Maximum Eigenvalues in Brownian Motion and Their Correlation with the Airy Process

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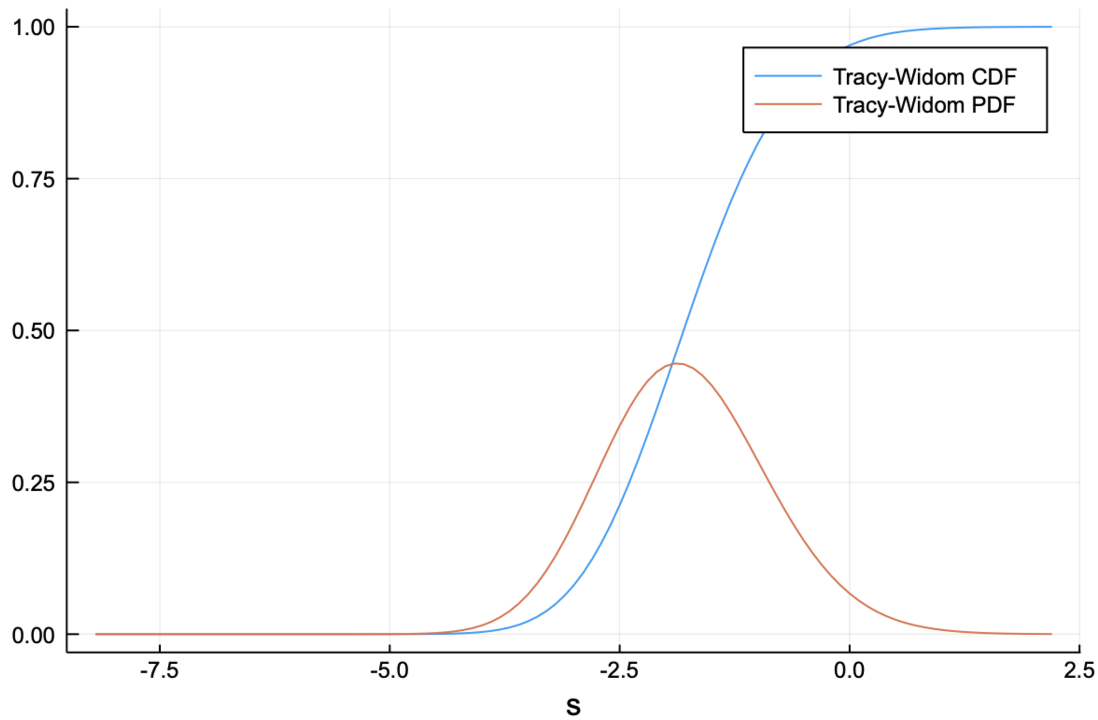
Abstract

The goal of this project was to empirically verify a a theoretical result that states that the Airy process correlates with a re-scaling of the maximum eigenvalue of Dyson Brownian Motion.

Introduction

Theory tells us that we can compute the probability $P(A(\tau_1) < s_1, A(\tau_2) < s_2, \ldots, A(\tau_m) < s_m)$ as a Fredholm determinant. Where A is the Airy Process and s is an appropriately chosen threshold. I will verify this theoretical result by running a Monte Carlo simulation that simulates the Airy Process as the rescaled largest eigenvalue of Dyson Brownian motion.

. Previous work by Edelman and Persson showed how to numerically compute the distribution of eigenvalues of β-ensembles. The Tracy-Widom Distribution for $\beta = 2$ or for the case of the GUE looks like the figure below.



The Airy Process

The Airy process is defined by an appropriate rescaling of the largest eigenvalue in the Dyson diffusion:

$$A(t) = \lim_{n\to\infty} \sqrt{2}n^{1/6}\Big(\lambda_{max}(n^{-1/3}t) - \sqrt{2n}\Big)$$

where λ_{max} denotes the maximum eigenvalue of the Dyson Brownian motion.

The Dyson Brownian motion, is a matrix generalization of one dimensional Brownian Motion:

1) $D_0 = 0$

2) D_t is continuous

3) D_t has independent increments

4) $D_t - D_s \sim \mathcal{N}(0, t-s)$ for $0 \le s \le t$

where $D_t \in \mathbb{R}^{n\times n}$ $\forall t$.

Note that each increment is a Gaussian Unitary Ensemble (GUE).

Methods

. In order to make sure the relationship between Airy Processes and Dyson Brownian motion eigenvalues holds I will empirically compute the probability

$$P(A(\tau_1) < \xi_1, A(\tau_2) < \xi_2, \ldots, A(\tau_m) < \xi_m)$$

where $\tau_1 < \tau_2 < \cdots < \tau_m$ and $\xi_1, \xi_2, \ldots, \xi_m$ are appropriately chosen thresholds. I will then compare it to it's theoretical result.

Fredholm Determinants

In mathematics, the Fredholm determinant is a complex-valued function which generalizes the determinant of a finite dimensional linear operator. The matrix determinant:

$$d(z) = det\left(I - z\begin{pmatrix} K_{11} & \ldots & K_{1N} \\ \vdots & \ddots & \vdots \\ K_{N1} & \ldots & K_{NN} \end{pmatrix}_{\restriction L^2(J_1)\oplus\cdots\oplus L^2(J_n)}\right)$$

can be approximated by:

$$d(z) = det\left(I - z\begin{pmatrix} A_{11} & \ldots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \ldots & A_{NN} \end{pmatrix}\right)$$

where:

$$(A_{ij})_{(p,q)} = w_{ip}^{1/2} K_{ij}(x_{ip}, x_{jq}) w_{jq}^{1/2},\ (p, q = 1, 2, \ldots, m)$$

I will empirically compute the probability: $P(A(\tau_1) < \xi_1, A(\tau_2) < \xi_2, \ldots, A(\tau_m) < \xi_m)$ and the determinant of $I - zA$ to get the following relationship:

$$P(A(\tau_1) < \xi_1, A(\tau_2) < \xi_2, \ldots, A(\tau_m) < \xi_m) = d(1) = det\left(I - 1\begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix}\right) \quad (1)$$

where the Extended Airy Kernel is defined as follows:

$$K_{ij}(x,y) = \int_0^\infty Ai(x+z)Ai(y+z)e^{-z(\tau_i - \tau_j)}dz, i \geq j \quad (2)$$

$$K_{ij}(x,y) = -\int_{-\infty}^0 Ai(x+z)Ai(y+z)e^{-z(\tau_i - \tau_j)}dz, i < j \quad (3)$$

Approximation the Infinite Matrix

. Because each of these A_ij matrices are being evaluated for the values of $x \in (s, \infty)$, for the purposes of numerical evaluation we must find a way to approximate it with finite points. Borneman provides such an approximation. This allows us to transform the matrix from covering the (s, ∞) to continuously covering $(0, 1)$. However for ease of computation, I changed the transformation so that it would cover $(-1, 1)$. Gauss-Legendre approximation is a simple way of approximating a function over the continuous interval from $(-1, 1)$.

$$d(z) = det\left(I - A_{s \upharpoonright L^2(s_1,\infty) \oplus \cdots \oplus L^2(s_2,\infty)}\right) = det\left(I - \tilde{A}_{s \upharpoonright L^2(-1,1) \oplus \cdots \oplus L^2(-1,1)}\right) \quad (4)$$

Where $\tilde{A}_s$ is defined as follows:

$$\tilde{A}_s u(\xi) = \int_{-1}^1 \tilde{K}_s(\xi, \eta)\, dx \quad (5)$$

$$\tilde{K}_s(\xi, \eta) = \sqrt{\phi_s'(\xi)\phi_s'(\eta)}K(\phi_s(\xi), \phi_s(\eta)) \quad (6)$$

and ϕ_s is defined as follows:

$$\phi_s(\xi) = s + 10\pi \tan(\pi(\xi + 1)/4) \quad (7)$$

Approximating the Infinite Integral

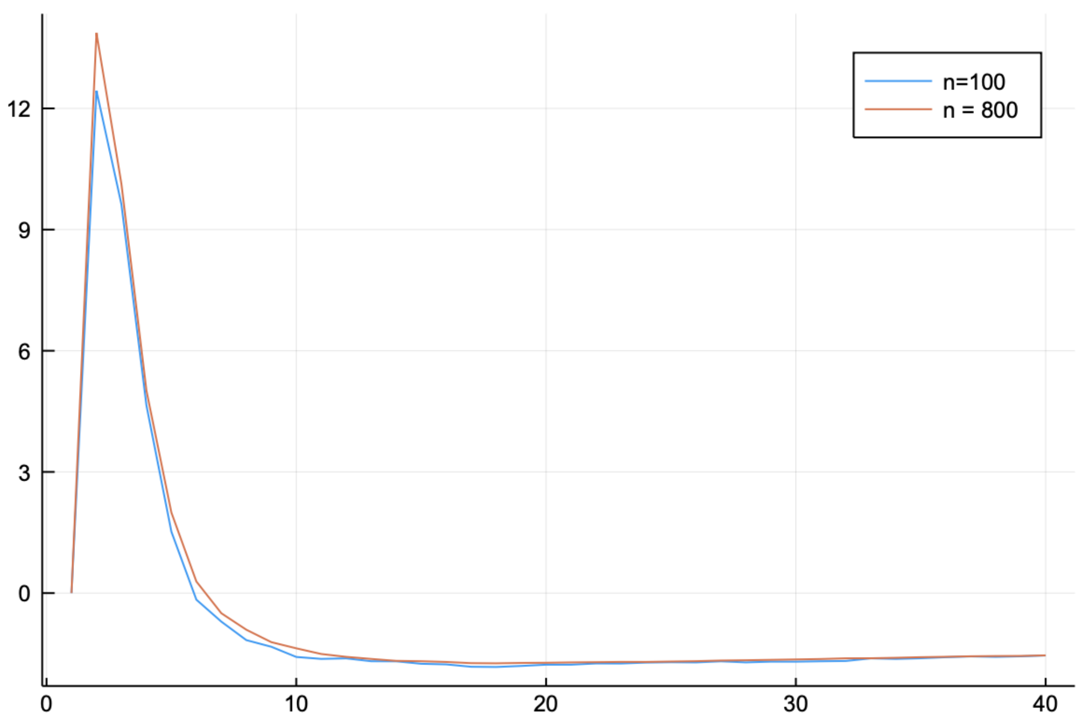
. Now that we know which points we will use to compute A_ij, we have to compute the infinite intervals in equations(2) and (3). These can be computed at the x values that are the transformed Gauss Legendre points. In order to compute (2) and (3), I used the Gauss Laguerre method. This approximates an integral from $(0, \infty)$ using n roots of the Laugerre polynomials.

$$\sum_{i=1}^{n} w_i f(x_i) = \int_0^{\infty} e^{-x} f(x) dx \tag{8}$$

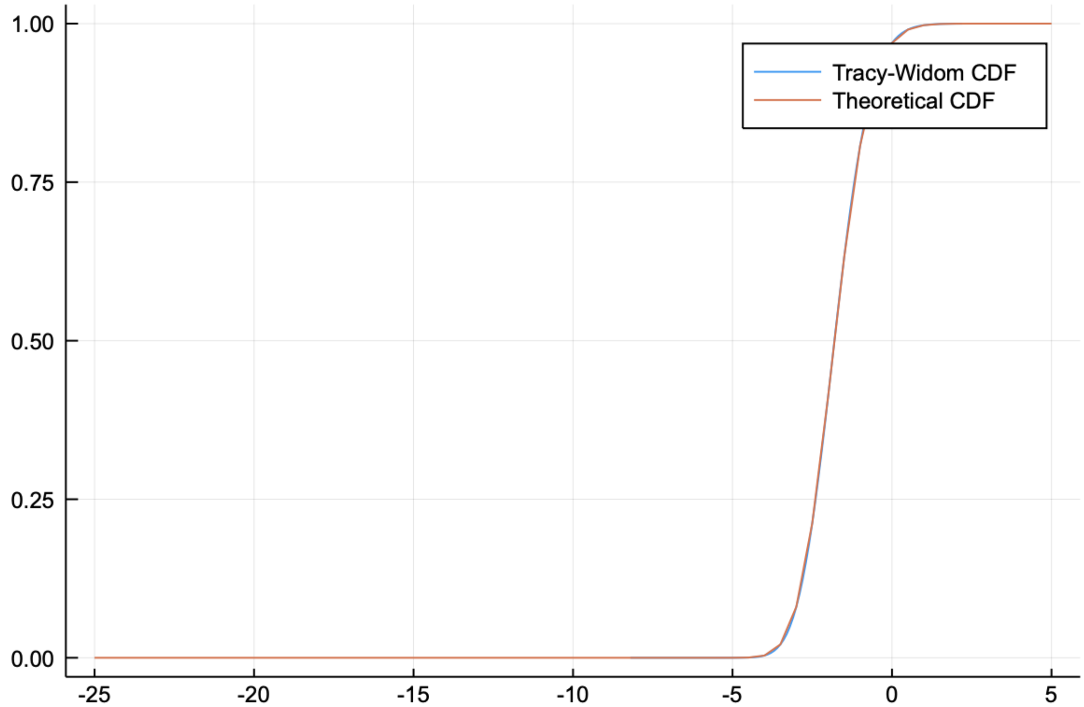
It is important to note that because Gauss Laguerre has an extra e^{-x} factor on the integral side, we must incorporate that into the computation process.

Results

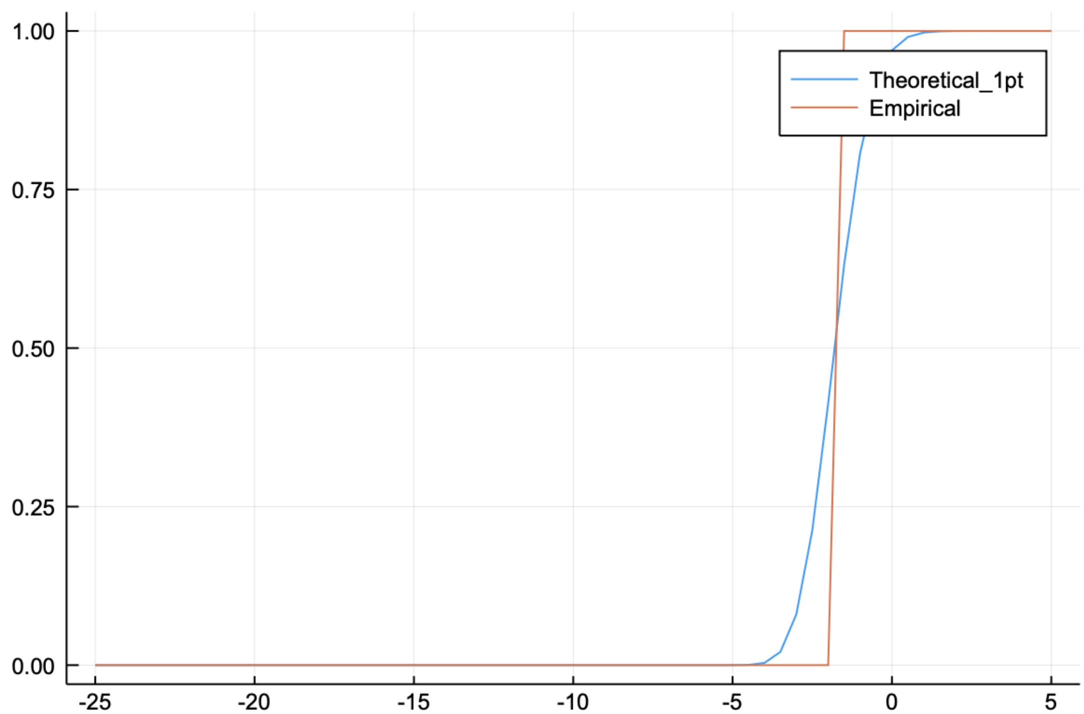
The figure shows my Airy Process Implementation as it evolves over time.The mean is close to the correct value but it has less variance than expected.



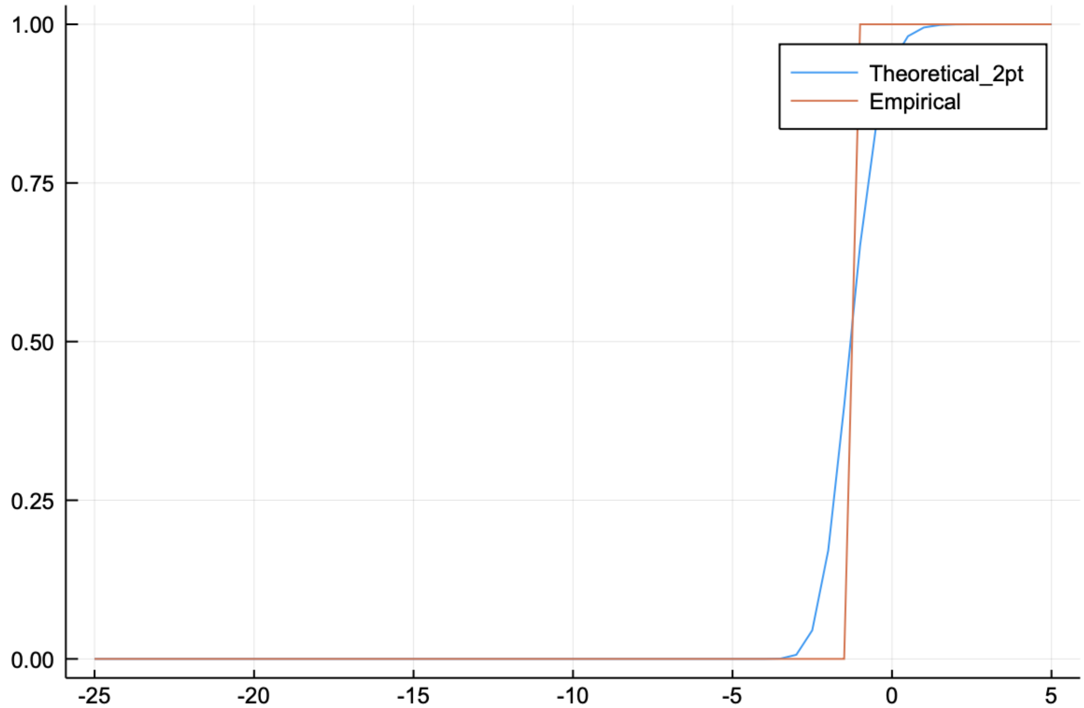
The first result I confirmed is that my theoretical computation for the CDF matched up with the simplification for the CDF of a single maximum eigenvalue of a GUE. I compared it to Edelman's code as shown in the figure below.



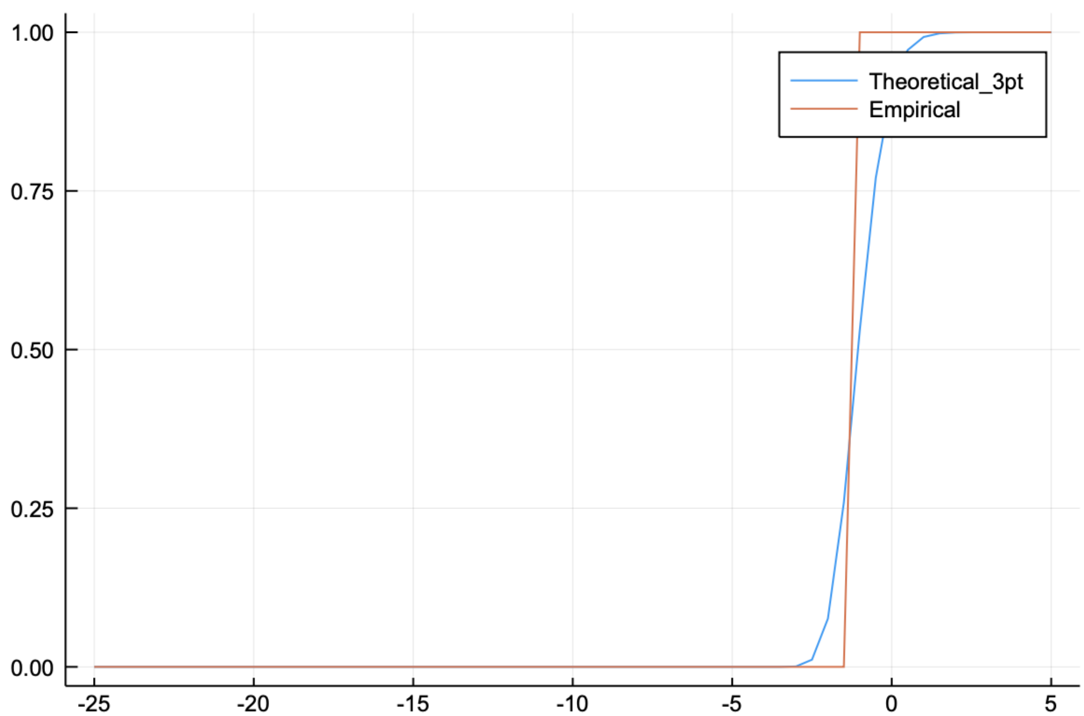
I also compared my empirical implementation of the Airy Process with this theoretical result for 1 point.



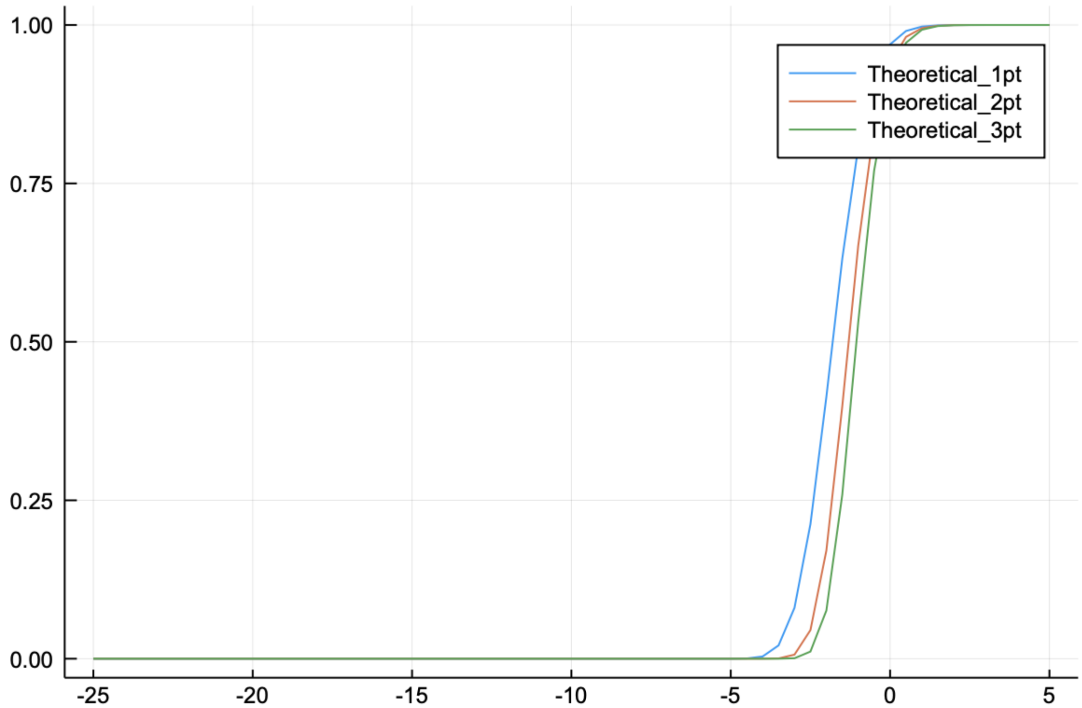
In addition I compared my implementation of the Airy Process Monte Carlo with the theoretical result for two points and three points.



Something we can note in the my three point case is that the empirical mean is shifting farther left of the theoretical mean.



Another result I looked at was how the theoretical CDF changes as we add more points. The one point distribution in the following figure is $P(A(20) < s)$. The two point distribution is $P(A(20) < s, A(9) < s)$ and the three point distribution is $P(A(20) < s, A(15) < s, A(9) < s)$. We can see that the distribution moves in smaller increments to the right and the CDF becomes steeper. If I had more computational power, it would be interesting to watch how this distribution evolves as the number of points goes to infinity.



Unfortunately I was unable to get the empirical variance to match the theoretical variance before the project was due. I will keep working on it in the next couple days to finish it.

Sources

[1] Folkmar Bornemann. ON THE NUMERICAL EVALUATION OF FREDHOLM DETERMINANTS. URL: https://arxiv.org/pdf/0804.2543.pdf

[2] Mark Adler and Pierre van Moerbeke. PDES FOR THE JOINT DISTRIBUTIONS OF THE DYSON, AIRY AND SINE PROCESSES. URL: https://arxiv.org/pdf/math/0403504.pdf.

[3] KURT JOHANSSON. DISCRETE POLYNUCLEAR GROWTH AND DETERMINANTAL PROCESSES, URL: https://arxiv.org/pdf/math/0206208.pdf.

[4] Michael Prahofer and Herbert Spohn. Scale Invariance of the PNG Droplet and the Airy Process. URL: https://arxiv.org/pdf/math/0105240.pdf.