Computation of Equilibrium Measure

Chun-Hei Lam
Department of Mathematics, *MIT*

Submitted in part fulfilment of the requirements for the course
18.338 Random Matrices Theory

This is an incomplete version of the report, compiled to make the marking procedures easier. Please look at the final Jupyter notebook for the details of computations!

1 Preliminaries

The project is to devise a method to obtain a equilibrium measure $d\mu = w(x)dx$ from a potential $e^{-V(x)}$. To understand what an equilibrium measure is, let us recall the joint density of eigenvalues of $n \times n$ Gaussian Unitary Ensembles (GUE) when all the entries are normally distributed (i.e. $V(x) = x^2/2$):

$$p_A(\lambda_1, ..., \lambda_n) \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 \exp\left(-\sum_i \frac{\lambda_i^2}{2}\right) d^n\lambda$$

We can scale up by $\lambda = \sqrt{n}x$ (like how we scale the eigenvalues during GUE experiments) to obtain

$$\begin{aligned} p_A(x_1, ..., x_n) &\propto \prod_{i<j} |x_i - x_j|^2 \exp\left(-n\sum_i \frac{x_i^2}{2}\right) d^n x \\ &= \exp\left(-\left(\sum_{i\neq j} \ln|x_i - x_j|^{-1} + n\sum_{i=1}^n V(x_i)\right)\right), \quad V(x) = x^2/2 \end{aligned}$$

Dyson has interpreted the term inside the bracket as the total potential of a gas of n electrons confined on a real axis with position $x_1, x_2, ..., x_n$:

$$\sum_{i\neq j} \underbrace{\ln|x_i - x_j|^{-1}}_{\text{potential from repulsion}} + n\sum_{i=1}^n \underbrace{V(x_i)}_{\text{exterior potential}}$$

If we define the counting measure μ_n of scaled position with density (with respect to Lebesgue measure) $w_n(x) = n^{-1}\sum_{i=1}^n \delta_{x_i}(x)$, then we can write the above term as $n^2 I^V(\mu_n)$, where $I^V(\mu)$ is a functional of all probability measures in $\mathbb{R}$ defined as

$$I^V(\mu) = \iint_{t\neq s} \ln|t - s|^{-1}\,\mu(dt)\,\mu(ds) + \int V(s)\,\mu(ds) \tag{1}$$

We expect that the rescaled positions x_i in the counting measure μ_n are chosen so that it minimises of $I^V(\mu)$ among all choices of positions x_i, and the counting measure μ_n will converge (weakly) to the global eigenvalue distribution μ (with density $w(x)$) as $n \to \infty$. Therefore μ (with density $w(x)$) is a minimiser of $I^V(\mu)$, known as the equilibrium measure of $V(x)$. **Remark**: we expect μ to have compact support.

One might wonder how can we experimentally determine $w_n(x)$ (i.e. the rescaled positions). Let's arrange x_i so that it becomes an increasing sequence. We can 'differentiate' the term inside the bracket with respect to x_i to obtain

$$\frac{2}{n}\sum_{j=1,\, j\neq i}^{n} \frac{1}{x_i - x_j} - V'(x_i) = 0 \quad (\forall i) \tag{2}$$

(formally we can use Euler Lagrange), which is equivalent to saying

$$\sum_{j=1,\, j\neq i}^{n} \frac{2}{\lambda_i - \lambda_j} - V'(\lambda_i) = 0 \quad (\forall i) \tag{3}$$

The following is a histogram of the positions x_i, and a comparison with eigenvalues of GUE.

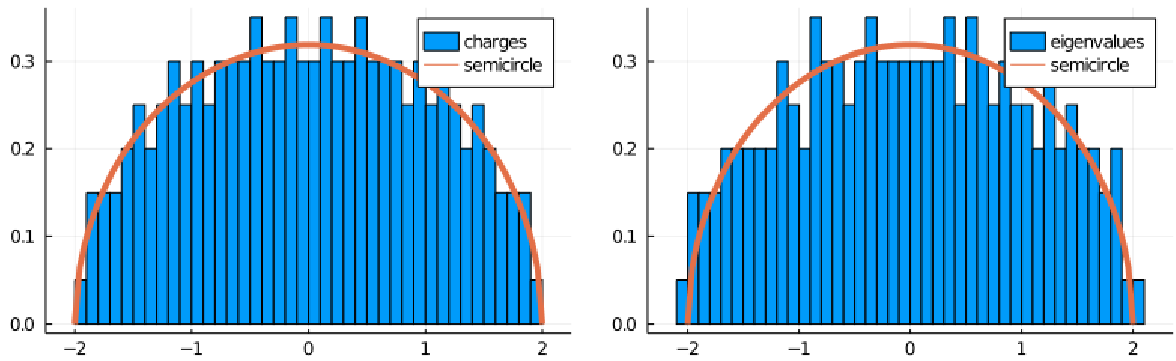

Figure 1: Semicircles!

We will start with the equilibrium systems to derive the equilibrium measure. Let us be reminded with the standard results of the Cauchy Transforms and Hilbert Transform.

2 Complex Number Primer

Without further specification we write f as a function from $\mathbb{C}$ to $\mathbb{C}$ and a piecewise-C^1 curve in $\mathbb{C}$ as γ. Recall the following definitions.

Definition 2.1: Cauchy Transform

The Cauchy Transform is the operator

$$\mathcal{C}_\gamma[f](z) = \frac{1}{2\pi i}\int_\gamma \frac{f(x)\,dx}{x - z} \tag{4}$$

Definition 2.2: Hilbert Transform

The Hilbert Transform is the operator

$$\mathcal{H}_\gamma[f](z) = \frac{1}{\pi}\text{PV}\int_\gamma \frac{f(x)\,dx}{x-z} \tag{5}$$

where the value of integral is considered as (Cauchy) Principal Value.

The Cauchy Transform comes from the Cauchy Theorem. In the case when γ is a closed curve, we have $\mathcal{C}_\gamma[f](z) = f(z)$ whenever z is inside γ, and $\mathcal{C}_\gamma[f](z) = 0$ whenever z is outside γ (the 'inside' and 'outside' regions comes from the definition of Jordan Curve Theorem). When z is on γ, we have $\mathcal{C}_\gamma[f](z) = f(z)/2$. This comes from a deformation argument. It is clear that there is jump for $\mathcal{C}_\gamma[f](z)$. As a convention we define the following limits at points when $f(z)$ is not defined (e.g. points on branch cuts).

Definition 2.3: Left and Right Limit

Define the following limits at $z = z_0$

$$f^+(z_0) = \lim_{z\to z_0^+} f(z), \quad \text{limit from the 'left'.}$$
$$f^-(z_0) = \lim_{z\to z_0^-} f(z), \quad \text{limit from the 'right'.}$$

Unfortunately the definition of 'left' and 'right' are subtle. Here we consider those limits when z_0 is on the curve γ where $f(z_0)$ cannot be properly defined:

- If γ is closed then we consider 'left' as region bounded by γ and 'right' as the complement of the bounded region.
- If γ is real line / subset of real line we consider 'left' as region above the real axis and 'right' as region below the real axis.
- Otherwise, we consider 'left' as region to the left of γ and 'right' as region to the right of γ.

For the case when γ is closed, we immediately obtain the following formula:

Theorem 2.4: Plemelj I

When $z_0 \in \gamma$ we have

$$\mathcal{C}_\gamma^+[f](z_0) - \mathcal{C}_\gamma^-[f](z_0) = f(z_0) \tag{6}$$

Of course, one might also think about the additive jump of the Cauchy Transform. Observe when $z_0 \in \gamma$ the following holds:

$$\mathcal{C}_\gamma^+[f](z_0) = \frac{1}{2}g(z_0) + \frac{1}{2\pi i}\int_\gamma \frac{g(x)}{x-z_0}$$
$$\mathcal{C}_\gamma^-[f](z_0) = -\frac{1}{2}g(z_0) + \frac{1}{2\pi i}\int_\gamma \frac{g(x)}{x-z_0}$$

So we obtain the following formula

Theorem 2.5: Plemelj II

When $z_0 \in \gamma$ we have

$$mathcalC_\gamma^+[f](z_0) + \mathcal{C}_\gamma^-[f](z_0) = \frac{1}{\pi i}\int_\gamma \frac{g(x)}{x-z_0} = -i\mathcal{H}_\gamma[f](z_0) \tag{7}$$

We go back to our problem. Observe that for all $z \neq x_i$

$$\frac{1}{n}\sum_{j=1}^n \frac{1}{z-x_j} = \int_{-\infty}^{\infty} \frac{w_n(x)}{z-x}$$

Moreover, for all $z = x_i$

$$\lim_{\epsilon\to 0}\left(\int_{-\infty}^{x_i-\epsilon} + \int_{x_i-\epsilon}^{\infty}\right)\frac{w_n(x)}{z-x}\,dx = \sum_{j=1,x_j<x_i}^n \frac{1}{z-x_j} + \sum_{j=1,x_j>x_i}^n \frac{1}{z-x_j} = \frac{1}{n}\sum_{j=1,\,j\neq i}^n \frac{1}{x_i-x_j}$$

Therefore for all $z = x_i$,

$$-2\pi\mathcal{H}_\gamma[w_n](x_i) - x_i = 0 \iff \mathcal{H}_\gamma[w_n](x_i) = -\frac{V'(x_i)}{2\pi}$$

If $n \to \infty$, by weak convergence,

$$\mathcal{H}_\gamma[w](z) = -\frac{V'(z)}{2\pi}$$

By Plemenj II,

$$(-2i\pi)\mathcal{C}_\gamma^+[w](z) + (-2i\pi)\mathcal{C}_\gamma^-[w](z) = -\pi\mathcal{H}_\gamma[w](z) = V'(z)$$

Define $\phi(z) = (-2i\pi)\mathcal{C}_\gamma[w](z)$. Then we have

$$\phi^+(z) + \phi^-(z) = V'(z) \tag{8}$$

Moreover,

$$\phi^+(z) - \phi^-(z) = (-2i\pi)(\mathcal{C}_\gamma^+[w](z) - \mathcal{C}_\gamma^-[w](z)) = (-2i\pi)w(z)$$

which implies

$$w(z) = \frac{i}{2\pi}(\phi^+(z) - \phi^-(z)) \tag{9}$$

The main task is to obtain ϕ. We have shown in simulation that μ (or w) must have compact support, so we have

$$\phi^+(z) + \phi^-(z) = V'(z), \quad z \in \text{supp}\mu$$

Remark. There might be more than one solution. We need to impose further condition: notice that if $w(x)$ is a 'well-behaved' density of probability measure on $\mathbb{R}$ then $w(x)$ integrates to 1. It follows that

$$\begin{aligned}\mathcal{C}_\gamma[w](z) &= -\frac{1}{2\pi i}\int_{\text{supp}\,\mu}\frac{w(x)\,dx}{z-x}\\ &= -\frac{1}{2\pi iz}\int_{\text{supp}\,\mu}\frac{w(x)\,dx}{1-(x/z)}\\ &= -\frac{1}{2\pi iz}\int_{\text{supp}\,\mu} w(x)\left(1+\frac{x}{z}+...\right)\\ &= -\frac{1}{2\pi iz}+...\end{aligned}$$

(if dominated convergence theorem applies). Therefore we must have

$$\phi(z) = \frac{1}{z}+...$$

2.1 The Chebyshev-Joukowsky Map

We introduce the Chebyshev-Joukowski Map:

$$J(z) = \frac{z+z^{-1}}{2} \tag{10}$$

It maps unit circle $\mathbb{T}$ ($z=e^{i\theta}$) onto a line segment $[-1,1]$. In addition, it maps non-unit circles to ellipses surrounding the interval $[-1,1]$. Therefore, if a function $f(z)$ (e.g. $f(z)=\sqrt{z-1}\sqrt{z+1}$) has jump on interval $[-1,1]$, then the function $J(z)$ will have jump on the unit circle. Therefore, if $\phi(z)$ satisfies the equation

$$\phi^+(z)-\phi^-(z) = f(z)$$

then the function $\psi(z) := \phi(J(z))$ satisfies the equation

$$\psi^+(z)-\psi^-(z) = f(J(z))$$

As we will see, the latter equation is much easier to be solved.

Inverse Maps: For $x \notin (-1,1)$, we may define two inverse maps of $J(x)$, namely $J_+^{-1}(x)$ and $J_-^{-1}(x)$.

$$J_+^{-1}(x) = x-\sqrt{x-1}\sqrt{x+1} \quad \text{(interior)} \tag{11a}$$

$$J_-^{-1}(x) = x+\sqrt{x-1}\sqrt{x+1} \quad \text{(exterior)} \tag{11b}$$

Notice that $J_+^{-1}(x) = (J_-^{-1}(x))^{-1}$. We can also define inverse map for $x \notin (-\infty,-1]\cup[1,\infty)$, namely

$$J_U^{-1}(x) = x+i\sqrt{1-x}\sqrt{x+1} \quad \text{(upper)} \tag{12a}$$

$$J_D^{-1}(x) = x-i\sqrt{1-x}\sqrt{x+1} \quad \text{(lower)} \tag{12b}$$

The first map maps $[-1,1]$ to the upper half of unit circle, while the second map maps $[-1,1]$ to the lower half of unit circle. Once again, we have the relation $J_U^{-1}(x) = (J_-^{-1}(D))^{-1}$.

Notice that both $J_+^{-1}(x)$ and $J_-^{-1}(x)$ have jump along $[-1,1]$, while $J_U^{-1}(x)$ and $J_D^{-1}(x)$ have jump along $(-\infty,-1]\cup[1,\infty)$.

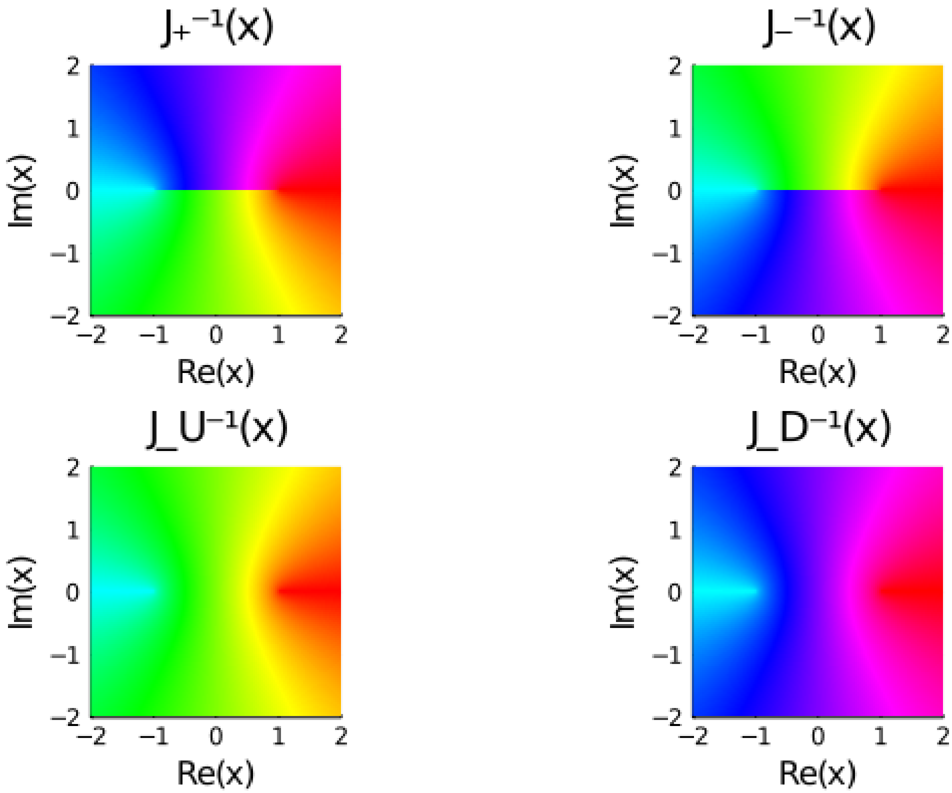

Figure 2: Phase Plots of four inverse maps

One can further infer the following from the phase plots:

$$J_+^{-1}(x) = J_D^{-1}(x), \quad \text{Im}(x) > 0 \tag{13a}$$

$$J_+^{-1}(x) = J_U^{-1}(x), \quad \text{Im}(x) < 0 \tag{13b}$$

$$J_-^{-1}(x) = J_U^{-1}(x), \quad \text{Im}(x) > 0 \tag{13c}$$

$$J_-^{-1}(x) = J_D^{-1}(x), \quad \text{Im}(x) < 0 \tag{13d}$$

3 A Canonical Example

We go back to the example $V(x) = x^2/2$, or $V'(x) = x$. Let's assume the equilibrium measure (or the density $w(x)$) has support $[-b,b]$ with $b > 0$. (The support must be symmetric since $V(x)$ is symmetric.) By rescaling $x \to bx$ we are solving the equation

$$\phi^+(x) + \phi^-(x) = bx, \quad x \in [-1,1] \tag{14}$$

Writing $\psi(z) = \phi(J(z))$ we have

$$\psi^+(z) + \psi^-(z) = \frac{b}{2}\left(z + \frac{1}{z}\right), \quad z \in \mathbb{T} \tag{15}$$

Notice the solution of this equation is simple, it is

$$\psi(z) = \begin{cases} bz/2 & |z| < 1 \\ b/2z & |z| > 1 \end{cases} \tag{16}$$

It remains for us to invert from $\psi(z)$ back to $\phi(x)$. I claim that the following function is our desired solution (before scaling back)

$$\phi(x) = \frac{\psi(J_+^{-1}(x)) + \psi(J_-^{-1}(x))}{2} \tag{17}$$

To show this we note that along $x \in [-1, 1]$,

$$\psi^+(J_+^{-1}(x)) = \psi^+(J_D^{-1}(x)) \tag{18a}$$

$$\psi^-(J_+^{-1}(x)) = \psi^-(J_U^{-1}(x)) \tag{18b}$$

$$\psi^+(J_-^{-1}(x)) = \psi^+(J_U^{-1}(x)) \tag{18c}$$

$$\psi^-(J_-^{-1}(x)) = \psi^-(J_D^{-1}(x)) \tag{18d}$$

AND

$$\psi^+(J_U^{-1}(x)) + \psi^-(J_U^{-1}(x)) = \psi^+(J_D^{-1}(x)) + \psi^-(J_D^{-1}(x)) = bx \tag{19}$$

Combining we have

$$\phi^+(x) + \phi^-(x) = bx$$

We are not done yet – we have not checked the decaying properties $\phi(x)$. We first simplify $\phi(x)$ by noting that $\psi(J_+^{-1}(x)) = bJ_+^{-1}(x)/2$ and $\psi(J_-^{-1}(x)) = b(J_-^{-1}(x))^{-1}/2 = bJ_+^{-1}(x)/2$. Therefore we have

$$\phi(x) = \frac{b}{2}\left(x - \sqrt{x-1}\sqrt{x+1}\right) = \frac{bx}{2}\left(1 - \sqrt{1-\frac{1}{x}}\sqrt{1+\frac{1}{x}}\right) \tag{20}$$

Rescaling back by $x \to x/b$ yields

$$\phi(x) = \frac{x}{2}\left(1 - \sqrt{1-\frac{b}{x}}\sqrt{1+\frac{b}{x}}\right) = \frac{b^2}{4}\frac{1}{x} + ... \tag{21}$$

For $\phi(x)$ to satisfy the decaying property we need $b^2/4 = 1$, or $b = 2$. Therefore the equilibrium measure has support $[-2, 2]$ and its density has Cauchy transform

$$\phi(x) = \frac{x}{2}\left(1 - \sqrt{1-\frac{2}{x}}\sqrt{1+\frac{2}{x}}\right) = \frac{1}{2}\left(x - \sqrt{x-2}\sqrt{x+2}\right) \tag{22}$$

To obtain the final density $w(x)$ we utilise Plemelj I and notice that when $x \in [-2, 2]$, we have

$$\begin{aligned}\left(\sqrt{x^2-4}\right)_+ - \left(\sqrt{x^2-4}\right)_- &= \sqrt{x+2}\left(\sqrt{x+2}\right)_+ - \left(\sqrt{x+2}\right)_- \\ &= 2i\sqrt{4-x^2}\end{aligned}$$

Therefore

$$w(x) = \frac{i}{2\pi}\left(\phi^+ - \phi^-\right) = \frac{1}{2\pi}\sqrt{4-x^2} \tag{23}$$

which is the semicircle law :)

4 Onto the Numerics

One may wonder if we generalise the above calculation to any convex potential $V(x)$ with only one minimum. We assume that the equilibrium measure has support (a, b). Let $V'(x) = f$. We would like to solve

$$\phi^+(z) + \phi^-(z) = f, \quad z \in (a, b)$$
$$\phi(z) = \int \frac{w(x)}{z - x} = \frac{2\pi}{i}\mathcal{C}_\gamma[w](z) \sim O(z^{-1})$$

One should note a simplification: there is a bijective map from (a, b) to $(-1, 1)$:

$$M_{(a,b)}(x) = \frac{2x - (a + b)}{b - a} \tag{24}$$

Therefore if ϕ satisfies $\phi^+ + \phi^- = f$ for $z \in (a, b)$ then the function $\tilde{\phi}(x) = \phi(M^{-1}_{(a,b)}(x))$ satisfies

$$\tilde{\phi}^+(x) + \tilde{\phi}^-(x) = f(M^{-1}_{(a,b)}(x)) := \tilde{f}(x), \quad z \in (-1, 1)$$

Of course, if we define $\psi(z) = \tilde{\phi}(J(z))$ then we must have

$$\psi^+(z) + \psi^-(z) = \tilde{f}(J(z)), \quad z \in \mathbb{T}$$

Magic happens when we can write $\tilde{f}(x)$ in Chebyshev expansion, i.e.

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \hat{f}_k T_k(x) \tag{25}$$

(here $\hat{f}_k$ depends on a and b!). Then

$$\tilde{f}(J(e^{i\theta})) = \hat{f}_0 + \frac{1}{2}\sum_{k=-\infty}^{\infty} \hat{f}_{|k|} e^{ik\theta} \tag{26}$$

In other words, for all $z \in \mathbb{T}$, we have

$$\tilde{f}(J(z)) = \hat{f}_0 + \frac{1}{2}\sum_{k=-\infty}^{\infty} \hat{f}_{|k|} z^k \tag{27}$$

> *Remark.* $\hat{f}_{|k|}$ is the Fourier coefficients of $\tilde{f}(J(e^{i\theta}))$, and you can estimate those by (Fast) Fourier Transform.

Again, if we define

$$F^+(z) = \hat{f}_0 + \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_{|k|} z^k, \quad F^-(z) = \frac{1}{2}\sum_{k=-\infty}^{-1} \hat{f}_{|k|} z^k \tag{28}$$

Then the function

$$\psi(z) = \tilde{\phi}(J(z)) = \begin{cases} F^+(z) & |z| < 1 \\ F^-(z) & |z| > 1 \end{cases} \tag{29}$$

satisfies $\psi^+(J(z))+\psi^-(J(z)) = \tilde{f}(J(z))$. Using previous argument, we may then guess the following function

$$\tilde{\phi}'(x) = \frac{\psi(J_+^{-1}(x)) + \psi(J_-^{-1}(x))}{2} \tag{30}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(z))^k \tag{31}$$

satisfies

$$\tilde{\phi}'^+(x) + \tilde{\phi}'^-(x) = \tilde{f}(x), \quad z \in [-1,1]$$

using the arguments above. Unfortunately $\tilde{\phi}'^+(\infty) = \hat{f}_0/2$ so $\tilde{\phi}'^+(\infty) = \hat{f}_0/2$ does not decay?! One can mitigate by subtracting a correction term

$$\frac{x\hat{f}_0 + C}{2\sqrt{x+1}\sqrt{x-1}}$$

notice that the function $(\sqrt{z+1}\sqrt{z-1})^{-1}$ only jumps outside $[-1,1]$ so does not affect the jump condition but enable the solution to decay in rate $O(x^{-1})$. By recaling back by substituting x with $M_{(a,b)}(x)$ we have the formula

$$\phi(x) = \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(M_{(a,b)}(x)))^k - \frac{M_{(a,b)}(x)\hat{f}_0 + C}{2\sqrt{M_{(a,b)}(x)+1}\sqrt{M_{(a,b)}(x)-1}}$$

We are not finished yet, since we don't know what is a and b. The decay behaviour for $\phi(x)$ will set up two equations for a, b. Firstly, we want $\hat{f}_0 = C = 0$ for $\phi(x)$ to be bounded. Secondly, notice by substituting the approximations:

$$J_+^{-1}(x) = \frac{1}{2x} + ..., \quad M_{(a,b)}(x) = \frac{2x}{b-a} + ...$$

yields

$$\phi(x) = \frac{1}{2}\hat{f}_1\left(\frac{1}{2\left(\frac{2x}{b-a}\right)}\right) + ... = \frac{b-a}{8}\,\hat{f}_1\frac{1}{x} + ...$$

Therefore we demand

$$\frac{b-a}{8}\,\hat{f}_1 = 1$$

We may then find a, b by solving these two equations, typically by Newton-Raphson, then perform the above procedure to obtain $\phi(x)$. We may then use Plemenlj I to conclude

$$w(x) = \frac{\sqrt{1 - M_{(a,b)(x)}}}{\pi}\sum_{k=1}^{\infty} \hat{f}_k U_{k-1}(M_{(a,b)}(x)) \tag{32}$$

The following is a result of the implementation of the algorithm:

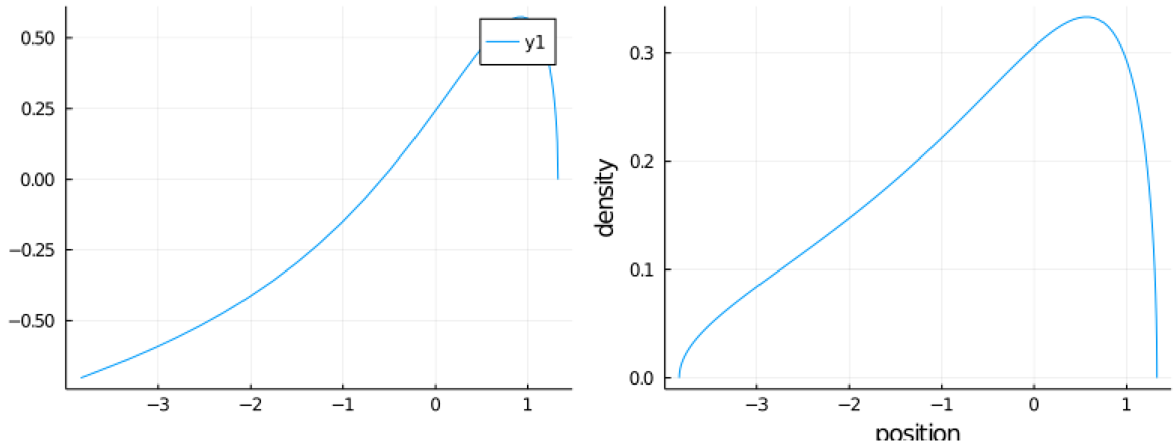

Figure 3: The computed equilibrium measures when $V(x) = e^x - x$. The left figure is from my code. The right figure is from the code in the EquilibriumMeasures.jl package.

Discrepancy arises since I have only used the first 20 coefficients. One can go further to increase efficiency. Other than that, the shape of the density is similar.

We can generalise the method to some other functions, and possibly predict some GUE with non-Gaussian entries and explore universality of GUE. However, the scaling will be different. One can obtain an empirical scaling by looking at how the maximum eigenvalues increase as n increases. This is a further direction to this project.

Declaration and Acknowledgements: I pledge that this essay and the work re-ported herein was our original work. Wherever works of others are involved, they havebeen cited clearly. We are grateful to Prof. Alan Edelman, Dr. Sheehan Olver and Mr. Bernie Wang for their guidance.