Painlevé Equations and the Sampling of Eigenvalue Statistics

Andrey Bryutkin
Department of Mathematics
bryutkin@mit.edu

Diego Chavez
Department of Mathematics
dichavez@mit.edu

Abstract

We investigated the Painlevé equations and their relationships to the eigenvalue statistics of the most common types of ensembles found in random matrix theory: Hermite ensembles, Laguerre ensembles, and Jacobi ensembles. We were able to form a collection of the relationships between the Fredholm determinants used to determine the eigenvalue distributions and the Painlevé transcendents that arise from the Painlevé equations. We were also able to reproduce some stable numerical results as well as test how unstable some of the others were. By using the relationship between the Painlevé differential equations and σ-forms, we also proposed a procedure for how we can use the standard equations to find solutions for the σ-forms that are much more nonlinear, provided we find consistent boundary conditions that generate those required for the σ-form to apply to random matrix theory.

1 Background

The journey to understanding the Painlevé equations, particularly within the domain of random matrix theory, begins with historical insights. The late 19th century saw groundbreaking work by Fuchs and Poincaré, who focused on first-order differential equations of the form $P(y', y, t) = 0$, where P is a polynomial in y' and y, with coefficients that are meromorphic in t. These equations are noted for their movable singularities, which are singular points whose position depends on the initial conditions of the equation [4].

One key aspect of this study was differentiating between equations that have movable poles and those with movable essential singularities. An example of the former is $\frac{dy}{dt} = y^2 + 1$, with the general solution $y = \tan(t + c)$. Here, all singularities are movable poles of the first order. In contrast, the equation $\frac{dy}{dt} = \frac{1}{\alpha y^{\alpha-1}}$ (for $\alpha = 2, 3, \ldots$) presents a general solution $y = (t - c)^{1/\alpha}$, showcasing movable branch points or essential singularities.

Fuchs and Poincaré successfully classified all equations of the first form, which are free from movable essential singularities. These equations could be simplified to either the differential equation of the Weierstrass $\mathcal{P}$-function, $\left(\frac{dy}{dt}\right)^2 = 4y^3 - g_2 y - g_3$, or the Riccati equation, $\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t)$, where a, b, c are analytic in t.

Building on this foundation, Painlevé explored the classification of second-order differential equations of the form $y'' = R(y', y, t)$, where R is a rational function in all its arguments. These equations are distinguished by their absence of movable essential singularities, a characteristic now known as the Painlevé property. Painlevé demonstrated that the only equations of this form, which could not be reduced to first-order equations or linear differential equations, are the ones we now recognize as the Painlevé equations.

In the realm of random matrix theory, these equations take on significant importance. The Painlevé transcendents, as solutions to these equations, are integral in modeling various phenomena in physics, particularly in the statistical study of large random matrices. This connection provides a fertile ground for applying the rich history and complex nature of Painlevé equations to contemporary mathematical and physical problems.

2 The Painlevé equations

We start by stating all relevant Painlave equation in their exact form and their σ-form to get a better overview and also to better keep track of the equations:

2.1 The normal form of Painlevé equations

$$
\begin{aligned}
&\text{PI } q'' = 6q^2 + t \\
&\text{PII } q'' = 2q^3 + tq + \alpha, \\
&\text{PIII } q'' = \frac{1}{q}\left(q'\right)^2 - \frac{1}{t}q' + \gamma q^3 + \frac{1}{t}\left(\alpha q^2 + \beta\right) + \frac{\delta}{q}, \\
&\text{PIV } q'' = \frac{1}{2q}\left(q'\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2\left(t^2 - \alpha\right)q + \frac{\beta}{q}, \\
&\text{PV } q'' = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(q'\right)^2 - \frac{1}{t}q' + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1}, \\
&\text{PVI } q'' = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(q'\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)q' \\
&+ \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right).
\end{aligned}
$$

2.2 The σ-form of Painlevé equations

The differential equations can also be defined in another form, the so-called σ-forms are given by:

$$
\begin{aligned}
&\sigma\text{PII} \quad \left(\sigma_{II}''\right)^2 + 4\sigma_{II}'\left(\left(\sigma_{II}'\right)^2 - t\sigma_{II}' + \sigma_{II}\right) - a^2 = 0, \\
&\sigma\text{PIII}' \quad \left(t\sigma_{III'}''\right)^2 - v_1 v_2\left(\sigma_{III'}'\right)^2 + \sigma_{III'}'\left(4\sigma_{III'}' - 1\right)\left(\sigma_{III'} - t\sigma_{III'}'\right) - \frac{1}{4^3}\left(v_1 - v_2\right)^2 = 0, \\
&\sigma\text{PIV} \quad \left(\sigma_{IV}''\right)^2 - 4\left(t\sigma_{IV}' - \sigma_{IV}\right)^2 + 4\sigma_{IV}'\left(\sigma_{IV}' + 2\alpha_1\right)\left(\sigma_{IV}' - 2\alpha_2\right) = 0, \\
&\sigma\text{PV} \quad \left(t\sigma_V''\right)^2 - \left(\sigma_V - t\sigma_V' + 2\left(\sigma_V'\right)^2 + \left(\nu_0 + \nu_1 + \nu_2 + \nu_3\right)\sigma_V'\right)^2 \\
&\quad + 4\left(\nu_0 + \sigma_V'\right)\left(\nu_1 + \sigma_V'\right)\left(\nu_2 + \sigma_V'\right)\left(\nu_3 + \sigma_V'\right) = 0, \\
&\sigma\text{PVI} \quad \sigma_{VI}'\left(t(1-t)\sigma_{VI}''\right)^2 + \left(\sigma_{VI}'\left(2\sigma_{VI} - (2t-1)\sigma_{VI}'\right) + v_1 v_2 v_3 v_4\right)^2 = \prod_{k=1}^{4}\left(\sigma_{VI}' + v_k^2\right)
\end{aligned}
$$

2.3 The Painlevé Hamiltonians and the connection to the σ-forms

To obtain the σ-forms one need additional structure, which is given by Hamiltonians. The connection between the Hamiltonians and the Painlevé equations is established in [4]. The Hamiltonians are given by:

$$
\begin{aligned}
H_I =& \frac{1}{2}p^2 - 2q^3 - tq, \\
H_{II} =& -\frac{1}{2}\left(2q^2 - p + t\right)p - \frac{v_1 - v_2}{2}q, \\
tH_{III'} =& q^2p^2 - \left(q^2 + v_1 q - t\right)p + \frac{1}{2}\left(v_1 + v_2\right)q, \\
H_{IV} =& (2p - q - 2t)pq - 2\left(v_1 - v_2\right)p + \left(v_3 - v_2\right)q, \\
tH_V =& q(q-1)^2p^2 - \left\{\left(v_1 - v_2\right)(q-1)^2 - 2\left(v_1 + v_2\right)q(q-1) + tq\right\}p \\
& + \left(v_3 - v_2\right)\left(v_4 - v_2\right)(q-1), \\
t(t-1)H_{VI} =& q(q-1)(q-t)p^2 - \left(\left(v_3 + v_4\right)(q-1)(q-t) + \left(v_3 - v_4\right)q(q-t)\right. \\
& \left. - \left(v_1 + v_2\right)q(q-1)\right)p + \left(v_3 - v_1\right)\left(v_3 - v_2\right)(q-t),
\end{aligned}
$$

the relations between the coefficients are:

PII	$v_1 + v_2 = 0, \quad \alpha = v_1 - \frac{1}{2},$
PIII'	$\alpha = -4v_2, \quad \beta = 4\left(v_1 + 1\right), \quad \gamma = 4, \quad \delta = -4,$
PIV	$v_1 + v_2 + v_3 = 0, \quad \alpha = 1 + 2v_3 - v_1 - v_2, \quad \beta = -2\alpha_1^2,$
PV	$v_1 + v_2 + v_3 + v_4 = 0, \alpha = \frac{1}{2}\left(v_3 - v_4\right)^2, \beta = -\frac{1}{2}\left(v_1 - v_2\right)^2, \gamma = 2v_1 + 2v_2 - 1, \delta = -\frac{1}{2},$
PVI	$\alpha = \frac{1}{2}\left(v_1 - v_2\right)^2, \quad \beta = -\frac{1}{2}\left(v_3 + v_4\right)^2, \quad \gamma = \frac{1}{2}\left(v_3 - v_4\right)^2, \delta = \frac{1}{2}\left(1 - \left(1 - v_1 - v_2\right)^2\right).$

Note that the Hamiltonians fulfill the following relations:

$$
q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}
$$

The essential idea is that the 2nd order ODE of q derived from the Hamiltonians is exactly the Painlevé equation in the exact form (except for PIII). Note that to obtain PIII from PIII' use the substitution $t \mapsto t^2, y \mapsto ty$. Now we define auxillary Hamiltonians, which are kind of a rescaled version of the Hamiltonians:

$$
\begin{aligned}
& h_{II}(t) = H_{II}, \\
& h_{III'}(t) = tH_{III'} + \frac{1}{4}v_1^2 - \frac{1}{2}t, \\
& h_{IV}(t) = H_{IV} - 2v_2 t, \\
& h_V(t) = tH_V + \left(v_3 - v_2\right)\left(v_4 - v_2\right) - v_2 t - 2v_2^2, \\
& h_{VI}(t) = t(t-1)H_{VI} + e_2\left[-v_1, -v_2, v_3\right]t, -\frac{1}{2}e_2\left[-v_1, -v_2, v_3, v_4\right],
\end{aligned}
$$

where $e_p\left[a_1, \ldots, a_s\right] := \sum_{1<j_1<\cdots<j_p<s} a_{j_1}a_{j_2}\cdots a_{j_p}$. Using these auxiliary Hamiltonians there is a direct relation to the σ-forms provided by:

$$
\begin{aligned}
\sigma_{II}(t) =& -2^{1/3}h_{II}\left(-2^{1/3}t\right)\Big|_{\left(v_1, v_2\right)=(a,-a)}, \\
\sigma_{III'}(t) =& -h_{III'}(t/4) + \frac{t}{8} + \frac{v_1 v_2}{4}, \\
\sigma_{IV}(t) =& \left.\left(h_{IV}(t) + 2v_2 t\right)\right|_{\left(1+v_3-v_1, v_1-v_2, v_2-v_3\right)=\left(\alpha_0, \alpha_1, \alpha_2\right)}, \\
\sigma_V(t) =& h_V(t) + v_2 t + 2v_2^2, \quad \nu_{j-1} = v_j - v_2 \quad (j = 1, \ldots, 4), \\
\sigma_{VI}(t) =& h_{VI}(t)
\end{aligned}
$$

As one can see from the definitions that the Painlevé equations are highly non-linear, especially the σ-forms are very unstable, so we propose a new numerical method to obtain the σ-forms solutions, without even solving the implicitly or explicitly.

We on purpose stated all the Hamiltonians, as we will use them to obtain the σ-forms from the exact Painlevé forms. We propose the following procedure: We first solve the Painlevé equations in the exact form, then we obtain from the Hamiltonian's the function $p(q', q, t)$ and plug in the results into the auxiliary Hamiltonian's, which give us the σ-forms. In our viewpoint this procedure is mathematically valid and will generate the desired forms. A structured formulation can be applied to each of the Painlevé equations. Unfortunately in most cases the problems in random matrix theory (RMT) are defined in the σ-formulation and the respective boundary conditions too. We tried multiple approaches and have not found a direct way to transform the boundary conditions into the exact form. Nevertheless, we wills state the calculated transformations for each of the equations and bring all the results together.

2.4 Numerical adjustments

For the numerical calculations we will need a specific fact, which we will use multiple times to set up a system of ODEs. In calculus, the Leibniz integral rule for differentiation under the integral sign states that for an integral where the bounds depend on the integrand, the derivative of this integral is expressible as

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x)) \cdot \frac{d}{dx}b(x) - f(x,a(x)) \cdot \frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)dt.$$

Since the random matrix theory probability densities tend to involve integrals with the Painlevé transcendents, we can also include them in the first-order system of differential equations to get more stable approximations of their values. This is also just more efficient since it removes the step of having to do a quadrature rule to estimate the integral after finding the discretized Painlevé transcendent.

2.5 The Painlevé system

In a lot of papers the connection between the distribution of the statistics of eigenvalues was established and related to the σ-forms of the Painlevé equation, but it was never written explicitly what kind of parameters you need to set up to get from the general Painlevé form to the required ODE for the RMT problems. We do a bit of bookkeeping and will show all the required facts about each of the differential equations needed to RMT problems. We explicitly calculated the forms of $p(q', q, t)$ and $q'(p, q, t)$ using Hamiltonian relations and as stated before the forms $p(q', q, t)$ and $q'(p, q, t)$ are essential to obtain direct transformation formulas for the equations. In theory, if we are able to solve for the solution q in the Painlevé differential equation, we can generate a solution for the σ-form differential equation by using the expression for p in terms of q and its derivatives and plugging both q and p into the auxiliary Hamiltonians that directly give the σ solutions.

2.6 Painlevé Equation II

The Painlevé equation II in the exact form appears in the description of the largest eigenvalue statistics of the Gaussian ensembles for the matrix size $N \to \infty$ (e.g for the Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE) and the Gaussian Symplectic Ensemble (GSE)). The useful form of the exact form, which appears in random matrix theory sets the constant $\alpha = 0$, which gives $v_1 = -v_2$ and $v_1 = \frac{1}{2}$ (note that $v_1 = a, v_2 = -a$) for the σ-form formulation. Summarized we have the following equations:

$$
\begin{aligned}
\text{PII} \quad & q'' = 2q^3 + tq \\
\sigma\text{PII} \quad & \left(\sigma_{II}''\right)^2 + 4\sigma_{II}'\left(\left(\sigma_{II}'\right)^2 - t\sigma_{II}' + \sigma_{II}\right) - \frac{1}{4} = 0 \\
H_{II} &= h_{II} = -\frac{1}{2}\left(2q^2 - p + t\right)p - \frac{1}{2}q \\
q' &= \frac{\partial H}{\partial p} = -\frac{1}{2}\left(2q^2 - p + t\right) + \frac{1}{2}p \\
&= \frac{1}{2}p + \frac{1}{2}p - q^2 - \frac{1}{2}t \\
p &= q' + q^2 + \frac{1}{2}t \\
\sigma_{II}(t) &= -2^{1/3}\left(-\frac{1}{2}\left(2q^2\left(-2^{1/3}t\right) - p\left(-2^{1/3}t\right) - 2^{1/3}t\right)p\left(-2^{1/3}t\right) - \frac{1}{2}q\left(-2^{1/3}t\right)\right) \\
&= -2^{1/3}\left(-\frac{1}{2}\left(2q^2\left(-2^{1/3}t\right) - q' - q^2 - \frac{1}{2}t - 2^{1/3}t\right)p\left(-2^{1/3}t\right) - \frac{1}{2}q\left(-2^{1/3}t\right)\right)
\end{aligned}
$$

2.7 Painlevé Equation III

Here for RMT the desired parameter choice is $v_1 = v_2 = \alpha$, where the α can be seen as the Laguerre parameter. The coefficients of $PIII$ are given by $\alpha = -4\alpha, \beta = 4(\alpha+1), \gamma = 4, \delta = -4$. So we would have:

$$
\begin{aligned}
\text{PIII} \quad q'' &= \frac{1}{q}\left(q'\right)^2 - \frac{1}{t}q' + 4q^3 + \frac{1}{t}\left(4(\alpha+1) - 4\alpha q^2\right) - \frac{4}{q} \\
\text{PIII}' \quad q'' &= \frac{1}{q}\left(q'\right)^2 - \frac{1}{t}q' + \frac{\alpha q^3}{4t^2} + \frac{1}{4t^2}\left(\beta q^2 + \gamma t\right) + \frac{\delta}{4q} \\
& t \leftrightarrow t^2 \quad q \leftrightarrow tq \quad III' \leftrightarrow III \\
\sigma\text{PIII}' \quad & \left(t\sigma''\right)^2 - \alpha^2\left(\sigma'\right)^2 + \sigma'\left(4\sigma' - 1\right)\left(\sigma - t\sigma'\right) = 0 \\
tH_{\text{III}} &= q^2p^2 - \left(q^2 + \alpha qp - t\right)p + \alpha q \\
h_{\text{III}}(t) &= tH_{\text{II}}{}' + \frac{1}{4}\alpha^2 - \frac{1}{2}t \\
tq' &= \frac{\partial tH}{\partial p} = 2q^2p - \left(q^2 + \alpha q - t\right) \\
p &= \frac{tq' + q^2 + \alpha q - t}{2q^2} \\
\sigma_{III}(t) &= -h_{III'}(t/4) + \frac{t}{8} + \frac{\alpha^2}{4}
\end{aligned}
$$

2.8 Painlevé Equation IV

Here for the RMT application we choose $\alpha_1 = N$ and $\alpha_2 = 0$. This gives $v_2 = v_3 = -\frac{N}{3}$ and $v_1 = \frac{2N}{3}$.

$$
\begin{aligned}
&\text{PIV } q'' = \frac{1}{2q}\left(q'\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2\left(t^2-\alpha\right)q + \frac{\beta}{q} \\
&\sigma\text{PIV} \quad \left(\sigma_{IV}''\right)^2 - 4\left(t\sigma_{IV}' - \sigma_{IV}\right)^2 + 4\sigma_{IV}'^2\left(\sigma_{IV}' + 2N\right) = 0 \\
&H_{IV} = (2p - q - 2t)pq - 2Np \\
&h_{IV}(t) = H_{IV} - 2v_2 t \\
&\sigma_{IV}(t) = \left(h_{IV}(t) + 2v_2 t\right)\big|_{(1+v_3-v_1, v_1-v_2, v_2-v_3)=(\alpha_0,\alpha_1,\alpha_2)} \\
&q' = \frac{\partial H}{\partial p} = -2N + (2p - q - 2t)q + 2pq \\
&\quad = -2N + 4pq - q^2 - 2qt \\
&p = \frac{1}{4q}\left(q' + q^2 + 2qt + 2N\right)
\end{aligned}
$$

2.9 Painlevé Equation V

For random matrix theory, we consider the σPV equation for $-\sigma$, with $\nu_0 = \nu_1 = 0$, $\nu_2 = n$, and $\nu_3 = n + \alpha$.

$$
\begin{aligned}
&\text{PV } q'' = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(q'\right)^2 - \frac{1}{t}q' + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1}, \\
&\sigma\text{PV} \ \left(t\sigma_V''\right)^2 - \left(\sigma_V - t\sigma_V' + 2\left(\sigma_V'\right)^2 + \left(\nu_0+\nu_1+\nu_2+\nu_3\right)\sigma_V'\right)^2 \\
&+ 4\left(\nu_0+\sigma_V'\right)\left(\nu_1+\sigma_V'\right)\left(\nu_2+\sigma_V'\right)\left(\nu_3+\sigma_V'\right) = 0 \\
&tH_V = q(q-1)^2p^2 - \left\{\left(v_1-v_2\right)(q-1)^2 - 2\left(v_1+v_2\right)q(q-1) + tq\right\}p \\
&+ \left(v_3-v_2\right)\left(v_4-v_2\right)(q-1) \\
&\text{PV} \quad v_1+v_2+v_3+v_4 = 0, \alpha = \frac{1}{2}\left(v_3-v_4\right)^2, \beta = -\frac{1}{2}\left(v_1-v_2\right)^2, \gamma = 2v_1+2v_2-1, \delta = -\frac{1}{2} \\
&h_V(t) = tH_V + \left(v_3-v_2\right)\left(v_4-v_2\right) - v_2t - 2v_2^2 \\
&\sigma_V(t) = h_V(t) + v_2t + 2v_2^2, \quad \nu_{j-1} = v_j - v_2 \quad (j = 1, \ldots, 4) \\
&tq' = \frac{\partial tH_V}{\partial p} = 2q(q-1)^2p - \left\{\left(v_1-v_2\right)(q-1)^2 - 2\left(v_1+v_2\right)q(q-1) + tq\right\} \\
&p = \frac{tq' + \left\{\left(v_1-v_2\right)(q-1)^2 - 2\left(v_1+v_2\right)q(q-1) + tq\right\}}{2q(q-1)^2}
\end{aligned}
$$

The Eigenvalue statistics

Int the following section we want to present all results coherently with all methods, which are known for us and the underlying experiments done for the respective cases. The main approaches are to present both the standard forms and the reduced tridiagonal forms for the respective ensembles, the approach of the Painlevé equations, the Fredholm determinant approach and the related DPP approach in the discretized case of the respective Kernel. We will repeat ourselves in specific subsection as we will state similar results, however, we think it is essential to have a full overview of the results at one glance.

3 Gaussian Ensemble Eigenvalue Statistics

3.1 The Tridiagonal Formulation

We sample from the Tridiagonal forms of our distributions and determine efficient statistics. We noted to our surprise that sampling the biggest eigenvalue from an ensemble of tridiagonal matrices leads to different normalisation coefficients, which we have tried to determine by plotting the normal stets and looking at the mean of the plots[1]. We know from class that the tridiagonal form, which represents the Gaussian ensemble is given by:

$$H_n^\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

The implementation is shown in Fig.1:

```
function generate_gaussian_beta_tridiagonal_matrix(n::Int, β::Float64)
    # Diagonal elements: N(0, 2)
    main_diag = [rand(Normal(0, sqrt(2))) for _ in 1:n]

    # Off-diagonal elements: Chi distribution
    off_diag = [rand(Chisq((n-i)*β)) for i in 1:n-1]

    # Creating the Tridiagonal matrix
    H = Tridiagonal(off_diag, main_diag, off_diag)
    return H
end
```

Figure 1: Gaussian Tridiagonal Ensemble

3.2 The biggest eigenvalue statistics $N \to \infty$

The Fredholm Determinant Approach Let us define the Airy-Kernel:

$$K_{Ai}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$$

Then the distribution of the biggest eigenvalue λ_{max} for $\beta = 2$ is given by:

$$\mathbb{P}_{GUE}(\lambda_{max} \leq s) = \det\left(I - K_{\mathrm{Ai}}|_{L^2(s,\infty)}\right)$$

This distribution is also called the Tracy-Widom distribution.

The Painlevé Equation Approach We can also get the Tracy-Widom distribution using the Painlevé-Equations. As stated in the previous chapter, we can get a formulation using PII in the exact form [2]:

$$F_2(s) \equiv \mathbb{P}_{GUE}\left(\lambda_{\max} \leq s\right) = \exp\left(-\int_s^\infty (t-s)q(t)^2\,\mathrm{d}t\right)$$

where $q(t)$ fulfills the PII equation with $\alpha = 0$ and the following boundary condition:

$$q(t) \sim \mathrm{Ai}(t), \quad \text{as } t \to \infty$$

[1] We determined the mean to perform a non-linear regression of the given statistics of the non-tridiagonal formulation

For $\beta = 1$ and $\beta = 4$ we get similar formulas for the largest eigenvalue distribution:

$$F_1(s) \equiv \mathbb{P}_{GOE}\left(\lambda_{\max} \le s\right) = \sqrt{\exp\left(-\int_s^\infty (t-s)q(t)^2 \, \mathrm{d}t\right)\exp\left(-\int_s^\infty q(t)\mathrm{d}t\right)}$$

$$F_4(s) \equiv \mathbb{P}_{GSE}\left(\lambda_{\max} \le s\right) = \sqrt{\exp\left(-\int_{2^{\frac{2}{3}}s}^\infty (t-s)q(t)^2 \, \mathrm{d}t\right)}\left(\cosh\left(\int_{2^{\frac{2}{3}}s}^\infty q(t)\mathrm{d}t\right)\right)$$

The numerical approach and sampling To solve for q in the PII ODE we consider that the pdf of the distribution is given by:

$$p(s) = \frac{\mathrm{d}}{\mathrm{d}s}F_2(s) = \frac{\mathrm{d}}{\mathrm{d}s}\exp\left(-\int_s^\infty (t-s)q(t)^2 \, \mathrm{d}t\right)$$

So one can enhance the stability by defining:

$$I(x) = \int_s^\infty (t-s)q(t)^2 dt$$

and writing then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ q' \\ I \\ I' \end{pmatrix} = \begin{pmatrix} q' \\ tq + 2q^3 \\ I' \\ q^2 \end{pmatrix},$$

which would make

$$p(s) = -I'(s)\exp(I(s)).$$

The MATLAB code used to solve the ODE was

```
%% Painleve II equation
t0 = 6;
tn = -8;
dydt = @(t, y) [y(2); t*y(1) + 2*y(1)^3; y(4); y(1)^2; -y(1)];
I0 = integral(@(x) airy(x).^2 .* (x - t0), t0, 10*t0);
J0 = integral(@(x) airy(x), t0, 10*t0);
y0 = [airy(t0); airy(1, t0); I0; airy(t0)^2; J0];
opts=odeset('reltol',1e-13,'abstol',1e-14);
[t, y] = ode45(dydt, [t0 tn], y0, opts);
```

Then, to test both the GOE ($\beta = 1$) and GUE ($\beta = 2$) cases, we ran

```
F2 = exp(-y(:, 3));
f2 = -y(:, 4) .* F2;

num_trials = 100;
n = 1000;
tic
eig_vec= zeros(num_trials, 1);
for i=1:num_trials
    A = randn(n, n) + 1i * randn(n, n);
    A_sym = (A + A')/2;
    eig_vec(i) = n^(1/6) * (max(eigs(A_sym)) - 2*sqrt(n));
end
toc
figure
histogram(eig_vec, 'Normalization', 'pdf')
hold on
plot(t, f2)

% beta = 1
```

```
F1 = sqrt(F2 .* exp(-y(:, 5)));
f1 = (f2 + y(:, 1) .* F2) .* exp(-y(:, 5)) ./ (2*F1);
eig_vec_2 = zeros(num_trials, 1);
tic
for i=1:num_trials
    A = randn(n, n);
    A_sym = (A + A')/(2);
    eig_vec_2(i) = n^(1/6) * (max(eigs(A_sym))*sqrt(2) - 2*sqrt(n));
end
toc
figure
histogram(eig_vec_2, 'Normalization', 'pdf')
hold on
plot(t, f1)
```

This gave the code output in Figures 2 and 3. To do the GOE or GSE cases, we can add another element to the differential vector by defining $J(s) = \int_s^\infty q(t)\,dt$, whose derivative is $-q$. This is used in the code above.

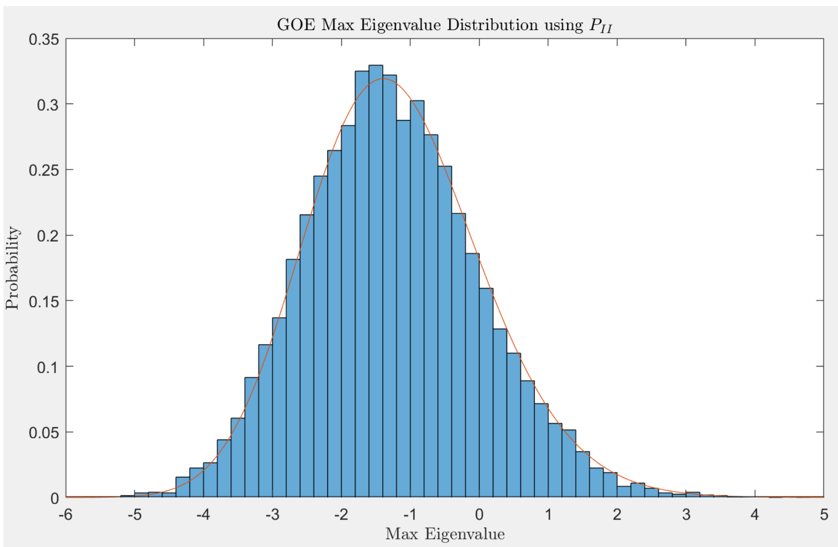

Figure 2: GOE maximum eigenvalue distribution against the pdf generated by the numerical solution to Painlevé II.

3.3 The biggest eigenvalue distribution for finite N case

In the special case if N does not tend to infinity there is a known formula, which relates a specific Painlave equation to the statistics, for the biggest eigenvalue distribution for $\beta = 2$, for $\beta = 1, 4$ we couldn't find a formula.

The Fredholm determinant approach Let us define for this case the Hermite Kernel for a fixed NxN matrix, which we simplify using the Darboux formula:

$$K_{\text{Herm}}^{(N)} = \sqrt{\frac{2}{n}}\frac{\pi_n(x)\pi_{n+1}(y) - \pi_n(y)\pi_{n+1}(x)}{y - x}$$

where

$$\pi_j(x) = \frac{H_j(x)}{(2^j\sqrt{\pi}j!)^{1/2}}$$

where H_j are Hermite polynomials. Then the largest eigenvalue distribution is given by [4]:

$$\mathbb{P}_{GUE}^{(N)}(\lambda_{\max} \leq s) = \det\left(I - K_{\text{Herm}}^{(N)}\Big|_{L^2(s,\infty)}\right)$$

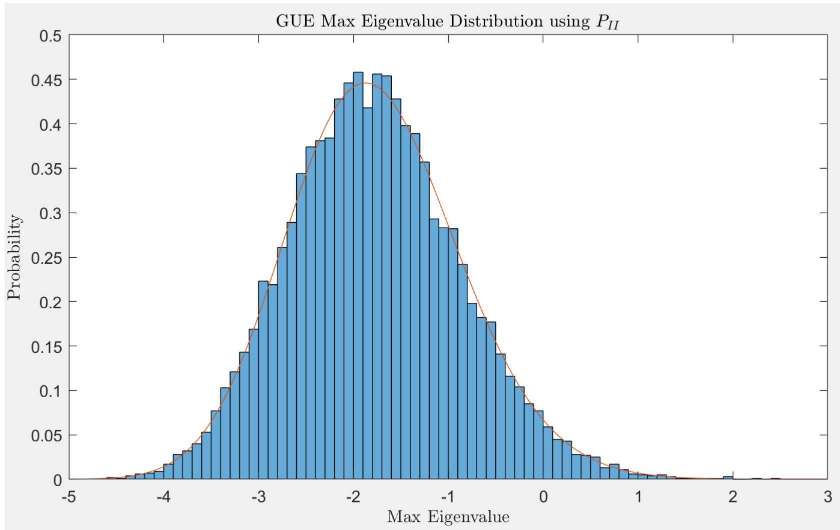

Figure 3: GUE maximum eigenvalue distribution against the pdf generated by the numerical solution to Painlevé II.

The Painlevé Equation Approach In this case, we also only could find the distribution for $\beta = 2$ but not for $\beta = 1, 4$. The biggest eigenvalue distribution is given by [1]:

$$\mathbb{P}^{(N)}_{GUE}(\lambda_{max} \leq s) = \exp\left(-\int_s^{\infty} \sigma(t)\, dt\right),$$

where σ is the solution to the σ-form of the Painlevé IV equation:

$$\sigma''^2 = 4(\sigma - t\sigma')^2 - 4\sigma'^2(\sigma' + 2n)$$

and

$$\sigma(x) \simeq \left(\mathrm{Ai}'(t)^2 - t\,\mathrm{Ai}(t)^2\right)$$

as $t \to \infty$.

The numerical approach The ODE system for the normal PIV form will be:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ q' \end{pmatrix} = \begin{pmatrix} q' \\ \frac{1}{2q}\left(q'\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2\left(t^2 - \alpha\right)q + \frac{\beta}{q} \end{pmatrix}$$

For the σ-form we will get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \sigma \\ \sigma' \\ I \end{pmatrix} = \begin{pmatrix} \sigma' \\ \sqrt{4\left(t\sigma' - \sigma\right)^2 - 4\sigma'^2(\sigma' + 2N)} \\ -\sigma \end{pmatrix}$$

where

$$I = \int_s^{\infty} \sigma(t) dt.$$

However, results for this simulation were very unstable. We see a tested case here:

3.4 The Bulk Limit Scaling

The Bulk scaling limit is a fundamental concept in random matrix theory, especially in the study of eigenvalue statistics of large random matrices. It describes the limiting behavior of the eigenvalue distribution in the "bulk" of the spectrum, i.e., away from the edges of the eigenvalue distribution. This limit is particularly interesting because it often reveals universal properties that are independent of the specific details of the matrix ensemble.

```
%% Painleve IV sigma form

N = 10;
dydt = @(t, y) [y(2); -2*sqrt((t.*y(2)-y(1)).^2 - 4*(y(2)).^2 .* (y(2) + 2*N)); -y(1); -y(2)];
t0 = 1.8*sqrt(N-1);
tn = 0;
asymp_func = @(x) 2^(N-1) .* (x.^(2*N-2)) .* exp(-x.^2) / sqrt(pi) / gamma(N);
I0 = integral(asymp_func, t0, 10*t0);
asymp_func_der = @(x) 2^(N-1) * ((2*N-2) .* (x.^(2*N-3)) - 2*(x.^(2*N-1))) .* exp(-x.^2) / sqrt(pi) / gamma(N);
y0 = [asymp_func(t0); asymp_func_der(t0); I0; -asymp_func(t0)];
opts=odeset('reltol',1e-6,'abstol',1e-8);
[t, sigma] = ode45(dydt, [t0, tn], y0, opts);

plot(t, real(sigma(:, 1)))
p_test = sigma(:, 4) .* exp(-real(sigma(:, 3)));
plot(t, p_test)
```

Figure 4: Testing with ode45 built into MATLAB. We also tried to test using the implicit solvers that MATLAB has but were not able to get it to work.

The Bulk scaling limit can be defined through various classical random matrix ensembles, such as the Hermite (Gaussian), Laguerre, and Jacobi ensembles. Each of these ensembles corresponds to a different type of random matrix with specific probability distributions for the matrix elements. However, despite these differences, they exhibit similar behavior in the bulk scaling limit, which is a testament to the universality of eigenvalue statistics in random matrix theory.

For instance, consider the Hermite or Gaussian ensemble, which consists of matrices with normally distributed independent entries. In the bulk scaling limit, as the size of the matrix goes to infinity, the local statistics of the eigenvalues (such as the spacing distribution between consecutive eigenvalues) in the central part of the spectrum tend to a universal form, known as the Wigner-Dyson distribution. This distribution is characterized by a repulsion between nearby eigenvalues and is independent of the specific details of the Gaussian distribution of the matrix entries.

Similarly, the Laguerre and Jacobi ensembles, which arise in different contexts (Laguerre in Wishart matrices and Jacobi in certain types of covariance matrices), show similar universal behavior in the bulk. In these ensembles, despite the differences in the distribution of matrix elements, the bulk scaling limit leads to the same type of eigenvalue statistics as in the Gaussian case.

The Fredholm Determinant Approach Again the obtained only results for $\beta = 2$ and are curious if there is also theory for $\beta = 1, 4$. We look at the bulk distribution as the limit of finite Hermite kernel, assume λ is an eigenvalue, then we can see the bulk distribution like a spacing of the eigenvalues.

We can consider a general interval (a, b), but a more appropriate interval to analyze is (0,s), as 0 is the limit of the spacing of eigenvalues. We define the sine-kernel:

$$K_{sin}(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$$

The complementary cdf is defined as [4]:

$$\mathbb{P}^{bulk}_{GUE}(\lambda \geq s) = \lim_{N\to\infty} \mathbb{P}^{(N)}_{\mathrm{GUE}}\left(\lambda \leq \frac{\pi a}{\sqrt{2N}} \vee s + \frac{\pi b}{\sqrt{2N}} \leq \lambda\right) = \det\left(I - K_{\sin}|_{L^2(0,s)}\right)$$

The Painlevé Equation Approach The pdf of the spacing is given by

$$p(s) = \frac{d^2}{ds^2}E(s)$$

where

$$E(s) = \exp\left(-\int_0^{\pi s} \frac{\sigma(t)}{t} dt\right)$$

and σ is the solution to the Painlevé V sigma form equation

$$(t\sigma'')^2 = 4(\sigma - t\sigma')(t\sigma' - \sigma - (\sigma')^2)$$

with the boundary condition

$$\sigma(t) \approx \frac{t}{\pi} + \left(\frac{t}{\pi}\right)^2$$

as $t \to 0$ [1]. Explicit differentiation gives

$$p(s) = \frac{1}{s^2}\left(\sigma(\pi s) + \sigma(\pi s)^2 - \pi s \sigma'(\pi s)\right) E(s)$$

Again we follow the same approach and wirte the second-order differential equation as a first-order system of differential equations:

$$\frac{d}{dt}\begin{pmatrix} \sigma \\ \sigma' \end{pmatrix} = \begin{pmatrix} \sigma' \\ \frac{2}{t}\sqrt{(\sigma - t\sigma')\left(t\sigma' - \sigma - (\sigma')^2\right)} \end{pmatrix}.$$

The solution to this problem is approached as an initial-value problem, beginning at a point where $t = t_0$, with t_0 being a very small positive number. It's necessary to bypass the value $t = 0$ due to the presence of division by t in the system of equations. Avoiding $t = 0$ is not an issue since the boundary condition furnishes a precise value for $\sigma(t_0)$ as well as for $E\left(\frac{t_0}{\pi}\right)$. Consequently, the boundary conditions for this system are established at $t = t_0$.

$$\begin{cases} \sigma(t_0) = \frac{t_0}{\pi} + \left(\frac{t_0}{\pi}\right)^2 \\ \sigma'(t_0) = \frac{1}{\pi} + \frac{2t_0}{\pi} \end{cases}$$

To enhance stability we add the variable

$$I(t) = \int_0^t \frac{\sigma(t')}{t'} dt'$$

is added to the system, as well as the equation $\frac{d}{dt} I = \frac{\sigma}{t}$. The corresponding initial value is

$$I(t_0) \approx \int_0^{t_0} \left(\frac{1}{\pi} + \frac{t}{\pi^2}\right) dt = \frac{t_0}{\pi} + \frac{t_0^2}{2\pi^2}.$$

Putting it all together, the final system is

$$\frac{d}{dt}\begin{pmatrix} \sigma \\ \sigma' \\ I \end{pmatrix} = \begin{pmatrix} \sigma' \\ \frac{2}{t}\sqrt{(\sigma - t\sigma')\left(t\sigma' - \sigma - (\sigma')^2\right)} \\ \frac{\sigma}{t} \end{pmatrix}$$

with boundary condition

$$\begin{pmatrix} \sigma(t_0) \\ \sigma'(t_0) \\ I(t_0) \end{pmatrix} = \begin{pmatrix} \frac{t_0}{\pi} + \left(\frac{t_0}{\pi}\right)^2 \\ \frac{1}{\pi} + \frac{t_0}{\pi_2} \\ \frac{t_0}{\pi} + \frac{t_0^2}{2\pi^2} \end{pmatrix}$$

The Numerical Approach The code used to implement the procedure above can be found below:

```
%% Painleve V sigma form

dydt = @(t, y) [y(2); ...
    (2/t).*sqrt((y(1) - t.*y(2)).*(t.*y(2) - y(1) - (y(2)).^2)); ...
    y(1) ./ t];
t0 = 10^(-12);
tn = 16;
y0 = [(t0/pi) + (t0/pi)^2; (1/pi) + (2*t0/pi^2); (t0/pi) + (t0^2/2*pi^2)];
opts=odeset('reltol',1e-13,'abstol',1e-14);
[t, sigma] = ode45(dydt, [t0, tn], y0, opts);
plot(t, sigma(:, 1))
E = exp(-sigma(:, 3));
p = -(pi./t).^2 .* (t .* sigma(:, 2) - sigma(:, 1) - (sigma(:, 1)).^2) .* E;
```

```
plot(t/pi, p)
hold on
plot(t/pi, E)

n=100;
nrep=10000;
beta=2;
ds=zeros(1,nrep*n/2);
for ii=1:nrep
    chi_vals = sqrt(chi2rnd((n-1:-1:1)'*beta)/2);
    A = diag(randn(n,1)) + diag(chi_vals, -1) + diag(chi_vals, 1);
    l = eig(A);
    d=diff(l(n/4:3*n/4))/beta/pi.*sqrt(2*beta*n-l(n/4:3*n/4-1).^2);
    ds((ii-1)*n/2+1:ii*n/2)=d;
end
figure
histogram(ds, 'Normalization', 'pdf')
hold on
plot(t/pi, p)
```

The code to generate the histogram was taken from the handout [7] as well. The figure generated can be found in Figure 5.

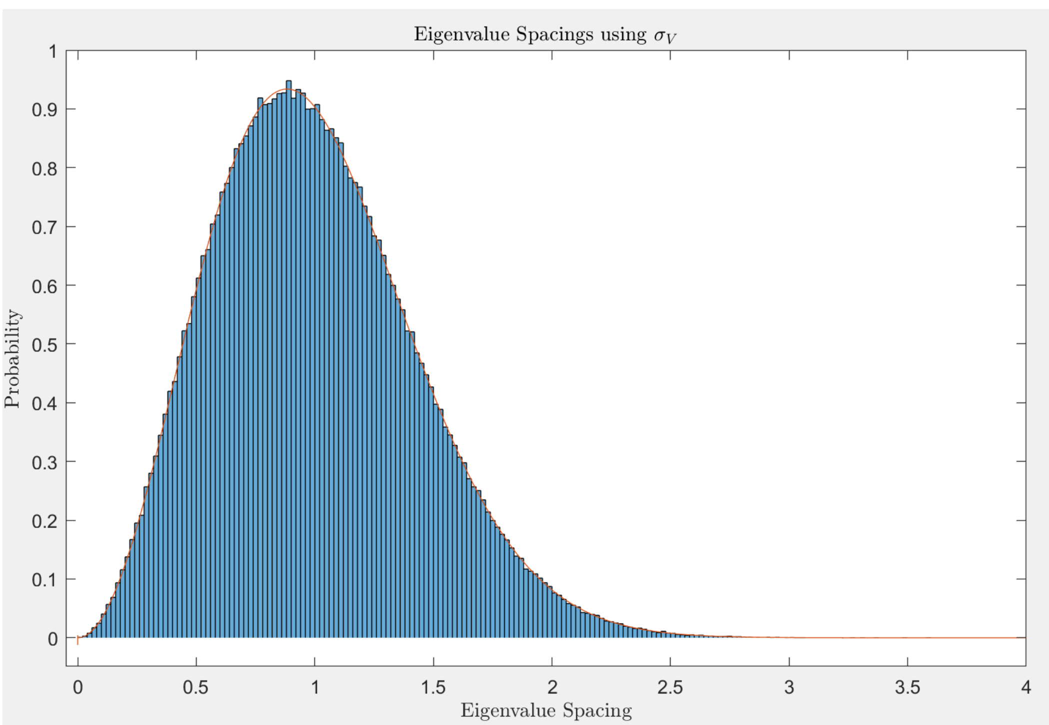

Figure 5: Bulk distribution using Painlevé V σ equation vs generated spacing.

4 Laguerre Ensemble Eigenvalue Statistics

4.1 The Tridiagonal Formulation

In the Laguerre case the tridiagonal formulation is done by defined bidiagonal matrices [2]:

$$L_n^\beta = B_n^\beta B_n^{\beta'}, \text{ where}$$

$$B_n^\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(n-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(n-1)} \end{pmatrix}$$

Note that we have the conditions $n \in \mathbb{N}, a \in \mathbb{R}, a > \frac{\beta}{2}(n-1)$. The code for implementation is given by Fig 6:

```
begin
    function generate_laguerre_B_bidiagonal_matrix(n::Int, β::Float64, a::Float64 =
    ceil(β/2 * (n+1)))
        # Ensuring 'a' satisfies the condition
        a = max(a, ceil(β/2 * (n+1)))

        # Diagonal elements: chi-distributed with degrees of freedom 2a - β(i-1)
        main_diag = [rand(Chisq(2 * a - β * (i - 1))) for i in 1:n]

        # Sub-diagonal elements: chi-distributed with degrees of freedom β(n-i)
        sub_diag = [rand(Chisq(β * (n - i))) for i in 1:n-1]

        # Creating the Bidiagonal matrix
        B = Bidiagonal(main_diag, sub_diag, :L)
        return B
    end

    function generate_laguerre_bidiagonal_matrix(n::Int, β::Float64, a::Float64 =
    ceil(β/2 * (n+1)))
        B = generate_B_bidiagonal_matrix(n, β, a)
        L = Tridiagonal(B * B')
        return L
    end
```

Figure 6: Laguerre Tridiagonal Ensemble

4.2 Largest Eigenvalue

In [1] and [2], there are discussions of how the soft edge of the Laguerre ensembles scales exactly like the Gaussian ensembles under the right renormalizations. For example, the Laguerre ensemble with $\beta = 1$ generated by an $m \times n$ matrix has (where $\xrightarrow{\mathcal{D}}$ means convergence in distribution)

$$\frac{\lambda_{\max} - (\sqrt{m-1} + \sqrt{n})^2}{(\sqrt{m-1} + \sqrt{n})\left(\frac{1}{\sqrt{m-1}} + \frac{1}{n}\right)} \xrightarrow{\mathcal{D}} F_1,$$

and for $\beta = 2$ we have

$$\frac{\lambda_{\max} - (\sqrt{m} + \sqrt{n})^2}{(\sqrt{m} + \sqrt{n})\left(\frac{1}{\sqrt{m}} + \frac{1}{n}\right)} \xrightarrow{\mathcal{D}} F_2,$$

with both results supposedly working if $m/n \to \gamma \geq 1$ as $n \to \infty$. The numerical results do show that the normalizations do give convergence to a distribution, but it doesn't seem to be exactly to

F_1 or F_2. This could be from n, m not being large enough or could be from some sort of error in normalization.

In the similar way to the Gaussian ensemble, we can obtain the Fredholm determinant representation of the largest eigenvalue for $\beta = 2$, which behaves in the limit like the Tracy-Widom distribution [10]:

$$\mathbb{P}_{LUE,\alpha}(\lambda_{max} \leq s) := \lim_{N\to\infty} \mathbb{P}^{(N)}_{\mathrm{LUE},\alpha}\left(\lambda_{max} \leq 4N + 2(2N)^{1/3}s\right)$$

where

$$\mathbb{P}^{(N)}_{\mathrm{LUE},\alpha}(\lambda_{max} \leq s) = \det\left(I - K^{(N)}_{\mathrm{Lag},\alpha}\Big|_{L^2(s,\infty)}\right)$$

and the Laguerre Kernel is defined by

$$K^{(N)}_{\mathrm{Lag},\alpha} = \frac{1}{\Gamma(\alpha+1)} \frac{L^{(\alpha)}_n(x)L^{(\alpha)}_{n+1}(y) - L^{(\alpha)}_{n+1}(x)L^{(\alpha)}_n(y)}{\frac{x-y}{n+1}\begin{pmatrix} n+\alpha \\ n \end{pmatrix}}, L^{(\alpha)}_n(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$$

where $L^{(\alpha)}_n$ are the generalised Laguerre polynomials. So the asymptotic distribution is given then by the Airy Kernel [8]:

$$\mathbb{P}_{LUE,\alpha}\left(\lambda_{\max} \leq s\right) = \det\left(I - K_{\mathrm{Ai}}|_{L^2(s,\infty)}\right)$$

We can see the results of simulation in Figures 7 and 8.

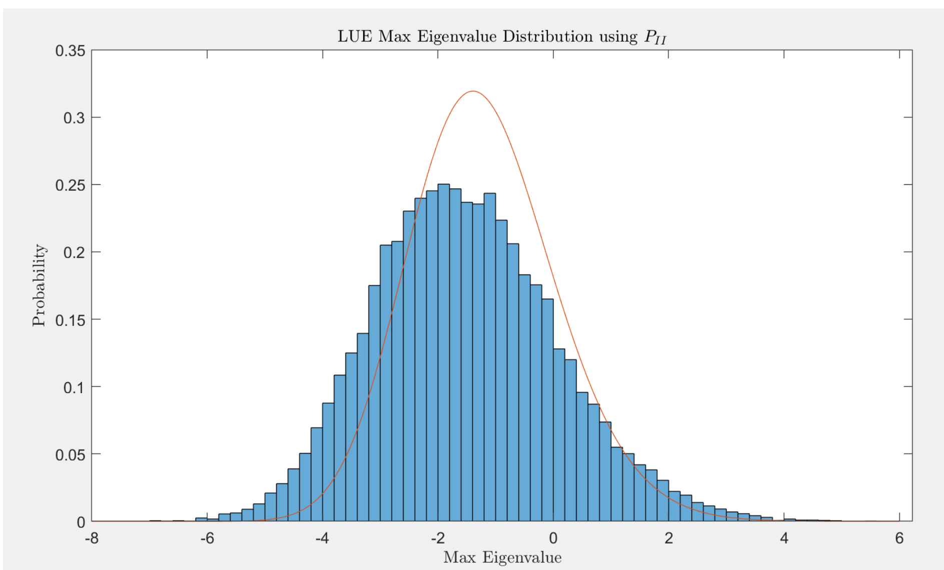

Figure 7: Max eigenvalues of 20000 Laguerre ensembles for $\beta = 1$ (disregard the title of the plot) generated by $m = 1200$ and $n = 1000$ matrices. The distribution does not seem to line up exactly.

4.3 Smallest Eigenvalue

As discussed in [1, 8], the hard edge at the large matrix limit for the smallest eigenvalue behaves as

$$1 - \mathbb{P}_{L\beta E,\alpha}(\lambda_{min} \leq s) = \begin{cases} \lim_{N\to\infty} 1 - \mathbb{P}^{(N)}_{L\beta E,\alpha}(\lambda_{min} \leq \frac{s}{4n}) & \beta = 1, 2 \\ \lim_{N\to\infty} 1 - \mathbb{P}^{(\frac{N}{2})}_{L\beta E,\alpha}(\lambda_{min} \leq \frac{s}{4n}) & \beta = 4 \end{cases}$$

using parameter $\alpha > -1$. Note that as done in [1] one can generalize it to k eigenvalues lying in an arbitrary interval, which will involve additional derivative with respective to the number of eigenvalues. We decided, to avoid the cubersome cases and focus on the (in our view) essential

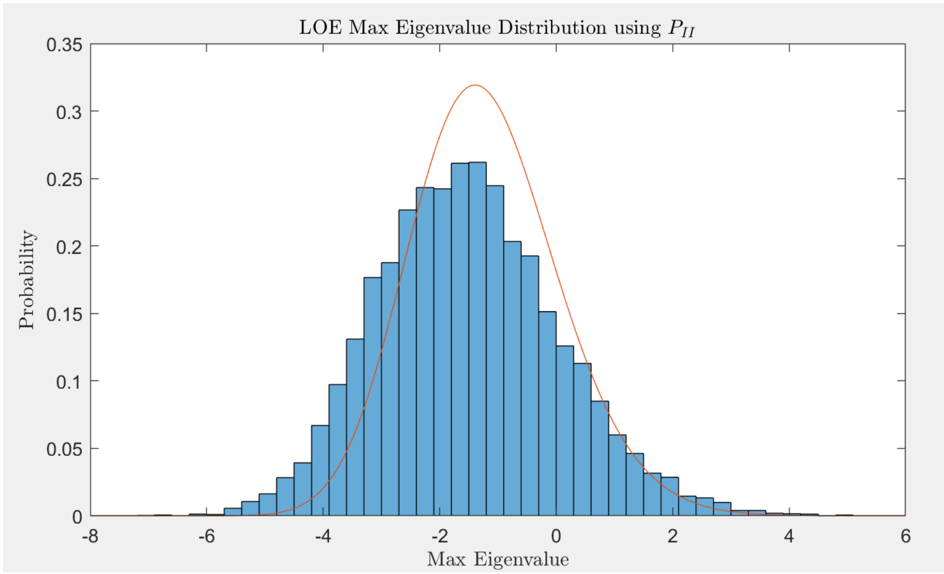

Figure 8: Max eigenvalues of 10000 Laguerre ensembles for $\beta = 1$ generated by $m = 1000$ and $n = 1000$ matrices. The distribution does not seem to line up exactly, but it does seem slightly closer than when $m = 1.2n$.

distributions. Then determinantal representations for the LUE can be also written with another representation of the Laguerre kernel [1]:

$$1 - \mathbb{P}^{(N)}_{L\beta E,\alpha}(\lambda_{min} \leq s) = \det\left(I - K^{(N)}_{\text{Lag},\alpha}\Big|_{L^2(s,\infty)}\right)$$

where the expression involving the Laguerre kernel

$$K_{n,\alpha}(x,y) = -\sqrt{n(n+\alpha)}\frac{\phi^{(\alpha)}_n(x)\phi^{(\alpha)}_{n-1}(y) - \phi^{(\alpha)}_n(y)\phi^{(\alpha)}_{n-1}(x)}{x-y},$$

where ϕ are constructed by L^2-orthogonality using the Laguerre polynomials:

$$\phi^{(\alpha)}_n(x) = \sqrt{\frac{k!}{\Gamma(k+\alpha+1)}}x^{\alpha/2}e^{-x/2}L^{(\alpha)}_k(x).$$

The scaling limit as $N \to \infty$ gives

$$1 - \mathbb{P}_{LUE,\alpha}(\lambda_{min} \leq s) = \det\left(I - K_{\text{Bess },\alpha}|_{L^2(0,s)}\right)$$

where the determinantal expression represents the complimentary cdf using the Bessel Kernel, which is defined as:

$$K_{Bess}(x,y) = \frac{\sqrt{y}J_\alpha(\sqrt{x})J'_\alpha(\sqrt{y}) - \sqrt{x}J_\alpha(\sqrt{y})J'_\alpha(\sqrt{x})}{2(x-y)}$$

4.4 The Painlave Equation Approach

In [1], we see that the limiting behavior of the hard edge gives rise to an expression for the determinantal expression in terms of a Painlevé equation:

$$1 - \mathbb{P}_{LUE}(\lambda_{min} \leq s) = \exp\left(-\int_0^s \frac{\sigma(t)}{t}dt\right),$$

where σ is the solution of the following σ form of the Painlevé III:

$$(t\sigma'')^2 = \alpha^2\sigma'^2 - \sigma'(\sigma - t\sigma')(4\sigma' - 1)$$

with boundary condition

$$\sigma(x) \approx \frac{z}{\Gamma(\alpha+1)\Gamma(\alpha+2)} \left(\frac{t}{4}\right)^{\alpha+1}$$

as $t \to 0^+$. In theory, this differential equation might be able to be solved around $z = 1$ for a few different values of z to be able to get a high-order numerical approximation of its derivative. Therefore, for fixed z, we can have

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \sigma \\ \sigma' \\ I \end{pmatrix} = \begin{pmatrix} \sigma' \\ -\frac{1}{t}\sqrt{\alpha^2 (\sigma')^2 - \sigma' (4\sigma' - 1)(\sigma - t\sigma')} \\ \frac{\sigma}{t} \end{pmatrix}.$$

The code used to attempt this for $z = 1$ is shown below, but it was not yielding very stable results.

```
%% Painleve III sigma
a = 0.5;
dydt = @(t, y) [y(2); ...
    -(1/t) .* sqrt((a*y(2))^2 - y(2)*(y(1) - t*y(2))*(4*y(2)-1)); y(1) / t];
t0 = 10^(-7);
tn = 10;
asymp_func = @(x, z) z / (gamma(a+1)*gamma(a+2)) * (x/4).^(a+1);
asymp_func_der = @(x, z) z / (gamma(a+1)^2) * x.^a / 4^(a+1);
I0 = 1/(gamma(a+2))^2 * (t0/4)^(a+1);
y0 = [asymp_func(t0, 1); asymp_func_der(t0, 1); I0];
opts=odeset('reltol',1e-3,'abstol',1e-4);
[t, sigma] = ode45(dydt, [t0, tn], y0, opts);

figure
plot(t, sigma(:, 1))

figure
plot(t, sigma(:, 3))
D_III = exp(sigma(:, 3));
figure
plot(t, D_III)
```

LUE hard edge for finite N According to [1], the determinantal expression can be expressed as the Painlave equation:

$$1 - \mathbb{P}^{(n)}_{\mathrm{LUE}}(\lambda_{min} \le s) = \exp\left(-\int_0^s \frac{\sigma(t)}{t}\, dt\right),$$

where σ is the solution to (a transformed version of) Painlevé V σ-form

$$(t\sigma'')^2 = (\sigma - t\sigma' - 2(\sigma')^2 + (2n+\alpha)\sigma')^2 - 4(\sigma')^2(\sigma' - n)(\sigma' - n - \alpha),$$

with boundary condition that

$$\sigma(t) \approx z \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)} t^{\alpha+1}$$

as $t \to 0^+$. The only difference between the Painlevé equation above and the one listed before is substituting $\sigma \mapsto -\sigma$. In theory, this differential equation might be able to be solved around $z = 1$ for a few different values of z to be able to get a high-order numerical approximation of its derivative, but we did not attempt to do this since the solver did not converge with our attempt of $z = 1$ itself. Attempting to solve this equation at $z = 1$ with the code below was extremely unstable.

```
%% Painleve V Laguerre

N = 20;
alpha = 0;
dydt = @(t, y) [y(2); ...
```

```
    -(1/t).*sqrt((y(1) - y(2).*(t + 2*y(2) - 2*N - alpha)).^2 - ...
                4*((y(2)).^2).*(y(2) - N).*(y(2) - N - alpha)); ...
    y(1) ./ t];
t0 = 10^(-12);
tn = 5;
y0 = [N*t0/2; N/2; N*t0/2];
%{
y0 = [gamma(N+alpha+1)/(gamma(N)*gamma(alpha+1)*gamma(alpha+2)) * t0^(alpha+1); ...
    gamma(N+alpha+1)/(gamma(N)*gamma(alpha+1)*gamma(alpha+2))*(alpha+1) * t0^alpha; ...
    -(t0/pi) - (t0^2/2*pi^2)];
%}
opts=odeset('reltol',1e-3,'abstol',1e-4);
[t, sigma] = ode45(dydt, [t0, tn], y0, opts);
figure
plot(t, sigma(:, 1))
figure
plot(t, exp(-sigma(:, 3)))
```

4.5 The Bulk Scaling Limit

Suprisingly the bulk limit is the same as for the Gaussian ensemble and we can refer the reader to the subsection 3.4. The scaling differs and was provided first by [6]. Nevertheless, the limit distribution is given by the sine-kernel.

5 Jacobi Ensemble Eigenvalue Statistics

5.1 Tridiagonal Formulation

For Jacobi ensemble representation the matrix takes a more complicated form. It is known, that the Jacobi ensemble matirces can be represented as a $2x2$ block matrices, the respective bidiagonal forms for each of the blocks are defined by [3]:

$$
\begin{aligned}
J^{\beta}_{a,b} &\sim \begin{bmatrix} B_{11}(\Theta,\Phi) & B_{12}(\Theta,\Phi) \\ B_{21}(\Theta,\Phi) & B_{22}(\Theta,\Phi) \end{bmatrix} \\
&= \left[\begin{array}{cccccccc}
c_n & -s_n c'_{n-1} & & & s_n s'_{n-1} & & & \\
 & c_{n-1} s'_{n-1} & \ddots & & c_{n-1} c'_{n-1} & s_{n-1} s'_{n-2} & & \\
 & & \ddots & -s_2 c'_1 & & \ddots & \ddots & \\
 & & & & & c_1 c'_1 & s_1 & \\
\hline
-s_n & -c_n c'_{n-1} & & & c_n s'_{n-1} & & & \\
 & -s_{n-1} s'_{n-1} & \ddots & & -s_{n-1} c'_{n-1} & c_{n-1} s'_{n-2} & & \\
 & & \ddots & -c_2 c'_1 & & \ddots & \ddots & \\
 & & & -s_1 s'_1 & & & -s_1 c'_1 & c_1
\end{array}\right] \\
&\beta > 0, a, b > -1
\end{aligned}
$$

$$
\begin{aligned}
\Theta &= (\theta_n, \dots, \theta_1) \in \left[0, \frac{\pi}{2}\right]^n & \Phi &= (\phi_{n-1}, \dots, \phi_1) \in \left[0, \frac{\pi}{2}\right]^{n-1} \\
c_i &= \cos\theta_i & c'_i &= \cos\phi_i \\
s_i &= \sin\theta_i & s'_i &= \sin\phi_i \\
c_i^2 &\sim \text{Beta}\left(\frac{\beta}{2}(a+i), \frac{\beta}{2}(b+i)\right) & (c'_i)^2 &\sim \text{Beta}\left(\frac{\beta}{2}i, \frac{\beta}{2}(a+b+1+i)\right)
\end{aligned}
$$

The code implementation for the decomposition is given by Fig. 9:

```
begin
function generate_J_matrix(n::Int, β::Float64, a::Float64, b::Float64)
    # Generating beta-distributed values and calculating angles
        c = [sqrt(rand(Beta(β/2*(a+i), β/2*(b+i)))) for i in 1:n]
        c_prime = [sqrt(rand(Beta(β/2*i, β/2*(a+b+1+i)))) for i in 1:n-1]
        s = sqrt.(c)
        s_prime = sqrt.(c_prime)

        # Constructing the sub-matrices as Bidiagonal
        B11 = Bidiagonal(reverse([c[1:end-1] .* s_prime; c[end]]), -reverse(s[2:end]
    .* c_prime), :U)
        B12 = Bidiagonal(reverse([s[1];s[2:end] .* s_prime]), reverse(c[1:end-1] .*
    c_prime), :L)
        B21 = Bidiagonal(-reverse([s[1:end-1] .* s_prime; s[end]]), -reverse(c[2:end]
    .* c_prime), :U)
        B22 = Bidiagonal(reverse([c[1]; c[2:end] .* s_prime]),-reverse(s[1:end-1] .*
    c_prime), :L)

        # Combining sub-matrices into the final matrix
        upper_half = [B11 B12]
        lower_half = [B21 B22]
        J = [upper_half; lower_half]
        return J
    end
```

Figure 9: Jacobi 2x2 Block Tridiagonal Ensemble

5.2 The Eigenvalue Statistics of The Jacobi ensemble

In this case the literature gets more involved and less connected to the Painlave equations, more feasible tend to be hypergeomtric functions, which describe explicit eigenvalue distributions. We start here by definining the finite Jacobi Kernel:

$$K^{(N)}_{Jac;a,b}(x,y) = \frac{(w(x)w(y))^{1/2}}{(p_{N-1},p_{N-1})_2}\frac{p_N(x)p_{N-1}(y)-p_{N-1}(x)p_N(y)}{x-y}$$

where in addition we need to define:

$$p_j(x) = (-1)^j j!\frac{\Gamma(a+b+j+1)}{\Gamma(a+b+2j+1)}P_j^{(a,b)}(1-2x) \quad w(x) = x^a(1-x)^b \quad (p_j,p_j)_2 := \int_0^1 (p_j(x))^2\, x^a(1-x)^b dx$$

$P_j^{(a,b)}$ are Jacobi polynomais. Now the distribution of the smalles eigenvalue for $\beta = 2$ is defined by [4]:

$$1 - \mathbb{P}^{(N)}_{Jac,a,b}(\lambda_{min} \leq s = \det\left(I - K^{(N)}_{\text{Jac},a,b}\Big|_{L^2(0,s)}\right)$$

For the appropriate scaling we obtain that the distribution of the smalles eigenvalue in the infinite dimensional limit is given by [9]:

$$1 - \mathbb{P}_{JUE,a,b}\left(\lambda_{\min} \leq s\right) = \lim_{N\to\infty} 1 - \mathbb{P}^{(N)}_{JUE,a,b}\left(\lambda_{\min} \leq \frac{s}{4n^2}\right) = \det\left(I - K_{\text{Bess},a}|_{L^2(0,s)}\right)$$

5.3 The bulk scaling limt

As in the case of the Hermite and Laguerre ensemble, in the Jacobi ensemble we obtain the same limiting bulk distribution, which is given by the sine kernel described in 3.4.

5.4 Eigenvalue Density

The density of the eigenvalues in [5] relates to the Painlevé VI σ-form directly. The sources gives that (for Jacobi weight $x^a(1-x)^b$). Specifically it gives the probability of having precisely N eigenvalues within the range $(t,1)$ for the given ensemble.

$$\mathbb{P}^{(N)}_{J\beta E,a,b}(\lambda_1,\cdots,\lambda_N\in(t,1))=\exp\left(\int_{1-t}^{1}\frac{ds}{s(1-s)}\left(\sigma(s)-\left(N+\frac{a+b}{2}\right)^2\left(s-\frac{1}{2}\right)-\frac{a^2+b^2}{8}\right)\right),$$

where σ satisfies the Painlevé VI σ-form

$$\begin{aligned}\sigma'(t(t-1)\sigma'')^2+&\left(\sigma'[2\sigma-(2t-1)\sigma']+\left(N+\frac{a+b}{2}\right)^2\frac{a^2-b^2}{4}\right)^2\\ &=\left(\sigma'+N+\frac{a+b}{2}\right)^2\left(\sigma'+\frac{a+b}{2}\right)\left(\sigma'+\frac{a-b}{2}\right),\end{aligned}$$

with boundary condition

$$\sigma(s;z)\approx s(1-s)^{b+1}zC_N(a,b)+\left(N+\frac{a+b}{2}\right)^2\left(s-\frac{1}{2}\right)+\frac{a^2+b^2}{8}$$

as $t\to 0^+$, where

$$C_N(a,b)=\frac{\Gamma(a+b+N+1)\Gamma(b+N+1)}{\Gamma(N)\Gamma(a+N)\Gamma(b+1)\Gamma(b+2)}.$$

6 Sampling from the Fredholm determinant and the continious DPP

By using the kernels

$$\begin{aligned}K_{\mathrm{Ai}}(x,y)&=\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}\\ K_{\sin}(x,y)&=\frac{\sin(\pi(x-y))}{\pi(x-y)}\\ K_{Bess}(x,y)&=\frac{\sqrt{y}J_\alpha(\sqrt{x})J'_\alpha(\sqrt{y})-\sqrt{x}J_\alpha(\sqrt{y})J'_\alpha(\sqrt{x})}{2(x-y)},\end{aligned}$$

in the infinite dimensional case, but also the finite dimensional kernels:

$$K^{(N)}_{\mathrm{Herm}}(x,y)=\frac{1}{\sqrt{2n}}\frac{\pi_n(x)\pi_{n+1}(y)-\pi_n(y)\pi_{n+1}(x)}{y-x},\quad \pi_j(x)=\frac{H_j^{Hermite}(x)}{\left(2^j\sqrt{\pi}j!\right)^{1/2}}$$

$$K^{(N)}_{\mathrm{Lag},\alpha}(x,y)=\frac{1}{\Gamma(\alpha+1)}\frac{L_n^{(\alpha)}(x)L_{n+1}^{(\alpha)}(y)-L_{n+1}^{(\alpha)}(x)L_n^{(\alpha)}(y)}{\frac{x-y}{n+1}\begin{pmatrix}n+\alpha\\ n\end{pmatrix}},\quad L_n^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$$

$$\tilde{K}^{(N)}_{Lag;\alpha}(x,y)=-\sqrt{n(n+\alpha)}\frac{\phi_n^{(\alpha)}(x)\phi_{n-1}^{(\alpha)}(y)-\phi_n^{(\alpha)}(y)\phi_{n-1}^{(\alpha)}(x)}{x-y},\quad \phi_n^{(\alpha)}(x)=\sqrt{\frac{k!}{\Gamma(k+\alpha+1)}}x^{\alpha/2}e^{-x/2}L_k^{(\alpha)}(x)$$

$$K^{(N)}_{Jac;a,b}(x,y)=\frac{(w(x)w(y))^{1/2}}{(p_{N-1},p_{N-1})_2}\frac{p_N(x)p_{N-1}(y)-p_{N-1}(x)p_N(y)}{x-y},\quad p_j(x)=(-1)^j j!\frac{\Gamma(a+b+j+1)}{\Gamma(a+b+2j+1)}P_j^{(a,b)}(1-2x)$$

where $w(x)=x^a(1-x)^b$, $(p_j,p_j)_2:=\int_0^1(p_j(x))^2x^a(1-x)^b dx$ and $P_j^{(a,b}$ are Jacobi polynomials, we can use the discrete & continuous DPPs constructed using these kernels to get a sample of the determinantal distribution. For the discrete and for the continious case we discretize the interval on which the kernels should be calculated. On the intervasl of interest we finite matrices K_n that we can use to approximate the original $K(x,y)$. By sampling from this DPP, we get an empirical estimate for the distributions of the determinantal expressions that help determine the distributions of the hard edge, soft edge, and bulk distributions. We can also use the other kernels corresponding to finite n matrices that were discussed throughout the report to produce empirical distributions of those eigenvalue distributions as well, but they tend to be much more complex. Note that in addition one should adapt the normalisation and scaling of the DPP distribution to obtain the exact function. The sampling will be done using the the projection DPP approach Fig. 10.

```julia
function DPPSampler()

    function randprojDPP(Y)
    n = size(Y, 2)
    J = fill(0, n)
    for k in 1:n
        p = mean(abs.(Y).^2, dims=2)
        J[k] = rand(Categorical(p[:]))
        Y = ( Y * qr(Y[J[k], :]).Q )[:, 2:end]
    end
        return sort(J)
    end

    function randDPP(Λ, Y)
       mask = rand.(Bernoulli.(Λ))
       return randprojDPP(Y[:, mask])#, (mask)
    end

        ########### Hermite KERNEL ################
    Γ = gamma
    Hj(j, x) = basis(Hermite, j)(x) # Hermite Polynomial
    ϕ(j, x) = exp(-x^2/2)*Hj(j, x)/( π^(1/4)*sqrt(Γ(j+1))*2^(j/2) )

    Kernel0_Hermite(j, x) = j*ϕ(j,x)^2-sqrt(j*(j+1))*ϕ(j-1,x)*ϕ(j+1,x)
    Kernel_Hermite(j, x, y) = x==y ? Kernel_Hermite(j, x) : sqrt(j/2) * (ϕ(j,x)*ϕ(j-
    1,y) - ϕ(j-1,x) * ϕ(j,y))/(x-y)
        ########### Sine KERNEL ################
    Kernel_sinc(x, y)    = x==y ? 1.0 : sin( π * (x - y) ) / ( π * (x - y) )
    Kernel0_sinc(x) = sin( π*x )/ (π*x)

    Ja = besselj
    Kernel_Bess(α, x, y) = x==y ? ( Ja(α, √x)^2 - Ja(α+1, √x) * Ja(α-1, √x) ) / 4 :
                                  ( Ja(α+1, √x) * √x * Ja(α, √y) - √y * Ja(α+1,
                         √y) * Ja(α, √x) ) / ( 2 * (x - y) )
        ########### Sine KERNEL ################
    airy_kernel(x, y) = x==y ? (airyaiprime(x))^2 - x * (airyai(x))^2 :
               (airyai(x) * airyaiprime(y) - airyai(y) * airyaiprime(x)) / (x - y)
    N_DPP = 15
    dx = 0.05
    x = -6:dx:6
    # For example
    K = [Kernel_Hermite(N_DPP, xi, xj) for xi in x, xj in x] * dx

    S, V = eigen(K);
    S[abs.(S).<1e-10] .= 0.0
    S[abs.(S).>1.0] .= 1.0
    r_DPP = [];
    for i in 1:10000
        append!(r_DPP, randDPP(S, V))
    end

    return histogram(r_DPP, normalized=true, bins=80)
end
```

Figure 10: The DPP sampling for the infinite dimensional kernels

7 Conclusion

From the class and the literature review we did for this project, it is clear that the Painlevé equations are deeply entangled with the different eigenvalue statistics and distributions of the Hermite, Laguerre, and even Jacobi ensembles. Our numerical results show that it is indeed difficult to get stable approximations for the Painlevé transcendants, but we were able to get stable results for the Painlevé II and V equations. The Painlevé differential equations are fairly nonlinear in the solution function and its first derivative, and the σ-forms are even nonlinear in the second derivative, which is the highest order. This on its own can lead to heavy instability. Coupled with the asymptotic boundary conditions that vanish at the limit (especially those to ∞ that vanish in all derivatives as well), we get highly unstable results that get worse with the numerical instability from low precision in the boundary condition.

It's possible that these issues could have been solved by using Julia which tends to have the superior differential equation solvers, but the lack of familiarity with Julia and the issues that we came across trying to implement the Painlevé V equation made MATLAB the ultimate choice for our specific implementations. To be more precise, we tried to solve the Painlevé V equation in Julia but could not get it to work, and the same code translated into MATLAB worked exactly; we could not find the differences in the two (pretty short) codes.

We also proposed a direct way to connect the standard Painlevé transcendents to the solutions to the σ-forms that are more commonly used for the eigenvalue distributions. We didn't test how viable this was due to the fact that we were not able to generate consistent boundary conditions that transferred to the correct ones for the σ-form solution, but it seems like a promising research direction.

Overall, it's possible that numerical integration techniques constructed specifically for each equation can be stable and perform well, but we did not get stable results ourselves with our implementations. We did get some positive numerical results, and though there were many more that were unstable, we gained a great perspective into these Painlevé differential equations and how capable they are in helping express the distributions of various different eigenvalue relationships in random matrix theory.

References

[1] Folkmar Bornemann. "On the Numerical Evaluation of Distribution in Random Matrix Theory: A Review". In: (2010). DOI: `https://doi.org/10.48550/arXiv.0904.1581`.

[2] A. Edelman and N.R. Rao. "Random Matrix Theory". In: *Acta Numerica* (2005), pp. 1–65. DOI: `10.1017/S0962492904000236`.

[3] Alan Edelman and Brian D Sutton. "The Beta-Jacobi Matrix Model, the CS Decomposition, and Generalized Singular Value Problems". In: *Foundations of Computational Mathematics* 8.2 (2008), pp. 259–285.

[4] P.J. Forrester. "Eigenvalue probabilities — Painlevé systems approach". In: *Log-Gases and Random Matrices (LMS-34)*. Princeton University Press, 2010, pp. 328–379. ISBN: 9780691128290. URL: `http://www.jstor.org/stable/j.ctt7t5vq.11` (visited on 12/12/2023).

[5] P.J. Forrester and N.S. Witte. "Random Matrix Theory and the Sixth Painlevé Equation". In: (2006). DOI: `https://doi.org/10.48550/arXiv.math/0610471`.

[6] Taro Nagao and Miki Wadati. "Correlation functions of random matrix ensembles related to classical orthogonal polynomials". In: *Journal of the Physical Society of Japan* 60.10 (1991), pp. 3298–3322.

[7] Per-Olof Persson. *Numerical Methods in Random Matrices - Handout #7*. Sept. 2004.

[8] Craig A Tracy and Harold Widom. "Fredholm determinants, differential equations and matrix models". In: *Communications in Mathematical Physics* 163.1 (1994), pp. 33–72.

[9] Craig A Tracy and Harold Widom. "Level spacing distributions and the Bessel kernel". In: *Communications in Mathematical Physics* 161.2 (1994), pp. 289–309.

[10] Craig A Tracy and Harold Widom. "Level-spacing distributions and the Airy kernel". In: *Communications in Mathematical Physics* 159 (1994), pp. 151–174.