DFT is therefore a special case of the chirp transform, obtained by taking $z = \omega_n$. Show how to evaluate the chirp transform in time $O(n \lg n)$ for any complex number $z$. (*Hint:* Use the equation

$$y_k = z^{k^2/2} \sum_{j=0}^{n-1} \left(a_j z^{j^2/2}\right) \left(z^{-(k-j)^2/2}\right)$$

to view the chirp transform as a convolution.)

## 30.3 Efficient FFT implementations

Since the practical applications of the DFT, such as signal processing, demand the utmost speed, this section examines two efficient FFT implementations. First, we shall examine an iterative version of the FFT algorithm that runs in $\Theta(n \lg n)$ time but can have a lower constant hidden in the $\Theta$-notation than the recursive version in Section 30.2. (Depending on the exact implementation, the recursive version may use the hardware cache more efficiently.) Then, we shall use the insights that led us to the iterative implementation to design an efficient parallel FFT circuit.

### An iterative FFT implementation

We first note that the **for** loop of lines 10–13 of RECURSIVE-FFT involves computing the value $\omega_n^k\, y_k^{[1]}$ twice. In compiler terminology, we call such a value a ***common subexpression***. We can change the loop to compute it only once, storing it in a temporary variable $t$.

```
for k = 0 to n/2 − 1
    t = ω y_k^[1]
    y_k = y_k^[0] + t
    y_{k+(n/2)} = y_k^[0] − t
    ω = ω ω_n
```

The operation in this loop, multiplying the twiddle factor $\omega = \omega_n^k$ by $y_k^{[1]}$, storing the product into $t$, and adding and subtracting $t$ from $y_k^{[0]}$, is known as a ***butterfly operation*** and is shown schematically in Figure 30.3.

We now show how to make the FFT algorithm iterative rather than recursive in structure. In Figure 30.4, we have arranged the input vectors to the recursive calls in an invocation of RECURSIVE-FFT in a tree structure, where the initial call is for $n = 8$. The tree has one node for each call of the procedure, labeled

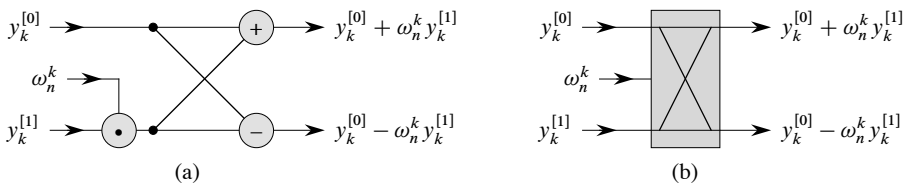


**Figure 30.3** A butterfly operation. **(a)** The two input values enter from the left, the twiddle factor $\omega_n^k$ is multiplied by $y_k^{[1]}$, and the sum and difference are output on the right. **(b)** A simplified drawing of a butterfly operation. We will use this representation in a parallel FFT circuit.

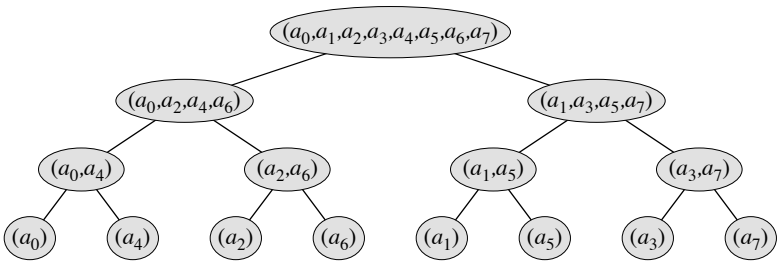


**Figure 30.4** The tree of input vectors to the recursive calls of the RECURSIVE-FFT procedure. The initial invocation is for $n = 8$.

by the corresponding input vector. Each RECURSIVE-FFT invocation makes two recursive calls, unless it has received a 1-element vector. The first call appears in the left child, and the second call appears in the right child.

Looking at the tree, we observe that if we could arrange the elements of the initial vector $a$ into the order in which they appear in the leaves, we could trace the execution of the RECURSIVE-FFT procedure, but bottom up instead of top down. First, we take the elements in pairs, compute the DFT of each pair using one butterfly operation, and replace the pair with its DFT. The vector then holds $n/2$ 2-element DFTs. Next, we take these $n/2$ DFTs in pairs and compute the DFT of the four vector elements they come from by executing two butterfly operations, replacing two 2-element DFTs with one 4-element DFT. The vector then holds $n/4$ 4-element DFTs. We continue in this manner until the vector holds two $(n/2)$-element DFTs, which we combine using $n/2$ butterfly operations into the final $n$-element DFT.

To turn this bottom-up approach into code, we use an array $A[0\,..\,n-1]$ that initially holds the elements of the input vector $a$ in the order in which they appear

in the leaves of the tree of Figure 30.4. (We shall show later how to determine this order, which is known as a bit-reversal permutation.) Because we have to combine DFTs on each level of the tree, we introduce a variable $s$ to count the levels, ranging from 1 (at the bottom, when we are combining pairs to form 2-element DFTs) to $\lg n$ (at the top, when we are combining two $(n/2)$-element DFTs to produce the final result). The algorithm therefore has the following structure:

```
1  for s = 1 to lg n
2      for k = 0 to n − 1 by 2^s
3          combine the two 2^(s−1)-element DFTs in
               A[k .. k + 2^(s−1) − 1] and A[k + 2^(s−1) .. k + 2^s − 1]
               into one 2^s-element DFT in A[k .. k + 2^s − 1]
```

We can express the body of the loop (line 3) as more precise pseudocode. We copy the **for** loop from the RECURSIVE-FFT procedure, identifying $y^{[0]}$ with $A[k \,..\, k + 2^{s-1} - 1]$ and $y^{[1]}$ with $A[k + 2^{s-1} \,..\, k + 2^s - 1]$. The twiddle factor used in each butterfly operation depends on the value of $s$; it is a power of $\omega_m$, where $m = 2^s$. (We introduce the variable $m$ solely for the sake of readability.) We introduce another temporary variable $u$ that allows us to perform the butterfly operation in place. When we replace line 3 of the overall structure by the loop body, we get the following pseudocode, which forms the basis of the parallel implementation we shall present later. The code first calls the auxiliary procedure BIT-REVERSE-COPY$(a, A)$ to copy vector $a$ into array $A$ in the initial order in which we need the values.

ITERATIVE-FFT$(a)$

```
 1  BIT-REVERSE-COPY(a, A)
 2  n = a.length                    // n is a power of 2
 3  for s = 1 to lg n
 4      m = 2^s
 5      ω_m = e^(2πi/m)
 6      for k = 0 to n − 1 by m
 7          ω = 1
 8          for j = 0 to m/2 − 1
 9              t = ω A[k + j + m/2]
10              u = A[k + j]
11              A[k + j] = u + t
12              A[k + j + m/2] = u − t
13              ω = ω ω_m
14  return A
```

How does BIT-REVERSE-COPY get the elements of the input vector $a$ into the desired order in the array $A$? The order in which the leaves appear in Figure 30.4

is a ***bit-reversal permutation***. That is, if we let $\text{rev}(k)$ be the $\lg n$-bit integer formed by reversing the bits of the binary representation of $k$, then we want to place vector element $a_k$ in array position $A[\text{rev}(k)]$. In Figure 30.4, for example, the leaves appear in the order $0, 4, 2, 6, 1, 5, 3, 7$; this sequence in binary is $000, 100, 010, 110, 001, 101, 011, 111$, and when we reverse the bits of each value we get the sequence $000, 001, 010, 011, 100, 101, 110, 111$. To see that we want a bit-reversal permutation in general, we note that at the top level of the tree, indices whose low-order bit is 0 go into the left subtree and indices whose low-order bit is 1 go into the right subtree. Stripping off the low-order bit at each level, we continue this process down the tree, until we get the order given by the bit-reversal permutation at the leaves.

Since we can easily compute the function $\text{rev}(k)$, the BIT-REVERSE-COPY procedure is simple:

```
BIT-REVERSE-COPY(a, A)
1  n = a.length
2  for k = 0 to n − 1
3      A[rev(k)] = a_k
```

The iterative FFT implementation runs in time $\Theta(n \lg n)$. The call to BIT-REVERSE-COPY$(a, A)$ certainly runs in $O(n \lg n)$ time, since we iterate $n$ times and can reverse an integer between 0 and $n - 1$, with $\lg n$ bits, in $O(\lg n)$ time. (In practice, because we usually know the initial value of $n$ in advance, we would probably code a table mapping $k$ to $\text{rev}(k)$, making BIT-REVERSE-COPY run in $\Theta(n)$ time with a low hidden constant. Alternatively, we could use the clever amortized reverse binary counter scheme described in Problem 17-1.) To complete the proof that ITERATIVE-FFT runs in time $\Theta(n \lg n)$, we show that $L(n)$, the number of times the body of the innermost loop (lines 8–13) executes, is $\Theta(n \lg n)$. The **for** loop of lines 6–13 iterates $n/m = n/2^s$ times for each value of $s$, and the innermost loop of lines 8–13 iterates $m/2 = 2^{s-1}$ times. Thus,

$$\begin{aligned} L(n) &= \sum_{s=1}^{\lg n} \frac{n}{2^s} \cdot 2^{s-1} \\ &= \sum_{s=1}^{\lg n} \frac{n}{2} \\ &= \Theta(n \lg n) \, . \end{aligned}$$

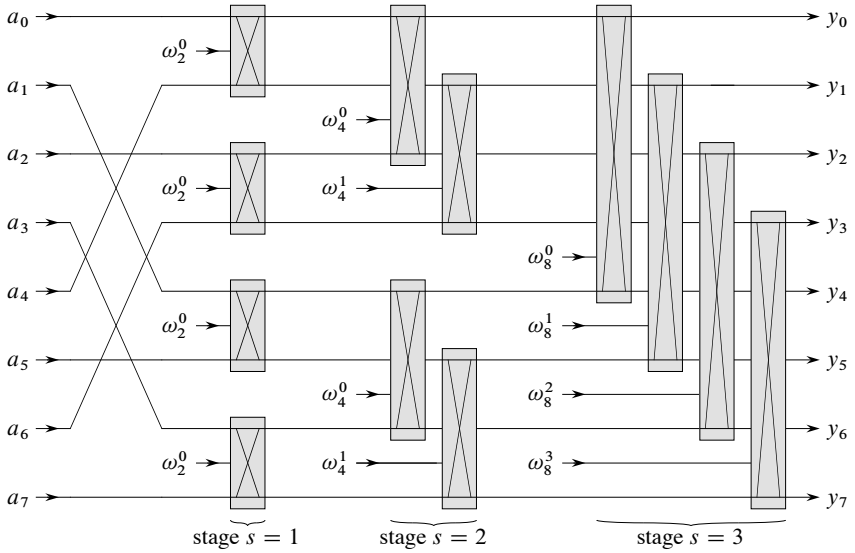


**Figure 30.5** A circuit that computes the FFT in parallel, here shown on $n = 8$ inputs. Each butterfly operation takes as input the values on two wires, along with a twiddle factor, and it produces as outputs the values on two wires. The stages of butterflies are labeled to correspond to iterations of the outermost loop of the ITERATIVE-FFT procedure. Only the top and bottom wires passing through a butterfly interact with it; wires that pass through the middle of a butterfly do not affect that butterfly, nor are their values changed by that butterfly. For example, the top butterfly in stage 2 has nothing to do with wire 1 (the wire whose output is labeled $y_1$); its inputs and outputs are only on wires 0 and 2 (labeled $y_0$ and $y_2$, respectively). This circuit has depth $\Theta(\lg n)$ and performs $\Theta(n \lg n)$ butterfly operations altogether.

## A parallel FFT circuit

We can exploit many of the properties that allowed us to implement an efficient iterative FFT algorithm to produce an efficient parallel algorithm for the FFT. We will express the parallel FFT algorithm as a circuit. Figure 30.5 shows a parallel FFT circuit, which computes the FFT on $n$ inputs, for $n = 8$. The circuit begins with a bit-reverse permutation of the inputs, followed by $\lg n$ stages, each stage consisting of $n/2$ butterflies executed in parallel. The ***depth*** of the circuit—the maximum number of computational elements between any output and any input that can reach it—is therefore $\Theta(\lg n)$.

The leftmost part of the parallel FFT circuit performs the bit-reverse permutation, and the remainder mimics the iterative ITERATIVE-FFT procedure. Because each iteration of the outermost **for** loop performs $n/2$ independent butterfly operations, the circuit performs them in parallel. The value of $s$ in each iteration within

ITERATIVE-FFT corresponds to a stage of butterflies shown in Figure 30.5. For $s = 1, 2, \ldots, \lg n$, stage $s$ consists of $n/2^s$ groups of butterflies (corresponding to each value of $k$ in ITERATIVE-FFT), with $2^{s-1}$ butterflies per group (corresponding to each value of $j$ in ITERATIVE-FFT). The butterflies shown in Figure 30.5 correspond to the butterfly operations of the innermost loop (lines 9–12 of ITERATIVE-FFT). Note also that the twiddle factors used in the butterflies correspond to those used in ITERATIVE-FFT: in stage $s$, we use $\omega_m^0, \omega_m^1, \ldots, \omega_m^{m/2-1}$, where $m = 2^s$.

### Exercises

***30.3-1***
Show how ITERATIVE-FFT computes the DFT of the input vector $(0, 2, 3, -1, 4, 5, 7, 9)$.

***30.3-2***
Show how to implement an FFT algorithm with the bit-reversal permutation occurring at the end, rather than at the beginning, of the computation. (*Hint:* Consider the inverse DFT.)

***30.3-3***
How many times does ITERATIVE-FFT compute twiddle factors in each stage? Rewrite ITERATIVE-FFT to compute twiddle factors only $2^{s-1}$ times in stage $s$.

***30.3-4*** ⋆
Suppose that the adders within the butterfly operations of the FFT circuit sometimes fail in such a manner that they always produce a zero output, independent of their inputs. Suppose that exactly one adder has failed, but that you don't know which one. Describe how you can identify the failed adder by supplying inputs to the overall FFT circuit and observing the outputs. How efficient is your method?

## Problems

***30-1 Divide-and-conquer multiplication***

***a.*** Show how to multiply two linear polynomials $ax + b$ and $cx + d$ using only three multiplications. (*Hint:* One of the multiplications is $(a + b) \cdot (c + d)$.)

***b.*** Give two divide-and-conquer algorithms for multiplying two polynomials of degree-bound $n$ in $\Theta(n^{\lg 3})$ time. The first algorithm should divide the input polynomial coefficients into a high half and a low half, and the second algorithm should divide them according to whether their index is odd or even.