18.338 Eigenvalues of Random Matrices

Problem Set 2

Due Date: Wed Oct. 13, 2021

Homework

Read cahpter 1, 2, 3 of and comment on `nb.mit.edu` (The lecture notes and the link to `nb` can be found in Piazza: `http://piazza.com/mit/fall2021/18338`.). Please give your feedback especially high level style and where things did not make sense, in addition to spelling or technical errors.

Do at least 7 out of the 11 problems (Computational/Mathematical problems are denoted as C/M. Exercises with numbers and pages are from the class notes.)

Concentration of Measure for Gaussian Ensembles

It is remarkable how well the semicircle describes the histogram for Gaussian ensembles and other Wigner-type matrices. These mathematical and computational problems investigate the semicircle, how good it is, and how far off we can get. Section 1.1 and section 5 of reference `http://www-math.mit.edu/~edelman/homepage/papers/flucts.pdf` are related to this question.

Take as given that the tridiagonal matrix T_n when normalized by $\sqrt{\beta n}$ (i.e. $H_n = T_n/\sqrt{\beta n}$) on Page 100 (Equation (6.3)) of the notes has the same eigenvalues as a Gaussian ensemble, where $\beta = 1$ is the GOE, $\beta = 2$ is the GUE, and any $\beta > 0$ is allowed. Make use of `Tridiagonal` in Julia, see `https://docs.julialang.org/en/v1/stdlib/LinearAlgebra/index.html`, and use the following code snippet as a reference.

```
using LinearAlgebra, Distributions
T(n,β) = SymTridiagonal(rand(Normal(0,√(2β)),n), [rand( Chi(i*β)) for i=(n-1):-1:1] )
B(m,n,β) = Bidiagonal( [rand(Chi(i*β)) for i=m:-1:(m-n+1)] , [rand(Chi(i*β)) for i=n-1:-1:1],:L)
```

1. (M) or (C). The first moment (and all odd moments) of the eigenvalues of the Gaussian ensembles has expected value 0. (This is a way of saying that $\mathbb{E}[\mathbf{Tr}(T_n)] = 0$). Mathematically or with a Monte Carlo simulation or both, conclude that $\mathbf{Tr}(T_n)$ is a scalar Gaussian. If you wish to access to Section 2.3.3 of Anderson, Guionnet, Zeitouni `http://www.wisdom.weizmann.ac.il/~zeitouni/cupbook.pdf` (book page 42, pdf page 56) you might compare 2.3.10. How close are they?

2. (M) or (C) The second moment is a factor of $n^2/2$ times a χ^2 random variable with $n(n-1)\beta/2 + n$ degrees of freedom. Prove this by using simple properties of chi-square. (The degrees of freedom add.) One might use approximations such as if X has the distribution of χ^2_k then $\sqrt{2X}$ is roughly normal with mean $\sqrt{2k-1}$ (or just $\sqrt{2k}$ with unit variance). Potentially compare the concentration of measure again.

3. (M) What would happen in Problem 1 and 2 if the matrices are Wigner matrices (i.e., diagonal has variance 1 and the off-diagonal has variance 2) as $n \to \infty$? (Hint: use the Central Limit Theorem.)

4. (C) Investigate how other odd moments deviate from 0 or how even moment deviate from the Catalan numbers.

5. (C) Try to investigate how the histograms themselves deviate from the semicircle. One can draw lots of pictures to see the semicircle. but what is interesting is to take averages and watch the fluctuations. See if you can estimate the fluctuations to the semicircle over various intervals using normals. One might start by taking the mean and seeing how far off finite n is from infinite n, or one can consider the variance.

6. (C) Perform Monte Carlo experiments on non-Gaussians carefully enough to predict the deviation.

Laguerre ensembles and Others

7. (M) or (C) Perform Monte Carlo experiments to explore the mean and variance of the sum of the singular values of the bidiagonal model (Page 100, Equation (6.4)) ($m = n$) for different n's. Furthermore, how does the sum change as a function of β (Page 112, Equation (6.7))? Is the mean monotonically going up or down? Where does it change? Again, use the Julia code in the first question as a reference.

8. (M) or (C) Investigate numerically or mathematically the sum of the singular values for Laguerre ensembles normalized properly and see whether they converge monotonically in an increasing or decreasing manner.

9. (C) Plot the histogram of the square singular values of (for different z's on the complex plane)

   $$\texttt{(randn(n,n) + im*randn(n,n)) / sqrt(2*n)} - z * \mathbb{I},$$

   and compare $|z| < 1$ with $|z| > 1$.

10. (C) Exercise 1.14

11. (M) Exercise 2.7