Random Quantum Density Operators: MIT 18.338 Final Project

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December 13, 2023

Abstract

This report studies random quantum density operators. These are random matrices which describe the state of a quantum system. The aim of the report is two-fold: (i) to precisely introduce quantum density operators, and (ii) to study (analytically or numerically) the eigenvalue distributions of several quantum density ensembles. First studied are random matrices arising from bipartite quantum systems (i.e. system+reservoir). The joint eigenvalue density of such ensemble is derived, and the asymptotic level-density is studied numerically. Second, we study random matrices arising from multipartite quantum systems. The eigenvalues of such matrices are related to the Fuss-Catalan numbers and the Fuss-Catalan distribution. An introduction to the mathematical formulation of this deeper topic is presented, and numerical results are presented.

1 Background and notation

This project concerns random matrices that arise from random states of a quantum system. Let us briefly introduce some notation. Random variables are displayed in sans serif, upright fonts; their realizations in serif, italic fonts. Vectors and matrices are denoted by bold lowercase and uppercase letters, respectively. The unitary group is denoted $U(d)$. Quantum states (vectors) are denoted using Dirac notation. A state $|\boldsymbol{\psi}\rangle \in \mathbb{C}^d$ is a unit-norm vector, and its conjugate transpose is denoted $\langle\boldsymbol{\psi}| \triangleq |\boldsymbol{\psi}\rangle^\dagger$. Each state can be associated with a *density operator*[1] $\boldsymbol{\Xi} \triangleq |\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}| \in \mathbb{C}^{d\times d}$, which is simply the outer product of $|\boldsymbol{\psi}\rangle$ with itself. If the state of a quantum system is only known probabilistically, then the associated density operator may be expressed as a convex combination of states:

$$\boldsymbol{\Xi} = \sum_k p_k \, |\boldsymbol{\psi}_k\rangle\langle\boldsymbol{\psi}_k| \in \mathbb{C}^{d\times d}$$

where $p_k \geqslant 0$ and $\sum_k p_k = 1$. Colloquially, p_k represents the probability that the system is in state $|\boldsymbol{\psi}_k\rangle$. The set of all quantum density operators in $\mathbb{C}^{d\times d}$ forms the closed convex set

$$\mathcal{D} = \{\, \boldsymbol{\Xi} \in \mathbb{C}^{d\times d} \,:\, \boldsymbol{\Xi} \succcurlyeq \mathbf{0},\, \mathrm{Tr}\{\boldsymbol{\Xi}\} = 1 \,\}.$$

Henceforth, we will simply refer to a density operator $\boldsymbol{\Xi}$ describing the state of a quantum system as simply *the* state of the system.

Given two quantum systems A and B in states $\boldsymbol{\Xi}^{\mathrm{A}} \in \mathbb{C}^{d_{\mathrm{A}}\times d_{\mathrm{A}}}$ and $\boldsymbol{\Xi}^{\mathrm{B}} \in \mathbb{C}^{d_{\mathrm{B}}\times d_{\mathrm{B}}}$, the state of the composite system may be written

$$\boldsymbol{\Xi}^{\mathrm{AB}} = \boldsymbol{\Xi}^{\mathrm{A}} \otimes \boldsymbol{\Xi}^{\mathrm{B}} \in \mathbb{C}^{d_{\mathrm{A}}d_{\mathrm{B}}\times d_{\mathrm{A}}d_{\mathrm{B}}}$$

where $\otimes$ denotes the Kronecker product. However, we will allow for arbitrary convex combinations of such product states so that the set of all valid quantum states of the composite system AB is

$$\mathcal{D}^{\mathrm{AB}} = \{\, \boldsymbol{\Xi}^{\mathrm{AB}} \in \mathbb{C}^{d_{\mathrm{A}}d_{\mathrm{B}}\times d_{\mathrm{A}}d_{\mathrm{B}}} \,:\, \boldsymbol{\Xi}^{\mathrm{AB}} \succcurlyeq \mathbf{0},\, \mathrm{Tr}\{\boldsymbol{\Xi}^{\mathrm{AB}}\} = 1 \,\}.$$

[1] The terminology "operator" is used to be consistent with the physics literature. Indeed, at the present, $\boldsymbol{\Xi}$ is simply a matrix and more grandiose terminology could be avoided.

One final piece of machinery is needed prior to discussing random matrices arising from quantum systems. The *partial trace* is introduced below, which is an operation that allows one to "forget" the subsystem B.

Definition 1 (Partial trace of a product state)**.** *Consider the product state* $\boldsymbol{\Xi}^{\mathrm{AB}} = \boldsymbol{\Xi}^{\mathrm{A}} \otimes \boldsymbol{\Xi}^{\mathrm{B}}$*, where* $\boldsymbol{\Xi}^{\mathrm{A}}$ *and* $\boldsymbol{\Xi}^{\mathrm{B}}$ *are density matrices. The partial trace of* $\boldsymbol{\Xi}^{\mathrm{AB}}$ *over subsystem B is denoted and defined by*

$$\mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\} \triangleq \mathrm{Tr}\{\boldsymbol{\Xi}^{\mathrm{B}}\}\,\boldsymbol{\Xi}^{\mathrm{A}}. \tag{1}$$

Directly from this definition we see that $\boldsymbol{\Xi}^{\mathrm{A}} = \mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\}$. This definition can be expanded by linearity to all density operators in $\mathcal{D}_{\mathrm{AB}}$. In particular, if

$$\boldsymbol{\Xi}^{\mathrm{AB}} = \sum_i \sum_j \eta_{ij}\, \boldsymbol{\Xi}_i^{\mathrm{A}} \otimes \boldsymbol{\Xi}_j^{\mathrm{B}} \tag{2}$$

for $\boldsymbol{\Xi}_i^{\mathrm{A}} \in \mathcal{D}_{\mathrm{A}}$ and $\boldsymbol{\Xi}_j^{\mathrm{B}} \in \mathcal{D}_{\mathrm{B}}$ then

$$\mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\} = \sum_i \sum_j \eta_{ij}\, \mathrm{Tr}\{\boldsymbol{\Xi}_j^{\mathrm{B}}\}\, \boldsymbol{\Xi}_i^{\mathrm{A}}\,. \tag{3}$$

This is not, however, a particularly useful definition of the partial trace because it requires expanding $\boldsymbol{\Xi}^{\mathrm{AB}}$ out in terms of tensor products. The following theorem gives an expression which is more direct.

Theorem 1. *Let* $\boldsymbol{I}_{d_{\mathrm{A}}}$ *be the* d_{A}*-dimensional identity and let* $\{\,|\boldsymbol{k}\rangle \,:\, k \in \mathbb{N}_{1:d_{\mathrm{B}}}\}$ *be an arbitrary orthonormal basis for* $\mathbb{C}^{d_{\mathrm{B}}}$*. For an arbitrary density* $\boldsymbol{\Xi}^{\mathrm{AB}} \in \mathcal{D}^{\mathrm{AB}}$ *the partial trace may be written*

$$\mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\} \;=\; \sum_{k=1}^{d_{\mathrm{B}}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes \langle \boldsymbol{k}|\right) \boldsymbol{\Xi}^{\mathrm{AB}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes |\boldsymbol{k}\rangle\right). \tag{4}$$

Proof. By definition of the product space $\mathbb{C}^{d_{\mathrm{A}} \times d_{\mathrm{A}}} \otimes \mathbb{C}^{d_{\mathrm{B}} \times d_{\mathrm{B}}}$, any density operator may be written as in (2). Therefore,

$$\begin{aligned}
\sum_{k=1}^{d_{\mathrm{B}}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes \langle \boldsymbol{k}|\right) \boldsymbol{\Xi}^{\mathrm{AB}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes |\boldsymbol{k}\rangle\right) &= \sum_{k=1}^{d_{\mathrm{B}}} \sum_i \sum_j \eta_{ij} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes \langle \boldsymbol{k}|\right) \boldsymbol{\Xi}_i^{\mathrm{A}} \otimes \boldsymbol{\Xi}_j^{\mathrm{B}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes |\boldsymbol{k}\rangle\right) \\
&= \sum_i \sum_j \sum_{k=1}^{d_{\mathrm{B}}} \eta_{ij}\, \boldsymbol{\Xi}_i^{\mathrm{A}}\, \langle \boldsymbol{k}|\boldsymbol{\Xi}_j^{\mathrm{B}}|\boldsymbol{k}\rangle
\end{aligned} \tag{5}$$

On the other hand, since the trace of a scalar is equal to the scalar itself,

$$\begin{aligned}
\sum_{k=1}^{d_{\mathrm{B}}} \langle \boldsymbol{k}|\boldsymbol{\Xi}_j^{\mathrm{B}}|\boldsymbol{k}\rangle &= \sum_{k=1}^{d_{\mathrm{B}}} \mathrm{Tr}\{\langle \boldsymbol{k}|\boldsymbol{\Xi}_j^{\mathrm{B}}|\boldsymbol{k}\rangle\} \\
\text{cyclic property of trace} \rightarrow \quad &= \sum_{k=1}^{d_{\mathrm{B}}} \mathrm{Tr}\{|\boldsymbol{k}\rangle\langle \boldsymbol{k}|\boldsymbol{\Xi}_j^{\mathrm{B}}\} \\
\text{resolution of the identity} \rightarrow \quad &= \mathrm{Tr}\{\boldsymbol{\Xi}_j^{\mathrm{B}}\}.
\end{aligned} \tag{6}$$

Combining the prior two equations,

$$\begin{aligned}
\sum_{k=1}^{d_{\mathrm{B}}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes \langle \boldsymbol{k}|\right) \boldsymbol{\Xi}^{\mathrm{AB}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes |\boldsymbol{k}\rangle\right) &= \sum_i \sum_j \eta_{ij}\, \mathrm{Tr}\{\boldsymbol{\Xi}_j^{\mathrm{B}}\}\boldsymbol{\Xi}_i^{\mathrm{A}} &(7)\\
&= \mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\}\,. &(8)
\end{aligned}$$

This proves the stated result. □

We emphasize that the orthonormal basis used in the prior theorem is arbitrary: any choice of basis results in the same partial trace. However, henceforth it may be useful to assume that $\{\, |\boldsymbol{k}\rangle \,:\, k \in \mathbb{N}_{1:d_{\mathrm{B}}} \,\}$ is the *standard basis*. For instance, in three dimensions

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{9}$$

The following theorem proves that the partial trace results in a matrix which is positive semidefinite with unit trace, and is therefore a valid quantum density matrix.

Theorem 2. *Let $\boldsymbol{\Xi}^{\mathrm{AB}} \in \mathcal{D}^{\mathrm{AB}}$ be a density operator of the composite system AB. The partial trace of $\boldsymbol{\Xi}^{\mathrm{AB}}$ over subsystem B has trace one and is positive semidefinite. Hence $\mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\}$ is a valid density operator for system A.*

Proof. Recall that $\boldsymbol{\Xi}^{\mathrm{AB}}$ may be written as (2). Since the trace is linear and $\mathrm{Tr}\{\boldsymbol{X} \otimes \boldsymbol{Y}\} = \mathrm{Tr}\{\boldsymbol{X}\}\,\mathrm{Tr}\{\boldsymbol{Y}\}$ for arbitrary matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, we conclude that

$$\begin{aligned} 1 &= \mathrm{Tr}\{\boldsymbol{\Xi}^{\mathrm{AB}}\} \\ &= \sum_i \sum_j \eta_{ij}\ \mathrm{Tr}\{\boldsymbol{A}_i\}\,\mathrm{Tr}\{\boldsymbol{B}_j\} \\ &= \mathrm{Tr}\left\{\, \mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\}\right\} \end{aligned}$$

where the final equality is due to (3). Therefore $\mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\}$ has unit trace. Define $\boldsymbol{M}_k \triangleq \boldsymbol{I}_{d_{\mathrm{A}}} \otimes \langle \boldsymbol{k}|$. According to the equivalent definition derived in Theorem 1, the partial trace may be expressed as

$$\begin{aligned} \mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\} &= \sum_{k=1}^{d_{\mathrm{B}}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes \langle \boldsymbol{k}|\right) \boldsymbol{\Xi}^{\mathrm{AB}} \left(\boldsymbol{I}_{d_{\mathrm{A}}} \otimes |\boldsymbol{k}\rangle\right) \\ &= \sum_{k=1}^{d_{\mathrm{B}}} \boldsymbol{M}_k\, \boldsymbol{\Xi}^{\mathrm{AB}} \boldsymbol{M}_k^{\dagger}. \end{aligned}$$

This immediately reveals that $\mathrm{Tr}^{\mathrm{B}}\{\boldsymbol{\Xi}^{\mathrm{AB}}\}$ is positive semidefinite. □

2 Random densities: d_{A}-system coupled to a d_{B}-reservoir

Consider, as before, a bipartite quantum system comprised of subsystems A and B with dimension d_{A} and d_{B}, respectively. We will consider density operators which arise from the following class of operations:

1. Let the total system be in an arbitrary pure state $|\boldsymbol{\psi}\rangle \in \mathbb{C}^{d_{\mathrm{A}}} \otimes \mathbb{C}^{d_{\mathrm{B}}}$.
2. Select a random unitary operator $\mathbf{U} \in U(d_{\mathrm{A}} \cdot d_{\mathrm{B}})$ uniformly according to the Haar measure and let $|\boldsymbol{\psi}\rangle \triangleq \mathbf{U}|\boldsymbol{\psi}\rangle$.
3. Let $\boldsymbol{\Xi}^{\mathrm{A}} \triangleq \mathrm{Tr}^{\mathrm{B}}\{|\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}|\}$ be the resulting state of subsystem A.

The result $\boldsymbol{\Xi}^{\mathrm{A}}$ is a random Hermitian positive semidefinite matrix with unit trace. In the future $\boldsymbol{\Xi}^{\mathrm{A}}$ will be simply denoted $\boldsymbol{\Xi}$ as the state of subsystem B is ignored. Clearly, the distribution of $\boldsymbol{\Xi}$ is determined by $\mathbf{U}$. On the other hand, the choice of $|\boldsymbol{\psi}\rangle$ is completely erased by the randomness of $\mathbf{U}$. As such, one may simply take $|\boldsymbol{\psi}\rangle$ to be the first standard basis vector $|1\rangle$ without any loss of generality. Various aspects of random matrices arising from the above procedure have been in the literature [8, 9, 12, 13]. The aim of the present study is to analyze the behavior of the eigenvalues of $\boldsymbol{\Xi}$. We summarize existing results while filling in mathematical details that are slightly terse in the literature.

The title of this section comes from the physics literature, and arises when one thinks of A as the system of interest and B as a reservoir to which A is coupled. If for instance the initial state of AB is the product state $|\boldsymbol{\psi}\rangle = |\boldsymbol{\psi}^{\mathrm{A}}\rangle \otimes |\boldsymbol{\psi}^{\mathrm{B}}\rangle$ then the random unitary $\mathbf{U}$ can have the effect of transforming $|\boldsymbol{\psi}\rangle$ to a state which cannot be written as a Kronecker product of two states. Such a state is called *entangled*. The partial trace will, in general, lead to a non-pure state on subsystem A.

2.1 Joint eigenvalue density

Here we will calculate the joint eigenvalue density of a random density operator generated according to the procedure stated above. We will need the following lemma.

Lemma 1 (See e.g. [4,6])**.** *Let $\boldsymbol{H} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix whose spectral decomposition is denoted $\boldsymbol{H} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^\dagger$ with $\boldsymbol{Q}$ unitary and $\boldsymbol{\Lambda}$ diagonal. Then the change of coordinates $\boldsymbol{H} \mapsto (\boldsymbol{Q}, \boldsymbol{\Lambda})$ satisfies*

$$(\mathrm{d}\boldsymbol{H})^\wedge = \prod_{1\leqslant i<j\leqslant n} (\lambda_i - \lambda_j)^2 \, (\mathrm{d}\boldsymbol{\Lambda})^\wedge \, (\boldsymbol{Q}^\dagger \mathrm{d}\boldsymbol{Q})^\wedge \tag{10}$$

where λ_i denotes the i-th diagonal element of $\boldsymbol{\Lambda}$.

The following definition and lemma concern the Dirac delta of a matrix argument, which will be important when deriving the joint eigenvalue density. In simple terms, the definition states that the delta function of many variables is simply the product of delta functions of the individual variables.

Definition 2. *Given $x \in \mathbb{R}$, let $\delta(x)$ denote the usual Dirac delta function, and for $z \in \mathbb{C}$ let $\delta(z) = \delta(\mathrm{Re}\{z\})\,\delta(\mathrm{Im}\{z\})$. This convention is used to directly generalize the Dirac delta to objects comprised of many variables. For instance, the Dirac delta of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is*

$$\delta(\boldsymbol{x}) = \prod_{i=1}^{n} \delta(\boldsymbol{x}_i) \tag{11}$$

and the Dirac delta of a Hermitian matrix $\boldsymbol{H} \in \mathbb{C}^{n\times n}$ is defined

$$\delta(\boldsymbol{H}) = \prod_{i=1}^{n} \delta(\boldsymbol{H}_{ii}) \prod_{1\leqslant i<j\leqslant n} \delta(\mathrm{Re}\{\boldsymbol{H}_{ij}\})\,\delta(\mathrm{Im}\{\boldsymbol{H}_{ij}\}). \tag{12}$$

Lemma 2 (See e.g. [11])**.** *For $\boldsymbol{X} \in \mathbb{C}^{n\times n}$ and Hermitian $\boldsymbol{H} \in \mathbb{C}^{n\times n}$ the relationship*

$$\delta(\boldsymbol{X}\boldsymbol{H}\boldsymbol{X}^\dagger) = \left|\det(\boldsymbol{X}\boldsymbol{X}^\dagger)\right|^{-n} \delta(\boldsymbol{H}) \tag{13}$$

holds.

Lemma 3. *Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be Hermitian. The Jacobian of the matrix transformation $\boldsymbol{X} \mapsto \boldsymbol{A}\boldsymbol{X}$ for $\boldsymbol{X} \in \mathbb{C}^{n\times k}$ is*

$$|\det \boldsymbol{A}|^{2k}. \tag{14}$$

Proof. First, we make use of the isomorphism

$$\boldsymbol{A}\boldsymbol{X} \;\mapsto\; \underbrace{\begin{bmatrix} \mathrm{Re}\boldsymbol{A} & -\mathrm{Im}\boldsymbol{A} \\ \mathrm{Im}\boldsymbol{A} & \mathrm{Re}\boldsymbol{A} \end{bmatrix}}_{\boldsymbol{B}} \underbrace{\begin{bmatrix} \mathrm{Re}\boldsymbol{X} \\ \mathrm{Im}\boldsymbol{X} \end{bmatrix}}_{\boldsymbol{Z}}. \tag{15}$$

Applying Theorem 10.1 of Edelman's book [5] proves that the Jacobian of the transformation $\boldsymbol{Z} \mapsto \boldsymbol{B}\boldsymbol{Z}$ is $|\det \boldsymbol{B}|^k$. A similarity transform of $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \underbrace{\begin{bmatrix} -\imath\boldsymbol{I} & \imath\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{P}} \begin{bmatrix} \boldsymbol{A}^* & \\ & \boldsymbol{A} \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\imath}{2}\boldsymbol{I} & \frac{1}{2}\boldsymbol{I} \\ -\frac{\imath}{2}\boldsymbol{I} & \frac{1}{2}\boldsymbol{I} \end{bmatrix}}_{\boldsymbol{P}^{-1}}. \tag{16}$$

Hence $\det \boldsymbol{B} = (\det \boldsymbol{A}^*)(\det \boldsymbol{A}) = (\det \boldsymbol{A})^2$, where the last equality follows because the eigenvalues of $\boldsymbol{A}$ are real. This completes the proof. □

The following result was stated with a proof sketch in [13]. Our proof follows this sketch; however, extra effort has been made to fill in the details.

Theorem 3. *Given $d_{\mathrm{A}} \leqslant d_{\mathrm{B}}$, the joint eigenvalue density for a random d_{A}-density operator coupled to a d_{B}-reservoir is*

$$f_{d_{\mathrm{A}},d_{\mathrm{B}}}(\lambda_1, \lambda_2, \ldots, \lambda_{d_{\mathrm{A}}}) \propto \delta\big(1 - \sum_{i=1}^{d_{\mathrm{A}}} \lambda_i\big) \prod_{i=1}^{d_{\mathrm{A}}} \lambda_i^{(d_{\mathrm{B}}-d_{\mathrm{A}})} \prod_{1 \leqslant i < j \leqslant d_{\mathrm{A}}} (\lambda_i - \lambda_j)^2 \tag{17}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{d_{\mathrm{A}}} \geqslant 0$.

Proof. Using Theorem 1, the reduced density $\boldsymbol{\Xi}$ may be written $\boldsymbol{\Xi} = \mathbf{X}\mathbf{X}^\dagger$ for a d_{A}-by-d_{B} random matrix $\mathbf{X}$ (this is explained further in Section 2.2). According to the procedure used to generate $\boldsymbol{\Xi}$, it follows that joint distribution of the entries of $\boldsymbol{\Xi}$ have distribution

$$f_{d_{\mathrm{A}},d_{\mathrm{B}}}(\boldsymbol{\Xi}) \propto \int \delta(\boldsymbol{\Xi} - \boldsymbol{X}\boldsymbol{X}^\dagger)\, \delta(\mathrm{Tr}\{\boldsymbol{X}\boldsymbol{X}^\dagger\} - 1)(\mathrm{d}\boldsymbol{X})^\wedge \tag{18}$$

where the argument $\boldsymbol{\Xi}$ is assumed positive definite without loss of generality. Consider now the change of variables $\boldsymbol{\Xi}^{1/2}\boldsymbol{Z} = \boldsymbol{X}$. According to Lemma 3 the Jacobian of this transformation is $|\det \boldsymbol{\Xi}|^{d_{\mathrm{B}}}$. Therefore

$$f_{d_{\mathrm{A}},d_{\mathrm{B}}}(\boldsymbol{\Xi}) \propto \int \delta(\boldsymbol{\Xi} - \boldsymbol{\Xi}^{1/2}\boldsymbol{Z}\boldsymbol{Z}^\dagger\boldsymbol{\Xi}^{1/2\,\dagger})\, \delta(\mathrm{Tr}\{\boldsymbol{\Xi}^{1/2}\boldsymbol{Z}\boldsymbol{Z}^\dagger\boldsymbol{\Xi}^{1/2\,\dagger}\} - 1)|\det \boldsymbol{\Xi}|^{d_{\mathrm{B}}}(\mathrm{d}\boldsymbol{Z})^\wedge \tag{19}$$

Employing Lemma 2 on the Dirac delta of a matrix argument and the cyclic property of the trace

$$\begin{aligned} f_{d_{\mathrm{A}},d_{\mathrm{B}}}(\boldsymbol{\Xi}) &\propto \int \delta(\boldsymbol{I} - \boldsymbol{Z}\boldsymbol{Z}^\dagger)\, \delta(\mathrm{Tr}\{\boldsymbol{\Xi}\boldsymbol{Z}\boldsymbol{Z}^\dagger\} - 1)|\det \boldsymbol{\Xi}|^{d_{\mathrm{B}}-d_{\mathrm{A}}}(\mathrm{d}\boldsymbol{Z})^\wedge & (20)\\ &= \int \delta(\boldsymbol{I} - \boldsymbol{Z}\boldsymbol{Z}^\dagger)\, \delta(\mathrm{Tr}\{\boldsymbol{\Xi}\} - 1)|\det \boldsymbol{\Xi}|^{d_{\mathrm{B}}-d_{\mathrm{A}}}(\mathrm{d}\boldsymbol{Z})^\wedge & (21)\\ &\propto \delta(\mathrm{Tr}\{\boldsymbol{\Xi}\} - 1)|\det \boldsymbol{\Xi}|^{d_{\mathrm{B}}-d_{\mathrm{A}}}. & (22) \end{aligned}$$

Employing Lemma 1 proves the stated result. □

2.2 Level density

Given the joint eigenvalue density derived in the prior section, one can perform the tedious task of integrating out all but one of the eigenvalues to obtain the level density [3]. Keeping the ratio $d_{\mathrm{B}}/d_{\mathrm{A}}$ constant, the rescaled $x \triangleq d_{\mathrm{A}}\lambda$ level density converges asymptotically to the distribution

$$f_{d_{\mathrm{A}},d_{\mathrm{B}}}(x) = \frac{\sqrt{(x - a_-)(a_+ - x)}}{2\pi x} \tag{23}$$

where

$$a_\pm = 1 + \frac{d_{\mathrm{B}}}{d_{\mathrm{A}}} \pm 2\sqrt{\frac{d_{\mathrm{B}}}{d_{\mathrm{A}}}}. \tag{24}$$

Our focus in this section will be numerically validating this distribution. To do this, there are two challenges for practical computation:

1. generating the large unitary matrices employed in Section 2 is impractical, and
2. the form of the partial trace derived in Theorem 1 requires a large amount of memory.

The first point is resolved by observing that the whole random unitary $\mathbf{U}$ is not needed—only its first column. Since $\mathbf{U}$ is distributed according to the Haar measure,

$$|\boldsymbol{\psi}\rangle = \mathbf{U}|\psi\rangle \sim \mathbf{U}|1\rangle \tag{25}$$

for any choice of state $|\psi\rangle$. The product $\mathbf{U}|1\rangle$ simply extracts the first column of the random matrix $\mathbf{U}$. Therefore, to generate random states $|\boldsymbol{\psi}\rangle$ of the composite system, we need only generate a random complex vector of unit length, which is trivial to do numerically:

$$\begin{aligned} \mathbf{u} &\leftarrow \mathrm{randn}(d_{\mathrm{A}} \cdot d_{\mathrm{B}}) + 1\mathrm{im} \cdot \mathrm{randn}(d_{\mathrm{A}} \cdot d_{\mathrm{B}}) \\ \mathbf{u} &\leftarrow \frac{\mathbf{u}}{\|\mathbf{u}\|}. \end{aligned}$$

The expression for the partial trace derived in Theorem 1 can be further simplified if $\boldsymbol{\Xi}^{\mathrm{AB}}$ represents a pure state. The procedure will be motivated using an example with $d_{\mathrm{A}} = 2$ and $d_{\mathrm{B}} = 3$, but we will see that this immediately generalizes to arbitrary dimensions. Let us first write the state of the composite system (using non-random notation, for simplicity):

$$|\boldsymbol{\psi}\rangle = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} . \tag{26}$$

The partial trace $\mathrm{Tr}^{\mathrm{B}}\{|\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}|\}$ depends on three quantities: $(\boldsymbol{I}_2 \otimes \langle 1|)|\boldsymbol{\psi}\rangle$, $(\boldsymbol{I}_2 \otimes \langle 2|)|\boldsymbol{\psi}\rangle$, and $(\boldsymbol{I}_2 \otimes \langle 3|)|\boldsymbol{\psi}\rangle$. These evaluate to

$$\left(\boldsymbol{I}_2 \otimes \langle 1|\right)|\boldsymbol{\psi}\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \\ d \end{bmatrix}, \tag{27}$$

$$\left(\boldsymbol{I}_2 \otimes \langle 2|\right)|\boldsymbol{\psi}\rangle = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}, \tag{28}$$

$$\left(\boldsymbol{I}_2 \otimes \langle 3|\right)|\boldsymbol{\psi}\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}. \tag{29}$$

Employing Theorem 1, the state of subsystem A is written

$$\mathrm{Tr}^{\mathrm{B}}\{|\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}|\} = \begin{bmatrix} a \\ d \end{bmatrix} \begin{bmatrix} a^* & d^* \end{bmatrix} + \begin{bmatrix} b \\ e \end{bmatrix} \begin{bmatrix} b^* & e^* \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix} \begin{bmatrix} c^* & f^* \end{bmatrix} \tag{30}$$

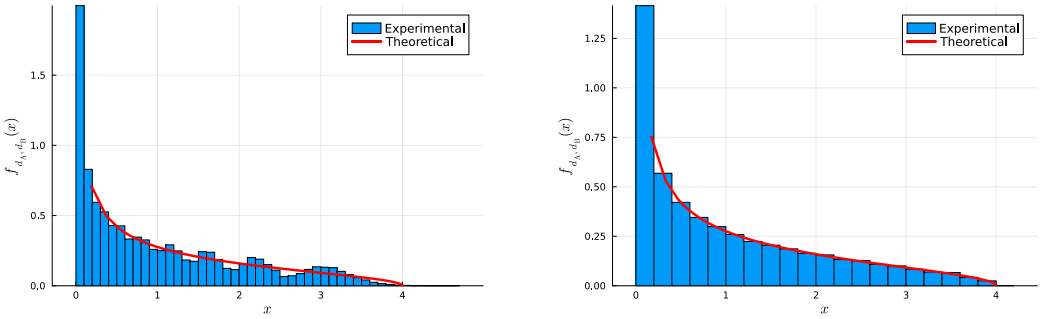
$$= \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} a^* & d^* \\ b^* & e^* \\ c^* & f^* \end{bmatrix} . \tag{31}$$

Thus, instead of computing the large density matrix $|\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}| \in \mathbb{C}^{6\times 6}$ for the composite system and then applying Theorem 1, one can instead simply reshuffle the elements of $|\boldsymbol{\psi}\rangle$ into the form

$$|\boldsymbol{\psi}\rangle = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \triangleq \boldsymbol{X} \in \mathbb{C}^{2\times 3} \tag{32}$$

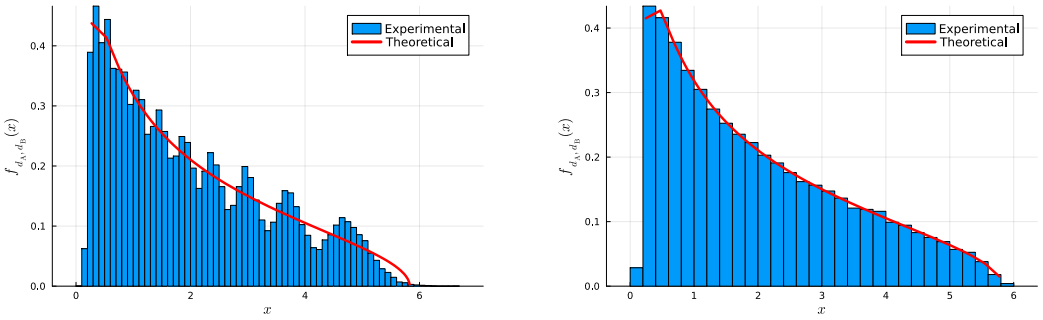
and compute the partial trace as $\mathrm{Tr}^{\mathrm{B}}\{|\boldsymbol{\psi}\rangle\langle\boldsymbol{\psi}|\} = \boldsymbol{X}\boldsymbol{X}^{\dagger}$. The generalization of this procedure is straightforward: reshape $|\boldsymbol{\psi}\rangle$ into the form $d_{\mathrm{A}} \times d_{\mathrm{B}}$ in row-major fashion.

Using the approach outlined in this section, we wrote Julia code to simulate the level density of the random density of a d_{A}-system coupled to a d_{B}-reservoir. A few results are depicted in Figure 1 for the case that $d_{\mathrm{B}}/d_{\mathrm{A}} = 1$ (i.e. the system and reservoir are of the same size), and in Figure 2 for the case that $d_{\mathrm{B}}/d_{\mathrm{A}} = 2$ (i.e. the reservoir is twice as large as



(a) $d_{\mathrm{A}} = 10$, $d_{\mathrm{B}} = 10$, and $n = 100,000$ (b) $d_{\mathrm{A}} = 100$, $d_{\mathrm{B}} = 100$, and $n = 100$

Figure 1: Experimental versus theoretical level density using sample size n when $d_{\mathrm{B}}/d_{\mathrm{A}} = 1$.



(a) $d_{\mathrm{A}} = 10$, $d_{\mathrm{B}} = 20$, and $n = 100,000$ (b) $d_{\mathrm{A}} = 100$, $d_{\mathrm{B}} = 200$, and $n = 100$

Figure 2: Experimental versus theoretical level density using sample size n when $d_{\mathrm{B}}/d_{\mathrm{A}} = 2$.

the system). Interestingly when d_{A} is small, the experimental density seems to oscillate in a well-behaved manner around the asymptotic distribution, which is labelled "Theoretical" in the plots. For moderate sized $d_{\mathrm{A}} = 100$, the experimental distribution already closely aligns with the asymptotic one.

3 Fuss-Catalan distribution

Another way to obtain a random density operator is to consider a collection of $2k$ subsystems, indexed by integers $1, 2, \ldots . 2k$. Let the initial state be $|\psi_0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \ldots \otimes |0\rangle_{2k}$ and consider k independent Haar random unitary operators $\mathbf{U}_{1,2}, \mathbf{U}_{3,4}, \ldots, \mathbf{U}_{2s-1,s} \in U(d)$. Define the random state $|\psi\rangle = \mathbf{U}_{1,2} \otimes \mathbf{U}_{3,4} \otimes \ldots \otimes \mathbf{U}_{2k-1,2k}|\psi_0\rangle$. Expanding this state in a product basis we obtain

$$|\boldsymbol{\psi}\rangle = \mathbf{U}_{1,2} \otimes \mathbf{U}_{3,4} \otimes \ldots \otimes \mathbf{U}_{2k-1,2k}|\psi_0\rangle \tag{33}$$

$$= \sum_{i_1,i_2,\ldots,i_{2s}} (\mathbf{G}_1)_{i_1,i_2}(\mathbf{G}_2)_{i_3,i_4} \ldots (\mathbf{G}_s)_{i_{2s-1},i_{2s}}|i_1, i_2, \ldots, i_{2k}\rangle \tag{34}$$

where $\mathbf{G}_i$ are Ginibre matrices with independent and identically distributed 0 mean unit variance complex Gaussian random variables. Then performing a projection onto a product of $s-1$ maximally entangled states[2]

$$\boldsymbol{P}_k \triangleq \boldsymbol{I}_1 \otimes |\Psi^+_{2,3}\rangle\langle\Psi^+_{2,3}| \otimes |\Psi^+_{4,5}\rangle\langle\Psi^+_{4,5}| \otimes \ldots \otimes |\Psi^+_{2k-2,2k-1}\rangle\langle\Psi^+_{2k-2,2k-1}| \otimes \boldsymbol{I}_{2k} \tag{35}$$

we obtain a pure state, describing the remaining two subsystems, of the form

$$|\boldsymbol{\varphi}\rangle = \boldsymbol{P}_k|\boldsymbol{\psi}\rangle = d^{1-k}\sum_{i,j}(\mathbf{G}_1\mathbf{G}_2 \ldots \mathbf{G}_s)_{ij}|i\rangle_1 \otimes |j\rangle_{2k}\,. \tag{36}$$

[2] $|\Psi^+_{i,i+1}\rangle$ is a maximally entangled state between subsystems i and $i+1$, $i = 1, 2, \ldots, 2k-1$. A possible choice is $|\Psi^+_{i,i+1}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle_i \otimes |d-j-1\rangle_{i+1}$.

Normalizing this state and taking a partial trace over the subsystem $2k$, we obtain the density operator describing the system left in the first subsystem as

$$\mathbf{\Xi}_k = \frac{\mathrm{Tr}^{2s}\,|\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}|}{\langle\boldsymbol{\varphi}|\boldsymbol{\varphi}\rangle} = \frac{\mathbf{G}_1\mathbf{G}_2\ldots\mathbf{G}_k\big(\mathbf{G}_1\mathbf{G}_2\ldots\mathbf{G}_s\big)^\dagger}{\mathrm{Tr}\left[\mathbf{G}_1\mathbf{G}_2\ldots\mathbf{G}_s\big(\mathbf{G}_1\mathbf{G}_2\ldots\mathbf{G}_k\big)^\dagger\right]}\,. \tag{37}$$

This type of construction is often times used to analyze a system of interest which interacts with an environment which is itself comprised of many sub-environments. The total system (system of interest + environment) evolves unitarily. A projective measurement is made using $\boldsymbol{P}_k$ and the environment is traced out to obtain the description of the system of interest. As the matrices $\mathbf{G}_1, \mathbf{G}_2, \ldots \mathbf{G}_k$ are square Ginibre matrices of size d and the normalized eigenvalue distribution ($x = d\lambda$) of $\mathbf{\Xi}_k$ is given by Fuss-Catalan distributions [10, 12]. Fuss Catalan distributions can be obtained through the Marchenko-Pastur distribution: denoting the Marchenko-Pastur distribution by $\pi^{(1)}$, the Fuss-Catalan distribution is $\pi^{(k)} = [\pi^{(1)}]^{\boxtimes k}$, where $\boxtimes$ denotes the free product. The S-transform can be used to obtain the distribution since the S transform of product of free random variables is the product of S transform of free random variables [2]. It can also be shown that, for a random square matrix $\mathbf{X}$ with zero mean, unit variance, and finite fourth moment entries, the distribution of the eigenvalues of $\mathbf{X}^k$ converges to $\pi^{(k)}$ [1]. The Marchenko-Pastur distribution has the moments as Catalan numbers. The moments of the Fuss-Catalan distribution are the Fuss-Catalan moments:

$$FC_k(n) = \frac{1}{kn+1}\binom{kn+n}{n} = \frac{(kn+n)!}{(kn+1)!\,n!} \tag{38}$$

for $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$.

3.1 Mellin Transform

The Mellin transform for a non-negative random variable x can be given as

$$\mathcal{M}_{\mathsf{x}}(s) \triangleq \mathbb{E}\{\mathsf{x}^s\} = \int_{-\infty}^{\infty} x^s dF_{\mathsf{x}}(x) \tag{39}$$

where s is a complex number and $\mathcal{M}_{\mathsf{x}}(s)$ is assumed finite [7]. The Mellin transform of a non-negative random variable x uniquely determines its distribution as we show next. First, observe that

$$\mathcal{M}_{\mathsf{x}}(\imath t) = \mathbb{E}\big\{\mathsf{x}^{\imath t}\big\} = \int_0^{\infty} x^{it} dF_{\mathsf{x}}(x) = \int_0^{\infty} e^{\imath t \ln(x)} dF_{\mathsf{x}}(x) \tag{40}$$

for real t. Hence (40) can be seen as the characteristic function of $\ln \mathsf{x}$. Noting that (i) the characteristic function of a random variable x uniquely determines its distribution, (ii) $\mathbb{P}\{\mathbf{x} \leqslant c\} = \mathbb{P}\{\ln \mathsf{x} \leqslant \ln c\}$ for $c \geqslant 0$, and (iii) the following fact for $a < b$

$$F_{\mathsf{x}}(b) - F_{\mathsf{x}}(a) = \lim_{u\to\infty} \frac{1}{2\pi}\int_{-u}^{u} \frac{e^{-\imath ua} - e^{-\imath ub}}{\imath u}\phi(u)du \tag{41}$$

where $\phi(u) = \mathbb{E}\{e^{\imath u\mathsf{x}}\}$ is the characteristic function of random variable x, then the Mellin tansform with parameter $\imath t$ uniquely determines the distribution of x. The inverse Mellin transform can be given as

$$f_{\mathsf{x}}(x) = \frac{1}{2\pi\imath}\int_{c-\infty}^{c+\infty} x^{-s}\mathcal{M}_{\mathsf{x}}(s)ds \tag{42}$$

where the line integral is over a vertical line in the complex field.

The Fuss-Catalan distribution of order k has the moments of Fuss-Catalan numbers given in (38). For any given s, there exists a density distribution $P_k(x)$ such that

$$\int_0^{K_k} x^n P_k(x)dx = FC_k(n), \qquad n = 0, 1, \ldots \tag{43}$$

where $K_k = (k+1)^{k+1}/k^k$. The solution of this moment problem associated with Fuss-Catalan numbers is unique [10]. For finding $P_k(x)$ the method of inverse Mellin transform is employed. First, we write the Fuss-Catalan numbers in terms of Gamma functions as

$$FC_k(n) = \frac{(kn+n)!}{(kn+1)!\,n!} = \frac{\Gamma(kn+n+1)}{\Gamma(n+1)\Gamma(kn+2)}. \tag{44}$$

Then we replace $n = s-1$ to extend the integer variable n to complex variable s for the Mellin transform $P_k(x)$ so that

$$FC_k(s) = \frac{\Gamma(ks-k+s)}{\Gamma(s)\Gamma(sk-k+2)} = \frac{\Gamma((k+1)(s-\frac{k}{k+1}))}{\Gamma(s)\Gamma(k(s-\frac{k-2}{k}))} \tag{45}$$

Using the Gauss-Legendre Formula twice, one can obtain [10]

$$FC_k(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{(k+1)^{k+1}}{k^k}\right]^s \frac{k^{k-3/2}}{(k+1)^{k+1/2}}\left[\prod_{j=0}^{k-1}\frac{\Gamma(k+\frac{j-k}{k+1})}{\Gamma(s+\frac{2+j-k}{k})}\right] \tag{46}$$

Meijer G-function of argument z can be written by the inverse Mellin transform as [10]

$$G_{p,q}^{m,n}\Big(z\Big|\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{matrix}\Big) = \mathcal{M}^{-1}\left[\frac{\prod_{j=1}^m\Gamma(\beta_j+s)\prod_{j=1}^n\Gamma(1-\alpha_j-s)}{\prod_{j=m+1}^q\Gamma(1-\beta_j-s)\prod_{j=n+1}^p\Gamma(\alpha_j+s)};z\right] \tag{47}$$

Then one can find the Fuss-Catalan distribution of order k as

$$P_k(x) = \frac{1}{\sqrt{2\pi}}\frac{k^{k-3/2}}{(k+1)^{k+1/2}}G_{k,k}^{k,0}\Big(z\Big|\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{matrix}\Big) \tag{48}$$

with $z = xk^k(k+1)^{-(k+1)}$, $\alpha_j = (1+j-k)/k$ and $\beta_j = (j-1-k/(k+1))$ for $j = 1,2,\ldots,k$. This can be written in the form

$$\begin{aligned}\pi^k(x) \;=\; & \sum_{i=1}^k \Lambda_{i,k}\, x^{\frac{i}{k+1}-1}\,{}_kF_{k-1}\Bigg(\Big[\Big\{1-\frac{1+j}{k}+\frac{i}{k+1}\Big\}_{j=1}^k\Big];\\ & \Big[\Big\{1+\frac{i-j}{k+1}\Big\}_{j=1}^{i-1}\Big];\Big\{1+\frac{i-j}{k+1}\Big\}_{j=i+1}^k;\frac{k^k}{(k+1)^{k+1}}x\Bigg)\end{aligned}$$

where $\Lambda_{i,k}$ is given by

$$\Lambda_{i,k} \triangleq k^{-3/2}\sqrt{\frac{k+1}{2\pi}}\Big(\frac{k^{k/(k+1)}}{k+1}\Big)\frac{\big[\prod_{j=1}^{i-1}\Gamma(\frac{j-i}{k+1})\big]\big[\prod_{j=i+1}^{k}\Gamma(\frac{j-i}{k+1})\big]}{\prod_{j=1}^k\Gamma(\frac{j+1}{k}-\frac{i}{k+1})} \tag{49}$$

with $x = N\lambda$ and ${}_pF_q\big([\{a_j\}_{j=1}^p\,;\,\{b_j\}_{j=1}^q];x\big)$ stands for the generalized hypergeometric function of the type ${}_pF_q$ as

$${}_pF_q\big([\{a_j\}_{j=1}^p\,;\,\{b_j\}_{j=1}^q];x\big) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_p)_n}{(b_1)_n(b_2)_n\ldots(b_q)_n}\frac{x^n}{n!} \tag{50}$$

Note that we obtained the form in (49) using the relationship between Meijer-G function and hypergeometric functions [10]:

$$\begin{aligned}G_{p,q}^{m,n}\Big(z\Big|\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{matrix}\Big) = & \sum_{k=1}^m \frac{\prod_{j=1}^m\Gamma(\beta_j-\beta_k)^*\prod_{j=1}^k\Gamma(1+\beta_k-\alpha_j)z^{b_k}}{\prod_{j=m+1}^n\Gamma(1+\beta_k-\beta_j)\prod_{j=m+1}^q\Gamma(1+\beta_k-\beta_j)\prod_{j=n+1}^p\Gamma(\alpha_j-\beta_k)}\\ & \times {}_pF_{q-1}(\{1+\beta_k-\alpha_j\}_{j=1}^p;\{1+\beta_k-\beta_j\}_{j=1}^q;(-1)^{p-m-n}z)\end{aligned}$$

3.2 Numerical Simulation of Fuss-Catalan Distributions

Here we present some numerical simulations for the Fuss-Catalan distributions. For obtaining the experimental results, k square complex Ginibre ensembles $\mathbf{G}_1,\mathbf{G}_2,\ldots,\mathbf{G}_k$ of size d are formed and multiplied together as $\mathbf{X}_k = \prod_{i=1}^k\mathbf{G}_i$. Then the eigenvalue distribution of $\frac{1}{d}\frac{\mathbf{X}_k\mathbf{X}_k^\dagger}{\mathrm{Tr}\{\mathbf{X}_k\mathbf{X}_k^\dagger\}}$ are plotted and compared with the theoretical distribution. The size $d = 1000$ and a random sample of 101 is used to generate the plots, which may be seen below. The experimental distribution aligns closely with the asymptotic distribution.

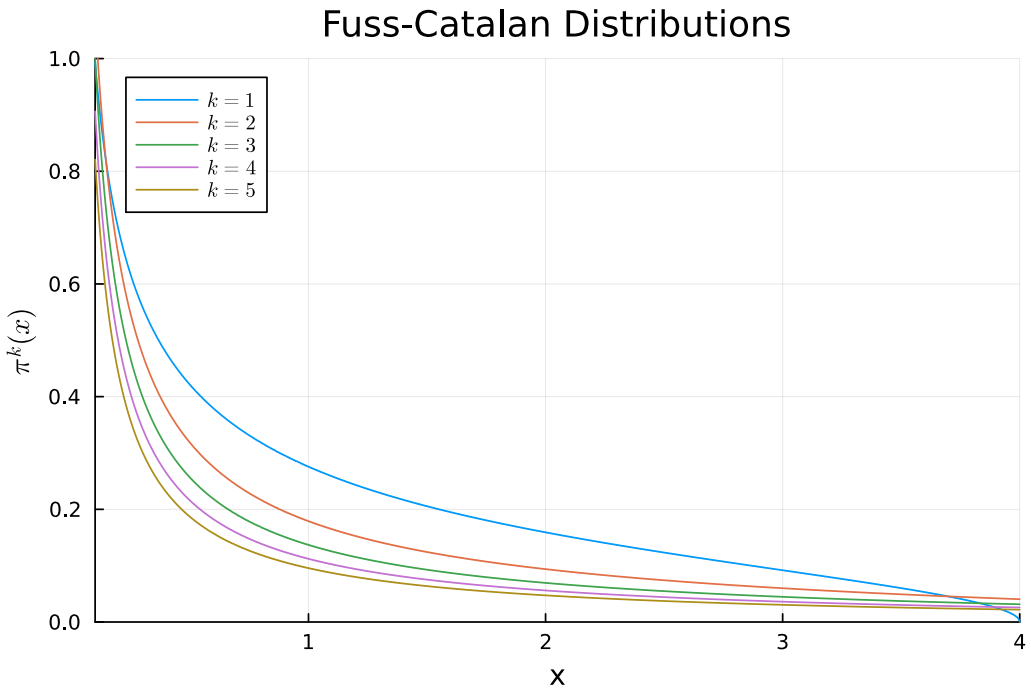


Figure 3: Fuss Catalan distributions for orders $k = 1, 2, 3, 4, 5$. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.

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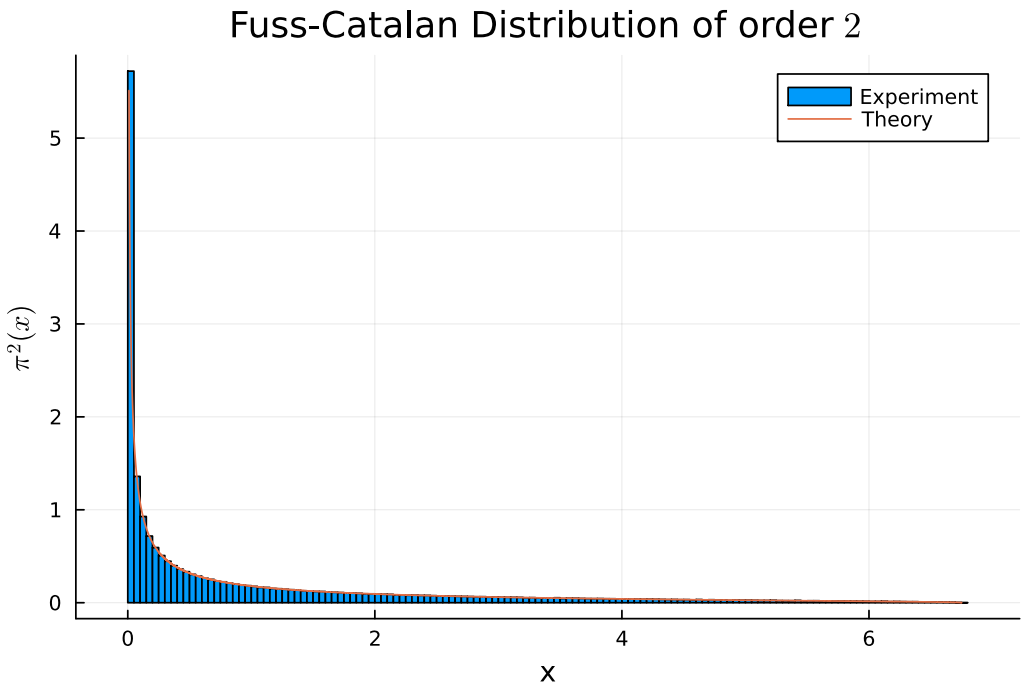

Figure 4: Fuss Catalan distribution of order 2.

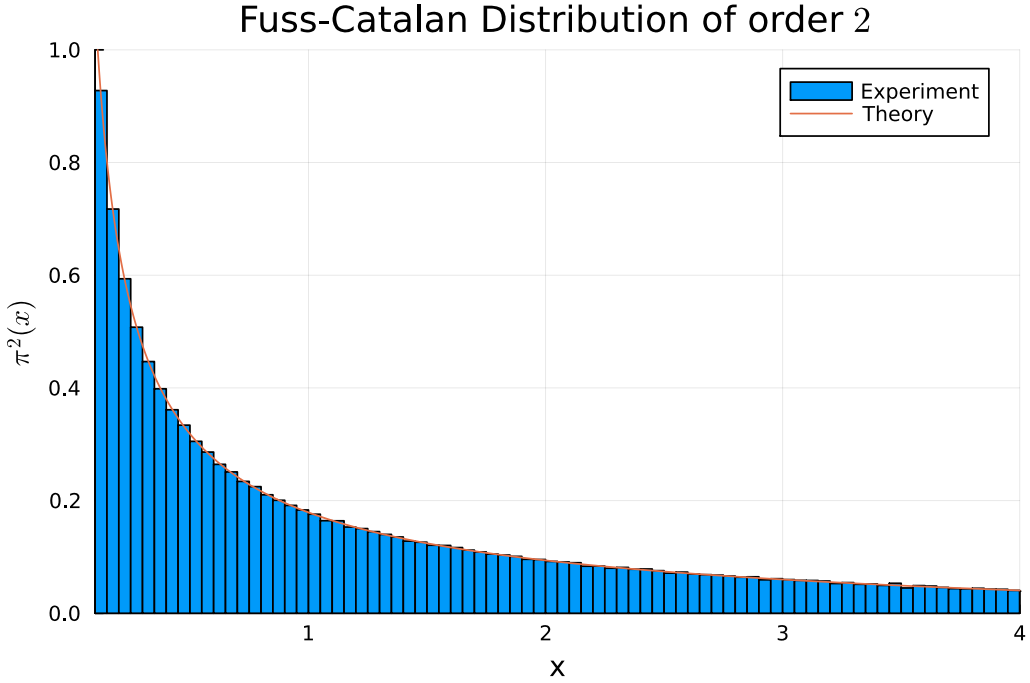

Figure 5: Fuss Catalan distribution of order 2. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.

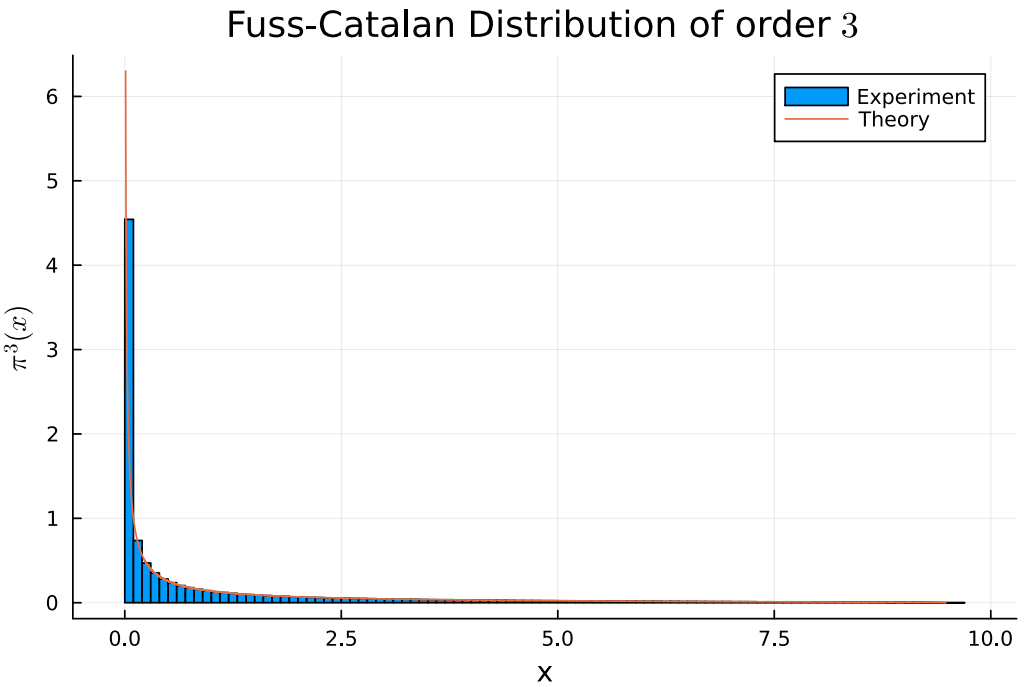

Figure 6: Fuss Catalan distribution of order 3.

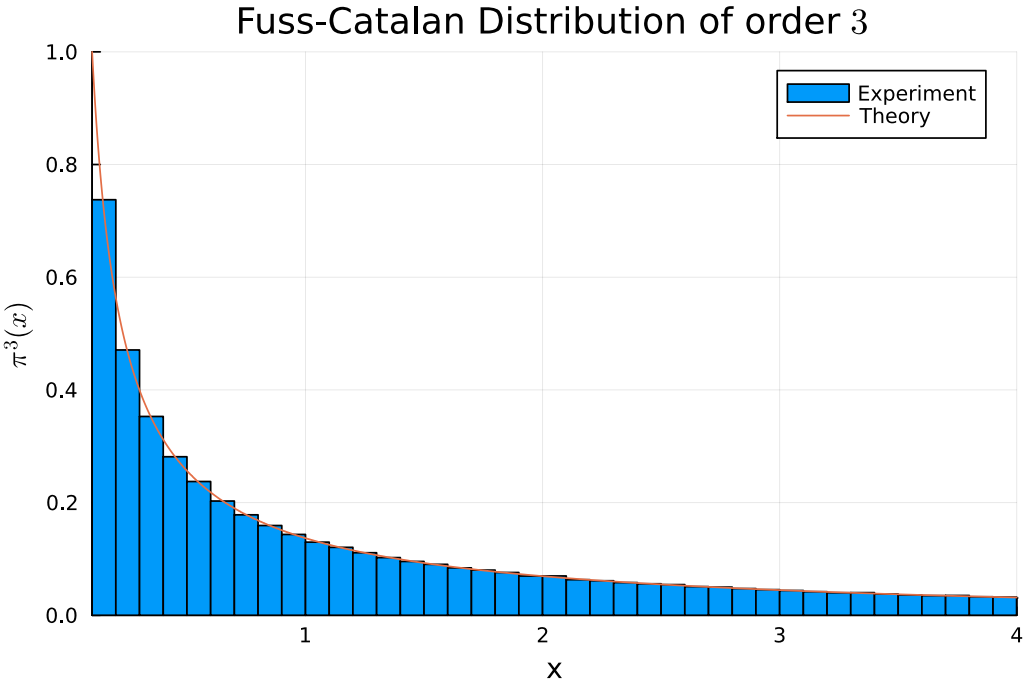

Figure 7: Fuss Catalan distribution of order 3. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.

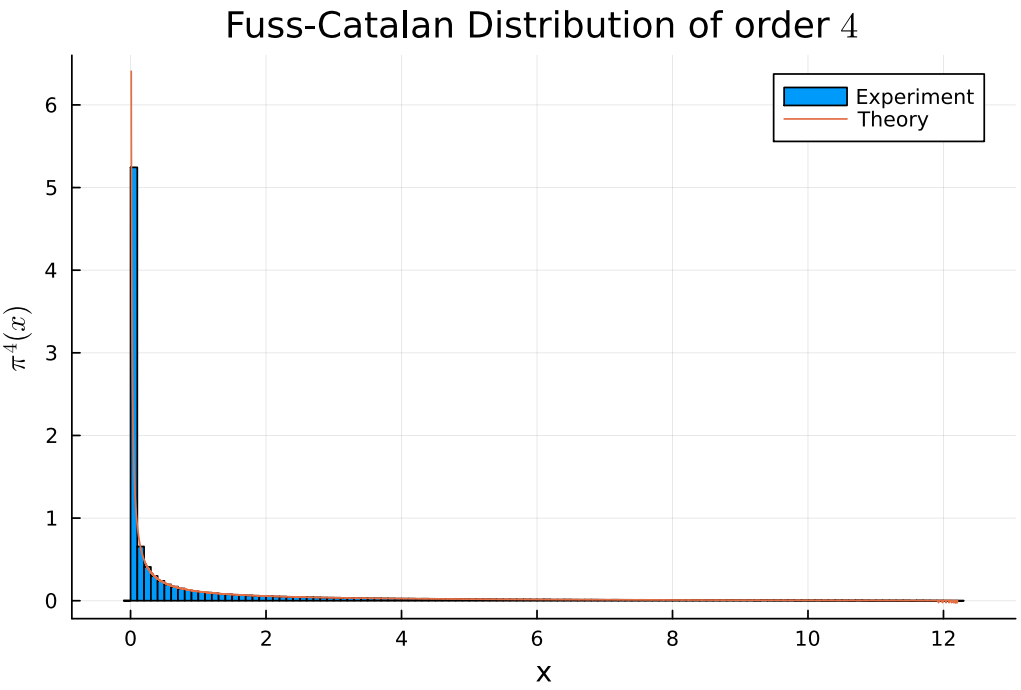

Figure 8: Fuss Catalan distribution of order 4.

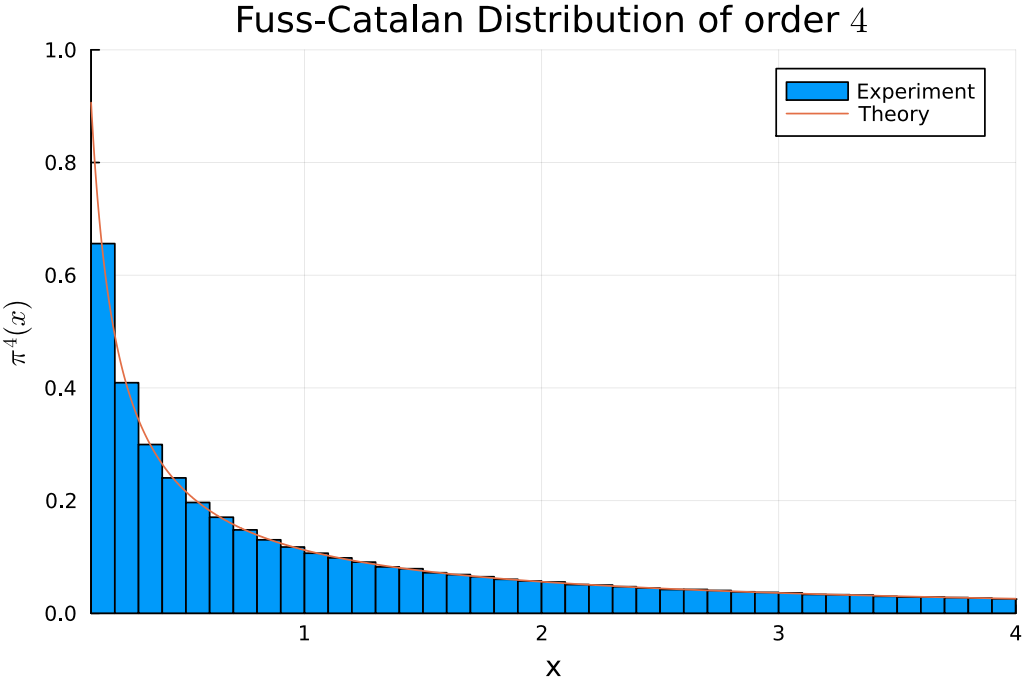

Figure 9: Fuss Catalan distribution of order 4. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.

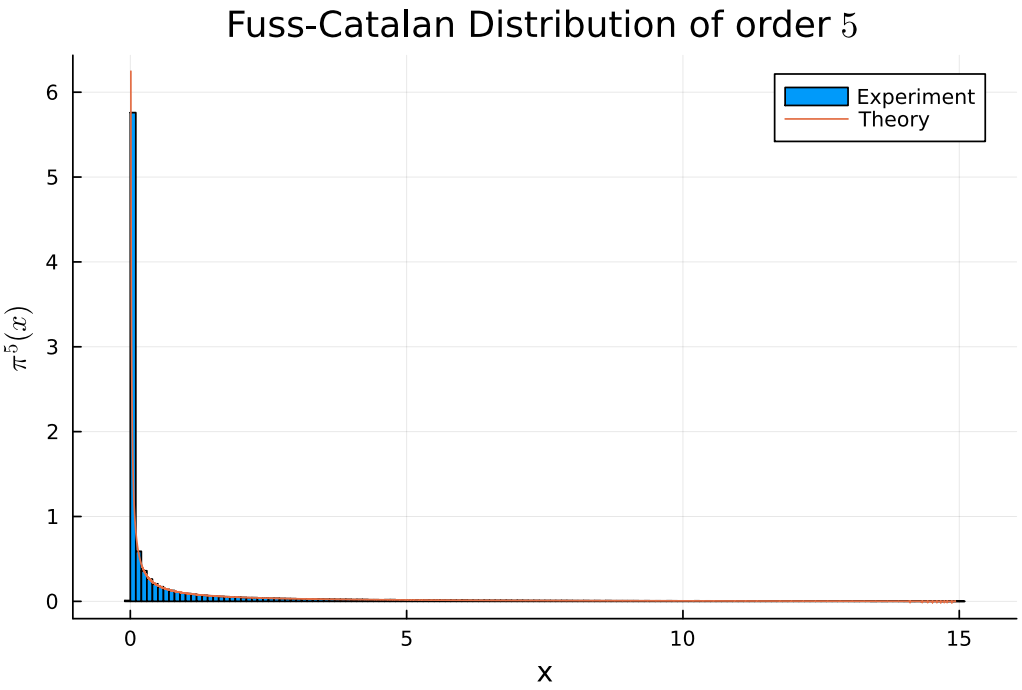

Figure 10: Fuss Catalan distribution of order 5.

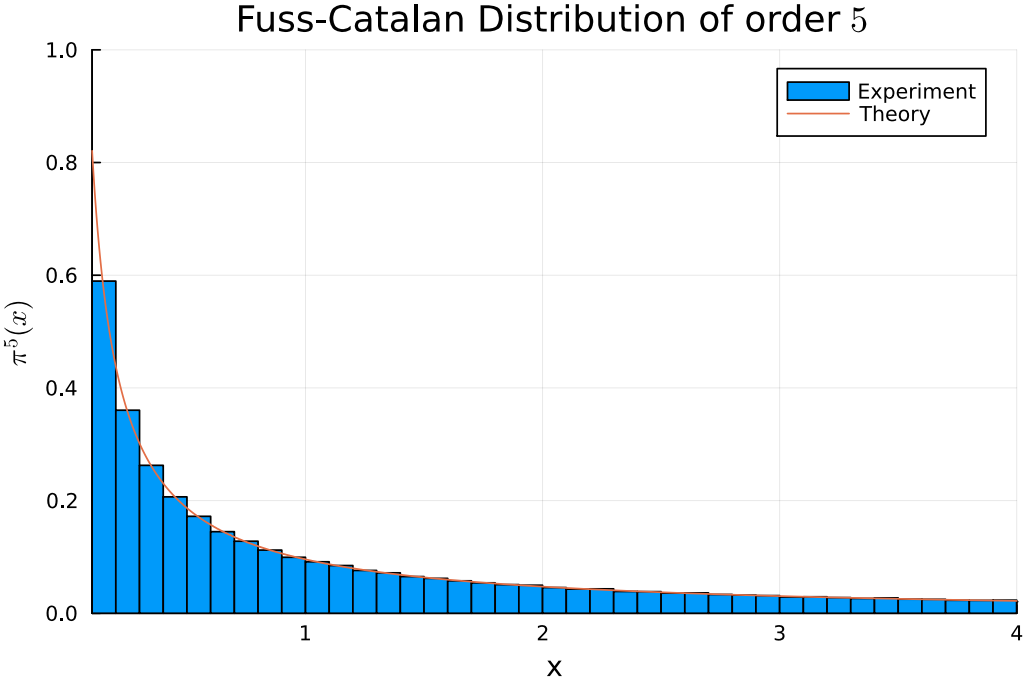

Figure 11: Fuss Catalan distribution of order 5. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.

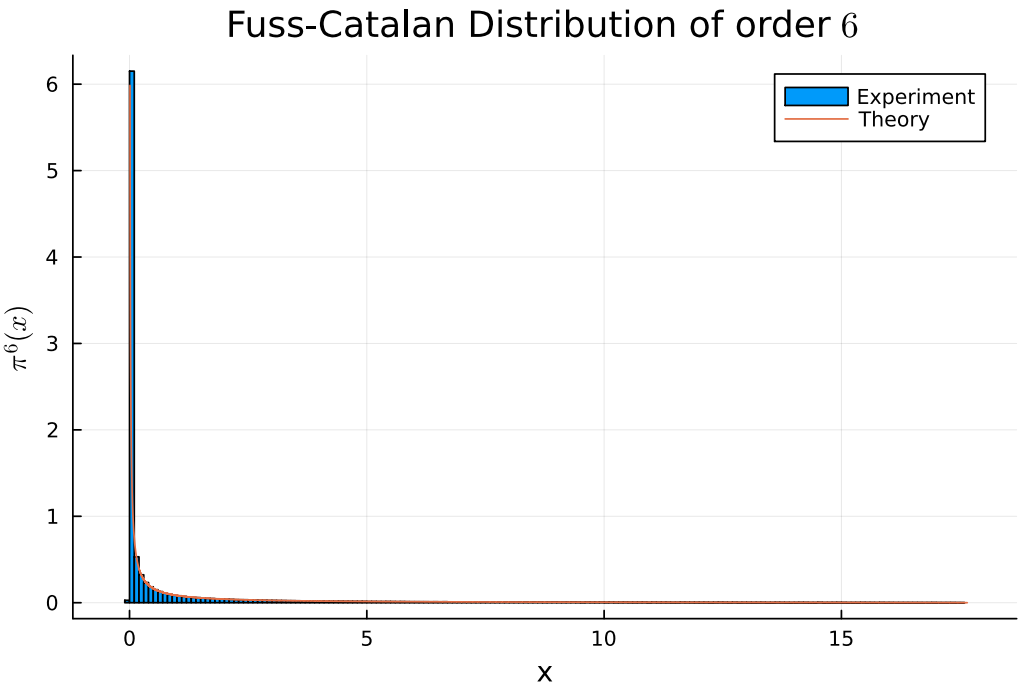

Figure 12: Fuss Catalan distribution of order 6.

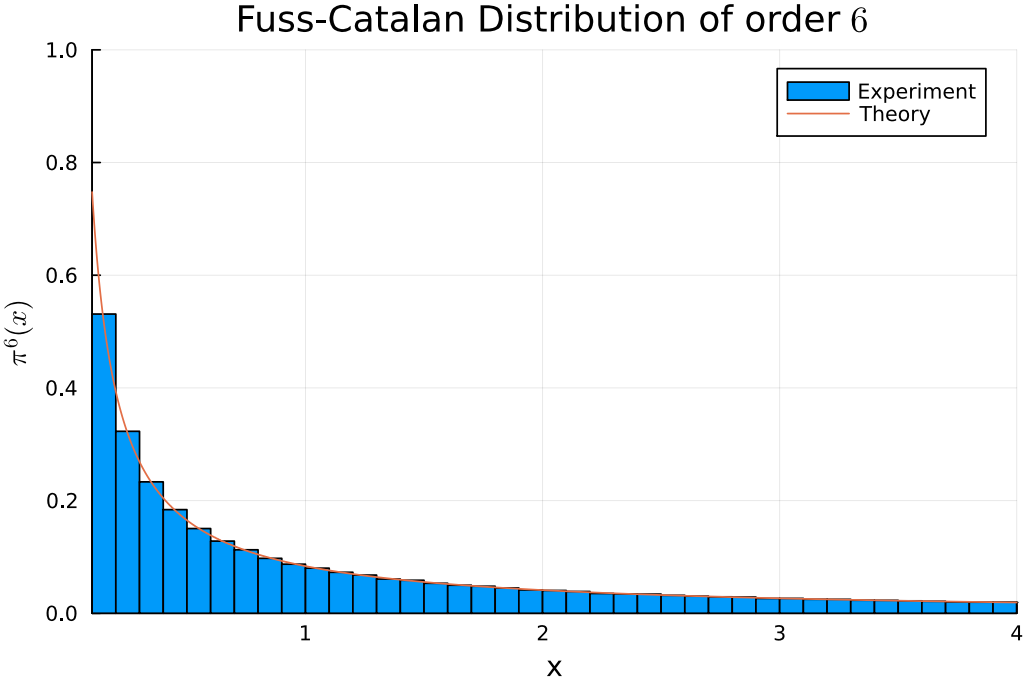

Figure 13: Fuss Catalan distribution of order 6. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.

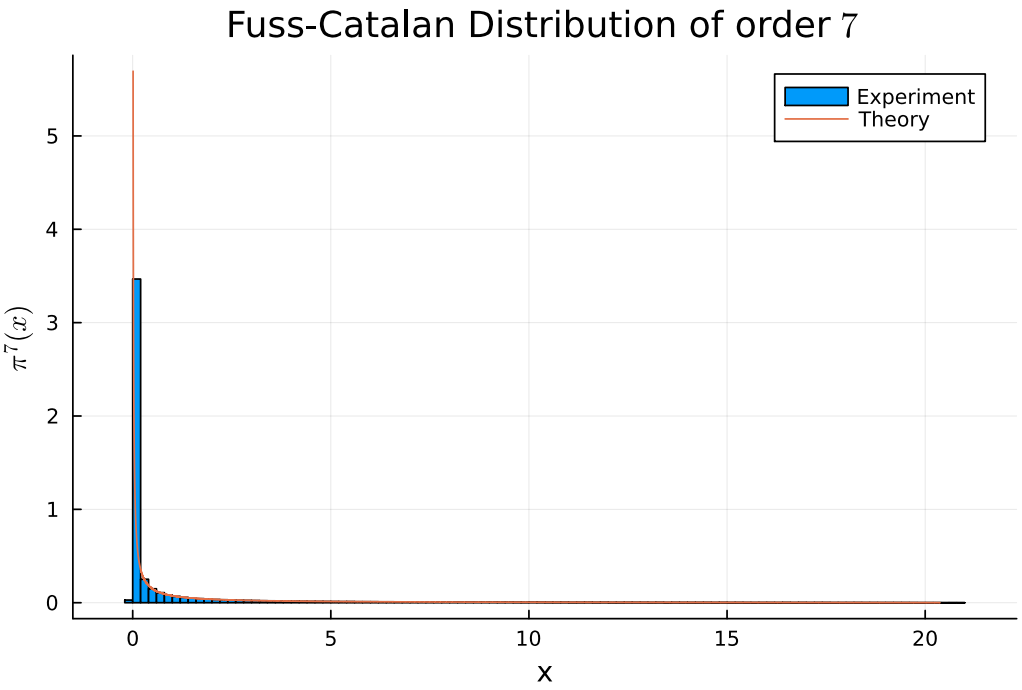

Figure 14: Fuss Catalan distribution of order 7.

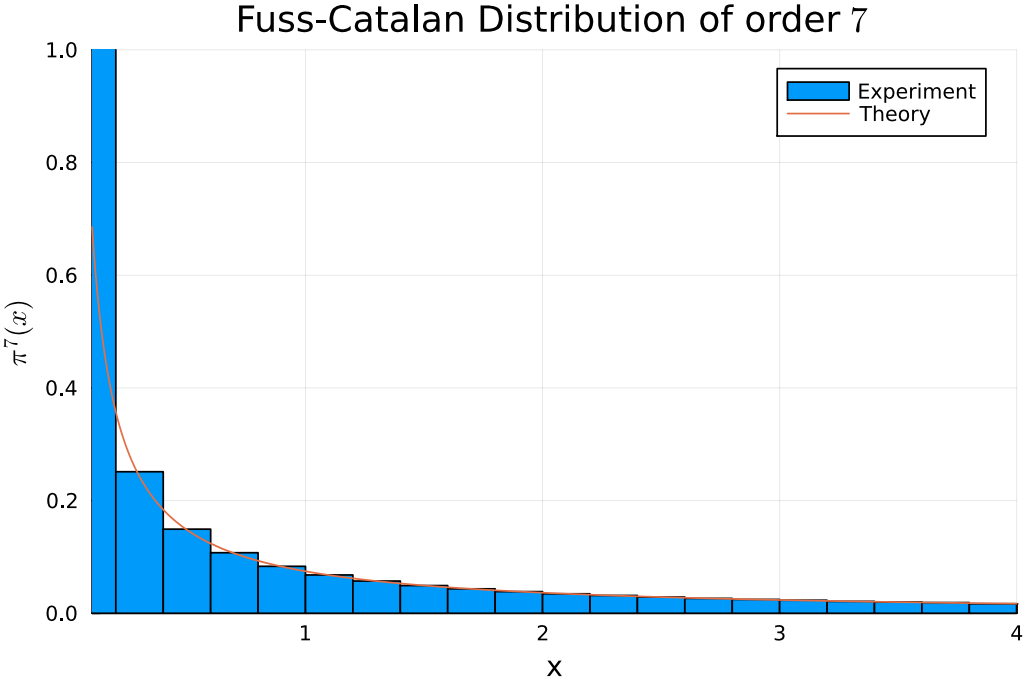

Figure 15: Fuss Catalan distribution of order 7. For better viewing the x-axis is constrained to the interval $[0.1, 4]$ and the y-axis is constrained to the interval $[0, 1]$.