Homework 3, Fall 2022 18338

Due 10/17 11:59pm

Please submit your homework via canvas.mit.edu.
If you are submitting .jl or .ipynb files, you must additionally submit .html or .pdf file that captures running notebook or code.

Reading and Notes

Read DPP notes, and comment on nb1.mit.edu (The lecture notes and the link to nb can be found in Piazza: http://piazza.com/mit/fall2022/18338.). Please give your feedback especially high level style and where things did not make sense, in addition to spelling or technical errors.

Problem sets

Solve 3 out of 16 exercises in the DPP notes. Then, do either Problem 1 or 2 below. (One out of the following two problems.)

1. Kernels related to Scaling limits

   (a) (M) The Christoffel-Darboux formula (also see equation (5.6) on page 81) states

   $$\sum_{j=0}^{n} \pi_i(x)\pi_j(y) = \frac{k_n}{k_{n+1}} \frac{\pi_n(x)\pi_{n+1}(y) - \pi_n(y)\pi_{n+1}(x)}{y - x}$$

   where k_n is the leading coefficient of π_n. Let

   $$\pi_j(x) = \frac{H_j(x)}{(2^j\sqrt{\pi}j!)^{1/2}}$$

   for Hermite polynomials H_j, then $k_n/k_{n+1} = \sqrt{n}$. Use the known asymptotics

   $$\lim_{m\to\infty} (-1)^m m^{1/4} \pi_{2m}(x) e^{-x^2/2} = \frac{\cos(\xi)}{\sqrt{\pi}}$$

   $$\lim_{m\to\infty} (-1)^m m^{1/4} \pi_{2m+1}(x) e^{-x^2/2} = \frac{\sin(\xi)}{\sqrt{\pi}}$$

   where $\xi = 2\sqrt{m}x$, to prove

   $$K_{2m}(x, y) = e^{-\frac{1}{2}(x^2+y^2)} \sum_{j=0}^{2m-1} \pi_j(x)\pi_j(y)$$

   converges to the sine kernel, i.e.,

   $$\frac{1}{\sqrt{m}} K_{2m}(x, y) \to = \frac{2\sin(\xi_1 - \xi_2)}{\pi(\xi_1 - \xi_2)},$$

   where $\xi_1 = 2\sqrt{m}x, \xi_2 = 2\sqrt{m}y$, as $m \to \infty$.

   (b) (C) Numerically verify the last convergence in Problem 1-(a). It would be enough to compare points $x, y \in [0, 1]^2$.

   (c) (C) Point out the numerical issues on the diagonal and corners. Probably on the diagonal, Christoffel-Darboux needs to be replaced by a derivative approximation. Suggest any alternatives.

(d) Numerically verify the convergence of K_n to the Airy kernel, i.e.,

$$\frac{1}{\sqrt{2}n^{1/6}}K_n(\sqrt{2n}+\frac{x}{\sqrt{2}n^{1/6}},\sqrt{2n}+\frac{y}{\sqrt{2}n^{1/6}})\to\frac{Ai(x)Ai'(y)-Ai'(x)Ai(y)}{x-y},$$

where $Ai(x)$ is the Airy function and as $n\to\infty$.

2. Narayana numbers, as introduced in equation (4.1) (page 52) of the class note, describes many combinatorial quantities on length $2n$ Dyck paths. Dyck paths of length $2n$ are all the paths consisted of $(1,1)$ steps and $(1,-1)$ steps, that start at $(0,0)$ and ends at $(2n,0)$, that never goes under x-axis. Incredibly, Narayana numbers describe the distribution of (1) number of peaks (2) double ascents (3) ascents in even positions (4) long non-final sequences, in $2n$ Dyck paths. In Kreweras's paper JOINT DISTRIBUTIONS OF THREE DESCRIPTIVE PARAMETERS OF BRIDGES, (where n-bridges are just length $2n$ Dyck paths) these are explained with details. Especially in random matrix context, it is known that moments of Laguerre ensembles are related to the number of ascents in even positions (Chapter 4). Prove that Narayana numbers count the number of ascents in even positions, by doing one of the followings.

(a) Krewaras, in his paper, wrote that the result is "implied by the set of papers [1], [2] and [3]." Read carefully the three references, and prove that Narayana numbers count the number of ascents in even positions.

(b) Prove by finding a bijection from the set of all length $2n$ Dyck paths to itself, such that the number of peaks are mapped to the number of even ascents.