Spacing of Riemann Zeta Zeros in Julia

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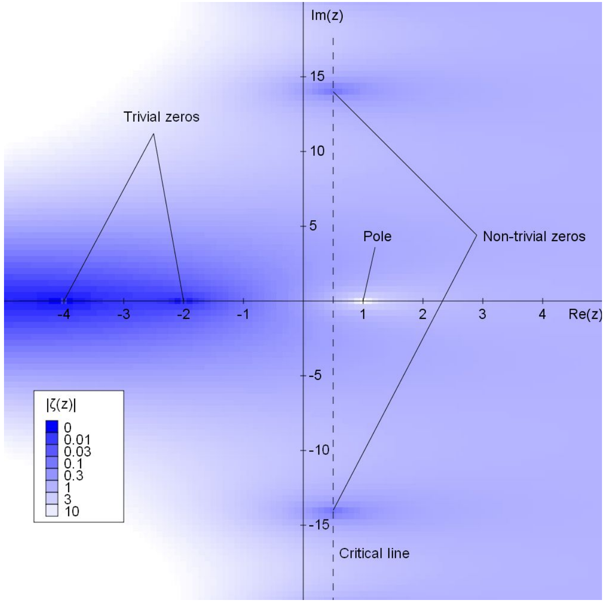
Riemann Siegel Z Function

For *s = 1/2 + it* on the critical line, extend the zeta function w/ the Riemann Siegel Z function

Z is purely real, and zeros of ζ occur at sign changes of *Z*



$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(s) = e^{-i\vartheta(t)} Z(t)$$

$$\vartheta(t) = \Im[\ln\Gamma(1/4 + it/2)] - \frac{1}{2} t \ln \pi$$

https://mathworld.wolfram.com/Riemann-SiegelFunctions.html

Riemann Siegel Formula

$$Z(t) = 2 \sum_{k=1}^{\lfloor \nu(t) \rfloor} \frac{1}{\sqrt{k}} \cos(\vartheta(t) - t \ln k) + R(t) + O(e^{-t/11})$$

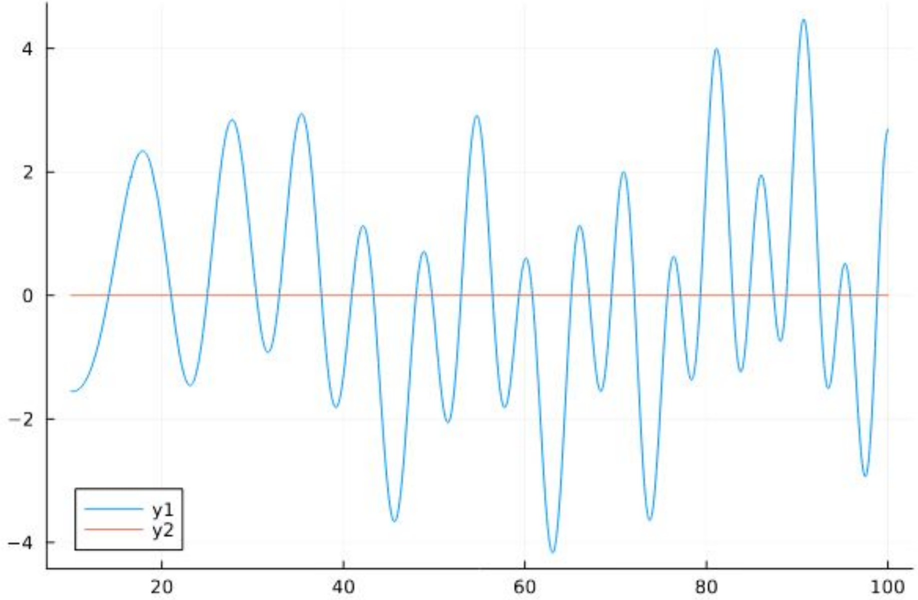
The c_k are entire functions, which can be approximated by their Taylor series

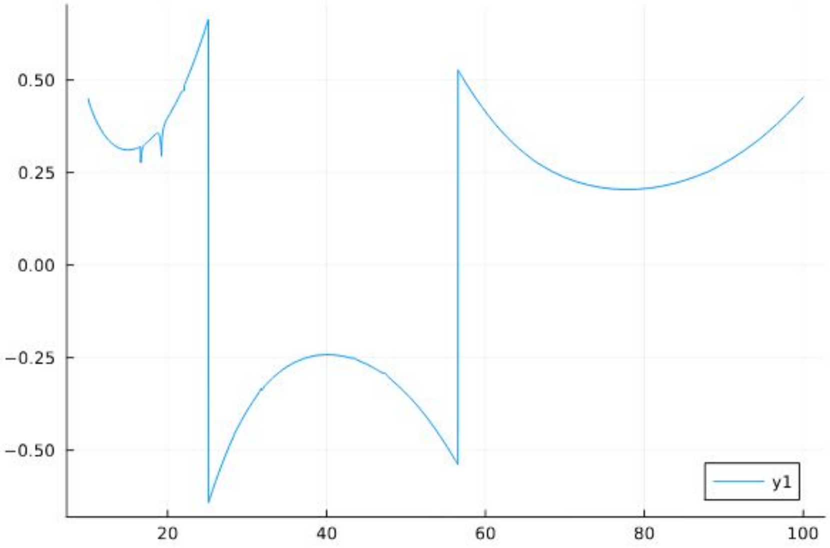
This formula gives an *O(sqrt(t))* method for computing the *Z* function

$$\nu(t) = \sqrt{t/2\pi}$$

$$R(t) = (-1)^{\lfloor \nu \rfloor - 1} \nu^{-1/2} \times \sum_{k=0}^{\infty} c_k(\mathrm{frac}(\nu)) \nu^{-k}$$

Z function and R remainder function





Gram Points

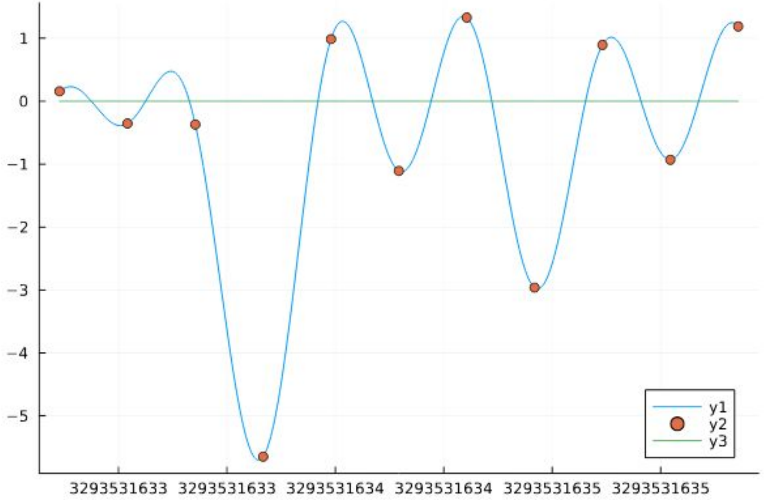
$$\zeta(s) = e^{-i\vartheta(t)} Z(t)$$

The *n*-th Gram point is the unique g_n such that $\vartheta(g_n) = \pi n$

and can be approximated as $$g_n \approx 2\pi \exp\left[1 + W\left(\frac{8n+1}{8e}\right)\right],$$

Zeros of the *Z* function tend to lie between Gram points:

Find zeros by using Gram points to find sign changes



Computed Values at height $t \sim 3*10^{10}$, $10^{11} < n < 10^{11}+10^5$

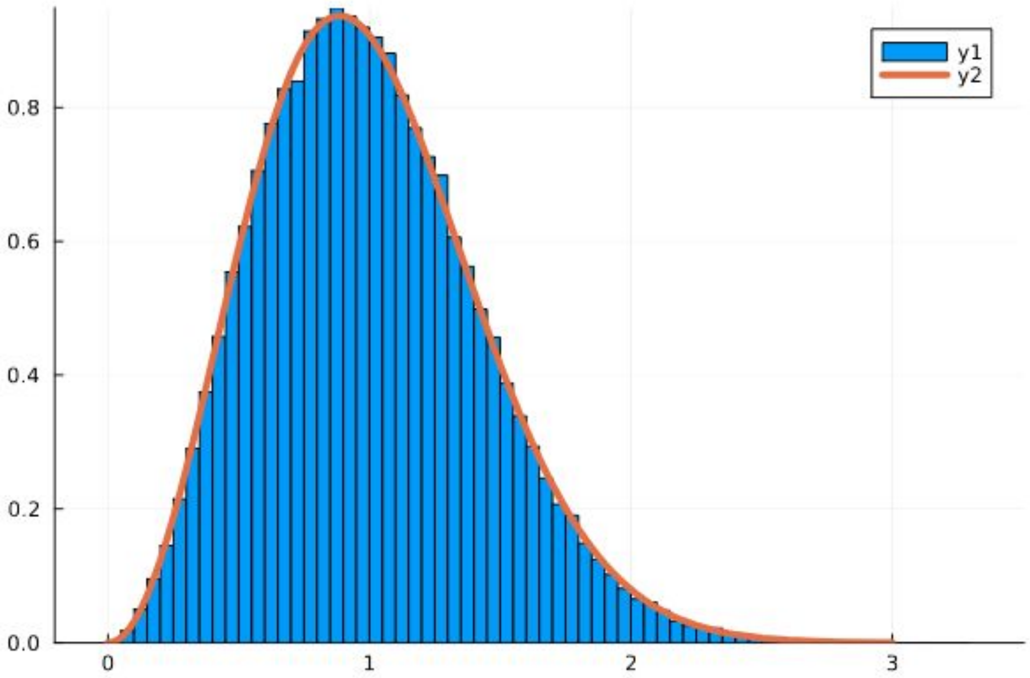
When comparing to database values, RMSE approximately 10^{-5}

Histogram (blue) of normalized spacings of zeta zeros, note that

$$\gamma_{i+1} - \gamma_i \asymp \frac{2\pi}{\log(\gamma_i/(2\pi))}$$

Orange line shows the GUE eigenvalue spacing (Wigner surmise)

$$p(x) \propto x^2 e^{-4x^2/\pi}$$



GUE correlation function

For the $N \times N$ GUE, the n-level correlation:

$$\rho_n(x_1 \cdots x_n) = \int_{\mathbb{R}^{N-n}} f_{GUE}(\vec{x}) dx_{N-n+1} \cdots dx_N$$
$$= \det[K_N(x_i, x_j)]_{i,j \in [n]}$$

where K_N is the DPP kernel; as $N \to \infty$

$$K_N(x,y) = \sum_{j=0}^{N-1} \phi_j(x)\phi_j(y) \asymp \frac{\sin(\xi_1 - \xi_2)}{\pi(\xi_1 - \xi_2)}$$

With ϕ_i being the oscillator wave functions

After normalization, pair correlation function $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$

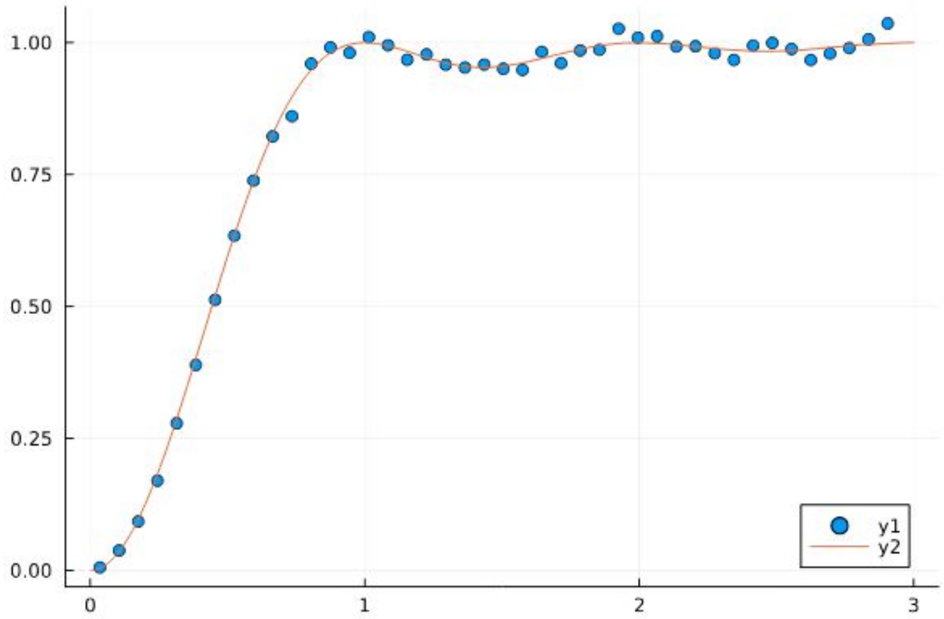
Montgomery Pair Correlation Conjecture

Normalized spacings of zeta zeros have same pair correlation function as the GUE as $N \to \infty$

Note that spacing of zeta zeros

$$\gamma_{i+1} - \gamma_i \asymp \frac{2\pi}{\log(\gamma_i/(2\pi))}$$

Plot shown is the empirical pair correlation function for first 100k roots at height 10^{11}, compare to the GUE prediction in orange



Background on Pair Correlation Conjecture

Take Fourier transform of differences, if RH:

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma')$$

$$F(\alpha) = \alpha + o(1) + (1+o(1))T^{-2\alpha}\log T$$

Let $r, \hat{r}$ be some Fourier pair, then

$$\sum_{0<\gamma,\gamma'\le T} r\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma') = \frac{T}{2\pi}\log T \int_{-\infty}^{\infty} \hat{r}(\alpha) F(\alpha)\, d\alpha$$

D.A. Goldston: https://arxiv.org/abs/math/0412313

Background on Pair Correlation Conjecture

Montgomery hypothesized that for fixed T, F(α) ≈ 1 for |α| > 1 based off the distribution of primes

Using the Fourier pair

$$k(u) = \left(\frac{\sin \pi\lambda u}{\pi\lambda u}\right)^2, \qquad \hat{k}(\alpha) = \frac{1}{\lambda}\max\left(1 - \frac{|\alpha|}{\lambda}, 0\right)$$

$$\left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma'-\gamma\leq \frac{2\pi\beta}{\log T}}} 1 \sim \int_0^\beta 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du$$