UNIT 4 DISCRETE PROBABILITY AND ADVANCED COUNTING TECHNIQUES

SIA GROUP

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. Define,

(a) Experiment

(b) Sample Space

(c) Event

(d) Finite Probability

Answer :

(a) Experiment

It is a procedure which contains one of the possible outcomes.

(b) Sample Space

It is a set of possible outcomes.

(c) Event

It is a subset of the sample space.

(d) Finite Probability

For a finitely many possible outcomes, if S is a finite non empty sample space of equally likely outcomes then

Probability of E, $P(E) = \frac{|E|}{|S|}$

where, E - Event subset of S.

Q2. Define probability. Model Paper-1, Q1(g)

Answer :

The probability of an event E is defined as a ratio of number of favourable events with respect to E, to the total number of events of a random experiment.

$$P(E) = \frac{\text{Number of favourable events}}{\text{Total number of events of the experiment}}$$

$$\therefore \quad P(E) = \frac{S}{N}$$

Q3. Define random variable.

Answer :

Random variable is defined as a function from the sample space (S) of the experiment to the set of real numbers.

The distribution of random variable X on s is the set of pairs $(r, P(X = r))\ \forall\, r \in X(s)$. Here, $P(X = r)$ is the probability that X takes the value r.

Example

Let $X(t)$ = Number of heads appear when t is outcome.

Then, $X(HHH) = 3$

$X(TTT) = 0$

$X(TTH) = X(THT) = X(HTT) = 1$

$X(HHT) = X(HTH) = X(THH) = 2$

Q4. Define probabilistic method.

Answer :

According to probabilistic method, an element in S exists, if a randomly chosen element from S does not have a particular property (or < 1).

If k is an integer with $k \geq 2$ then

$R(k/k) \geq 2^{k/2}$

where,

$R(k, k)$ = Ramsey number

k - Mutual elements

Q5. Define pairwise and mutual independence.

Model Paper-2, Q1(g)

Answer :

Pairwise Independence

The events $E_1, E_2, \ldots\ldots E_n$ are said to be pairwise independent if and only if,

$$P(E_i \cap E_j) = P(E_i)\,P(E_j)\,;\ 1 \leq i < j \leq n$$

Mutual Independence

The events $E_1, E_2, \ldots\ldots E_n$ are said to be mutually independent if and only if,

$$P(E_{i_1} \cap E_{i_2} \cap E_{i_m}) = P(E_{i_1})\,P(E_{i_2}) \ldots\ldots P(E_{i_m})\,;\ 1 \leq i_1 < i_2 < \ldots\ldots i_m \leq n\,;\ m \geq 2$$

Q6. What is the probability that a card selected at random from a standard deck of 52 cards is an ace?

Answer :

Let E be an event of randomly selecting an ace.

As a deck of 52 cards has 4 aces

$\Rightarrow n(E) = 4$

Let S be the total number of cards

$\Rightarrow n(S) = 52$

The probability of selecting an ace from the total number of cards is given as,

$$P_{ace} = \frac{n(E)}{n(S)} = \frac{4}{42}$$

$$= \frac{1}{13}$$

$$\therefore \quad P_{ace} = \frac{1}{13}$$

Q7. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?

Answer :

Given that,

Number of integers = 100

Number of positive integers not exceeding 100 and divisible by 3 = 33 i.e., 3, 6, 9..........99

[$\because \frac{100}{3}$ = 33.333....... then there are 33 occurrences]

The probability that a positive integer not exceeding 100 selected at random is divisible by 3 is obtained as,

$$\text{Probability, } P = \frac{\text{Number of integers divisible by 3}}{\text{Total number of integers}}$$

$$P = \frac{33}{100} = 0.33$$

$\therefore P = 0.33$

Q8. Write about expected value.

Model Paper-3, Q1(g)

Answer :

If X is a random variable on the sample space S then, its expected value is given by,

$$E(X) = \sum_{s \in S} P(s)\, X(s)$$

or

$$E(X) = \sum_{i=1}^{n} P(x_i)\, X(x_i)$$

Where,

$S = \{x_1, x_2, \ldots\ldots x_n\}$

Deviation of $X = X(S) - E(S)$

If $P(X=r)$ is the probability such that

$$P(X=r) = \sum_{s \in S} X(S) = r^{P(s)} \text{ then,}$$

$$E(X) = \sum_{r \in X(S)} P(X=r)r$$

It is also called as expectation or mean.

Q9. What is the expected number of heads that come up when a fair coin is flipped 10 times?

Model Paper-1, Q1(h)

Answer :

Given that,

A fair coin is flipped 10 times

$\Rightarrow n = 10$

Let, P be the probability of success in each coin.

$\Rightarrow P = \frac{1}{2}$

From Bernoulli trails, the expected value is given as,

Expected value $= n \times P$

$= 10 \times \frac{1}{2}$

$= 5$

$\therefore$ The expected number of heads that come up when a fair coin is flipped 10 times is 5.

Q10. Suppose that a Bayesian spam filter is trained on a set of 10,000 spam messages and 5000 messages that are not spam. The word "enhancement" appears in 15000 spam message and 20 messages that are not spam, while the word 'herbal' appears in 800 spam messages and 200 messages that are not spam. Estimate the probability that a received message containing both the words "enhancement" and "herbal" is spam. Will the message be rejected as spam if the threshold for rejecting spam is 0.9?

Answer : Model Paper-4, Q1(a)

Let, $p(\text{enhancement}) = \dfrac{1500}{10000} = 0.15$

$$q(\text{enhancement}) = \frac{20}{5000} = 0.004$$

$$p(\text{herbal}) = \frac{800}{10000} = 0.08$$

$$q(\text{herbal}) = \frac{200}{5000} = 0.04$$

The probability that the message is spam is given as,

$$r(\text{enhancement, herbal}) = \frac{p(\text{enhancement}).p(\text{herbal})}{p(\text{enhancement}).p(\text{herbal}) + q(\text{enhancement})q(\text{herbal})}$$

$$= \frac{(0.15)(0.08)}{(0.15)(0.08) + (0.004)(0.04)}$$

$$= 0.987 > 0.9$$

As r(enhancement, herbal) is greater than the threshold 0.9, an incoming message containing 'enhancement' and 'herbal' will be rejected.

Q11. Suppose that we roll a pair of fair dice until the sum of the numbers on the dice is seven. What is the expected number of times we roll the dice?

Answer :

Given that,

A pair of fair dice is rolled until the sum of numbers is 7.

A dice has 6 faces

Total number of outcomes = 6 × 6 = 36

Favourable cases for 7 = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

The probability that the sum of numbers is 7 is,

$$P[\text{Sum of numbers is } 7] = \frac{\text{Favourable cases for 7}}{\text{Total number of outcomes}}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

Expected value is given as,

$$E(X) = \frac{1}{P}$$

$$\Rightarrow \quad E(X) = \frac{1}{1/6} = 6$$

$$\therefore \quad E(X) = 6$$

Q12. Show that V(X + Y) = V(X) + V(Y) + 2 Cov(X, Y).

Model Paper-3, Q1(g)

Answer :

Let X, Y be any two random variables

The variance of a random variable X is given as,

$$V(X) = E(X^2) - E(X)^2$$

$$\Rightarrow \quad V(X+Y) = E((X+Y)^2) - E(X+Y)^2$$

$$= E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2$$

$$= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2$$

$$= [E(X^2) - E(X)^2] + 2[E(XY) - E(X)E(Y)] + [E(Y^2) - E(Y)^2]$$

$$\Rightarrow \quad V(X+Y) = V(X) + 2\,\text{Cov}(X, Y) + V(Y) \qquad [\because \text{Cov}(X, Y) = E(XY) - E(X)E(Y)]$$

$$\therefore \quad V(X+Y) = V(X) + 2\,\text{Cov}(X, Y) + V(Y)$$

Q13. Write the uses of recurrence relations.

Model Paper-3, Q1(h)

Answer :

The recurrence relations are used to model a large number of problems which are as follows,

1. Identifying the number of movements in the tower of Hanoi problem.
2. Calculating the compound interest.
3. Calculating rabbits on island.
4. Calculating bit strings.

Q14. Show that $a_{n-1} = a_n - \nabla a_n$.

Model Paper-2, Q1(h)

Answer :

Let $\{a_n\}$ be the sequence of real numbers

The first difference is given as,

$$\nabla a_n = a_n - a_{n-1}$$

Consider,

$$a_n - \nabla a_n = a_n - (a_n - a_{n-1})$$

$$= a_n - a_n + a_{n-1}$$

$$= a_{n-1}$$

$$\therefore \quad a_{n-1} = a_n - \nabla a_n$$

Q15. State master theorem.

Model Paper-4, Q1(h)

Answer :

Let f represents an increasing function which satisfies the following recurrence relation.

$$f(n) = af\left(\frac{n}{b}\right) + cn^d$$

whenever $n = b^k$

where k – positive integer

$a \geq 1$

b - Integer > 1

c, d - Real numbers

Then,

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Q16. Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.

Answer :

There are exactly 50 odd positive integers not exceeding 100.

$\Rightarrow \quad |A_1| = 50$

10 squares i.e., $1^2, 2^2, \ldots\ldots 10^2$

Further, half of these square integers are odd

i.e., $1^2, 3^2, 5^2, 7^2, 9^2$.

$\Rightarrow \quad |A_1 \cap A_2| = 5$

From Inclusion–Exclusion principle,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 50 + 10 - 5 = 55$$

Q17. How many ways are there to distribute six different toys to three different children such that each child gets at least one toy?

Model Paper-5, Q1(h)

Answer :

The number of ways to distribute 6 different toys to 3 children is the number of onto functions from 6 to 3 i.e.,

$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m + \ldots (-1)^{n-1} C(n, n-1).\ 1^m$

Here,

$n = 3, m = 6$

$\Rightarrow \quad 3^6 - 3C_1 2^6 - 3C_2 1^6$

$= 729 - 3(64) - 3(1)$

$= 729 - 189$

$= 540$ functions.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

4.1 DISCRETE PROBABILITY

4.1.1 An Introduction to Discrete Probability, Probability Theory

Q18. Give relation of Bernoulli trials with binomial distribution.

Answer :

Bernoulli trial represents the possible outcomes of an experiment.

The probability of exactly k successes in n independent Bernoulli trials is given by,

$$C(n, k)\, p^k\, q^{n-k}$$

where,

C - Number of trials

p - success probability

q - failure probability $= 1 - p$

For binomial distribution,

$$b(k; n, p) = C(n, k)\, p^k\, q^{n-k}$$

The sum of probabilities of k successes in n independent Bernoulli trials is given by,

$$\sum_{k=0}^{n} C(n, k)\, p^k q^{n-k} = (p+q)^n = 1$$

Q19. State and prove probabilities of complements and union of events.

Answer :

Probability of complements

Statement

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by,

$$P(\overline{E}) = 1 - P(E)$$

Proof :

Let E be an event in a sample space S such that,

$$\overline{E} = S - E$$

$$\Rightarrow \quad |\overline{E}| = |S| - |E|$$

Then, from finite probability,

$$P(\overline{E}) = \frac{|\overline{E}|}{|S|}$$

$$= \frac{|S| - |E|}{|S|}$$

$$= 1 - \frac{|E|}{|S|}$$

$$= 1 - P(E)$$

$$\therefore \quad P(\overline{E}) = 1 - P(E)$$

Union of Events

Statement

Let E_1 and E_2 be events in the sample space S. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof

Let E_1 and E_2 be events in S.

Then,

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

From finite probability,

$$P(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|}$$

$$= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$\therefore \quad P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Q20. What are the conditions to be met in probability theory?

Model Paper

Answer :

The following are the conditions to be met in probability theory are,

(i) Probability of each outcome is a non-negative real number and not greater than 1.

i.e., $0 \leq P(s) \leq 1$ for each $s \in S$

(or)

$0 \leq P(x_i) \leq 1$ for each $i = 1, 2, \ldots\ldots n$

(ii) Sum of the probabilities of all possible outcomes should be 1.

i.e., $\sum_{s \in S} P(s) = 1$

(or)

$\sum_{i=1}^{n} P(x_i) = 1$ (Probability distribution)

Where,

x or s - outcome

S - Sample space

n - number of possible outcomes.

(iii) For a uniform distribution, each element of s is assigned with the probability $1/n$.

(iv) $P(E) = \sum_{s \in E} P(s)$

where, E - infinite set event

$\sum_{s \in E} P(s)$ - convergent infinite series.

(v) If $E_1, E_2, \ldots\ldots$ is a sequence of pair wise disjoint events in a sample space S, then

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

(vi) If E and F are events with $P(F) > 0$ then the conditional probability is defined as,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Q21. What is the probability that a five-card poker hand contains at least one ace?

Answer :

Let $P(X \geq 1)$ denotes the probability of getting atleast one ace.

Probability that a five-card poker hand contains atleast one ace is given as,

$$P(X \geq 1) = 1 - P(X = 0) \quad \ldots(1)$$

Let, $(P(X = 0)$ denotes probability of a hand containing no aces.

Consider,

$$P(X = 0) = \frac{48C_5}{52C_5}$$

$$= \frac{\frac{48!}{5!43!}}{\frac{52!}{5!47!}} \qquad \left[\because {}^nC_r = \frac{n!}{r!(n-r)!}\right]$$

$$= \frac{48!5!47!}{52!5!43!}$$

$$= \frac{47.46.45.44}{52.51.50.49}$$

$$= 0.659$$

$$\therefore \quad P(X = 0) = 0.659 \quad \ldots(2)$$

Substituting equation (2) in equation (1),

$$P(X \geq 1) = 1 - 0.659$$

$$= 0.341$$

$$\therefore \quad P(X \geq 1) = 0.341$$

Q22. What is the probability that a five-card poker hand contains card of five different kinds and does not contain in a flush or a straight?

Model Paper-2, Q8(a)

Answer :

A standard deck consists of 52 cards. A five-card poker hand selects 5 cards among 52 cards.

The number of hands in total is obtained as,

$${}^{52}C_5 = \frac{52!}{5!(52-5)!}$$

$$= \frac{52!}{5!47!}$$

$$= 2{,}598{,}960$$

5 cards are to be selected from 13 kinds (since 13 cards are of each suit (heart, diamond, spades, clubs)).

$$\Rightarrow \quad {}^{13}C_5 = \frac{13!}{5!(13-5)!}$$

$$= \frac{13!}{5!8!}$$

$$= 1287$$

1 among 4 cards is selected from each of 5 kinds.

For first kind,

$${}^4C_1 = \frac{4!}{1!(4-1)!}$$

$$= \frac{4!}{1!3!}$$

$$= \frac{24}{6}$$

$$= 4$$

Similarly, remaining cards are also selected.

$\Rightarrow \quad 4 \times 4 \times 4 \times 4 \times 4 = 4^5$

Number of hands containing 5 different hands is obtained as,

$= {}^{13}C_5\, 4^5$

$= \frac{13 \times 12 \times 11 \times 10 \times 9}{5!}\, 4^5$

$= 1317888$ hands of this type.

1 card among 4 suits is selected

$\Rightarrow {}^4C_1 = 4$

5 cards are selected from 13 cards i.e.,

${}^{13}C_5 = 1287$

Number of hands containing a flush is obtained as,

$4 \times 1287 = 5148$

Since the straight can start with any card from the set {A, 2, 3, 4, 5, 6, 7, 8, 9, 10}

So there are ${}^{10}C_1 = 10$ ways

And there are 4 cards of each suit to make each of these 5 choices = 45

Number of hands containing a straight is obtained as,

$10.4^5 = 10{,}240$

Further, there are 40 hands that are both straights and flushes.

$\therefore$ The total number of hands that are flushes or straights is obtained as,

|Straight or flush| = |Straight| + | flush| – |Straight and flush|

$\Rightarrow \quad 51 + 8 + 10240 - 40 = 15348$

The number of hands containing five different kinds and no flush nor straight is obtained as,

|Five different kinds and no flush nor straight| = |Five different kinds| – |Straight or flush|

$= 1317888 - 15348$

$= 1302540$

The probability that a five-card poker hand contains cards of five different kinds and does not contain a flush or a straight is obtained as,

$$P(\text{Five different kinds and no flush/straight}) = \frac{|\text{Straight or flush}|}{\text{Number of hands in total}}$$

$$= \frac{1302540}{{}^{52}C_5} = \frac{1302540}{2598960} = 0.50$$

$\therefore \quad P(\text{Five different kinds and no flush/straight}) = 0.50$

Q23. What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?

Answer :

Given that,

Total number of integers = 40

Number of integers chosen correctly = 5

Number of integers chosen = 6

Total number of ways to choose 6 cards out of 40 is ${}^{40}C_6$.

Out of 6 cards, only 5 cards are chosen correctly i.e., ${}^6C_5$

$\therefore$ The probability is $\frac{{}^6C_5}{{}^{40}C_6}$

But, only one number has to be chosen out of 34 (= 40 – 6)

$\therefore$ Probability = ${}^6C_5$

$= 5.3147 \times 10^{-5}$

$\therefore$ Probability = $\frac{{}^6C_5 . {}^{34}C_1}{{}^{40}C_6}$

$= 5.3147 \times 10^{-5}$

$\therefore$ Probability = 5.3147×10^{-5}

Q24. What is the probability that Bo, Colleen, Jeff, and Rohini win the first, second, third and fourth prizes, respectively, in a drawing if 50 people enter a contest and

(a) no one can win more than one prize

(b) winning more than one prize is allowed.

Answer : **Model Paper-3, Q8(a)**

Given that,

For a contest

Number of people = 50

(a) 4 winners are selected from 50 people

Repetition of winners is not allowed.

The number of ways to select the winners is obtained as,

$${}^{50}P_4 = \frac{50!}{(50-4)!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$= \frac{50!}{46!}$$

$$= 47.48.49.50$$

$$= 5,527,200$$

In only one drawing among 5,527,200 Bo, Colleen, Jeff and Rohini win the first, second, third and fourth prize respectively.

$\therefore$ The probability that no one can win more than one prize in a contest is obtained as,

$$P_{(\text{Bo, colleen, Jeff and Rohini})} = \frac{1}{5527200}$$

$$= 1.809 \times 10^{-7}$$

(b) 4 winners are selected from 50 people.

Repetition of winners is allowed.

The number of ways to select the winners is obtained as,

$50^4 = 6,250,000$

In only one drawing among 6250000, Bo, Colleen, Jeff and Rohini win the first, second, third and fourth prize respectively.

The probability that winning more than one prize is allowed in a contest is obtained as,

$$P_{(\text{Bo, Colleen, Jeff and Rohini})} = \frac{1}{6250000}$$

$$= 1.6 \times 10^{-7}$$

Q25. Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

Answer :

Rolling a total of 8 when 2 dice are rolled

There are $6^2 = 36$ equally likely possible outcomes.

There are 5 ways to get a total of 8 when 2 dice are rolled i.e., (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)

$\therefore$ Probability $= \frac{5}{36} = 0.1388$

Rolling a total of 9 when 3 dice are rolled

There are $6^3 = 216$ equally likely possible outcomes.

There are 21 ways to get a total of 8 when 3 dice are rolled i.e.,

(6, 1, 1) (5, 2, 1) (5, 1, 2) (4, 1, 3) (4, 2, 2) (4, 3, 1) (3, 1, 4) (3, 2, 3) (3, 3, 2) (3, 4, 1) (2, 1, 5) (2, 2, 4) (2, 3, 3), (2, 4, 2) (2, 5, 1) (1, 1, 6) (1, 2, 5) (1, 3, 4) (1, 4, 3) (1, 5, 2) (1, 6, 1).

$\therefore$ Probability $= \frac{21}{216} = 0.0972$

$\Rightarrow$ $0.1388 > 0.0972$

Thus, rolling a total of 8 is more likely when 2 dice are rolled when compared to 3 dice.

Q26. (a) Find the probability of rolling at least one six when a fair die is rolled four times.

(b) Find the probability that a double six comes up at least once when a pair of dice is rolled 24 times asking

(c) Is it more likely that a six comes up at least once when a fair die is rolled four times or that a double six comes up at least once when a pair of dice is rolled 24 times?

Answer :

(a) Given that,

A fair die is rolled 4 times.

Let P_1 be the probability of rolling atleast one 6.

Number of possible outcomes when a die is rolled 4 times $= 6^4$

There are 5^4 outcomes in which one 6 does not appear.

$\Rightarrow$ The probability of not rolling one 6 $\overline{P_1} = \frac{5^4}{6^4}$.

The probability of rolling atleast one 6 is given as,

$$P_1 = [1 - \overline{P_1}])$$

$$= 1 - \frac{5^4}{6^4} = \frac{671}{1296}$$

$\therefore$ $P_1 = 0.518$

(b) Given that,

A pair of dice is rolled 24 times.

Let P_2 be the probability that a double 6 appears.

Number of possible outcomes when 2 dice are rolled 24 times $= 36^{24}$.

There are 35^{24} outcomes in which a double 6 does not appear.

$\Rightarrow$ The probability of not rolling double 6, $\overline{P_2} = \frac{35^{24}}{36^{24}}$

The probability of rolling atleast one double 6 is given as,

$$P_2 = 1 - \overline{P_2} = 1 - \frac{35^{24}}{36^{24}}$$

$$= 0.491$$

(c) From parts (a) and (b)

$P_1 = 0.516$

$P_2 = 0.491$

Here, $0.516 > 0.491$

$\Rightarrow \quad P_1 > P_2$

$\therefore$ Rolling atleast one 6 when a fair die is rolled 4 times is more likely when compared to pair of dice rolled 24 times.

Q27. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.

Model Paper-4, Q8(a)

Answer :

Given that,

A loaded dice is rolled.

The possible outcomes are 1, 2, 3, 4, 5, 6.

Let x denotes the probability of rolling 1.

$\Rightarrow \quad P(1) = x$

1 will appear as 2, 4, 5 and 6

$\Rightarrow \quad P(1) = P(2) = P(4) = P(5) = P(6) = x$

3 will appear twice as twice.

$\Rightarrow \quad P(3) = 2P(1)$

$= 2x$

The total probability is equal to 1

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

$\Rightarrow \quad x + x + 2x + x + x + x = 1$

$\Rightarrow \quad 7x = 1$

$\Rightarrow \quad x = \frac{1}{7}$

Hence, the probability of each outcome is,

$P(1) = \frac{1}{7}$

$P(2) = \frac{1}{7}$

$P(3) = \frac{2}{7}$

$P(4) = \frac{1}{7}$

$P(5) = \frac{1}{7}$

$P(6) = \frac{1}{7}$

Q28. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

(a) The first 13 letters of the permutation are in alphabetical order.

(b) a is the first letter of the permutation and z is the last letter.

(c) a and z are next to each other in the permutation.

(d) a and b are not next to each other in the permutation.

(e) a and z are separated by at least 23 letters in the permutation.

(f) z precedes both a and b in the permutation.

Answer :

Let S represents the set of all the permutations of the 26 lower case letters of English alphabet.

Number of permutations of the lower case letters is,

$$|S| = 26!$$

(a) Let E represents the event that the first 13 letters of the permutation are in alphabetical order.

The remaining 13 letters can be selected in 13! ways.

$\Rightarrow \quad |E| = 13!$

The probability that the first 13 letters of the permutation are in alphabetical order is given as,

$$P(E) = \frac{|E|}{|S|}$$

$$= \frac{13!}{26!}$$

$$= 1.544 \times 10^{-17}$$

$\therefore \quad P(E) = 1.544 \times 10^{-17}$

(b) Let E be the event that a is the first letter of the permutation and z is the last letter.

If the first letter is a and last letter is z, then 24 letters are to be selected in 24! ways.

$\Rightarrow \quad |E| = 24!$

The probability that a is the first letter of the permutation and z is the last letter is given as,

$$P(E) = \frac{|E|}{|S|}$$

$$= \frac{24!}{26!}$$

$$= 1.538 \times 10^{-3}$$

$\therefore \quad P(E) = 1.538 \times 10^{-3}$

(c) Let E be the event that a and z are next to each other in the permutation.

$\Rightarrow$ az or za can be included in permutation.

Number of possible outcomes = 2!

Number of successful outcomes is,

$$|E| = 25!\,2!$$

The probability that a and z are next to each other in the permutation is given as,

$$P(E) = \frac{|E|}{|S|}$$

$$= \frac{25!2!}{26!}$$

$$= \frac{2.25!}{26.25!}$$

$$= \frac{1}{13}$$

$\therefore \quad P(E) = \frac{1}{13}$

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(d) Let E be the event that a and b are not next to each other in the permutation.

From Part (c), the probability that a and b are next to each other in permutation is,

$$P(E) = \frac{1}{13}$$

The probability that a and b are not next to each other in the permutation is given as,

$$P(\overline{E}) = 1 - P(E)$$

$$= 1 - \frac{1}{13}$$

$$= \frac{2}{13}$$

$$\therefore \quad P(\overline{E}) = \frac{2}{13}$$

(e) Let E be the event that a and z are separated by at least 23 letters in the permutation.

If a and z are separated by at least 23 letters, then they can be placed as

"$a - 23 - z - 1$", "$a - 24 - z$", "$1 - a - 23 - z$", "$z - 23 - a - 1$", "$z - 24 - a$", "$1 - z - 23 - a$"

$\Rightarrow$ There are 6 ways

$$\therefore \quad |E| = 6 \times 23!$$

The probability that a and z are separated by at least 23 letters in the permutation is given as,

$$P(E) = \frac{|E|}{|S|}$$

$$= \frac{6 \times 23!}{26!}$$

$$= \frac{1}{2600}$$

(f) Let E be the event that z precedes both a and b in the permutation.

z precedes both a and b in $\frac{1}{3}$ of the permutations.

$\Rightarrow$ $\frac{26!}{3}$ of 26! permutations will have z preceding both a and b.

$$\Rightarrow \quad |E| = \frac{26!}{3}$$

The probability that z precedes both a and b in the permutation is given as,

$$P(E) = \frac{|E|}{|S|}$$

$$= \frac{\frac{26!}{3}}{26!}$$

$$= \frac{1}{3}$$

$$\therefore \quad P(E) = \frac{1}{3}$$

Q29. Show that if E and F are events, then $p(E \cap F) \geq p(E) + p(F) - 1$.

Answer :

Given that,

E and F are two events

The probability of the union of 2 events is given as,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Also, probability range lies between 0 and 1.

i.e., $0 \leq P(E \cup F) \leq 1$.

Substituting $P(E \cup F) \leq 1$ in equation (1),

$$P(E) + P(F) - P(E \cap F) \leq 1$$

$\Rightarrow \quad 1 \geq P(E) + P(F) - P(E \cap F)$

$\Rightarrow \quad P(E \cap F) \geq P(E) + P(F) - 1$

$\therefore \quad P(E \cap F) \geq P(E) + P(F) - 1$

The above inequality is known as Bonferroni's inequality.

4.1.2 Bayes Theorem, Expected Value and Variance

Q30. State and prove Bayes Theorem.

Model Paper-I, Q

Answer :

Bayes Theorem

If A and B are events from a sample space S such that

$P(A) \neq 0$ and $P(B) \neq 0$. Then

$$P(B/A) = \frac{P(A/B)\,P(B)}{(A/B)\,P(B) + P(A/\overline{B})P\overline{B}}$$

Proof

From the definition of conditional probability,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A/B)\,P(B)$$

and $\quad P(B/A) = \dfrac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B/A)\,P(A)$

Since, $P(A \cap B) = P(B \cap A)$.

Equating equations (1) and (2),

$$P(A/B)\,P(B) = P(B/A)\,P(A)$$

$\Rightarrow \quad P(B/A) = \dfrac{P(A/B)P(B)}{P(A)}$

Further, $A \cap B$ and $A \cap \overline{B}$ are disjoint

Then,

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$
$$= P(A/B)\,P(B) + P(A/\overline{B})\,P(\overline{B})$$

Substituting equation (2) in equation (1),

$\therefore \quad P(B/A) = \dfrac{P(A/B)\,P(B)}{P(A/B)\,P(B) + P(A/B)\,P(\overline{B})}$

Similarly,

$$P(A/B) = \frac{P(B/A)\,P(A)}{P(B/A)\,P(A) + P(B/\bar{A})\,P(\bar{A})}$$

Generalized Bayes theorem

If A is an event from a sample space S and $B_1, B_2, \ldots\ldots B_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$. Then,

$$P(B_j/A) = \frac{P(A/B_j)P(B_j)}{\sum_{i=1}^{n} P(A/B_j)P(B_i)}; \quad P(A) \neq 0, \quad P(B_i) \neq 0$$

Use

Based on Bayes theorem, Bayesian spam filters are used to eliminate spam developed in electronic mailboxes or messages.

Q31. Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.9% of people with the disease test positive and only 0.02% who do not have the disease test positive.

(a) What is the probability that someone who tests positive has the genetic disease?

(b) What is the probability that someone who tests negative does not have the disease?

Answer :

Given that,

Total number of people = 10,000

Percentage of people with the disease test positive = 99.9

Percentage of people who do not have the disease test positive = 0.02

(a) Let D be the event that a person has a rare genetic disease.

$\Rightarrow \quad P(D) = \frac{1}{10,000} = 0.0001.$

$\Rightarrow \quad P(\bar{D}) = 0,9999$

Let P be the person with the disease test positive.

$\Rightarrow P(P/D) = 0.999$

and $\bar{P}$ be the person who do not have disease test positive.

$\Rightarrow P(P/\bar{D}) = 0.0002$

From the above values,

$P(\bar{P}/D) = 0.001$

$P(\bar{P}/\bar{D}) = 0.9998$

The probability that someone who tests positive has the genetic disease test positive is obtained by using Bayes theorem,

$$P(D/P) = \frac{P(P/D)P(D)}{P(P/D).P(D) + P(P/\bar{D})P(\bar{D})}$$

$$= \frac{(0.999)(0.0001)}{(0.999)(0.0001) + (0.0002)(0.9999)}$$

$\therefore \quad P(D/P) = 0.3331$

(b) The probability that someone who tests negative does not have the disease is given as,

$$P(\bar{D}/\bar{P}) = \frac{P(\bar{P}/\bar{D})P(\bar{D})}{P(\bar{P}/\bar{D})P(\bar{D}) + P(\bar{P}/D)P(D)}$$

$$= \frac{(0.9998)(0.9999)}{(0.9998)(0.9999) + (0.001)(0.0001)}$$

$\therefore \quad P(\bar{D}/\bar{P}) = 0.9999$

Q32. An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predicts either the success or the failure of the product. Of new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes, while 40% of failed products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.

Answer :

Given that,

For an electronic company.

Percentage of products that are successes = 60

Percentage of successful products that were predicted to be successes = 70

Percentage of failed products that were predicted to be successes = 40

Let S be the event that a product is a success.

$\Rightarrow \quad P(S) = 0.6$

$\Rightarrow \quad P(\overline{S}) = 1 - 0.6 = 0.4$

Let P be the event that a product is predicted to be successful.

$\Rightarrow \quad P(P/S) = 0.7$

Let $\overline{P}$ be the event that a failed product is predicted to be success.

$\Rightarrow \quad P(P/\overline{S}) = 0.4$

The probability that the new camera phone will be successful if its success has been predicted is obtained by using Bayes Theorem.

$$P(S/P) = \frac{P(P/S)P(S)}{P(P/S)P(S)+P(P/\overline{S})P(\overline{S})} = \frac{(0.7)(0.6)}{(0.7)(0.6)+(0.4)(0.4)} = 0.724$$

$\therefore \quad P(S/P) = 0.724$

Q33. Suppose that E, F_1, F_2 and F_3 are events from a sample space S and that F_1, F_2, and F_3 are pairwise disjoint and their union is S. Find $p(F_2 \mid E)$ if $p(E \mid F_1) = 2/7$, $p(E \mid F_2) = 3/8$, $p(L \mid F_3) = 1/2$, $p(F_1) = 1/6$, $p(F_2) = 1/2$ and $p(F_3) = 1/3$.

Answer : Model Paper-4, Q8(a)

Given that,

E, F_1, F_2 and F_3 are events from a sample space. S.

F_1, F_2 and F_3 are pairwise disjoint and their union is S.

$$p(E/F_1) = \frac{2}{7},\ p(E/F_2) = \frac{3}{8},\ p(E/F_3) = \frac{1}{2}$$

$$p(F_1) = \frac{1}{6},\ p(F_2) = \frac{1}{2},\ p(F_3) = \frac{1}{3}$$

From Bayes theorem,

$$p(F_2/E) = \frac{p(E/F_2)p(F_2)}{p(E/F_2)p(F_2)+p(E/F_1)p(F_1)+p(E/F_3)p(F_3)}$$

Substituting the corresponding values in above equation,

$$p(F_2/E) = \frac{\frac{3}{8}\cdot\frac{1}{2}}{\frac{3}{8}\cdot\frac{1}{2}+\frac{2}{7}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{3}} = \frac{7}{15}$$

$$= 0.4666$$

$\therefore \quad p(F_2/E) = 0.466$

Q34. Suppose that E, F_1, F_2 and F_3 are events from a sample space S and that F_1, F_2 and F_3 are pairwise disjoint and their union is S. Find $p(F_1 \mid E)$ if $p(E \mid F_1) = 1/8$, $p(E \mid F_2) = 1/4$, $p(E \mid F_3) = 1/6$, $p(F_1) = 1/4$, $p(F_2) = 1/4$ and $p(F_3) = 1/2$.

Answer :

Given that,

E, F_1, F_2 and F_3 are events from a sample space S.

F_1, F_2 and F_3 are pairwise disjoint and their union is S.

$$p(E/F_1) = \frac{1}{8}, \; p(F_1) = \frac{1}{4}$$

$$p(E/F_2) = \frac{1}{4}, \; p(F_2) = \frac{1}{4}$$

$$p(E/F_3) = \frac{1}{6}, \; p(F_3) = \frac{1}{2}$$

From Bayes theorem,

$$p(F_1/E) = \frac{p(E/F_1)p(F_1)}{p(E/F_1).p(F_1) + p(E/F_2).p(F_2) + p(E/F_3).p(F_3)}$$

Substituting the corresponding values in above equation,

$$p(F_1/E) = \frac{\frac{1}{8}\cdot\frac{1}{4}}{\frac{1}{8}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{2}}$$

$$= \frac{3}{17}$$

$$= 0.1764$$

Q35. Prove that,

$$P(F_j/E) = \frac{P(E/F_j)P(F_j)}{\sum_{i=1}^{n} P(E/F_i)P(F_i)}$$

Answer :

Let E be an event from a sample space S.

Let $F_1, F_2, \ldots\ldots F_n$ are mutually exclusive events such that,

$$\bigcup_{i=1}^{n} F_i = S \quad \ldots(1)$$

Let $P(E) \neq 0$ and $P(F_i) \neq 0$ for $i = 1, 2, 3, \ldots\ldots n$.

Conditional probability is given by,

$$P(F_j/E) = P(E \cap F_j)/P(E) \quad \ldots(2)$$

and $$P(E/F_j) = P(E \cap F_j)/P(F_j) \quad \ldots(3)$$

$\therefore$ $$P(E \cap F_j) = P(F_j/E)P(E) \quad \ldots(4)$$

and $$P(E \cap F_j) = P(E/F_j).P(F_j)$$

Equating equations (3) and (4),

$$P(F_j/E).P(E) = P(E/F_j).P(F_j)$$

Dividing on both sides with $P(E)$,

$$P(F_j/E) = \frac{P(E/F_j).P(F_j)}{P(E)}$$

Consider,

$$E = E \cap S$$

$$= E \cap \left(\bigcup_{i=1}^{n} F_i\right) \quad [\because \text{From equation (1)}]$$

$$= \bigcup_{i=1}^{n} (E \cap F_i)$$

$$P(E) = \bigcup_{i=1}^{n} [P(E \cap F_i)]$$

$$= P(E \cap F_1) + P(E \cap F_2) + \ldots\ldots + P(E \cap F_n)$$

$$= P(E/F_1)\, P(F_1) + P(E/F_2).\, P(F_2) + \ldots\ldots + P(E/F_n).P(F_n) \; [\because E \cap F_1, E \cap F_2 \ldots\ldots E \cap F_n \text{ are mutually disjoint}]$$

$$\therefore P(E) = \sum_{i=1}^{n} P(E/F_i) P(F_i) \quad \ldots(5)$$

Substituting equations (4) and (5) in equation (2),

$$P(F_j/E) = \frac{P(E/F_j).P(F_j)}{\sum_{i=1}^{n} P(E/F_i).P(F_i)}$$

$$\therefore \quad P(F_j/E) = \frac{P(E/F_j)P(F_j)}{\sum_{i=1}^{n} P(E/F_i).P(F_i)}$$

Q36. Suppose that a Bayesian spam filter is trained on a set of 500 spam messages and 200 messages that are not spam. The word "exciting" appears in 40 spam messages and in 25 messages that are not spam. Would an incoming message be rejected as spam if it contains the word "exciting" and the threshold for rejecting spam is 0.9?

Answer :

Given that,

For a Bayesian spam filter,

Number of messages that are spam = 500

Number of messages that are not spam = 200.

Number of messages that are spam in which the word 'Exciting' appears = 40

Number of messages that are not spam in which the word 'Exciting' appears = 25.

Threshold for rejecting spam = 0.9

Let, $p(\text{exciting}) = \dfrac{40}{500} = 0.08$

and $q(\text{exciting}) = \dfrac{25}{200} = 0.125$

The probability that the message is spam is given as,

$$r(\text{exciting}) = \frac{p(\text{exciting})}{p(\text{exciting}) + q(\text{exciting})}$$

$$= \frac{0.08}{0.08 + 0.125}$$

$$= 0.3902 < 0.9$$

Because r(exciting) is less than the threshold 0.9, an incoming message containing 'exciting' would not be rejected.

Q37. Suppose that a Bayesian spam filter is trained on a set of 10,000 spam messages and 5000 messages that are not spam. The word "enhancement" appears in 15000 spam message and 20 messages that are not spam, while the word 'herbal' appears in 800 spam messages and 200 messages that are not spam. Estimate the probability that a received message containing both the words "enhancement" and "herbal" is spam. Will the message be rejected as spam if the threshold for rejecting spam is 0.9?

Answer :

Let. $p(\text{enhancement}) = \frac{1500}{10000} = 0.15$

$$q(\text{enhancement}) = \frac{20}{5000} = 0.004$$

$$p(\text{herbal}) = \frac{800}{10000} = 0.08$$

$$q(\text{herbal}) = \frac{200}{5000} = 0.04$$

The probability that the message is spam is given as,

$$r(\text{enhancement, herbal}) = \frac{p(\text{enhancement})\,p(\text{herbal})}{p(\text{enhancement})\,p(\text{herbal}) + q(\text{enhancement})\,q(\text{herbal})}$$

$$= \frac{(0.15)(0.08)}{(0.15)(0.08) + (0.004)(0.04)}$$

$$= 0.987 > 0.9$$

As r (enhancement, herbal) is greater than the threshold 0.9, an incoming message containing 'enhancement' and 'herbal' will be rejected.

Q38. Derive the expected number of success based on Bernoulli trials.

Answer :

Consider X as a random variable and n as mutually independent Bernoulli trials.

Then from Bernoulli trials, the probability of exactly k successes is given by,

$$P(X = k) = C(n, k)\, p^k q^{n-k} \quad \text{...(1)}$$

Then,

$$E(X) = \sum_{k=1}^{n} kP(X = k)$$

$$= \sum_{k=1}^{n} k(C(n, k)\, p^k q^{n-k}) \quad [\because \text{From equation (2)}]$$

$$= \sum_{k=1}^{n} n\, C(n-1, k-1)\, p^k q^{n-k}$$

$$= np \sum_{k=1}^{n} C(n-1, k-1)\, p^{k-1} q^{n-k}$$

$$= np \sum_{j=0}^{n-1} C(n-1, j)\, p^j q^{n-1-j}$$

$$= np(p+q)^{n-1} \quad [\because \text{From Binomial theorem}]$$

$$\therefore \quad E(X) = np \quad [\because p + q = 1].$$

Q39. If X_i, i = 1, 2, ... n with n a positive number are random variables on S, and if a and b are real numbers, then prove that.

(i) $E(x_1 + x_2 + \ldots\ldots + x_n) = E(x_1) + E(x_2) + \ldots. + E(x_n)$

(ii) $E(aX + b) = a\,E(X) + b$

Answer :

Given that,

X_i is a random variable on S with $i = 1, 2, ... n$

a and b are real numbers.

(i) Let $n = 2$

Considering L. H. S,

$$\Rightarrow \quad E(X_1 + X_2) = \sum_{s \in S} P(s)(X_1(s) + X_2(s))$$

$$= \sum_{s \in S} P(s)X_1(s) + \sum_{s \in S} P(s)X_2(s)$$

$$\therefore \quad E(X_1 + X_2) = E(X_1) + E(X_2)$$

Similarly,

$$E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ...+ E(X_n)$$

Hence proved.

(ii) Considering L. H. S

$$\Rightarrow \quad E(aX + b) = \sum_{s \in S} P(s)\,(aX(s) + b)$$

$$= \sum_{s \in S} P(s)\,(aX(s)) + \sum_{s \in S} P(s)b$$

$$= a\sum_{s \in S} P(s)\,X(s) + b\sum_{s \in S} P(s)$$

$$\therefore \quad E(aX + b) = aE(X) + b \qquad \left[\because \sum_{s \in S} P(s) = 1\right]$$

Q40. Mention some of the properties of expected value.

Answer :

The following are some of the properties of expected value of a random variable.

1. Let X be a random variable and a_j be set of possible inputs. Then, the average case complexity of the algorithm is,

$$E(X) = \sum_{j=0}^{n} P(a_j)X(a_j)$$

2. A random variable has a geometrical distribution with parameter p if,

$$P(X = k) = (1 - p)^{k-1}p\,;\, k = 1, 2, 3, ... 0 \leq p \leq 1$$

Then, $E(X) = \dfrac{1}{p}$

3. If X and Y are independent random variables on sample space S, then,

$$P(X = r_1 \text{ and } Y = r_2) = P(X = r_1).\,P(Y = r_2)$$

or

$$E(XY) = E(X)E(Y).$$

Q41. If X and Y are independent random variable on a sample space S, then show that E(XY) = E(X)E(

Answer :

Given that,

X and Y are independent random variables on S.

Let r be a value such that,

$X = r_1$ and $X = r_2$

with $r = r_1 r_2$

Then,

$$E(XY) = \sum_{r \in XY(S)} r\, P(XY = r)$$

$$= \sum_{r_1 \in X(S),\, r_2 \in Y(S)} r_1 r_2 P(X = r_1 \text{ and } Y = r_2)$$

$$= \sum_{r_1 \in X(S)} \sum_{r_2 \in Y(S)} r_1 r_2 P(X = r_1) . P(Y = r_2)$$

$$= \sum_{r_1 \in X(S)} r_1 P(X = r_1) . \sum_{r_2 \in Y(S)} r_2 P(Y = r_2)$$

$\therefore \quad E(XY) = E(X)E(Y)$

Q42. Define variance of a random variable.

Answer :

If X is a random variable on a sample space S, then its variance is given by,

$$V(X) = \sum_{s \in S} (X(S) - E(X))^2 P(s)$$

or

$$V(X) = E(X^2) - E(X)^2$$

Standard deviation of X i.e., $\sigma(X) = \sqrt{V(X)}$

→ According to Bienayme's formula, for two independent random variables X and Y,

$$V(X + Y) = V(X) + V(Y)$$

For X_i, $i = 1, 2, ...n$

$$V(X_1 + X_2 + ... + X_n) = V(X_1) + V(X_2) + ... + V(X_n)$$

→ According to Chebyshev's inequality, for a random variable X on S and r as a positive real number,

$$P(|X(S) - E(X)| \geq r) \leq \frac{V(X)}{r^2}$$

Q43. Prove that $V(X) = E((X - \mu)^2)$

Answer :

Let X be a random variable on a sample space 'S'.

Then its expectation is given by,

$$E(X) = \mu$$

Considering $E((X - \mu)^2)$

Then,

$$E((X - \mu)^2) = E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - E(2\mu X) + E(\mu^2)$$

$$= E(X^2) - 2\mu\, E(X) + \mu^2$$

$$= E(X^2) - 2\mu.\mu + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

$\therefore \quad E(X - \mu)^2) = V(X)$

Q44. What is the expected sum of the numbers that appear when three fair dice are rolled?

Answer :

Given that,

Three fair dice are rolled

$\Rightarrow n = 3$

Let, X_1 = Number appearing on the first die

X_2 = Number appearing on the second die

X_3 = Number appearing on the third die.

The expected value of X_1 is obtained as,

$$E(X_1) = \frac{1+2+3+4+5+6}{6}$$

$$= \frac{21}{6}$$

$$= \frac{7}{2}$$

All the three dice have same expectation as they are fair.

$$\Rightarrow \quad E(X_1) = E(X_2) = E(X_3) = \frac{7}{2}$$

The expected value of the sum is given as,

$$E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3)$$

$$= \frac{7}{2} + \frac{7}{2} + \frac{7}{2}$$

$$= \frac{21}{2}$$

$\therefore$ The expected sum of numbers that appear when 3 fair dice are rolled is 10.5.

Q45. Let X be the number appearing on the first die when two fair dice are rolled and let Y be the sum of the numbers appearing on the two dice. Show that E(X) E(Y) ≠ E(XY).

Answer :

Given that,

X = Number appearing on first die when two dice are rolled.

Y = Sum of numbers appearing on the two dice.

The expected value of X is given as,

$$E(X) = \frac{1+2+3+4+5+6}{6}$$

$$= \frac{21}{6}$$

$$= \frac{7}{2}$$

Let Z be the number appearing on the second dice.

$$\Rightarrow \quad E(Z) = \frac{7}{2}$$

Then,

$$E(Y) = E(X) + E(Z)$$

$$= \frac{7}{2} + \frac{7}{2}$$

$$\therefore \quad E(Y) = 7$$

Consider,

$$E(X).\ E(Y) = \frac{7}{2}(7) = \frac{49}{2}$$

$$E(X)E(Y) = \frac{49}{2}$$

For example let (3, 4) be the outcome.

$\Rightarrow$ $X = 3$ and $Y = 3 + 4 = 7$

So $XY = 21$

$\Rightarrow$ $XY = X(X + Y)$

The table below represents the XY values for all outcomes when two dice are rolled.

X\Y	1	2	3	4	5	6
1	2	3	4	5	6	7
2	6	8	10	12	14	16
3	12	15	18	21	24	27
4	20	24	28	32	36	40
5	30	35	40	45	50	55
6	42	48	54	60	66	72

$$E(XY) = \frac{2+3+4+\ldots\ldots+72}{36}$$

$$\therefore \quad E(XY) = \frac{987}{36} = \frac{329}{12}$$

$$\Rightarrow \quad E(X).\ E(Y) \neq E(XY)$$

$$\Rightarrow \quad \frac{49}{2} \neq \frac{329}{12}$$

$$\therefore \quad E(X)\ E(Y) \neq E(XY)$$

Q46. What is the variance of the number of heads that come up when a fair coin is flipped 10 times?

Answer :

Given that,

A fair coin is flipped 10 times.

$\Rightarrow n = 10$

When a fair coin is flipped, the probability of getting number of heads is $\frac{1}{2}$

$$\Rightarrow \quad p = \frac{1}{2}$$

From Bernoulli trails,

$$\text{Mean} = E(X)$$

$$= np = 10.\frac{1}{2} = 5$$

$$\therefore \quad E(X) = 5$$

The variance *is* given as,

$$\text{Variance} = V(X)$$

$$= np(1-p) = 5\left(1-\frac{1}{2}\right)$$

$$= \frac{5}{2}$$

$$\Rightarrow \quad V(X) = 2.5$$

$\therefore$ The variance of number of heads that come up when a fair coin is flipped 10 times is 2. 5.

Q47. Show that if X and Y are independent random variables, then V(XY) = E(X)² V(Y) + E(Y)² V(X) + V(X) V(Y).

Answer :

Given that,

X and Y are independent random variables.

Consider, $V(XY) = E(X^2Y^2) - E(XY)^2$

$$= E(X^2)E(Y^2) - E(X)^2 E(Y)^2$$

$$= E(X^2)E(Y^2) - V(X)V(Y) - E(X)^2 E(Y)^2 + V(X)V(Y) \quad [\because \text{Adding and subtracting } V(X)V(Y)]$$

$$= E(X^2)E(Y^2) - [E(X^2) - E(X)^2][E(Y^2) - E(Y)^2] - E(X)^2 E(Y)^2 + V(X)V(Y) \quad [\because V(X) = E(X^2) - E(X)^2]$$

$$= E(X^2)E(Y^2) - E(X^2)E(Y^2) + E(X^2)E(Y)^2 + E(X)^2 E(Y^2) - E(X)^2E(Y)^2 - E(X)^2E(Y)^2 + V(X)V(Y)$$

$$= E(X^2)E(Y^2) + E(X)^2 E(Y^2) - 2E(X)^2E(Y)^2 + V(X)V(Y)$$

$$= E(X)^2E(Y^2) - E(X)^2 E(Y)^2 + E(X^2)E(Y)^2 - E(X)^2 E(Y)^2 + V(X)V(Y)$$

$$= E(X)^2[E(Y^2) - E(Y)^2] + E(Y)^2[E(X^2) - E(X)^2] + V(X)V(Y)$$

$$\Rightarrow \quad V(XY) = E(X)^2V(Y) + E(Y)^2 V(X) + V(X)V(Y) \quad [\because V(X) = E(X^2) - E(X)^2]$$

$$\therefore \quad V(XY) = E(X)^2V(Y) + E(Y)^2 V(X) + V(X)V(Y)$$

Q48. Let X be a random variable on a sample space S such that X(s) ≥ 0 for all s ∈ S. Show that p(X(s) ≥ a) ≤ E(X)/a for every positive real number a.

Model Paper-2, Q

Answer :

Given that,

X is a random variable on a sample space S such that $X(s) \geq 0 \ \forall s \in s$

Let the event $A \subseteq S$ defined by

$$A = \{s \in S \mid X(s) \geq a\}$$

The expected value of a random variable X is given as,

$$E(X) = \sum_{s \in S} P(s)X(s)$$

$$= \sum_{s \in A} P(s)X(s) + \sum_{s \notin A} P(s)X(s)$$

$$\geq \sum_{s \in A} P(s)X(s) \quad [\because X(s) \geq 0 \ \forall \ s \in S]$$

$$\geq \sum_{s \in A} P(s)a$$

$$\geq a\sum_{s \in A} P(s)$$

$$\geq a.P(A)$$

$$\therefore \quad E(X) \geq a.P(A) \quad [\because X(s) \geq a \ \forall \ s \in A]$$

Dividing on both sides by a,

$$\frac{E(X)}{a} \geq \frac{aP(A)}{a}$$

$$\Rightarrow \quad P(A) \leq \frac{E(X)}{a}$$

$$\therefore \quad P[X(s) \geq a] \leq \frac{E(X)}{a}$$

The above inequality is known as Markov's inequality.

Q49. Suppose that the number of tin cans recycled in a day at a recycling center is a random variable with an expected value of 50,000 and a variance of 10,000.

(a) Use Markov's inequality to find an upper bound on the probability that the center will recycle more than 55,000 cans on a particular day.

(b) Use Chebyshev's inequality to provide a lower bound on the probability that the center will recycle 40,000 to 60,000 cans on a certain day.

Answer :

(a) Given that,

For a recycling center,

Expected value, $E(X) = 50,000$

Let, $a = 55,000$

Markov's Inequality is given by,

$$P[X(s) \geq a] \leq \frac{E(X)}{a}$$

$$\Rightarrow \quad P[X \geq 55,000] \leq \frac{50,000}{55,000}$$

$$\Rightarrow \quad P[X \geq 55,000] \leq \frac{10}{11}$$

$\therefore$ Upper bound on the probability that the center will recycle more than 55,000 cans $= \frac{10}{11}$

(b) Given that,

Mean $= \mu = 50,000$

Variance $= \sigma^2 = 10,000$

$$\Rightarrow \sigma = \sqrt{10,000} = 100$$

Chebyshev's Inequality is given by,

$$P[|X - \mu| \geq k\sigma] \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow \quad P[-k\sigma \leq X - \mu \leq k\sigma] \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow \quad P[\mu - k\sigma \leq X \leq \mu + k\sigma] \geq 1 - \frac{1}{k^2}$$

Here,

$$k = \frac{X - \mu}{\sigma}$$

$$= \frac{60,000 - 50,000}{100}$$

$$= \frac{10,000}{100}$$

$$\Rightarrow \quad k = 100$$

$$\Rightarrow \quad P[50,000 - 100\,(100) \leq X \leq 50,000 + 100\,(100)] \geq 1 - \frac{1}{10,000}$$

$$\Rightarrow \quad P[50,000 - 10,000 \leq X \leq 50,000 + 10,000] \geq \frac{9999}{10,000}$$

$$\Rightarrow \quad P[40,000 \leq X \leq 60,000] \geq 0.9999$$

$\therefore$ Lower bound on the probability that the center will recycle 40,000 to 60,000 cans $= 0.9999$

Q50. Show that Cov(X, Y) = E(XY) – E(X) E(Y), and use this result to conclude that Cov(X, Y) = 0 if X and Y are independent random variables.

Answer :

Let X, Y be any two random variables

The covariance between X and Y is given as,

$$\text{Cov}(X, Y) = E[(X - E(X))\,(Y - E(Y))]$$
$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$
$$= E[XY] - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)$$

$\therefore \quad \text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

If X and Y are independent random variables then,

$$E(XY) = E(X)E(Y)$$

$\Rightarrow \quad \text{Cov}(X, Y) = E(XY) - E(XY) = 0$

$\therefore \quad \text{Cov}(X, Y) = 0$

4.2 ADVANCED COUNTING TECHNIQUES

4.2.1 Recurrence Relations, Solving Recurrence Relations

Q51. Explain about recurrence relation.

Answer :

Recurrence Relation

Recurrence is defined as a process in which information of any specific term can be found by considering the knowledge of previous terms.

Recurrence relation for any sequence $\{a_n\}$ is an equation that defines 'a_n' term based on their preceding sequence of terms like $a_0, a_1, a_2, \ldots\ldots a_{n-1}$. Here, $a_0, a_1, a_2, \ldots\ldots a_{n-1}$ represent integers set with $n \geq N$. Where, N denotes non-negative integer.

Moreover, if each term which is present in a sequence satisfies recurrence relation then this sequence is known as a solution of recurrence relation.

The concept of recurrence relation is also known as difference equation. It is basically occur in the following situations.

1. In error correction codes.
2. In multiple counting problems.
3. During the analysis of programming problems.
4. During the analysis of discrete-time systems.
5. During algorithm analysis.

Recurrence relation can be calculated by using the following formula,

$$a_n = a_{n-1} + a_{n-2} \text{ where, } n \geq 2, a_1 = 0, a_2 = 1.$$

So, $a_3 = 1 + 0 = 1$

In general, $a_n = \dfrac{n^2 - n}{2}$

Properties

The properties of recurrence relation are as follows,

1. A recurrence relation is said to be of degree k if $\{a_n\}$ is expressed as a function of a_{n-1} ---- a_{n-k} appears in the relation.
2. If n and k are non-negative integers, a recurrence relation is of the form,

   $C_0(n)a_n + C_1(n)\, a_{n-1} + \ldots + C_k(n)a_{n-k} = f(n);\ n > k$ ... (1)

   Where, $C_0(n) \ldots C_k(n)$ and $f(n)$ are functions of n and is called as a linear recurrence relation.
3. If $f(n)$ is identically zero in equation (1), then the recurrence relation is called homogeneous recurrence relation, else it is called as non-homogeneous.
4. If $C_0(n), C_1(n), \ldots, C_k(n)$ are constants in equation (1), then the relation is called as linear relation, else it is called as non-linear relation.

Q52. (a) Find a recurrence relation for the number of permutations of a set with n elements.

(b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

Answer :

Let $\{a_1, a_2, a_3, ... a_n\}$ represents the set with n elements.

(a) For $n = 1$, the element is $\{a_1\}$

$\therefore$ The number of possible permutations is $P_1 = 1$

For $n = 2$, the elements are $\{a_1, a_2\}$

The possible ways are a_1a_2, a_2a_1

$\therefore$ The number of possible permutations is $P_2 = 2.1$

$= 2P_1$

$\therefore P_2 = 2P_1$

For $n = 3$, the elements are $\{a_1, a_2, a_3\}$

The possible ways are $a_1a_2a_3$, $a_1a_3a_2$, $a_2a_3a_1$, $a_3a_1a_2$, $a_2a_1a_3$, $a_3a_2a_1$

$\therefore$ The number of possible permutations is,

$P_3 = 6$

$= 3.2$

$= 3.P_2$

Similarly,

For n elements, the number of possible permutations is,

$P_n = nP_{n-1}$

$\therefore$ The recurrence relation for the number of permutations of a set with n elements is $p_n = p_{n-1}$

(b) From part (a),

First iteration, $P_1 = 1$

$\Rightarrow P_1 = 1!$

Second Iteration, $P_2 = 2P_1$

$\Rightarrow P_2 = 2.1!$

$\Rightarrow P_2 = 2!$

Third iteration, $P_3 = 3P_2$

$\Rightarrow P_3 = 3.2!$

$= 3!$

Similarly n^{th} iteration, $P_n = nP_{n-1}$

$\Rightarrow P_n = n(n-1)!$

$= n!$

Hence, the number of permutations of a set with n elements is, $P_n = n!$

Q53. (a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, ... a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for j = 1, 2,....k –1

(b) What are the initial conditions?

(c) How many sequences of the type described in (a) are there when n is an integer with n ≥ 2?

Answer :

(a) Given sequence is,

$a_1, a_2 ... a_k$

$a_1 = 1$

$a_k = n$

$a_j < a_{j+1}$ for $j = 1, 2.... k - 1$

Let P_n be the number of strictly increasing sequence of positive integers with $a_1 = 1$ $a_k = n$

The following cases are considered

Case (i)

Any sequence contains $n-1$ as last integer. So, there are $n-1$ possible ways.

Hence, the number of possible sequences is P_{n-1}

Case (ii)

If $n-1$ is not present in sequence, then n is the last integer. So, there are $n-1$ possible ways

Hence, the number of possible sequences is P_{n-1}

The recurrence relation is obtained by adding the number of possible sequences in the above cases

i.e., $P_n = P_{n-1} + P_{n-1}$

$= 2P_{n-1}$

Hence, the recurrence relation is, $P_n = 2P_{n-1}$

(b) If $n = 1$, then there exists exactly 1 strictly increasing sequence of positive integers with $a_1 = 1$ and $a_k = 1$

$\Rightarrow \quad P_1 = 1$

If $n = 2$, then there exists exactly, 1 strictly increasing sequence of positive integers with $a_1 = 1$ and $a_k = 2$

$\Rightarrow \quad P_2 = 1$

$= 2^0$

If $n = 3$, then there are exactly 2 strictly increasing sequence of positive integers with $a_1 = 1$ and $a_k = 3$

$\Rightarrow \quad P_3 = 2P_{3-1}$

$= 2P_2$

$= 2.1$

$= 2^1 = 2^{3-2}$

Similarly,

$P_4 = 2P_{4-1}$

$= 2P_3$

$= 2.2$

$= 2^2 = 2^{4-2}$

In general, $P_n = 2P_{n-1}$

$= 2^{n-2}$

$\therefore \; P_n = 2^{n-2}$

(c) If $n \geq 2$, then the first integer in the sequence is $a_1 = 2$ as 1 is not present in the sequence. So, there are $n-2$ possible ways

Hence, the number of sequence is $n-2$

Q54. (a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

(b) What are the initial conditions?

(c) How many bit strings of length seven do not contain three consecutive 0s?

Answer : Model Paper-3, Q8(b)

Given that,

Length of bit string $= n$

Let a_n represents the number of bit strings which do not contain three consecutive 0s

(a) If the string starts with 1, then there exist $n-1$ bit strings that do not contain 3 consecutive 0s

Number of sequences $= a_{n-1}$

If the string starts with 01, then there are $n-2$ bit strings that do not contain 3 consecutive 0s.

Number of sequences $= a_{n-2}$

If the string starts with 001, then there are $n-3$ bit strings that do not contain three consecutive 0s

$\therefore$ Number of sequences $= a_{n-3}$

$\therefore$ The recurrence relation is obtained by adding all the above sequences

i.e., $a_n = a_{n-1} + a_{n-2} + a_{n-3} \; \forall n \geq 3$

(b) If the string doesn't contain any digit, then there exists only 1 different bit string that doesn't contain 3 consecutive zeros

$\Rightarrow \quad a_0 = 1$

If the string contains 1 bit, then there exist 2 different bit strings i.e., 0, 1 that do not contain 3 consecutive zeros

$\Rightarrow \quad a_1 = 2$

If the string contains 2 bits, then there exist 4 different bit strings i.e., 00, 01, 10, 11 do not contain 3 consecutive zeros.

$\Rightarrow \quad a_2 = 4$

(c) Given that,

Length of the bit string, $n = 7$.

The recurrence relation is,

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

For $n = 3$,

$$a_3 = a_{3-1} + a_{3-2} + a_{3-3}$$

$$\Rightarrow \quad a_3 = a_2 + a_1 + a_0$$

$$= 4 + 2 + 1 = 7$$

$$\therefore \quad a_3 = 7$$

For $n = 4$,

$$a_4 = a_{4-1} + a_{4-2} + a_{4-3}$$

$$\Rightarrow \quad a_4 = a_3 + a_2 + a_1$$

$$= 7 + 4 + 2 = 13$$

$$\therefore \quad a_4 = 13$$

For $n = 5$,

$$a_5 = a_{5-1} + a_{5-2} + a_{5-3}$$

$$\Rightarrow \quad a_5 = a_4 + a_3 + a_2$$

$$= 13 + 7 + 4 = 24$$

$$\therefore \quad a_5 = 24$$

For $n = 6$,

$$a_6 = a_{6-1} + a_{6-2} + a_{6-3}$$

$$\Rightarrow \quad a_6 = a_5 + a_4 + a_3$$

$$= 24 + 13 + 7 = 44$$

$$\therefore \quad a_6 = 44$$

For $n = 7$,

$$a_7 = a_{7-1} + a_{7-2} + a_{7-3}$$

$$\Rightarrow \quad a_7 = a_6 + a_5 + a_4$$

$$= 44 + 24 + 13 = 81$$

$$\therefore \quad a_7 = 81$$

Hence, there are 81 bit strings of length 7 that do not contain 3 consecutive 0s.

Q55. (a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.

(b) What are the initial conditions ?

(c) How many ternary strings of length six do not contain two consecutive 0s or two consecutive 1s?

Answer :

Let, a_n be the number of bit strings of length 'n' that contain a pair of consecutive 0's

The following cases are considered.

Case (i)

If the sequence ends in 1, then the bit string of length $n-1$ consists of a_{n-1} possible strings that contain a pair of consecutive 0's

$\therefore$ Number of sequences $= a_{n-1}$

Case (ii)

If sequence ends in 2, then the bit string of length $n-1$ consists of a_{n-1} possible strings

$\therefore$ Number of sequences $= a_{n-1}$

Case (iii)

If the sequence ends in 10, then the bit string of length $n-2$ consists of a_{n-2} possible strings

$\therefore$ Number of sequences $= a_{n-2}$

Case (iv)

If a sequence ends in 20, then the bit string of length $n-2$ consists of a_{n-2} possible strings

$\therefore$ Number of sequences $= a_{n-2}$

Case (v)

If a sequence ends in 00, then the bit string of length $n-2$ consists of 3^{n-2} possible strings

$\therefore$ Number of sequences $= 3^{n-2}$

The recurrence relation is obtained by adding all the above sequences.

i.e., $a_n = a_{n-1} + a_{n-1} + a_{n-2} + a_{n-2} + 3^{n-2}$

$= 2a_{n-1} + 2a_{n-2} + 3^{n-2}$

Hence, the recurrence relation is,

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

(b) For $n = 0$, there exist 0 bit strings since, it is an empty string

$\Rightarrow$ $a_0 = 0$

For $n = 1$, there exist 0 bit string since, it consists of only 1 element for a pair of consecutive 0's

$\Rightarrow$ $a_1 = 1$

For $n = 2$, there exist exactly 1 bit strings with a pair of consecutive 0's i.e., 00

Hence, the initial conditions are $a_0 = 0$, $a_1 = 0$

(c) $n = 6$

The recurrence relation is,

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

For $n = 2$,

$a_2 = 2a_1 + 2a_0 + 3^{2-2}$

$= 0 + 0 + 1 = 1$ $\left[\because \begin{array}{l} a_0 = 0 \\ a_1 = 0 \end{array}\right]$

$\therefore$ $a_2 = 1$

For $n = 3$,

$$a_3 = 2a_2 + 2a_1 + 3^{3-2}$$
$$= 2 + 0 + 3 = 5$$

$\therefore \quad a_3 = 5$

For $n = 4$,

$$a_4 = 2a_3 + 2a_2 + 3^{4-2}$$
$$= 10 + 2 + 9 = 21$$

$\therefore \quad a_4 = 21$

For $n = 5$,

$$a_5 = 2a_4 + 2a_3 + 3^{5-2}$$
$$= 42 + 10 + 27 = 79$$

$\therefore \quad a_5 = 79$

For $n = 6$,

$$a_6 = 2a_5 + 2a_4 + 3^{6-2}$$
$$= 158 + 42 + 81 = 281$$

$\therefore \quad a_6 = 281$

Thus, there are 281 ternary strings of length 6 that contain a pair of consecutive 0's

Q56. (a) Find the recurrence relation satisfied by S_n, where S_n is the number of regions into which three-dimensional space is divided by n planes if every three of the planes meet in one point, but no four of the planes go through the same point.

(b) Find S_n using iteration.

Answer :

(a) Given that,

S_n is the number of regions into which 3 dimensional space is divided by 'n' planes if every 3 of the planes meet in one point, but no 4 of the planes go through the same point

Let S_{n-1} be the number of regions, into which 3 dimensional space is divided by $n-1$ planes

If the n^{th} plane is drawn then it will pass through exactly 2 of $n-1$ planes

$\therefore$ There exist ${}^{n-1}C_2$ ways to choose the planes through which the nth plane passes

The recurrence relation is obtained as,

$$S_n = S_{n-1} + {}^{n-1}C_2$$
$$= S_{n-1} + \frac{(n-1)!}{2!(n-3)!}$$
$$= S_{n-1} + \frac{(n-1)(n-2)}{2}$$

$$\Rightarrow \quad S_n = S_{n-1} + \frac{n^2-n+2}{2}$$

$\therefore$ The recurrence relation is,

$$S_n = S_{n-1} + \frac{n^2-n+2}{2}$$

(b) For $n = 0$, $S_0 = 1$ since there is 1 region for 0 planes.

Here, $S_n = S_{n-1} + \dfrac{n^2-n+2}{2}$

From iteration method.

$$S_n = S_{n-2} + \frac{(n-1)^2-(n-1)+2}{2} + \frac{n^2-n+2}{2}$$

$$= S_1 + \frac{2^2-2+2}{2} + ... + \frac{(n-1)^2-(n-1)+2}{2} + \frac{n^2-n+2}{2}$$

$$= S_0 + \frac{1^2-1+1}{2} + \frac{2^2-2+2}{2} + ... + \frac{(n-1)^2-(n-1)+2}{2} + \frac{n^2-n+2}{2}$$

$$= 1 + \frac{1^2-1+1}{2} + \frac{2^2-2+2}{2} + ... + \frac{(n-1)^2-(n-1)+2}{2} + \frac{n^2-n+2}{2}$$

Above equation is of the form,

$$S_n = \frac{1}{2}\left[\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k + \sum_{k=1}^{n} 2\right]$$

$$S_n = 1 + \frac{1}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k + \frac{1}{2}\sum_{k=1}^{n} 2$$

$$= 1 + \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\cdot\frac{n(n+1)}{2} + \frac{1}{2}(2n)$$

$$\left[\because \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},\quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right]$$

$$= 1 + \frac{n(n+1)(2n+1)}{12} - \frac{n(n+1)}{4} + n$$

$$\Rightarrow \quad S_n = \frac{12+(n^2+n)(2n+1)-3(n^2+n)+12n}{12}$$

$$= \frac{12+2n^3+3n^2+n-3n^2-3n+12n}{12}$$

$$= \frac{2n^3+10n+12}{12}$$

$$\Rightarrow \quad S_n = \frac{n^3+5n+6}{6}$$

$$\therefore \quad S_n = \frac{n^3+5n+6}{6}$$

Q57. Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7,$.....together with the initial conditions $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$,. Use this recurrence relation to show that f_{5n} is divisible by 5, for n = 1, 2, 3, ...

Model Paper-1, Q9(a)

Answer :

Given recurrence relation is,

$$f_n = 5f_{n-4} + 3f_{n-5};\; n = 5, 6, 7, ...$$

Initial conditions are,

$$f_0 = 0$$
$$f_1 = 1$$
$$f_2 = 1$$
$$f_3 = 2$$
$$f_4 = 3$$

For $n = 5$,

$$f_5 = 5f_{5-4} + 3f_{5-5}$$
$$= 5f_1 + 3f_0$$
$$= 5(1) + 3(0)$$
$$= 5$$
$$\therefore \quad f_5 = 5$$

For $n = 6$,

$$f_6 = 5f_{6-4} + 3f_{6-5}$$
$$= 5f_2 + 3f_1$$
$$= 5(1) + 3(1)$$
$$= 8$$

$\therefore \quad f_5 = 8$

For $n = 7$,

$$f_7 = 5f_{7-4} + 3f_{7-5}$$
$$= 5f_3 + 3f_2$$
$$= 5(2) + 3(1)$$
$$= 10 + 3$$
$$= 13$$

$\therefore \quad f_7 = 13$

For $n = 8$,

$$f_8 = 5f_{8-4} + 3f_{8-5}$$
$$= 5f_4 + 3f_3$$
$$= 5(3) + 3(2)$$
$$= 15 + 6$$
$$= 21$$

$\therefore \quad f_8 = 21$

The sequence is, 0, 1, 1, 2, 3, 5, 8, 13, 21.....

Hence, the recurrence relation is satisfied by Fibonacci numbers.

Consider,

$$f_{5n}$$

If $n = 1 \Rightarrow f_{5n} = f_5 \qquad [\because f_n = 5f_{n-4} + 3f_{n-5}]$

$$= 5f_{5-4} + 3f_{5-5}$$
$$= 5(1) + 3(0) = 5$$

$\Rightarrow \quad f_5$ is divisible by 5

If $n = 2 \Rightarrow f_{5n} = f_{10}$

$$= 5f_{10-4} + 3f_{10-5}$$
$$= 5f_6 + 3f_5$$
$$= 5(8) + 3(5) = 55$$

$[\because f_6 = 5f_2 + 3f_1 = 5 + 3 = 8]$

$\Rightarrow \quad f_{10}$ is divisible by 5

If $n = 3 \Rightarrow f_{5n} = f_{15}$

$$= 5f_{15-4} + 3f_{15-5}$$
$$= 5f_{11} + 3f_{10}$$
$$= 5(89) + 3(55)$$

$[\because f_{11} = 5f_7 + 3f_6 = 5(13) + 3(8) = 89]$

$$= 610$$

$\Rightarrow \quad f_{15}$ is divisible by 5

Let, $\quad f_{5k} = 5t$

Where $t \in N$ $\qquad$ ...(1)

$$f_{5k+1} = 5f_{5k+1-4} + 3f_{5k+1-5}$$
$$= 5f_{5k-3} + 3f_{5k-4}$$

For $n = k + 1$, $\qquad$ ...(2)

$$f_5(k+1) = f_{5k+5}$$
$$= 5f_{5k+5-4} + 3f_{5k+5-5}$$
$$= 5f_{5k+1} + 3f_{5k}$$
$$= 5(5f_{5k-3} + 3f_{5k-4}) + 3(5t)$$
$$= 5(5f_{5k-3} + 3f_{5k-4} + 3t)$$

From the principle of mathematical induction, $f_{5(k+1)}$ is divisible by 5.

Hence, f_{5n} is divisible by 5 for $n = 1, 2, 3, ...$

Q58. Find ∇a_n for the sequence $\{a_n\}$, where

(a) $\quad a_n = 4$

(b) $\quad a_n = 2n$

(c) $\quad a_n = n^2$

(d) $\quad a_n = 2^n$

Answer : $\qquad$ **Model Paper-4, Q8(b)**

Given sequence is,

$$\{a_n\}$$

The first difference is given as,

$$\nabla a_n = a_n - a_{n-1}$$

(a) $\quad a_n = 4$

$$a_{n-1} = 4$$

$\Rightarrow \quad \nabla a_n = a_n - a_{n-1}$

$$= 4 - 4$$

$\Rightarrow \quad \nabla a_n = 0$

$\therefore \quad \nabla a_n = 0$

(b) $\quad a_n = 2n$

$$a_{n-1} = 2(2n - 1)$$

$\Rightarrow \quad \nabla a_n = 2n - 2(n - 1)$

$$= 2n - 2n + 2$$

$\Rightarrow \quad \nabla a_n = 2$

$\therefore \quad \nabla a_n = 2$

(c) $\quad a_n = n^2$

$$a_{n-1} = (n - 1)^2$$

$\Rightarrow \quad \nabla a_n = n^2 - (n - 1)^2$

$$= n^2 - n^2 - 1 + 2n$$

$\Rightarrow \quad \nabla a_n = 2n - 1$

$\therefore \quad \nabla a_n = 2n - 1$

(d) $a_n = 2^n$

$$a_{n-1} = 2^{n-1}$$

$$\Rightarrow \nabla a_n = 2^n - 2^{n-1}$$

$$= 2^n\left[1-\frac{1}{2}\right]$$

$$\Rightarrow \nabla a_n = 2^{n-1}$$

$$\therefore \nabla a_n = 2^{n-1}$$

Q59. Find $\nabla^2 a_n$ for the sequence $\{a_n\}$ where

(a) $a_n = 4$

(b) $a_n = 2n$

(c) $a_n = n^2$

(d) $a_n = 2^n$

Answer :

For answer refer Unit-IV, Q58.

Let,

The $(k+1)^{th}$ difference is given as,

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$$

For $k = 1$,

$$\nabla^2 a_n = \nabla^1 a_n - \nabla^1 a_{n-1}$$

$$\Rightarrow \nabla^2 a_n = \nabla a_n - \nabla a_{n-1}$$

(a) $a_n = 4$

$$\nabla a_n = 0$$

$$\nabla a_{n-1} = 0$$

$$\Rightarrow \nabla^2 a_n = 0 - 0$$

$$= 0$$

$$\therefore \nabla^2 a_n = 0$$

(b) $a_n = 2n$

$$\nabla a_n = 2$$

$$\nabla a_{n-1} = 2$$

$$\Rightarrow \nabla^2 a_n = 2 - 2$$

$$= 0$$

$$\therefore \nabla^2 a_n = 0$$

(c) $a_n = n^2$

$$\nabla a_n = 2n - 1$$

$$\nabla a_{n-1} = 2(n-1) - 1$$

$$= 2n - 2 - 1$$

$$= 2n - 3$$

$$\Rightarrow \nabla^2 a_n = (2n-1) - (2n-3)$$

$$= 2n - 1 - 2n + 3$$

$$= 2$$

$$\therefore \nabla^2 a_n = 2$$

(d) $a_n = 2^n$

$$\nabla a_n = 2^{n-1}$$

$$\nabla a_{n-1} = 2^{(n-1)-1}$$

$$\Rightarrow \nabla^2 a_n = 2^{n-1} - 2(n-1) - 1$$

$$= 2^{n-1}[1 - 2^{-1}]$$

$$= 2^{n-1}\left[1-\frac{1}{2}\right]$$

$$= 2^{n-1}.\, 2^{-1}$$

$$= 2^{n-1-1}$$

$$= 2^{n-2}$$

$$\therefore \nabla^2 a_n = 2^{n-2}$$

Q60. Show that $a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$.

Answer : Model Paper-3, Q...

Let $\{a_n\}$ be the sequence of real numbers

The first difference is given as,

$$\nabla a_n = a_n - a_{n-1}$$

The $(k+1)^{th}$ difference is given as,

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}$$

Consider,

$$a_n - 2\nabla a_n + \nabla^2 a_n = a_n - 2\nabla a_n + \nabla a_n - \nabla a_{n-1}$$

[$\because$ From equation ...]

$$= a_n - \nabla a_n - \nabla a_{n-1}$$

$$= a_n - (a_n - a_{n-1}) - (a_{n-1} - a_{n-2})$$

[$\because$ From equation ...]

$$= a_n - a_n + a_{n-1} - a_{n-1} + a_{n-2}$$

$$= a_{n-2}$$

$$\therefore a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$$

Q61. Prove that a_{n-k} can be expressed in terms of a_n, $\nabla a_n, \nabla^2 a_n, \ldots, \nabla^k a_n$.

Answer :

Let $\{a_n\}$ be the sequence of real numbers

The first difference is given as,

$$\nabla a_n = a_n - a_{n-1}$$

$$\Rightarrow a_{n-1} = a_n - \nabla a_n$$

$$a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$$

Similarly,

$$a_{n-3} = a_n + (-3)\nabla a_n + \frac{(-3)(-3+1)}{2!}.\nabla^2 a_n + \frac{(-3)(-3+1)(-3+2)}{3!}.\nabla^3 a_n$$

$$a_{n-3} = a_n - 3\nabla a_n + 3\nabla^2 a_n - \nabla^3 a_n$$

From Newton's Backward difference formula,

$$a_{n-k} = a_n - k\nabla a_n + \frac{(-k)(-k+1)}{2!}\nabla^2 a_n + ...+ \frac{(-k)(-k+1)....(-k+k-1)}{k!}.\nabla^k a_n$$

$$= a_n - k\nabla a_n + \frac{k(k-1)}{2!}\nabla^2 a_n +....+ \frac{k(k-1)....3.2.1}{k!}\nabla^k a_n$$

$$= a_n - k\nabla a_n + kC_2\nabla^2 a_n - C_3\nabla^3 a_n ...... + kC_k\nabla^k a_n$$

$$\Rightarrow \quad a_{n-k} = a_n - k\nabla a_n + kC_2\nabla^2 a_n ....(-1)^n\nabla^k a_n$$

Hence, a_{n-k} can be expressed in terms of a_n, ∇a_n, $\nabla^2 a_n$......$\nabla^k a_n$

Q62. Express the recurrence relation $a_n = a_{n-1} + a_{n-2}$ in terms of a_n, ∇a_n, and $\nabla^2 a_n$.

Answer :

Given recurrence relation is,

$$a_n = a_{n-1} + a_{n-2} \qquad ...(1)$$

The first difference is given as,

$$\nabla a_n = a_n - a_{n-1} \qquad ...(2)$$

$$\Rightarrow \quad a_{n-1} = a_n - \nabla a_n \qquad ...(3)$$

Similarly,

$$a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$$

Substituting equations (2) and (3) in equation (1),

$$a_n = (a_n - \nabla a_n) + (a_n - 2\nabla a_n + \nabla^2 a_n)$$

$$= 2a_n - 3\nabla a_n + \nabla^2 a_n$$

$$\Rightarrow \quad 3\nabla a_n - \nabla^2 a_n = 2a_n - a_n$$

$$\therefore \quad a_n = 3\nabla a_n - \nabla^2 a_n.$$

Q63. What are the various forms of solution for the recurrence relation?

Answer :

The various forms of solution for the recurrence relation is based on the following conditions (i.e., nature of the roots).

Condition 1

The sequence 'a_n' is a solution for the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$

If r_1 and r_2 are two different roots then the sequence 'a_n' is a solution for the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$

If $a_n = A_1 r_1^n + A_2 r_2^n$ when $n = 0, 1, 2, 3, ...$

Here, A_1 and A_2 are arbitrary constants which can be identified by using the initial conditions.

Condition 2

If r is only single root then the sequence 'a_n' is a solution for the recurrence relation

$a_n = C_1 a_{n-1} + C_2 a_{n-2}$ iff $a_n = A_1 r^n + A_2 r^n$ when $n = 0, 1, 2, ...$

Here, A_1 and A_2 are arbitrary constants which be identified by using the initial conditions.

Condition 3

If $r_1, r_2, r_3, ..., r_n$ are k-different roots then a sequence 'a_n' is a solution for the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2} + ... + C_k a_{n-k}$ iff $a_n = A_1 r_1^n + A_2 r_2^n + ... + A_k r_k^n$ when $n = 0, 1, 2, ...$ Here, $A_1, A_2, ... A_k$ are arbitrary constants which can be identified by using the initial conditions.

Condition 4

If $r_1, r_2, r_3, \ldots\ldots, r_n$ are different roots with $m_1, m_2, m_3, \ldots\ldots, m_n$ as multiplicities and $m_p \geq 1$, when $p = 1, 2, 3, \ldots, n$ and $m_1 + m_2 + m_3 + \ldots\ldots = k$.

Then a sequence 'a_n' is a solution for the recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k}$

Iff $a_n = (A_{1,0} + A_{1,1} n + \ldots\ldots + A_{1,m} - 1\ nm_{1-1}) r_1^n + (A_{2,0} + A_{1,1} n + \ldots\ldots + A_2\ m_{2-1}\ nm_{2-1}) r_2^n + \ldots + (A_{t,0} + A_{t,1} n + \ldots + A_{p,m_t-1}\ n^{m_t-1}) r_t^n$

When $n = 0, 1, 2, \ldots\ldots$ Here, A_{pq} are arbitrary constants which can be identified by using the initial conditions when $1 \leq p \leq t$ and $0 \leq q \leq m_p - 1$.

Condition 5

If r_1 and r_2 are two complex conjugate roots then modulus amplitude form of r_1 and r_2 are given as follows.

$r_1 = r(\cos\theta + i\sin\theta)$ and

$r_2 = r(\cos\theta - i\sin\theta)$.

Then, the solution for the recurrence relation is $a_n = r^n (A_1 \cos\theta + A_2 \sin\theta)$.

Q64. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

(a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

(b) $a_n = 2na_{n-1} + a_{n-2}$

(c) $a_n = a_{n-1} + a_{n-4}$

(d) $a_n = a_{n-1} + 2$

(e) $a_n = a^2_{n-1} + a_{n-2}$

(f) $a_n = a_{n-2}$

(g) $a_n = a_{n-1} + n$

Answer :

(a) Given recurrence relation is,

$$a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3} \quad \ldots(1)$$

Equation (1) is of the form,

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} \quad \ldots(2)$$

Comparing equations (1) and (2),

$$c_1 = 3, c_2 = 4, c_3 = 5$$

Equation (1) linear, since all the terms i.e., a_i have power 1

Equation (1) is homogeneous because it does not have constant terms

It has degree 3 because the longest k value for which a_{n-k} occurs in the relation is, a_{n-3}

Hence, the given relation is a linear homogeneous recurrence relation with constant coefficients with degree 3.

(b) Given recurrence relation is,

$$a_n = 2na_{n-1} + a_{n-2} \quad \ldots(1)$$

Equation (1) does not have constant coefficients since coefficient of a_{n-1} is $2n$ which is non-constant

It is a homogeneous relation since, it does not have any constant terms

Hence the given relation is not a linear homogeneous recurrence relation.

(c) Given recurrence relation is,

$$a_n = a_{n-1} + a_{n-4} \quad \ldots(1)$$

This is linear, homogeneous with constant coefficients

Equation (1) is of the form,

$$a_n = c_1na_{n-1} + c_4a_{n-4} \quad \ldots(2)$$

Comparing equations (1) and (2),

$$C_1 = 1, C_4 = 1$$

It has degree 4 because the largest k value for which a_{n-k} occurs in the relation is, a_{n-4}

Hence, the given relation is a linear homogeneous recurrence relation with constant coefficients with degree 4.

Given recurrence relation is,

(ii) $a_n = a_{n-1} + 2$ ...(1)

Equation is not homogeneous because of 2.

Hence, the given relation is not linear homogeneous recurrence relation.

Given recurrence relation is,

(iii) $a_n = a^2_{n-1} + a_{n-2}$ ...(1)

Equation (1) is not linear since the power of a^2_{n-1} is 2

Hence, the given relation is not linear homogeneous recurrence relation.

Given recurrence relation is,

(iv) $a_n = a_{n-2}$ ...(1)

Equation (1) is of the form,

$a_n = c_2 a_{n-2}$ ...(2)

Comparing equations (1) and (2),

$c_2 = 1$

Equation (1) is linear, homogeneous with constant coefficient

It has degree 2 because the largest k value occurring in the relation is, a_{n-2}

Hence, the given relation is a linear homogeneous recurrence relation with constant coefficient with degree 2

Given recurrence relation is,

(v) $a_n = a_{n-1} + n$ ...(1)

Equation (1) is not homogeneous because of n

Hence, the given relation is linear but non homogeneous recurrence relation.

Q65. Solve these recurrence relations together with the initial conditions given.

(a) $a_n = a_{n-1} + 6a_{n-2}$ **for** $n \geq 2$, $a_0 = 3$, $a_1 = 6$

(b) $a_n = 7a_{n-1} - 10a_{n-2}$ **for** $n \geq 2$, $a_0 = 2$, $a_1 = 1$

(c) $a_n = 6a_{n-1} + 8a_{n-2}$ **for** $n \geq 2$, $a_0 = 4$, $a_1 = 10$

(d) $a_n = 2a_{n-1} - a_{n-2}$ **for** $n \geq 2$, $a_0 = 4$, $a_1 = 1$

(e) $a_n = a_{n-1}$ **for** $n \geq 2$, $a_0 = 5$, $a_1 = -1$

(f) $a_n = -6a_{n-1} - 9a_{n-2}$ **for** $n \geq 2$, $a_0 = 3$, $a_1 = -3$

(g) $a_{n+2} = -4a_{n+1} + 5a_n$ **for** $n \geq 0$, $a_0 = 2$, $a_1 = 8$

Answer :

(a) Given recurrence relation is,

$a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$ ...(1)

Initial conditions are,

$a_0 = 3, a_1 = 6$

Equation (1) is of the form,

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$ ...(2)

Comparing equations (1) and (2),

$c_1 = 1, c_2 = 6$

The characteristic equation is,

$r^2 - c_1 r - c_2 = 0$

$\Rightarrow \quad r^2 - r - 6 = 0$

The roots are obtained as,

$$r^2 - r - 6 = 0 \Rightarrow r^2 - 3r + 2r - 6 = 0$$
$$\Rightarrow r(r-3) + 2(r-3) = 0$$
$$\Rightarrow (r+2)(r-3) = 0$$
$$\Rightarrow r = -2, 3$$

The solution to the recurrence relation is of the form,

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$= \alpha_1(-2)^n + \alpha_2 . 3^n$$

$$\therefore \quad a_n = (-2)^n \alpha_1 + 3^n \alpha_2 \qquad \text{... (3)}$$

For $n = 0$,

$$a_0 = 3 = (-2)^0 \alpha_1 + 3^0 \alpha_2 \qquad \text{... (4)}$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 = 3$$

For $n = 1$,

$$a_1 = 6 = (-2)^1 \alpha_1 + 3^1 \alpha_2 \qquad \text{... (5)}$$

$$-2\alpha_1 + 3\alpha_2 = 6$$

Solving equations (4) and (5),

$$\begin{array}{r} 3\alpha_1 + 3\alpha_2 = 9 \\ \underset{+}{-}2\alpha_1 \underset{-}{+} 3\alpha_2 \underset{-}{=} 6 \\ \hline 5\alpha_1 = 3 \end{array}$$

$$\Rightarrow \alpha_1 = \frac{3}{5}$$

At, $\alpha_1 = \frac{3}{5}, \alpha_2 = \frac{12}{5}$

Substituting the corresponding values in equation (3),

$$a_n = (-2)^n \frac{3}{5} + 3^n \frac{12}{5}$$

Hence, the solution of recurrence relation is,

$$a_n = (-2)^n \frac{3}{5} + 3^n \frac{12}{5}$$

(b) Given recurrence relation is,

$$a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2 \qquad \text{... (1)}$$

Initial conditions are,

$$a_0 = 2, a_1 = 1$$

Equation (1) is of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \qquad \text{... (2)}$$

Comparing equations (1) and (2),

$$c_1 = 7, c_2 = -10$$

The characteristic equation is,

$$r^2 - cr - c_2 = 0$$
$$\Rightarrow \quad r^2 - 7r + 10 = 0$$
$$\Rightarrow \quad r^2 - 2r - 5r + 10 = 0$$
$$\Rightarrow \quad r(r-2) - 5(r-2) = 0$$
$$\Rightarrow \quad (r-2)(r-5) = 0$$
$$\therefore \quad r = 2, 5$$

The solution of recurrence relation is,

$$a_n = r_1^n \alpha_1 + r_2^n \alpha_2$$

$$\therefore \quad a_n = 2^n \alpha_1 + 5^n \alpha_2 \qquad \text{... (3)}$$

For $n = 0$,

$$a_0 = 2 = \alpha_1 + \alpha_2$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 = 2 \qquad \text{...(4)}$$

For $n = 1$,

$$a_1 = 1 = 2\alpha_1 + 5\alpha_2$$

$$\Rightarrow \quad 2\alpha_1 + 5\alpha_2 = 1 \qquad \text{...(5)}$$

Solving equations (4) and (5),

$$\begin{array}{r} 2\alpha_1 + 2\alpha_2 = 4 \\ \underset{+}{2}\alpha_1 \underset{-}{+} 5\alpha_2 \underset{-}{=} 1 \\ \hline -3\alpha_2 = 3 \end{array}$$

$$\Rightarrow \alpha_2 = -1$$

At $\alpha_2 = -1, \alpha_1 = 3$

Substituting the corresponding values in equation (3),

$$a_n = 2^n(3) + (-1)5^n$$

Hence, the solution of recurrence relation is,

$$a_n = -2^n(3) + (-1)5^n$$

(c) Given recurrence relation is,

$$a_n = 6a_{n-1} - 8a_{n-2} \text{ for } n \geq 2 \qquad \text{...(1)}$$

Initial conditions are,

$$a_0 = 4, a_1 = 10$$

Equation (1) is of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \qquad \text{...(2)}$$

Comparing equations (1) and (2),

$$c_1 = 6, c_2 = -8$$

The characteristic equation is,

$$r^2 - c_1 r - c_2 = 0$$
$$\Rightarrow \quad r^2 - 6r + 8 = 0$$
$$\Rightarrow \quad r^2 - 4r - 2r + 8 = 0$$
$$\Rightarrow \quad r(r-4) - 2(r-4) = 0$$
$$\Rightarrow \quad (r-2)(r-4) = 0$$
$$\therefore \quad r = 2, 4$$

The solution of recurrence relation is,

$$a_n = r_1^n \alpha_1 + r_2^n \alpha_2$$

$$\therefore \quad a_n = 2^n \alpha_1 + 4^n \alpha_2 \qquad \text{...(3)}$$

For $n = 0$,

$$a_0 = 4 = \alpha_1 + \alpha_2$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 = 4 \qquad \text{...(4)}$$

For $n = 1$,

$$a_1 = 10 = 2\alpha_1 + 4\alpha_2$$

$$\Rightarrow \quad 2\alpha_1 + 4\alpha_2 = 10 \qquad \text{...(5)}$$

Solving equations (4) and (5),

$$\alpha_1 + \alpha_2 = 4$$
$$\alpha_1 + 2\alpha_2 = 5$$
$$- \quad - \quad -$$
$$\alpha_2 = 1$$

At $\alpha_2 = 1, \alpha_1 = 3$

Substituting the corresponding values in equation (3),

$$a_n = 3.2^n + 1.4^n$$

Hence, the solution of recurrence relation is,

$$\therefore \quad a_n = 3.2^n + 1.4^n$$

(d) Given recurrence relation is,

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{... (1)}$$

Initial conditions are,

$$a_0 = 4, a_1 = 1$$

Equation (1) is of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$c_1 = 2, c_2 = -1$$

The characteristic equation is,

$$r - c_1 r - c_2 = 0$$
$$\Rightarrow \quad r^2 - 2r + 1 = 0$$
$$\Rightarrow \quad r^2 - r - r + 1 = 0$$
$$\Rightarrow \quad (r-1)(r-1) = 0$$
$$\therefore \quad r = 1 \text{ is the root}$$

The solution to the recurrence relation is of the form,

$$a_n = r_1^n \alpha_1 + r_2^n \alpha_2$$
$$\Rightarrow \quad a_n = \alpha_1 . 1^n + \alpha_2 . n . 1^n$$
$$\therefore \quad a_n = \alpha_1 + n\alpha_2 \quad \text{... (3)}$$

For $n = 0$,

$$a_0 = 4 = \alpha_1$$
$$\Rightarrow \quad \alpha_1 = 4$$

For $n = 1$,

$$a_1 = 1 = \alpha_1 + \alpha_2$$
$$\Rightarrow \quad \alpha_2 = -3$$

Substituting the corresponding values in equation (3),

$$a_n = 4 + n(-3)$$
$$\therefore \quad a_n = 4 - 3^n.$$

(e) Given recurrence relation is,

$$a_n = a_{n-2} \quad \text{... (1)}$$

Initial conditions are,

$$a_0 = 5, a_1 = -1$$

Equation (1) is of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$c_1 = 0, c_2 = 1$$

The characteristic equation is,

$$r^2 - c_1 r - c_2 = 0$$
$$\Rightarrow \quad r^2 - 1 = 0$$
$$\Rightarrow \quad r^2 = \pm 1$$

The solution of recurrence relation is,

$$a_n = r_1^n \alpha_1 + r_2^n \alpha_2$$
$$\Rightarrow \quad a_n = \alpha_1 . 1^n + \alpha_2 (-1)^n$$
$$\therefore \quad a_n = \alpha_1 + (-1)^n \alpha_2 \quad \text{... (3)}$$

For $n = 0$,

$$a_0 = 5 = \alpha_1 + \alpha_2$$
$$\Rightarrow \quad \alpha_1 + \alpha_2 = 5 \quad \text{... (4)}$$

For $n = 1$,

$$a_1 = -1 = \alpha_1 - \alpha_2$$
$$\Rightarrow \quad \alpha_1 - \alpha_2 = -1$$

Solving equations (1) and (2),

$$\alpha_1 + \alpha_2 = 5$$
$$\alpha_1 - \alpha_2 = -1$$
$$+ \quad - \quad -$$
$$2\alpha_1 = 4$$
$$\Rightarrow \alpha_1 = 2$$

At $\alpha_1 = 2, \alpha_2 = 3$

Substituting the corresponding values in equation (3),

$$a_n = 2 + (-1)^n . 3$$

Hence, the solution of recurrence relation is,

$$\therefore \quad a_n = 2 + (-1)^n . 3$$

(f) Given recurrence relation is,

$$a_n = -6a_{n-1} - 9a_{n-2} \quad \text{... (1)}$$

Initial conditions are,

$$a_0 = 3, a_1 = -3$$

Equation (1) is of the form,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$c_1 = 0, c_2 = 1$$

The characteristic equation is,

$$r^2 - c_1 r - c_2 = 0$$

$\Rightarrow$ $r^2 + 6r + 9 = 0$

$\Rightarrow$ $r^2 + 3r + 3r + 9 = 0$

$\Rightarrow$ $(r+3)(r+3) = 0$

$\Rightarrow$ $r = -3$ is the root

The solution of recurrence relation is,

$$a_r = r_1^r \alpha_1 + r_2^r \alpha_2$$

$\therefore$ $a_r = \alpha_1(-3)^r + \alpha_2 n(-3)^r$ ... (3)

For $n = 0$,

$$a_0 = 3 = \alpha_1 + 0.\alpha_2$$

$\Rightarrow$ $\alpha_1 = 3$

For $n = 1$,

$$a_1 = -3 = -3\alpha_1 - 3\alpha_2$$

$\Rightarrow$ $\alpha_1 + \alpha_2 = 1$

At $\alpha_1 = 3, \alpha_2 = -2$

Substituting the corresponding values in equation (3),

$$a_r = 3.(-3)^r - 2n(-3)^r$$

$$= (-3)^r[3 - 2n]$$

Hence, the solution of recurrence relation is,

$\therefore$ $a_r = = (-3)^r[3 - 2n]$

(g) Given recurrence relation is,

$$a_{r+2} = -4a_{r+1} + 5a_r \quad \text{... (1)}$$

Initial conditions are,

$$a_0 = 2, a_1 = 8$$

Equation (1) is of the form,

$$a_r = c_1 a_{r-1} + c_2 a_{r-2} \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$c_1 = -4, c_2 = 5$$

The characteristic equation is,

$$r^2 - c_1 r - c_2 = 0$$

$\Rightarrow$ $r^2 + 4r - 5 = 0$

$\Rightarrow$ $(r+5)(r-1) = 0$

$\therefore$ $r = 5, -1$

The solution of recurrence relation is,

$$a_n = r_1^n \alpha_1 + r_2^n \alpha_2$$

$\Rightarrow$ $a_{n+2} = \alpha_1(-5)^{n+2} + \alpha_2$ ... (3)

For $n = 0$,

$$a_0 = 2$$

$\Rightarrow$ $a_{-2+2} = \alpha_1(-5)^{-2+2} + \alpha_2$ $[\because a_0 = 2]$

$\Rightarrow$ $\alpha_1 + \alpha_2 = 2$ ... (4)

For $n = 1$,

$\Rightarrow$ $a_1 = 8$

$\Rightarrow$ $a_{-1+2} = a_1(-5)^{-1+2} + \alpha_2$

$$-5\alpha_1 + \alpha_2 = 8 \quad \text{...(5)}$$

Solving equations (1) and (2),

$$5\alpha_1 + 5\alpha_2 = 10$$

$$-5\alpha_1 + \alpha_2 = 8$$

$$6\alpha_2 = 18$$

$\Rightarrow$ $\alpha_2 = 3$

At $\alpha_2 = 3, \alpha_1 = -1$

Substituting the corresponding values in equation (3),

$$a_{n+2} = (-1)(-5)^{n+2} + 3$$

Hence, the solution of recurrence relation is,

$\therefore$ $a_{n+2} = -25.5^n + 3$

Q66. The Lucas numbers satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2}$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

(a) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, ...,$ where f_n is the nth Fibonacci number.

(b) Find an explicit formula for the Lucas numbers.

Answer :

(a) Given recurrence relation is,

$$L_n = L_{n-1} + L_{n-2}$$

Initial conditions are,

$$L_0 = 2, L_1 = 1$$

[The fibonacci series is given as 0, 1, 1, 2, 3, 5, 8,.....
$f_0, f_1, f_2, f_3, f_4, f_5, f_6.....$]

Proof by Induction

Let $P(n)$ represents the equation

$$L_n = f_{n-1} + f_{n+1}$$

Basis step

For $n = 2$,

$$P(2) = L_2 = L_1 + L_0$$

$$= 1 + 2$$

$$= 3$$

$$L_2 = 3$$

$$f_1 + f_3 = 1 + (f_2 + f_1)$$
$$= 1 + (1 + 1)$$
$$\Rightarrow f_1 + f_3 = 3.$$
$$\Rightarrow L_2 = f_1 + f_3$$
$$L_2 = f_{2-1} + f_{2+1}$$

$\therefore$ $P(2)$ is true.

Inductive step

Suppose $P(2), P(3) \ldots\ldots P(k-1), P(k)$ is true.

$$L_{k-1} = f_{k-2} + f_k$$
$$L_k = f_{k-1} + f_{k+1} \quad \ldots(1)$$

For $n = k + 1$, ...(2)

$$P(k+1) = L_{k+1} = L_{k+1-1} + L_{k+1-2}$$
$$= L_k + L_{k-1}$$
$$= (f_{k-1} + f_{k+1}) + (f_{k-2} + f_k)$$
$$= (f_{k-1} + f_{k-2}) + (f_{k+1} + f_k)$$
$$= f_k + f_{k+2}$$
$$= f_{k+1-1} + f_{k+1+1}$$

$\Rightarrow$ $P(k+1)$ is true.

From the principle of mathematical induction, $P(n)$ is true for all positive integers with $n \geq 2$.

(b) The recurrence relation for Lucas numbers is given as,

$$L_n = L_{n-1} + L_{n-2} \quad \ldots(3)$$

Initial conditions are

$$L_0 = 2, L_1 = 1$$

Equation (3) is of the form,

$$L_n = C_1 L_{n-1} + C_2 L_{n-2} \quad \ldots(4)$$

Comparing equations (3) and (4),

$$C_1 = 1, C_2 = 1$$

The characteristic equation is,

$$r^2 - C_1 r - C_2 = 0$$
$$\Rightarrow r^2 - r - 1 = 0$$
$$\Rightarrow r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$
$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$$

The solution of recurrence relation is given as,

$$L_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$L_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \ldots(5)$$

For $n = 0 \Rightarrow L_0 = 2$

$$= \alpha_1 + \alpha_2$$
$$\therefore \quad \alpha_1 + \alpha_2 = 2$$

For $n = 1 \Rightarrow L_1 = 1$ ...(6)

$$= \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)$$
$$\therefore \quad L_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)$$

Solving equations (6) and (7). ...(7)

$$\alpha_2 = 2 - \alpha_1 \quad [\because \text{From equation (6)}]$$
$$\Rightarrow L_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + (2-\alpha_1)\left(\frac{1-\sqrt{5}}{2}\right)$$
$$\Rightarrow \alpha_1 = \frac{\alpha_1 + \alpha_1\sqrt{5} + 2 - 2\sqrt{5} - \alpha_1 + \alpha_1\sqrt{5}}{2}$$
$$\Rightarrow 1 = \frac{2[\alpha_1\sqrt{5} + 1 - \sqrt{5}]}{2}$$
$$\Rightarrow \alpha_1\sqrt{5} + 1 - \sqrt{5}$$
$$\Rightarrow 1 = \alpha_1\sqrt{5} = 1 - 1 + \sqrt{5}$$
$$\Rightarrow \alpha_1 = \frac{\sqrt{5}}{\sqrt{5}}$$
$$\Rightarrow \alpha_1 = 1$$
$$\Rightarrow \alpha_2 = 2 - \alpha_1 = 2 - 1$$
$$= 1$$
$$\therefore \alpha_2 = 1$$

Substituting the corresponding values in equation (7),

$$L_n = \left[1 \times \left(\frac{1+\sqrt{5}}{2}\right)^n\right] + \left[1 \times \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\therefore$ The explicit formula for the Lucas numbers is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Q67. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$ and $a_3 = 8$.

Answer : Model Paper-2, Q9(a)

Given recurrence relation is,

$$a_n = 5a_{n-2} - 4a_{n-4} \quad \ldots(1)$$

Initial conditions are,

$$a_0 = 3, a_1 = 2, a_2 = 6, a_3 = 8$$

Equation (1) is of the form,

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + C_3 a_{n-3} + C_4 a_{n-4} \quad \ldots(2)$$

Comparing equations (1) and (2),

$$C_1 = 0, C_2 = 5, C_3 = 0, C_4 = -4$$

The characteristic equation is,

$$r^4 - C_1 r^3 - C_2 r^2 - C_3 r + C_4 = 0$$

$$\Rightarrow \quad r^4 - 5r^2 + 4 = 0$$

$$\Rightarrow \quad r^4 - 4r^2 - r^2 + 4 = 0$$

$$\Rightarrow \quad r^2(r^2 - 4) - 1(r^2 - 4) = 0$$

$$\Rightarrow \quad (r^2 - 1)(r^2 - 4) = 0$$

Hence, the roots are $-1, +1, -2, +2$

The general solution is given as,

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \alpha_4 r_4^n$$

$$a_n = \alpha_1(-1)^n + \alpha_2 + \alpha_3(-2)^n + \alpha_4(2)^n \quad \text{...(3)}$$

For $n = 0 \Rightarrow a_0 = 3$

$$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\therefore \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3 \quad \text{...(4)}$$

For $n = 1 \Rightarrow a_1 = 2$

$$= -\alpha_1 + \alpha_2 - 2\alpha_3 + 2\alpha_4$$

$$\therefore -\alpha_1 + \alpha_2 - 2\alpha_3 + 2\alpha_4 = 2 \quad \text{...(5)}$$

For $n = 2 \Rightarrow a_2 = 6$

$$= \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$$

$$\therefore \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4 = 6 \quad \text{...(6)}$$

For $n = 3 \Rightarrow a_3 = 8$

$$= -\alpha_1 + \alpha_2 - 8\alpha_3 + 8\alpha_4$$

$$\therefore -\alpha_1 + \alpha_2 - 8\alpha_3 + 8\alpha_4 = 8 \quad \text{...(7)}$$

Solving equations (4), (5), (6) and (7)

$$(4) \Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3$$

$$(5) \Rightarrow \quad -\alpha_1 + \alpha_2 - 2\alpha_3 + 2\alpha_4 = 2$$

$$2\alpha_2 - \alpha_3 + 3\alpha_4 = 5 \quad \text{...(8)}$$

$$(4) \Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 3$$

$$(7) \Rightarrow \quad -\alpha_1 + \alpha_2 - 8\alpha_3 + 8\alpha_4 = 8$$

$$2\alpha_2 - 7\alpha_3 + 9\alpha_4 = 11 \quad \text{...(9)}$$

From equations (8) and (9),

$$(8) \Rightarrow \quad 2\alpha_2 - \alpha_3 + 3\alpha_4 = 5$$

$$(9) \Rightarrow \quad 2\alpha_2 - 7\alpha_3 + 9\alpha_4 = 11$$

$$6\alpha_3 - 6\alpha_4 = -6$$

$$\Rightarrow \quad \alpha_3 - \alpha_4 = -1 \quad \text{...(10)}$$

From equations (6) and (7),

$$\alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4 = 6$$

$$-\alpha_1 + \alpha_2 - 8\alpha_3 + 8\alpha_4 = 8$$

$$2\alpha_2 - 4\alpha_3 + 12\alpha_4 = 14 \quad \text{...(11)}$$

From equations (9) and (11),

$$2\alpha_2 - 7\alpha_3 + 9\alpha_4 = 11$$

$$2\alpha_2 - 4\alpha_3 + 12\alpha_4 = 14$$

$$-3\alpha_3 - 3\alpha_4 = -3$$

$$\Rightarrow \alpha_3 + \alpha_4 = 1 \quad \text{...(12)}$$

From equations (10) and (12),

$$\alpha_3 - \alpha_4 = -1$$

$$\alpha_3 + \alpha_4 = 1$$

$$2\alpha_3 = 0$$

$$\Rightarrow \alpha_3 = 0$$

From equation (12),

$$0 + \alpha_4 = 1$$

$$\Rightarrow \quad \alpha_4 = 1$$

From equation (8),

$$2\alpha_2 - 0 + 3(1) = 5$$

$$\Rightarrow 2\alpha_2 = 2$$

$$\Rightarrow \alpha_2 = 1$$

From equation (4),

$$\alpha_1 + 1 + 0 + 1 = 3$$

$$\Rightarrow \alpha_1 = 1$$

$$\therefore \quad \alpha_1 = 1$$

Substituting the corresponding values in equation (3),

$$a_n = 1(-1)^n + 1 + 0(-2)^n + 1(2)^n$$

$$= (-1)^n + 1 + (2)^n$$

Hence, the explicit formula for Lucas numbers is,

$$a_n = 1 + (-1)^n + (2)^n$$

Q68. Consider the non homogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.

(a) Show that $a_n = n2^n$ is a solution of this recurrence relation.

(b) Find all solutions of this recurrence relation.

(c) Find the solution with $a_0 = 2$.

Answer :

(a) Given recurrence relation is

$$a_n = 2a_{n-1} + 2^n \quad \text{...(1)}$$

If $a_n = n2^n$ then

$$a_{n-1} = (n-1)\,2^{n-1} \quad \text{...(2)}$$

Consider,

$$2a_{n-1} + 2^n$$
$$= 2(n-1)\,2^{n-1} + 2^n \qquad [\because \text{From equation (2)}]$$
$$= 2^n(n-1) + 2^n$$
$$= 2^n[n-1+1]$$

Hence, $a_n = n2^n$ is a solution of recurrence relation.

The associated linear homogeneous equation of,

$$a_n = 2a_{n-1} + 2^n \text{ is}$$
$$a_n = 2a_{n-1}$$

Its solution is,

$$a_n^{(h)} = \alpha\, 2^n$$

where α – constant

The solution of recurrence relation is given as,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Here, $a_n^{(p)} = n2^n$

$$\therefore \quad a_n = \alpha 2^n + n2^n$$
$$= 2^n(\alpha + n)$$

$$a_0 = 2$$
$$a_n = 2^n(\alpha + n)$$

For $n = 0 \Rightarrow a_0 = 2$

$$= 2^0(\alpha + 0)$$
$$\Rightarrow \alpha = 2$$
$$\Rightarrow a_n = 2^n(2 + n)$$

Hence, the solution of recurrence relation with $a_0 = 2$ is, $a_n = 2^n(2 + n)$.

4.2.2 Divide and Conquer Algorithms and Recurrence Relations, Generating Functions

Q69. Explain about divide and conquer algorithm.

Answer :

Divide and conquer algorithm breaks the problem (say to size n) into one or more sub problems (say a problems each of size n/b) and then combines the results of these sub problems to create the solution of original problem. Hence, if $f(n)$ be the total number of operations needed to find the solution of the original problem, then the recurrence relation obtained is as follows,

$$f(n) = af\left(\frac{n}{b}\right) + g(n)$$

This recurrence relation is called "divide and conquer recurrence relation" where $g(n)$ be the additional operations needed in combining the sub problem solutions to find the solution of original problem.

The applications of divide and conquer algorithm are as follows,

1. Binary Search

The recurrence relation of binary search is,

$$f(n) = f\left(\frac{n}{b}\right) + 2 \text{ if } n \text{ is even.}$$

Where,

$f(n)$ represents the total number of comparisons needed to search for an element in a set of n elements.

2. Merge Sort

The recurrence relation of finding the maximum and minimum of a sequence is,

$$f(n) = 2f\left(\frac{n}{2}\right) + 2$$

Here,

$f(n)$ indicates number of comparisons required to find the maximum and minimum of a sequence of size n.

The recurrence relation of merge sort is,

$$M(n) = 2M\left(\frac{n}{2}\right) + 2$$

3. Fast Multiplication of Integers

Fast multiplication algorithm breaks the two $2n$ bit integers into two blocks of size n bits. By this, the original multiplication of two $2n$ bit integers is changed to three multiplications of n bit integers together with addition, subtraction and shifts.

Let,

$$p = (p_{2n-1}, p_{2n-2}, \ldots, p_1 p_0)_2$$
$$q = (q_{2n-1}, q_{2n-2}, \ldots, q_1 q_0)_2$$

Also assume,

$$p = 2^n p_1 + p_0$$
$$q = 2^n q_1 + q_0$$
$$p_1 = (p_{2n-1} \ldots p_{n+1} p_n)_2$$
$$p_0 = (p_{n-1} \ldots p_1 p_0)_2$$
$$q_1 = (q_{2n-1} \ldots q_{n+1} q_n)_2$$
$$q_0 = (q_{n-1} \ldots q_1 q_0)_2$$

Consider pq,

$$pq = (2^n p_1 + p_0)(2^n q_1 + q_0)$$
$$\Rightarrow \quad pq = 2^{2n} p_1 q_1 + 2^n p_1 q_0 + 2^n p_0 q_1 + p_0 q_0 \qquad \ldots (1)$$

Adding and subtracting $2^n p_1 q_1$ and $2n\, p_0 q_0$

$$pq = 2^{2n} p_1 q_1 + 2^n p_1 q_1 - 2^n p_1 q_1 + 2^n p_1 q_0 + 2^n p_0 q_1 + 2^n p_0 q_0 - 2^n p_0 q_0 + p_0 q_0$$
$$= p_1 q_1(2^{2n} + 2^n) + 2^n p_1(q_0 - q_1) - 2^n p_0(-q_1 + q_0) + p_0 q_0(2^n + 1)$$

According to fast multiplication algorithm, pq can be written as,

$$pq = p_1 q_1(2^{2n} + 2^n) + 2^n(p_1 - p_0)(q_0 - q_1) + p_0 q_0(2^n + 1)$$

From equation (2), multiplication of two $2n$ bit integers is reduced to three multiplications of n bit integers.

Hence, if $f(n)$ represents the number of comparisons required for the multiplication of two n bit integers, then

$$f(2n) = 3f(n) + cn$$

Q70. How many comparisons are needed for a binary search in a set of 64 elements ?

Answer :

(a) Given that,

Number of elements = 64

Let $f(n)$ be the number of comparisons needed in a binary search of a list of n elements.

Then, $f(n) = f\left(\frac{n}{2}\right) + 2$; when n is even

$$\Rightarrow \quad f(64) = f\left(\frac{64}{2}\right) + 2$$

$= f(32) + 2$

$= f(16) + 4$

$= f(8) + 6$

$= f(4) + 8$

$= f(2) + 10 \qquad [\because f(1) = 2]$

$= f(1) + 12 = 2 + 12$

$= 14$

$\therefore f(64) = 14$

Q71. Suppose that f(n) = f(n/5) + 3n² when n is a positive integer divisible by 5 and f(1) = 4. Find

(a) f(5)

(b) f(125)

(c) f(3125).

Model Paper-3, Q9(a)

Answer :

Given that,

$$f(n) = f\left(\frac{n}{5}\right) + 3n^2$$

n is a positive integer divisible by 5

$f(1) = 4$

(a) $f(5) = f\left(\frac{5}{5}\right) + 3(5)^2$

$= f(1) + 3(25)$

$= 4 + 75 = 79$

$\therefore f(5) = 79$

(b) $f(25) = f\left(\frac{25}{5}\right) + 3(25)^2$

$= f(5) + 3(25)^2$

$= 79 + 1875$

$= 1954$

$\therefore f(25) = 1954$

(c) $f(3125) = f\left(\frac{3125}{5}\right) + 3(3125)^2$

$= f(625) + 3(3125)^2$

$f(625) = f(125) + 3(625)^2$

$f(125) = f(25) + 3(125)^2$

$= 1954 + 46875$

$= 48,829$

$\therefore f(625) = 48829 + 1171875 = 1220704$

$f(3125) = 1220704 + 29296875$

$= 30517579$

$\therefore f(3125) = 30517579$

Q72. Suppose that the function f satisfies the recurrence relation f(n) = 2f(√n) + 1 whenever n is a perfect square greater than 1 and f(2) = 1.

(a) Find f(16)

(b) Give a big - O estimate for f(n). [Hint : Make the substitution m = log n]

Answer :

Given that,

The function f satisfies the recurrence relation, $f(n) = 2f(\sqrt{n}) + 1$.

and $f(2) = 1$

(a) $f(16) = 2f(\sqrt{16} + 1)$

$= 2f(4) + 1$

$= 2\,[2f(2) + 1] + 2$

$= 2\,[2.1 + 1] + 1$

$= 7$

$\therefore f(16) = 7$

(b) Consider,

$f(n) = 2f(\sqrt{n} + 1)$

Let, $m = \log n$

$\Rightarrow n = 2^m$

$\therefore f(2^m) = 2f(\sqrt{2^m} + 1)$

From Master theorem,

$f(n) = O(n^{\log_b a})$ if $a > b^d$

Here, $a = 2, b = 2, c = 1, d = 1$

$\Rightarrow \quad f(2^m) = O(m \log_2 2)$

$= O(m)$

$\Rightarrow \quad f(n) = O(\log n)$

$\therefore$ The big-O estimate for $f(n)$ is $f(n) = O(\log n)$

Q73. Show that if a = bᵈ and n is a power of b, then

$f(n) = f(1)n^d + cn^d \log_b n$.

Answer :

Given that,

$a = b^d$

n is a power of b

$f(n) = f(1)n^d + cn^d \log_b n$

Let, $n = b^k$ then $k = \log_b n$

$f(b^k) = f(1)\,(b^k)^d + c(b^k)^d k$ [$\because$ From equation (1)]

For $k = 0$,

$\Rightarrow f(1) = f(1) + 0$

$\Rightarrow f(1) = f(1)$

$\therefore f(1)$ is true for $k = 0$

From the principle of mathematical induction,

$f(b^k) = f(1)\,(b^k)^d + c(b^k)^d k$ is true.

For $k = k + 1$

$\Rightarrow f(b^{k+1}) = f(1)\,(b^{k+1})^d + c(b^{k+1})^d (k+1)$

By using the recurrence relation for $f(n)$ in terms of $f(n/b)$,

$f(b^{k+1}) = b^d f(b^k) + c(b^{k+1})^d$

Consider,

$$\begin{aligned} b^d f(b^k) + c(b^{k+1})^d &= b^d\,[f(1)\,(b^k)^d + c(b^k)^d k] + c(b^{k+1})^d \\ &= f(1)b^{kd+d} + cb^{kd+d} + c(b^{k+1})^d \\ &= f(1)\,(b^{k+1})^d + c(b^{k+1})^d k + c(b^{k+1})^d \end{aligned}$$

$$f(b^{k+1}) = f(1)\,(b^{k+1})^d + c(b^{k+1})^d (k+1)$$

$\therefore f(b^{k+1})$ is true for $k = k + 1$

From the inductive hypothesis,

$f(n)$ is true $\forall\ k = 0, 1, 2, \ldots, n$.

$\therefore f(n) = f(1)n^d + cn^d \log_b n$

Q74. Show that if $a \neq b^d$ and n is a power of b, then $f(n) = C_1 n^d + C_2 n^{\log_b a}$, where $C_1 = b^d c/(b^d - a)$ and $C_2 = f(1) + b^d c/(a - b^d)$.

Answer : **Model Paper-1, Q9(b)**

Given that,

$a \neq b^d$

n is a power of b

$f(n) = C_1 n^d + C_2 n^{\log_b a}$

where, $C_1 = b^d c/(b^d - a)$

$C_2 = f(1) + b^d c/(a - b^d)$

For $n = 1$,

$$f(1) = C_1 + C_2$$

$$= \frac{b^d c}{b^d - a} + f(1) + \frac{b^d c}{a - b^d} = \frac{b^d c}{b^d - a} + f(1) - \frac{b^d c}{b^d - a}$$

Hence, $f(n)$ is true for $n = 1$

Suppose that the function $f(n)$ is true.

For $n = b^k$

$$f(n) = \frac{b^d c}{b^d - a} n^d + \left[f(1) + \frac{b^d c}{a - b^d}\right] n^{\log_b a}$$

For $n = b^{k+1}$,

$$f(n) = af\left(\frac{n}{b}\right) + cn^d$$

$$= a\left[\frac{b^d c}{b^d - a}\left(\frac{n}{b}\right)^d + \left(f(1) + \frac{b^d c}{a - b^d}\right)\left(\frac{n}{b}\right)^{\log_b a}\right] + cn^d$$

$$= \frac{b^d c}{b^d - a} \cdot n^d \cdot \frac{a}{b^d} + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a} + cn^d$$

$$= n^d\left[\frac{a_c}{b^d - a} + \frac{c(b^d - a)}{b^d - a}\right] + \left[f(1) + \frac{b^d c}{a - b^d}\right] n^{\log_b a}$$

$$= \frac{b^d c}{b^d - a} \cdot n^d + \left[f(1) + \frac{b^d c}{a - b^d}\right] n^{\log_b a}$$

$$= c_1 n^d + c_2 n^{\log_b a}$$

$\therefore f(n)$ is the true for $n = b^{k+1}$

$\therefore f(n) = c_1 n^d + c_2 n^{d \log_b a}$ is true for all values of n.

Q75. Define generating function and give its properties in binomial.

Answer :

Generating function is defined as a function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real number as the infinite series.

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

→ If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

(a) $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

(b) $f(x)\,g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} a_j b_{k-j}\right) x^k$

Generating functions are extended to binomial theorem.

→ If u is a real number and k is a non-negative integer, then the extended binomial coefficient is given by,

$$\binom{u}{k} = \begin{cases} \dfrac{u(u-1)\ldots\ldots(u-k+1)}{k!} & ; \text{if } k > 0 \\ 1 & ; \text{if } k = 0 \end{cases}$$

→ For $|x| < 1$, where x is a real number, the extended binomial theorem is given by,

$$(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Q76. List some of the generating functions.

Answer :

The following are some of the useful generating functions.

G(x)	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + ...+ x^n$	$C(n, k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)\, a^2x^2 + ...+ a^n x^n$	$C(n, k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)\, x^{2r} + ...+x^{rn}$	$C(n, k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + ... + x^n$	1 if $k \leq n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + ...$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + ...$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + ...$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + ...$	$k + 1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + ...$	$C(n+k-1, k) = C(n+k-1, n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)\, x + C(n+1,2)\, x^2 -...$	$(-1)^k C(n+k-1, k) = (-1)^k C(n+k-1, n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)\, a^2x^2 + ...$	$C(n+k-1, k)a^k = C(n+k-1, n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$	$1/k!$
$ln(1+k) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$	$(-1)^{k+1}/k$

Q77. Find the generating function for the finite sequence 1, 4, 16, 64, 256.

Answer :

Given finite sequence is,

1, 4, 16, 64, 256

The generating function is given by

$$G(x) = 1 + 4x + 16x^2 + 64x^3 + 256x^4$$

$$= 1 + 4x + (4x)^2 + (4x)^3 + (4x)^4$$

The generating function is obtained from the formula,

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Where, a and r are real numbers.

Here, $a = 1, n = 4, r = 4x$

$$G(x) = \frac{(4x)^{4+1} - 1}{4x - 1} = \frac{(4x)^5 - 1}{4x - 1}$$

$\Rightarrow$

Hence, the generating function is,

$$G(x) = \frac{(4x)^5 - 1}{4x - 1} \text{ where } |4x| > 1$$

Q72. Find a closed form for the generating function for the sequence $\{a_n\}$, where

(a) $a_n = 5$ for all $n = 0, 1, 2, \ldots$

(b) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$

(c) $a_n = 2$ for $n = 3, 4, 5, \ldots$ and $a_0 = a_1 = a_2 = 0$

(d) $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$

(e) $a_n = \binom{8}{n}$ for all $n = 0, 1, 2, \ldots$

(f) $a_n = \binom{n+4}{n}$ for all $n = 0, 1, 2, \ldots$

Model Paper-4, Q9(a)

Answer :

Given sequence is,

$\{a_n\}$

(a) $a_n = 5 \ \forall n = 0, 1, 2 \ldots$

The generating function for $a_n = 1$ is given as,

$$G(x) = \frac{1}{1-x}$$

For $a_n = 5$,

$$G(x) = \frac{5}{1-x}$$

Hence, the generating function is, $G(x) = \frac{5}{1-x}$.

(b) $a_n = 3^n \ \forall n = 0, 1, 2, \ldots$

The generating function for the given sequence is obtained from the relation,

$$\sum_{n=0}^{\infty} a^n x^n = 1 + ax + a^2x^2 + \ldots\ldots\ldots = \frac{1}{1-ax}$$

Here $a = 3$,

$$G(x) = \frac{1}{1-3x}$$

Hence the generating function is $G(x) = \frac{1}{1-3x}$

(c) $a_n = 2 \ \forall n = 3, 4, 5$ and $a_0 = a_1 = a_2 = 0$

The generating function for the sequence $a_n = 1$ is given as,

$$\Rightarrow \quad G(x) = \frac{1}{1-x}$$

For $n = 2$, $G(x)$ is of the form,

$$G(x) = \frac{1 + 2x + 2x^2 + 2x^3}{1-x}$$

But $a_0 = 9$, $a_1 = 0$

$$\Rightarrow \quad G(x) = \frac{0+0+0+2x^3}{1-x}$$

$$= \frac{2x^3}{1-x}$$

Hence, the generating function is, $G(x) = \frac{2x^3}{1-x}$

(d) $a_n = 2n + 3 \; \forall n = 0, 1, 2, \ldots\ldots$

Here, $2n + 3 = 2(n + 1) + 1$

The generating function for the sequence $(n + 1)$ is, $\frac{1}{(1-x)^2}$

The generating function for the sequence $a_n = 1$ is, $\frac{1}{1-x}$

$\therefore$ The generating function is, $G(x) = \frac{2}{(1-x)^2} + \frac{1}{1-x}$

$$\Rightarrow \quad G(x) = \frac{2+(1-x)}{(1-x)^2}$$

$$= \frac{2+1-x}{(1-x)^2}$$

$$= \frac{3-x}{(1-x)^2}$$

Hence, the generating function is, $G(x) = \frac{3-x}{(1-x)^2}$

(e) $a_n = \binom{8}{n} \; \forall n = 0, 1, 2\ldots\ldots$

From the definition of generating function,

$$G(x) = \binom{8}{0} + \binom{8}{1}x + \binom{8}{2}x^2 + \ldots\binom{8}{8}x^8 + 0x^9 + 0x^{10} + \ldots.$$

$$= \binom{8}{0} + \binom{8}{1}x + \binom{8}{2}x^2 + \ldots + \binom{8}{8}x^8$$

$$= \sum_{k=0}^{\infty}\binom{8}{k}x^k$$

$$= (1+x)^8 \qquad \left[\because \sum_{k=0}^{\infty}\binom{n}{k}x^k = (1+x)^n\right]$$

(f) $a_n = \binom{n+4}{n} \; \forall n = 0, 1, 2\ldots\ldots$

From the definition of generating function,

$$G(x) = \sum_{k=0}^{\infty}(n+k-1, k)x^k = 1 + c(n, 1)x + c(n+1, 2)x^2 + \ldots..$$

$$= \frac{1}{(1-x)^5}$$

Here, $n = 5$

$$\Rightarrow G(x) = \frac{1}{(1-x)^5}$$

$\therefore$ Hence the generating function is, $G(x) = \frac{1}{(1-x)^5}$

Q79. Find the coefficient of x^{10} in the power series of each of these functions

(a) $1/(1-2x)$

(b) $1/(1-x)^2$

(c) $1/(1-x)^3$

(d) $1/(1+2x)^4$

(e) $x^4/(1-3x)^3$.

Model Paper-5, Q9(a)

Answer :

(a) Given generating function is,

$$G(x) = \frac{1}{(1-2x)}$$

The expansion of above function is,

$$\frac{1}{(1-2x)} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} 2^n x^n$$

$\Rightarrow$ Coefficient of $x^{10} = 2^{10} = 1024$

Hence, the coefficient of x^{10} is, 1024

(b) Given generating function is,

$$G(x) = \frac{1}{(1+x)^2}$$

The expansion of above function is,

$$\frac{1}{(1+x)^2} = 1 - {}^2C_1 x + {}^{2+1}C_2 x^2 - \ldots\ldots$$

$$= \sum_{k=0}^{\infty} {}^{2+k-1}C_k (-1)^k x^k$$

$\Rightarrow$ Coefficient of $x^{10} = (-1)^{10}\, {}^{2+10-1}C_{10} x^{10}$

$$= {}^{11}C_{10} = 11$$

Hence, the coefficient of x^{10} is, 11.

(c) Given generating function is,

$$G(x) = \frac{1}{(1-x)^3}$$

The expansion of $\frac{1}{(1-x)^n}$ is given as,

$$\frac{1}{(1-x)^n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} {}^{k+n-1}C_k (-1)^k x^k$$

Coefficient of $x^{10} = {}^{3+10-1}C_{10}$

$$= {}^{12}C_{10} = 66$$

Hence, the coefficient of x^{10} is, 66

(d) Given generating function is,

$$G(x) = \frac{1}{(1+2x)^4}$$

The expansion of $\frac{1}{(1-ax)^n}$ is,

$$\frac{1}{(1-ax)^n} = \sum_{n=0}^{\infty} {}^{k+n-1}C_k (-1)^k x^k$$

Here $n = 4$, $k = 10$

$$\Rightarrow \frac{1}{(1+2x)^4} = \sum_{n=0}^{\infty} C(4+k-1, k)(-2)^k x^k$$

$\Rightarrow$ Coefficient of $x^{10} = (-2)^n C(n+3, 3)$

$$= (-2)^{10} C(10+3, 3)$$

$$= (-2)^{10}\, {}^{13}C_3 = 292864$$

$\therefore$ Coefficient of x^{10} is, 292864

(e) Given generating function is,

$$G(x) = \frac{x^4}{(1-3x)^3}$$

Since x^4 is present, the coefficient of x^6 gives the coefficient of x^{10}

i.e., $x^{10} = x^4 . x^6$

The expansion of $\frac{1}{(1-a_x)^n}$ is,

$$\frac{1}{(1-ax)^n} = \sum_{n=0}^{\infty} C(k+n-1, k) a^k x^k$$

Here, $n = 3$, $k = 6$

Coefficient of $x^6 = C(6+3-1, 6)(3)^6$

$$= {}^8C_6 3^6$$

$$= \frac{8!}{2!6!} 729$$

$$= 20{,}412$$

$\therefore$ Coefficient of x^{10} is, 20,412

Q80. (a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 = k$ when x_1, x_2 and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

(b) Use your answer to part (a) to find a_6.

Answer :

(a) Given sequence is,

$\{a_k\}$

a_k is the number of solutions of $x_1 + x_2 + x_3 = k$

where $x_1 \geq 2$, $0 \leq x_2 \leq 3$, $2 \leq x_3 \leq 5$

Let $G(x)$ be the generating function.

For $x_1 \geq 2$,

The corresponding term in $G(x)$ is,

$\Rightarrow x^2 + x^3 + x^4 + \ldots\ldots$ ...(1)

For $0 \leq x_2 \leq 3$

The corresponding term in $G(x)$ is

$1 + x + x^2 + x^3$ ...(2)

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For $2 \leq x_3 \leq 5$

The corresponding term in $G(x)$ is,

$x^2 + x^3 + x^4 + x^5$

The product of these 3 equations gives the generating function, i.e.,

$$G(x) = (x^2 + x^3 + x^4 + ...)\,(1 + x + x^2 + x^3)\,(x^2 + x^3 + x^4 + x^5)$$

$$= x^2\,(1 + x + x^2 + ...)\,(1 + x + x^2 + x^3)\,x^2(1 + x + x^2 + x^3)$$

$$= x^4\,(1 + x + x^2 + ...)\,(1 + x + x^2 + x^3)^2$$

$$\therefore \quad G(x) = \frac{x^4(1+x+x^2+x^3)^2}{1-x} \qquad \left[\because\ 1+x+x^2+.........=\frac{1}{1-x}\right]$$

Hence, the generating function is, $G(x) = \dfrac{x^4(1+x+x^2+x^3)^2}{1-x}$.

(b) Consider the generating function,

$$G(x) = \frac{x^4(1+x+x^2+x^3)^2}{1-x}$$

$\because$ x^4 is present, the coefficient of x^6 is the coefficient of x^2 in the expansion

$$\Rightarrow \quad \frac{(1+x+x^2+x^3)^2}{(1-x)} = \frac{1+2x+3x^2+\text{terms of higher order}}{1-x}$$

$$a_6 = 1 + 2 + 3 = 6$$

$$\therefore \quad a_6 = 6$$

Q81. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

Answer :

(a) Given recurrence relation is,

$$a_k = 3a_{k-1} + 2$$

$$a_0 = 1$$

Let, $G(x) = \sum\limits_{k=0}^{\infty} a_k x^k$

Multiplying by x on both sides,

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$$

$$= \sum_{k=1}^{\infty} a_{k-1} x^k$$

Consider,

$$G(x) - 3x\,G(x) = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1} x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k$$

$$= a_0 + \sum_{k=1}^{\infty} 2x^k$$

$$= 1 + \frac{2}{1-x} - 2 = \frac{2}{1-x} - 1$$

$$= \frac{1+x}{1-x}$$

Applying the initial condition $a_0 = 1$

$$\sum_{k=0}^{\infty} 2x^k = \frac{2}{1-x}$$

Thus,

$$G(x)(1-3x) = \frac{1+x}{1-x}$$

$$\Rightarrow \quad G(x) = \frac{1+x}{(1-x)(1-3x)}$$

Applying partial fractions,

$$\frac{1+x}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

$A = 2$ and $B = -1$

$$\therefore \quad G(x) = \frac{2}{1-3x} + \frac{-1}{1-x}$$

$$= 2(1-3x)^{-1} - (1-x)^{-1}$$

$$= 2\sum_{k=0}^{\infty}(3x)^k - \sum_{k=0}^{\infty} x^k$$

$$= 2\sum_{k=0}^{\infty} 2.3^k\, n^k - \sum_{k=0}^{\infty} x^k$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty}(2.3^k - 1)\, x^k$$

Comparing the coefficients of x^k on both sides, $a_k = (2.3^k - 1)$

$\therefore \quad a_k = 2.3^k - 1$

4.2.3 Inclusion-Exclusion, Applications of Inclusion-Exclusion

Q82. State and prove principle of inclusion-exclusion.

Answer : *Model Paper-2, Q9(b)*

The sum rule which is applied to the non-disjoint sets is called 'principle of inclusion-exclusion'. This principle is also called as 'Sieve method'. Principle of inclusion and exclusion is based on the following two statements.

1. **Statement**

If A and B are two subsets of any set S(universal) then,

$$|A \cup B| = |A| + |B| - |A \cap B| \quad \ldots(1)$$

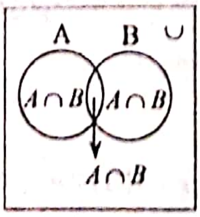
Proof

Given that,

For any set S

$A, B \leq S$

i.e., A and B are subsets of S



Figure(1): Venn Diagram

$|A \cup B|$ = All the elements of A and B

$|A| = |A \cap \bar{B}| + |A \cap B|$ [∵ From Venn diagram]

$|B| = |\bar{A} \cap B| + |A \cap B|$ [∵ From Venn diagram]

Similarly,

$A \cap B$ contains the elements which belong to both sets A and B.

$$\therefore \quad |A| + |B| = |A \cap \bar{B}| + |A \cap B| + |\bar{A} \cap B| + |A \cap B|$$

$$|A| + |B| = |A \cap \bar{B}| + 2|A \cap B| + |\bar{A} \cap B|$$

Subtracting $|A \cap B|$ from both the sides of above equation,

$$|A| + |B| - |A \cap B| = |A \cap \bar{B}| + 2|A \cap B| + |\bar{A} \cap B| - |A \cap B|$$

$$= |A \cap \bar{B}| + |\bar{A} \cap B| + |A \cap B| \qquad [\because \text{From Venn diagram}]$$

$$= |A \cup B|$$

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

2. **Statement**

If A, B, C are any three subsets of set S then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Proof

Given A, B, C are three subsets of set S

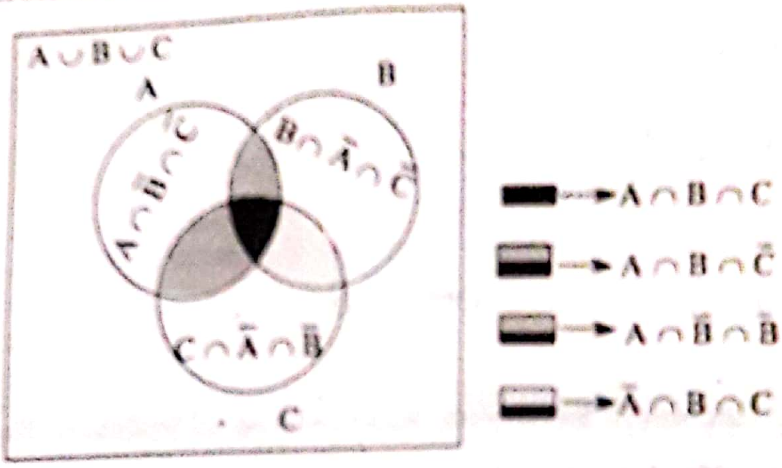


Figure (2): Venn Diagram

$$|A| = |A \cap \bar{B} \cap \bar{C}| + |A \cap B \cap \bar{C}| + |A \cap C \cap \bar{B}| + |A \cap B \cap C| \qquad [\because \text{From Venn diagram}]$$

$$|B| = |B \cap \bar{A} \cap \bar{C}| + |A \cap B \cap \bar{C}| + |\bar{A} \cap B \cap C| + |A \cap B \cap C| \qquad [\because \text{From Venn diagram}]$$

$$|C| = |C \cap \bar{A} \cap \bar{B}| + |\bar{A} \cap B \cap C| + |A \cap C \cap \bar{B}| + |A \cap B \cap C| \qquad [\because \text{From Venn diagram}]$$

$$\therefore |A| + |B| + |C| = |A \cap \bar{B} \cap \bar{C}| + |B \cap \bar{A} \cap \bar{C}| + |C \cap \bar{A} \cap \bar{B}| + |A \cap B \cap \bar{C}| + |A \cap C \cap \bar{B}| +$$
$$|\bar{A} \cap B \cap C| + |A \cap B \cap \bar{C}| + |A \cap C \cap \bar{B}| + |\bar{A} \cap B \cap C| + |A \cap B \cap C| +$$
$$|A \cap B \cap C| + |A \cap B \cap C|$$

From Venn diagram,

$$|A \cup B \cup C| = |A \cap \bar{B} \cap \bar{C}| + |A \cap B \cap \bar{C}| + |A \cap C \cap \bar{B}| + |\bar{A} \cap \bar{C} \cap B| + |B \cap C \cap \bar{A}|$$
$$+ |\bar{A} \cap \bar{B} \cap C| + |A \cap B \cap C|$$

Substituting equation (3) in equation (2),

$$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B \cap \bar{C}| + |A \cap B \cap C| + |\bar{A} \cap B \cap C| + |A \cap \bar{B} \cap C| + |A \cap B \cap C|$$

$$\Rightarrow |A| + |B| + |C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap \bar{B} \cap C|$$

$$[\because |A \cap B \cap C| + |\bar{A} \cap B \cap C| = |B \cap C| \text{ and } |A \cap B \cap \bar{C}| + |A \cap B \cap C| = |A \cap B|]$$

Adding $|A \cap B \cap C|$ on both sides of equation (4),

$$|A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap \bar{B} \cap C| + |A \cap B \cap C|$$

$$\Rightarrow |A| + |B| + |C| + |A \cap B \cap C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |C \cap A|$$

$$[\because |A \cap \bar{B} \cap C| + |A \cap B \cap C| = |C \cap A|$$

$$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Q83. State and prove generalized theorem of principle of inclusion-exclusion.

Answer :

General Statement of the Principle of Inclusion-exclusion

If A_i are finite subsets of a universal set $\cup$, then,

$$|A_1 \cup A_2 \cup \ldots\ldots \cup A_n| = \sum_{i=0}^{n} |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n| \quad \ldots (1)$$

Here, the second summation is taken over all 2-combinations $\{i, j\}$ of the integers $\{1, 2, \ldots\ldots n\}$, and the third summation is taken over all 3-combinations $\{i, j, k\}$ of $\{1, 2, \ldots\ldots n\}$ and so on.

Consider the equation,

$$\sum |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap A_3 \ldots\ldots \cap A_n|$$

Proof

This theorem can be proved by mathematical induction. Let the theorem be true for $n = 1$. Assume the theorem to be true for any n subsets of S. Suppose, that there exist $n + 1$ sets for 2 sets repeatedly in the proof.

Also consider $A_1 \cup A_2 \cup \ldots\ldots \cup A_n \cup A_{n+1}$ as the union of the sets $A_1 \cup A_2 \cup \ldots\ldots \cup A_n$ and A_{n+1}.

Then $|(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) \cup A_{n+1}| = |A_1 \cup \ldots\ldots \cup A_n)| + |A_{n+1}| \quad \ldots (2)$

As the intersection is distractive over unions, it can be written as follows,

$$|A_1 \cup A_2 \cup \ldots \cup A_n) \cap A_{n+1}| = |(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \ldots \cup (A_n \cap A_{n+1})| \quad \ldots (3)$$

Applying induction to 2 of the 3 sets in equation (3),

$$|A_1 \cup A_2 \cup \ldots\ldots \cup A_n| = \sum_{i,j \le n} |A_i \cap A_j| + \sum_{i,j \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} . |A_1 \cap A_2 \cap \ldots\ldots \cap A_n| \quad \ldots (4)$$

$$= |(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \ldots\ldots \cup (A_n \cap A_{n+1})|$$

$$= \sum_{i=1}^{n} |A_i \cap A_{n+1}| - \sum_{i,j} |(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})| + \sum_{i,j \le n} |A_i \cap A_{n+1}| \cap (A_j \cap A_{n+1}) \cap (A_k \cap A_{n+1})|$$

$$+ \ldots + (-1)^{n-1} |A_1 \cap A_{n+1} \cap (A_2 \cap A_{n+1}) \cap \ldots\ldots \cap (A_n \cap A_{n+1})| \quad \ldots (5)$$

Substituting equations (4) and (5) in equation (2),

$$(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1} \dot{\cup} (A_n \cup A_{n+1}) = A_i \cap A_j \cap A_k \cap A_{n+1},$$

$$- \sum_{i=1}^{n} |A_i \cap A_{n+1}| + \sum_{i,j \le n} |A_i \cap A_j \cap A_{n+1}| + (-1)\, n - 1 |A_1 \cap A_2 \cap \ldots A_i| + |A_{n+1} + \ldots\ldots + (-1)\, n - 1|$$

$$A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n \cap A_{n+1}| \quad \ldots (6)$$

In $\left(\sum_{i,j \le n} |A_i \cap A_j| + \sum_{i=1}^{n} A_i \cap A_{n+1}| \right)$, the first sum is taken over j, the 2-combinations $\{i, j\}$ of $\{1, 2, \ldots\ldots n\}$, and the second sum is taken over the i-combination of the form $\{i, n + 1\}$, where $i \in \{1, 2, \ldots. n\}$ can be simplified to $\Sigma A_i \cap A_j$ where this sum is taken over all 2-combinations of $\{1, 2, \ldots. n, n + 1\}$, since it is the set of all 2-combinations of $\{1, 2 \ldots\ldots n, n + 1\}$. Similarly, $i \cup j$ sum the sums of $\Sigma |A_i \cap A_j \cap A_k| + \Sigma |A_i \cap A_j \cap A_{n+1}|$

Where the first sum is taken over all 3-combinations of $\{1, 2, \ldots n\}$, and the second is taken over 3-combinations of the form $\{i, j, n + 1\}$ of $\{1, 2, \ldots\ldots n, n + 1\}$, can be simplified to $\Sigma_{i,j,k}$ $|A_i \cap A_j \cap A_k|$ where this sum is taken over all 3-combinations of $\{1, 2 \ldots\ldots n, n + 1\}$

$\therefore$ Equation (6) becomes,

$$|A_1 \cup A_2 \dot{\cup} \ldots\ldots \cup A_n \cup A_{n+1}| = \sum_{i=1}^{n+1} |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \ldots\ldots + (-1)^n |A_1 \cap A_2 \ldots\ldots \cap A_{n+1}| \quad \ldots (7)$$

Where the second sum is taken over all 2-combinations of $\{1, 2, ..., n+1\}$, the third sum is taken at 3-combinations of $\{1, 2, ..., n, n+1\}$ and so on. Equation (7) = Equation (1) with $n = n+1$.

$\therefore$ The theorem is true for $n = n+1$ by the principle of mathematical induction.

From De-Morgan's law,

$$\overline{(A_1 \cup A_2 \cup A_3 ... \cup A_{n-1} \cup A_n)} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap ... \bar{A}_{n-1} \cap \bar{A}_n$$

As $|\bar{A}| = |U| - |A|$ if $A \subseteq U$, then

$$|\bar{A}_1 \cap \bar{A}_2 \cap ... \bar{A}_n| = \overline{(A_1 \cup A_2 \cup A_3 \cup ... A_n)}$$

$$= |U| - |(A_1 \cup A_2 \cup ... \cup A_n)|$$

Substituting equation (7) in the above equation, the following equation is obtained.

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap ... \cap \bar{A}_n| = |U| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$$

The above equation is another version of principle of inclusion and exclusion.

Q84. How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2 and

a. $A_1 \cap A_2 = \phi$? **b. $|A_1 \cap A_2| = 1$?**

c. $|A_1 \cap A_2| = 6$? **d. $A_1 \subseteq A_2$?**

Answer :

Given that,

$|A_1| = 12$

$|A_2| = 18$

From Inclusion-Exclusion principle

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$= 12 + 18 - |A_1 \cap A_2|$$

$$= 30 - |A_1 \cap A_2|$$

a. $A_1 \cap A_2 = \phi$

$\Rightarrow$ $|A_1 \cap A_2| = 0$

$\Rightarrow$ $|A_1 \cap A_2| = 30 - 0 = 30$

$\therefore$ $|A_1 \cup A_2| = 30$

b. $|A_1 \cap A_2| = 1$

$\Rightarrow$ $|A_1 \cup A_2| = 30 - 1 = 29$

$\therefore$ $|A_1 \cup A_2| = 29$

c. $|A_1 \cap A_2| = 6$

$\Rightarrow$ $|A_1 \cup A_2| = 30 - 6 = 24$

$\therefore$ $|A_1 \cup A_2| = 24$

d. $A_1 \subseteq A_2$

If $A_1 \subseteq A_2$ then, $A_1 \cap A_2 = A_1$

$\Rightarrow$ $|A_1 \cap A_2| = |A_1| = 12$

$\Rightarrow$ $|A_1 \cup A_2| = 30 - 12 = 18$

$\therefore$ $|A_1 \cup A_2| = 18$

Q85. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1, 1000 in A_2, and 10, 000 in A_3 if,

(a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$

(b) the sets are pairwise disjoint.

(c) there are two elements common to each pair of sets and one element in all three sets.

Model Paper-3, Q9(b)

Answer :

Given that,

$|A_1| = 100$

$|A_2| = 1000$

$|A_3| = 10,000$

(a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$

$\Rightarrow$ $A_1 \cap A_2 = A_1$ and $A_2 \cap A_3 = A_2$

$\Rightarrow$ $A_1 \cap A_3 = A_1$ $[\because A_1 \subseteq A_3]$

From Inclusion–Exclusion principle,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$$

$$= 100 + 1000 + 10,000 - 100 - 1000 - 100 + 100$$

$$= 10,000$$

$\therefore$ $|A_1 \cup A_2 \cup A_3| = 10,000$

(b) If the set are pairwise disjoint, then the cardinality of the union is the sum of cardinalities

i.e., $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$

$$= 100 + 1000 + 10,000$$

$$= 11,100$$

$\therefore$ $|A_1 \cup A_2 \cup A_3| = 11,100$

(c) $|A_1 \cap A_2| = |A_2 \cap A_3| = 2$

$|A_1 \cap A_2 \cap A_3| = 1$

From Inclusion–Exclusion principle,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 100 + 1000 + 10,000 - 2 - 2 - 2 + 1$$

$$= 11,095$$

$\therefore$ $|A_1 \cup A_2 \cup A_3| = 11095$

Q86. How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?

Answer :

Given that,

$A_1 = 50$

$A_2 = 60$

$A_3 = 70$

$A_4 = 80$

$A_1 \cap A_2 = 5$

$A_2 \cap A_3 = 5$

$A_3 \cap A_4 = 5$

$A_1 \cap A_3 = 5$

$A_1 \cap A_4 = 5$

$A_2 \cap A_4 = 5$

$A_1 \cap A_2 \cap A_3 = 1$

$A_1 \cap A_3 \cap A_4 = 1$

$A_2 \cap A_3 \cap A_4 = 1$

$A_1 \cap A_2 \cap A_4 = 1$

$A_1 \cap A_2 \cap A_3 \cap A_4 = 0$

By Inclusion–Exclusion principle,

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

Substituting the corresponding values in above equation,

$$= 50 + 60 + 70 + 80 - 6(5) + 4(1) - 0 = 234.$$

$\therefore \quad |A_1 \cup A_2 \cup A_3 \cup A_4| = 234$

Q87. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, either all are odd, all are divisible by 3, or all are divisible by 5.

Model Paper-4, Q9(b)

Answer :

The probability that 4 numbers chosen from 1 to 100 has ${}^{100}C_4$ ways.

There are 50 odd numbers from 1 to 100.

The probability of choosing 4 odd numbers is ${}^{50}C_4$.

$\therefore \quad$ Total probability, $\dfrac{|A_1|}{|S|} = \dfrac{{}^{50}C_4}{{}^{100}C_4}$

Similarly, there are 33 multiples of 3 from 1 to 100.

The probability of choosing 4 numbers is $\dfrac{|A_2|}{|S|} = \dfrac{{}^{33}C_4}{{}^{100}C_4}$

There are 20 multiples of 5 from 1 to 100.

The probability of choosing 4 numbers, $\dfrac{|A_3|}{|S|} = \dfrac{{}^{20}C_4}{{}^{100}C_4}$

Numbers that are both odd & divisible by 3 = 17

Probability $\dfrac{|A_1 \cap A_2|}{|S|} = \dfrac{{}^{17}C_4}{{}^{100}C_4}$

Numbers that are both odd & divisible by 5 = 10.

Probability $\dfrac{|A_2 \cap A_3|}{|S|} = \dfrac{{}^{10}C_4}{{}^{100}C_4}$

Numbers that are both divisible by 3 and 5 = 6

Probability $\dfrac{|A_1 \cap A_3|}{|S|} = \dfrac{{}^{6}C_4}{{}^{100}C_4}$

Finally, the numbers satisfying all the 3 conditions $\dfrac{|A_1 \cap A_2 \cap A_3|}{|S|} = 0$

Since, there are only 3 numbers, as we have to choose 4, so the probability is 0.

$\therefore \quad$ By Inclusion-Exclusion principle,

$$P|A_1 \cup A_2 \cup A_3| = \frac{|A_1|}{|S|} + \frac{|A_2|}{|S|} + \frac{|A_3|}{|S|} - \frac{|A_1 \cap A_2|}{|S|} - \frac{|A_2 \cap A_3|}{|S|} - \frac{|A_1 \cap A_3|}{|S|} - \frac{|A_1 \cap A_2 \cap A_3|}{|S|}$$

$$= \frac{^{50}C_4}{^{100}C_4} + \frac{^{33}C_4}{^{100}C_4} + \frac{^{20}C_4}{^{100}C_4} - \frac{^{17}C_4}{^{100}C_4} - \frac{^{10}C_4}{^{100}C_4} - \frac{^{6}C_4}{^{100}C_4} + 0$$

$$= \frac{230300 + 40920 + 4845 - 2380 - 210 - 15}{3921225}$$

$$= \frac{273460}{3921225}$$

$$= \frac{4972}{71295}$$

$$\simeq 0.0697$$

$$\therefore \quad P(A_1 \cup A_2 \cup A_3) = 0.0697$$

Q88. State the applications of inclusion-exclusion principle.

Answer :

Inclusion-exclusion principle is used in solving the counting problems. There are several counting problems out of these problems some of the most famous counting problems are as follows,

(a) The sieve of eratosthenes

(b) The number of onto functions

(c) Derangements.

(a) The Sieve of Eratosthenes

The sieve of eratosthenes is used to identify the number of primes less than a given positive integer.

Example

To find the primes less than a given positive integer the principle of inclusion -exclusion can be used. This can be computed by considering a composite integer which is divisible by a prime and is less than its square root.

Let,

Positive integer = 100

Composite integers less than 100 must have a prime factor less than 10 (square root of 100).

The series of primes that should not exceed 10 are 2, 3, 5 and 7. Hence, it can be said that the obtained primes (2, 3,5 and 7) are less than the given positive integer (i.e., 100)

$$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = U - |A \cup B \cup C \cup D| \quad \text{... (1)}$$

Where,

U = Set of all integers greater than 1 and less than 100

$\therefore \quad U = 99$

A = A positive integer divisible by 2

B = A positive integer divisible by 3

C = A positive integer divisible by 5

D = A positive integer divisible by 7.

Expanding equation (1)

$$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = U - |A| - |B| - |C| - |D| + |A \cap B| + |A \cap C| + |B \cap C| + |A \cap D| + |B \cap D| + |C \cap D|$$
$$- |A \cap B \cap C| - |A \cap B \cap D| - |B \cap C \cap D| - |A \cap C \cap D| + |A \cap B \cap C \cap D| \quad \text{... (2)}$$

$|A|$ = Integers divisible by 2 $= \left[\frac{100}{2}\right] = 50$

$|B|$ = Integers divisible by 3 $= \left[\frac{100}{3}\right] = 33$

$|C|$ = Integers divisible by 5 = $\left[\frac{100}{5}\right] = 20$

$|D|$ = Integers divisible by 7 = $\left[\frac{100}{7}\right] = 14$

$|A \cap B|$ = Integers divisible by 2 and 3 = $\left[\frac{100}{2.3}\right] = 16$

$|A \cap C|$ = Integers divisible by 2 and 5 = $\left[\frac{100}{2.5}\right] = 10$

$|A \cap D|$ = Integers divisible by 2 and 7 = $\left[\frac{100}{2.7}\right] = 7$

$|B \cap C|$ = Integers divisible by 3 and 5 = $\left[\frac{100}{3.5}\right] = 6$

$|B \cap D|$ = Integers divisible by 3 and 7 = $\left[\frac{100}{3.7}\right] = 4$

$|C \cap D|$ = Integers divisible by 5 and 7 = $\left[\frac{100}{5.7}\right] = 2$

$|A \cap B \cap C|$ = Integers divisible by 2, 3 and 5 = $\left[\frac{100}{2.3.5}\right] = 3$

$|A \cap B \cap D|$ = Integers divisible by 2, 3 and 7 = $\left[\frac{100}{2.3.7}\right] = 2$

$|B \cap C \cap D|$ = Integers divisible by 3, 5 and 7 = $\left[\frac{100}{3.5.7}\right] = 0$

$|A \cap C \cap D|$ = Integers divisible by 2, 5 and 7 = $\left[\frac{100}{2.5.7}\right] = 1$

$|A \cap B \cap C \cap D|$ = Integers divisible by 2, 3, 5 and 7 = $\left[\frac{100}{2.3.5.7}\right] = 0$

Substituting all the above values in equation (2),

$$|\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}| = 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 0 - 1 + 0$$

$$|\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}| = 21$$

The series of primes (2, 3, 5 and 7) less than 100 is expressed as,

$$4 + |\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}|$$

$\Rightarrow \quad 4 + 21 = 25$

$\therefore$ There are 25 prime less than the given positive integer 100.

(b) The Number of Onto Functions

The number of onto functions from a finite set of 'r' elements to a finite set of 'S' elements can be determined using the principle of inclusion exclusion.

Example

Let the elements in the codomain be $a_r = a_1, a_2, a_3$ and A, B and C be the properties such that a_1, a_2 and a_3 are not in a function range. A function is said to be an onto function if it does not have any of the properties such as A, B and C.

According to the principle of inclusion -exclusion the number of onto functions from a set of 6 elements to a set of 3 elements is,

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = U - [|A| + |B| + |C|] + [|A \cap B| + |A \cap C| + |B \cap C| - A \cap B \cap C \quad \text{...(1)}$$

Where,

U = Total number of functions from a set of 6 elements to one with 3 elements

$\therefore \quad U = 3^6$

$|A| + |B| + |C|$ = The number of functions that does not have 'a_i' in the function range i.e., 2^6. Thus, there are C(3, 1) elements of this type.

$\therefore \quad |A| + |B| + |C| = C(3, 1)\, 2^6$

$|A \cap B| + |A \cap C| + |B \cap C|$ = The number of functions that does not have 'a_i' and 'a_j' in the function range i.e., $1^6 = 1$. Thus, there are $C(3, 2)$ elements of this type.

$$|A \cap B| + |A \cap C| + |B \cap C| = C(3, 2)\ 1^6$$

$\therefore$ $|A \cap B \cap C| = 0$ Since, the number of functions that does not have any of $a_r(a_1, a_2, a_3)$ in the function range

Substituting the above values in equation (1),

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 3^6 - C(3, 1)\ 2^6 + C(3, 2)\ 1^6$$

$$= 729 - ({}^3C_1)\ 2^6 + ({}^3C_2)\ 1^6$$

$$= 729 - (3)\ 64 + (3)1$$

$$= 729 - 192 + 3$$

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 540$$

$\therefore$ The number of onto functions from a set of 6 elements to a set of 3 elements is 540.

(c) Derangements

Derangement is a process of scrambling objects in such a way that none of the objects are placed in its right position. The concept of inclusion-exclusion principle can be used to count the permutation of 'k' objects, such that none of the objects are placed at their respective positions.

Example

The Hat Check Problem

An employee collects the hats of 'k' people at a mall and these collected hats get scrambled (mixed). When the customer asks for their own hats, the employee returns the hat which is randomly picked from the rest of the hats. The probability of how many customers get their own hat back is not known. The solution to this problem is the number of ways in which the hats are placed such that none of the hats are at there right position divided by $k!$ and the number of permutations of 'k' hats.

Q89. Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?

Answer :

Let,

A_1 = Apples that have worms

A_2 = Apples that have bruises

$|A_1 \cup A_2| = 100$

$|A_1| = 20; |A_2| = 15$

$|A_1 \cap A_2| = 10$

From Inclusion-Exclusion principle,

$$|A_1 \cup A_2| - |A_1| - |A_2| + |A_1 \cap A_2| = 100 - 20 - 15 + 10$$

$$= 75$$

$\therefore$ 75 out of 100 apples can be sold.

Q90. How many onto functions are there from a set with seven elements to one with five elements?

Answer :

Let m and n are positive integer with $m = n$

The number of onto function from a set with m elements to a set with n elements is given as,

$$n^m - C(n, 1)\ (n-1)^m + C(n, 2)(n-2)^m + \ldots (-1)^{n-1}\ C(n, n-1).\ 1^m$$

Here, $n = 5$

$m = 7$

Number of onto functions from a set with seven elements to a set with five elements,

$= 5^7 - C(5, 1)4^7 + C(5, 2)3^7 - C(5, 3)2^7 + C(5, 4)1^7$

$= 78125 - 5(16384) + 10(2187) - 10(128) + 5(1)$

$= 100000 - 83200$

$= 16800$

$\therefore$ 16800 onto functions are there from seven to five.

Q91. In how many ways can seven different jobs be assigned to four different employees so that each employee is assigned at least one job and the most difficult job is assigned to best employee?

Model Paper-3, Q9

Answer :

Given that,

Number of elements = 7

The number of derangements of a set with n elements is given as,

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + (-1)^n \frac{1}{n!}\right]$$

Here, $n = 7$

The number of derangements of a set with 7 elements is

$$D_7 = 7!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right]$$

$= 5040 - 5040 + 2520 - 840 + 210 - 42 + 7 - 1$

$= 1854$

$\therefore$ $D_7 = 1854$

EXERCISE QUESTIONS

1. **What is the probability that a fair die comes up six when it is rolled?**

   Ans : [illegible]

2. **What is the probability that two people chosen at random were born during the same month of the year?**

   Ans : $\frac{1}{12}$

3. **What is the expected number of times a 6 appears when a fair die is rolled 10 times?**

   Ans : [illegible]

4. **Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.**

   Ans : $a_n = (n^2 + 3n + 5)(-1)^n$

5. **How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?**

   Ans : 50,13[illegible]

6. **How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where x_1, x_2 and x_3 are nonnegative integers less than 6?**

   Ans : [illegible]