The Weingartens function for $\beta = 1, 2, 4$ and its possible generalizations using Jack polynomials

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Schur polynomials

Derivation of the Weingarten formula for $U(N)$

Generalization for $\beta \in \{1, 4\}$

Jack polynomials

Results

Problem setting

- Let $\mathcal{U}(N)$ be a unitary group. Find the matrix moment

$$\int_{\mathcal{U}(N)} dU\, U_{i_1 j_1} \dots U_{i_n j_n} \cdot \overline{U_{i'_1 j'_1} \dots U_{i'_n j'_n}},$$

where dU is the normalized Haar measure.

- Let $\mathcal{O}(N)$ be an orthogonal group. Find the matrix moment

$$\int_{\mathcal{O}(N)} dU\, O_{i_1 j_1} \dots O_{i_{2n} j_{2n}}.$$

In both the cases, one can easily show that other matrix moments for these groups are equal to 0.

Schur polynomials

- For any partition $\lambda = (\lambda_1, \dots, \lambda_d) \vdash n$ denote by $p_\lambda(x)$ the *power sum polynomial*

$$p_\lambda(x_1, \dots, x_m) = (x_1^{\lambda_1} + \dots + x_m^{\lambda_1}) \cdot \ldots \cdot (x_1^{\lambda_d} + \dots + x_m^{\lambda_d}).$$

  The polynomials $p_\lambda(x)$, $\lambda \vdash n$ form a basis of the space of symmetric polynomials of degree n.
- For any $\pi \in S_n$, denote by $[\pi]$ the partition corresponding to the cyclical decomposition of π. Also, denote by $p_\pi(x) = p_{[\pi]}(x)$ the corresponding symmetric polynomial.
- Let $\chi^\lambda \colon S_n \to \mathbb{Z}$ be an irreducible character of S_n corresponding to the partition λ. Denote by $s_\lambda(x)$ the *Schur polynomial* defined as follows:

$$s_\lambda(x) = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu(x) = \sum_{\pi \in S_n} \chi^\lambda(\pi) p_{[\pi]}, \quad z_\mu = \prod_i i^{t_i} t_i!, \mu = (1^{t_1} 2^{t_2} \dots)$$

Properties of Schur polynomials

- Define the Hall inner product: $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\mu$. The Schur polynomials forms **an orthonormal basis**: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.
- We can represent the power sum polynomials in the Schur polynomials basis as follows:
$$p_\mu = \sum_{\lambda \vdash n} \langle p_\mu, s_\lambda \rangle s_\lambda = \sum_{\lambda \vdash n} \chi^\lambda(\mu) s_\lambda.$$
- **Triangularity:** $s_\lambda = \sum_{\mu \preceq \lambda} c_{\lambda\mu} m_\mu$;

The Schur polynomials are uniquely defined by the properties (1) and (3).

Connection with the unitary group

Definition

Let $p(x_1, \ldots, x_n)$ be a symmetric polynomial, and let A be a $n \times n$ matrix. Define

$$p(A) = p(\alpha_1, \ldots, \alpha_n),$$

where α_i are the eigenvalues of the matrix A.

- Group theoretic fact: s_λ are irreducible characters of the unitary group, hence orthogonal
$$\int_{U(N)} dU \; s_\lambda(U)\overline{s_\mu(U)} = \delta_{\lambda\mu}.$$
- Generalization of the previous fact: for any matrices A and B we have
$$\int_{U(N)} dU \; s_\lambda(AU)\overline{s_\mu(BU)} = \delta_{\lambda\mu}\frac{s_\lambda(AB^*)}{s_\lambda(I_N)}.$$

Weingarten function for $U(N)$

Given real matrices A and B, let's find the integral $\int_{U(N)} dU \; p_{(1^n)}(AU)\overline{p_{(1^n)}(B^T U)}$.

We have $p_{(1^n)}(A) = \sum \chi^\lambda(1^n) \cdot s_\lambda(A)$; therefore,

$$\int_{U(N)} dU \; p_{(1^n)}(AU) \cdot \overline{p_{(1^n)}(B^T U)}$$
$$= \int_{U(N)} dU \left(\sum_{\lambda \vdash n} \chi^\lambda(1^n) \cdot s_\lambda(AU) \right) \cdot \left(\sum_{\lambda \vdash n} \chi^\lambda(1^n) \cdot \overline{s_\lambda(B^T U)} \right)$$
$$= \sum_{\lambda \vdash n} \chi^\lambda(1^n)^2 \cdot \frac{s_\lambda(AB)}{s_\lambda(I_N)}$$

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For any partition $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n$, we can write an explicit formula for $p_\lambda(A)$:

$$p_\lambda(A) = \mathrm{Tr}(A^{\lambda_1}) \cdot \ldots \cdot \mathrm{Tr}(A^{\lambda_d}) = \sum_{i_1, \ldots, i_n} A_{i_1 i_{\pi(1)}} \cdot \ldots \cdot A_{i_n i_{\pi(n)}},$$

where π is any permutation such that $[\pi] = \lambda$.

- $p_{(1^n)}(AU) = \mathrm{Tr}(AU)^n = \left(\sum_{i,j} A_{ji} U_{ij}\right)^n$;
- $\overline{p_{(1^n)}(B^T U)} = p_{(1^n)}(B^T \overline{U}) = \mathrm{Tr}(B^T \overline{U})^n = \left(\sum_{i,j} B_{i'j'} \overline{U}_{i'j'}\right)^n$;
- $$s_\lambda(AB) = \frac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) \cdot p_\pi(AB)$$
  $$= \frac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) \cdot \sum_{i_1, \ldots, i_n} \sum_{j_1, \ldots, j_n} A_{j_1 i_1} B_{i_1 j_{\pi(1)}} \cdot \ldots \cdot A_{j_n i_n} B_{i_n j_{\pi(n)}}.$$

Derivation of the Weingarten formula for $U(N)$

Comparing the coefficients of the term $A_{j_1 i_1} \dots A_{j_n i_n} \cdot B_{i'_1 j'_1} \dots B_{i'_n j'_n}$ of both left- and right-hand side of the equality

$$\int_{U(N)} dU \; p_{(1^n)}(AU) \cdot \overline{p_{(1^n)}(B^T U)} = \sum_{\lambda \vdash n} \chi^\lambda(1^n)^2 \cdot \frac{s_\lambda(AB)}{s_\lambda(I_N)}$$

we conclude that

$$\begin{aligned} (n!)^2 \int_{\mathcal{U}(N)} dU \, U_{i_1 j_1} \dots U_{i_n j_n} \cdot \overline{U_{i'_1 j'_1} \dots U_{i'_n j'_n}} \\ &= \frac{\partial^n}{\partial A_{j_1 i_1} \dots A_{j_n i_n}} \frac{\partial^n}{\partial B_{i'_1 j'_1} \dots B_{i'_n j'_n}} \int_{U(N)} dU \; p_{(1^n)}(AU) \cdot \overline{p_{(1^n)}(B^T U)} \\ &= \frac{\partial^n}{\partial A_{j_1 i_1} \dots A_{j_n i_n}} \frac{\partial^n}{\partial B_{i'_1 j'_1} \dots B_{i'_n j'_n}} \cdot \sum_{\lambda \vdash n} \chi^\lambda(1^n)^2 \cdot \frac{s_\lambda(AB)}{s_\lambda(I_N)} \\ &= \sum_{\lambda \vdash n} \frac{\chi^\lambda(1^n)^2}{s_\lambda(I_N)} \cdot \sum_{\sigma, \tau \in S_n} \chi^\lambda(\tau^{-1}\sigma) \delta^\sigma_{i,i'} \delta^\tau_{j,j'}, \text{where } \delta^\sigma_{a,b} = \delta_{a_1 b_{\sigma(1)}} \dots \delta_{a_n b_{\sigma(n)}} \end{aligned}$$

Weingarten function for $U(N)$

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Denote by

$$\mathrm{Wg}^U = \frac{1}{(n!)^2} \sum_{\lambda \vdash n} \frac{\chi^\lambda(1^n)^2}{s_\lambda(I_N)} \cdot \chi^\lambda$$

This function is called *the Weingarten function*.

We derived the formula:

$$\int_{\mathcal{U}(N)} dU\, U_{i_1 j_1} \dots U_{i_n j_n} \cdot \overline{U_{i'_1 j'_1} \dots U_{i'_n j'_n}} = \sum_{\sigma, \tau \in S_n} \mathrm{Wg}^U(\tau^{-1}\sigma) \delta^\sigma_{i,i'} \delta^\tau_{j,j'}$$

Question: can this formula be generalized for other compact Lie groups?

Pair partitions set

Let M_{2n} be the set of all pair partitions on $\{1, 2, \ldots, 2n\}$. Each pair partition $\mathfrak{m}$ in M_{2n} is uniquely expressed by the form

$$\{\{\mathfrak{m}(1), \mathfrak{m}(2)\}, \{\mathfrak{m}(3), \mathfrak{m}(4)\}, \ldots, \{\mathfrak{m}(2n-1), \mathfrak{m}(2n)\}\},$$

where $\mathfrak{m}(2k-1) < \mathfrak{m}(2k)$ and $\mathfrak{m}(1) < \mathfrak{m}(3) < \cdots < \mathfrak{m}(2n-1)$.

For each partition $\sigma \in S_{2n}$, consider the graph $\Gamma(\sigma)$ whose vertex are $\{1, 2, \ldots, 2n\}$ and whose edge set consists of $\{2i-1, 2i\}$ and $\{\sigma(2i-1), \sigma(2i)\}$. Then, $\Gamma(\sigma)$ consists of connected parts of even lengths $(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_d)$. Then denote by **the coset type** $[\sigma]$ the partition $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n$.

In addition, denote $M_{2n}^U \subset M_{2n}$ the set of such pair matchings between sets $\{1, 3, \ldots, 2n-1\}$ and $\{2, 4, \ldots, 2n\}$. Also, $\delta_{\mathfrak{m}}(i) = \prod \delta_{i_{\mathfrak{m}(2k-1)} i_{\mathfrak{m}(2k)}}$ and $\Delta_{\mathfrak{m}}(i) = \prod \left(\delta_{i_{\mathfrak{m}(2k-1)}+N, i_{\mathfrak{m}(2k)}} - \delta_{i_{\mathfrak{m}(2k-1)}, i_{\mathfrak{m}(2k)}+N}\right)$

Generalization of the Weingarten function

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See [CM09, Nov14, GK21]

- **The case** $\beta = 1$. The corresponding Lie group is $O(N)$;

$$\int_{O(N)} dO \; O_{i_1 j_1} \dots O_{i_{2n} j_{2n}} = \sum_{\mathfrak{m}, \mathfrak{n} \in M_{2n}} \mathrm{Wg}^{O}([\mathfrak{m}^{-1}\mathfrak{n}]) \cdot \delta_{\mathfrak{m}}(i) \delta_{\mathfrak{n}}(j);$$

- **The case** $\beta = 2$. The corresponding Lie group is $U(N)$;

$$\int_{U(N)} dU \; U_{i_1 j_1} \overline{U}_{i_2 j_2} \dots U_{i_{2n-1} j_{2n-1}} \overline{U}_{i_{2n} j_{2n}} = \sum_{\mathfrak{m}, \mathfrak{n} \in M^{U}_{2n}} \mathrm{Wg}^{U}([\mathfrak{m}^{-1}\mathfrak{n}]) \cdot \delta_{\mathfrak{m}}(i) \delta_{\mathfrak{n}}(j);$$

- **The case** $\beta = 4$. The corresp. Lie group is $Sp(N) = Sp(2N, \mathcal{C}) \cap U(2N)$;

$$\int_{Sp(2N)} dS \, S_{i_1 j_1} \dots S_{i_{2n} j_{2n}} = \sum_{\mathfrak{m}, \mathfrak{n} \in M_{2n}} \mathrm{Wg}^{Sp}([\mathfrak{m}^{-1}\mathfrak{n}]) \cdot \Delta_{\mathfrak{m}}(i) \Delta_{\mathfrak{n}}(j) \cdot (-1)^{\mathfrak{m}^{-1}\mathfrak{n}}$$

Jack polynomials

For real α, consider the inner product on the space of symmetric polynomials:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} \alpha^{l(\lambda)} z_\lambda.$$

This inner product generalizes **the Hall inner product** for $\alpha = 1$.

The family of symmetric polynomials $J_\lambda^{(\alpha)}(x)$ is the unique family that satisfies:

- **Orthogonality:** $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = 0$ if $\lambda \neq \mu$;
- **Triangularity:** $J_\lambda^{(\alpha)} = \sum_{\mu \preceq \lambda} c_{\lambda\mu}^{(\alpha)} m_\mu$;
- **Normalization:** $[m_{1^n}] J_\lambda^{(\alpha)} = n!$.

For $\alpha = 1$, Jack polynomials generalizes the Schur polynomials up to normalization coefficients: $J_\lambda^{(1)} = H_\lambda \cdot s_\lambda$, where H_λ is *the hook product*.

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Jack polynomials are zonal spherical polynomials

The following equalities hold [Mac98, VII, Examples in §4, 5, 6]:

- For all symmetric matrices A, B: $\displaystyle\int_{O(N)} dO\, J_\lambda^{(2)}(AOBO^T) = \frac{J_\lambda^{(2)}(A) J_\lambda^{(2)}(B)}{J_\lambda^{(2)}(I_N)}$;
- For all Hermitean A, B: $\displaystyle\int_{U(N)} dU\, J_\lambda^{(1)}(AUBU^*) = \frac{J_\lambda^{(1)}(A) J_\lambda^{(1)}(B)}{J_\lambda^{(1)}(I_N)}$;
- Denote by $\mathfrak{gl}(N, \mathbb{H})$ the set of $2N \times 2N$ complex matrices of the form $\begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix}$. For any symmetric polynomial f the value ${}^H f(A)$ is the following: ${}^H p_\lambda(A) := p_\lambda(A)/2^{l(\lambda)}$, for other polynomials by linearity. Then for all Hermitean $A, B \in \mathfrak{gl}(n, \mathbb{H})$:

$$\int_{Sp(N)} dS\, {}^H J_\lambda^{(1/2)}(ASBS^*) = \frac{{}^H J_\lambda^{(1/2)}(A) \cdot {}^H J_\lambda^{(1/2)}(B)}{J_\lambda^{(1/2)}(I_N)}$$

Exact formula for Weingarten function

Represent power sum polynomials in the basis of Jack polynomials:

$$p_\mu = \sum_{\lambda \vdash n} \omega_\lambda(\mu; \alpha) J_\lambda^\alpha.$$

Then one can (???) derive the following formulas for Weingarten functions:

$$\mathrm{Wg}^O(\mu) = \sum_{\lambda \vdash n} \frac{\omega_\lambda(\mu, 2/1)}{J_\lambda^{(2/1)}(I_N)},$$

$$\mathrm{Wg}^U(\mu) = \sum_{\lambda \vdash n} \frac{\omega_\lambda(\mu, 2/2)}{J_\lambda^{(2/2)}(I_N)},$$

$$\mathrm{Wg}^{Sp}(\mu) = (-4)^n \cdot (-1)^{l(\mu)} \cdot \sum_{\lambda \vdash n} \frac{\omega_\lambda(\mu, 2/4)}{J_\lambda^{(2/4)}(I_N)}(????).$$

Possible generalizations for other β

Question 1 (orthogonal group generalization). Is it possible to find the measure $d^{\,\beta}O$ on the space of $N \times N$ matrices such that

$$\int_{{}^\beta O(N)} d^{\,\beta}O\, J_\lambda^{(2/\beta)}(A^{\,\beta}OB^{\,\beta}O^T) = \frac{J_\lambda^{(2/\beta)}(A)J_\lambda^{(2/\beta)}(B)}{J_\lambda^{(2/\beta)}(I_N)},$$

$$\int_{{}^\beta O(N)} d^{\,\beta}O\ {}^\beta O_{i_1j_1}\dots{}^\beta O_{i_{2n}j_{2n}} = \sum_{\mathfrak{m},\mathfrak{n}\in M_{2n}} \mathrm{Wg}^\beta([\mathfrak{m}^{-1}\mathfrak{n}])\cdot\delta_{\mathfrak{m}}(i)\delta_{\mathfrak{n}}(j);$$

for all symmetric (diagonal) matrices A and B?

Possible generalizations for other β

Question 2 (unitary group generalization). Is it possible to find the measure $d^{\,\beta}U$ on the space of $N \times N$ matrices such that

$$\int_{{}^{\beta}U(N)} d^{\,\beta}U\, J_\lambda^{(2/\beta)}(A^{\,\beta}UB^{\,\beta}U^T) = \frac{J_\lambda^{(2/\beta)}(A)J_\lambda^{(2/\beta)}(B)}{J_\lambda^{(2/\beta)}(I_N)},$$

$$\int_{{}^{\beta}U(N)} d^{\,\beta}U \; {}^{\beta}U_{i_1j_1}\overline{{}^{\beta}U}_{i_2j_2}\dots{}^{\beta}U_{i_{2n-1}j_{2n-1}}\overline{{}^{\beta}U}_{i_{2n}j_{2n}}$$
$$= \sum_{\mathfrak{m},\mathfrak{n}\in M_{2n}^U} \mathrm{Wg}^{\beta}([\mathfrak{m}^{-1}\mathfrak{n}])\cdot\delta_{\mathfrak{m}}(i)\delta_{\mathfrak{n}}(j);$$

It can be derived that in both the questions **the only possible formula for** Wg^{β} **is the following:**

$$\mathrm{Wg}^{\beta}(\mu) = \sum_{\lambda\vdash n}\frac{\omega_\lambda(\mu, 2/\beta)}{J_\lambda^{(2/\beta)}(I_N)}$$

How to compute Jack characters?

First, represent $J_\lambda^{(\alpha)}$ is the power sum basis:

$$J_\lambda^{(\alpha)} = \sum_{\mu \vdash n} \theta_\lambda(\mu) p_\mu.$$

Denote $j_\lambda(\alpha) = \langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha = \sum_{\mu \vdash n} \theta_\lambda(\mu)^2 \cdot z_\mu \cdot \alpha^{l(\mu)}$. Then

$$p_\mu = \sum_{\lambda \vdash n} \frac{\langle p_\mu, J_\lambda^{(\alpha)} \rangle_\alpha}{\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \rangle_\alpha} \cdot J_\lambda^{(\alpha)} = \sum_{\lambda \vdash n} \frac{\theta_\lambda(\mu) \cdot z_\mu \cdot \alpha^{l(\mu)}}{j_\lambda(\alpha)} \cdot J_\lambda^{(\alpha)}.$$

This formula generalizes the corresponding representation for Schur polynomials.

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How to compute Jack characters?

- Find the translation coefficients $\theta_\lambda(\mu)$ using the Pieri formula [Las09, Sec. 3]:

$$\theta_\lambda(\mu_{\downarrow(1)}) = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\theta_{\lambda^{(i)}}(\mu),$$

$$\sum_{r\geq 1}(m_r(\rho)+1)\theta_\lambda(\mu_{\downarrow(r+1)}) = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)(\lambda_i-(i-1)/\alpha)\theta_{\lambda^{(i)}}(\mu),$$

$$c_i(\lambda) = \frac{1}{\alpha\lambda_i + l(\lambda) - i + 2}\prod_{j=1,j\neq i}^{l(\lambda)+1}\frac{\alpha(\lambda_i-\lambda_j)+j-i+1}{\alpha(\lambda_i-\lambda_j)+j-i+1}.$$

- Find $\theta_\lambda(\mu)$ using chromatic quasisymmetric functions [HW17, Corollary 4.3.1]:

$$J_\lambda^{(\alpha)} = \sum_{H\subset G_{\lambda'}^+}(-1)^H p_{\lambda(H)} \prod_{\{u,v\}\in H\setminus G_{\lambda'}} -\mathrm{hook}_{\lambda'}^{(\alpha)}(u),$$

Results of numerical experiments

- Both methods for computing characters of Jack polynomials were implemented. These methods produce identical coefficients, so, probably, the expressions for characters are correct.
- The first method can be used for $|\lambda| \leq 20$, and the second one can be used for $|\lambda| \leq 7$.
- The answers for both **question 1** and **question 2** are negative: the formulas are not consistent even for diagonal matrices A and B.

References I

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Thank you for your attention!