Massachusetts Institute of Technology

18.338: Eigenvalues of Random Matrices

Project Report - Harry Walden

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"The present line of attack would lead to very complicated integrals, but it may be hoped that some other approach will furnish more information about the distribution of the number of real roots of (the) equation."

- *Mark Kac (1943)*

Contents

1 Introduction

In class we briefly discussed the problem of finding the expected number of real roots of a random polynomial by a geometric argument, reducing the problem to finding the length of a certain curve [1]. This report will begin by reviewing this problem and some simple variations on it, before moving towards finding densities of real and complex roots using real geometry (in particular integral geometry, also known as geometric probability), in contrast to most literature on random polynomials where it is most common to invoke complex analysis [2]. Finally, we will discuss the possible application of results in integral geometry to the distribution of eigenvalues of random matrices. Accompanying the review and application of theory are a series of numerical experiments and some commentary on practical aspects of the computations.

2 The Tools of Integral Geometry

This section serves as a brief review of the geometry used in the method of root enumeration by curve length on a sphere, and to state a generalisation of this result to manifolds on a sphere of arbitrary dimension. The latter will not be proven here, taken from the field of integral geometry as a black box simplification of certain computations. A substantial proportion of this report will describe how various problems in random polynomials can be reinterpreted geometrically to meet the assumptions of the theorem. The final step of such computations will also require finding the measure of certain manifolds projected onto the unit sphere in the underlying space, for which it is convenient to work with the logarithmic derivative, a useful method for what is formally computing pullbacks of differential forms in the corresponding projective space. This tool has value here since the theorem we seek to apply holds for submanifolds of spheres, whereas the natural choice of manifold will ordinarily need to be projected onto the sphere; the logarithmic derivative thus allows us to evaluate measures without an explicit change of coordinates.

2.1 Equators of the Sphere

A more detailed explanation than is provided here can be found in [1].

In short, the problem of finding roots of a polynomial can be viewed as finding the points on a certain curve which are orthogonal to the vector of polynomial coefficients, or equivalently the intersections of the orthogonal plane to the coefficient vector with the curve. Section 3 elaborates on the possible choices of the curve and the root finding problem, but for the purposes of this section we are simply considering the intersections of certain planes and curves. In particular, one of the properties which make the i.i.d. Gaussian coefficient problem more tractable is that projecting this geometry onto the unit sphere does not affect the number of intersections, and the projection of the coefficient vector is uniformly distributed over the sphere. Figure 1 shows an example of this construction in $\mathbb{R}^3$.

Next, we observe that an equivalent problem to asking how many intersections some random plane has with a curve is to ask for the measure of random planes intersecting some

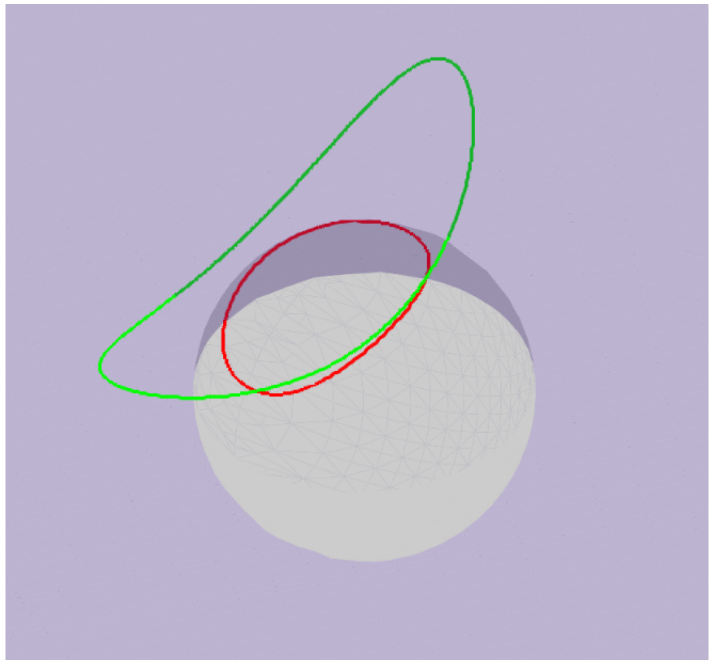


Figure 1. The projection of a curve in $\mathbb{R}^3$ onto S^2. An arbitrary plane passes through the curves to illustrate that to count intersections it is sufficient to work only on the sphere.

point (or infinitesimal segment) of the curve. In particular in this setting we identify planes with their normal, a uniformly random unit vector, and the set of such points orthogonal to some point $\gamma(t)$ on the curve is the orthogonal great circle. As we move along curve γ the great circle sweeps out some area over the sphere. Eventually for any interesting choice of γ we expect to sweep over the same area more than once, so in fact we are counting areas 'with multiplicity', which in this setting corresponds to covering a certain point once for each of its real roots. The final step is to argue that the swept area (counting multiplicity) is proportional to the length of the curve, with some scaling which depends on the dimension of the underlying space. Thus the problem of the expected number of real roots to such a polynomial is reduced to finding the length of a curve, up to a known scaling. This is a deliberately more schematic presentation of the method than we discussed in class, intended to abstract out the crucial features of the construction which make this work.

2.2 Expected Intersection Theorem

In the above we consider a curve γ on the sphere (say S^n) and its intersection with a random uniform choice of n-dimensional hyperplane, whose projection onto S^n is thus $n-1$-dimensional. The generalisation of this theorem is for submanifolds M and N of S^{m+n} of dimension m and n respectively. Let Q be an orthogonal transformation chosen uniformly randomly (under the Haar measure), and use QN to denote the image of N under the random transformation. Then the following is true:

$$\mathbb{E}\left[\#(M \cap QN)\right] = 2\frac{|M||N|}{|S^m||S^n|}$$

where # denotes cardinality of the set, almost surely finite due to the assumptions on the dimensions of N and M, and $|P|$ denotes the volume of a manifold P.
In particular for M the curve γ and N *any* plane of appropriate dimension we recover the result of swept great circles, including the proportionality constant.

2.3 The Logarithmic Derivative

In the following we will often need to consider projections of manifolds onto the unit sphere, which we can equivalently think of as working in a projective space. For example, for the real roots problem we have a curve $\Gamma(t)$ in $\mathbb{R}^n$, which we normalise down to $\gamma(t) = \Gamma(t)/||\Gamma(t)||_2$, a curve on the unit sphere. Finding the length element can be tedious due to the term in the denominator, but the logarithmic derivative provides a shortcut for the computation. For the example of a curve:

$$||\gamma'(t)||_2^2 = \frac{\partial^2}{\partial x \partial y} \log(\Gamma(x) \cdot \Gamma(y))\Big|_{x=y=t}$$

$$\text{or } \mathrm{d}\ell = \sqrt{\frac{\partial^2}{\partial x \partial y} \log(\Gamma(x) \cdot \Gamma(y))\Big|_{x=y=t}}\,\mathrm{d}t$$

This can also be extended to the projective space produced by taking the quotient of a matrix space by the equivalence relation of having the same column span, i.e. the space of subspaces of a given dimension or the Grassmannian. Suppose $M(t)$ is a curve in matrix space, then the length element on the Grassmannian is found:

$$\mathrm{d}\ell = \sqrt{\frac{\partial^2}{\partial x \partial y} \log\det(M(x)^\top M(y))\Big|_{x=y=t}}\,\mathrm{d}t$$

Likewise this formula can be extended over the independent variables. Let M be a chart for some smooth manifold embedded in a matrix space (or as a special case Euclidean space) and $\mathrm{d}S$ the volume element in the domain of the chart, then the logarithmic derivative can be used to evaluate the transformation to volume element $\mathrm{d}\tilde{S}$ on the image of the manifold in projective space:

$$\mathrm{d}\tilde{S}_{M(\mathbf{z})} = \sqrt{\det(\nabla_{\mathbf{x}} \nabla_{\mathbf{y}} \log\det(M(\mathbf{x})^\top M(\mathbf{y})))\Big|_{\mathbf{x}=\mathbf{y}=\mathbf{z}}}\,\mathrm{d}S_{\mathbf{z}}$$

These results will be used without proof for the purposes of this report, although it is not so difficult to prove them. Their real power comes when applied to manifolds where the Cartesian components are difficult to work with but the pairwise inner products of columns have sensible closed forms. We will see a prime example of this in §3.2, where the components are weighted in such a way that the inner product is just a binomial expansion, which results in an explicit form for the root distribution.

3 Real Roots of Random Polynomials

Throughout this report we will work with normally distributed coefficients. The normally distributed polynomial was first studied by Mark Kac in 1943 [3], but the problem of uniformly distributed coefficients had been considered previously by other authors [4], albeit only asymptotically. Kac used what we might consider the 'obvious' choice of basis when we consider constructing a polynomial, the monomial basis, but as we will see this basis does not prove terribly easy to work with. An alternative basis, motivated physically by some authors [5], is to weight the basis set by the square-roots of binomial coefficients, rendering many computations much more simple. We will see this results in a closed form for the expected number of real roots, without resorting to asymptotics as Kac did.

While we could simply ask for the expected number of real roots, it is no additional work to ask for the density of roots, and it is through this density that we will be able to leverage results from integral geometry by asking for the set of polynomials which have some $t \in \mathbb{R}$ as a root, rather than the set of $t \in \mathbb{R}$ which are roots of a particular polynomial.

3.1 Monomial Basis and Generically I.I.D. Coefficients

For the problem of i.i.d. standard normal coefficients, Kac deduced the following number density for roots along the real line:

$$\rho(n,t) = \frac{1}{\pi}\sqrt{\frac{1}{(1-t^2)^2} - \frac{(n+1)^2 t^{2n}}{(1-t^{2n+2})^2}}$$

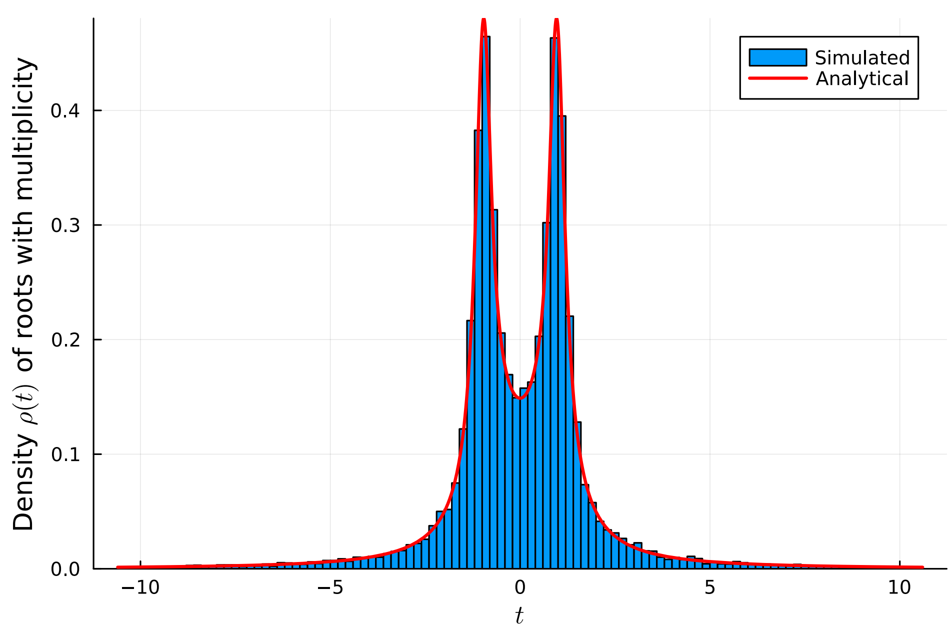


Figure 2. A comparison of numerical experiments against the theoretical density (with multiplicity). A total of 10,000 polynomials of degree 10 were generated in the monomial basis and are compared here to Kac's formula for root density [3].

This isn't very nice! Figure 2 shows how the number density varies along the real line for degree 10 polynomials of this type, comparing simulations to Kac's analytically derived result [3]. This is problem we saw in class, where we sketched how this density might be derived as the line element of a certain curve, namely the following:

$$\gamma(t) = \frac{\Gamma(t)}{||\Gamma(t)||_2}$$
$$\text{for } \Gamma(t) = (1, t, t^2, ..., t^n)$$

For this entire class of problems we usually derive a number density for real roots, which we can view as the expected number of real roots multiplying the probability density of a uniformly chosen real root. The curve length method is covered in detail in Edelman and Kostlan's paper [1]. While the form of the density leaves much to be desired it is perhaps still surprising that it can be written down explicitly in such a form. One can show that the leading order contribution to the expected number of real zeroes is $2\log(n)/\pi$.

3.2 Binomial Basis

It is natural to ask whether the monomials are the best choice of coordinates for the curve Γ, after all they are rarely the natural choice in other problems such as polynomial interpolation or more generally as the basis of some Hilbert space for a particular application. This motivates the investigation of different bases for n^{th} degree polynomials.

There is one particularly nice choice of basis for this problem. It is not so difficult to work in an arbitrary basis, as we will see in §3.3, but in fact very few choices of basis will turn out to be nicer than the monomials. The particular choice is what I will refer to as the binomial basis is to weight the monomials by the square roots of binomial coefficients:

$$\begin{aligned}
\Gamma_{\text{monomial}}(t) &= (1, t, t^2, ..., t^n) \\
\Gamma_{\text{binomial}}(t) &= \left(1, \sqrt{n}t, \sqrt{\frac{n(n-1)}{2}}t^2, ..., t^n\right) \\
&= \left(1, \binom{n}{1}^{1/2} t, \binom{n}{2}^{1/2} t^2, ..., t^n\right) \\
\Gamma_{\text{binomial}}(t) \cdot \Gamma_{\text{binomial}}(s) &= (1+ts)^n
\end{aligned}$$

Now we see the utility of the logarithmic derivative: the log of this function is very simple.

$$\begin{aligned}
\log \Gamma_{\text{binomial}}(t) \cdot \Gamma_{\text{binomial}}(s) &= n\log(1+ts) \\
\frac{\partial^2}{\partial s \partial t} \log \Gamma_{\text{binomial}}(t) \cdot \Gamma_{\text{binomial}}(s) &= \frac{n}{(1+st)^2} \\
\mathrm{d}\ell &= \frac{\sqrt{n}}{1+t^2}\mathrm{d}t
\end{aligned}$$

This special case sees the number density decouple into the expected number of roots and the marginal root density. The average number of roots is exactly $\sqrt{n}$ and probability density is exactly that of a Cauchy random variable. For comparison the number of expected roots in the monomial basis does not have a closed form, and behaves asymptotically very differently.

If we consider the $n = 1$ (linear) case of both this and the original problem, they are the same and involve solving $At + B = 0$ for A, B independent standard normals. One definition of the Cauchy distribution is as ratio of two standard normal variables, so we see that if we want to consider the class of weighted monomial bases then the only density which can possibly decouple from n is a scaled Cauchy density, so this is indeed a distinguished choice. I believe it should be true that this choice of basis is unique up to some symmetries (scaling, reordering, etc.) but I have not proven this fact. As we will see in later sections it is in general difficult to reconstruct a choice of basis given a root density.

I think this result is more special the longer you look at it: e.g. if you had some system where you believe roots of Gaussian random polynomials occur, and that system is independent of dimension, then the roots must be marginally Cauchy. Equivalently, if your polynomial has jointly Gaussian entries then there is a strong restriction on the form of the covariance.

3.3 Arbitrary Basis / Correlated Coefficients

While the binomial basis leads to a particularly nice form of the density, every choice of basis corresponds with a choice of curve Γ. The following procedure illustrates how a choice of basis is equivalent to a choice of coefficient covariance in the monomial basis, or in general between any two bases:

$$\begin{aligned} \Gamma_0(t) &= (1, t, t^2, ..., t^n) \\ \mathbf{a} &\sim \mathcal{N}_{n+1}(\mathbf{0}, \Sigma) \\ \text{Can write } a = \Sigma^{1/2}\mathbf{b}, \;\; \mathbf{b} &\sim \mathcal{N}_{n+1}(\mathbf{0}, I) \\ \text{then } \mathbf{a} \cdot \Gamma_0 = 0 \iff \mathbf{b} \cdot \Sigma^{1/2}\Gamma_0 &= 0 \end{aligned}$$

Through the lens of this construction we can see the binomial basis as just the monomial basis with independent Gaussian coefficients with mean zero and binomial coefficients as the variances. For a generic covariance matrix C it is relatively straightforward to derive the density:

$$\rho(n, t; C) = \frac{\sqrt{[\Gamma_0(t)' \cdot C \cdot \Gamma_0(t)'][\Gamma_0(t) \cdot C \cdot \Gamma_0(t)] - [\Gamma_0(t) \cdot C \cdot \Gamma_0(t)']^2}}{\Gamma_0(t) \cdot C \cdot \Gamma_0(t)}$$

One method to generate covariance (i.e. symmetric PSD) matrices is to randomly draw a Wishart matrix. Several instances of this process are shown in Figure 3. One might even consider asking for the expected number of real roots of the mixture: a polynomial where the coefficients are Gaussian with a Wishart distributed covariance.

There are many more extensions of these results, including non-zero coefficient means and roots of linear combinations of other functions, discussed by various authors [1][2][6].

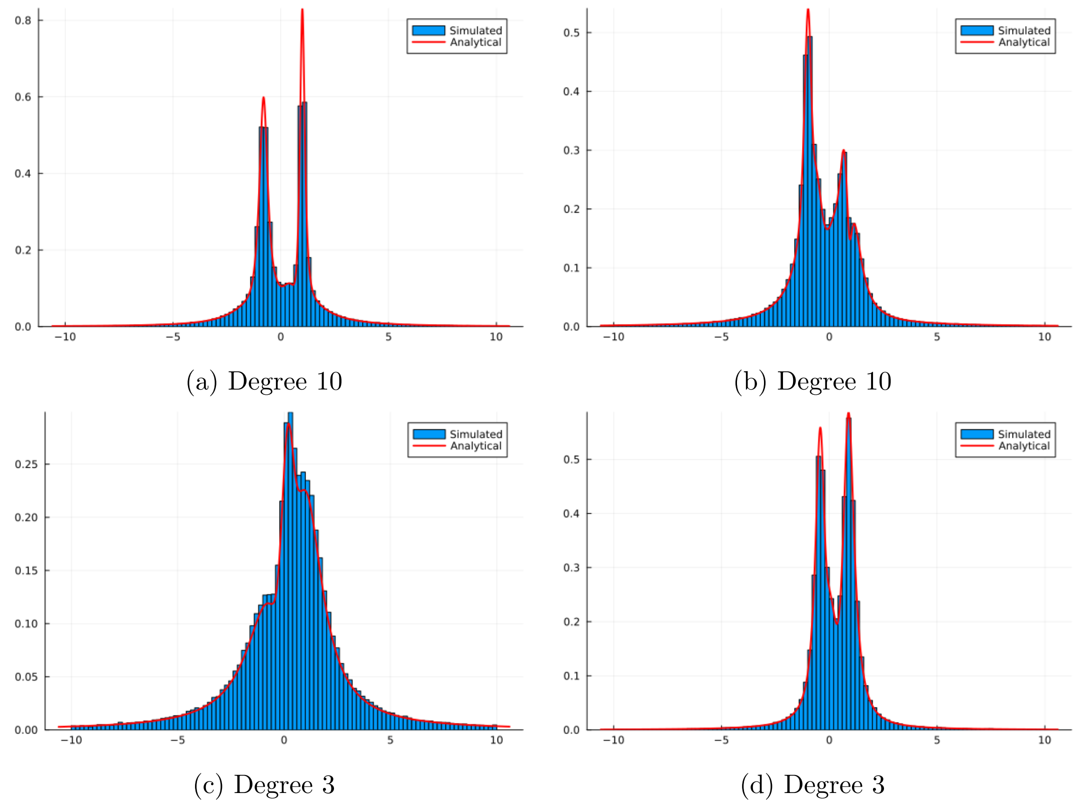


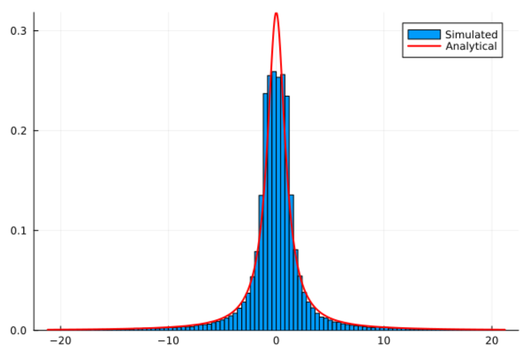
Figure 3. A collection of instances using a Wishart distributed covariance for coefficients of polynomials of varying degree. The marginal density is plotted alongside a histogram (empirical density) of roots generated from 100,000 randomly polynomials with fixed Wishart covariance.

3.4 Universality

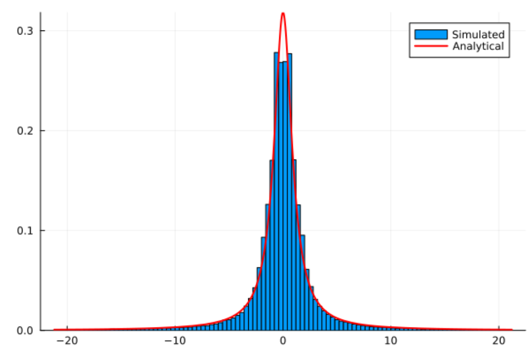
While this report deals for the most part with Gaussian coefficients, it is worth remarking on the related problem with an arbitrary coefficient distribution. Littlewood, among other authors around the same time, was able to obtain asymptotic results as $n \to \infty$ [4]. To what extent are the properties discussed here generic to other coefficient distributions? It is well established in the literature that, under some mild conditions (such as mean zero, sub-logarithmic tails etc.), the roots of polynomials with i.i.d. coefficients converge to the unit circle in the complex plane (note that the peaks observed in the monomial basis are at the intersection of this circle with the real line) [7][8][9].

The interpretation of these results which I would like to present here relates them back to the geometry dealt with above. Ibragimov [7] presents a dense twenty page proof to show convergence (in fairness in the more general complex setting), but I'd like to argue that we can see universality from the projection of coefficient vectors onto the sphere. The claim, which I have only verified numerically, is that as n gets large the generic coefficient vector becomes in some sense isotropic. Evidently this is not the usual sense of isotropic, e.g. for Bernoulli coefficients there is still strong axis alignment, but we hypothesise that the growing number of dimensions allows the projected density on the sphere to approach uniform. Recall that this was the motivation for the spherical projection in the first place, leveraging isotropy of a standard Gaussian vector, so we may apply the same theory using curve lengths to obtain root densities for more general coefficients, for which we pay the price of only asymptotic correctness as $n \to \infty$. This is in agreement with the aforementioned universality results.

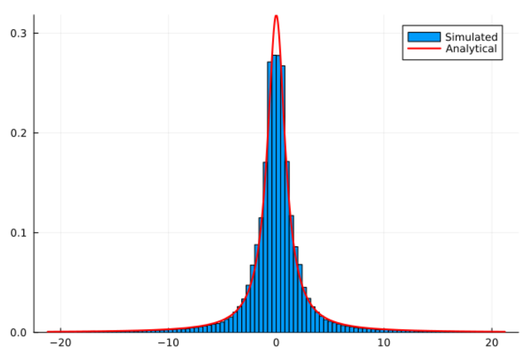
If this is indeed the correct geometric intuition behind universality, then one consequence would be that not only the monomial basis should behave asymptotically like the Gaussian case, but in fact so should any choice of basis. Our favourite choice of basis is the binomial, which returns Cauchy distributed roots in the Gaussian case (§3.2). Figures 4 and 5 compare empirical density functions from 100,000 trials of uniform and Bernoulli coefficients respectively to the Cauchy density, and seem to demonstrate convergence. Even cubics with uniform coefficients appear a very good match while the discrete version converges more slowly, as we might expect for a less smooth distribution. These results support my intuitive picture for why we find universal results in this setting.



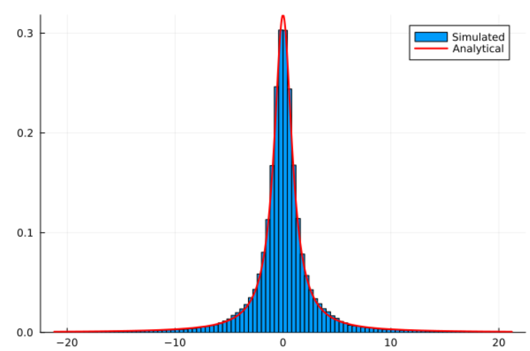
(a) Degree 1



(b) Degree 2

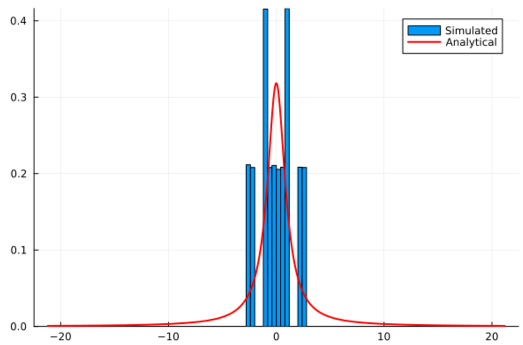


(c) Degree 3

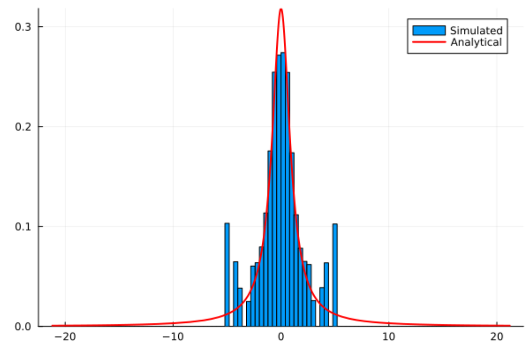


(d) Degree 10

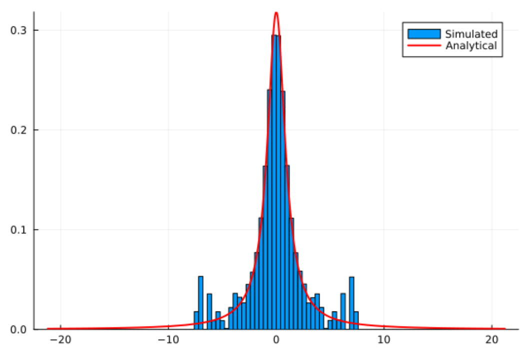
Figure 4. Numerical verification of the universality law in the binomial basis using coefficients uniformly distributed on $[-1, 1]$ for various degrees of polynomial. The p.d.f. of the Cauchy distribution is shown for comparison.



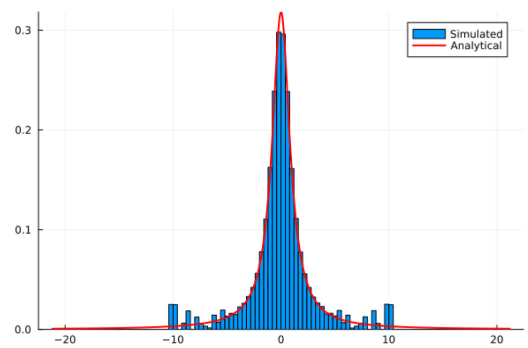
(a) Degree 3



(b) Degree 10



(c) Degree 20



(d) Degree 40

Figure 5. Numerical verification of the universality law in the binomial basis using coefficients taking values uniformly on $\{-1, 1\}$ for various degrees of polynomial. The p.d.f. of the Cauchy distribution is shown for comparison.

4 Complex Roots of Random Polynomials

We now look instead at complex roots. Shepp and Vanderbei [2] obtain a solution by leveraging results from complex analysis for a fairly broad class of such problems. Edelman and Kostlan [1] discuss a general formulation via complex integral geometry which encompasses this problem. The aim of this section is to illustrate how the expected intersection theorem provides a method to find root densities off the real line. This will include a discussion on tractable methods to evaluate the density.

The original motivation for solving the problem this way was to count complex roots without using the fundamental theorem of algebra, which is traditionally proved with basic complex analysis. The hope was that if the average number of roots could be computed geometrically by a certain area integral then this might provide an alternative proof (albeit with some caveat of almost-sureness) of the fundamental theorem of algebra, broadly representative of a general pattern in mathematics of being able to study phenomena through lens of different fields, each giving their own insights.

4.1 Real Coefficients

If we seek to directly extend the real case to the complex case we might let the entries of vector Γ be complex ($t \in \mathbb{C}$) and for the purposes of finding lengths just treat $\mathbb{C}^{n+1}$ as $\mathbb{R}^{2n+2}$. However we now have a problem when we seek to take the inner product with the coefficient vector: since the raw coefficient vector $\mathbf{a} \in \mathbb{R}^{n+1}$ is still real the appropriate inner product is taken with $(\mathbf{a}, \mathbf{a}) \in \mathbb{R}^{2n+2}$. In doing this we have broken one of the crucial assumptions of the theorem: that the coefficient vector (strictly meaning its orthogonal great circle) is uniformly distributed over the sphere (strictly over the space of great circles, the Grassmannian $\mathrm{Gr}(n+1, n) \cong \mathrm{Gr}(n+1, 1)$). We are really constrained to work in the 'natural' space for the random elements of the problem, which in this case is the coefficient vector in $\mathbb{R}^{n+1}$. We will see that in the case of complex coefficients we truly can work in the larger space, so in some sense the most difficult case is the intermediate case, which is reflected in the simplicity of the respective solutions.

So, given that we may only probe orthogonality/coincidence in $\mathbb{R}^{n+1}$, what is the appropriate geometry? We require mutual orthogonality of $\mathbf{a}$ with the real *and* imaginary parts of the monomial 'curve', which when embedded in real space is really a 2-manifold, e.g. parametrised by the real and imaginary parts of the root. The pair (t, s) corresponding to $z = t + is$ are thus a root exactly under the following condition:

$$\mathbf{a} \in \langle \Re(\Gamma), \Im(\Gamma) \rangle^{\perp}$$

Another way to say this is that we are asking $\mathbf{a}$ to lie in some $(n-1)$-dimensional subspace of $\mathbb{R}^{n+1}$, which varies over the two parameters s and t, which can thus be thought of either as an $(n-3)$-submanifold of S^n or as an element of the Grassmannian $\mathrm{Gr}(n+1, n-1)$. The second interpretation is useful as it allows us to use the more general form of the logarithmic derivative to evaluate the pullback of the metric, once again removing the need to explicitly work in projective space. One final useful fact is that we may identify $\mathrm{Gr}(n+1, n-1)$ with $\mathrm{Gr}(n+1, 2)$ [10] and work with the orthogonal complement in $\mathbb{R}^{n+1}$, which is more convenient as we have the spanning vectors $\Re(\Gamma)$ and $\Im(\Gamma)$ already. We may then read off one form of the number density for arbitrary Γ:

$$M(t, s) = \begin{pmatrix} | & | \\ \Re(\Gamma(t, s)) & \Im(\Gamma(t, s)) \\ | & | \end{pmatrix}$$

$$\rho(n, t, s) = \sqrt{\det \nabla_{(t,s)} \nabla_{(u,v)} \log \det(M(t, s)^{\top} M(u, v))}\Bigg|_{(u,v)=(s,t)}$$

I wrote code to evaluate this using symbolic math in MATLAB in the case $n = 2$. In fact in the particular case $n = 2$ the orthogonal subspace is a line in $\mathbb{R}^3$ (point on S^2), in which case the Grassmannian is isomorphic to S^2, representatives of which can be obtained from the cross product of the real and imaginary parts of Γ. Figure 6 shows a plot of the density function off the real line, with the expected symmetries. Complex universality results tell us that as dimension increases the two peaks should condense strongly onto the unit circle [7]. Of particular interest is the total number of complex roots. Integrating the number density

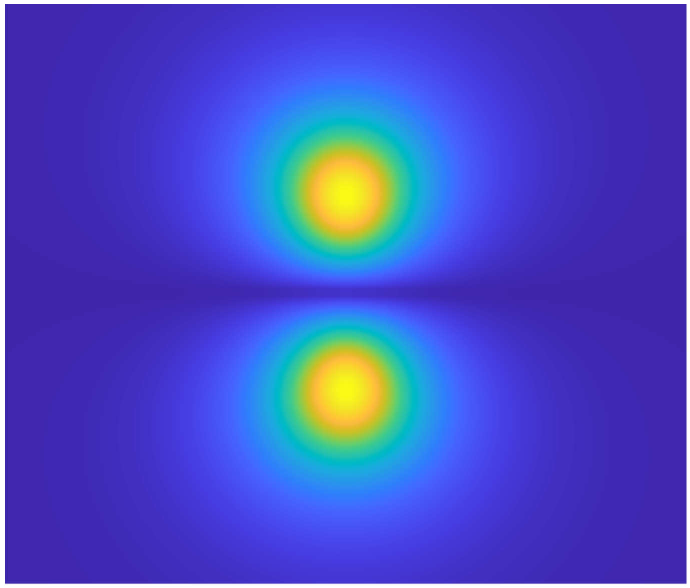


Figure 6. A density plot in the complex plane for roots of quadratics in the monomial basis.

(still specialised to the $n = 2$ case) over the whole plane and scaling as per the expected intersection theorem returns an average of 0.7030 complex roots. We know there ought to be 2 roots, so what went wrong? Careful inspection of the method reveals that the projection of the imaginary part of Γ is not well-defined on the real line. A rigorous treatment would require us to take the upper and lower half planes and the real line as separate cases. In terms of computation this means that to count correctly we need to include a copy of the expected number of real roots, the rescaled curve length 1.2970. By combining the counts of real and non-real roots we obtain a total count which is 2 to seven decimal places. I consider this a good proof of concept for these methods, and constitutes a numerical proof of the fundamental theorem of algebra for degree two.

To remark briefly on the computation, while I used symbolic math I think there is a strong case for these calculations to be carried out using autodiff software. The functions involved are all products, sums and compositions of elementary functions and the logarithmic derivative and Grassmannian equivalence tricks reduce to a relatively small number of computations. The inner determinant in the Grassmannian formulation can also be reduced by converting $\log\det$ to $\operatorname{tr}\log$, which is more convenient for taking derivatives.

While the integrals can be computed numerically, the motivation was to find a proof for the fundamental theorem of algebra, so we might hope for an integral which can be evaluated by hand. I have not as yet found a choice of basis which vastly simplifies the density, like the binomial basis did in the real case. We know of course that in the binomial basis we should have $n - \sqrt{n}$ complex roots on average, so perhaps there is a way to transform the integral into a 'nice' form that I haven't come across so far.

4.2 Complex Coefficients

As mentioned previously, the case of complex coefficients is in many ways easier than that of real coefficients. I will not cover the construction in such detail, as we have already motivated many aspects of it in earlier sections. The appropriate base space is now $\mathbb{R}^{2n+2}$, respecting the degrees of freedom afforded by $n+1$ complex coefficients. If $(\mathbf{a}, \mathbf{b})$ are the real and imaginary parts of the coefficients concatenated then we have the following condition for a complex root:

$$
\begin{aligned}
(\mathbf{a}, \mathbf{b}) \cdot (\Re(\Gamma), \Im(\Gamma)) = 0 \\
(\mathbf{a}, \mathbf{b}) \cdot (-\Im(\Gamma), \Re(\Gamma)) = 0
\end{aligned}
$$

One interpretation of this is using the representation of $\mathbb{C}$ as a subalgebra of $\mathrm{GL}_2(\mathbb{R})$ to move from complex to real space. This can equivalently be stated in terms of a 2-manifold on the Grassmannian, as in the real coefficient case. While it is useful to be able to work with this problem only in real space, Edelman and Kostlan [1] present a theorem which specialises in this case to the following:

$$\rho(n, z = t + is) = \det \left[\frac{\partial^2}{\partial z \partial \bar{z}} \log \Gamma(z)^\top \Gamma(\bar{z}) \right]$$

We therefore see that the binomial basis still has utility in this setting and leads to the following number density:

$$\rho(n, t, s) = \frac{n}{(1 + t^2 + s^2)^2}$$

So, it is straightforward to verify the (almost-sure) fundamental theorem of algebra over complex polynomials by direct integration. Figure 7 compares a histogram of root magnitudes drawn from an $n = 5$ ensemble to the corresponding theoretical density, obtained from the above by a simple transformation.

4.3 Root Correlations

As a final note, the same machinery could in principle be used to obtain k-point correlation functions of real roots, up to some scaling. Rather than the element of the Grassmannian associated to the span of real and imaginary parts, we can consider instead the span of real curve elements $\Gamma(t_i)$ for $i = 1, ..., k$. Orthogonality to this subspace is equivalent to each of $t_1, ..., t_k$ being roots. The same volume element computation can be carried out with M now having k columns rather than 2. Taking $k = n$ it would therefore be possible to obtain a joint density over all the roots, although some care would need to be taken to correctly interpret the results, since this will only count cases in which all roots are real. The complex analog can again be computed using Edelman and Kostlan's more general theorem, and may admit a simpler solution with fewer caveats since over $\mathbb{C}$ there will always be n roots. I have not produced simulations for this construction.

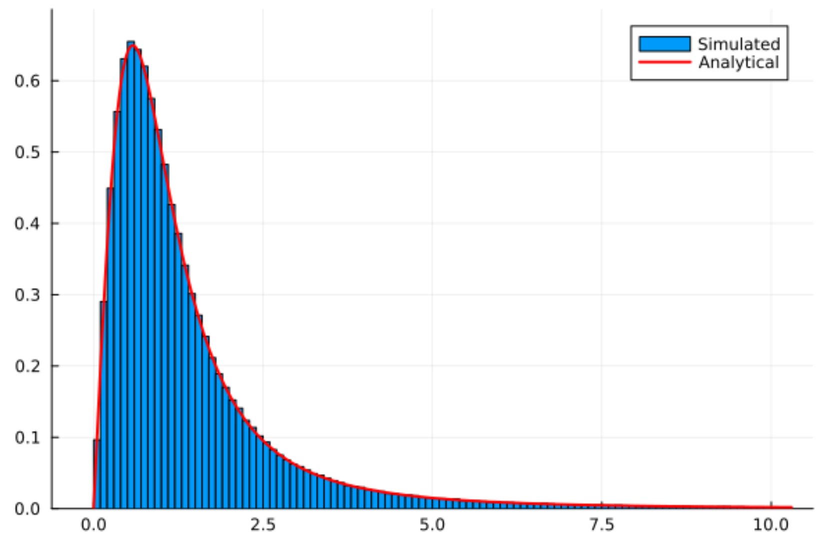


Figure 7. Empirical root magnitude density generated from a sample of 100,000 degree 5 Gaussian complex polynomials in the binomial basis, compared to the theoretical density obtained by marginalising over the argument.

5 Random Matrices

One of the secondary goals of this project was to apply integral geometry to finding eigenvalues of random matrices. I remain convinced that geometry should provide a solution, but I have not made any great progress towards it. In this section I will discuss connections between the problems of random polynomials and random matrices, and why some of the obvious constructions fail to solve the problem.

5.1 Parallels with Random Polynomials

Firstly, as a piece of trivia more than anything, when you ask Julia or MATLAB to find the roots of a polynomial what actually happens is they will find the eigenvalues of a certain matrix, known as the companion matrix and which has the first subdiagonal all ones and the final column the negative coefficients. So, in a way, we have been solving a random matrix problem all along. Further, the companion matrix is diagonalised by conjugating by the Vandermonde matrix of the roots. In random matrix theory, certain formulations allow us to see eigenvalue repulsion in finite RMT as an artifact of a Vandermonde determinant, so we might think of the occurrence here as being related to root repulsion.

On a related subject, just as we can compute the Jacobian of an eigendecomposition to find eigenvalue densities, we can likewise compute the Jacobian of the transformation from coefficients to roots, which also results in a Vandermonde determinant factor. This is the approach taken by Bogomolny et al. in a related problem where correlations between roots are of greater interest [5].

5.2 Potential Constructions

It is not immediately clear what the appropriate base space should be for the problem of, for example, $n \times n$ matrices drawn from the GOE. By insisting on symmetry we guarantee that the eigenvalues are real and leave $n(n+1)/2$ independent components, so we might consider an $n(n+1)/2$-dimensional base space. A trivial construction we could consider is to take N to be zero dimensional, a vectorisation of the random matrix, and take manifold M to be a parametrisation of symmetric matrices by eigendecomposition. Then we may consider infinitesimal portions of M to obtain the density. Therefore, obtaining the volume term in the expected intersection theorem amounts to computing the Jacobian of the eigendecomposition, and we see that this construction is just the geometric realisation of the usual method in finite random matrix theory.

We might instead consider M to be the manifold of diagonal matrices, and N to be the random manifold of matrices orthogonally similar to a sampled matrix from the GOE. This construction seems more in line with the approach taken for random polynomials. However, we can no longer apply the expected intersection theorem because we have no guarantee that randomly rotating the manifold is equivalent to randomly drawing the GOE sample. One could instead consider the manifold of *all* similar matrices, and apply Sylvester's law of inertia to obtain sign information from the eigenvalues, but the same complication prohibits us from applying the theorem.

We saw in §4 that we will never be able to apply the theorem if we make the wrong choice of base space. Phrased in terms of symmetry groups, the real coefficient vector has $O(n)$ symmetry, and the complex $O(2n)$, and so our choice of base space must reflect this so that choosing N randomly is equivalent to a deterministic choice under some random orthogonal transformation. The GOE draw on the other hand has symmetries which are a subgroup of $O(n) \otimes O(n) \leq O(n^2)$, which does not correspond to the symmetry group of the sphere in any dimension, so in fact an approach this direct can never work. This feels right in a way, because a matrix space is more than just Euclidean space and matrix multiplication is central to what it means to be an eigenvalue. This also suggests however that integral geometry might provide a solution if we are willing to find a generalisation of the expected intersection theorem to more general manifolds. Formulated this way, it may even be as easy to recover the Laguerre ensemble by using that the symmetries of the singular value problem are the entire group $O(n) \otimes O(n)$.

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