18.338 Final Project: The KPZ Equation and Fixed Point

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Abstract

In this write up I will explain what the KPZ equation and KPZ universality class are. Additionally I will emphasize the connection between ASEP and the KPZ equation. Another goal is to explain what it means to solve the KPZ equation through a Hopf-Cole transform. A lot of this material can be found in [Qua11] and [Rez19].

1 Introduction

The Kardar-Parisi-Zhang (KPZ) equation can be written as

$$\begin{aligned} &\partial_t h = -\lambda(\partial_x h)^2 + \nu \partial_x^2 h + \sqrt{D}\xi \\ &h(0,x) = h_0(x) \end{aligned} \tag{1.1}$$

We want to solve for a random height function $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. $\lambda, \nu, D > 0$ are physical constants, and ξ is Gaussian space time white noise. Naively we think of ξ as a random field with standard Gaussian values at any point $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and

$$\mathbf{E}[\xi(t_1,x_1)\xi(t_2,x_2)] = \delta(t_1 - t_2)\delta(x_1 - x_2) \tag{1.2}$$

We will describe what ξ is more precisely in section 2.1.

The KPZ equation is part of a large class of models in the *KPZ Universality Class*. Random growth models in this universality class have the feature that

$$\frac{h(Tt, cT^{2/3}x) - C(T,t,x)}{T^{1/3}} \to \mathrm{h}_{\mathrm{FP}}(t,x) \tag{1.3}$$

as $T \to \infty$, where $c, C(T,t,x)$ are deterministic and the *KPZ Fixed Point* $\mathrm{h}_{\mathrm{FP}}(t,x)$ is a random field which only depends on the initial data $h_0(x)$ and not on which model from the universality class $h(t,x)$ evolves according to. The scaling in (1.3) is usually referred to as the $1:2:3$ scaling. This choice of scaling will be explained later.

One of the first models predicted to have such a scaling behavior is the Ballistic Deposition model shown in Figure 1. The model consists of square blocks falling into columns indexed by the integers. Each column has an independent mean 1 exponential waiting time before a new block falls and the wait time resets. The block either lands on top of the previously highest block or sticks to the side of the highest block in the two columns adjacent to it, whichever comes first. Shockingly this model is completely unsolved, i.e. it's fluctuations are not proven to be at the $t^{1/3}$ scale.

Most of the tractable models in the KPZ universality class are integrable, meaning they have enough structure that explicit formulas can be obtained. Some of those models are TASEP, Polynuclear Growth (PNG), and the six vertex model. Formulas for the KPZ fixed point are usually obtained through TASEP as it is probably the easiest model to solve. Sometimes other models can be studied by comparison to integrable models, for instance [QS22] established that a large class of asymmetric exclusion processes belong to the KPZ Universality class by comparing the dynamics of those models to TASEP. Since the KPZ equation is a certain scaling limit for the weakly asymmetric simple exclusion process the Quastel and Sarkar also showed KPZ universality for the KPZ equation.

Regardless of integrablity, the common features of all models in, or believed to be in the KPZ universality class are
(1) The dynamics is local, i.e. the dynamics of the height function at far away points do not effect each other.
(2) Some smoothing or averaging mechanism

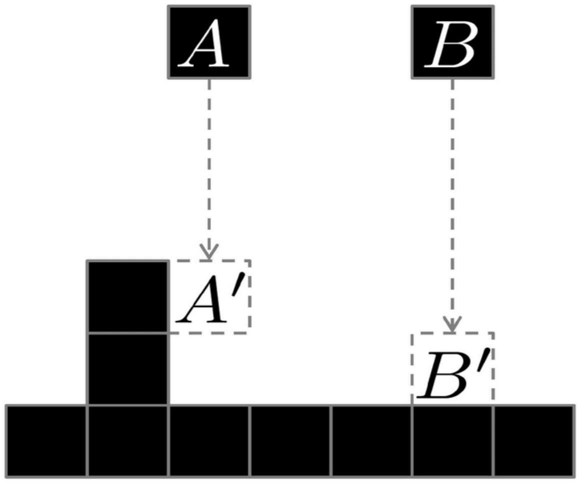

Figure 1: Dynamics for Ballistic Deposition Model.

(3) There is lateral or outward growth. Can also be thought of as slope dependent growth.
(4) Space-time randomness/white noise.

In the next subsection we will see that the KPZ equation has all of these features.

1.1 Physical Intuition

The KPZ equation can model the growth of an interface between two mediums. Specifically the $\nu\partial_x^2 h$ term in (1.1) represents a smoothing or diffusion mechanism. The ξ term represents some random noise. Finally the most important term $-\lambda(\partial_x h)^2$ represents lateral or outward growth. To see this, we first note that such growth should be given as a function of the slope $F(\partial_x h)$. After drawing a picture, basic geometric considerations suggest the candidate $F(x) = c(1+x^2)^{-1/2}$. After replacing $h(t,x)$ with $h(t,x)-ct$, WLOG $F(0) = 0$. Moreover any reasonable choice of F should be an even function so $F'(0) = 0$. Therefore Taylor expanding we may expect $F(x) \approx Cx^2$ leading to (1.1), moreover for our geometric guess $C = F''(0)/2 < 0$. However this derivation is bogus since $\partial_x h$ should not be small in general! Nevertheless it seems to work!

1.2 Rescaling

After a re-scaling of the form

$$h(t,x) \to \epsilon^a h(\epsilon^{-bt,\epsilon^{-1}t}) \tag{1.4}$$

WLOG $\lambda = \nu = 1/2$ and $D = 1$.

1.3 ASEP and the KPZ equation

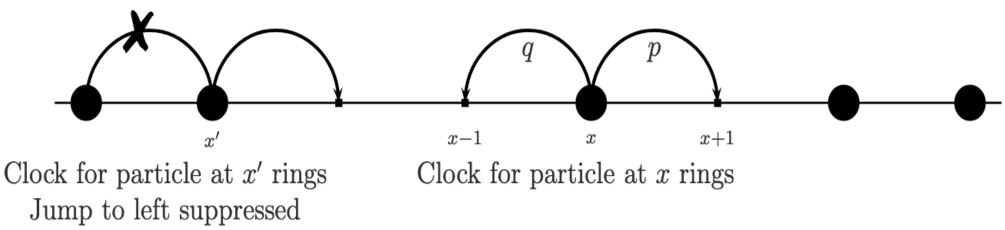

Figure 2: Dynamics for ASEP.

The Asymmetric Simple Exclusion Process with parameters $p, q \geq 0$ with $p + q = 1$ (Shown in Figure 2) can be described as follows. Suppose we have a collection of particles which can occupy points on the lattice $\mathbb{Z}$. Each particle has a mean 1 exponential wait time associated to it. All these wait times are i.i.d. and after the timer for a specific particle goes off, it attempts to jump 1 unit to the right with probability p, and one unit to the left with probability q. If there is another particle where the first one attempts to jump, the jump is blocked, otherwise the jump is completed, and in either case the wait time for the first particle is reset. This process is a continuous time Markov Process on the state space $S = \{0,1\}^{\mathbb{Z}}$ (i.e. 0 represents an empty space and 1 represents a particle).

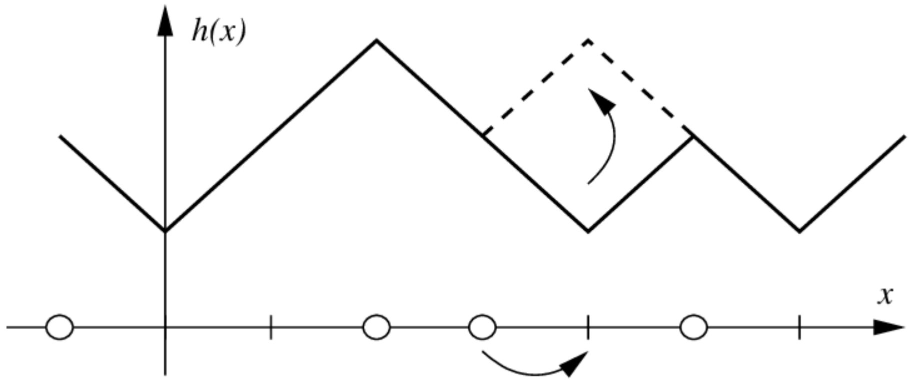

Figure 3: Height function for ASEP.

To each $\eta \in S$ which evolves as $\eta(t)$ under the ASEP dynamics, we can associate a height function h defined by

$$h(t,x) = \begin{cases} 2N(t) - \sum_{0<y\le x} \hat{\eta}(t,y), & x > 0, \\ 2N(t) \quad x = 0, \\ 2N(t) + \sum_{x<y\le 0} \hat{\eta}(t,y), & x < 0, \end{cases} \tag{1.5}$$

where $\hat{\eta}(x) = 2\eta(x) - 1$ and $N(t)$ is the number of particles which have crossed from site 0 to 1 by time t. This height function is initially only defined for $x \in \mathbb{Z}$ but we can extend to $x \in \mathbb{R}$ via linear interpolation. The dynamics of this height function are rather simple. Again each $x \in \mathbb{Z}$ gets a mean 1 exponential wait time. When it goes off, if h has a local minimum at x it flips up to $h(x) \mapsto h(x) + 2$ to become a local maximum with probability p and if h has a local maximum at x it flips up to $h(x) \mapsto h(x) - 2$ to become a local minimum with probability q. In either case the waiting time resets. An example is shown in Figure 3.

If $p = 1$ this model is the Totally Asymmetric Simple Exclusion Process (TASEP). Lets look locally at the dynamics of the height function for TASEP at some $x \in \mathbb{Z}$. In the interval $[x-1, x+1]$ the height function can look the following four ways $\vee, \wedge, /, \backslash$ and the dynamics of TASEP is approximately

$$\mathrm{d}h_t = 2\mathbf{1}_{\vee}\, \mathrm{d}X_t \tag{1.6}$$

where $X_t = \mathbf{1}_{\tau_x}$ and τ_x is the waiting time associated to the point in space x. Note we can write $\mathrm{d}X_t \approx \mathrm{d}t + \mathrm{d}M_t$ for a martingale M_t (note that $\lim_{t\to 0} \frac{\mathbf{E}[X_t]}{t} = 1$).

Now defining the forward and backwards difference operators $\nabla^+ h(x) := h(x+1) - h(x)$ and $\nabla^- h(x) := h(x) - h(x-1)$, and the discrete laplacian $\Delta := \nabla^+\nabla^-$, we can compute

$$\begin{aligned} \nabla^+ h &= \mathbf{1}_{/} + \mathbf{1}_{\vee} - \mathbf{1}_{\backslash} - \mathbf{1}_{\wedge} \\ \nabla^- h &= \mathbf{1}_{/} + \mathbf{1}_{\wedge} - \mathbf{1}_{\backslash} - \mathbf{1}_{\vee} \\ \Delta h &= 2(\mathbf{1}_{\vee} - \mathbf{1}_{\wedge}) \\ 1 &= \mathbf{1}_{/} + \mathbf{1}_{\wedge} + \mathbf{1}_{\backslash} + \mathbf{1}_{\vee} \end{aligned} \tag{1.7}$$

Therefore $-(\nabla^+ h)(\nabla^- h) = \mathbf{1}_{\vee} + \mathbf{1}_{\wedge} - \mathbf{1}_{/} - \mathbf{1}_{\backslash}$ and hence the local dynamics of TASEP look like

$$\mathrm{d}h_t = (-\frac{1}{2}(\nabla^+ h)(\nabla^- h) + \frac{1}{2} + \frac{1}{2}\Delta h) + \mathrm{d}M_t \tag{1.8}$$

So we see that TASEP resembles a discrete version of the KPZ equation. Unfortunately TASEP does not actually scale correctly to have the KPZ equation as a scaling limit, but ASEP does for $p \approx q \approx 1/2$.
The precise scaling is as follows. If $q - p = \epsilon^{1/2}$, $\nu_\epsilon = p + q - 2\sqrt{qp} = \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)$ and $\lambda_\epsilon = \frac{1}{2}\log(q/p) = \epsilon^{1/2} + \frac{1}{3}\epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2})$, then the rescaled height function

$$h_\epsilon(t,x) := \lambda_\epsilon h(\epsilon^{-2}t, \epsilon^{-1}x) + \epsilon^{-2}\nu_\epsilon t \to h(t,x) \tag{1.9}$$

as $\epsilon \to 0$ where $h(t,x)$ solves the KPZ equation. Or a bit more precisely under certain conditions on the initial data of $z_\epsilon(t,x) := h_\epsilon$, $e^{-h_\epsilon(t,x)} \to z(t,x)$ weakly where $z(t,x)$ solves the stochastic heat equation (1.10). The

proof of this is due to Bertini and Giacomin [BG97]. Their proof can roughly be divided into two steps.
(1) Show that $z_\epsilon(t,x)$ solves a discretized stochastic heat equation.
(2) Prove that the family $z_\epsilon(t,x)$ is tight in the uniform topology.

These two steps together with Prokhorov's theorem imply the result. Step (2) is very challenging!

One very important consequence of this convergence result is that we can find an invariant measure for the KPZ equation. Let TSSRW denote the law of a two sided 1D simple random walk, and TSBM the law of a two sided 1D Brownian motion. One can show that Leb $\times$ TSSRW is an invariant measure for the evolution of a height function under ASEP (this is sketched in the appendix), so we may so we may suspect that Leb $\times$ TSBM is an invariant measure of the KPZ equation. This turns out to be true, and so we may expect solutions of (1.1) to locally look like Brownian motion. This shows just how ill-posed (1.1) is because $\partial_x h$ should then locally look like white noise which has $H^{-1/2-\delta}_{\text{loc}}$ regularity for $\delta > 0$ (see the appendix), so we have little hope of making sense of $(\partial_x h)^2$.

Another important consequence of the convergence of ASEP to the KPZ equation is exact formulas! ASEP is exactly solvable through the Bethe Ansatz method. So taking limits of the formulas for the height function in ASEP, **the solution to the KPZ equation has exact formulas in terms of Fredholm determinants!** (For certain initial conditions).

1.4 Hopf-Cole Transform

Suppose for the moment that ξ was smooth nonrandom function and $z > 0$ solved the following Stochastic Heat Equation.

$$\begin{aligned}\partial_t z &= \frac{1}{2}\partial_x^2 z - \xi z \\ z(0,x) &= z_0(x)\end{aligned} \tag{1.10}$$

Then we could set $h = -\log z$ and compute

$$\partial_t h = -\frac{\partial_t z}{z}, \quad (\partial_x h)^2 = \frac{(\partial_x z)^2}{z^2}, \quad \partial_x^2 h = \frac{(\partial_x z)^2 - z\partial_x^2 z}{z^2} \tag{1.11}$$

So using (1.10) we have

$$\begin{aligned}\partial_t h + \frac{1}{2}(\partial_x h)^2 - \frac{1}{2}\partial_x^2 h &= -\frac{\partial_t z}{z} + \frac{(\partial_x z)^2}{2z^2} - \frac{(\partial_x z)^2 - z\partial_x^2 z}{2z^2} \\ &= \frac{-\partial_t z + \frac{1}{2}\partial_x^2 z}{z} = \frac{z\xi}{z} = \xi\end{aligned} \tag{1.12}$$

Which means that h satisfies the KPZ equation. So from now on we will call $h = -\log z$ a solution to the KPZ equation provided $z > 0$ and z solves (1.10).

1.5 The KPZ Fixed Point Revisited

At this point let's justify the 1:2:3 scaling for the KPZ universality class

Replace the height function $h(t,x)$ satisfying the KPZ equation with $h_T(t,x) = T^{-a}h(Tt, T^b x)$ Plugging back into the KPZ equation and noting the slightly less trivial homogeneity $\xi(Tt, T^b x) = T^{-\frac{1+b}{2}}\xi(t,x)$, we have

$$T^{a-1}\partial_t h_T = -\frac{1}{2}T^{2a-2b}(\partial_x h_T)^2 + \frac{1}{2}T^{a-2b}\partial_x^2 h_T + T^{-\frac{1+b}{2}}\xi \tag{1.13}$$

We would like the time evolution term $\partial_t h_T$ to scale the same way as the lateral growth term $(\partial_x h_T)^2$ so we take $a - 1 = 2a - 2b$.

- If we take the choice $a = b = 1$ we have

$$\partial_t h_T = (\partial_x h_T)^2 + \frac{1}{2}T^{-1}\partial_x^2 h_T + T^{-1}\xi \tag{1.14}$$

we hope as T converges to a solution of the Hamilton-Jacobi equation

$$\partial_t \mathrm{h} = -\frac{1}{2}(\partial_x \mathrm{h})^2 \tag{1.15}$$

however this equation doesn't necessarily posses classical solutions. Nevertheless if you smooth out the Hamiltion Jacobi equation a little by taking

$$\partial_t \mathrm{h} = -\frac{1}{2}(\partial_x \mathrm{h})^2 + \epsilon \partial_x^2 \mathrm{h} \tag{1.16}$$

we can apply the Hopf-Cole transformation to obtain a solution, and send $\epsilon \to 0$ we obtain a vanishing viscosity solution known given by the Hopf-Lax-Oleinik formula

$$\mathrm{h}(t,x) = \sup_y \left(\mathrm{h}(0,y) + \frac{(x-y)^2}{2t} \right) \tag{1.17}$$

This limit is of course deterministic.

- The most interesting case is when we also impose that $2a = b$. This is because we expect solutions of the KPZ equation to look locally like a brownian motion (in the space variable) since two sided Brownian motion is an invariant measure for the equation. Thus the scaling for the height function should obey $2a = b$ like for a Brownain motion. In this case $a = 1/3$ and $b = 2/3$, the KPZ 1:2:3 scaling! And h_T satisfies

$$\partial_t h_T = (\partial_x h_T)^2 + \frac{1}{2}T^{-1/3}\partial_x^2 h_T + T^{-1/6}\xi \tag{1.18}$$

Notice in this case the white noise dominates the smoothing term so it seems reasonable that we will not get the same vanishing viscosity solution (1.17). Nevertheless we can still describe the KPZ fixed point by a variational formula which resembles (1.17),

$$\mathrm{h}_{\mathrm{FP}}(t,x) = \mathrm{h}(t,x) = \sup_y \left(\mathrm{h}(0,y) + \frac{(x-y)^2}{2t} + \mathcal{A}\left(\frac{x}{t^{2/3}}, \frac{y}{t^{2/3}}\right)\right) \tag{1.19}$$

where $\mathcal{A}(x,y)$ is the Airy sheet which is a random field which is a scaled Airy 2 process in one variable whenever the other is fixed.

1.6 Solving ASEP

For doing explicit computation it is more informative to change our state space for ASEP to $E_n = \{(x_1, ..., x_n) \in \mathbb{Z}^n : x_1 < x_2 < ... < x_n\}$ where the x_i represent the order of the particles. The power of such a state space is that it keeps track of the fact that particles cannot change order. For simplicity let $p = 1, q = 0$ for now so that we're studying TASEP. In this setting, again letting P_t be the Markov semi-group for TASEP on E_n, we can let it act on the Banach space of bounded functions on $\mathbb{Z}^n$, $C_0(\mathbb{Z}^n)$. And in this setting, by a similar computation to (3.1), the generator L has domain $D(L) := \{f \in C_0(\mathbb{Z}^n) : \lim_{t\downarrow 0} \frac{P_t f - f}{t} \text{ exists}\} = C_0(\mathbb{Z}^n)$ and it acts by

$$Lf(x) = \sum_{i=1}^n \mathbf{1}_{X_i \in E_n} f(X_i) - f(X) \tag{1.20}$$

where for $X = (x_1, ...x_i, .., x_n) \in \mathbb{Z}^n$, $X_i = (x_1, ..., x_i + 1, ..., x_n)$. The Kolmogorov Backward and Forward Equation still hold in this setting. Moreover suppose $X, Y \in E_N$. Then setting $\mu_0 = \delta_Y$, $f = \mathbf{1}_X$ in the forward equation, we have

$$\frac{d}{dt}p(X,Y,t) = L^* p(X,Y,t) \tag{1.21}$$

Additionally a simple summation by parts yields

$$\begin{aligned} \int_{E_n} Lf \, \mathrm{d}\mu &= \sum_{X\in E_n}\sum_{i=1}^n \mathbf{1}_{X_i \in E_n}(f(X_i) - f(X))\mu(X) \\ &= \sum_{X\in E_n}\sum_{i=1}^n \mathbf{1}_{X^i \in E_n} f(X)(\mu(X^i) - \mu(X)) \end{aligned} \tag{1.22}$$

where for $X = (x_1, ...x_i, .., x_n) \in \mathbb{Z}^n$, $X_i = (x_1, ..., x_i - 1, ..., x_n)$. So we have the formula for L^*,

$$L^*\mu(X) = \sum_{i=1}^n \mathbf{1}_{X^i \in E_n}(\mu(X^i) - \mu(X)) \tag{1.23}$$

Finally plugging into (1.21) we have

$$\frac{d}{dt}p(X,Y,t) = \sum_{i=1}^{n} \mathbf{1}_{X^i \in E_n}(p(X^i,Y,t) - P(X,Y,t)) \tag{1.24}$$

By "solving TASEP" we hope, for now, hope to solve for p(X,Y,t) in the equation above. The strategy of attack will be what's known as the *Bethe Ansatz.* This involves three steps:
1. Ignore the "boundary condition" and solve the free equation

$$\frac{d}{dt}p(X,Y,t) = \sum_{i=1}^{n}(p(X^i,Y,t) - P(X,Y,t)) \tag{1.25}$$

2. Solve the equation with the Neumann type boundary condition of 0 difference at the boundary.

3. Modify the formula in the last part to satisfy (1.24)

The first step is actually quite simple. Fix Y and suppress it from our notation. We look for solutions of the form $p(X,t) = \prod_{i=1}^{n} p_i(x_i - y_i, t)$. Then (1.25) reduces to

$$\begin{cases} \frac{d}{dt}p_i(x,t) = p_i(x+1,t) - p_i(x,t) \\ p(x,0) = \mathbf{1}_{x=0} \end{cases} \tag{1.26}$$

We solve this via generating functions i.e. letting $\phi(z,t) = \sum_{x=-\infty}^{\infty} p_i(x,t)z^x$, (1.26) reduces to

$$\frac{d\phi}{dt}(z,t) = (z-1)\phi(z,t), \quad \phi(z,0) = 1 \tag{1.27}$$

which has the unique solution $\phi(z,t) = e^{t(z-1)}$ and thus by the residue theorem

$$p_i(x,t) = \oint_{|z|=1} z^{-x-1}e^{t(z-1)}\,\mathrm{d}z \tag{1.28}$$

completing step (1).

Steps (1) and (2) are a bit trickier, but the key two satisfying the condition in part (2) is more or less to take a determinant with entries which are solutions to part (1), a formula due to Karlin and McGregor.

At the end of all of these steps we obtain Schütz's formula:

Theorem 1.1. *[Sch97] For $X,Y \in E_n$, and $p(X,Y,t)$ the transition probability for TASEP, we have*

$$(2\pi i)^{-n}\oint_{\gamma_1}\ldots\oint_{\gamma_n}\det[z_j^{-x_i+y_j-1}(1-z_j^{-1})^{j-i}]_{i,j=1}^{n}e^{t\sum_{j=1}^{n}(z_j-1)}\prod_{j=1}^{n}\mathrm{d}z_j \tag{1.29}$$

where we can just take each γ_i to be a counterclockwise circle with radius large than 1.

Remark 1.2. *Three remarks are in order:*
(1) Although (1.29) is a bit tricky to derive, it is actually quite straight forward to prove by using elementary properties of the determinant after plugging into the Kolmogorov equation.
(2) A similar, but slightly more complicated formula exists for ASEP.
(3) Although remarkable, this formula is a disaster for finding large n asymptotics for the height function. There are a series of works which lead to more managable formulas, see for instance [BFPS07] and [MQR21].

1.7 KPZ Fixed Point Revisited II

Now that we know TASEP is solvable we can access formulas for the KPZ fixed point by taking the appropriate scaling limits. Here are what these scaling limits are in the context of the KPZ equation (I believe this has only been confirmed for the KPZ equation recently by at least one group, for instance by Quastel and Sarkar's universality result [QS22].

(1) In the case of wedge/step initial data (in the SHE this corresponds to $z(0,x) = \delta_0$),

$$-2^{1/3}t^{-1/3}(h(t,2^{1/3}t^{2/3}x) - 2^{-1/3}t^{1/3}x^2 - \frac{t}{24} - \log\sqrt{2\pi t}) \to \mathcal{A}_2(x) \tag{1.30}$$

where $\mathcal{A}_2(x)$ is the Airy 2 process which models the fluctuations in the largest eigenvalues of a matrix evolving according to the GUE Dyson Brownian motion; i.e. if $A(0)$ starts off GUE and evolves according to the following Ornstein-Uhlenbeck process given by

$$\mathrm{d}A(x) = -\frac{1}{2N}A(x)\,\mathrm{d}x + \mathrm{d}B(x) \tag{1.31}$$

with $B_{i,i}(x)$ $\Re B_{i,j}(x), \Im B_{i,j}(x)$ $(i<j)$ all real Brownian motions, the first with diffusion coefficient 1 and the others with 1/2. Then the largest eigenvalue of $\lambda_N^{\max}(x) \sim 2N + N^{1/3}A_2(x)$.

(2) In the case of flat initial data (in the SHE this corresponds to $z(0,x)=1$),

$$-2^{1/3}t^{-1/3}(h(t,2^{1/3}t^{2/3}x) - 2^{-1/3}t^{1/3}x^2 - \frac{t}{24}) \to \mathcal{A}_1(x) \tag{1.32}$$

where $\mathcal{A}_1(x)$ is the Airy 1 process. The Airy 1 process has one point distribution function given by the GOE Tracy-Widom distribution, but it is **not** the limit of a largest eigenvalue process on GOE matrices.

(3) In the case of equilibrium initial data (in the SHE this corresponds to $z(0,x)=e^{B(x)}$ where $B(x)$ is a two sided Brownian motion),

$$-2^{1/3}t^{-1/3}(h(t,2^{1/3}t^{2/3}x) - 2^{-1/3}t^{1/3}x^2 - \frac{t}{24}) \to \mathcal{A}_{\mathrm{stat}}(x) \tag{1.33}$$

where $\mathcal{A}_{\mathrm{stat}}(x)$ is the Airy stationary process. The Airy stationary process is simply a height shifted two sided Brownian motion, however the height shift is dependent on the two sided Brownian motion and highly non-trivial (i.e. the marginals involve Fredholm determinants with kernels involving Airy functions as for $\mathcal{A}_1(x)$ and $\mathcal{A}_2(x)$).

There are more initial data which are mixes of the first three which also lead to other Airy processes.

For quite a large class of initial data, there are more general Fredholm determinant formulas [MQR21]. Recently there are also multi-time formulas which compute $\mathbf{P}(\mathrm{h}_{\mathrm{FP}}(t_1,x_1) \le h_1, ..., \mathrm{h}_{\mathrm{FP}}(t_n,x_n) \le h_n)$ for flat and wedge initial data by Zhipeng Liu [Liu22]. Certain asymptotics for these formulas were investigated by Ruxian Zhang and myself [NZ22], and by Zhipeng Liu and Yizao Wang [LW22].

2 Analysis of Stochastic Heat Equation

Suppose we want to solve the SHE

$$\begin{aligned} \partial_t z &= \frac{1}{2}\partial_x^2 z - \xi z \\ z(0,x) &= z_0(x) \end{aligned} \tag{2.1}$$

and assume for the moment replace $-z\xi$ with a deterministic $f \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ so that we instead try to solve the deterministic heat equation.

$$\begin{aligned} \partial_t z &= \frac{1}{2}\partial_x^2 z + f \\ z(0,x) &= z_0(x) \end{aligned} \tag{2.2}$$

Well then we can first solve the homogeneous problem (for instance via Fourier Analysis)

$$\begin{aligned} \partial_t z_1 &= \frac{1}{2}\partial_x^2 z_1 \\ z_1(0,x) &= z_0(x) \end{aligned} \tag{2.3}$$

to obtain $z_1(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)\,\mathrm{d}y$ where $p(t,x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ is the Heat kernel.

Then we solve the inhomogeneous part with 0 initial data:

$$\begin{aligned} \partial_t z_2 &= \frac{1}{2}\partial_x^2 z_2 + f \\ z_2(0,x) &= 0 \end{aligned} \tag{2.4}$$

If P^t is the generator for the homogeneous problem, i.e. $P^t z_0 = z_1$. Then using Duhamel's principle we know that $z_2(t,x) = \int_0^t P^{t-s}f(s,x)\,\mathrm{d}s = \int_0^t \int_{\mathbb{R}} p(t-s,x-y)f(s,y)\,\mathrm{d}y\,\mathrm{d}s$ (this is easy to prove).

So finally we obtain

$$
\begin{aligned}
z(t,x) &= z_1(t,x) + z_2(t,x) \\
&= \int_{\mathbb{R}} p(t,x-y) z_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} p(t-s,x-y) f(s,y)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned} \tag{2.5}
$$

and so finally assuming everything is Kosher we plug $f = -z\xi$ back in to obtain the integral equation

$$
z(t,x) = \int_{\mathbb{R}} p(t,x-y) z_0(y)\,\mathrm{d}y - \int_0^t \int_{\mathbb{R}} p(t-s,x-y) z(s,y)\xi(s,y)\,\mathrm{d}y\,\mathrm{d}s \tag{2.6}
$$

From now on we say that z is *Mild Solution* of (2.1) if z satisfies (2.6). At this point we still need to define ξ and make sense of the integrals involving white noise.

2.1 White Noise

For the moment let's define white noise on any open subset $U \subset \mathbb{R}^d$.

Take an orthonormal basis of $L^2(U;\mathbb{R})$. Let $\{Z_k\}$ be a sequence of i.i.d. standard Gaussians defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. We would like to define white noise by

$$
\xi = \sum_{k=1}^{\infty} Z_k f_k \tag{2.7}
$$

but this typically will not make sense as a function. The way around this is to think of ξ as a distribution.

For $\varphi \in C_c^\infty\infty(U)$ we can define

$$
(\xi, \varphi) := \sum_{k=1}^{\infty} Z_k (f_k, \varphi)_{L^2} \tag{2.8}
$$

whenever the sum above converges.
In the Appendix I will show that $\xi \in H_{\mathrm{loc}}^{-d/2-\delta}(U)$ a.s. however I will not need this going forward.

For any fixed $\varphi \in L^2(U)$ and $m < n$ we can directly compute $\mathbf{E}\left[\left(\sum_{k=m+1}^n Z_k (f_k,\varphi)_{L^2}\right)^2\right] = \sum_{k=m+1}^n (f_k,\varphi)_{L^2(U)} \to 0$ as $m,n \to \infty$ since $\varphi \in L^2(U)$. Therefore (2.8) is a convergent series in $L^2(\Omega)$, so (2.8) makes sense for $\varphi \in L^2(U)$. This is surprising given the regularity of ξ!

Moreover for $\varphi_1, \varphi_2 \in L^2(U)$ we can compute

$$
\begin{aligned}
\mathbf{E}[(\xi,\varphi_1)(\xi,\varphi_2)] &= \lim_{n\to\infty} \mathbf{E}\left[\left(\sum_{k=1}^n Z_k (f_k,\varphi_1)_{L^2}\right)\left(\sum_{k=1}^n Z_k (f_k,\varphi_2)_{L^2}\right)\right] \\
&= \sum_{k=1}^{\infty} (f_k,\varphi_1)(f_k,\varphi_2) = (\varphi_1,\varphi_2)_{L^2(U)}
\end{aligned} \tag{2.9}
$$

In particular if $\mathrm{supp}(\varphi_1) \cap \mathrm{supp}(\varphi_2) = \emptyset$, then $\mathbf{E}[(\xi,\varphi_1)(\xi,\varphi_2)] = 0$.

If we take $||\varphi_1||_{L^2(U)} = ||\varphi_2||_{L^2(U)}$ and let $\varphi_1^{(\epsilon)}(x) = \epsilon^{-d/2}\varphi_1(\epsilon^{-1}x)$, $\varphi_2^{(\epsilon)}(x) = \epsilon^{-d/2}\varphi_2(\epsilon^{-1}x)$, then $\mathbf{E}[(\xi,\varphi_1^{(\epsilon)}(x-a)),(\xi,\varphi_1^{(\epsilon)}(x-b))] \to \delta(a-b)$ giving justification to the statement

$$
\mathbf{E}[\xi(a)\xi(b)] = \delta(a-b) \tag{2.10}
$$

2.2 Integration With Respect to space-time White Noise

For this section fix $U = \mathbb{R}_+ \times \mathbb{R}$.

We can simply define for deterministic $\phi \in L^2(\mathbb{R})$,

$$
\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t := (\xi, f) \tag{2.11}
$$

To extend this definition to random functions, we first define a filtration

$$
\mathscr{F}_t = \sigma(\{\int_{\mathbb{R}_+\times\mathbb{R}} \phi(x)\mathbf{1}_{(0,s]}(w)\xi(w,x)\,\mathrm{d}x\,\mathrm{d}w : \phi \in L^2(\mathbb{R}), s \le t\}) \tag{2.12}
$$

and isolate a class of simple functions

$$\mathcal{S} = \{\sum_{i=1}^{n} X_i(\omega)\phi_i(x)\mathbf{1}_{(a_i,b_i]}(t) : \phi_i \in L^2(\mathbb{R}), a_1 < b_1 < a_2 < ... < a_n < b_n, \text{ and } X_i \in \mathscr{F}_{a_i}\} \tag{2.13}$$

Then we can naturally define the integral on $\mathcal{S}$ as

$$\int_{\mathbb{R}_+\times\mathbb{R}} \sum_{i=1}^{n} X_i\phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t := \sum_{i=1}^{n} X_i \int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{2.14}$$

This integral is easily checked to be linear.
The key to extending the integral to a larger class of integrands is the following isometry property.

Lemma 2.1. *For any $f \in \mathcal{S}$,*

$$\boldsymbol{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} f(\omega,t,x)\xi(\omega,t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^2\right] = \boldsymbol{E}\left[||f||^2_{L^2(\mathbb{R}_+\times\mathbb{R})}\right] \tag{2.15}$$

Proof. Write $f(\omega,t,x) = \sum_{i=1}^{n} X_i\phi_i(x)\mathbf{1}_{(a_i,b_i]}$, then

Our space of integrands we use is $L^2(\Omega\times\mathbb{R}_+\times\mathbb{R}, \sigma(\mathcal{S})\times\mathscr{B}(\mathbb{R}_+\times\mathbb{R}))$. Going forward I will simply write $L^2(\Omega\times\mathbb{R}_+\times\mathbb{R})$.

$$\left(\int_{\mathbb{R}_+\times\mathbb{R}} f(\omega,t,x)\xi(\omega,t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^2 = \sum_{i,j=1}^{n} X_iX_j\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_j(x)\mathbf{1}_{(a_j,b_j]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right) \tag{2.16}$$

Taking the expectation of one of the terms on the RHS, if $i < j$, we use the fact that $\{\mathscr{F}_t\}$ is a filtration and properties of conditional expectation to deduce

$$\begin{aligned}
&\mathbf{E}\left[X_iX_j\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_j(x)\mathbf{1}_{(a_j,b_j]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)\right]\\
&= \mathbf{E}\left[X_iX_j\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)\mathbf{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_j(x)\mathbf{1}_{(a_j,b_j]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)|\mathscr{F}_{a_j}\right]\right]\\
&= \mathbf{E}\left[X_iX_j\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)\mathbf{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_j(x)\mathbf{1}_{(a_j,b_j]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)\right]\right] = 0
\end{aligned} \tag{2.17}$$

Of course the terms when $i > j$ are 0 by symmetry. If $i = j$ then we have

$$\begin{aligned}
&\mathbf{E}\left[X_i^2\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^2\right]\\
&= \mathbf{E}\left[X_i^2\mathbf{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^2|\mathscr{F}_{a_i}\right]\right]\\
&= \mathbf{E}\left[X_i^2\mathbf{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} \phi_i(x)\mathbf{1}_{(a_i,b_i]}\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^2\right]\right]\\
&= \mathbf{E}[X_i^2||\phi_i(x)\mathbf{1}_{(a_i,b_i]}||^2_{L^2(\mathbb{R}_+\times\mathbb{R})}]
\end{aligned} \tag{2.18}$$

So

$$\begin{aligned}
&\mathbf{E}\left[\left(\int_{\mathbb{R}_+\times\mathbb{R}} f(\omega,t,x)\xi(\omega,t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^2\right] = \sum_{i=1}^{n}\mathbf{E}[X_i^2||\phi_i(x)\mathbf{1}_{(a_i,b_i]}||^2_{L^2(\mathbb{R}_+\times\mathbb{R})}]\\
&= \sum_{i,j=1}^{n}\mathbf{E}[X_iX_j(\phi_i(x)\mathbf{1}_{(a_i,b_i]}, \phi_j(x)\mathbf{1}_{(a_j,b_j]})_{L^2(\mathbb{R}_+\times\mathbb{R})}] = \mathbf{E}[||f||_{L^2(\mathbb{R}_+\times\mathbb{R})}]
\end{aligned} \tag{2.19}$$

□

Lemma 2.2. *$\mathcal{S}$ is dense in $L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R})$.*

Proof. See Quastel's notes [Qua11]. □

Together with the isometry property and the density of $\mathcal{S}$ in $L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ we can naturally extend the integral so that $\int_{\mathbb{R}\times\mathbb{R}_+} f(\omega,t,x)\xi(\omega,t,x)\,\mathrm{d}x\,\mathrm{d}t \in L^2(\Omega)$ makes sense for all $f \in L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R})$.

2.3 Solving the Stochastic Heat Equation

We would like to solve the Stochastic Heat Equation (1.10). What we really hope to find is a mild solution solving (2.6). Through the last section we now have a mathematically well defined problem.

The first approach one may take is to apply a Picard Iteration to (2.6) formally obtaining the infinite series solution

$$z(t,x) = \int_{\mathbb{R}} p(t,x-y)z_0(y)\,\mathrm{d}y + \sum_{n=1}^{\infty}\int_{0\le s_1\le\ldots\le s_n\le t}\int_{\mathbb{R}^{n+1}} p(t-s_n,x-y_n)p(s_n-s_{n-1},y_n-y_{n-1})\ldots p(s_1,y_1-y_0)z_0(y_0)\xi(s_n,y_n)\ldots\xi(s_1,y_1)\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{s} \tag{2.20}$$

The iterated integrals can be defined inductively since at each stage we have a progressively measurable L^2 stochastic process. The representation (2.20) is known as a **Weiner Chaos Expansion**.

We would like to verify that (2.20) converges in some space and actually solves (2.6). If we assume $\sup_{x\in\mathbb{R}} \mathbf{E}[|z_0(x)|^2] < \infty$, then it is quite easy to prove that for each t, x the series (2.20) converges to a solution of (2.6) in $L^2(\Omega)$. This proof is given in [Qua11].

The issue is that the condition $\sup_{x\in\mathbb{R}} \mathbf{E}[|z_0(x)|^2] < \infty$ is too restrictive. In particular it isn't satisfied by the two initial conditions we care most about, $z_0 = \delta_0$ and $z_0(x) = e^{-B(x)}$ for a two sided Brownian motion $B(x)$. In [Qua11], Jeremy Quastel claims to resolve the issue with slightly stronger estimates while imposing the initial condition that z_0 is a random signed measure with $\mathbf{E}\left[\sup_{x\in[-n,n]} z_0\right] < ce^{cn}$ for all n and one fixed constant c (this condition can be checked easily for $z_0 = \delta_0$ and $z_0(x) = e^{-B(x)}$). Unfortunately his proof doesn't seem to be correct! He claims that $\limsup_{s\to 0} s^{1/2}\mathbf{E}\left[\left(\int_{\mathbb{R}} p(s,x-y)z_0(y)\,\mathrm{d}y\right)\right] < \infty$ for all x, but this is clearly not true taking $x = 0$ and $z_0 = \delta_0$.

Tracking down the original reference [BC95], Bertini and Cancrini take a different approach. He assumes the initial data is deterministic (this is not restrictive as we could always condition both sides of (2.6) with respect to $\sigma(z_0)$), and that $\limsup_{s\to 0} s^{1/2}\int_{\mathbb{R}} p(s,x-y)z_0(y)\,\mathrm{d}y < \infty$ which is easy to verify for instance for $z_0 = \delta_0$. The approach is based on the Feynman-Kac formula which could be used to solve the deterministic equation

$$\partial_t z = \frac{1}{2}\partial_x^2 z - f(t,x)z \tag{2.21}$$

where f isn't random. The formula breaks down for random rough f such as white noise. So they smooth out the white noise with a mollifier. Then they are able to solve the corresponding smoothed out mild stochastic heat equation with a renormalized Feynman-Kac solution. The hardest part of the analysis is then showing that we can obtain an L^2 limit of these solutions to the smoothed out equation which actually solves (2.6). This analysis is all carried out in [BC95], but it is significantly harder than what Quastel attempts to do in [Qua11].

Going back to the chaos expansion (2.20), one may expect to recover a chaos expansion for the solution to the KPZ equation $h(t,x)$. Unfortunately, this is not the case; there is no known chaos expansion for $h(t,x)$.

2.4 Further Properties of Solutions to the Stochastic Heat Equation

There are regularity results for the solution to the stochastic heat equation (1.10).

Theorem 2.3.

One question of fundamental importance is whether starting with initial data $z_0(x)$ satisfying $z_0(x) \ge 0$ and $\int_{\mathbb{R}} z_0(x)\,\mathrm{d}z > 0$ guarantees that $z(t,x) > 0$ for $t > 0$. This is important because we would like our Hopf-Cole solution to the KPZ equation $h(t,x) := -\log z(t,x)$ to be well defined. It turns out to be true [Mue91].

Theorem 2.4. *Let $z(t,x)$ solve (2.6) with initial data $z_0(x)$ satisfying $z_0(x) \geq 0$ and $\int_{\mathbb{R}} z_0(x)\,\mathrm{d}z > 0$. Then for all $t > 0$, $z(t,x) > 0$ for all $x \in \mathbb{R}$ with probability* 1 **??**

One of the key steps in the proof is the following comparison principle
If z_1 solves

$$\partial_t z_1 = \frac{1}{2}\partial_x^2 z_1 + f_1(z_1) + \sigma(z_1)\xi \tag{2.22}$$

and z_2 solves

$$\partial_t z_2 = \frac{1}{2}\partial_x^2 z_2 + f_2(z_2) + \sigma(z_2)\xi \tag{2.23}$$

Lemma 2.5. *Suppose σ, f_1, f_2 are Lipschitz functions with $f_1(z) \leq f_2(z)$. Further suppose z_1 solves (2.22) and z_2 solves (2.23) with the same white noise and initial data satisfying*

$$z_1(0,x) \leq z_2(0,x), \quad x \in \mathbb{R} \tag{2.24}$$

then with probability 1*, for all $t \geq 0$,*

$$z_1(t,x) \leq z_2(t,x), \quad x \in \mathbb{R} \tag{2.25}$$

The proof of the comparison lemma involves going to a discretization of the stochastic heat equation and taking the limit to the continuum. The process of taking the limit requires a few pages of analysis. The proof that the comparison result holds in the discretization is "almost obvious" according to Jeremy Quastel [Qua11]. After looking at the original paper [Mue91], I would not say it's almost obvious. The proof relies on a more elementary comparison theorem for Ito processes.

Unfortunately I didn't have time for the next two sections.

2.5 Continuum Directed Polymer

2.6 Directed Polymer in Random Environment

3 Appendix A: ASEP as a Markov Process

Recall that ASEP is a continuous time Markov Process on the state space $S = \{0,1\}^{\mathbb{Z}}$. For a *local function* f, (i.e. a function which only depends on finitely many coordinates $x_1, ..., x_n$), we can compute the action of the generator, L, on f

$$\begin{aligned}
Lf(\eta) &:= \lim_{t \downarrow t} \frac{\mathbf{E}[f(\eta_t)|\eta_0 = \eta] - f(\eta)}{t} \\
&= \lim_{t\downarrow 0} \frac{e^{-nt}f(\eta) + (1-e^{-t})e^{-(n-1)t}\sum_{x\in\mathbb{Z}}[pf(\eta^{x,x+1}) + qf(\eta^{x,x+1})] + O(t^2) - f(\eta)}{t} \\
&= \sum_{x\in\mathbb{Z}}[pf(\eta^{x,x+1}) + qf(\eta^{x,x+1})] - nf(\eta) \\
&= \sum_{x\in\mathbb{Z}}[(p\eta(x)(1-\eta(x+1)) + q\eta(x+1)(1-\eta(x)))(f(\eta^{x,x+1}) - f(\eta))]
\end{aligned} \tag{3.1}$$

Where $\eta^{x,x+1}$ is the particle configuration with coordinates x and $x+1$ switched.

I will now briefly review some theory of continuous Markov processes. The Markov semigroup P_t associated to this process acts on the Banach space of bounded functions $C_0(S)$ with sup norm $||f|| = \sup_{\eta\in S}|f(\eta)|$ by

$$P_t f(\eta) = \mathbf{E}[f(\eta_t|\eta_0 = \eta)] \tag{3.2}$$

On this domain L is an unbounded operator so we must specify its domain. We take the domain of the generator L to be exactly

$$\mathcal{D} := \{f \in C_0(S) : Lf := \lim_{t\downarrow 0} \frac{P_t f - f}{t} \text{ exists}\} \tag{3.3}$$

P_t obeys the Markov semigroup property $P_{t+s} = P_t P_s = P_s P_t$ (which comes from the properties of conditional expectation), so for any $f \in \mathcal{D}$,

$$\frac{P_{t+h}f - P_t f}{h} = P_t \frac{P_h f - f}{h} \to P_t L f \tag{3.4}$$

and this implies that $P_t f \in \mathcal{D}$ with

$$\frac{P_{t+h} f - P_t f}{h} = \frac{P_h P_t f - P_t f}{h} \to L P_t f \tag{3.5}$$

Thus we get the differential equation

$$\frac{d}{dt} P_t f = P_t L f = L P_t f \tag{3.6}$$

(3.6) is often known as the Kolmogorov Backward Equation.

If we take an initial measure on S, μ_0 and let $\mu_t = (P_t)^* \mu_0$ be the evolution of μ_0 under the dynamics our Markov Process, then for $f \in C_0(S)$, by definition

$$\begin{aligned} \int_S f \, \mathrm{d}\mu_t &= \mathbf{E}^{\mu_0}[f(\eta(t))] \\ &= \int_S \mathbf{E}^{\eta_0}[f(\eta(t))] \mu_0(\mathrm{d}\eta) = \int_S P_t f(\eta) \mu_0(\mathrm{d}\eta) \end{aligned} \tag{3.7}$$

Now suppose $f \in D(L)$. Differentiating the equation above and applying the bounded convergence theorem we have

$$\frac{d}{dt} \int_S f \, \mathrm{d}\mu_t = \int_S P_t L f(\eta) \mu_0(\mathrm{d}\eta) = \int_S L f(\eta) \mu_t(\mathrm{d}\eta) = \int_S f(\eta) \, \mathrm{d}L^* \mu_t \tag{3.8}$$

For short we can write

$$\frac{d}{dt} \mu_t = L^* \mu_t \tag{3.9}$$

(3.9) is often known as the Kolmogorov Forward Equation.

Definition 3.1. *A measure is **invariant** under a Markov process on a state space S with semigroup operator P_t if $\int_S P_t f \, \mathrm{d}\mu \int f \, \mathrm{d}\mu$ for all $f \in C_0(S)$*

If we're a bit more careful about the calculation (3.1) making sure we get uniform bounds over all $\eta \in S$, we can see that the set of local functions, $\mathcal{D}_0$ is a subset of the domain $\mathcal{D}$.

There is a certain amount of functional analysis nonsense one needs to go through in order to show that for any $\rho \in [0,1]$ the Bernoulli product measure π_ρ with $\mathbf{P}(\eta(x) = 1) = \rho$, $\mathbf{P}(\eta(x) = 0) = 1 - \rho$, $x \in \mathbb{Z}$, is invariant for ASEP. The following can all be found in [Lig85].

Recall that for an unbounded operator $L : D(L) \to \mathcal{B}$, $D(L) \subset \mathcal{B}$, L is *closed* if the graph of L, $\Gamma(L) := \{(x, Lx) \in \mathcal{B} \times \mathcal{B} : x \in D(L)\}$ is a closed subset of $\mathcal{B} \times \mathcal{B}$. An *extension* of L is an operator $\tilde{L} : D(\tilde{L}) \to \mathcal{B}$ such that $D(L) \subset D(\tilde{L})$ and $L \equiv \tilde{L}$ on $D(L)$. L is *closable* if it has a closed extension. If L is closable, the *closure* of L is the closed extension with the smallest domain.

Definition 3.2. *A **core** for a Markov Process with generator L on a domain $\mathcal{D}$ is a subspace $\mathcal{D}_0 \subset \mathcal{D}$ such that L is the closure of its restriction to $\mathcal{D}_0$*

We need the following two lemmas proven in [Lig85].

Lemma 3.3. *The set of local functions D_0 forms a core for L.*

Lemma 3.4. *A measure μ is invariant for a Markov Process L iff $\int_S f \, \mathrm{d}\mu = 0$ for all $f \in \mathcal{D}_0$ (where $\mathcal{D}_0$ is a core for L).*

The only if direction in the last lemma is easy to prove by differentiating under the integral sign (justified by the bounded convergence theorem) and substituting in the backward Kolmogorov equation. The other direction relies on the Hille-Yosida theorem in functional analysis which gives an explicit correspondence between Markov Processes and Markov generators.

Using these two lemmas it is quite straightforward to check that

Theorem 3.5. *For any $\rho \in [0,1]$ the Bernoulli product measure π_ρ on $\{0,1\}^{\mathbb{Z}}$ with $\pi_\rho(\eta(x) = 1) = \rho$ and $\pi_\rho(\eta(x) = 0) = 1 - \rho$ is invariant for ASEP*

Proof. A proof is given in [Qua11] □

For $\rho = 1/2$ in the language of height functions this says that up to the height at 0, the distribution of height functions given by a two sided random walk TSRW is invariant. Or more precisely $\mathrm{Leb}_{\mathbb{Z}} \times \mathrm{TSRW}$ is invariant for ASEP where the first coordinate just determines the height at 0.

4 Appendix B: Regularity of White Noise

4.1 Preliminaries

Recall the following definitions

Definition 4.1. *The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is given by*

$$\hat{f}(\lambda) := \int_{\mathbb{R}^d} e^{-2\pi \mathrm{i}\lambda\cdot x} f(x)\,\mathrm{d}x \tag{4.1}$$

Remark 4.2. *(i) $f \in L^1(\mathbb{R}^d)$ implies that $f \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $||\hat{f}||_{L^\infty} \leq ||f||_{L^1}$.*
(ii) The Fourier transform $\mathscr{F}$ extends to a bounded operator $\mathscr{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ which is an isometry, i.e. $||\hat{f}||_{L^2} = ||f||_{L^2}$

By differentiation under the integral sign and integration by parts respectively,

$$\begin{aligned} \partial^\alpha \hat{f}(\lambda) &= (-2\pi \mathrm{i}\lambda)^\alpha \hat{f}(\lambda) \\ \widehat{\partial^\alpha f}(\lambda) &= (2\pi \mathrm{i}\lambda)^\alpha \hat{f}(\lambda) \end{aligned} \tag{4.2}$$

holds for any multi-index α.

Definition 4.3. *The Schwartz Space $\mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) : ||x^\alpha \partial^\beta f||_{L^\infty(\mathbb{R}^d)} < \infty\}$*

By (4.2) the Fourier transform maps $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

$\mathcal{S}(\mathbb{R}^d)$ has semi-norms $||f||_{m,n} := \sup_{|\alpha|\leq m, |\beta|\leq n} ||x^\alpha \partial^\beta f||_{L^\infty}$. Then enumerating the seminorms $||\cdot||_1, ||\cdot||_2, ||\cdot||_3, ...$ we can turn $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space with metric given by

$$\mathrm{d}(f,g) := \sum_{n=1}^\infty 2^{-n} \frac{||f-g||_n}{1+||f-g||_n} \tag{4.3}$$

Definition 4.4. *The space of Tempered Distributions $\mathcal{S}'(\mathbb{R}^d)$ is the dual space of $\mathcal{S}(\mathbb{R}^d)$, (i.e. $\mathcal{S}'(\mathbb{R}^d) := \{u : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} | u$ is linear and continuous$\}$*

Definition 4.5. *For $u \in \mathcal{S}'(\mathbb{R}^d)$ we can define the Fourier transform $\hat{u}$ by $(\hat{u}, f) := (u, \hat{f})$ for $f \in \mathcal{S}(\mathbb{R}^d)$.*

Notation: The Japanese bracket $\langle\lambda\rangle := (1+|\lambda|^2)^{1/2}$
For any $u \in L^2(\mathbb{R}^d)$ and any function $|s| < \infty$, we can define the linear function $\langle x\rangle^n u : f \mapsto (\langle x\rangle^n u(x), f(x))_{L^2}$. It is not hard to check that $\langle x\rangle^n u \in \mathcal{S}'(\mathbb{R}^d)$.

Definition 4.6. *For any $s \in \mathbb{R}$, we can define the Sobolev space $H^s(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d) : \langle\lambda\rangle^s u \in L^2(\mathbb{R}^d)\}$*

Definition 4.7. *For any $s \in \mathbb{R}$ and open subset $U \subset \mathbb{R}^d$, we can define $H^s_{\mathrm{loc}}(U) := \{u \in \mathcal{D}'(U) : \chi u \in H^s(\mathbb{R}^d)$ for all $\chi \in C_c^\infty(U)\}$*

It is important to note that $H^s(\mathbb{R}^d)$ is a Banach space with norm $||u||_{H^s(\mathbb{R}^d)} = ||\langle x\rangle^s u(x)||_{L^2(\mathbb{R}^d)}$. Moreover we can turn $H^s_{\mathrm{loc}}(U)$ into a complete metric space. Take a sequence of compact sets K_i which exhaust U, and let $\chi_i \in C_c^\infty(U)$ with $\chi_i = 1$ on K_i. Then $||u||_i = ||u_i||$ define a sequence of seminorms, and we can turn $H^s_{\mathrm{loc}}(U)$ into a complete metric space with metric

$$\mathrm{d}(u,v) := \sum_{n=1}^\infty 2^{-n} \frac{||u-v||_n}{1+||u-v||_n} \tag{4.4}$$

4.2 Proof

The goal of this subsection is to prove

Theorem 4.8. *$\xi := \sum_{k=1}^\infty Z_k f_k$ almost surely converges in $H^{-d/2-\delta}_{\mathrm{loc}}(U)$ for all $\delta > 0$*

Define $\xi_n = \sum_{k=1}^n Z_k f_k \in L^2(U)$ a.s.
It suffices to prove that for a fixed $\chi \in C_c^\infty(U)$ and fixed $\delta > 0$, $\chi\xi_n$ a.s. converges to a limit in $H^{-d/2-\delta}(U)$ since we only need to check this for the countable family $\{\chi_i\}_{i=1}^\infty$ and $\delta \in \{1/n\}_{n=1}^\infty$.

And at this point the goal is equivalent to showing that a.s. ξ_n is Cauchy in $H^{-d/2-\delta}(\mathbb{R}^d)$ which occurs iff $\langle\lambda\rangle^{-d/2-\delta}\widehat{\xi_n}(\lambda)$ is Cauchy, or converges, in $L^2(\mathbb{R}^d)$. We will check this via a generalization of the Kolmogorov Two Series Theorem

Lemma 4.9. *Suppose $\mathcal{H}$ is a Hilbert space and $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$. Furthermore take a sequence of independent random variables $\{X_k\}_{k\geq 1}$. Then*

$$\sum_{k=1}^{\infty} \boldsymbol{E}[X_k]g_k \text{ converges in } \mathcal{H} \text{ and } \sum_{k=1}^{\infty} \boldsymbol{E}[X_k^2]||g_k||_{\mathcal{H}}^2 < \infty \Rightarrow \sum_{k=1}^{\infty} X_k g_k \text{ converges a.s. in } \mathcal{H} \tag{4.5}$$

Proof of Theorem 4.8 given Lemma 4.9:

We would like to apply lemma 4.9 with $\mathcal{H} = L^2(\mathbb{R}^d)$, $X_n = Z_n$, and $g_n = (\lambda) = \langle\lambda\rangle^{-d/2-\delta}\widehat{\chi f_n}(\lambda)$. $\mathbf{E}Z_k = 0$ so the first condition is trivial

For any fixed $\lambda \in \mathbb{R}^d$

$$||\chi||_{L^2}^2 = ||e^{2\pi i\lambda\cdot x}\chi||_{L^2}^2 = \sum_{k=1}^{\infty} |(e^{2\pi i\lambda\cdot x}\chi, f_k)_{L^2}|^2 = \sum_{k=1}^{\infty} |\widehat{\chi f_k}(\lambda)|^2 \tag{4.6}$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{E}[X_k^2]||g_n||_{L^2}^2 &= \sum_{n=1}^{\infty} ||g_n||_{L^2}^2 \\ &= \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \langle\lambda\rangle^{-d-2\delta}|\widehat{\chi f_k}(\lambda)|^2 \,\mathrm{d}\lambda \\ &= ||\chi||_{L^2}^2 \int_{\mathbb{R}^d} \langle\lambda\rangle^{-d-2\delta} \,\mathrm{d}\lambda < \infty \end{aligned} \tag{4.7}$$

This completes the proof mod Lemma 4.9!

Proof of Lemma 4.9:

Replacing X_k with $X_k - \mathbf{E}X_k$ and using the fact that $\sum_{k=1}^{\infty} \mathbf{E}[X_k]g_k$ converges, WLOG $\mathbf{E}X_k = 0$.

The goal is now to show that $\sum_{k=1}^{n} X_k g_k$ is a.s. Cauchy.

Fix an $m \geq 1$, and consider the filtration $\mathscr{F}_n = \sigma(X_{m+1}, ..., X_{m+n})$ and the sequence of random variables $M_n = ||\sum_{k=m+1}^{m+n} X_k g_k||_{\mathcal{H}}^2$. Then

Claim: M_k is a submartingale with respect to the filtration $\mathscr{F}_k$ and $\sup_{n\geq 1} \mathbf{E}M_n < \sum_{k=m+1}^{\infty} \mathbf{E}[X_k^2]||g_k||_{\mathcal{H}}^2 =: A_m$.

The proof of this claim is an easy calculation.

Now we can apply Doob's inequality on M_k to deduce that for any $C > 0$,

$$\mathbf{P}(\sup_{n>m} ||\sum_{k=m+1}^{n} X_k g_k||_{\mathcal{H}} > C) < \frac{A_m}{C} \tag{4.8}$$

and thus by the triangle inequality

$$\mathbf{P}(\sup_{m,n:m>n>r} ||\sum_{k=n}^{m} X_k g_k||_{\mathcal{H}} > C) < \frac{2A_r}{C} \tag{4.9}$$

We're done if we can prove that $\mathbf{P}(\lim_{r\to\infty} \sup_{m,n:m>n>r} ||\sum_{k=n}^{m} X_k g_k||_{\mathcal{H}} > \epsilon) = 0$.
But $\lim_{r\to\infty} \sup_{m,n:m>n>r} ||\sum_{k=n}^{m} X_k g_k||_{\mathcal{H}} > \epsilon$ implies that for any subsequence r_j, $\sup_{m,n:m>n>r_j} ||\sum_{k=n}^{m} X_k g_k||_{\mathcal{H}} > \frac{\epsilon}{2}$ infinitely often. But then choosing a subsequence so that $\sum_{j=1}^{\infty} A_{n_j} < \infty$, the Borel Canteli lemma tells us that this occurs with probability 0. This completes the proof!

5 References

[BC95] Lorenzo Bertini and Nicoletta Cancrini. The stochastic heat equation: Feynman-kac formula and intermittence. *Journal of Statistical Physics*, 78(5-6):1377–1401, 1995.

[BFPS07] Alexei Borodin, Patrik L. Ferrari, Michael Prähofer, and Tomohiro Sasamoto. Fluctuation properties of the tasep withnbsp;periodicnbsp;initialnbsp;configuration. *Journal of Statistical Physics*, 129(5-6):1055–1080, 2007.

[BG97] Lorenzo Bertini and Giambattista Giacomin. Stochastic burgers and kpz equations from particle systems. *Communications in Mathematical Physics*, 183(3):571–607, 1997.

[Joh00] Kurt Johansson. Shape fluctuations and random matrices. *Communications in Mathematical Physics*, 209(2):437–476, Feb 2000.

[KS99] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus.* Springer, 1999.

[KS12] Leonid B. Koralov and Yakov G. Sinai. *Theory of probability and random processes.* Springer, 2012.

[Lig85] T. M. Liggett. *Interacting particle systems.* Springer, 1985.

[Liu22] Zhipeng Liu. Multipoint distribution of tasep. *The Annals of Probability*, 50(4), 2022.

[LW22] Zhipeng Liu and Yizao Wang. A conditional scaling limit of the kpz fixed point with height tending to infinity at one location, 2022.

[MQR21] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The kpz fixed point. *Acta Mathematica*, 227(1):115–203, 2021.

[Mue91] Carl Mueller. On the support of solutions to the heat equation with noise. *Stochastics and Stochastic Reports*, 37(4):225–245, 1991.

[NZ22] Ron Nissim and Ruixuan Zhang. Asymptotics of the one-point distribution of the kpz fixed point conditioned on a large height at an earlier point, 2022.

[QS22] Jeremy Quastel and Sourav Sarkar. Convergence of exclusion processes and the kpz equation to the kpz fixed point. *Journal of the American Mathematical Society*, 36(1):251–289, 2022.

[Qua11] Jeremy Quastel. Introduction to kpz. *Current Developments in Mathematics*, 2011(1), 2011.

[Rez19] Fraydoun Rezakhanlou. Stochastic growth and kpz equation (lecture notes), 2019.

[Sch97] Gunter M. Schütz. Exact solution of the master equation for the asymmetric exclusion process. *Journal of Statistical Physics*, 88(1-2):427–445, 1997.