18.338 Eigenvalues of Random Matrices

Problem Set 3

Due Date: Wed Nov. 3, 2021

Reading and Notes

Read the DPP notes (available on Piazza, under Resources, Lecture Notes, `dpp_notes.pdf`) and provide feedback.

Homework

Do the following exercises plus 3 out of the 16 exercises in the DPP notes.

1. (M) The Christoffel-Darboux formula (also see equation (5.6) on page 81) states

$$\sum_{j=0}^{n} \pi_i(x)\pi_j(y) = \frac{k_n}{k_{n+1}} \frac{\begin{vmatrix} \pi_n(x) & \pi_n(y) \\ \pi_{n+1}(x) & \pi_{n+1}(y) \end{vmatrix}}{y-x},$$

where k_n is the leading coefficient of π_n. Let

$$\pi_j(x) = \frac{H_j(x)}{(\sqrt{\pi} j! 2^j)^{1/2}}$$

for Hermite Polynomials, then $k_n/k_{n+1} = \sqrt{n}$.
Use the known asymptotics

$$\lim_{m\to\infty} (-1)^m m^{1/4}\ \pi_{2m}(x) e^{-x^2/2} = \frac{\cos(\xi)}{\sqrt{\pi}}$$
$$\lim_{m\to\infty} (-1)^m m^{1/4}\ \pi_{2m+1}(x) e^{-x^2/2} = \frac{\sin(\xi)}{\sqrt{\pi}}$$

where $x = \xi/(2\sqrt{m})$ to prove

$$K_{2m}(x,y) = e^{-\frac{1}{2}(x^2+y^2)} \sum_{j=0}^{2m-1} \pi_j(x)\pi_j(y) \tag{1}$$

converges to the sine kernel, i.e.,

$$\frac{1}{\sqrt{m}} K_{2m}(x,y) \to \frac{2}{\pi} \frac{\sin(x-y)}{x-y}.$$

2. (C) Do a numerical experiment to "see" the convergence in Problem 1. There are numerical issues on the diagonal and corners. Probably on the diagonal, Christoffel-Darboux needs to be replaced by a derivative approximation. See if you can make it better.

3. (C) Obtain the Airy Process limit by taking numerically

$$\frac{1}{\sqrt{2}\, n^{1/6}} K_n\Big(\sqrt{2n} + \frac{x}{\sqrt{2}\, n^{1/6}}, \sqrt{2n} + \frac{y}{\sqrt{2}\, n^{1/6}}\Big) \to \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y},$$

where $Ai(x)$ is the Airy function and K_n is defined in (1).