# A NOVEL EXTENSION OF GAMMA-LINDLEY DISTRIBUTION WITH APPLICATIONS IN RELIABILITY ENGINEERING

**Mahfooz Alam[1*], Vaibhavi Admane[1], Deepesh Lama[1], Suhas Bhagwan Chavan[2] and Jayendra Jadhav[3]**

[1]Department of Mathematics and Statistics, Faculty of Science and Technology, Vishwakarma University, Pune-411 048, India;
[2]Department of Computer Engineering, Vishwakarma University, Pune-411 048, India;
[3]Department of Artificial Intelligence,, Vishwakarma University, Pune-411 048, India;
Email Id: *mahfooz.stats@gmail.com, vaibhaviadmane20@gmail.com
lamadeepesh780@gmail.com, chavan.suhas18@gmail.com and jayendra.jadhav@vupune.ac.in

**Abstract**

In this paper, we present the Survival Weighted Gamma-Lindley (SWGaL) distribution, a new two-parameter probability model created by including the survival function into the gamma-Lindley family. The SWGaL model's probability, cumulative, survival, and hazard functions, as well as higher-order moments and mean residual life, are examined in order to examine its analytical structure. With the aid of numerical optimisation, the maximum likelihood approach is used to estimate the model parameters. AIC, BIC, AICC, HQIC, and the Kolmogorov–Smirnov test are among the information-based metrics used to assess the model's performance. Two reliability datasets are used to test the model. According to the results, the suggested SWGaL distribution is a promising tool for reliability and survival assessments in applied statistics and provides more flexibility for characterising different kinds of lifetime data.

**Keywords.** Survival Weighted function, Moments, Estimates, Application, Reliability Data.

**Mathematics Subject Classifcation.** 62G30, 62E10, 62E15, 62F10.

## 1 Introduction

In recent decades, researchers have introduced numerous extensions and modifications of statistical models to address real-world data from diverse fields. Traditional probability distributions often fail to adequately represent certain forms of data, motivating the development of new families by either generalizing existing models or constructing hybrids that incorporate additional parameters for enhanced adaptability. Classical distributions, in particular, have attracted significant attention due to their flexibility and broad range of applications.

Let $X$ be a non-negative random variable with the PDF $g(x)$ and suppose non-negative weight function is $w(x)$, then the PDF of the weighted random variable $X_w$ is given by

$$g_w(x) = \frac{w(x)\, f(x)}{E(w(x))}, \quad x > 0,$$

*Corresponding author e-mail address: mahfooz.stats@gmail.com

where $w(x)$ be a non-negative weight function and and exits

$$E(w(x)) = \int_{-\infty}^{+\infty} w(x)g(x)dx \quad -\infty < x < \infty \tag{1}$$

Let $w(x)$ denote a non-negative weighting function for which the expectation $E[w(x)]$ is finite. Rao [8] introduced a particular form of weighted distributions, referred to as size biased distributions, by selecting $w(x) = x^{\beta}$. This formulation is known as the size-biased distribution of order $\beta$. Specifically, when $\beta = 1$, it is identified as the length biased distribution, and when $\beta = 2$, it is termed the area-biased distribution, for more details [1], [2] [3], [7], [9], [10].

Nedjar and Zeghdoudi [4, 5] developed the gamma–Lindley distribution. The probability density function with the random variable $X$ in this distribution is expressed as,

$$f_{GaL}(x|\theta,\, \beta) = \frac{\theta^2((\beta + \beta\,\theta - \theta)x + 1)\, e^{-\theta x}}{\beta(1+\theta)}, \qquad x,\, \theta > 0,\, \beta > 0, \tag{2}$$

and the survival function $(sf)$ is

$$S_{GaL}(x|\theta,\, \beta) = \frac{\theta^2((\beta + \beta\,\theta - \theta)(\theta x + 1) + \theta)\, e^{-\theta x}}{\beta(1+\theta)}, \qquad x,\, \theta > 0,\, \beta > 0, \tag{3}$$

where $S_{GaL}(x) = 1 - F_{GaL}(x)$. From (1) and (3), we get

$$E(S_{GaL}(x)) = \int_0^{+\infty} \frac{\theta^2((\beta + \beta\,\theta - \theta)(\theta x + 1) + \theta)\, e^{-\theta x}}{\beta(1+\theta)} f_{GaL}(x|\theta,\, \beta)dx$$

$$E(S_{GaL}(x)) = \frac{\theta^4}{\beta^2(1+\theta)^2} \int_0^{+\infty} [(\beta + \beta\,\theta - \theta)(\theta x + 1) + \theta][(\beta + \beta\,\theta - \theta)x + 1]\, e^{-2\theta x} dx.$$

After solving the above equation, we get

$$E(S_{GaL}(x)) = \frac{\theta^4}{\beta^2(1+\theta)^2} \left( \frac{(\beta + \beta\,\theta - \theta)}{(2\theta)^3} + \frac{2(\beta + \beta\,\theta - \theta) + (\beta + \beta\,\theta - \theta)^2}{(2\theta)^2} + \frac{\beta(1+\theta)}{2\theta} \right)$$

Let

$$\phi(\theta, \beta) = \frac{\theta^4}{\beta^2(1+\theta)^2} \left( \frac{(\beta + \beta\,\theta - \theta)}{(2\theta)^3} + \frac{2(\beta + \beta\,\theta - \theta) + (\beta + \beta\,\theta - \theta)^2}{(2\theta)^2} + \frac{\beta(1+\theta)}{2\theta} \right).$$

Paper is organized as follows. In Section 2, The Survival Weighted Gamma Lindley distribution is derived, followed by a discussion on its mathematical properties, including survival and hazard functions, quantile function, moments and mean residual life (MRL) in Section 3. The estimation of parameters using the maximum likelihood (ML) approaches is presented in Section 4. Evaluates the performance of the proposed estimators, and illustrates the practical utility of the model through applications to real-life datasets in Section 5. The concluding remarks are summarized in Section 6.

## 2 Survival Weighted Gamma Lindley (SWGaL) Distribution

In this section, we are consider the $w(x) = S(x)$, survival function. The SWGaL model is introduced with the idea that, although its application may primarily focus on the extreme values of a distribution, it remains straightforward to implement. Its expressions for the mean, variance, mean deviation, and quantile function are presented in a concise manner, making them practical tools for obtaining fast approximations in various situations.

The *pdf* of SWGaL distribution are given by

$$f_{SWGaL}(x|\theta,\,\beta) = \frac{\theta^4[(\beta+\beta\,\theta-\theta)(\theta x+1)+\theta][(\beta+\beta\,\theta-\theta)x+1]\,e^{-2\theta x}}{\phi(\theta,\beta)\,\beta^2(1+\theta)^2}, \quad \theta>0,\,\beta>0, \tag{4}$$

and the distribution function (*df*) is

$$F_{SWGaL}(x|\theta,\,\beta) = \Phi(\theta,\beta)$$
$$\times\left(\frac{\gamma(3,2\theta\,x)}{(2\theta)^3} + \frac{[2(\beta+\beta\,\theta-\theta)+(\beta+\beta\,\theta-\theta)^2]\gamma(2,2\theta\,x)}{(2\theta)^2} + \frac{\beta(1+\theta)[1-e^{2\theta x}]}{2\theta}\right) \tag{5}$$

where

$$\Phi(\theta,\beta) = \frac{\theta^4}{\phi(\theta,\beta)\,\beta^2(1+\theta)^2}$$

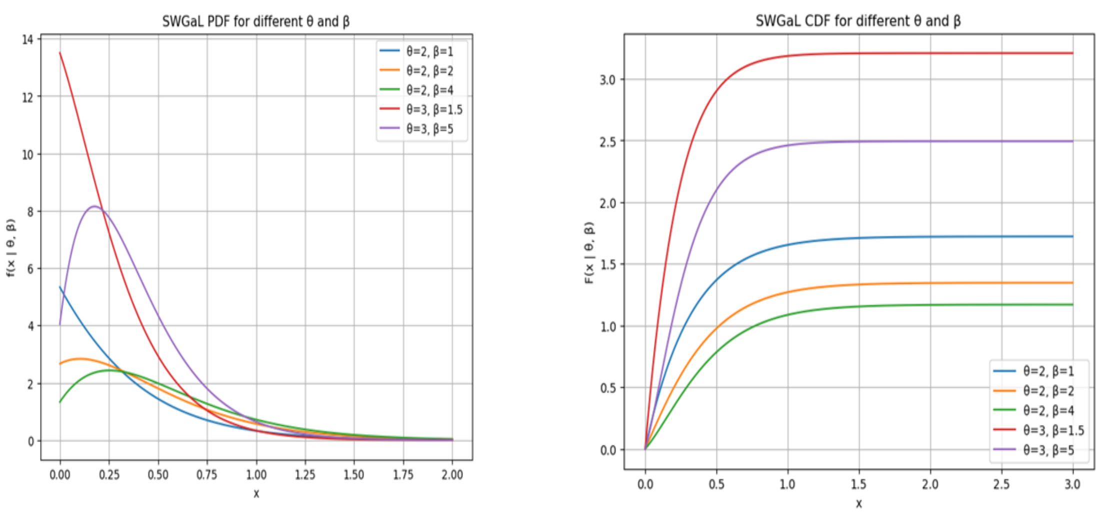


Figure 1: Behaviour of the WSGL distribution based on *pdf* and *cdf* at different values of different parameters

## 3 Statistical Properties

In the following section, analytical formulations are presented for key distributional characteristics, encompassing the survival and hazard measures, quantile representation, raw moment structure and the mean residual life function.

### 3.1 Survival and Reverse Hazard functions

The survival function characterizes lifetime distribution and is essential in modelling time-to-event data. It quantifies the event probability over time and its functional form is given as

$$S(x) = P(X > x) = \int_x^\infty f(x)dx$$

$$S(x) = 1 - \Phi(\theta,\beta)\left(\frac{\gamma(3,2\theta\, x)}{(2\theta)^3} + \frac{[2(\beta+\beta\,\theta-\theta)+(\beta+\beta\,\theta-\theta)^2]\gamma(2,2\theta\, x)}{(2\theta)^2} + \frac{\beta(1+\theta)[1-e^{2\theta x}]}{2\theta}\right) \tag{6}$$

The reverse hazard function is given by

$$\bar{h}(x) = \frac{f(x)}{F(x)} = \frac{[(\beta+\beta\theta-\theta)(\theta x+1)+\theta^2][(\beta+\beta\theta-\theta)x+1]e^{-2\theta x}}{\frac{\gamma(3,2\theta x)}{(2\theta)^3} + \frac{[2(\beta+\beta\theta-\theta)+(\beta+\beta\theta-\theta)^2]\gamma(2,2\theta x)}{(2\theta)^2} + \frac{\beta(1+\theta)[1-e^{2\theta x}]}{2\theta}} \tag{7}$$

In Figure 2, a few forms of the reverse hazard and survival functions are displayed.

### 3.2 Quantile Functions

The quantile function $Q(p)$ is defined as

$$Q(p) = F^{-1}(x) = \inf\{x \in \mathbb{R} : p \leq F(x_p)\}, \quad 0 < p < 1. \tag{8}$$

where $F(.)$ denotes in (5). Solving for $F(x_p) = p$ we obtain the *cdf* for the distribution is given by:

$$F(x_p) = \Phi(\theta,\beta)\left(\frac{\gamma(3,2\theta x_p)}{(2\theta)^3} + \frac{[2(\beta+\beta\theta-\theta)+(\beta+\beta\theta-\theta)^2]\gamma(2,2\theta x_p)}{(2\theta)^2} + \frac{\beta(1+\theta)[1-e^{2\theta x_p}]}{2\theta}\right).$$

The quantile function, we need to solve the following equation for $x_p$,

$$p = \Phi(\theta,\beta)\left(\frac{\gamma(3,2\theta x_p)}{(2\theta)^3} + \frac{[2(\beta+\beta\theta-\theta)+(\beta+\beta\theta-\theta)^2]\gamma(2,2\theta x_p)}{(2\theta)^2} + \frac{\beta(1+\theta)[1-e^{2\theta x_p}]}{2\theta}\right).$$

This equation cannot be solved algebraically for $x_p$ because the incomplete gamma functions, $\gamma(a,z)$, and the exponential term are transcendental functions. Therefore, the quantile function, $Q(p) = x_p$, does not have a simple closed-form expression. The value of $x_p$ must be obtained numerically. Let $p = 0.5$, we have the median, $x_m$ say, as

$$0.5 = \Phi(\theta,\beta)\left(\frac{\gamma(3,2\theta x_m)}{(2\theta)^3} + \frac{[2(\beta+\beta\theta-\theta)+(\beta+\beta\theta-\theta)^2]\gamma(2,2\theta x_m)}{(2\theta)^2} + \frac{\beta(1+\theta)[1-e^{2\theta x_m}]}{2\theta}\right)$$

Table 1: Quantile Function $Q(p;\theta,\beta)$ for selected $p$, $\theta$ and $\beta$

| $p$ | $\theta=0.5,\beta=1$ | $\theta=0.5,\beta=2$ | $\theta=1,\beta=1$ | $\theta=1,\beta=2$ | $\theta=2,\beta=1$ | $\theta=2,\beta=2$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.3257 | 0.3972 | 0.0963 | 0.1403 | 0.0334 | 0.0514 |
| 0.2 | 0.6180 | 0.7096 | 0.1990 | 0.2708 | 0.0705 | 0.1045 |
| 0.3 | 0.9077 | 1.0140 | 0.3106 | 0.4029 | 0.1124 | 0.1612 |
| 0.4 | 1.2109 | 1.3353 | 0.4345 | 0.5439 | 0.1605 | 0.2234 |
| 0.5 | 1.5425 | 1.6952 | 0.5758 | 0.7012 | 0.2170 | 0.2938 |
| 0.6 | 1.9219 | 2.1250 | 0.7427 | 0.8858 | 0.2857 | 0.3768 |
| 0.7 | 2.3821 | 2.6871 | 0.9504 | 1.1180 | 0.3736 | 0.4804 |
| 0.8 | 2.9937 | 3.5629 | 1.2326 | 1.4470 | 0.4962 | 0.6232 |
| 0.9 | 3.9756 | 6.4959 | 1.6947 | 2.0760 | 0.7030 | 0.8687 |

### 3.3 Moments

Moments are essential instruments for characterising a probability distribution's structure. They offer a consistent framework for describing important statistical characteristics, such as skewness, kurtosis, variability, and central tendency. Moments are essentially numerical descriptors that highlight significant aspects of a distribution.
The $k$-th, order moment about the origin of (2) is given by

$$\begin{aligned}\mathbb{E}(x)^k &= \int_0^\infty x^k\, f_{SWGaL}(x|\theta,\,\beta)dx \qquad (9)\\ &= \Phi(\theta,\beta)\int_0^\infty x^k\,[(\beta+\beta\,\theta-\theta)(\theta x+1)+\theta][(\beta+\beta\,\theta-\theta)x+1]\,e^{-2\theta x}\,dx\end{aligned}$$

after solving the above equation and using the gamma function, we get

$$\mathbb{E}(x)^k = \Phi(\theta,\beta)\left[A^2\theta\,\frac{\Gamma(k+3)}{(2\theta)^{k+3}} + B\,\frac{\Gamma(k+2)}{(2\theta)^{k+2}} + C\,\frac{\Gamma(k+1)}{(2\theta)^{k+1}}\right]$$

When $k=1,2,3,4$, we have find the moments as

$$\mathbb{E}(x) = \Phi(\theta,\beta)\left[\frac{6\,A^2\theta}{(2\theta)^4} + \frac{2\,B}{(2\theta)^3} + \frac{C}{(2\theta)^2}\right]$$

$$\mathbb{E}(x)^2 = \Phi(\theta,\beta)\left[\frac{24\,A^2\theta}{(2\theta)^5} + \frac{6\,B}{(2\theta)^4} + \frac{2\,C}{(2\theta)^3}\right]$$

$$\mathbb{E}(x)^3 = \Phi(\theta,\beta)\left[\frac{120\,A^2\theta}{(2\theta)^6} + \frac{24\,B}{(2\theta)^5} + \frac{6C}{(2\theta)^4}\right]$$

$$\mathbb{E}(x)^4 = \Phi(\theta,\beta)\left[\frac{720\,A^2\theta}{(2\theta)^7} + \frac{120\,B}{(2\theta)^6} + \frac{24\,C}{(2\theta)^5}\right]$$

and the variance as $\mathbb{V}(x)=\mathbb{E}(x)^2-(\mathbb{E}(x))^2$

$$\mathbb{V}(x) = \Phi(\theta,\beta)\left[\left\{\frac{24\,A^2\theta}{(2\theta)^5} + \frac{6\,B}{(2\theta)^4} + \frac{2\,C}{(2\theta)^3}\right\} - \left\{\frac{6\,A^2\theta}{(2\theta)^4} + \frac{2\,B}{(2\theta)^3} + \frac{C}{(2\theta)^2}\right\}^2\right]$$

where $A = (\beta + \beta\,\theta - \theta)$, $B = A(\beta + \beta\,\theta + \theta)$ and $C = (\beta + \beta\,\theta)$.
The skewness and excess kurtosis can be obtained as

$$\gamma_1 = \frac{(\mathbb{E}(x)^3)}{\sqrt{(\mathbb{E}(x)^2)^3}}$$

$$\gamma_2 = \frac{(\mathbb{E}(x)^4)}{(\mathbb{E}(x)^2)^2} - 3$$

Table 2: SWGaL Distribution Moments and Variance ($\theta = 1.5$)

| $\beta$ | $\mathbb{E}(x)$ | $\mathbb{E}(x)^2$ | $\mathbb{E}(x)^3$ | $\mathbb{E}(x)^4$ | $\mathbb{V}(x)$ |
|---|---|---|---|---|---|
| 1 | 0.569231 | 0.523077 | 0.687179 | 1.162393 | 0.199053 |
| 2 | 0.915294 | 0.983529 | 1.433725 | 2.614379 | 0.145766 |
| 3 | 1.096552 | 1.227586 | 1.834483 | 3.402299 | 0.025161 |
| 4 | 1.207509 | 1.377474 | 2.081456 | 3.889268 | 0.000000 |
| 5 | 1.282353 | 1.478733 | 2.248567 | 4.219206 | 0.000000 |
| 6 | 1.336232 | 1.551691 | 2.369082 | 4.457327 | 0.000000 |
| 7 | 1.376867 | 1.606747 | 2.460080 | 4.637216 | 0.000000 |
| 8 | 1.408606 | 1.649766 | 2.531213 | 4.777882 | 0.000000 |
| 9 | 1.434081 | 1.684305 | 2.588341 | 4.890882 | 0.000000 |
| 10 | 1.454979 | 1.712645 | 2.635227 | 4.983642 | 0.000000 |

Table 3: SWGaL Distribution Moments and Variance ($\beta = 2.0$)

| $\theta$ | $\mathbb{E}(x)$ | $\mathbb{E}(x)^2$ | $\mathbb{E}(x)^3$ | $\mathbb{E}(x)^4$ | $\mathbb{V}(x)$ |
|---|---|---|---|---|---|
| 1 | 1.326531 | 2.183673 | 4.836735 | 13.346939 | 0.423990 |
| 2 | 0.693878 | 0.551020 | 0.596939 | 0.811224 | 0.069554 |
| 3 | 0.463221 | 0.240557 | 0.171637 | 0.154186 | 0.025983 |
| 4 | 0.345631 | 0.133010 | 0.070631 | 0.047330 | 0.013549 |
| 5 | 0.274905 | 0.083941 | 0.035471 | 0.018944 | 0.008369 |
| 6 | 0.227882 | 0.057641 | 0.020219 | 0.008974 | 0.005711 |
| 7 | 0.194438 | 0.041965 | 0.012580 | 0.004776 | 0.004159 |
| 8 | 0.169471 | 0.031890 | 0.008345 | 0.002767 | 0.003170 |
| 9 | 0.150138 | 0.025041 | 0.005814 | 0.001711 | 0.002500 |
| 10 | 0.134735 | 0.020177 | 0.004210 | 0.001114 | 0.002024 |

### 3.4 Mean Residual Life

The projected remaining longevity of a unit, contingent on its survival up to time $x$, is represented by the mean residual life (MRL) function in reliability studies. It can be denoted as

$$MLR = \frac{1}{S(x)} \int_x^\infty u f(u) du - x. \tag{10}$$

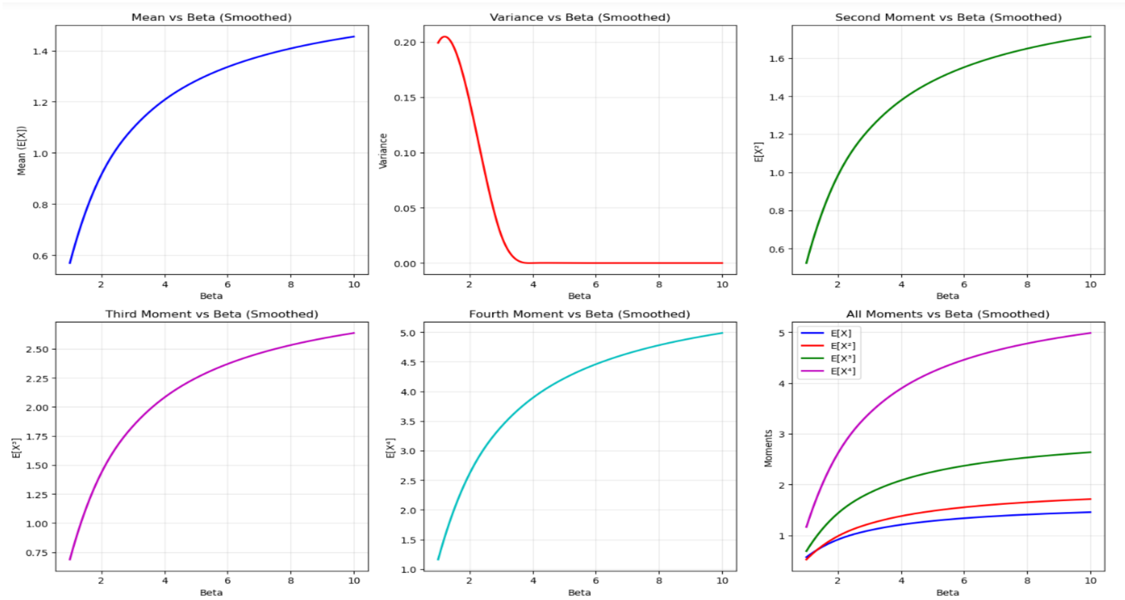


Figure 2: Moments propertires of SWGaL distribution for different values of $\beta$ and $\theta$

Using the (4), we get

$$MLR = \frac{\Phi(\theta,\beta)}{S(x)}\left[\frac{A^2\theta}{(2\theta)^4}\,\Gamma(4,\, x/2\theta) + \frac{B}{(2\theta)^3}\Gamma(3,\, x/2\theta) + \frac{C}{(2\theta)^2}\Gamma(2,\, x/2\theta)\right] - x.$$

and $\Phi(\theta,\beta)$ and $S(x)$ mentioned above.

# 4 Estimation

The process of identifying numerical values that define a statistical model from observed data is known as parameter estimation. Models are widely developed in fields like physics, biology, engineering, and economics to explain the relationships between variables. These models rely on parameters, which are quantifiable measurements that control their composition and actions. Finding the values that best describe the data is the goal of parameter estimation, which guarantees that the model appropriately captures the underlying relationships between the variables being studied. In this section, we are using method of Maximum Likelihood Estimation to estimates the parameters.

## 4.1 Maximum Likelihood Estimation and Score Equations

In this section, we concentrate on estimating the parameters of the SWGaL distribution using the maximum likelihood approach. Assume that $x_1, x_2, \ldots, X_n$ represents a random sample of size $n$ taken from a population that follows the SWGaL distribution, with observed values, and that $x_1, x_2, \ldots, x_n$ represents a random sample of size $n$ taken from a population that follows the SWGaL distribution. The log-likelihood function is

$$\ell(\theta,\beta) = \sum_{i=1}^{n} \ln f_{SWGaL}(x_i \mid \theta,\beta)$$

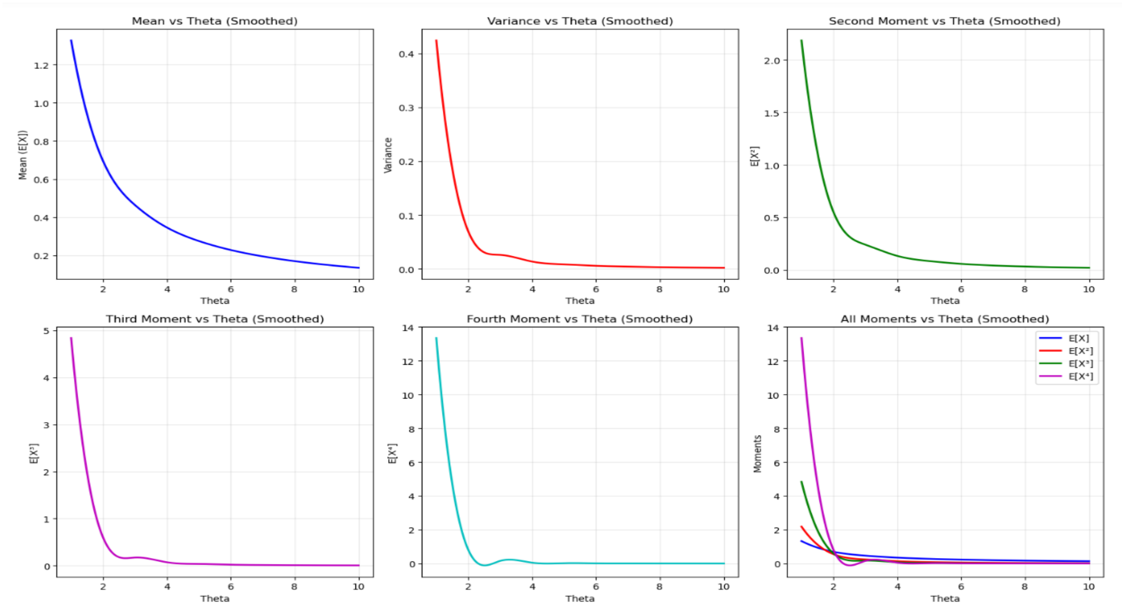


Figure 3: Moments propertires of SWGaL distribution for different values of $\beta$ and $\theta$

or equivalently,

$$\ell(\theta,\beta) = 4n\ln\theta - n\ln\phi(\theta,\beta) - 2n\ln\beta - 2n\ln(1+\theta) + \sum_{i=1}^{n}\ln\left[(\beta+\beta\theta-\theta)(\theta x_i+1)+\theta\right]$$

$$+\sum_{i=1}^{n}\ln\left[(\beta+\beta\theta-\theta)x_i+1\right] - 2\theta\sum_{i=1}^{n}x_i. \tag{11}$$

## 4.2 Score Equations

The score equations are obtained by differentiating $\ell(\theta,\beta)$ with respect to $\theta$ and $\beta$.

### (a) Derivative with respect to $\theta$

$$\frac{\partial\ell}{\partial\theta} = \frac{4n}{\theta} - n\frac{\phi_\theta(\theta,\beta)}{\phi(\theta,\beta)} - \frac{2n}{1+\theta} + \sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ln\left[(\beta+\beta\theta-\theta)(\theta x_i+1)+\theta\right] + \sum_{i=1}^{n}\frac{\partial}{\partial\theta}\ln\left[(\beta+\beta\theta-\theta)x_i+1\right] - 2\sum_{i=1}^{n}x_i.$$

### (b) Derivative with respect to $\beta$

$$\frac{\partial\ell}{\partial\beta} = -n\frac{\phi_\beta(\theta,\beta)}{\phi(\theta,\beta)} - \frac{2n}{\beta} + \sum_{i=1}^{n}\frac{\partial}{\partial\beta}\ln\left[(\beta+\beta\theta-\theta)(\theta x_i+1)+\theta\right] + \sum_{i=1}^{n}\frac{\partial}{\partial\beta}\ln\left[(\beta+\beta\theta-\theta)x_i+1\right].$$

The maximum likelihood estimators ($MLEs$) $\hat{\theta}$ and $\hat{\beta}$ are obtained by solving the nonlinear equations

$$\frac{\partial\ell}{\partial\theta} = 0 \quad \text{and} \quad \frac{\partial\ell}{\partial\beta} = 0.$$

## 5 Application and Evaluation indicators

Based on the real data, this section provides a useful assessment of the recently created SWGaL distribution. For maximum accuracy and computational performance, parameter estimation is done in the Python environment. A number of well-known information criteria, including the Kapaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (AICC), Hannan–Quinn Information Criterion (HQIC), Kolmogorov–Smirnov (K-S) test, and associated p-values, are used to evaluate the adequacy of a model.

$$
\begin{aligned}
&\text{AIC} = 2k - 2\log L,\\
&\text{BIC} = k\log n - 2\log L,\\
&\text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1},\\
&\text{HQIC} = -2\log L + 2k\log(\log(n)),\\
&\text{K-S} = \max_{i=1,2,\dots,n}\left\{\frac{i}{n} - \hat{F}(x_{(i)}),\, \hat{F}(x_{(i)}) - \frac{i-1}{n}\right\}
\end{aligned}
$$

where $k$ the number of parameters, $n$ sample size of the data, $\hat{F}(x_{(i)})$ the estimated CDF of the distribution under the ordered data and $(-\log L)$ indicates to , maximized Likelihood value of the fitted model respectively. Table 5 reports the estimated parameters together with the values of the model selection criteria. Smaller values of AIC, BIC, AICC, HQIC, and $-2\log L$ suggest a superior model fit. In addition, Figure 4 and Figure 5 displays the fitted probability density functions for Data Set 1 and Data Set 2 respectively.

### 5.1 Wang's data: Failure time data-I

Wang's [11] 18 Electronic Devices data:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 11 | 21 | 31 | 46 | 75 | 98 | 122 | 145 | 165 | 195 | 224 | 245 | 293 | 321 | 330 | 350 | 420 |

### 5.2 Proschan's data: Failure time data-II

Proschan [6] provided the duration, expressed in operating hours, between the subsequent breakdowns of 13 aircraft's air conditioning systems in order to examine their ageing characteristics. The following information relates to plane number three:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 90 | 10 | 60 | 186 | 61 | 49 | 14 | 24 | 56 | 20 | 79 | 84 | 44 | 59 | 29 | 118 | 25 | 156 |
| 310 | 76 | 26 | 44 | 23 | 62 | 130 | 208 | 70 | 101 | 208 | | | | | | | |

Table 4: Summary of the data

| Data | Min. | 1st Qu. | Median | Mean | 3rd Qu. | Max. | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|---|
| Wang's data | 5 | 53.25 | 155 | 172.055 | 281.0 | 420 | 0.316 | $-1.149$ |
| Proschan's data | 10 | 29.00 | 61 | 83.517 | 101.0 | 310 | 1.522 | 1.988 |

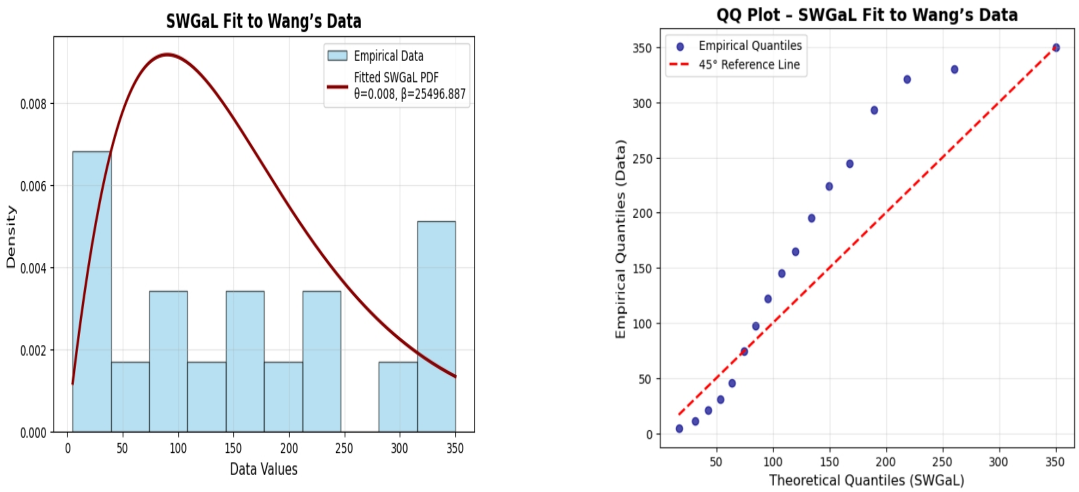


Figure 4: Fitted and QQ plots behaviour of the WSGaL distribution for the Wang's dataset

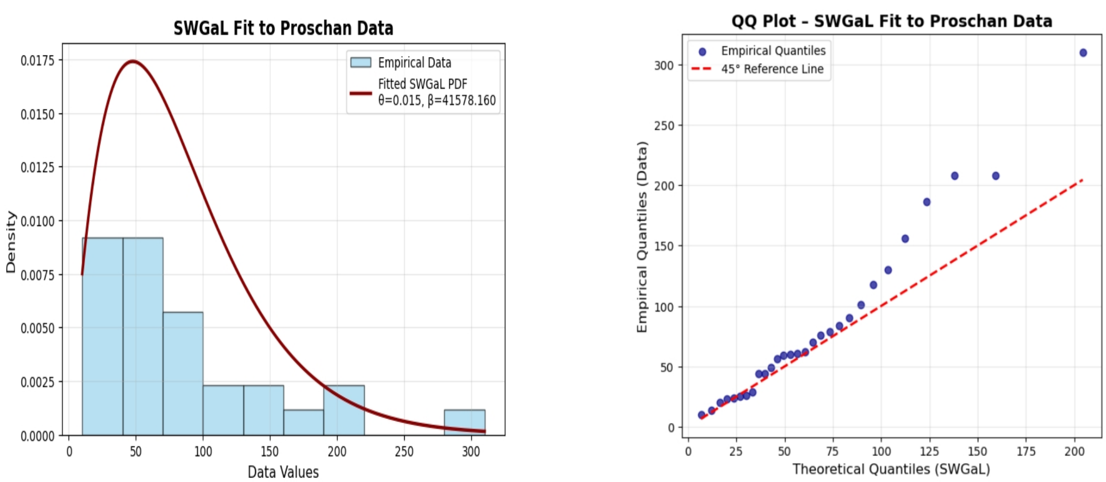


Figure 5: Fitted and QQ plots behaviour of the WSGaL distribution for the Proschan's dataset

Table 5: Performance of the SWGaL distributions for the different reliability data sets

| Data | Wang's data | Proschan's data |
|---|---|---|
| Estimates | $\hat{\theta} = 0.0078, \hat{\beta} = 25496.8870$ | $\hat{\theta} = 0.0148, \hat{\beta} = 41578.1597$ |
| log-likelihood | $-93.7494$ | $-136.2322$ |
| AIC | 191.4989 | 276.4644 |
| BIC | 193.1653 | 279.1990 |
| AICC | 192.3560 | 276.9260 |
| HQIC | 191.6645 | 277.33209 |
| KS Test | 0.2184 | 0.1166 |
| P-value | 0.3412 | 0.7830 |

## 6 Conclusions

An expansion of the Gamma Lindley model, the SWGaL distribution has two parameters and is organised using a unique weighting system. This distribution is appropriate for reliability analysis and survival estimation processes since both the survival function and the hazard function can be analytically derived. The method of maximum likelihood is used to estimate parameters, and mathematical analysis is used to look at the model's statistical characteristics. The SWGaL distribution exhibits good empirical fitting when used on real datasets. These findings demonstrate that SWGaL is a flexible and useful option for simulating real-world occurrences.

# Declarations

**Conflicts of Interest:** The authors declare no competing interests.

**Consent for Publication:** I consent to the publication of the data and materials.

**Data Availability** This study did not generate or utilize any new datasets; therefore, data sharing is not applicable.

**Code Availability** We are using the software an Python to calculate the numerical values and graphics.

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