integer n, for any $a, b \ge 1$, we have

$$\frac{a}{a+b} - (a+b)\sum_{j=0}^{n-2}\left(\frac{\sqrt{a(a+b-1)}}{a+b}\right)^{2j+4} N_{j+1}\left(\frac{b}{a(a+b-1)}\right) =$$

$$\frac{1}{(a+b)^{2n-1}}\left[a^n(a+b)^{n-1} + \sum_{r=1}^{n-1}(a+b)^{n-1-r}\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right)\right].$$

This gives a new way to express the limiting case of the moments of MANOVA matrices (where the LHS is the known formula, and the RHS is the new way). Also this shows us that the $c_{n,r,i}$ are the numbers on page 195 of `math.mit.edu/~edelman/publications/infinite-random.pdf`. Specifically, $c_{n,r,i}$ is the ith number from the right in the rth row from the bottom in the $n-1$st triangle on that page.

Proof. We proceed by induction on n. The base case $n = 1$ is clear (as the nasty sums become empty, so both sides equal $\frac{a}{a+b}$). Now for the inductive step, to go from n to $n+1$. Let L_n and R_n denote the LHS and RHS of the identity for n, and analogous for $n+1$. We know $L_n = R_n$, and we want $L_{n+1} = R_{n+1}$. It suffices to show $(a+b)^{2n+1}(L_n - L_{n+1}) = (a+b)^{2n+1}(R_n - R_{n+1})$. Now $(a+b)^{2n+1}(L_n - L_{n+1}) = (a(a+b-1))^{n+1}N_n(\frac{b}{a(a+b-1)})$.

What is $(a+b)^{2n+1}(R_n - R_{n+1})$? We have $(a+b)^{2n+1}R_n = a^n(a+b)^{n+1} + \sum_{r=1}^{n-1}(a+b)^{n-r}\left((a+b)\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right)$ and $(a+b)^{2n+1}R_{n+1} = a^{n+1}(a+b)^n + \sum_{r=1}^{n}(a+b)^{n-r}\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n+1,r,i}a^{n+1-i}b^i\right)$.

Now let us do a useful sub-computation.

Lemma. For $1 \le r \le n-1$, we have $(a+b)\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right) - \sum_{i=1}^{r}(-1)^{r-i}c_{n+1,r,i}a^{n+1-i}b^i = \sum_{i=1}^{r+1}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i$.

Proof. We can see that $(a+b)\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right) = \sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n+1-i}b^i + \sum_{i=2}^{r+1}(-1)^{r-i+1}c_{n,r,i-1}a^{n+1-i}b^i = \sum_{i=1}^{r+1}(-1)^{r-i}(c_{n,r,i} - c_{n,r,i-1})a^{n+1-i}b^i$ (recalling that $c_{n,r,i} = 0$ whenever it is not the case that $1 \le i \le r \le n-1$). Thus $(a+b)\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right) - \sum_{i=1}^{r}(-1)^{r-i}c_{n+1,r,i}a^{n+1-i}b^i = \sum_{i=1}^{r+1}(-1)^{r-i}(c_{n,r,i} - c_{n,r,i-1} - c_{n+1,r,i})a^{n+1-i}b^i$.

Now, for $2 \le i \le r$, we have $c_{n,r,i}, c_{n,r,i-1}, c_{n+1,r,i} \ne 0$. We can check from the definition that $\frac{c_{n,r,i-1}}{c_{n,r,i}} = \frac{i(i-1)}{(n-i+1)(r-i+1)}$ and $\frac{c_{n+1,r,i}}{c_{n,r,i}} = \frac{(n+1)n}{(n-r)(n-i+1)}$. Then $c_{n+1,r,i} + c_{n,r,i-1} - c_{n,r,i} = c_{n,r,i}\left(\frac{i(i-1)}{(n-i+1)(r-i+1)} + \frac{(n+1)n}{(n-r)(n-i+1)} - 1\right) = c_{n,r,i}\frac{(i(i-1)(n-r)+(n+1)n(r-i+1)-(n-i+1)(r-i+1)(n-r))}{(n-i+1)(r-i+1)(n-r)}$. Now, by Wolfram Alpha, we have $i(i-1)(n-r)+(n+1)n(r-i+1)-(n-i+1)(r-i+1)(n-r) = -ir^2 - ir + nr^2 + nr + r^2 + r$. Thus $c_{n+1,r,i} + c_{n,r,i-1} - c_{n,r,i} = c_{n,r,i}\frac{r(r+1)(n-i+1)}{(n-i+1)(r-i+1)(n-r)} = c_{n,r,i}\frac{r(r+1)}{(r-i+1)(n-r)} = \frac{n!(n-1)!}{(n-r-1)!(i-1)!(r-i)!i!(n-i)!(r+1)r}\cdot\frac{r(r+1)}{(r-i+1)(n-r)} = \frac{n!(n-1)!}{(n-r)!(i-1)!(r-i+1)!i!(n-i)!} = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}$.

Now we should handle the $i = 1$ and $i = r + 1$ cases separately.

For $i = 1$, we have $c_{n+1,r,i} + c_{n,r,i-1} - c_{n,r,i} = c_{n+1,r,1} - c_{n,r,1}$. Now note that $c_{n,r,1} = \frac{n!(n-1)!}{(n-r-1)!(r-1)!(n-1)!(r+1)r} = \frac{n!}{(n-r-1)!(r+1)!}$; similarly $c_{n+1,r,1} = \frac{(n+1)!}{(n-r)!(r+1)!}$, so $c_{n+1,r,1} - c_{n,r,i} = ((n+1)-(n-r))\frac{n!}{(n-r)!(r+1)!} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r} = \frac{1}{n}\binom{n}{1}\binom{n}{0}\binom{n}{n-r}$.

For $i = r+1$, we have $c_{n+1,r,i}+c_{n,r,i-1}-c_{n,r,i} = c_{n,r,r} = \frac{n!(n-1)!}{(n-r-1)!(r-1)!r!(n-r)!(r+1)r} = \frac{1}{n}\binom{n}{r+1}\binom{n}{r} = \frac{1}{n}\binom{n}{r+1}\binom{n}{r}\binom{n-r}{n-r}$.

Thus in any case, $c_{n+1,r,i} + c_{n,r,i-1} - c_{n,r,i} = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}$. Thus $(a + b)\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right) - \sum_{i=1}^{r}(-1)^{r-i}c_{n+1,r,i}a^{n+1-i}b^i = \sum_{i=1}^{r+1}(-1)^{r-i}(c_{n,r,i} - c_{n,r,i-1} - c_{n+1,r,i})a^{n+1-i}b^i = \sum_{i=1}^{r+1}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i$, as claimed.
■

Now $(a + b)^{2n+1}(R_n - R_{n+1}) = a^n(a + b)^{n+1} - a^{n+1}(a + b)^n + \sum_{r=1}^{n-1}(a + b)^{n-r}\left((a + b)\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n,r,i}a^{n-i}b^i\right) - \sum_{i=1}^{r}(-1)^{r-i}c_{n+1,r,i}a^{n+1-i}b^i\right) - \sum_{r=n}^{n}(a + b)^{n-r}\left(\sum_{i=1}^{r}(-1)^{r-i}c_{n+1,r,i}a^{n+1-i}b^i\right) = a^n b(a + b)^n + \sum_{r=1}^{n-1}(a+b)^{n-r}\left(\sum_{i=1}^{r+1}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i\right) - \sum_{i=1}^{n}(-1)^{n-i}c_{n+1,n,i}a^{n+1-i}b^i$.

Now note that, for $1 \le i \le n$, we have $c_{n+1,n,i} = \frac{(n+1)!n!}{(i-1)!(n-i)!i!(n+1-i)!(n+1)n} = \frac{1}{n}\binom{n}{i}\binom{n}{i-1}$.

Also, $\sum_{r=1}^{n-1}(a+b)^{n-r}\left(\sum_{i=1}^{r+1}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i\right) = \sum_{1\le i\le r+1\le n;(i,r)\neq(1,0)}(a + b)^{n-r}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i$. Now note that if we took $(i, r) = (1, 0)$ in that inner term, we get $(a + b)^n a^n b$. Thus $a^n b(a + b)^n + \sum_{r=1}^{n-1}(a + b)^{n-r}\left(\sum_{i=1}^{r+1}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i\right) = \sum_{1\le i\le r+1\le n}(a + b)^{n-r}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i$.

Now $(a + b)^{2n+1}(R_n - R_{n+1}) = \sum_{1\le i\le r+1\le n}(a+b)^{n-r}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i + \sum_{i=1}^{n}(-1)^{n-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}a^{n+1-i}b^i = \sum_{1\le i\le r+1\le n+1;(i,r)\neq(n+1,n)}(a + b)^{n-r}(-1)^{r-i+1}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\binom{n-i+1}{n-r}a^{n+1-i}b^i = \sum_{i=1}^{n}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}a^{n+1-i}b^i\left(\sum_{r=i-1}^{n}\binom{n-i+1}{n-r}(a + b)^{n-r}(-1)^{r-i+1}\right) = \sum_{i=1}^{n}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}a^{n+1-i}b^i(a + b - 1)^{n+1-i} = (a(a + b - 1))^{n+1}\sum_{i=1}^{n}\frac{1}{n}\binom{n}{i}\binom{n}{i-1}\left(\frac{b}{a(a+b-1)}\right)^i = (a(a + b - 1))^{n+1}N_n(\frac{b}{a(a+b-1)})$. Thus we have $(a+b)^{2n+1}(L_n - L_{n+1}) = (a+b)^{2n+1}(R_n - R_{n+1})$, as desired. Thus the Theorem holds. QED.