EXOTIC OPTION PRICING AND STRATEGIES

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Introduction

In this project we aim to price Rainbow, Chooser and Forex options. For every type of option, we use two models to validate the prices. The following methods have been used for pricing the options respectively:

- Monte Carlo Simulation with Correlated GBM using Cholesky decomposition & Ornstein Uhlenbeck process
- Cox Ross Rubinstein model
- Kou Jump Diffusion model
- Heston Model
- Garman Kohlhagen Model

Additionally, we implement hedging strategies to observe how the priced options can be used in practicality. In the first strategy, we try to explore the min max parity argument in rainbow options using static portfolio replication. In the second strategy, Forex options were hedged against Chooser options.

1) Rainbow Options (multi-asset options)

Rainbow options, or basket options, are exotic options dependent on multiple underlying assets. They allow investors to speculate on the relative performance of these assets based on their correlations and manage diversified portfolio risks.Developed during the 1980s and 1990s alongside advances in computational finance, rainbow options required new methods for pricing due to their complexity. Research has focused on numerical techniques like Monte Carlo simulations to handle their sophisticated correlation structures. Studies have been conducted on such options ranging back since 1978 by Stulz[1], Johnson[2] and Margrabe[3]

In this paper we focus on 2 main types of rainbow options

- **"Call on max"** option, giving the holder the right to purchase the maximum asset at the strike price at expiry (Stulz 1982), (Johnson 1987)
  The payoff for a Call on max option is given by **max(max(S1, S2, . . . , Sn) - K, 0)**
- **"Call on min"** option, giving the holder the right to purchase the minimum asset at the strike price at expiry (Stulz 1982), (Johnson 1987)
  The payoff for a Call on min option is given by **max(min(S1, S2, . . ., Sn) - K, 0)**

1.1 Methodology

We will be pricing both types of options by means of MonteCarlo Simulation with two different types of asset dynamics to see if both of them reconcile their price estimates over time. This will serve as model validation as OTC exotic derivatives rarely have consistent market data.We explore divergences in pricing and posit reasons for such. The underlyings for both the underlyings will be 2 fixed income ETFs. The data was procured from bloomberg for US Short Term Treasury Bond Index and US Corporate Bond Index. The risk free rate we will be using is the US 3 Year Treasury bond rate. Date range was defined between 2019 and 2023 to capture effects of tail events on pricing estimates. The strike price for such options has been decided to be the average of both the prices. This can lead to bloated prices when pricing underlyings with varying price ranges. Volatility is calculated on a rolling 90-day window. We will also explore a static replication (min-max parity argument) posited by Stulz which says having a two asset rainbow maximum call and the corresponding two asset rainbow minimum call is just the same as having two vanilla calls on the two assets.

Cmax(S1, S2, K) + Cmin(S1, S2, K) ≈ C(S1,K) + C(S2,K)

1.2 Asset Dynamics

Correlated GBM dynamics

$$\frac{dS}{S} = (r - q)dt + AdW$$

Correlated GBM Analytical Solution

$$S_{t+1} = S_t \times e^{(r - \frac{1}{2} \times \sigma^2)\Delta t + (\sigma \times \sqrt{\Delta t} \times Z_t)}$$

$\boldsymbol{S_t}$: Asset price at time
$\boldsymbol{r}$: Risk-free rate
$\boldsymbol{\sigma}$: Volatility vector for the assets
q: Dividend Yield
A: Cholesky decomposition of Variance-Covariance matrix Σ
$\boldsymbol{\Delta_t}$: Time increment
Z - Matrix of independent standard normal random variables, size 2xN for two assets over N simulation steps.

Ornstein-Uhlenbeck Dynamics

$$dX_t = \kappa(\theta - X_t)dt + \sigma dWt$$

Where:
X_t: The process value at time t.
κ: The rate of mean reversion, indicating how quickly Xt returns to its mean level, θ.
θ: The long-term mean level to which Xt reverts.
σ: The volatility of the process, denoting the effect of random fluctuations.

Analytical Solution of the OU Process

$$X_t = X_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dW_s$$

1.3 Pricing and Replication Visualizations

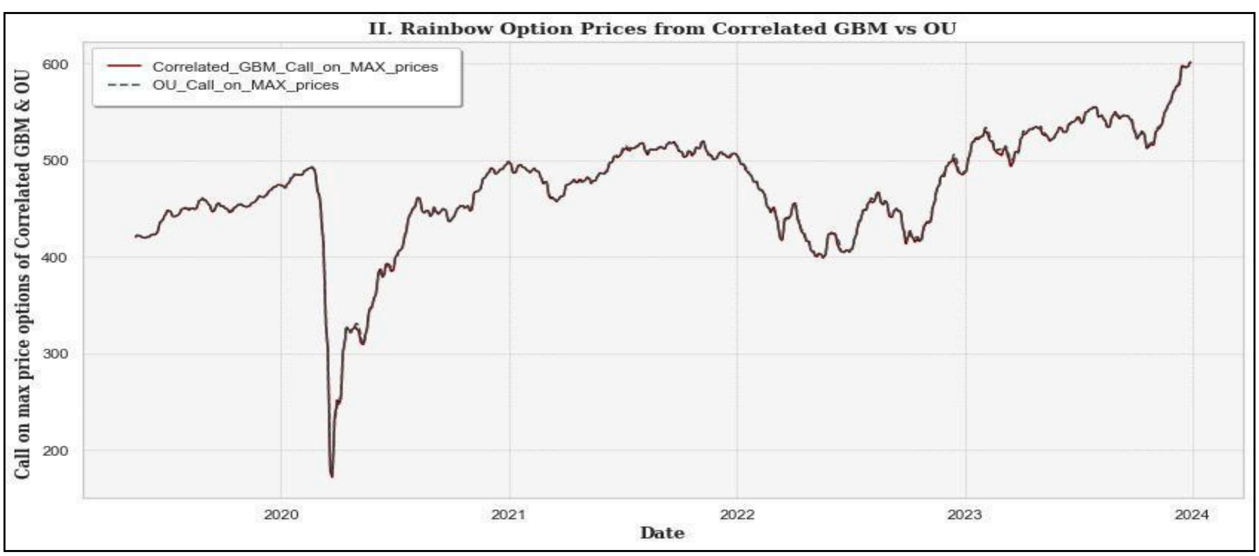

Fig: 1.1) Simulated daily option prices seem to agree with no major differences noticed.

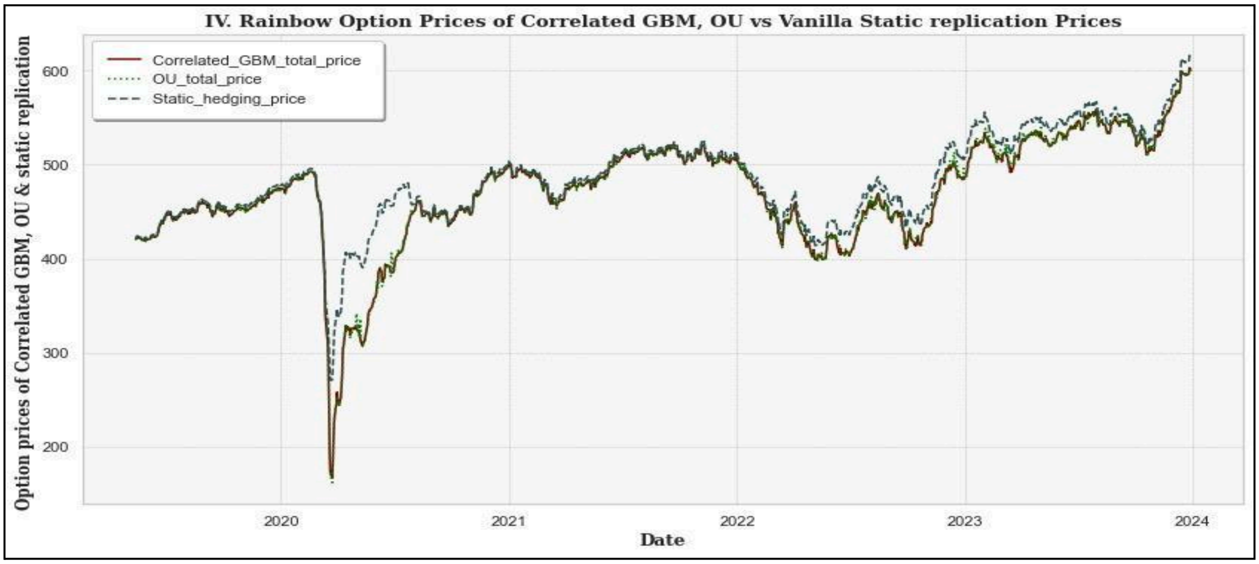

Fig: 1.2) Verification of the min-max parity argument.

We see that argument holds except in certain timeframes. 2020 divergence can be attributed to increased volatility during pandemic and 2022-2023 divergence can be attributed to debt ceiling crisis affecting treasury yields.

2) FX Option Pricing

Foreign exchange (forex) options are financial instruments that provide investors and traders the right, but not the obligation, to exchange money denominated in one currency into another currency at a pre-agreed exchange rate on a specified date. These options are vital tools for managing risk and speculating in the volatile forex markets. The risk factor is high as the exchange rates between currencies can be influenced by global economic indicators, interest rates, and geopolitical events. In today's forex landscape, characterized by significant fluctuations due to emerging market dynamics, shifts in monetary policy among major economies, and unforeseen global challenges, understanding and effectively pricing forex options has become increasingly critical. In this section, we delve into the application of the Heston model and Garman Kohlhagen Model to price forex options. We use the fluctuations in USD-to-Yen exchange rate and the interest rates differentials to price the options.

Data & Preprocessing

The common data used by both the models includes exchange rate, USD interest rate(domestic) and Japan's interest rate (foreign). We have used the 1 year treasury bond rates of the USA and Japan as the risk free interest rates. This data was taken from FRED and the Ministry of Finance (Japan). The data for the forex option prices was taken from Bloomberg. We used the exchange rates to find the log returns. Next, we calculated the historical volatility using a rolling standard deviation of the log returns over a 90-day window. Then this was annualized by multiplying by the square root of 252 (the typical number of trading days in a year). The time to expiration for the options in both the models is taken to be 1 year.

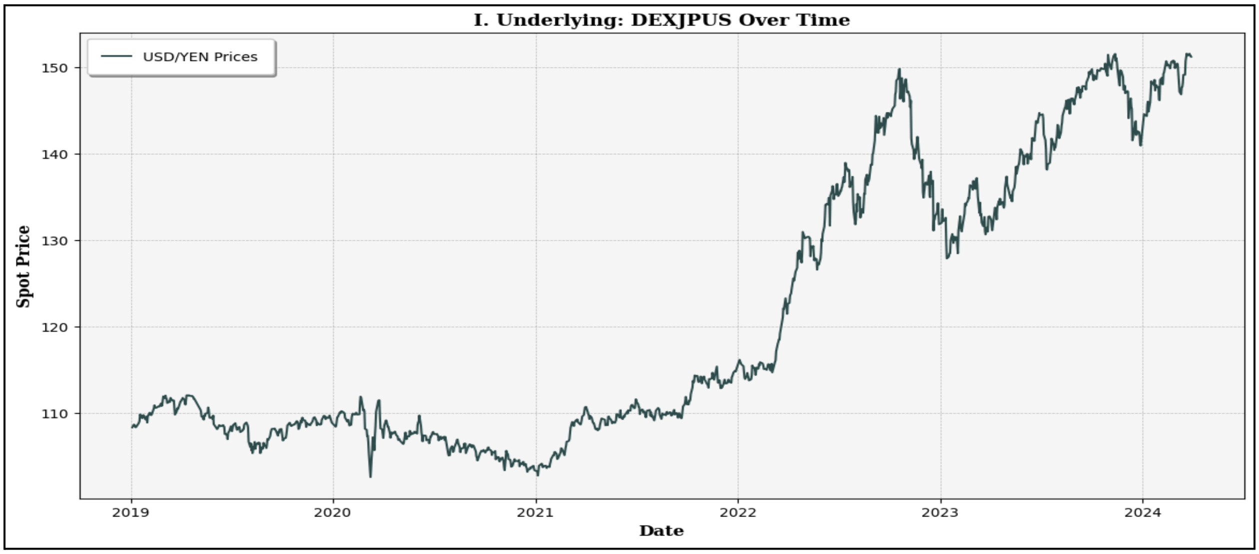

Fig: 2.1) Movement of underlying- USD/YEN over time

2.1) Garman Kohlhagen Model

The Garman Kohlhagen model is an extension of the Black Scholes Model. Its main purpose is to help price forex options. The model incorporates the domestic and foreign risk-free interest rates corresponding to the currencies involved in the option. The domestic rate is used to discount the option's payoff, while the foreign rate is used in the calculation of the forward price of the currency.
Similar to the Black-Scholes model, the Garman-Kohlhagen model assumes that the logarithm of exchange rates is normally distributed and that the markets are frictionless (no transaction costs or restrictions on short selling). It also assumes constant volatility and interest rates. This model is for European options.

Parameters of the model:

S: Current spot exchange rate (number of domestic currency units required to buy one unit of foreign currency).
K:Strike price of the option (in terms of domestic currency).
$\mathbf{r_d}$**:** Domestic risk-free interest rate.
$\mathbf{r_f}$ **:** Foreign risk-free interest rate.
T**:** Time to expiration of the option, in years.
$\boldsymbol{\sigma}$: Volatility of the exchange rate.
N**(·) :** cumulative distribution function of the standard normal distribution

The price of a European put option P on foreign exchange is given by:

$$P = K \cdot e^{-rd.T} \cdot N(-d_2) \ - S \cdot e^{-rf.T}.N(-d_1)$$

$$d_1 = \frac{(ln\,(\frac{S}{K}) + (\,rd - rf + \frac{\sigma 2}{2}\,))}{\boldsymbol{\sigma}\sqrt{T}}$$

$$d_2 = d_1 - \boldsymbol{\sigma}\sqrt{T}$$

2.2) Heston Model

The Heston model was chosen for its capability to model stochastic volatility, crucial for accurately pricing forex options. Unlike the traditional Black-Scholes model with its constant volatility assumption, the Heston model treats volatility as a dynamic factor. This feature is particularly vital in the forex market, where volatility frequently shifts due to macroeconomic factors.

This dynamic volatility modeling is essential for forex options, which are highly sensitive to changes in volatility due to underlying currency pair movements. The Heston model's ability to reflect these changes offers a more accurate and realistic framework for option pricing, capturing the complex dynamics of the forex market. This makes it an invaluable tool for our analysis, providing robust insights into forex options pricing under varied market conditions and multiple strikes.

The Heston model specifies that the underlying asset price follows a stochastic process described by two equations: one for the underlying asset price itself and another for the variance of the asset price. The model is defined by the following system of stochastic differential equations:

1. **Exchange Rate Dynamics:**

$$dSt = \mu \, St \, dt + vt \, St \, dWSt$$

St : is the asset price at time tt
μ : is the drift rate of the asset price
vt : is the instantaneous variance
dWSt : is a Brownian motion process affecting the asset price

2. **Variance Dynamics:**

$$dvt = \kappa(\theta - vt)dt + \sigma \, vt \, dWvt$$

κ : is the rate of mean reversion
θ : is the long-term variance
σ : is the volatility of volatility
dWvt : is another Brownian motion process that is correlated with *dWSt*

The correlation in the two Brownian motion process' is defined by:

$$dWSt.dWvt = \rho dt$$

The implementation of the Heston model for the purposes of our project, involved simulating the paths for the spot exchange rate under the assumption of stochastic volatility. The Heston model was used to define the dynamics of the exchange rate and its variance using the above set of stochastic differential equations. For our simulation, the model parameters—such as the long-term variance (theta), volatility of volatility (sigma), and the correlation between the asset price and its volatility (rho)—play crucial roles in shaping the volatility structure and, consequently, the pricing of the options.

To ensure that our model accurately reflects the real market conditions and to improve the reliability of our simulation results, we calibrated the Heston model parameters against real market data. Calibration involved adjusting the model parameters to minimize the difference between the market observed option prices and those predicted by our model.

We used the following specific approach for the calibration:

- **Objective Function:** The objective function was designed to quantify the discrepancy between the theoretical prices generated by the Heston model and the observed market prices. This function sums the squared differences across all options considered.

- **Optimization Method:** We applied numerical optimization techniques, such as the minimize function from SciPy with bounds on parameters to find the optimal set of parameters that minimizes our objective function. The bounds ensure that the parameters stay within realistic limits.
- **Characteristic Function:** The Heston model's characteristic function, which is crucial for option pricing, was used within the pricing model to evaluate the price of the options under the current parameter set.

Once the calibrated parameters were generated, they were plugged back into the Heston model to estimate the expected payoffs of call options at different strike prices. By applying a discount factor, which reflects the risk-free rate, these payoffs were then used to estimate the fair prices of the options. The variability in these prices in the table below across different simulations highlights the impact of stochastic volatility as modeled by the Heston framework.

Results

In the plotted graph below comparing FX option prices with the Heston and Garman-Kohlhagen models, we see clear differences. The Heston model, using stochastic volatility, shows dynamic pricing, while the Garman-Kohlhagen model, with constant volatility, yields smoother curves. Although the Heston model captures market dynamics more precisely, its computational complexity may slow pricing compared to the Garman-Kohlhagen model, highlighting the trade-off between accuracy and computational efficiency in FX option pricing.

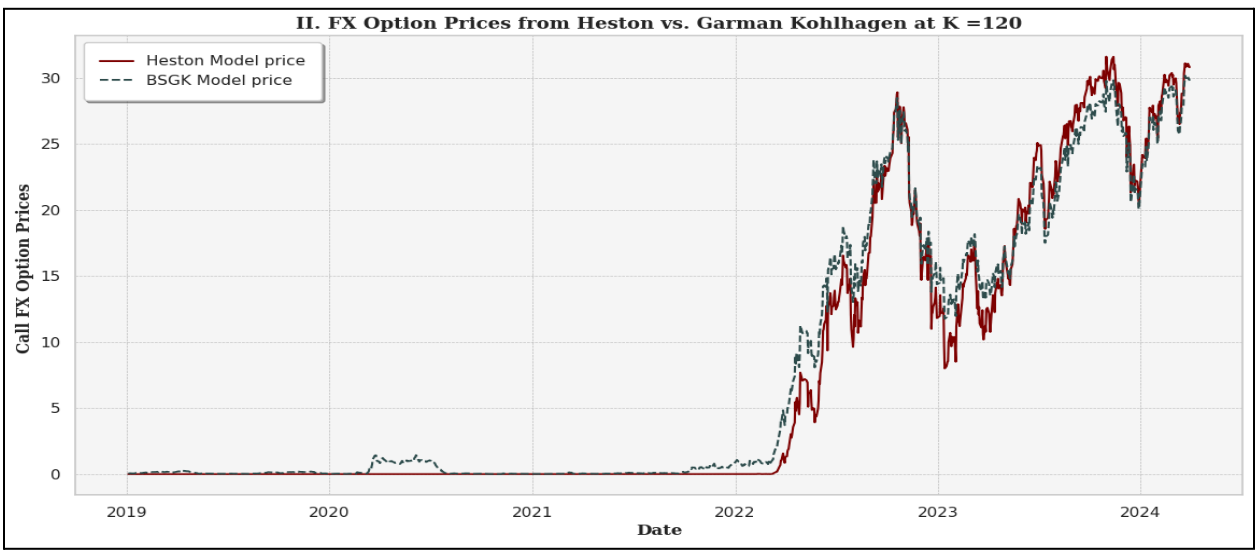

Fig: 2.2) Comparison between FX option prices from Heston vs. Garman Kohlhagen Model

Both models differ not only in their volatility modeling and computational intensity but also in how they handle interest rate differentials. The Heston model focuses on asset prices and volatility changes over time, while also considering currency interest rate variations. In contrast, the Garman-Kohlhagen model closely examines interest rate discrepancies between currencies in the option, factoring in their impact on option pricing. Although both models address interest rates, they do so differently, with Garman-Kohlhagen offering a more detailed analysis. Overall, the Heston model provides flexibility in capturing market dynamics but is computationally intensive, while Garman-Kohlhagen offers simplicity but may lack some nuances. The choice between the two depends on specific needs and the balance between accuracy and computational efficiency.

3) Chooser Option Pricing

Chooser options are a type of exotic financial derivative that provides exceptional flexibility by allowing the holder to determine whether the option will function as a call or a put at a later date. This versatility makes them especially beneficial in volatile markets, where the investor can select the option type with the greatest potential benefit according to the current market conditions. Chooser options usually have a single strike price and expiration date, and they are organized as European options, which means they can only be exercised when they expire.

Despite their versatility, chooser options typically have greater pricing than ordinary options due to their exotic character and increased risk of counterparty default, as they are frequently traded on alternative platforms. The pricing of these options is driven by the same characteristics as vanilla options, but the ability to pick between a call or a put at a later date can result in greater premiums, particularly in conditions with high implied volatility. While chooser options provide significant strategic value, investors should be mindful of the complexities and costs connected with them, especially when assessing their potential in near-term expirations and specific market circumstances.

Underlying Data:

The Bloomberg U.S. Universal Total Return Index Value Unhedged U is a financial index that tracks the total return performance of a broad range of U.S. dollar-denominated bonds. As of the end of March 2023, the index is primarily composed of Treasury (35%), Corporate (25%), and Mortgage (25%) sectors. The index value has decreased slightly, indicating a drop in the overall performance of the bonds included in the index.

Parameters:

- **S**: Current price of the US Universal ETF (Spot)
- **K**: Strike price of the option (assume $550)
- **T**: Time to expiry of the option (1 year)
- **T1**: Time until the choice must be made (6 month)
- **r**: Risk-free interest rate (3 month treasury rates)
- **sigma**: Volatility of the underlying asset (90 day window)

3.1) Cox Ross Rubinstein Model

The model operates under the risk-neutral measure, meaning that the world is assumed to be risk-neutral and the expected return of the stock is the risk-free rate.

Binomial Tree Construction: The first step in the CRR model is to construct a binomial tree for the underlying asset prices. This tree represents possible paths that the price of the underlying asset can take over the life of the option. Each node in the tree represents a possible price of the asset at a given point in time. The up and down movements in the tree are calculated using the following formulas:

$$u = e^{(\sigma \sqrt{\Delta T})} \qquad d = e^{(-\sigma \sqrt{\Delta T})}$$

u: up factor,
d: down factor,
σ: volatility of the underlying asset
ΔT: length of the time step

Option Values at Maturity: The next step is to calculate the call and put option values at maturity for each node at the last level of the tree. The values of call and put options at maturity are given by:

$$C = \max(0, S-K) \qquad P = \max(0, K-S)$$

where:

C: value of the call option

P: value of the put option

S: spot price of the underlying asset

K: strike price of the option

Backward Induction: Finally, option values at earlier nodes are calculated by backward induction. The value of the option at any given node is the discounted expected value of the option at the next time step. For a call and put option, this is given by:

$$C = e^{-r\Delta t}[pC_u + (1 - p)\, C_d] \qquad P = e^{-r\Delta t}[pP_u + (1 - p)\, P_d]$$

where:

C_u and C_d are the values of the call option in the up and down states, respectively,

P_u and P_d are the values of the put option in the up and down states, respectively,

r is the risk-free interest rate, and

p is the risk-neutral probability, given by:

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

3.2) Kou Jump-Diffusion Model

The methodology involves simulating asset paths using a combination of normal, uniform, and Poisson random variables, and then pricing a chooser option using these simulated paths.

Asset Path Simulation: The first step is to simulate the paths of the underlying asset. This is done using a combination of normal, uniform, and Poisson random variables:

Normal Random Variable: A normal random variable is used to model the continuous component of asset price changes, often referred to as the geometric Brownian motion. The change in asset price (dS) at any time (t) is given by:

$$dS = \mu S dt + \sigma S dW_t$$

where μ is the expected return, σ is the standard deviation of returns (volatility), S is the asset price, and dW_t is a standard Wiener process.

Uniform Random Variable: A uniform random variable is used to determine the direction of the jumps. This could be implemented as a binary variable that takes a value of 1 (upward jump) or -1 (downward jump) with equal probability.

Poisson Random Variable: A Poisson random variable is used to model the jump count, i.e., the number of jumps that occur in a given time interval. The probability of observing (k) jumps in an interval is given by the Poisson distribution:

$$P(k;\ \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

where λ is the average rate of jumps per unit of time.

Chooser Option Pricing: Once the asset paths have been simulated, they can be used to price a chooser option. At the chooser time, both call and put option values are calculated for each simulated path. The value of the chooser option is then taken as the maximum of these two values:

$$C_{chooser} = max(C_{call}, C_{put})$$

where:

$C_{chooser}$ is the value of the chooser option

C_{call} and C_{put} are the values of the call and put options, respectively.

This methodology allows for the pricing of chooser options in a more realistic market setting that includes both continuous price changes and jumps. However, it's important to note that the accuracy of the results will depend on the appropriateness of the chosen distributions and parameters for the asset price simulation.

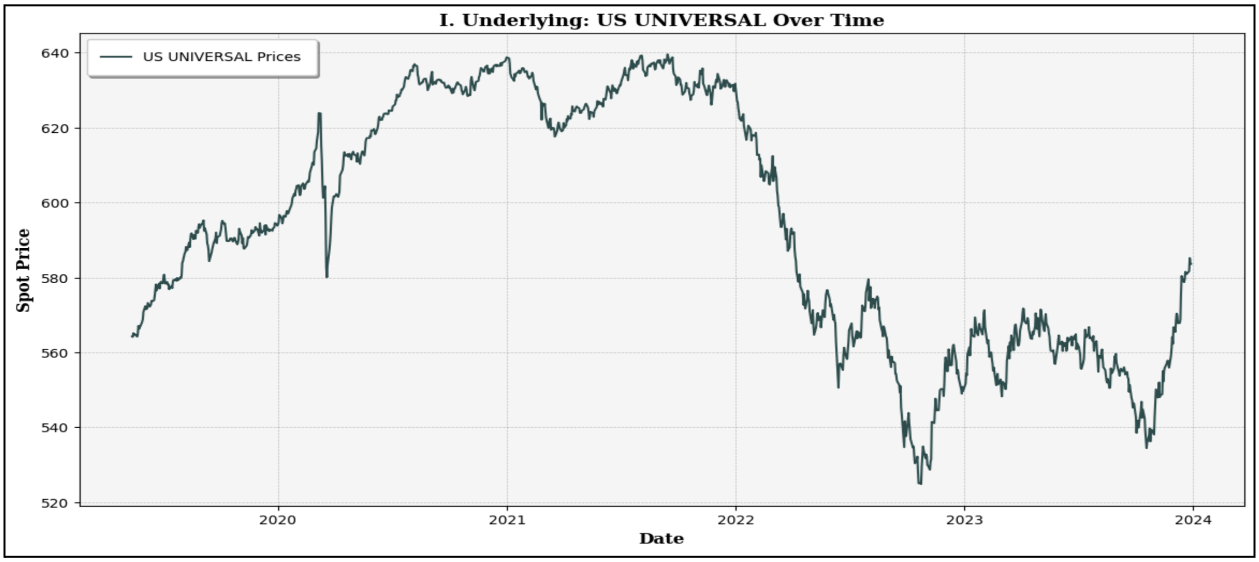

Fig: 3.1) Movement of Underlying - US UNIVERSAL Over Time

The graph depicts the movement of US UNIVERSAL prices from 2020 until well before 2024. It begins with a large increase in pricing until early 2021, peaking just around 620. Then, prices plummet below 560. Prices fluctuate for a bit, but mainly move lower, reaching their lowest position above 520 around late 2022 or early 2023. Following this, there is a clear recovery, with prices rising sharply towards the end of the timeline. This pattern depicts the numerous market and economic factors that influenced price movements throughout this time period.

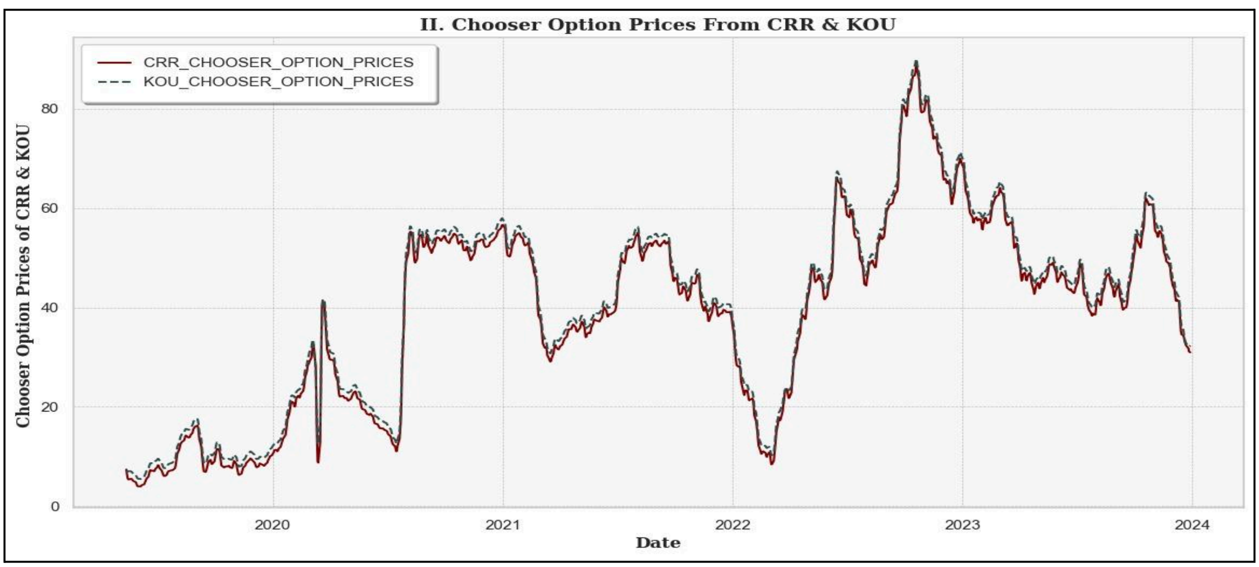

Fig: 3.2) Model Prices - Chooser Option Prices From CRR & KOU

The graph displays the Chooser Option Prices from 2019 to just before 2024, comparing two pricing models: the Cox-Ross-Rubinstein (CRR) model, shown by a solid line, and the Kou Jump-Diffusion (KOU) model, shown by a dashed line. Both lines exhibit significant volatility over time, reflecting fluctuations in option prices due to different modeling assumptions: the CRR model uses a binomial approach for price changes, while the KOU model incorporates both continuous and jump processes, capturing more extreme market movements.

3.3) Chooser-FX Trading Strategy

The trading strategy is designed to leverage the insights provided by both the Heston model for FX option pricing and the KOU model for chooser option pricing. By combining these models, we aim to capitalize on the nuanced dynamics of the FX and chooser option markets.

- **Projecting Future Payoffs:**
  The first step in our strategy entails forecasting the future payoffs for one year of both the FX and chooser options.
- **Comparative Analysis for Position Selection:**
  Once we have meticulously projected the future payoffs of both the FX and chooser options, the next step in our strategy involves conducting a comprehensive comparative analysis to determine the optimal position for the chooser option at the time of six months. Our analysis revolves around evaluating which position—call or put—offers the highest potential payoff under each scenario envisioned.
- **Determining FX and Chooser Option Positions:**
  After identifying the optimal position for the chooser option through our comparative analysis, we further refine our strategy by analyzing the difference between the projected payoff and the option price. In our approach, we consistently maintain a Long Position in the FX option, while the position for the chooser option is contingent upon this difference (profit margin). A negative margin prompts a short position in the chooser option, hedging against downside risk. Conversely, a positive margin indicates Long positions in both options for balanced exposure.
- **Performance Evaluation:**
  One of the key advantages of our strategy is its risk mitigation mechanism. By carefully considering the relationship between payoffs and option prices, we aim to ensure that our positions are structured to limit downside risk. Our strategy's Profit/Loss graph illustrates this risk management approach, demonstrating that the lowest points do not dip below zero, thereby safeguarding against potential losses.

Fig: 3.3) Profit/Loss for Chooser-FX trading strategy over time

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