29.4-6

Which result from Chapter 26 can be interpreted as weak duality for the maximum-flow problem?

29.5 The initial basic feasible solution

In this section, we first describe how to test whether a linear program is feasible, and if it is, how to produce a slack form for which the basic solution is feasible. We conclude by proving the fundamental theorem of linear programming, which says that the SIMPLEX procedure always produces the correct result.

Finding an initial solution

In Section 29.3, we assumed that we had a procedure INITIALIZE-SIMPLEX that determines whether a linear program has any feasible solutions, and if it does, gives a slack form for which the basic solution is feasible. We describe this procedure here.

A linear program can be feasible, yet the initial basic solution might not be feasible. Consider, for example, the following linear program:

$$\text{maximize} \quad 2x_1 - x_2 \tag{29.102}$$

subject to

$$2x_1 - x_2 \leq 2 \tag{29.103}$$

$$x_1 - 5x_2 \leq -4 \tag{29.104}$$

$$x_1, x_2 \geq 0 \; . \tag{29.105}$$

If we were to convert this linear program to slack form, the basic solution would set $x_1 = 0$ and $x_2 = 0$. This solution violates constraint (29.104), and so it is not a feasible solution. Thus, INITIALIZE-SIMPLEX cannot just return the obvious slack form. In order to determine whether a linear program has any feasible solutions, we will formulate an ***auxiliary linear program***. For this auxiliary linear program, we can find (with a little work) a slack form for which the basic solution is feasible. Furthermore, the solution of this auxiliary linear program determines whether the initial linear program is feasible and if so, it provides a feasible solution with which we can initialize SIMPLEX.

Lemma 29.11

Let L be a linear program in standard form, given as in (29.16)–(29.18). Let x_0 be a new variable, and let L_{aux} be the following linear program with $n + 1$ variables:

$$\text{maximize} \quad -x_0 \tag{29.106}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i \quad \text{for } i = 1, 2, \ldots, m \,, \tag{29.107}$$

$$x_j \geq 0 \quad \text{for } j = 0, 1, \ldots, n \,. \tag{29.108}$$

Then L is feasible if and only if the optimal objective value of L_{aux} is 0.

Proof Suppose that L has a feasible solution $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. Then the solution $\bar{x}_0 = 0$ combined with $\bar{x}$ is a feasible solution to L_{aux} with objective value 0. Since $x_0 \geq 0$ is a constraint of L_{aux} and the objective function is to maximize $-x_0$, this solution must be optimal for L_{aux}.

Conversely, suppose that the optimal objective value of L_{aux} is 0. Then $\bar{x}_0 = 0$, and the remaining solution values of $\bar{x}$ satisfy the constraints of L. ■

We now describe our strategy to find an initial basic feasible solution for a linear program L in standard form:

INITIALIZE-SIMPLEX(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint
    and setting the objective function to $-x_0$
let (N, B, A, b, c, ν) be the resulting slack form for L_{aux}
$l = n + k$
// L_{aux} has $n + 1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, \nu) = \text{PIVOT}(N, B, A, b, c, \nu, l, 0)$
// The basic solution is now feasible for L_{aux}.
iterate the **while** loop of lines 3–12 of SIMPLEX until an optimal solution
    to L_{aux} is found
if the optimal solution to L_{aux} sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_{aux}, remove x_0 from the constraints and
        restore the original objective function of L, but replace each basic
        variable in this objective function by the right-hand side of its
        associated constraint
    return the modified final slack form
else return "infeasible"

INITIALIZE-SIMPLEX works as follows. In lines 1–3, we implicitly test the basic solution to the initial slack form for L given by $N = \{1, 2, \ldots, n\}$, $B = \{n+1, n+2, \ldots, n+m\}$, $\bar{x}_i = b_i$ for all $i \in B$, and $\bar{x}_j = 0$ for all $j \in N$. (Creating the slack form requires no explicit effort, as the values of A, b, and c are the same in both slack and standard forms.) If line 2 finds this basic solution to be feasible—that is, $\bar{x}_i \geq 0$ for all $i \in N \cup B$—then line 3 returns the slack form. Otherwise, in line 4, we form the auxiliary linear program L_{aux} as in Lemma 29.11. Since the initial basic solution to L is not feasible, the initial basic solution to the slack form for L_{aux} cannot be feasible either. To find a basic feasible solution, we perform a single pivot operation. Line 6 selects $l = n + k$ as the index of the basic variable that will be the leaving variable in the upcoming pivot operation. Since the basic variables are $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, the leaving variable x_l will be the one with the most negative value. Line 8 performs that call of PIVOT, with x_0 entering and x_l leaving. We shall see shortly that the basic solution resulting from this call of PIVOT will be feasible. Now that we have a slack form for which the basic solution is feasible, we can, in line 10, repeatedly call PIVOT to fully solve the auxiliary linear program. As the test in line 11 demonstrates, if we find an optimal solution to L_{aux} with objective value 0, then in lines 12–14, we create a slack form for L for which the basic solution is feasible. To do so, we first, in lines 12–13, handle the degenerate case in which x_0 may still be basic with value $\bar{x}_0 = 0$. In this case, we perform a pivot step to remove x_0 from the basis, using any $e \in N$ such that $a_{0e} \neq 0$ as the entering variable. The new basic solution remains feasible; the degenerate pivot does not change the value of any variable. Next we delete all x_0 terms from the constraints and restore the original objective function for L. The original objective function may contain both basic and nonbasic variables. Therefore, in the objective function we replace each basic variable by the right-hand side of its associated constraint. Line 15 then returns this modified slack form. If, on the other hand, line 11 discovers that the original linear program L is infeasible, then line 16 returns this information.

We now demonstrate the operation of INITIALIZE-SIMPLEX on the linear program (29.102)–(29.105). This linear program is feasible if we can find nonnegative values for x_1 and x_2 that satisfy inequalities (29.103) and (29.104). Using Lemma 29.11, we formulate the auxiliary linear program

$$\text{maximize} \quad -x_0 \tag{29.109}$$

subject to

$$2x_1 - x_2 - x_0 \leq 2 \tag{29.110}$$

$$x_1 - 5x_2 - x_0 \leq -4 \tag{29.111}$$

$$x_1, x_2, x_0 \geq 0 .$$

By Lemma 29.11, if the optimal objective value of this auxiliary linear program is 0, then the original linear program has a feasible solution. If the optimal objective value of this auxiliary linear program is negative, then the original linear program does not have a feasible solution.

We write this linear program in slack form, obtaining

$$\begin{array}{rcrcrcrcl} z &=& & & & & &-& x_0 \\ x_3 &=& 2 &-& 2x_1 &+& x_2 &+& x_0 \\ x_4 &=& -4 &-& x_1 &+& 5x_2 &+& x_0 \; . \end{array}$$

We are not out of the woods yet, because the basic solution, which would set $x_4 = -4$, is not feasible for this auxiliary linear program. We can, however, with one call to PIVOT, convert this slack form into one in which the basic solution is feasible. As line 8 indicates, we choose x_0 to be the entering variable. In line 6, we choose as the leaving variable x_4, which is the basic variable whose value in the basic solution is most negative. After pivoting, we have the slack form

$$\begin{array}{rcrcrcrcl} z &=& -4 &-& x_1 &+& 5x_2 &-& x_4 \\ x_0 &=& 4 &+& x_1 &-& 5x_2 &+& x_4 \\ x_3 &=& 6 &-& x_1 &-& 4x_2 &+& x_4 \; . \end{array}$$

The associated basic solution is $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = (4, 0, 0, 6, 0)$, which is feasible. We now repeatedly call PIVOT until we obtain an optimal solution to L_{aux}. In this case, one call to PIVOT with x_2 entering and x_0 leaving yields

$$\begin{array}{rcrcrcrcl} z &=& &-& x_0 & & & & \\ x_2 &=& \frac{4}{5} &-& \frac{x_0}{5} &+& \frac{x_1}{5} &+& \frac{x_4}{5} \\ x_3 &=& \frac{14}{5} &+& \frac{4x_0}{5} &-& \frac{9x_1}{5} &+& \frac{x_4}{5} \; . \end{array}$$

This slack form is the final solution to the auxiliary problem. Since this solution has $x_0 = 0$, we know that our initial problem was feasible. Furthermore, since $x_0 = 0$, we can just remove it from the set of constraints. We then restore the original objective function, with appropriate substitutions made to include only nonbasic variables. In our example, we get the objective function

$$2x_1 - x_2 = 2x_1 - \left(\frac{4}{5} - \frac{x_0}{5} + \frac{x_1}{5} + \frac{x_4}{5}\right) \; .$$

Setting $x_0 = 0$ and simplifying, we get the objective function

$$-\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \; ,$$

and the slack form

$$
\begin{aligned}
z &= -\frac{4}{5} + \frac{9x_1}{5} - \frac{x_4}{5} \\
x_2 &= \frac{4}{5} + \frac{x_1}{5} + \frac{x_4}{5} \\
x_3 &= \frac{14}{5} - \frac{9x_1}{5} + \frac{x_4}{5} \ .
\end{aligned}
$$

This slack form has a feasible basic solution, and we can return it to procedure SIMPLEX.

We now formally show the correctness of INITIALIZE-SIMPLEX.

Lemma 29.12

If a linear program L has no feasible solution, then INITIALIZE-SIMPLEX returns "infeasible." Otherwise, it returns a valid slack form for which the basic solution is feasible.

Proof First suppose that the linear program L has no feasible solution. Then by Lemma 29.11, the optimal objective value of L_{aux}, defined in (29.106)–(29.108), is nonzero, and by the nonnegativity constraint on x_0, the optimal objective value must be negative. Furthermore, this objective value must be finite, since setting $x_i = 0$, for $i = 1, 2, \ldots, n$, and $x_0 = |\min_{i=1}^{m} \{b_i\}|$ is feasible, and this solution has objective value $-|\min_{i=1}^{m} \{b_i\}|$. Therefore, line 10 of INITIALIZE-SIMPLEX finds a solution with a nonpositive objective value. Let $\bar{x}$ be the basic solution associated with the final slack form. We cannot have $\bar{x}_0 = 0$, because then L_{aux} would have objective value 0, which contradicts that the objective value is negative. Thus the test in line 11 results in line 16 returning "infeasible."

Suppose now that the linear program L does have a feasible solution. From Exercise 29.3-4, we know that if $b_i \geq 0$ for $i = 1, 2, \ldots, m$, then the basic solution associated with the initial slack form is feasible. In this case, lines 2–3 return the slack form associated with the input. (Converting the standard form to slack form is easy, since A, b, and c are the same in both.)

In the remainder of the proof, we handle the case in which the linear program is feasible but we do not return in line 3. We argue that in this case, lines 4–10 find a feasible solution to L_{aux} with objective value 0. First, by lines 1–2, we must have

$$b_k < 0 \ ,$$

and

$$b_k \leq b_i \quad \text{for each } i \in B \ . \tag{29.112}$$

In line 8, we perform one pivot operation in which the leaving variable x_l (recall that $l = n + k$, so that $b_l < 0$) is the left-hand side of the equation with minimum b_i, and the entering variable is x_0, the extra added variable. We now show that after this pivot, all entries of b are nonnegative, and hence the basic solution to L_{aux} is feasible. Letting $\bar{x}$ be the basic solution after the call to PIVOT, and letting $\widehat{b}$ and $\widehat{B}$ be values returned by PIVOT, Lemma 29.1 implies that

$$\bar{x}_i = \begin{cases} b_i - a_{ie}\widehat{b}_e & \text{if } i \in \widehat{B} - \{e\} , \\ b_l/a_{le} & \text{if } i = e . \end{cases} \tag{29.113}$$

The call to PIVOT in line 8 has $e = 0$. If we rewrite inequalities (29.107), to include coefficients a_{i0},

$$\sum_{j=0}^{n} a_{ij}x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m , \tag{29.114}$$

then

$$a_{i0} = a_{ie} = -1 \quad \text{for each } i \in B . \tag{29.115}$$

(Note that a_{i0} is the coefficient of x_0 as it appears in inequalities (29.114), not the negation of the coefficient, because L_{aux} is in standard rather than slack form.) Since $l \in B$, we also have that $a_{le} = -1$. Thus, $b_l/a_{le} > 0$, and so $\bar{x}_e > 0$. For the remaining basic variables, we have

$$\begin{aligned} \bar{x}_i &= b_i - a_{ie}\widehat{b}_e && \text{(by equation (29.113))} \\ &= b_i - a_{ie}(b_l/a_{le}) && \text{(by line 3 of PIVOT)} \\ &= b_i - b_l && \text{(by equation (29.115) and } a_{le} = -1) \\ &\ge 0 && \text{(by inequality (29.112))} , \end{aligned}$$

which implies that each basic variable is now nonnegative. Hence the basic solution after the call to PIVOT in line 8 is feasible. We next execute line 10, which solves L_{aux}. Since we have assumed that L has a feasible solution, Lemma 29.11 implies that L_{aux} has an optimal solution with objective value 0. Since all the slack forms are equivalent, the final basic solution to L_{aux} must have $\bar{x}_0 = 0$, and after removing x_0 from the linear program, we obtain a slack form that is feasible for L. Line 15 then returns this slack form. ■

Fundamental theorem of linear programming

We conclude this chapter by showing that the SIMPLEX procedure works. In particular, any linear program either is infeasible, is unbounded, or has an optimal solution with a finite objective value. In each case, SIMPLEX acts appropriately.

Theorem 29.13 (Fundamental theorem of linear programming)
Any linear program L, given in standard form, either

1. has an optimal solution with a finite objective value,
2. is infeasible, or
3. is unbounded.

If L is infeasible, SIMPLEX returns "infeasible." If L is unbounded, SIMPLEX returns "unbounded." Otherwise, SIMPLEX returns an optimal solution with a finite objective value.

Proof By Lemma 29.12, if linear program L is infeasible, then SIMPLEX returns "infeasible." Now suppose that the linear program L is feasible. By Lemma 29.12, INITIALIZE-SIMPLEX returns a slack form for which the basic solution is feasible. By Lemma 29.7, therefore, SIMPLEX either returns "unbounded" or terminates with a feasible solution. If it terminates with a finite solution, then Theorem 29.10 tells us that this solution is optimal. On the other hand, if SIMPLEX returns "unbounded," Lemma 29.2 tells us the linear program L is indeed unbounded. Since SIMPLEX always terminates in one of these ways, the proof is complete. ■

Exercises

29.5-1
Give detailed pseudocode to implement lines 5 and 14 of INITIALIZE-SIMPLEX.

29.5-2
Show that when the main loop of SIMPLEX is run by INITIALIZE-SIMPLEX, it can never return "unbounded."

29.5-3
Suppose that we are given a linear program L in standard form, and suppose that for both L and the dual of L, the basic solutions associated with the initial slack forms are feasible. Show that the optimal objective value of L is 0.

29.5-4
Suppose that we allow strict inequalities in a linear program. Show that in this case, the fundamental theorem of linear programming does not hold.

29.5-5

Solve the following linear program using SIMPLEX:

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & - & x_2 & \le & 8 \\
& -x_1 & - & x_2 & \le & -3 \\
& -x_1 & + & 4x_2 & \le & 2 \\
& & x_1, x_2 & & \ge & 0 \; .
\end{array}
$$

29.5-6

Solve the following linear program using SIMPLEX:

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & - & 2x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & + & 2x_2 & \le & 4 \\
& -2x_1 & - & 6x_2 & \le & -12 \\
& & & x_2 & \le & 1 \\
& & x_1, x_2 & & \ge & 0 \; .
\end{array}
$$

29.5-7

Solve the following linear program using SIMPLEX:

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & 3x_2 & & \\
\text{subject to} & & & & & \\
& -x_1 & + & x_2 & \le & -1 \\
& -x_1 & - & x_2 & \le & -3 \\
& -x_1 & + & 4x_2 & \le & 2 \\
& & x_1, x_2 & & \ge & 0 \; .
\end{array}
$$

29.5-8

Solve the linear program given in (29.6)–(29.10).

29.5-9

Consider the following 1-variable linear program, which we call P:

$$
\begin{array}{lrcl}
\text{maximize} & tx & & \\
\text{subject to} & & & \\
& rx & \le & s \\
& x & \ge & 0 \; ,
\end{array}
$$

where r, s, and t are arbitrary real numbers. Let D be the dual of P.

State for which values of r, s, and t you can assert that

1. Both P and D have optimal solutions with finite objective values.
2. P is feasible, but D is infeasible.
3. D is feasible, but P is infeasible.
4. Neither P nor D is feasible.

Problems

29-1 Linear-inequality feasibility

Given a set of m linear inequalities on n variables $x_1, x_2, \ldots, x_n$, the ***linear-inequality feasibility problem*** asks whether there is a setting of the variables that simultaneously satisfies each of the inequalities.

a. Show that if we have an algorithm for linear programming, we can use it to solve a linear-inequality feasibility problem. The number of variables and constraints that you use in the linear-programming problem should be polynomial in n and m.

b. Show that if we have an algorithm for the linear-inequality feasibility problem, we can use it to solve a linear-programming problem. The number of variables and linear inequalities that you use in the linear-inequality feasibility problem should be polynomial in n and m, the number of variables and constraints in the linear program.

29-2 Complementary slackness

Complementary slackness describes a relationship between the values of primal variables and dual constraints and between the values of dual variables and primal constraints. Let $\bar{x}$ be a feasible solution to the primal linear program given in (29.16)–(29.18), and let $\bar{y}$ be a feasible solution to the dual linear program given in (29.83)–(29.85). Complementary slackness states that the following conditions are necessary and sufficient for $\bar{x}$ and $\bar{y}$ to be optimal:

$$\sum_{i=1}^{m} a_{ij}\bar{y}_i = c_j \text{ or } \bar{x}_j = 0 \quad \text{for } j = 1, 2, \ldots, n$$

and

$$\sum_{j=1}^{n} a_{ij}\bar{x}_j = b_i \text{ or } \bar{y}_i = 0 \quad \text{for } i = 1, 2, \ldots, m\ .$$