UNIT 1

THE FOUNDATIONS: LOGIC AND PROOFS

SIA GROUP

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Define proposition.

Model Paper-I, Q1(a)

Answer :

Proposition

A proposition is defined as a declarative sentence which can either be true or false, but it cannot be both. In other words a proposition is a sentence which is used for declaring a fact. It is also referred as a statement.

Examples

1. New Delhi is the capital of India.
2. Sun rises from the west.

Here, the proposition 1 is true, whereas 2 is false.

Q2. Write short notes on,

(i) Formal proof

(ii) Sound argument

(iii) Valid conclusion

(iv) Valid argument.

Answer :

(i) **Formal Proof**

The proof in which a conclusion is derived from a set of premises by applying certain implication equivalence of reasoning is known as formal proof or deduction.

(ii) **Sound Argument**

An argument is said to be a sound argument if,

(a) Conclusions are accepted to be true when the set of premises are accepted to be true.

(b) Reasoning that is applied in deducing the conclusion follows certain rule of inferences.

(iii) **Valid Conclusion**

The conclusion that is derived using rules of inference is called a valid conclusion.

(iv) **Valid Argument**

The argument that use rules of inference inorder to derive a conclusion is called as a valid argument.

Q3. Write the contrapositive of the implication. "If it is sunday then it is a holiday".

Answer :

Given statement is,

"If it is sunday then it is a holiday".

Let,

$P(x)$ denotes if it is sunday

$Q(x)$ denotes it is a holiday

Symbolically the given statement can be written as, $P \to Q$

The contrapositive of the symbolic form is given as,

$\sim Q \to \sim P$

Which means, "If it is not a holiday, then it is not sunday".

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Q4. Find the truth table for p → q.

Answer :

The truth table for $p \to q$ is as follows,

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Table: Truth Table

The conditional statement $p \to q$ takes the truth value false only when p is true and q is false otherwise, it is always true.

Q5. Construct the truth table for P → ~Q.

Answer :

The truth table for $P \to \sim Q$ is as follows,

P	Q	~Q	P → ~Q
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

Table: Truth Table

Q6. Construct the truth table for the following P ∧ (P ∨ Q).

Answer : Model Paper-2, Q1(a)

Given that,

$P \wedge (P \vee Q)$.

The truth table for $P \wedge (P \vee Q)$ is as follows,

P	Q	P ∨ Q	P ∧ (P ∨ Q)
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Table: Truth Table

∴ It can be observed that $P \wedge (P \vee Q)$ is equivalent to P.

Q7. Give the truth value of T ↔ T ∧ F.

Answer :

Given that,

$T \leftrightarrow T \wedge F$

$= T \leftrightarrow F$

[∵ From conjunction of two propositions]

$= F$ [∵ From biconditional $p \leftrightarrow q$]

∴ The truth value of $T \leftrightarrow T \wedge F = F$.

Q8. Define Tautology with an example.

Answer : Model Paper-3, Q1(a)

Tautology

A statement formula is said to be a tautology or universally valid formula if all the resultant truth values are true, for every possible truth values of the statement variables.

Examples

(i) $p \vee \sim p$

(ii) $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

(iii) $[(p \wedge \sim q) \to r] \to [p \to (q \vee r)]$.

Q9. Show that p → q and ¬p∨q are logically equivalent.

Answer :

Given compound propositions are,

$p \to q$

$\neg p \vee q$

The truth table for the above propositions is,

p	q	¬p	p → q	¬p ∨ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It can be seen from the truth table, that the truth values of $p \to q$ and $\neg p \vee q$ are same.

Hence, $p \to q$ and $\neg p \vee q$ are logically equivalent.

Q10. What are the contrapositive, the converse and the inverse of the conditional statement "The home team wins whenever it is raining."?

Answer : Model Paper-4, Q1(a)

Given statement is,

"The home team wins whenever it is raining."

Contrapositive

"If the home team doesn't win, then it is not raining."

Converse

"If the home team wins, then it is raining."

Inverse

"If it is not raining, then the home team doesn't win."

Q11. What is the negation of each of these propositions?

(a) Today is Thursday

(b) There is no population in New Jersey

(c) 2 + 1 = 3

(d) The summer in Maine is hot and sunny.

Answer :

(a) Given proposition is,

Today is Thursday

Negation is,

Today is not Thursday

Given proposition is,

There is no population in New Jersey

Negation is,

There is population in New Jersey

Given proposition is,

$2+1=3$

Negation is,

$2+1\neq 3$

Given proposition is,

The summer in Maine is hot and sunny

Negation is,

The summer in Maine is not hot or it is not sunny.

Q2. Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

Model Paper-1, Q1(b)

Answer :

Given compound proposition is,

$(p \vee \neg q) \rightarrow (p \wedge q)$

The truth table for the above proposition is,

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Q3. How many rows appear in a truth table for each of these compound propositions ?

(a) $(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$

(b) $(p \vee \neg t) \wedge (p \vee \neg s)$

(c) $(p \rightarrow r) \vee (\neg s \rightarrow \neg t) \vee (\neg u \rightarrow v)$

(d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

Answer :

A truth table needs 2^n rows when there are n variables

(a) Given compound proposition is,

$(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$

Number of variables, $n = 2$

$\Rightarrow 2^n = 2^2 = 4$

$\therefore$ 4 rows appear in truth table

(b) Given compound proposition is,

$(p \vee \neg t) \wedge (p \vee \neg s)$

Number of variables, $n = 3$

$\Rightarrow 2^n = 2^3 = 8$

$\therefore$ 8 rows appear in a truth table

(c) Given compound proposition is,

$(p \rightarrow r) \vee (\neg s \rightarrow \neg t) \vee (\neg u \rightarrow v)$

Number of variables, $n = 6$

$\Rightarrow 2^n = 2^6 = 64$

$\therefore$ 64 rows appear in a truth table

(d) Given compound proposition is,

$(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$

Number of variables $n = 5$

$\Rightarrow$ $2^n = 2^5 = 32$

$\therefore$ 32 rows appear in a truth table.

Q14. Find the bitwise OR, bitwise AND and bitwise XOR of the bit strings 0110110110 and 1100011101.

Model Paper-5, Q1(a)

Answer :

Given bit strings are,

01 1011 0110

11 0001 1101

Bitwise OR

01 1011 0110

11 0001 1101

11 1011 1111

$[\because 1 + 0 = 1, 1 + 1 = 1, 0 + 0 = 0]$.

Bitwise AND

01 1011 0110

11 0001 1101

01 0001 0100

$[\because 0.1 = 0, 1.0 = 0, 1.1 = 1]$

Bitwise XOR

01 1011 0110

11 0001 1101

10 1010 1011

$[\because 1 \oplus 1 = 0, 0 \oplus 0 = 0, 1 \oplus 0 = 1, 0 \oplus 1 = 1]$

Q15. Use De Morgan's laws to find the negation of each of the following statements.

(a) Kwame will take a job in industry or go to graduate school

(b) Yoshiko knows Java and calculus

(c) James is young and strong

(d) Rita will move to Oregon or Washington

Answer :

(a) Given statement is,

Kwame will take a job in industry or go to graduate school

The negation of above statement is,

Kwame will not take a job in industry or not go to graduate school

(b) Given statement is,

Yoshiko knows Java and calculus

The negation of above statement is

Yoshiko doesn't know Java or doesn't know Calculus

(c) Given statement is,

James is young and strong

The negation of above statement is

James is not young or not strong

(d) Given statement is,

Rita will move to Oregon or Washington

The negation of above statement is,

Rita will not move to Oregon or will not move to Washington.

Q16. Show that $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

Model Paper-2, Q1(b)

Answer :

Given conditional statements are,

$\neg p \to (q \to r)$

$q \to (p \vee r)$

Consider,

$$\neg p \to (q \to r) \equiv \neg p \to (\neg q \vee r) \qquad [\because p \to q \equiv \neg p \vee q]$$
$$\equiv \neg(\neg p) \vee (\neg q \vee r) \qquad [\because p \to q \equiv \neg p \vee q]$$
$$\equiv p \vee (\neg q \vee r) \qquad [\because \neg(\neg p) \equiv p]$$
$$\equiv (p \vee \neg q) \vee r \qquad [\because p \vee (q \vee r) \equiv (p \vee q) \vee r]$$
$$\equiv p \vee \neg q \vee r$$
$$\equiv \neg q \vee (p \vee r) \qquad [\because (p \vee q) \equiv (q \vee p)]$$
$$\Rightarrow \neg p \to (q \to r) \equiv \neg q \vee (p \vee r)$$

Hence, $\neg p \to (q \to r)$ and $q \to (p \vee r)$ are logically equivalent.

Q17. Translate these statements into English, where the domain for each variable consists of all real numbers.

(a) $\exists x \forall y(xy = y)$

(b) $\forall x \forall y(((x \geq 0) \wedge (y < 0)) \to (x - y > 0))$

(c) $\forall x \forall y \exists z(x = y + z)$

Model Paper-3, Q1(b)

Answer :

(a) Given statement is,

$\exists x \forall y(xy = y)$

The domain of each variable consists of all real numbers

The above statement can be expressed as,

"There is a real number x such that for every real number y, the product of x and y is equal to y."

(b) Given statement is,

$\forall x \forall y(((x \geq 0) \wedge (y < 0)) \to (x - y > 0))$

The above statement can be expressed as,

"For every real number x and every real number y, if x is greater than or equal to zero and y is less than zero, then their difference $x - y$ is greater than zero."

(c) Given statement is,

$\forall x \forall y \exists z(x = y + z)$

The above statement can be expressed as,

"For every real number x and every real number y, there is a real number z such that the sum of y and z is equal to x."

Q18. What is wrong with this argument? Let H(x) be "x is happy." Given the premise $\exists x H(x)$, we conclude that H(Lola). Therefore, Lola is happy.

Model Paper-4, Q1(b)

Answer :

Given proposition is,

$H(x)$: x is happy

Premise, $\exists x H(x)$

H(Lola)

Conclusion is "Lola is Happy"

From the premise, there exists some x that is happy."

The premise, $\exists x H(x)$ makes the proposition $H(x)$ true.

Though $H(x)$ is true, it cannot be concluded that Lola is one such x.

Q19. Prove that if m + n and n + p are even integers where m, n and p are integers then m + p is even. What kind of proof did you use?

Answer :

Given that,

$m+n, n+p$ are even integers.

m, n, p are integers.

Let, $m+n$ and $n+p$ are even integers.

Then,

$m+n = 2k$ ... (1)

$n+p = 2l$ for some integers k and l ... (2)

Adding equations (1) and (2),

$m+n+n+p = 2k+2l$

$\Rightarrow$ $m+2n+p = 2k+2l$

$\Rightarrow$ $m+p = 2k+2l-2n$

$\Rightarrow$ $m+p = 2(k+l-n)$

As, k, l, n are integers, $(k+l-n)$ is also an integer.

$\Rightarrow$ $m+p$ is even

$\therefore$ $m+p$ is an even integer

Q20. Use a direct proof to show that the sum of two odd integers is even.

Answer : Model Paper-5, Q1(b)

Let m and n represent two odd integers given as,

$m = 2y+1$ ... (1)

$n = 2z+1$ ... (2)

Where,

y, z integers

Adding equations (1) and (2),

$m+n = (2y+1)+(2z+1)$

$= 2y+1+2z+1$

$= 2y+2z+2$

$= 2(y+z+1)$

Since, y and z are integers, $y+z+1$ is also an integer.

$\Rightarrow$ $m+n$ is even

Hence, the sum of two odd integers is even.

Q21. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Answer :

Consider the integers,

8 and 9

$8 = 2^3$

$\Rightarrow$ 8 is a perfect cube

$9 = 3^2$

$\Rightarrow$ 9 is a perfect square

8 and 9 are a pair of consecutive integers such that 8 is a perfect cube and 9 is a perfect square

Hence, there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Q22. Prove or disprove that if a and b are rational numbers then a^b is also rational.

Answer :

Given that,

a and b are rational numbers

Let, $a = 2$ and $b = \frac{1}{2}$

Consider,

$a^b = 2^{\frac{1}{2}}$

$= \sqrt{2}$

$\Rightarrow$ $\sqrt{2}$ is irrational

Hence, if a and b are rational numbers then a^b is not necessarily a rational number.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

1.1 PROPOSITIONAL LOGIC, APPLICATIONS OF PROPOSITIONAL LOGIC

Q23. Define proposition. Explain briefly about propositional variables and propositional logic.

Answer :

Proposition

A proposition is defined as a declarative sentence which can either be true or false, but it cannot be both. In other words, proposition is a sentence which is used for declaring a fact. It is also referred as a statement.

Example

New Delhi is the capital of India.

Sun rises from the west.

Here, the proposition 1 is true, whereas 2 is false.

Propositional Variables

A proposition or statement is usually represented by using the variables p, q, r, s etc. called the propositional variables. The concept of propositional variable is usually used for connecting two or more statements by making use of the connectives.

Example

P : New Delhi is the capital of India.

Here, the propositional variable P is used to represent a statement "New Delhi is the Capital of India".

Propositional Calculus/Propositional Logic

Propositional calculus or propositional logic is a formal system, which calculates all the propositions of two primitive propositions using the logical connectives such as $\vee$ (OR), $\wedge$ (AND), and $\rightarrow$ (Implies) etc. If p and q are two primitive propositions, then all the propositions obtained from p and q using the logical connectives must be boolean algebra.

The propositional calculus system or language consists of the following two things,

1. A set of primitive symbols which are called atomic formulae (Example: a well formed formula).
2. A set of operator symbols or logical connectives.

These formulae and logical connectives are used to obtain the propositions by applying the various rules of inference.

The main objective of propositional calculus in logical application is to determine the relationship of logical equivalents along with the propositional formulae. And these relationships are determined by using the transformation rules, whose sequences are called proof.

Q24. What are connectives? Explain different types of connectives for the statements with their truth table.

Answer :

Connectives

A word or expression which is used for connecting two or more statements is called as a connective. The basic connectives used while applying mathematical logic are 'NOT', 'AND', 'OR' are called logical connectives. These connectives are referred in objective language as follows,

1. Negation
2. Conjunction
3. Disjunction
4. Implication
5. Biconditional statement.

1. **Negation (NOT)**

The negative of a given statement (or) complementary of the given statement is represented using the negation (NOT) connective.

Explanation

If p is any statement then the negation of p is given by, 'not' p.

Notation

Negation of p is denoted by $\sim p$.

$\boxed{\sim \rightarrow \text{not}}$

Example

p: Onida is a good company.

$\sim p$: 'Onida is {**not**} a good company'.

The truth table for negation is as follows,

p	~p
T	F
F	T

Table 1: Truth Table for Negation

The negation of p takes the complementary truth values of that of p.

2. Conjunction (AND)

The compound statement formed by connecting two statements by using the 'AND' connective is called conjunction of two statements.

Explanation

If p, q are two statements then conjunction of p, q is given as p AND q.

Notation

Conjunction of p and q is represented as $p \wedge q$ i.e.,

$\boxed{\wedge \rightarrow \text{and}}$

Example

p: Japan is a country.

q: Onida is a good company.

$p \wedge q$: Japan is a country **and** Onida is a good company.

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Table 2: Truth Table for Conjunction

Conjunction $p \wedge q$ takes the truth value T only when both p and q are true otherwise, it is always false (F).

3. Disjunction (OR)

The compound statement formed using the 'OR' connective is called disjunction of two statements.

Explanation

If p, q are two statements then disjunction of p, q is given as p OR q.

Notation

Disjunction of p OR q is denoted as $p \vee q$ i.e.,

$\boxed{\vee \rightarrow \text{OR}}$

Example

p: Japan is a country.

q: Onida is a good company.

$p \vee q$: Japan is a country **OR** Onida is a good company.

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

Table 3: Truth Table for Disjunction

Disjunction $p \vee q$ of two statements is true even if any one of the statement is true and it is false only if both p and q are false.

4. Implication (→)

The compound statement formed by connecting two statements using 'if then' connective is called implication of two statement.

Explanation

If p and q are two statements then the p and q implication statement is 'if p then q'.

Notation

Implication statement 'if p then q' is represented as '$p \rightarrow q$ or $p \Rightarrow q$' i.e., p implies q.

$\boxed{\rightarrow \text{implies}}$

Example

p : Japan is a country

q : Onida is a good company.

$p \rightarrow q$: **If** Japan is a country, **then** Onida is a good company.

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Truth Table for Implication

The implication statement $p \rightarrow q$ takes the truth value false only when p is true and q is false otherwise, it is always true.

Here, p is called antecedent and q is called consequent.

5. Biconditional Statement

The compound statement formed using the 'if and only if' connective is called biconditional statement.

Explanation

If p and q are two statements then biconditional statement formed using p and q is given by p 'if and only if' q.

...biconditional statement of two statements p and q is ...by $p \rightleftarrows q$ or $p \leftrightarrow q$ (i.e., p iff q).

... $\rightleftarrows$ If and only if (IFF)

Example

p : Japan is a country.

q : Onida is a good company.

$p \rightleftarrows q$: Japan is a country if and only if Onida is a good company.

p	q	$p \rightleftarrows q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 5: Truth Table for Biconditional Statement

The biconditional statement $p \rightleftarrows q$ takes the truth value ...(T) only when both p and q have the same truth value ...erwise it is false (F).

Q25. Explain in detail about the following connectives along with their truth tables.

(i) NAND

(ii) NOR.

Answer :

(i) NAND

The logical NAND is a combination of two logical ...nectives NOT and AND which means P NAND Q is equivalent ...NOT (P AND Q). NAND is denoted by the symbol $\uparrow$. Therefore, P NAND Q is written as $P \uparrow Q$.

The truth table for NAND i.e., P NAND Q is,

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

Table (1): Truth Table of $P \uparrow Q$

The logical connective NAND produces the value as true if one of it's operands is false and it produces the value as false if both of its operands are true.

(ii) NOR

NOR is also a logical connective used for performing logical operations on two logical values.

The logical "NOR" is a combination of two logical connectives NOT and OR which means P NOR Q is equivalent to NOT (P OR Q). NOR is denoted by the symbol $\downarrow$. Therefore, P NOR Q is written as $P \downarrow Q$.

The truth table for P NOR Q is,

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

Table (2): Truth table of $P \downarrow Q$

NOR produces a true value if both its operands are false, otherwise it produces a false value.

Q26. Discuss in brief about converse, inverse and contrapositive of an implication.

Answer :

Converse of an Implication

The converse of an implication is obtained by interchanging the two statements of the implication i.e., suppose there are two statements A and B, then, $A \rightarrow B$ is a logical implication. Converse of this logical implication can be denoted as,

$$B \rightarrow A$$

Example

A : You sleep early

B : You wake up early.

Given implication is $A \rightarrow B$

Its converse can be written as, $B \rightarrow A$. That is,

If you wake up early, then you will sleep early.

Inverse of an Implication

Inverse of an implication can be obtained by finding the negation of the implication i.e., if $A \rightarrow B$ is an implication, then the inverse of this implication can be defined as,

$$\sim A \rightarrow \sim B$$

Inverse of the above example can be written as $B \rightarrow A$. That is,

If you do not sleep early, then you will not wake up early.

Contrapositive of an Implication

Contrapositive of an implication can be obtained by finding the negation of converse or in other words, it is obtained by finding the opposite of converse i.e., $\sim B \rightarrow \sim A$.

Contrapositive of the above example can be written as $\sim B \rightarrow \sim A$.

If you do not wake up early, then you will not sleep early.

Q27. Write each of these propositions in the form "p if and only if q" in English

(a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside

(b) For you to win the contest it is necessary and sufficient that you have the only winning ticket.

(c) You get promoted only if you have connections, and you have connections only if you get promoted

(d) If you watch television your mind will decay, and conversely.

(e) The trains run late on exactly those days when I take it.

Answer :

(a) Given proposition is,

If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside

The above proposition can be written as,

You buy an ice cream cone if and only if it is hot outside

(b) Given proposition is,

For you to win the contest it is necessary and sufficient that you have the only winning ticket

The above proposition can be written as,

You win the contest if and only if you hold the only winning ticket.

(c) Given proposition is,

You get promoted only if you have connections, and you have connections only if you get promoted

The above proposition can be written as,

You get promoted if and only if you have connections

(d) Given proposition is,

If you watch television your mind will decay and conversely

The above proposition can be written as,

Your mind will decay if and only if you watch television

(e) Given proposition is,

The trains run late on exactly those days when I take it.

The above proposition can be written as,

The trains run late if and only if it is a day I take the train.

Q28. State the converse, contrapositive, and inverse of each of these conditional statements.

(a) If it snows today, I will ski tomorrow

(b) I come to class whenever there is going to be a quiz

(c) A positive integer is a prime only if it has no divisors other than 1 and itself.

Answer :

(a) Given conditional statement is,

If it snows today, I will ski tomorrow

Converse:

I will ski tomorrow only if it snows today

Contrapositive:

If I do not ski tomorrow, then it will not have snowed today

Inverse:

If it does not snow today, then I will not ski tomorrow

(b) Given conditional statement is,

I come to class whenever there is going to be a quiz

Converse:

If I come to class, then there will be a quiz

Contrapositive:

If I do not come to class, then there will not be a quiz

Inverse:

If there is not going to be a quiz, then I don't come to class

Given conditional statement is,

) A positive integer is a prime only if it has no divisors other than 1 and itself

Converse:

A positive integer is a prime if it has no divisors other than 1 and itself

Contrapositive:

If a positive integer has a divisor other than 1 and itself, then it is not a prime

Inverse:

If a positive integer is not prime, then it has a divisor other than 1and itself.

29. Let p, q and r be the propositions

p : You have the flu

q : You miss the final examination

r : You pass the course

Express each of these propositions as an English sentence

(a) $p \to q$

(b) $\neg q \leftrightarrow r$

(c) $q \to \neg r$

(d) $p \vee q \vee r$

(e) $(p \to \neg r) \vee (q \to \neg r)$

(f) $(p \wedge q) \vee (\neg q \wedge r)$

Answer :

Given propositions are,

p : You have the flu

q : You miss the final examination

r : You pass the course

(a) Given proposition is,

$$p \to q$$

The sentence is,

If you have the flu, then you miss the final exam

(b) Given proposition is,

$$\neg q \leftrightarrow r$$

The sentence is,

You do not miss the final exam if and only if you pass the course.

(c) Given proposition is,

$$q \to \neg r$$

The sentence is,

If you miss the final exam, then you do not pass the course.

(d) Given proposition is,

$$p \vee q \vee r$$

The sentence is,

You have the flu or miss the final exam or pass the course.

(e) Given proposition is,

$$(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$$

The sentence is,

It is either the case that if you have the flu then you don't pass the course or the case if you miss the final exam then you do not pass the course.

(f) Given proposition is,

$$(p \wedge q) \vee (\neg q \wedge r)$$

The sentence is,

Either you have the flu and miss the final exam or you don't miss the final exam and do pass the course.

Q30. Prove the implication $p \rightarrow (q \rightarrow r) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$ by using truth table.

Answer : **Model Paper-1, Q2(a)**

To prove the implication $p \rightarrow (q \rightarrow r) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$, construct a truth table as follows,

p	q	r	p → q	q → r	p → r	p → (q → r)	(p → q) → (p → r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence from the columns (8 and 9) of the truth table $p \rightarrow (q \rightarrow r) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$.

Q31. Construct a truth table for each of these compound propositions.

(a) $p \rightarrow \neg p$

(b) $p \leftrightarrow \neg p$

(c) $p \oplus (p \vee q)$

(d) $(p \wedge q) \rightarrow (p \vee q)$

(e) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

(f) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

Answer :

(a) Given compound proposition is,

$$p \rightarrow \neg p$$

The truth table for the above proposition is,

p	¬p	p → ¬p
T	F	F

F	T	T

(b) Given compound proposition is,

$p \leftrightarrow \neg p$

The truth table for the above proposition is,

p	¬p	p ↔ ¬p
T	F	F
F	T	F

(c) Given compound proposition is,

$p \oplus (p \vee q)$

The truth table for the above proposition is,

p	q	p ∨ q	p ⊕ (p ∨ q)
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

(d) Given compound proposition is,

$(p \wedge q) \rightarrow (p \vee q)$

The truth table for the above proposition is,

p	q	p ∧ q	p ∨ q	(p ∧ q) → (p ∨ q)
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(e) Given compound proposition is,

$(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

The truth table for the above proposition is,

p	q	¬p	q → ¬p	p ↔ q	(q → ¬p) ↔ (p ↔ q)
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

(f) Given compound proposition is,

$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

The truth table for the above proposition is,

p	q	¬q	p ↔ q	p ↔ ¬q	(p ↔ q) ⊕ (p ↔ ¬q)
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	T

Q32. Construct a truth table for $((p \rightarrow q) \rightarrow r) \rightarrow s$.

Answer :

Given compound proposition is,

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$((p \to q) \to r) \to s$

The truth table for the above proposition is,

p	q	r	s	p → q	(p → q) → r	((p → q) → r) → s
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	F	F	F	T	F	T

Q33. Construct a truth table for each of these compound propositions

(a) $(p \vee q) \vee r$

(b) $(p \vee q) \wedge r$

(c) $(p \wedge q) \vee r$

(d) $(p \wedge q) \wedge r$

(e) $(p \vee q) \wedge \neg r$

(f) $(p \wedge q) \vee \neg r$

Answer :

(a) Given compound proposition is,

$(p \vee q) \vee r$

The truth table for the above proposition is,

p	q	r	p ∨ q	(p ∨ q) ∨ r
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(b) Given compound proposition is,

$(p \vee q) \wedge r$

The truth table for the above proposition is,

p	q	r	p ∨ q	(p ∨ q) ∧ r
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(c) Given compound proposition is,

$(p \wedge q) \vee r$

The truth table for the above proposition is,

p	q	r	p ∧ q	(p ∧ q) ∨ r
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

(d) Given compound proposition is,

$(p \wedge q) \wedge r$

The truth table for the above proposition is,

p	q	r	p ∧ q	(p ∧ q) ∧ r
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(e) Given compound proposition is,

$(p \vee q) \wedge \neg r$

The truth table for the above proposition is,

p	q	r	¬r	p ∨ q	(p ∨ q) ∧ ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	F	F

Given compound proposition is,

$(p \wedge q) \vee \neg r$

The truth table for the above proposition is,

p	q	r	¬r	p∧q	(p∧q)∨¬r
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

Construct a truth table for each of these compound propositions

(a) $p \rightarrow (\neg q \vee r)$

(b) $\neg p \rightarrow (q \rightarrow r)$

(c) $(p \rightarrow q) \vee (\neg p \rightarrow r)$

(d) $(p \rightarrow q) \wedge (\neg p \rightarrow r)$

(e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

(f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

ver :

Given compound proposition is,

$p \rightarrow (\neg q \vee r)$

The truth table for the above proposition is,

p	q	r	¬q	¬q∨r	p→(¬q∨r)
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Given compound proposition is,

$\neg p \rightarrow (q \rightarrow r)$

The truth table for the above proposition is,

p	q	r	¬p	q→r	¬p→(q→r)
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Given compound proposition is,

$(p \to q) \vee (\neg p \to r)$

The truth table for the above proposition is,

p	q	r	¬p	p → q	¬p → r	(p → q) ∨ (¬p → r)
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Given compound proposition is,

$(p \to q) \wedge (\neg p \to r)$

The truth table for the above proposition is,

p	q	r	¬p	p → q	¬p → r	(p → q) ∧ (¬p → r)
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

Given compound proposition is,

$(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

The truth table for the above proposition is,

p	q	r	¬q	p ↔ q	¬q ↔ r	(p ↔ q) ∨ (¬q ↔ r)
T	T	T	F	T	F	T
T	T	F	F	T	T	T
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

Given compound proposition is,

$(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

The truth table for the above proposition is,

p	q	r	¬p	¬q	¬p ↔ ¬q	q ↔ r	(¬p ↔ ¬q) ↔ (q ↔ r)
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

Q35. Determine whether each of these conditional statements is true or false

(a) If 1 + 1 = 2, then 2 + 2 = 5

(b) If 1 + 1 = 3, then 2 + 2 = 4

(c) If 1 + 1 = 3, then 2 + 2 = 5

(d) If monkeys can fly, then 1 + 1 = 3.

Model Paper-2, Q2(a)

Answer :

(a) Given conditional statement is,

If $1 + 1 = 2$, then $2 + 2 = 5$

$1 + 1 = 2$ - True

$2 + 2 = 5$ - True

First statement is true but the second statement is false

Hence, the conditional statement is false.

(b) Given conditional statement is,

If $1 + 1 = 3$, then $2 + 2 = 4$

$1 + 1 = 3$ - False

$2 + 2 = 4$ - True

First statement is false but the second statement is true.

Hence, the conditional statement is true

(c) Given conditional statement is,

If $1 + 1 = 3$, then $2 + 2 = 5$

$1 + 1 = 3$ - False

$2 + 2 = 5$ - False

$\Rightarrow$ Both the statements are false

Hence, the conditional statement is true

(d) Given conditional statement is,

If monkeys can fly, then $1 + 1 = 3$

Monkeys can fly - False

$1 + 1 = 3$ - False

$\Rightarrow$ Both the statements are false

Hence, the conditional statement is true.

Q36. Determine whether these biconditionals are true or false

(a) 2 + 2 = 4 if and only if 1 + 1 = 2

(b) 1 + 1 = 2 if and only if 2 + 3 = 4

(c) 1 + 1 = 3 if and only if monkeys can fly

(d) 0 > 1 if and only if 2 > 1.

Answer :

(a) Given biconditional is,

$2 + 2 = 4$ if and only if $1 + 1 = 2$

$2 + 2 = 4$ - True

$1 + 1 = 2$ - True

$\Rightarrow$ Both statements are true

Hence, the biconditional is true

(b) Given biconditional is,

$1 + 1 = 2$ if and only if $2 + 3 = 4$

$1 + 1 = 2$ - True

$2 + 3 = 4$ - False

$\Rightarrow$ First statement is true but the second statement is false

Hence, the biconditional is false

Given biconditional is,

$1+3$ if and only if monkeys can fly

$1+1=3$ - False

Monkeys can fly - False

$\Rightarrow$ Both statements are false

Hence, the biconditional is true

Given biconditional is,

$0>1$ if and only if $2>1$

$0>1$ - False

$2>1$ - True

First statement is false but the second statement is true.

Hence, the biconditional is false.

...7. Construct a truth table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$. Model Paper-4, Q2(a)

...swer :

Given compound proposition is,

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$

The truth table for above proposition is,

p	q	r	s	$p \leftrightarrow q$	$r \leftrightarrow s$	$(p \leftrightarrow q) \rightarrow (r \leftrightarrow s)$
T	T	T	T	T	T	T
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	T	F	F	T	T	T
T	F	T	T	F	T	F
T	F	T	F	F	F	T
T	F	F	T	F	F	T
T	F	F	F	F	T	F
F	T	T	T	F	T	F
F	T	T	F	F	F	T
F	T	F	T	F	F	T
F	T	F	F	F	T	F
F	F	T	T	T	T	T
F	F	T	F	T	F	F
F	F	F	T	T	F	F
F	F	F	F	T	T	T

Q38. Construct a truth table for each of these compound propositions.

(a) $(p \vee q) \rightarrow (p \oplus q)$

(b) $(p \oplus q) \rightarrow (p \wedge q)$

(c) $(p \vee q) \oplus (p \wedge q)$

(d) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

(e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

(f) $(p \oplus q) \rightarrow (p \oplus \neg q)$

Answer :

(a) Given compound proposition is,

$(p \vee q) \rightarrow (p \oplus q)$

The truth table for the above proposition is,

p	q	$p \vee q$	$p \oplus q$	$(p \vee q) \rightarrow (p \oplus q)$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

(b) Given compound proposition is,

$(p \oplus q) \rightarrow (p \wedge q)$

The truth table for the above proposition is,

p	q	$p \oplus q$	$p \wedge q$	$(p \oplus q) \rightarrow (p \wedge q)$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(c) Given compound proposition is,

$(p \vee q) \oplus (p \wedge q)$

The truth table for the above proposition is,

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \oplus (p \wedge q)$
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

(d) Given compound proposition is,

$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

The truth table for the above proposition is,

p	q	$\neg p$	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	F	T	F	T
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	T	F	T

(e) Given compound proposition is,

$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

The truth table for the above proposition is,

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
T	T	T	F	F	T	T	F
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	T	F	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	F

Given compound proposition is,

$(p \oplus q) \rightarrow (p \oplus \neg q)$

The truth table for the above proposition is,

p	q	$\neg q$	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \rightarrow (p \oplus \neg q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

Q8. Evaluate each of these expressions

(a) 11000 ∧ (01011 ∨ 11011)

(b) (01111 ∧ 10101) ∨ 01000

(c) (01010 ⊕ 11011) ⊕ 01000

(d) (11011 ∨ 01010) ∧ (10001 ∨ 11011)

Answer : **Model Paper-3, Q2(a)**

(a) Given expression is,

11000 ∧ (01011 ∨ 11011)

Consider,

(01011 ∨ 11011)

⇒ 01011
11011
11011

Consider,

11000 ∧ 11011

⇒ 11000
11011
11000

∴ 11000 ∧ (01011 ∨ 11011) = 11000

(b) Given expression is,

(01111 ∧ 10101) ∨ 01000

Consider,

01111 ∨ 10101

⇒ 01111
10101
00101

Consider,

00101 ∨ 01000

⇒ 00101
01000
01101

∴ (01111 ∧ 10101) ∨ 01000 = 01101

:) Given expression is,

$$(01010 \oplus 11011) \oplus 01000$$

Consider,

$$01010 \oplus 11011$$

$$\Rightarrow \begin{array}{r} 01010 \\ \underline{11011} \\ \underline{10001} \end{array}$$

Consider,

$$10001 \oplus 01000$$

$$\Rightarrow \begin{array}{r} 10001 \\ \underline{01000} \\ \underline{11001} \end{array}$$

$\therefore \quad (01010 \oplus 11011) \oplus 01000 = 11001$

(d) Given expression is,

$$(11011 \vee 01010) \wedge (10001 \vee 11011)$$

Consider,

$$11011 \vee 01010$$

$$\Rightarrow \begin{array}{r} 11011 \\ \underline{01010} \\ \underline{11011} \end{array}$$

Consider,

$$10001 \vee 11011$$

$$\Rightarrow \begin{array}{r} 10001 \\ \underline{11011} \\ \underline{11011} \end{array}$$

Consider,

$$11011 \wedge 01010$$

$$\Rightarrow \begin{array}{r} 11011 \\ \underline{11011} \\ \underline{11011} \end{array}$$

$\therefore \quad (11011 \vee 01010) \wedge (10001 \vee 11011) = 11011$

Q40. Find the bitwise OR, bitwise AND and bitwise XOR of each of these pairs of bit strings

(a) 101 1110, 0100001

(b) 1111 0000, 1010 1010

(c) 00 0111 0001, 10 0100 1000

(d) 11 1111 1111, 00 0000 0000.

Answer :

(a) 1011110, 0100001

Given bit strings are,

1011110
0100001

Bitwise OR

$$\begin{array}{r} 101\ 1110 \\ \underline{010\ 0001} \\ \underline{111\ 1111} \end{array} \qquad [\because 1+0=1, 0+1=1]$$

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Bitwise AND

$$\begin{array}{r} 101\ 1110 \\ \underline{010\ 0001} \\ \underline{000\ 0000} \end{array} \qquad [\because 1.0 = 0, 0.1 = 0]$$

Bitwise XOR

$$\begin{array}{r} 101\ 1110 \\ \underline{010\ 0001} \\ \underline{111\ 1111} \end{array} \qquad [\because 1 \oplus 0 = 1, 0 \oplus 1 = 1]$$

(b) 1111 0000, 1010 1010

Given bit strings are,

1111 0000

1010 1010

Bitwise OR

$$\begin{array}{r} 1111\ 0000 \\ \underline{1010\ 1010} \\ \underline{1111\ 1010} \end{array} \qquad [\because 1 + 1 = 1, 1 + 0 = 1, 0 + 0 = 0, 0 + 1 = 1]$$

Bitwise AND

$$\begin{array}{r} 1111\ 0000 \\ \underline{1010\ 1010} \\ \underline{1010\ 0000} \end{array} \qquad [\because 1.1 = 1, 1.0 = 0, 0.1 = 0, 0.0 = 0]$$

Bitwise XOR

$$\begin{array}{r} 1111\ 0000 \\ \underline{1010\ 1010} \\ \underline{0101\ 1010} \end{array} \qquad [\because 1.1 = 0, 0.0 = 0, 1.0 = 1, 0.1 = 0]$$

(c) 00 0111 0001, 10 0100 1000

Given bit strings are,

00 0111 0001

10 0100 1000

Bitwise OR

$$\begin{array}{r} 00\ 0111\ 0001 \\ \underline{10\ 0100\ 1000} \\ \underline{10\ 0111\ 1001} \end{array} \qquad [\because 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 1, 0 + 0 = 0]$$

Bitwise AND

$$\begin{array}{r} 00\ 0111\ 0001 \\ \underline{10\ 0100\ 1000} \\ \underline{00\ 0100\ 0000} \end{array} \qquad [\because 0.1 = 0, 0.0 = 0, 1.0 = 0, 1.1 = 1]$$

Bitwise XOR

$$\begin{array}{r} 00\ 0111\ 0001 \\ \underline{10\ 0100\ 1000} \\ \underline{10\ 0011\ 1001} \end{array} \qquad [\because 0 \oplus 1 = 1, 1 \oplus 0 = 1, 0 \oplus 0 = 0, 1 \oplus 1 = 0]$$

(d) 11 1111 1111, 00 0000 0000

Given bit strings are,

11 1111 1111

00 0000 0000

Bitwise OR

11 1111 1111
00 0000 0000 $[\because 1+0=1]$
11 1111 1111

Bitwise AND

11 1111 1111
00 0000 0000 $[\because 1.0=0]$
00 0000 0000

Bitwise XOR

11 1111 1111
00 0000 0000 $[\because 1 \oplus 0=1]$
11 1111 1111

1.2 PROPOSITIONAL EQUIVALENCE

Q41. Define tautology, contradiction and contingency with examples.

Answer :

Tautology

A statement formula is said to be a tautology or universally valid formula if all the resultant truth values are true, for every possible truth values of the statement variables.

Examples

(i) $p \vee \sim p$

(ii) $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

(iii) $[(p \wedge \sim q) \rightarrow r] \rightarrow [p \rightarrow (q \vee r)]$.

Contradiction

A statement formula is said to be a contradiction or absurdity if all the resultant truth values are false for every possible truth values of the statement variables.

Example

$p \wedge \sim p$

Contingency

A statement formula which is neither logically true nor absurdity is referred to as contingency. It can have both true and false values.

Examples

(i) $[p \rightarrow (p \vee q)]$

(ii) $[(\sim q \rightarrow p) \wedge q]$.

Q42. Explain about logically equivalent statement with an example. List the different logical equivalence formulae.

Answer :

Equivalence

Two compound statements are said to be logically equivalent if they have the same truth values for all 2^n components.

Explanation

If M, N are two compound statements and if the truth values in the last column of M are same as that of truth values of last column of N then M is said to be logically equivalent to N.

Notation

$\boxed{\text{logically equivalent is} \Leftrightarrow}$

The usual notation to express that the two statements M and N are logically equivalent is,

$M \Leftrightarrow N$ and if $M \not\Leftrightarrow N$ then M is 'not logically equivalent to' N.

Example

If p and q are any two atomic statements then,

$\neg(\neg p) \Leftrightarrow p$

$p \vee \neg p \Leftrightarrow q \vee \neg q$

Standard Equivalent Formulae

The various formulae which define the logical equivalences are given below.

Equivalent Formulae

1. Idempotent Laws : $p \vee p \Leftrightarrow p$
   $p \wedge p \Leftrightarrow p$
2. Associative Laws : $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$;
   $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
3. Commutative Laws : $(p \wedge q) \Leftrightarrow (q \wedge p)$
   $(p \vee q) \Leftrightarrow (q \vee p)$
4. Distributive Laws : $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
5. Absorption Laws : $p \wedge (p \vee q) \Leftrightarrow p$
   $p \vee (p \wedge q) \Leftrightarrow p$
6. De Morgan's Laws : $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
   $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$
7. Identity Laws : $p \wedge T \Leftrightarrow p$
   $p \vee F \Leftrightarrow p$
8. Domination Laws : $p \vee T \Leftrightarrow T$
   $p \wedge F \Leftrightarrow F$
9. Double negation Laws : $\sim(\sim p) \Leftrightarrow p$
10. Negation Laws : $p \vee \sim p \Leftrightarrow T$
    $p \wedge \sim p \Leftrightarrow F$.

Apart from the standard logical equivalence formulae, there are other logical equivalence formulae which consist of the conditional statements.

These formulae are given below.

$p \to q \Leftrightarrow \neg p \vee q$

$p \to q \Leftrightarrow \neg q \to \neg p$

$p \vee q \Leftrightarrow \neg p \to q$

$p \wedge q \Leftrightarrow \neg(q \to \neg p)$

$\neg(p \to q) \Leftrightarrow p \wedge \neg q$.

Moreover, there are few logical equivalence formulae which consist of biconditionals. They are as follows,

$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \Leftrightarrow \neg p \to \neg q$

$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$.

Write short notes on,

(i) Tautological implications

(ii) Logical implication.

Answer :

(i) Tautological Implication

If p and q are any two statements (compound or atomic) such that if $p \to q$ is a tautology, then it can be said that p tautologically implies q. The tautological implication between two statements p and q is denoted by using the symbol '$p \Rightarrow q$'.

Example

The truth table for $P \wedge Q \Rightarrow P$ is shown below.

P	Q	P ∧ Q	(P ∧ Q) → P
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Table: Tautological Implication

Since, all the entries in the last column are true, $(P \wedge Q) \to P$ is a tautology, hence $(P \wedge Q) \Rightarrow P$.

Properties

$X \Rightarrow Y$ states that $X \to Y$ is a tautology.

$X \Rightarrow Y$ guarantees that, Y has a truth value T whenever X has the truth value T.

(ii) Logical Implication

If p and q are two compound statements then p is said to be logical implication of q, if q is true for all the true values of p.

Examples

(i) $p \leftrightarrow q \Rightarrow p \to q$

(ii) $p \Rightarrow p \vee q$

Q44. Show that $\neg$ and $\wedge$ form a functionally complete collection of logical operators.

Model Paper-1, Q2(b)

Answer :

Given logical operators are,

$\neg$ and $\wedge$

$\neg$ and $\wedge$ form a functionally complete collection of logical operators if the operators $\vee$, $\to$ and $\leftrightarrow$ are expressed as combination of $\neg$ and $\wedge$

From De Morgan's law,

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \quad \text{... (1)}$$

$$p \vee q \equiv \neg(\neg(p \vee q))$$

$$\equiv \neg(\neg p \wedge \neg q) \quad [\because \text{From (1)}]$$

$$\therefore \quad p \vee q \equiv \neg(\neg p \wedge \neg q) \quad \text{... (2)}$$

From the rules of logical equivalence,

$p \to q \equiv \neg p \vee q$

$\equiv \neg((\neg(\neg p) \wedge (\neg q))$ [$\because$ From (2)]

$\equiv \neg(p \wedge (\neg q))$ [$\because \neg(\neg p) = p$]

$\therefore \quad p \to q \equiv \neg(p \wedge (\neg q))$ ... (3)

From the rules of logical equivalence,

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\equiv (\neg(p \wedge (\neg q)) \wedge (\neg(q \wedge (\neg p)))$ [$\because$ From (3)]

$\therefore \quad p \leftrightarrow q \equiv (\neg(p \wedge (\neg q)) \wedge (\neg(q \wedge \neg p)))$

Hence, $\neg$ and $\wedge$ form a functionally complete collection of logical operators.

Q45. Determine whether $(\neg p \wedge (p \to q)) \to \neg q$ is tautology.

Answer :

Given conditional statement is,

$(\neg p \wedge (p \to q)) \to \neg q$

The truth table for the above statement is,

p	q	¬p	¬q	p → q	¬p ∧ (p → q)	(¬p ∧ (p → q)) → ¬q
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

It can be seen from the truth table that the truth values in sixth and seventh columns are not same.

Hence, $(\neg p \wedge (p \to q)) \to \neg q$ is not a tautology.

Q46. How many of the disjunctions $p \vee \neg q$, $\neg p \vee q$, $q \vee r$, $q \vee \neg r$ and $\neg q \vee \neg r$ can be made simultaneously true by an assignment of truth values to p, q and r?

Answer :

Given disjunctions are,

$p \vee \neg q$, $\neg p \vee q$, $q \vee r$, $q \vee \neg r$ and $\neg q \vee \neg r$

The truth table for the above disjunctions is,

p	q	r	¬p	¬q	¬r	p ∨ ¬q	¬p ∨ q	q ∨ r	q ∨ ¬r	¬q ∨ ¬r
T	T	T	F	F	F	T	T	T	T	F
T	T	F	F	F	T	T	T	T	T	T
T	F	T	F	T	F	T	F	T	F	T
T	F	F	F	T	T	T	F	F	T	T
F	T	T	T	F	F	F	T	T	T	F
F	T	F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	T	T	T	F	T
F	F	F	T	T	T	T	T	F	T	T

It can be seen from truth table that the disjunctions $p \vee \neg q$, $\neg p \vee q$, $q \vee r$ and $q \vee \neg r$ are made true by assigning same truth values to p, q, r.

The disjunction $\neg q \vee \neg r$ is made true by assigning atleast one false value to q, r.

Q47. Show that $\neg(p \oplus q)$ and $p \leftrightarrow q$ are logically equivalent. Model Paper-5, Q2(a)

Answer :

Given propositions are,

$\neg(p \oplus q)$

$p \leftrightarrow q$

The truth table for the above propositions is,

p	q	$p \ominus q$	$\neg(p \ominus q)$	$p \leftrightarrow q$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

It can be seen from truth table that the truth values in fourth and fifth columns are same.

Hence, $\neg(p \ominus q)$ and $p \leftrightarrow q$ are logically equivalent.

Q48. Show that each of these conditional statements is a tautology by using truth tables.

(a) $(p \wedge q) \rightarrow p$

(b) $p \rightarrow (p \vee q)$

(c) $\neg p \rightarrow (p \rightarrow q)$

(d) $(p \wedge q) \rightarrow (p \rightarrow q)$

(e) $\neg(p \rightarrow q) \rightarrow p$

(f) $\neg(p \rightarrow q) \rightarrow \neg q$.

Model Paper-2, Q2(b)

Answer :

(a) Given conditional statement is,

$$(p \wedge q) \rightarrow p$$

The truth table for the above statement is,

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

It can be seen from truth table that the truth values in the last column are true

Hence, $(p \wedge q) \rightarrow p$ is a tautology

(b) Given conditional statement is,

$$p \rightarrow (p \vee q)$$

The truth table for the above statement is,

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

It can be seen from truth table that the truth values in the last column are true

Hence, $p \rightarrow (p \vee q)$ is a tautology.

(c) Given conditional statement is,

$$\neg p \rightarrow (p \rightarrow q)$$

The truth table for the above statement is,

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

It can be seen from truth table that the truth values in last column are true

Hence, $\neg p \rightarrow (p \rightarrow q)$ is a tautology

(d) Given conditional statement is,

$(p \wedge q) \rightarrow (p \rightarrow q)$

The truth table for the above statement is,

p	q	p ∧ q	p → q	(p ∧ q) → (p → q)
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

It can be seen from truth table that the truth values in last column are true

Hence, $(p \wedge q) \rightarrow (p \rightarrow q)$ is a tautology

(e) Given conditional statement is,

$\neg(p \rightarrow q) \rightarrow p$

The truth table for the above statement is,

p	q	p → q	¬(p → q)	¬(p → q) → p
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

It can be seen from truth table that the truth values of last column are true

Hence, $\neg(p \rightarrow q) \rightarrow p$ is a tautology.

(f) Given conditional statement is,

$\neg(p \rightarrow q) \rightarrow \neg q$

The truth table for the above statement is

p	q	¬q	p → q	¬(p → q)	¬(p → q) → ¬q
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T
F	F	T	T	F	T

It can be seen from truth table that the truth values of last column are true

Hence, $\neg(p \rightarrow q) \rightarrow \neg q$ is a tautology.

Q49. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Answer :

Given compound propositions are,

$\neg(p \vee (\neg p \wedge q))$

$\neg p \wedge \neg q$

Consider,

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ [$\because$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$ De Morgan's law]

$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$ [$\because$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$ De Morgan's law]

$\equiv \neg p \wedge (p \vee \neg q)$ [$\because$ $\neg(\neg p) \equiv p$ Double Negation law]

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ [$\because$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ Distributive law]

$\equiv F \vee (\neg p \wedge \neg q)$ [$\because$ $\neg p \wedge p \equiv F$ Negation law]

$\equiv (\neg p \wedge \neg q) \vee F$ [$\because$ $p \vee q \equiv q \vee p$ Commutative law for disjunction]

$\equiv \neg p \wedge \neg q$ [$\because$ $p \vee F \equiv p$ Identity law]

$\therefore$ $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent

Q50. Show that { | } is a functionally complete collection of logical operators.

Answer :

Given logical operator is,

{ | }

{ | } forms a functionally complete collection of logical operators if the operators $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$ are expressed as a combination of { | }

Consider,

$$p|q \equiv \neg(p \wedge q) \quad \text{... (1)}$$

Consider,

$$\neg p \equiv (p \wedge p)$$

$$\equiv p|p$$

$$\therefore \quad \neg p \equiv p|p \quad \text{... (2)}$$

$$p \wedge q \equiv \neg(\neg(p \wedge q))$$

$$\equiv \neg(p|q) \quad [\because \text{From (1)}]$$

$$\equiv (p|q)|(p|q) \quad [\because \text{From (2)}]$$

$$\therefore \quad p \wedge q \equiv (p|q)|(p|q) \quad \text{... (3)}$$

Consider,

$$p \vee q \equiv \neg(\neg(p \vee q))$$

$$\equiv \neg(\neg p \wedge \neg q)$$

$$\equiv \neg p | \neg q \quad [\because \text{From (1)}]$$

$$\equiv (p|p)|(q|q) \quad [\because \text{From (2)}]$$

$$\therefore \quad p \vee q \equiv (p|p)|(q|q) \quad \text{... (4)}$$

Consider,

$$p \rightarrow q \equiv \neg p \vee q$$

$$\equiv \neg(p \wedge \neg q)$$

$$\equiv p | \neg q \quad [\because \text{From (1)}]$$

$$\equiv p|(q|q) \quad [\because \text{From (2)}]$$

$$\therefore \quad p \rightarrow q \equiv p|(q|q)$$

Consider,

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$\equiv (\neg p \vee q) \wedge (\neg q \vee p) \quad [\because p \rightarrow q \equiv \neg p \vee q]$$

$$\equiv (\neg p \wedge \neg q) \vee (p \wedge q)$$

$$\equiv \neg[(p \vee q) \wedge (p \wedge q)]$$

$$\equiv (p \vee q) \,|\, (\neg(p \wedge q))$$

$$\equiv [(p|p)|(q|q)]|(p|p) \quad [\because \text{From (4) and (1)}]$$

$$\therefore \quad p \leftrightarrow q \equiv [(p|p)|(q|q)]|(p|p)$$

Hence, { | } is a functionally complete collection of logical operators.

Q51. Show that p|(q|r) and (p|q)|r are not equivalent, so that the logical operator | is not associative.

Answer :

Given statements are,

$p|(q|r)$

$(p|q)|r$

The truth table for the above statement is,

p	q	r	q|r	p|q	p|(q|r)	(p|q)|r
T	T	T	F	F	T	T
T	T	F	T	F	F	T
T	F	T	T	T	F	F
T	F	F	T	T	F	T
F	T	T	F	T	T	F
F	T	F	T	T	T	T
F	F	T	T	T	T	F
F	F	F	T	T	T	T

It can be seen from truth table that the truth values in sixth and seventh column are not same.

Hence, $p|(q|r)$ and $(p|q)|r$ are not equivalent i.e., the operator | is not associative.

Q52. Show that p|q is logically equivalent to $\neg(p \wedge q)$.

Answer :

Given propositions are,

$p|q$

$\neg(p \wedge q)$

The truth table for the above propositions is,

p	q	$p \wedge q$	p|q	$\neg(p \wedge q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

It can be seen from truth table that the truth values in fourth and fifth column are same.

Hence, $p|q$ and $\neg(p \wedge q)$ are logically equivalent.

Q53. Find the dual of each of these compounds propositions

(a) $p \wedge \neg q \wedge \neg r$

(b) $(p \wedge q \wedge r) \vee s$

(c) $(p \vee F) \wedge (q \vee T)$

Answer :

(a) Given compound proposition is,

$p \wedge \neg q \wedge \neg r$

The dual of above proposition is obtained by replacing operators

$\wedge$ by $\vee$ and

$\vee$ by $\wedge$

$\Rightarrow \quad p \vee \neg q \vee \neg r$

Hence, the dual of $p \wedge \neg q \wedge \neg r$ is $p \vee \neg q \vee \neg r$.

(a) Given compound proposition is,

$(p \wedge q \wedge r) \vee s$

The dual of above proposition is obtained by replacing operators $\wedge$ by $\vee$ and $\vee$ by $\wedge$

$\Rightarrow \quad (p \vee q \vee r) \wedge s$

Hence, the dual of $(p \wedge q \wedge r) \vee s$ is $(p \vee q \vee r) \wedge s$

Given compound proposition is,

(c) $(p \vee F) \wedge (q \vee T)$

The dual of above proposition is obtained by replacing operators $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F and F by T.

$\Rightarrow \quad (p \wedge T) \vee (q \wedge F)$

Hence, the dual of $(p \vee F) \wedge (q \vee T)$ is $(p \wedge T) \vee (q \wedge F)$

Q54. Show that $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

Answer :

Given statements are,

$(p \wedge q) \rightarrow r$

$(p \rightarrow r) \wedge (q \rightarrow r)$

The truth table for the above statement is,

p	q	r	$p \wedge q$	$p \rightarrow r$	$q \rightarrow r$	$(p \wedge q) \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

It can be seen from truth table that the truth values in seventh and eighth column are not same.

Hence, $(p \wedge q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are not logically equivalent.

Q55. Show that each of these conditional statements is a tautology by using truth tables.

(a) $[\neg p \wedge (p \vee q)] \rightarrow q$

(b) $[p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

(c) $[p \wedge (p \rightarrow q)] \rightarrow q$

(d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

Answer :

(a) Given conditional statement is,

$[\neg p \wedge (p \vee q)] \rightarrow q$

The truth table for the above statement is,

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

It can be seen from truth table that the truth values in last column are true

Hence, $[\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology

(b) Given conditional statement is,

$[p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

The truth table for the above statement is,

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

It can be seen from truth table that the truth values in last column are true

Hence, $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology

(c) Given conditional statement is,

$$[p \wedge (p \to q)] \to q$$

The truth table for the above statement is,

p	q	$p \wedge q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

It can be seen from truth table that the truth values in last column are true.

Hence, $[p \wedge (p \to q)] \to q$ is a tautology

(d) Given conditional statement is,

$$[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$$

The truth table for the above statement is,

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \wedge (p \to r)$	$(p \vee q) \wedge (p \to q) \wedge (q \to r)$	$[(p \vee q) \wedge (r \to q) \wedge (q \to r)] \to r$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F	T
F	F	T	F	T	T	F	F	T
F	F	F	F	T	T	F	F	T

It can be seen from truth table that the truth values in the last column are true.

Hence, $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is a tautology.

1.3 PREDICATES AND QUANTIFIERS

Q56. Discuss about predicates.

Answer :

Predicate Logic

Symbolic logic is restricted to the consideration of statements and its formulae. It fails to express that two atomic statements may have certain common features. Therefore, to overcome this failure, concept of predicate in an atomic statement has been introduced. The logic which depends upon the analysis of predicates in any statement is referred as the predicate logic.

Example

Consider two statements,

(a) Jack is a Doctor

(b) Jill is a Doctor.

Here, "is a doctor" is expressed using a symbol. If a method joins it with symbols indicating the names of persons, then it is possible to express common feature about any person who can be a doctor. The part "is a doctor" is called predicate. If P stands for "is a doctor", then $P(x)$ stands for "x is a doctor". Here, $P(x)$ is not a statement until x is assigned a name. If x = Jack, then $P(x)$ is a statement.

N-place Predicate

If a predicate requires N names of object, then it is called N-place predicate.

Examples

1. Jack is a bachelor (1-place predicate)
   Subject predicate

2. Jack is taller than Jill (2-place predicate)
   predicate
   Subject

3. Rama is in between Raja nd Hari (3-place predicate)
   predicate
   Subject

Q57. Define quantifiers. Explain the different types of quantifiers.

Answer :

Quantifiers

Quantifiers are the words like all, some, none, atleast one, most, etc which indicate the quantity of the subject.

There are two types of quantifiers. They are,

1. Existential quantifier
2. Universal quantifier.

1. Existential Quantifier

It is denoted as $\exists x$ and refers to the word "some". It can be expressed in any of the following phrases.

(i) There is atleast one x

(ii) There exist an x such that

(iii) For some x such that

(iv) Some x is such that.

2. Universal Quantifier

It is denoted as $\forall x$, and refers to the word "all". It can be expressed in any of the following phrases.

(i) For all x,....

(ii) For each x,.....

(iii) For every x,......

(iv) For any x,.......

Sometimes it may be necessary to use more than one quantifier in a statement. For example,

$$s(x, y) : x \text{ is intelligent than } y.$$

Consider the statement given below.

For any x and y, if x is intelligent than y, then it is not true that y is intelligent than x.

The above statement can be symbolized as,

$$\forall x\ \forall y[s(x, y) \rightarrow \neg s(y, x)]$$

Q58. Explain about variables, free and bound variables in the context of predicate logic. Also discuss about statement function in predicate calculus.

Answer :

Variables in Predicate Logic

Variables are the letters used as a placeholder and as object names. Let us consider a predicate (S) as "is a student", then $S(x)$, $S(y)$, $S(z)$ represent statements and x, y, z represent object names. It is possible to write these statement in a common form as $S(P)$ for P is a student. $S(P)$ is not a statement and P is an object. Variable can be replaced by appropriate object name.

Free and Bound Variables

The occurrence of any variable in a logical formula "P" is said to be a bound occurrences if the variable appear immediately after the quantifiers $\forall$ and $\exists$ or in between the quantifiers. However, when the occurrence of a variable is not a bound occurrence then it is called a free occurrence.

The occurrences of the variable are basically determined based on the scope of quantifier. The scope of the quantifier is defined as the formula which immediately follows the quantifier. If the scope of a quantifier is an atomic formula, then in such situation, it is not necessary to enclose the formula within the parenthesis.

Examples for Bound and Free Occurrences of the Variables

(i) **$(x)(P(x) \rightarrow Q(x))$:** Here the formula $P(x) \rightarrow Q(x)$ defines the scope of universal quantifier. The occurrences of variable x are bound occurrences.

(ii) **$(x)\ P(x, y)$:** Here $P(x, y)$ defines the scope of quantifiers the occurrence of x are bound occurrences whereas the occurrence of variable 'y' is free occurrence.

(iii) **$(\exists x)\ P(x) \wedge Q(x)$:** Here the last occurrence of variable x in $Q(x)$ is a free occurrence.

Whereas, the occurrence of variable x in $P(x)$ is bound occurrence.

Statement Function

Statement function is defined as an expression which consists of a predicate symbol and a variable. It is possible to convert the given statement function into a statement by substituting the variable with an object name. The statement which is generated after substitution is referred to as "substitution instance" of a statement function. Such statement functions are called simple statement functions.

Example 1

$H(b)$: b is a bachelor.

In addition to simple statement functions, there exist compound statement functions that are generated after combining one or more simple statement functions by making use of logical connectives.

Example 2

Consider the following simple statement function.

$A(x)$: x is an apple

$G(x)$: x is green.

Compound statement function which is generated after combining the above two statement functions with '$\wedge$' connective is given as, $A(x) \wedge G(x)$.

Q59. Define the following.

(i) Substitution Instance

(ii) Prime formula

(iii) Logically equivalent statement

(iv) Logical implication

(v) Contrapositive, converse and inverse of universally quantified statement

$\forall x[p(x) \to q(x)]$.

Answer :

(i) Substitution Instance: The statement that is obtained by substituting a formula in a statement function is called as substitution instance of the statement function.

(ii) Prime Formula: A predicate formula without the sentential connectives is called a prime formula.

(iii) Logically Equivalent: If $p(x)$ and $q(x)$ are two open statements, then $p(x)$ is logically equivalent to $q(x)$ [i.e., $\forall x\ p(x) \Leftrightarrow q(x)$] if $p(a) \rightleftarrows q(a)$ is true for each replacement by an element (say 'a') from the given universe.

(iv) Logical Implication: If $p(x)$ and $q(x)$ are two open statements, then $p(x)$ logically implies $q(x)$ [i.e., $\forall x\ p(x) \Rightarrow q(x)$] if $p(a) \to q(a)$ is true for each replacement by an element (say 'a') from the given universe.

(v) Contrapositive, Converse and Inverse of Universally Quantified Statement $\forall x\ [p(x) \to q(x)]$

Contrapositive : $\forall x[p(x) \to q(x)] \Leftrightarrow \forall x[\neg q(x) \to \neg p(x)]$

Converse : $\forall x[p(x) \to q(x)] \Leftrightarrow \forall x[q(x) \to p(x)]$

Inverse : $\forall x[p(x) \to q(x)] \Rightarrow \forall x[\neg p(x) \to \neg q(x)]$.

Formulae of Existentially, Universally Quantified Statements

(i) $\exists x[p(x) \wedge q(x)] \Rightarrow [\exists x\ p(x) \wedge \exists x\ q(x)]$

(ii) $\exists x[p(x) \vee q(x)] \Leftrightarrow [\exists x\ p(x) \vee \exists x\ q(x)]$

(iii) $\forall x[p(x) \wedge q(x)] \Leftrightarrow [\forall x\ p(x) \wedge \forall x\ q(x)]$

(iv) $[\forall x\ p(x) \vee \forall x\ q(x)] \Rightarrow \forall x[p(x) \vee q(x)]$

(v) $\forall x\ \neg(\neg p(x)) \Leftrightarrow \forall x\ p(x)$

(vi) $\forall x\ \neg[p(x) \wedge q(x)] \Leftrightarrow \forall x[\neg p(x) \vee \neg q(x)]$

(vii) $\forall x\ \neg[p(x) \vee q(x)] \Leftrightarrow \forall x[\neg p(x) \wedge \neg q(x)]$.

Q60. Show that $\exists x\ P(x) \wedge \exists x\ Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.

Answer : Model Paper-4, Q7(b)

Given quantified statement,

$$\exists x\ P(x) \wedge \exists x\ Q(x) \text{ and } \exists x\ [(P(x) \wedge Q(x)]$$

Inorder to prove that,

$$\exists x\ [P(x) \wedge Q(x)] \not\Leftrightarrow \exists x\ P(x) \wedge \exists x\ Q(x)$$

Let, us verify whether any of the following implications are false.

(i) $\exists x\ [P(x) \wedge Q(x)] \Rightarrow \exists x P(x) \wedge \exists x Q(x)$

(ii) $\exists x\ P(x) \wedge \exists x Q(x) \Rightarrow \exists x[P(x) \wedge Q(x)]$

If any of the implication is false, then the given quantified statements are not logically equivalent.

(i) Considering, $\exists x\ [P(x) \wedge Q(x)] \Rightarrow \exists x P(x) \wedge \exists x Q(x)$

Consider L.H.S,

$\exists x[P(x) \wedge Q(x)]$

$\Rightarrow$ $P(s) \wedge Q(s)$ for some $s \in U$ where 'U' is the universe

$\Rightarrow$ $P(s)$ for some $s \in U$ and $Q(s)$ for some $s \in U$

$\Rightarrow$ $\exists x P(x) \wedge \exists x Q(x)$

$\therefore$ $\exists x\ [P(x) \wedge Q(x)] \Rightarrow \exists x P(x) \wedge \exists x Q(x)$ is true. ... (1)

(ii) Considering, $\exists x\ P(x) \wedge \exists x\ Q(x) \Rightarrow \exists x[P(x) \wedge Q(x)]$

Consider L.H.S,

$\exists x P(x) \Rightarrow P(s)$ for some $s \in U$

$\exists x Q(x) \Rightarrow P(t)$ for some $t \in U$

$\therefore$ $\exists x P(x) \wedge \exists x Q(x) \Rightarrow P(s) \wedge Q(t)$

$\not\Leftrightarrow P(s) \wedge Q(s)$ $[\because s \not\Leftrightarrow t]$

$\therefore$ $\exists x P(x) \wedge \exists x Q(x) \Rightarrow \exists x[P(x) \wedge Q(x)]$ is false ... (2)

$\therefore$ From equations (1) and (2), it can be concluded that,

$$\exists x[P(x) \wedge Q(x)] \not\Leftrightarrow \exists x P(x) \wedge \exists x Q(x).$$

Q61. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

(a) $(\forall x\ P(x)) \vee A \equiv \forall x(P(x) \vee A)$

(b) $(\exists x(P(x)) \vee A \equiv \exists x(P(x) \vee A)$

Answer : Model Paper-1, Q3(a)

Given logical equivalence is,

$$(\forall x\ P(x)) \vee A \equiv \forall x(P(x) \vee A)$$

x does not occur as a free variable in A

The following cases are considered

Case (i)

If A is true, then $(\forall x(P(x)) \vee A$ will be true

Similarly, $\forall x$ as $P(x) \vee A$ is true

$\forall x(P(x) \vee A)$ is also true.

$\Rightarrow$ $(\forall xP(x)) \vee A \equiv \forall x(P(x) \vee A)$

$\Rightarrow$ $(\forall xP(x)) \vee A \equiv \forall x(P(x) \vee A)$

$\therefore$ $(\forall xP(x) \vee A \equiv \forall x(P(x) \vee A)$

Case (ii)

If A is false, then $P(x)$ will be true $\forall x$

$\Rightarrow$ $(\forall x(P(x)) \vee A$ is true .

Similarly, $\forall x$ as $P(x) \vee A$ is true,

Then $\forall x(P(x) \vee A)$ is also true

$\Rightarrow$ $(\forall x(P(x)) \vee A \equiv \forall x(P(x) \vee A)$

If $P(x)$ is false $\forall x$, then

$(\forall x(P(x)) \vee A$ is false

Similarly, $\forall x(P(x) \vee A)$ is also false

$\Rightarrow$ $(\forall x(P(x)) \vee A \equiv \forall x(P(x) \vee A)$

$\therefore$ $(\forall x(P(x) \vee A \equiv \forall x(P(x) \vee A)$

(b) Given logical equivalence is,

$(\exists x(P(x) \vee A \equiv \exists x(P(x) \vee A)$

The following cases are considered

Case(i)

If A is true then, $\exists x(P(x) \vee A$ is also true

Similarly, $\forall x$ as $P(x) \vee A$ is true, then

$\exists x(P(x) \vee A)$ is also true

$\Rightarrow$ $\exists x(P(x) \vee A \equiv \exists x(P(x) \vee A)$

Case (ii)

If A is false, then

When $P(x)$ is true $\forall x$,

$\Rightarrow$ $(\exists x(P(x)) \vee A$ is also true

Similarly, $\forall x$ as $P(x) \vee A$ is true, then $\exists x(P(x) \vee A)$ is also true

$\Rightarrow$ $(\exists x(P(x)) \vee A \equiv \exists x(P(x) \vee A)$

If $P(x)$ is false $\forall x$, then $(\exists x(P(x)) \vee A$ is false

Similarly, $\exists x(P(x) \vee A)$ is also false

$\Rightarrow$ $(\exists x(P(x)) \vee A \equiv \exists x(P(x) \vee A)$

$\therefore$ $(\exists x(P(x)) \vee A \equiv \exists x(P(x) \vee A)$

Q62. Suppose the domain of the propositional function P(x, y) consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

(a) $\exists xP(x, 3)$

(b) $\forall yP(1, y)$

(c) $\exists y\neg P(2, y)$

(d) $\forall x\neg P(x, 2)$

Model Paper-4, Q2(b)

Answer :

Given that,

$P(x, y)$ consists of pairs x and y

x is 1, 2 or 3

y is 1, 2 or 3

(a) Given proposition is,

$\exists xP(x, 3)$

The above proposition can be expressed using disjunction as,

$P(1, 3) \vee p(2, 3) \vee P(3, 3)$

$\therefore$ $\exists xP(x, 3) = P(1, 3) \vee P(2, 3) \vee P(3, 3)$

(b) Given proposition is,

$\forall yP(1, y)$

The above proposition can be expressed using conjunction as,

$P(1, 1) \wedge P(1, 2) \wedge P(1, 3)$

$\therefore$ $\forall yP(1, y) = P(1, 1) \wedge P(1, 2) \wedge P(1, 3)$

(c) Given proposition is,

$\exists y \neg P(2, y)$

The above proposition is expressed using disjunction as,

$\neg P(2, 1) \vee \neg P(2, 2) \vee \neg P(2, 3)$

$\therefore$ $\exists y \neg P(2, y) = \neg P(2, 1) \vee \neg P(2, 2) \vee \neg P(2, 3)$

(d) Given proposition is,

$\forall x \neg P(x, 2)$

The above proposition is expressed using conjunction as,

$\neg P(1, 2) \wedge \neg P(2, 2) \wedge \neg P(3, 2)$

$\therefore$ $\forall x \neg P(x, 2) = \neg P(1, 2) \wedge \neg P(2, 2) \wedge \neg P(3, 2)$

Q63. Find a counter example, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers

(a) $\forall x(x^2 \neq x)$

(b) $\forall x(x^2 \neq 2)$

(c) $\forall x(|x| > 0)$

Answer :

(a) Given statement is,

$\forall x(x^2 \neq x)$

If $x = 0$, then

$0^2 = 0$

$\Rightarrow \quad x^2 = x$

If $x = 1$, then

$1^2 = 1$

$\Rightarrow \quad x^2 = x$

$\Rightarrow \quad \forall x(x^2 \neq x)$ is not true for $x = 0, 1$

Hence, the counter example for $\forall x(x^2 \neq x)$ is $x = 0, 1$

(b) Given statement is,

$\forall x(x^2 \neq 2)$

If $x = \pm\sqrt{2}$, then

$(\pm\sqrt{2})^2 = 2$

$\therefore \quad \forall x\,(x^2 \neq 2)$ is not true for $x = \pm\sqrt{2}$

Hence, the counter example for $\forall x(x^2 \neq 2)$ is $x = \pm\sqrt{2}$

(c) Given statement is,

$\forall x(|x| > 0)$

If $x = 0$, then

$|0| = 0 \not> 0$

$\Rightarrow \quad \forall x(|x| > 0)$ is not true for $x = 0$

Hence, the counter example for $\forall x(|x| > 0)$ is $x = 0$.

Q64. Suppose that the domain of the propositional function P(x) consists of the integers –2, –1, 0, 1 and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

(a) $\exists x P(x)$

(b) $\forall x P(x)$

(c) $\exists x \neg P(x)$

(d) $\forall x \neg P(x)$

(e) $\neg \exists x P(x)$

(f) $\neg \forall x P(x)$

Answer :

Given that,

$P(x)$ consists of integers –2, –1, 0, 1 and 2.

Domain = {–2, 1, 0, 1, 2}

(a) Given proposition is,

$\exists x P(x)$

Since the domain is {–2, –1, 0, 1, 2}, the proposition $\exists x P(x)$ is same as the disjunction ($\vee$).

The proposition $\exists x P(x)$ can be expressed as,

$\therefore \quad \exists x P(x) = P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)$

Given proposition is,

$\forall x P(x)$

Since, the domain is $\{-2, -1, 0, 1, 2\}$, the proposition is same as conjunction

The proposition $\forall x P(x)$ can be expressed as,

$\forall x P(x) = P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)$

$\therefore \quad \forall x P(x) = P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)$

Given proposition is,

$\exists x \neg P(x)$

From De Morgan's law,

$\exists x \neg P(x) = \neg \forall x P(x)$

The proposition $\neg \forall x P(x)$ can be expressed as,

$\neg[P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)]$

$\therefore \quad \exists x \neg P(x) = \neg[P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)]$

Given proposition is,

$\forall x \neg P(x)$

From De Morgan's law,

$\forall x \neg P(x) = \neg \exists x P(x)$

The proposition $\neg \exists x P(x)$ can be expressed as,

$\neg \exists x P(x) = \neg[P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)]$

$\therefore \quad \forall x \neg P(x) = \neg[P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)]$

Given proposition is,

$\neg \exists x P(x)$

Since, the domain is $\{-2, 1, 0, 1, 2\}$

The proposition is same as the negation of $\exists x P(x)$

$\Rightarrow \quad \neg \exists x P(x) = \neg[P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)]$

$\therefore \quad \neg \exists x P(x) = \neg[P(-2) \vee P(-1) \vee P(0) \vee P(1) \vee P(2)]$

Given proposition is,

$\neg \forall x P(x)$

Since, the domain is $\{-2, -1, 0, 1, 2\}$

The proposition is same as the negation of $\forall x P(x)$.

$\Rightarrow \quad \neg \forall x P(x) = \neg[P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)]$

$\therefore \quad \neg \forall x P(x) = \neg[P(-2) \wedge P(-1) \wedge P(0) \wedge P(1) \wedge P(2)]$

Q65. Translate these statements into English, where R(x) is "x is a rabbit" and H(x) is "x hops" and the domain consists of all animals.

(a) $\forall x(R(x) \rightarrow H(x))$

(b) $\forall x(R(x) \wedge H(x))$

(c) $\exists x(R(x) \rightarrow H(x))$

(d) $\exists x(R(x) \wedge H(x))$

Answer :

Given statements are,

$R(x) =$ "x is a rabbit"

$H(x) =$ " x hops"

The domain consists of all animals

(a) Given statement is,

$\forall x(R(x) \to H(x))$

$\Rightarrow$ For all animals if it is a rabbit, then it hops

$\therefore$ Hence, the statement is "Every rabbit hops".

(b) Given statement is,

$x(R(x) \wedge (H(x))$

$\Rightarrow$ x is a rabbit and x hops

$\therefore$ Hence, the statement is, "All animals are rabbits and all animals hop."

(c) Given statement is,

$\exists x(R(x) \to H(x))$

$\Rightarrow$ As there is an animal such that if it is a rabbit, then it hops.

$\therefore$ Hence, the statement is, "There exists an animal such that if it is a rabbit then it hops."

(d) Given statement is,

$\exists x(R(x) \wedge H(x))$

$\Rightarrow$ There is an animal that is a rabbit and hops.

$\therefore$ Hence, the statement is, "There exists rabbit that hops."

Q66. State the value of x after the statement if P(x) then x := 1 is executed, where P(x) is the statement "x > 1," if the value of x when this statement is reached is

(a) x = 0

(b) x = 1

(c) x = 2.

Answer : **Model Paper-5, Q2(b)**

Given statements are,

If $P(x)$ then $x := 1$ is executed

$P(x) = x > 1$

(a) Given that,

$x = 0$

$\Rightarrow$ $x > 1$ does not hold

$\Rightarrow$ $P(x)$ is false

$\Rightarrow$ $x := 1$ is not executed

$\Rightarrow$ $x = 0$ is unchanged

$\therefore$ $x = 0$

(b) Given that,

$x = 1$

$\Rightarrow$ $x > 1$ does not hold

$\Rightarrow$ $P(x)$ is false

$\Rightarrow$ $x := 1$ is not executed

$\Rightarrow$ $x = 1$ is unchanged

$\therefore$ $x = 1$

(c) Given that,

$x = 2$

$\Rightarrow$ $x > 1$ holds

$\Rightarrow$ $P(x)$ is true

$\Rightarrow$ $x := 1$ is executed

$\Rightarrow$ $x = 1$ is obtained

$\therefore$ $x = 1$

Q67. What are the negations of the statement "There is an honest politician" and "All Americans eat cheese burgers"?

Answer :

Given statements are,

There is an honest politician

All American eat cheese burgers.

Let $H(x)$ represent the statement "x is honest"

The statement "There is an honest politician" is represented as,

$\exists x\, H(x)$

Where, the domain consists of all politicians.

The negation of above statement is,

$\neg \exists x\, H(x)$

$\Rightarrow$ $\forall x\, \neg H(x)$

The above negation can be expressed as,

"Every politician is dishonest".

Let $C(x)$ represents the statement

"x eats cheeseburgers".

The statement " All Americans eat cheese burgers" is given by $\forall x C(x)$

Where, the domain consists of all Americans

The negation of above statement is,

$\neg \forall x\, C(x)$

$\Rightarrow$ $\exists x \neg C(x)$

The above negation can be expressed as,

" There exist an American who does not eat cheese burgers."

Q68. What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Answer :

Given statements are,

$\forall x(x^2 > x)$

$\exists x(x^2 = 2)$

Consider,

$\forall x(x^2 > x)$

The negation of above statement is,

$\neg(\forall x(x^2 > x))$

$\Rightarrow \exists x \neg(x^2 > x)$

$\Rightarrow \exists x (x^2 \leq x)$

$\therefore$ The negation of $\forall x(x^2 > x)$ is $\exists x (x^2 \leq x)$

Consider,

$\exists x(x^2 = 2)$

The negation of above statement is,

$\neg(\exists x(x^2 = x))$

$\Rightarrow \forall x \neg(x^2 = 2)$

$\Rightarrow \forall x (x^2 = 2)$

1.4 NESTED QUANTIFIERS

Q69. What are nested quantifiers?

Answer :

Nested Quantifiers

Two quantifiers are said to be nested if either of them is present within the scope of the other one. The concept of nested quantifiers was introduced for representation of complex sentences.

Example

$\forall a \forall b$ Sister$(a, b) \Rightarrow$ Sibling(a, b).

The above example states that "sisters are siblings". It can also be written as,

$\forall a, b$ Sister$(a, b) \Rightarrow$ Sibling(a, b). ... (1)

More complex statements are represented using the combination of quantifiers that are of different types. For example, for every person p in a family, there is a toothbrush t such that p brushes their teeth with t.

$\forall p \exists t$ Brushes with(p, t). ... (2)

The above example statement means that initially, a person is selected and then the toothbrush is chosen. Thus, the choice of the toothbrush depends upon the choice of the person. If the order of quantifiers is shuffled, that is,

$\exists t \forall p$ Brushes with(p, t). ... (3)

the meaning will be changed as follows,

In this statement (i.e., statement 3), a toothbrush 't' is chosen first and then, it is asserted that every person uses this tooth brush t. Therefore, the order in which the quantification is performed must be carefully chosen. The quantification order followed in statement 3 is applicable to a situations where a single object is shared among several people. For example, there is a house h, such that every person p in a family resides in the house h. The statement for this example is given below,

$\exists h \forall p$ resides In(p, h).

While proving a statement with nested quantifiers, care must be taken in plotting the quantifiers if they are of different types (i.e., a mixture of existential and universal). This is because, their quantification order has a direct impact on the meaning of the statement. However, if the quantifiers are of the same type i.e., either existential or universal, then their quantification order does not effect the meaning of the sentence.

Q70. Let Q(x, y) be the statement "x + y = x – y". If the domain for both variables consists of all integers, what are the truth values?

(a) Q(1, 1)

(b) Q(2, 0)

(c) $\forall y$ Q(1, y)

(d) $\exists x$ Q(x, 2)

(e) $\exists x \exists y$ Q(x, y)

(f) $\forall x \exists y$ Q(x, y)

(e) $\exists y \forall x$ Q(x, y)

(h) $\forall y \exists x$ Q(x, y)

(i) $\forall x \forall y$ Q(x, y)

Answer :

Given statement is,

$Q(x, y): x + y = x - y$

The domain consists of all integers

(a) Q(1, 1)

$x = 1, y = 1$

$x + y = x - y$

$1 + 1 \neq 1 - 1$

$\Rightarrow 2 \neq 0$

$\therefore Q(1,1)$ is false

(b) Q(2, 0)

$x = 2, y = 0$

$2 + 0 = 2 - 0$

$\Rightarrow 2 = 2$

$\therefore Q(2, 0)$ is true

(c) $\forall y$ Q(1, y)

$x = 1, y = y$

$1 + y = 1 - y$

If $y = 2$, then

$1 + 2 \neq 1 - 2$

$\Rightarrow 3 \neq -1$

$\therefore \forall y\, Q(1, y)$ is false

(d) $\exists x Q(x, 2)$

$$x = x, y = 2$$
$$x + 2 = x - 2$$

If $x = 0$, then

$$0 + 2 = 0 - 2$$
$$\Rightarrow \quad 2 \neq -2$$
$$\therefore \quad \exists x\, Q(x, 2) \text{ is false}$$

(e) $\exists x \exists y\, Q(x, y)$

$$x + y = x - y$$

If $x = 1$ and $y = 0$, then

$$1 + 0 = 1 - 0$$
$$\Rightarrow \quad 1 = 1$$
$$\therefore \quad \exists x \exists y\, Q(x, y) \text{ is true}$$

(f) $\forall x \exists y\, Q(x, y)$

$$x + y = x - y$$

If $y = 0$, then,

$$x + 0 = x - 0$$
$$\Rightarrow \quad x = x$$
$$\therefore \quad \forall x \exists y\, Q(x, y) \text{ is true}$$

(g) $\exists y \forall x\, Q(x, y)$

$$x + y = x - y$$

If $y = 0$, then

$$x - 0 = x - 0$$
$$\Rightarrow \quad x = x$$
$$\therefore \quad \exists y \forall x\, Q(x, y) \text{ is true}$$

(h) $\forall y \exists x\, Q(x, y)$

$$x + y = x - y$$

If $x = 0$, then

$$0 + y = 0 - y$$
$$y \neq -y$$
$$\therefore \quad \forall y \exists x\, Q(x, y) \text{ is false}$$

(i) $\forall x \forall y\, Q(x, y)$

$$x + y = x - y$$

If $x = 2$ and $y = 3$ then,

$$2 + 3 = 2 - 3$$
$$\Rightarrow \quad 5 \neq -1$$
$$\therefore \quad \forall x \forall y\, Q(x, y) \text{ is false}$$

Q71. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

(a) $\forall x \exists y \left(x = \frac{1}{y}\right)$

(b) $\forall x \exists y (y^2 - x < 100)$

(c) $\forall x \forall y (x^2 \neq y^3)$

Answer :

(a) Given statement is,

$$\forall x \exists y \left(x = \frac{1}{y}\right)$$

The domain for all variables consists of all integers

If $x = 0$, then

$$y = \infty \notin Z$$
$$\therefore \quad \forall x \exists y \left(x = \frac{1}{y}\right) \text{ is false}$$

(b) Given statement is,

$$\forall x \exists y (y^2 - x < 100)$$

If $x = -100$, then,

$$y^2 - (-100) < 100$$
$$\Rightarrow \quad y^2 + 100 < 100$$
$$\Rightarrow \quad y^2 + 100 - 100 < 100 - 100$$
$$\Rightarrow \quad y^2 < 0$$

It is impossible for any $y \in Z$

$$\therefore \quad \forall x \exists y (y^2 - x < 100) \text{ is false}$$

(c) Given statement is,

$$\forall x \forall y (x^2 \neq y^3)$$

If $x = 8$ and $y = 4$, then,

$$x^2 = 64$$
$$y^3 = 64$$
$$\Rightarrow \quad x^2 = y^3 = 64$$
$$\therefore \quad \forall x \forall y (x^2 \neq y^3) \text{ is false}$$

Q72. Let Q(x, y) denote "x+y = 0". What are the truth values of the quantifications $\exists y \forall x\, Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Answer :

Given statement is,

$$Q(x, y) = x + y = 0$$

The quantifications are,

$$\exists y \forall x\, Q(x, y)$$
$$\forall x \exists y\, Q(x, y)$$

The domain for all variables consists of all real numbers

The quantification $\exists y \forall x\, Q(x, y)$ represents the proposition

"There exists a real numbers such that for every real number x, $Q(x, y)$

There exists only one x value such that $x + y = 0$ and is independent of y values

As there is no real number y for which $x + y = 0$ for all real numbers x

$\Rightarrow$ $\exists y \forall x\, Q(x, y)$ is false

Hence, the quantification $\exists y \forall x\, Q(x, y)$ is false

The quantification $\forall x \exists y\, Q(x, y)$ represents the proposition

"For every real number x, there exists a real number y for which $Q(x, y)$"

For a real number, "there exists a real number y for which $x + y = 0$ i.e., $y = -x$

$\Rightarrow$ $\forall x \exists y\, Q(x, y)$ is true

Hence, the quantification $\forall x \exists y\, Q(x, y)$ is true.

Q73. Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world".

Model Paper-2, Q3(a)

Answer :

Given statement is,

"There does not exist a woman who has taken a flight on every airline in the world".

Let $P(w, f)$ represents the statement "w has taken f".

Let $Q(f, a)$ represents "f is a flight on a"

The given statement is expressed as,

$$\neg \exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$$

$$\Rightarrow \quad \neg \exists w \forall a \exists f (P(w, f) \wedge Q(f, a)) \equiv \forall w \neg \forall a \exists f (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \exists a \neg \exists f (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a))$$

$$\equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a)) \quad [\because \text{From De-Morgan's law}]$$

$$\therefore \quad \neg \exists w \forall a \exists f \equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a))$$

The meaning of above statement is,

"For every woman there exists an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline".

Q74. Use quantifiers and predicates with more than one variable to express these statements.

(a) There is a student in this class who can speak Hindi

(b) Every student in this class plays some sport

(c) Some student in this class has visited Alaska but has not visited Hawaii

(d) All students in this class have learned at least one programming language

(e) There is a student in this class who has taken every course offered by one of the departments in this school

(f) Some student in this class grew up in the same town as exactly one other student in this class

(g) Every student in this class has chatted with at least one other student in at least one chat group.

Answer :

(a) Given statement is,

"There is a student in this class who can speak Hindi".

The domain is the set of all students

Let $C(x)$ denotes "x is a student in the class".

Let $H(x)$ denotes "x speaks Hindi".

$C(x) \wedge H(x)$

$\Rightarrow$ The student in the class and he can speak Hindi

$\therefore$ The statement can be expressed as,

$\exists x(C(x) \wedge H(x))$.

(b) Given statement is,

"Every student in this class plays some sport"

Let $C(x)$ denotes "x is a student in the class"

Let $P(x)$ denotes "x plays some sport"

$C(x) \wedge P(x) \Rightarrow$ The student in the class and he plays some sport

$\therefore$ The statement can be expressed as,

$\forall x (C(x) \wedge P(x))$

(c) Given statement is,

"Some student in this class has visited Alaska but has not visited Hawaii".

Let $C(x)$ denotes "x is a student in the class"

Let $V(x)$ denotes "x has visited Alaska"

Let $H(x)$ denotes "x has visited Hawaii"

Then,

$\neg H(x)$ denotes "x has not visited Hawaii"

$C(x) \wedge V(x) \wedge \neg H(x) \Rightarrow$ A student in the class visited Alaska but not visited Hawaii

$\therefore$ The statement can be expressed as,

$\exists x (C(x) \wedge V(x) \wedge \neg H(x))$

(d) Given statement is,

"All students in this class have learned at least one programming language".

Let $C(x)$ denotes "x is a student in the class".

Let $L(x)$ denotes "x has learnt atleast one programming language"

$C(x) \wedge L(x) \Rightarrow$ Students in the class have learned atleast one programming language

$\therefore$ The statement can be expressed as,

$\forall x (C(x) \wedge L(x))$

(e) Given statement is,

"There is a student in this class who has taken every course offered by one of the departments in this school"

Let $C(x)$ denotes "x is a student in the class"

Let $D(x)$ denotes "x has taken every course offered by one of the departments in this school."

$C(x) \wedge D(x) \Rightarrow$ Students in the class have taken every course offered by one of the departments in this school.

The statement can be expressed as,

$\forall x (C(x) \wedge D(x))$.

(f) Given statement is,

"Some student in this class grew up in the same town as exactly one another student in this class"

Let $C(x)$ denotes "x is a student in the class"

Let $G(x)$ denotes "x grew up in a town"

For x and y in the domain,

$C(x, y) \Rightarrow x$ and y are students in the class

$G(x, y) \Rightarrow x$ and y grew up in a town

The statement can be expressed as,

$\therefore$ $\exists x \exists y\, C(x,y) \wedge G(x,y))$

Given statement is,

"Every student in this class has chatted with atleast one other student in atleast one chat group"

Let $C(x)$ denotes "x is a student in the class"

Let $A(x)$ denotes "x chatted with atleast one student"

Let $E(x)$ denotes "x chatted in atleast one group"

For every x and y in domain,

$C(x, y) \Rightarrow x$ and y are students in the class

$A(x,y) \Rightarrow x$ and y chatted with atleast one student

$G(x,y) \Rightarrow x$ and y chatted with atleast one group

Every student has chatted with atleast one student $\Rightarrow$ For every x in the class there is y in the class such that $A(x) \wedge E(y)$ is true

$\therefore$ The statement can be expressed as,

$\forall x \exists y\, C(x,y) \wedge A(x, y) \wedge E(x, y)$

Q75. Show that $\forall x P(x) \wedge \exists x Q(x)$ is equivalent to $\forall x \exists y (P(x) \wedge Q(y))$.

Model Paper-4, Q3(a)

Answer :

Given statement is

$$\forall x P(x) \wedge \exists x Q(x) \Leftrightarrow \forall x \exists y (P(x) \wedge Q(y))$$

Assume that $\forall x P(x) \wedge \exists x Q(x)$ is true.

This implies that $P(x)$ is true, $\forall x$. Also assume there exist an element y for which $Q(y)$ is true.

$P(x)$ and $Q(y)$ are true.

$P(x) \wedge Q(y)$ is true $\forall x$ there is a element y for which $Q(y)$ is true, therefore, $\forall x\, \exists y (P(x) \wedge Q(y))$ is true.

Similarly also assume that the second proposition i.e., $\forall x\, \exists y (P(x) \wedge Q(y))$ is true.

Let x be any element in the domain, then there is an element y such that $Q(y)$ is true.

$\therefore$ $\exists x\, Q(x)$ is also true.

Since $\forall x\, P(x)$ is also true, it results that the first proposition i.e., $\forall x\, P(x) \wedge \exists x\, Q(x)$ is true.

As both the propositions are true, therefore it can be concluded that $\forall x\, P(x) \wedge \exists x\, Q(x)$ is equivalent to $\forall x\, \exists y\, (P(x) \wedge Q(y))$.

1.5 RULES OF INFERENCE

Q76. What are the rules of inference? List the different kinds of rules of inference for propositional logic along with their logical implication.

Answer :

Rules of Inference

Rules of inference determine the validity of an argument. Consider the following example of an implication statement.

$$(p_1 \wedge p_2 \wedge p_3 \wedge .... \wedge p_n) \rightarrow q$$

Here, the statements $p_1, p_2, \ldots, p_n$ are referred to as premises and q is said to be the conclusion for the argument. An argument is said to be valid, if every individual premise is true then it automatically means that the conclusion is true. An alternative way of determining whether an argument is valid or not is to prove that the given propositional statement is a tautology.

The following are different kinds of rules of inference.

Name of the Rule	Rule of Inferences	Related Logical Implication
1. Rule of detachment (Modus ponens)	p $p \to q$ $\therefore\ q$	$[p \wedge (p \to q)] \to q$
2. Law of the syllogism	$p \to q$ $q \to r$ $\therefore\ p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
3. Modus Tollens	$p \to q$ $\sim q$ $\therefore\ \sim p$	$[(p \to q) \wedge \sim q] \to \sim p$
4. Rule of conjunction syllogism	p q $\therefore\ p \wedge q$	
5. Rule of disjunctive syllogism	$p \vee q$ $\sim p$ $\therefore\ q$	$[(p \vee q) \wedge \sim p] \to q$
6. Rule of contradiction	$\sim p \to F$ $\therefore\ p$	$(\sim p \to F) \to p$
7. Rule of conjunctive simplifiction	$p \wedge q$ $\therefore\ p$	$(p \wedge q) \to p$
8. Rule of disjunctive amplification	p $\therefore\ p \vee q$	$p \to (p \vee q)$
9. Rule of conditional proof	$p \wedge q$ $p \to (q \to r)$ $\therefore\ r$	$[(p \wedge q) \wedge [p \to (q \to r)]] \to r$
10. Rule for proof by cases	$p \to r$ $q \to r$ $\therefore\ (p \vee q) \to r$	$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

Table

Q77. Define and list down all the rules of inference for quantified statements.

Answer :

Rules of Inference for Quantified Statements

The rules of inference for quantified statements are the rules which are used in the statements consisting of the quantifiers. These rules are usually used in various mathematical arguments. These rules of inference are classified into four types. They are.

1. Rule of Universal Instantiation/Specification
2. Rule of Universal Generalization
3. Rule of Existential Instantiation/Specification

4. Rule of Existential Generalization.

Rule of Universal Instantiation/Specification

An open statement $p(x)$, defined as a universal statement is said to be true for 'each specific' element (say $p(a)$) in the universe, if and only if, the universally quantified statement $p(x)$ is true for all substitution performed by the arbitrarily chosen elements in a universe [i.e., $\forall x\, p(x)$].

Rule of Universal Generalization

A universally quantified statement [$\forall x\, p(x)$] is said to be true, if and only if the open statement $p(x)$ is true for replacement by any arbitrarily chosen element [$p(a)$] in a universe.

Rule of Existential Instantiation/Specification

This rule of inference specifies that if there exist an element x such that $\exists x\, P(x)$ is true, then there is a element 'a' in the universe such that $P(a)$ is true. Here, an arbitrary value for 'a' cannot be chosen, but instead the value of 'a' must be such that $P(a)$ is true.

Rule of Existential Generalization

This rule of inference specifies that if $P(a)$ is true for some element 'a' in the universe, then there exist x such that $\exists x\, P(x)$ is true.

These rules of inference can be written in notation form as follows,

	Name of the Rule	Rule of Inference
1.	Universal instantiation/specification	$\dfrac{\forall x P(x)}{\therefore P(a)}$
2.	Universal generalization	$\dfrac{P(a) \text{ for any arbitrary } a}{\therefore \forall x P(x)}$
3.	Existential instantiation/specification	$\dfrac{\exists x P(x)}{\therefore\ P(a) \text{ for some element } a}$
4.	Existential generalization	$\dfrac{P(a) \text{ for some element } a}{\therefore \exists x P(x)}$

Table

Q78. Use rules of inference to obtain the conclusion of the following arguments:

"Doug is a student in this class, knows how to write programmes in JAVA". "Everyone who knows how to write programmes in JAVA can get a high-paying job". Therefore, "Someone in this class can get a high-paying job".

Answer :

Given statement is,

"Doug is a student in this class, knows how to write programmes in JAVA". "Everyone who knows how to write programmes in JAVA can get a high-paying job". Therefore, "Someone in this class can get a high-paying job".

To obtain the conclusion that "someone in this class can get a high-paying job", initially translate the given arguments to symbolic notation.

Let, $C(x)$ denotes x is in this class

$P(x)$ denotes x knows how to write programmes in JAVA

$j(x)$ denotes x can get a high paying job.

Then the given arguments will be of the form,

$C(\text{Doug}), P(\text{Doug}), \forall x(P(x) \rightarrow j(x))$.

Conclusion: $\exists x(C(x) \wedge j(x))$.

To obtain the conclusion from the given arguments, consider the steps given below,

S.No	Statement	Reason
1.	$C(\text{Doug})$	Premise
2.	$P(\text{Doug})$	Premise
3.	$\forall x(P(x) \to J(x))$	Premise
4.	$P(\text{Doug}) \to J(\text{Doug})$	Universal Instantiation [3]
5.	$J(\text{Doug})$	Modus ponens [2, 4]
6.	$C(\text{Doug}) \wedge J(\text{Doug})$	Conjunction [1, 6]
7.	$\exists x(C(x) \wedge J(x))$	Existential generalization [6]

Table

∴ "Someone in this class can get a high-paying job" is a conclusion obtained from the given arguments.

Q79. For the following set of premises, explain which rules of inferences are used to obtain conclusion from the premises. "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution".

Answer :

Given statement is,

"Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution".

To obtain the conclusion that "there is a person in this class who cares about ocean pollution", initially translate the given premises to symbolic notation.

Let, $c(x)$ denotes x is in this class

$w(x)$ denotes x enjoys whale watching

$o(x)$ denotes x cares about ocean pollution.

Then the given arguments will be of the form,

$$\exists x(c(x) \wedge w(x)),\ \forall x(w(x) \to o(x))$$

Conclusion: $\exists x(c(x) \wedge o(x))$

To explain the rules of inferences used to obtain the conclusion from the given premises, consider the steps given below,

S.No	Step	Reason
1.	$\exists x(c(x) \wedge w(x))$	Premise
2.	$c(a) \wedge w(a)$	Existential instantiation [1]
3.	$w(a)$	Simplification [2]
4.	$\forall x(w(x) \to o(x))$	Premise
5.	$w(a) \to o(a)$	Universal instantiation [4]
6.	$o(a)$	Modus ponen [3, 5]
7.	$c(a)$	Simplification [2]
8.	$c(a) \wedge o(a)$	Conjunction [6, 7]
9.	$\exists x(c(x) \wedge o(x))$	Existential generalization [8]

Table

∴ **In the given set of premises, rules of inference for quantified statements are used to obtain the conclusion that there is a person in this class who cares about ocean pollution.**

Q. Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. To show that $100^2 < 2^{100}$ by using universal modus ponens.

Answer : Model Paper-1, Q2(b)

Given statement is,

"For all positive integers n, if n is greater than 4, then n^2 is less than 2^n".

To prove that $100^2 < 2^{100}$, initially translate the given statements to symbolic form. That is,

Let, $P(n)$ denotes n is greater than 4 (i.e., $n > 4$)

$Q(n)$ denotes n^2 is less than 2^n (i.e., $n^2 < 2^n$)

Then the given statement will be of the form,

$\forall n(P(n) \to Q(n))$

It is also given that the above given statement is assumed as true, that is $\forall n(P(n) \to Q(n))$ is true.

If $n = 100$ then $P(100)$ is true, since $100 > 4$.

Now, by using universal modus ponens, as $P(n)$ is true, then $Q(n)$ is also true.

That is, $100^2 < 2^{100}$ is also true.

$\therefore$ $100^2 < 2^{100}$.

Q51. Show that the premises "A student in this class has not read the book" and "Everyone in this class passed the first examination" imply the conclusion "someone who passed the first examination has not read the book".

Answer :

To show that the given premises lead to the conclusion "Someone who passed the first exam has not read the book", initially translate the given arguments to symbolic notation.

Let, $c(x)$ denotes x is in this class

$b(x)$ denotes x has read the book

$p(x)$ denotes x has passed the first exam.

Symbolically the given arguments will be of the form,

$\exists x(c(x) \wedge \neg b(x)), \forall x(c(x) \to p(x))$.

Conclusion: $\exists x(p(x) \wedge \neg b(x))$.

Inorder to obtain the conclusion from the given arguments considers the steps given below,

S.No	Statement	Reason
1.	$\exists x(c(x) \wedge \neg b(x))$	Premise
2.	$c(a) \wedge \neg b(a))$	Existential instantiation [1]
3.	$c(a)$	Simplification [2]
4.	$\forall x(c(x) \to p(x))$	Premise
5.	$c(a) \to p(a))$	Universal instantiation [5]
6.	$p(a)$	Modus ponens [3, 5]
7.	$\neg b(a)$	Simplification [2]
8.	$p(a) \wedge \neg b(a)$	Conjunction [6, 7]
9.	$\exists x(p(x) \wedge \neg b(x))$	Existential generalization [8]

Table

$\therefore$ "Someone who passed the first exam has not read the book" is a conclusion obtained/implied from the given premises.

Q82. Show that the equivalence $p \wedge \neg p \equiv F$ can be derived using resolution together with the fact that a conditional statement with a false hypothesis is true. [Hint: Let $q = r = F$ in resolution]

Answer : Model Paper-5, Q3(a)

Given equivalence is,

$$p \wedge \neg p \equiv F$$

Consider the rule of inference resolution.

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

or

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$$

A conditional statement with a false hypothesis is always true.

$\Rightarrow$ $F \rightarrow x$ is always true irrespective of truth value of x.

In this case, it is required to make the compound proposition $(p \vee q) \wedge (\neg p \vee r)$ into the form $p \wedge \neg p$.

Let, $q = r = F$

Where, F - False statement or contradiction consider, $(p \vee q)$.

Replacing q with F,

$$p \vee F \equiv p \vee F$$

$$\equiv p \qquad [\because \text{Identity law}]$$

$$\therefore \quad p \vee F \equiv p$$

Consider, $\neg p \vee r$

Replacing r with F

$$\neg p \vee r \equiv \neg p \vee F$$

$$\equiv \neg p \qquad [\because \text{Identity law}]$$

$$\therefore \quad \neg p \vee r \equiv \neg p$$

From equations (1) and (3),

$$(p \vee q) \wedge (\neg q \vee r) \equiv p \wedge \neg p$$

Consider $q \vee r$,

Replacing q and r with F,

$$q \vee r \equiv F \vee F$$

$$\equiv F \qquad [\because \text{Idempotent law}]$$

$$\therefore \quad q \vee r \equiv F$$

By (4) and (5), (1) becomes a valid statement in view of (i).

$\Rightarrow$ $p \wedge \neg p \rightarrow F$ is a valid statement

$\Rightarrow$ $p \wedge \neg p \equiv F$ $\qquad [\because \text{Negation law}]$

$\therefore$ $p \wedge \neg p \equiv F$

Q83. Use rules of inference to show that if $\forall x(P(x) \vee (Q(x))$ and $\forall x(\neg P(x) \wedge Q(x)) \rightarrow (R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

Answer :

Given premises are,

$\forall x(P(x) \vee Q(x))$

$\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$

$\forall x(\neg R(x) \rightarrow P(x))$

The domains of all quantifiers are same.

	Steps	Reasons
1.	$\forall x(P(x) \vee Q(x))$	Premise
2.	$P(a) \vee Q(a)$	Universal Instantiation from step 1
3.	$\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$	Premise
4.	$(\neg P(a) \wedge Q(a)) \rightarrow R(a)$	Universal Instantiation from step 3
5.	$\neg(\neg P(a) \wedge Q(a)) \vee R(a)$	Equivalent form of implication on step 4 $[(p \rightarrow q) \equiv \neg(p \vee q)]$
6.	$(P(a) \vee \neg Q(a)) \vee R(a)$	De Morgan's law on step 5
7.	$P(a) \vee (\neg Q(a) \vee R(a)$	Associative law on step 6
8.	$P(a) \vee P(a) \vee R(a)$	Resolution from steps 2 and 7
9.	$P(a) \vee R(a)$	Idempotent on step 8
10.	$P(a) \vee \neg(\neg R(a))$	Double negation on step 9
11.	$\neg R(a) \rightarrow P(a)$	Equivalent form of implication on step 10.
12.	$\forall x(\neg R(x) \rightarrow P(x))$	Universal generalization on step 11.

Table

Hence, if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true then $\forall x(\neg R(x) \rightarrow P(x))$ is also true.

Q84. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on", "If the sailing race is held, then the trophy will be awarded" and "The trophy was not awarded" imply the conclusion "It rained".

Answer : **Model Paper-3, Q3(a)**

Given hypotheses are,

"If it does not rain or if it is not foggy, then the sailing race will be held and the life saving demonstration will go on".

"If the sailing race is held, then the trophy will be awarded".

"The trophy was not awarded" imply the conclusion "It rained".

Let,

r: It rains

f: It is foggy

s: The sailing race will be held

l: The lifesaving demonstration will go on

t: The trophy will be awarded.

Then $\neg r$: It does not rain

$\neg f$: It is not foggy

$\neg t$: The trophy was not awarded.

The premises can be expressed as,

$$(\neg r \vee \neg f) \rightarrow (s \wedge l),\ s \rightarrow t \text{ and } \neg t.$$

The conclusion part is r.

	Steps	Reasons
1.	$\neg t$	Hypothesis
2.	$s \rightarrow t$	Hypothesis
3.	$\neg s$	Modus tollens using $\neg t$ and $s \rightarrow t$
4.	$(\neg r \vee \neg f) \rightarrow (s \wedge l)$	Hypothesis
5.	$(\neg(s \wedge l)) \rightarrow \neg(\neg r \vee \neg f)$	Contrapositive of $(\neg r \vee \neg f) \rightarrow (s \wedge l)$
6.	$(\neg s \vee \neg l) \rightarrow (r \wedge f)$	De Morgan's law and double negation
7.	$\neg s \vee \neg l$	Addition using $\neg s$
8.	$r \wedge f$	Modus ponens using $(\neg s \vee \neg l) \rightarrow (r \wedge f)$ and $\neg s \vee \neg l$
9.	r	Simplification using $r \wedge f$

Table

Q85. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

(a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

(b) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.

(c) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

Answer :

(a) Given argument is,

If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$ then $n > 1$.

Let,

$p : n > 1$

$q : n^2 > 1$

The argument can be written as,

$$\begin{array}{ll} & p \rightarrow q \\ & q \\ \hline \therefore & p \end{array}$$

$\Rightarrow \quad ((p - q) \wedge q) \rightarrow p$

But $((p - q) \wedge q) \rightarrow p$ is false when p is false and q is true.

$\Rightarrow \quad ((p - q) \wedge q) \rightarrow p$ is not a tautology.

$\therefore$ The given argument is incorrect using the fallacy of affirming the conclusion.

(b) Given argument is,

If n is a real number with $n > 3$ then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.

Let,

$p : n > 3$

$q : n^2 > 9$

The argument can be written as,

$p \to q$

$\neg q$

$\neg p$

$\Rightarrow$ $((p \to q) \wedge \neg q) \to p$

$\Rightarrow$ The proposition $((p \to q) \wedge \neg q) \to \neg p$ is valid since it is constructed using modus tollens.

$\therefore$ The given argument is valid.

Given argument is,

If n is a real number with $n > 2$,

then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

Let,

$p : n > 2$

$q : n^2 > 4$

The argument can be written as,

$p \to q$

$\neg p$

$\therefore$ $\neg q$

$\Rightarrow$ $((p \to q) \wedge \neg p) \to \neg q$

But $((p \to q) \wedge \neg p) \to \neg q$ is false when p is false and q is true.

$\Rightarrow$ $((p \to q) \wedge \neg p) \to \neg q$ is not a tautology.

$\therefore$ The given argument is incorrect using the fallacy of denying the hypothesis.

6. For each of these arguments, explain which rules of inference are used for each step.

(a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket".

(b) "Each of the five room mates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five room mates can take a course in algorithms next year".

(c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

(d) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

Answer :

(a) Given argument is,

"Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, some in this class has gotten a speeding ticket".

Let,

$c(x)$: x is in this class

$r(x)$: x owns a red convertible

$t(x)$: x has gotten speeding ticket

Taking the premises as $c(\text{Linda})$, $r(\text{Linda})$ $\forall x(r(x) \to t(x))$

The conclusion is $\exists x\, (c(x) \wedge t(x))$

The steps to establish the conclusion from the premises are as follows,

	Steps	Reasons
1.	$\forall x(r(x) \to t(x))$	Hypothesis
2.	$r(\text{Linda}) \to t(\text{Linda})$	Universal instantiation using step 1
3.	$r(\text{Linda})$	Hypothesis
4.	$t(\text{Linda})$	Modus ponens using steps 2 and 3
5.	$c(\text{Linda})$	Hypothesis
6.	$c(\text{Linda}) \wedge t(\text{Linda})$	Conjunction using steps 4 and 5.
7.	$\exists x(c(x) \wedge t(x))$	Existential generalization using step 6

Table

(b) Given argument is,

Each of five room mates Melisa, Aaron, Ralph, Veneesha and Keeshawn has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five room mates can take a course in algorithms next year".

Let,

$r(x)$: x is one of the room mates listed

$d(x)$: x has taken a course in discrete mathematics

$a(x)$: x can take course in algorithms

Taking the premises as $\forall x(r(n) \to d(x))$ and $\forall x(d(x) \to a(x))$

The conclusion is,

$$\forall x(r(x) \to a(x))$$

Let y denotes an arbitrary person.

The steps to establish the conclusion from the premises are as follows,

	Steps	Reasons
1.	$\forall x(r(x) \to d(x))$	Hypothesis
2.	$r(y) \to d(y)$	Universal instantiation using step 1
3.	$\forall x(d(x) \to a(x))$	Hypothesis
4.	$d(y) \to a(y)$	Universal instantiation using step 3
5.	$r(y) \to a(y)$	Hypothetical syllogism using steps 2 and 4.
6.	$\forall x(r(x) \to a(x))$	Universal generalization using step 6

Table

(c) Given argument is,

All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners".

Let,

$s(x)$: x is a movie produced by Sayles

$c(x)$: x is a movie about coal miners

$w(x)$: movie x is wonderful

Taking the premises as

$$\forall x(s(x) \to w(x)) \text{ and } \exists x(s(x) \wedge c(x))$$

The conclusion is,

$\exists x(c(x) \wedge w(x))$

Let y denotes an unspecified particular movie.

The steps to establish the conclusion from the premises are,

	Steps	Reasons
1.	$\exists x(s(x) \wedge c(x))$	Hypothesis
2.	$s(y) \wedge c(y)$	Existential instantiation using step 1
3.	$s(y)$	Simplification using step 2
4.	$\forall x(s(x) \rightarrow w(x))$	Hypothesis
5.	$s(y) \rightarrow w(y)$	Universal instantiation using step 4
6.	$w(y)$	Modus ponens using steps 3 and 5
7.	$c(y)$	Simplification using step 2
8.	$w(y) \wedge c(y)$	Conjunction using steps 6 and 7
9.	$\exists x(c(x) \wedge w(x))$	Existential generalization using step 8

Table

(d) Given argument is,

"There is someone in this class who has been to France. Everyone who goes to France visits Louvre. Therefore, someone in this class has visited the Louvre".

Let,

$c(x)$: x is in this class

$f(x)$: x has been to France

$l(x)$: x has visited the Louvre

Taking the premises as

$\exists x(c(x) \wedge f(x))$, $\forall x(f(x) \rightarrow l(x))$

The conclusion is, $\exists x(c(x) \wedge l(x))$

The steps to establish the conclusion from the premises are as follows,

	Steps	Reasons
1.	$\exists x(c(x) \wedge f(x))$	Hypothesis
2.	$c(y) \wedge f(y)$	Existential instantiation using step 1
3.	$f(y)$	Simplification using step 2
4.	$c(y)$	Simplification using step 2
5.	$\forall x(f(x) \rightarrow l(x))$	Hypothesis
6.	$f(y) \rightarrow l(y)$	Universal instantiation using step 5
7.	$l(y)$	Modus ponens using steps 3 and 6
8.	$c(y) \wedge l(y)$	Conjunction using steps 4 and 7
9.	$\exists x(c(x) \wedge l(x))$	Existential generalization using step 8.

Table

1.6 INTRODUCTION TO PROOFS

Q87. When is a set of formulae is said to be consistent and inconsistent? Also discuss the direct and indirect method of proof.

Answer :

Let $S_1, S_2, S_3, \ldots, S_n$ represents a set of formulas,

If conjunction is applied to these set of formulas if the values of all the formulas are true then $S_1, S_2, S_3, \ldots S_n$ is said to be consistent.

If conjunction is applied to a set of formulas and any of the value of the formulas is false then $S_1, S_2, S_3, \ldots S_n$ is said to be inconsistent.

Direct Proof of p → q

In direct proof, assumption is made that, the statement p is true. Then, based on this assumption and also by considering other available information which is generated from frame of valid reference, q is proved to be true.

Example

If x is a number such that,

$$x^2 - 9x + 20 = 0, \text{ then } x = 4 \text{ or } x = 5$$

Let $p: x^2 - 9x + 20 = 0$ be true.

The frame of reference must include all the rules of algebra.

∴ From rules of algebra,

$$x^2 - 9x + 20 = 0$$

$$\Rightarrow x^2 - 4x - 5x + 20 = 0$$

$$\Rightarrow x(x-4) - 5(x-4)$$

$$\Rightarrow (x-4)(x-5) = 0.$$

Since, it is known that if product of two numbers say a, b is zero, then either a is zero or b is zero.

$$\Rightarrow x - 4 = 0 \quad \text{or} \quad x - 5 = 0$$

$$\Rightarrow x = 4 \quad \text{or} \quad x = 5.$$

Indirect Proof of p → q

Indirect proof is direct proof of a contrapositive statement (i.e. $p \to q \Leftrightarrow \sim q \to \sim p$).

In indirect proof, assumption is made that q is false. Then based on this assumption and other available information generated from frame of reference p is proved to be false..

Example

If the product of two integers x and y is even then either x or y is even.

Let p: xy is even

q: x is even or y is even

$\sim q$: x is odd and y is odd.

$\therefore x = 2m + 1, y = 2n + 1$

Where m, n are integers.

But, $xy = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + mn + n) + 1$

$\Rightarrow$ xy is odd.

Therefore, from the assumption $\sim q$ we conclude $\sim p$.

Q88. Discuss in brief about the rule of conditional proof (Rule CP).

Answer :

Conditional Proof

The proof that is made by asserting a conditional using only the known information is referred to as a conditional proof. It takes the form "If 'A' then 'B'" i.e., if certain condition(s) A is met then it is possible to deduce the desirable conclusion "B".

Rule of Conditional Proof

For a conditional statement $A \to B$ if the conclusion "B" can be derived from the statement "A" and a set of premises, then the conditional statement $(A \to B)$ is deductable from the set of premises alone. Conditional proof involves the following steps,

(i) Consider the set of premises Pi as the known information.

(ii) Deduce the statement in conditional form, $A \to B$ from the set of premises.

(iii) Consider premise Pa as added premises, whenever required and treat it as known information.

(iv) Deduce the conclusion 'B' using the known information (premises where $i = 1, 2, 3, \ldots n$) along with the added premise (Pa).

Q89. Using proof by contradiction, prove that $\sqrt{2}$ is irrational.

Answer :

Let $p : \sqrt{2}$ is irrational

$\sim p : \sqrt{2}$ is rational

Contradiction Method

Initially, assume $\sim p$ is true. If $\sqrt{2}$ is rational then there exist two integer values x and y such that $x/y = \sqrt{2}$. As both these values does not have common factors.

$$x/y = \sqrt{2}$$

Squaring on both sides,

$$\left(\frac{x}{y}\right)^2 = (\sqrt{2})^2$$

$$\Rightarrow \frac{x^2}{y^2} = 2$$

$$\Rightarrow x^2 = 2y^2 \quad \ldots (1)$$

From equation (1), it can be concluded that x^2 is even which implies that x is also even.

As x is even, by using the definition of an integer,

$x = 2z$ where z is some integer.

Substituting the value of x in equation (1),

$$(2z)^2 = 2y^2$$

$$\Rightarrow 4z^2 = 2y^2$$

$$\Rightarrow 2z^2 = y^2 \quad \ldots (2)$$

From equation (2), it can be concluded that y^2 is even, which implies that y is also even.

Moreover, from equations (1) and (2) it can be proved that p is false.

Therefore, x and y have no common factors and 2 divides x and y.

As both these statements contradict each other. It can be and that, $p \rightarrow (q \wedge \neg q)$ where q is a statement that x and y are integers with no common factors.

Therefore, $\neg p$ is false.

That is $\sqrt{2}$ is irrational.

Hence, $\sqrt{2}$ is irrational.

Q90. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Model Paper-2, Q3(b)

Answer :

(i) Assume that,

$$m^2 = n^2$$

$\Rightarrow$ $m = n$ or $m = -n$

Assume that $m \neq n$ or $m \neq -n$.

$\Rightarrow$ $m - n \neq 0$ or $m + n \neq 0$

Consider the product,

$$(m - n)(m + n) \neq 0$$

$\Rightarrow$ $m^2 - mn + mn - n^2 \neq 0$

$\Rightarrow$ $m^2 - n^2 \neq 0$

$\Rightarrow$ $m^2 \neq n^2$

This is a contradiction to assumption $m^2 = n^2$.

Hence, if $m^2 = n^2$ then $m = n$ or $m = -n$.

(ii) Conversely assume that

$$m^2 \neq n^2$$

$\Rightarrow$ $m^2 - n^2 \neq 0$ $[\because m^2 \neq n^2]$

$(m + n)(m - n) \neq 0$

$\Rightarrow$ $(m + n) \neq 0$ and

$\Rightarrow$ $(m - n) \neq 0$

$\Rightarrow$ $m \neq -n$ and $m \neq n$

This is a contradiction to assumption $m = n$ or $m = -n$.

Hence, if $m = n$ or $m = -n$ then $m^2 = n^2$

Therefore, $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Q91. Prove that if n is an integer and 3n + 2 is even, then n is even using

(a) a proof by contraposition.

(b) a proof by contradiction.

Answer :

Given that,

n is an integer

$3n + 2$ is even

(a) Let,

$p : n$ is an integer and $3n + 2$ is even

$q : n$ is even.

The contraposition of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

$\Rightarrow$ $\neg p : n$ is an integer and $3n + 2$ is odd

$\neg q : n$ is odd

Suppose that n is an odd integer.

Then $n = 2k + 1$ for some integer k.

Consider,

$$3n + 2 = 3(2k + 1) + 2$$
$$= 6k + 3 + 2$$
$$= 6k + 5$$
$$= 6k + 4 + 1$$
$$= 2(3k + 2) + 1$$

$\Rightarrow$ $2(3k + 2) + 1$ is odd.

$\Rightarrow$ $(3n + 2)$ is odd.

$\Rightarrow$ If n is an integer, n is odd then $3n + 2$ is odd.

Hence, by contraposition if n is an integer and $3n + 2$ is even then n is even.

(b) Assume that $3n + 2$ is even and n is an odd integer.

As n is odd, the product of two odd integers is odd.

$\Rightarrow$ $3n$ is odd then $3n + 2$ is odd.

Hence, the assumption that $3n + 2$ is even and n is odd, is false.

This is a contradiction.

Hence, if n is an integer and $3n + 2$ is even then n is even.

Q92. Prove or disprove that the product of a non-zero rational number and an irrational number is irrational.

Answer :

Let x represents a non zero rational number.

Let y represents an irrational number.

Let $k = x.y$ ... (1)

Assume that k is a rational number.

Then there are integers p and q with $q \neq 0$ such that,

$$k = \frac{p}{q} \quad \text{... (2)}$$

From equations (1) and (2),

$$x.y = \frac{p}{q}$$

$\Rightarrow$ $y = \dfrac{p}{xq}$

As x is a non zero rational number and q is a non zero integer then xq is a non zero rational number and p is an integer.

$\Rightarrow \quad \frac{p}{xq}$ is a rational number.

But y is an irrational number.

This is a contradiction.

$\Rightarrow \quad x.y$ is irrational

Hence, the product of a non zero rational number and an irrational number is irrational.

Q93. Prove that these four statements about the integer n are equivalent

(i) n^2 is odd

(ii) $1 - n$ is even

(iii) n^3 is odd

(iv) $n^2 + 1$ is even

Answer : **Model Paper-4, Q3(b)**

Given statements are,

n^2 is odd ... (i)

$1 - n$ is even ... (ii)

n^3 is odd ... (iii)

$n^2 + 1$ is even ... (iv)

Assume n^2 is odd

$\Rightarrow \quad n : n$ is odd

$\Rightarrow \quad n$ is odd [$\because$ The product of two odd integers is odd]

Let $n = 2k + 1$ for some integer k

$\Rightarrow \quad 1 - n = -2k$

$\Rightarrow \quad 1 - n = 2(-k)$ [$\because k$ is integer $\Rightarrow -k$ is also an integer]

$\Rightarrow \quad 1 - n$ is even

$\therefore \quad n^2$ is odd $\Rightarrow 1 - n$ is even

Assume that $1 - n$ is even.

$\Rightarrow \quad 1 - n = 2k$ for some integer k

$\Rightarrow \quad n = -2k + 1$

$\Rightarrow \quad n = 2(-k) + 1$ [$\because -k$ is an integer]

$\Rightarrow \quad n$ is odd [$\because$ odd integer is of the form $2k + 1$]

$\Rightarrow \quad n.n$ is odd

$\Rightarrow \quad n^2$ is odd

$\Rightarrow \quad n^2.n$ is odd [$\because$ The product of two odd integers is odd]

$\Rightarrow \quad n^3$ is odd

$\therefore \quad 1 - n$ is even $\Rightarrow n^3$ is odd

Assume that n^3 is odd

$\Rightarrow \quad n^2.n$ is odd [$\because n^3 = n^2.n$]

$\Rightarrow \quad n^2$ is odd [$\because$ Product of two odd integers is odd]

$\Rightarrow \quad n^2 = 2k + 1$ for some integer k

Adding 1 on both sides,

$n^2 + 1 = 2k + 1 + 1$

$\Rightarrow \quad n^2 + 1 = 2k + 2$

$\Rightarrow \quad n^2 + 1 = 2(k + 1)$

$\Rightarrow \quad n^2 + 1$ is even

$\therefore \quad n^3$ is odd $\Rightarrow n^2 + 1$ is even

Assume that, $n^2 + 1$ is even

$\Rightarrow \quad n^2 + 1 = 2k$ for some integer k

$\Rightarrow \quad n^2 = 2k - 1$ is odd [$\because$ odd number is of the form $2k - 1$]

$\Rightarrow \quad n^2 + 1$ is even $\Rightarrow n^2$ is odd

Hence, the given statements are equivalent

1.7 PROOF METHODS AND STRATEGY

Q94. State and explain the proof methods.

OR

Discuss briefly the six standard methods of proving theorem with example.

Answer :

The different standard methods of proving theorems are,

1. Vacuous proof
2. Trivial proof
3. Direct proof
4. Existence proof
5. Indirect proof
6. Proof by cases
7. Exhaustive proof.

1. **Vacuous Proof**

If P and Q are two propositions, then the proof of implication $P \rightarrow Q$ is true based on the fact that P is false, irrespective of whether Q is true or false.

Example

Suppose 'x' is the proposition such that "–2 is a positive integer" and y is the proposition that "0 is a whole number". Then the implication is true, because x is false. Here, the truth value of y is not considered.

2. **Trivial Proof**

If P and Q are two propositions then the proof of implication $P \rightarrow Q$ is true based on the fact that Q is true irrespective of the truth value of P.

Example

Suppose x is the proposition that "–2 is a negative integer" and y is the proposition that "0 is a whole number". Then the implication $x \rightarrow y$ is true, because y is true. In this case, the truth value of x is not considered.

3. **Direct Proof**

In direct proof the implication proposition of P and Q can be proved by assuming that P as true. Thus, from the known facts, if it is established that Q is true, then $P \rightarrow Q$ is proved to be true.

Example

Use the direct method to prove that "the product of any two odd integers is an odd integer".

Proof

Consider any two odd integers a and b, where,

$a = 2x + 1,\ x$ is an integer or $x \in Z$

$b = 2y + 1, y$ is an integer or $y \in Z$

$$\Rightarrow \quad a \times b = (2x+1) \times (2y+1)$$
$$= 4xy + 2x + 2y + 1$$
$$= 2(2xy + x + y) + 1$$
$$= 2k + 1, \text{ where } k = 2xy + x + y \text{ is an integer.}$$

From this it can be inferred that ab is an odd integer.

4. Existence Proof

In mathematics, many theorems are based on assertions that object of some type exist. This type of theorems include ∃ and can be proved by a method called 'Existence proof'. There are two types of existence proof,

(i) Constructive existence proof

(ii) Non-constructive existence proof.

(i) Constructive Existence Proof

A theorem of the form $\exists x\, P(x)$ can be proved by assuming that there exist an object 'O' for which $P(0)$ is true. This method of theorem proving is known as constructive existence proof.

Example

Prove that a positive integer can be expressed in two different forms as the sum of two squares.

Proof

Let a be an integer that has the required properties.

$\therefore$ Let $a = 625$.

The two different forms of 'a' as the sum of two squares are,

$$a = 7^2 + 24^2$$
$$a = 15^2 + 20^2$$

(ii) Non-constructive Existence Proof

If the theorem $\exists x P(x)$ is proved by assuming that there is no such object 'O' such that $P(0)$ is true then such method of theorem proving is known as non-constructive existence proof.

Example

Prove that there exist irrational numbers a^b such that a and b are also irrational.

Proof

Let $a = b = \sqrt{2}$ $\quad [\because \sqrt{2}$ is irrational$]$

Consider that a^b is rational,

$$\Rightarrow \quad a^b = \sqrt{2}^{\sqrt{2}}$$

Then, there will be a rational number in the desired from. But if $\sqrt{2}^{\sqrt{2}}$ is considered irrational then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$

$$\Rightarrow \quad a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$
$$= (\sqrt{2})^{(\sqrt{2})(\sqrt{2})}$$
$$= (\sqrt{2})^2 = 2$$

'2' is obviously a rational number.

Since, here any irrational numbers a and b do not exist such that a^b is rational, therefore this proof is called a non-constructive existence proof. Instead, here it has been established that '∃' a rational number a^b such that a and b are irrational. This means, among the two pairs i.e., $a = \sqrt{2}$, $b = \sqrt{2}$ and $a = (\sqrt{2})^{\sqrt{2}}$, $b = \sqrt{2}$, only one will have the desired property.

Indirect Proof

There are two types of indirect proof. They are,

(i) Contrapositive proof

(ii) Contradiction proof.

Contrapositive Proof

(i) Consider the implication $P \rightarrow Q$.

$$P \rightarrow Q = \sim Q \rightarrow \sim P \qquad [\because \text{By De Morgan's law}]$$

Therefore, to prove $P \rightarrow Q$ is true, it is sufficient to establish $\sim Q \rightarrow \sim P$ is true.

In this method, initially assume that Q is false. Based on this assumption and other information it can be obtained that P is also false, thus proving the theorem $P \rightarrow Q$ is true.

Example

Using indirect proof method it can be proved that, if x^2 is an odd integer then x is also odd integer.

Proof

Let x be an integer.

Given that x^2 is an odd integer. It should be proved that, x is also odd using indirect proof (contrapositive). Assume that x is not an odd (i.e., conclusion is false).

$\Rightarrow$ x can be considered as an even integer.

$\therefore$ Let, $x = 2k$ (Where k is any integer)

Squaring on both the sides,

$$x^2 = (2k)^2$$

$$x^2 = 4k^2$$

$x^2 = 2(2k^2)$, which is an even integer.

$\therefore$ The hypothesis x^2 is an odd integer is false.

Hence by contrapositive law, the assumption that x is an even integer is false. This mean x is an odd integer. Thus, the theorem if x^2 is an odd integer then x is also an odd integer is true.

(ii) **Contradiction Proof**

In this method of proof, the implication $P \rightarrow Q$ can be proved to be true by proving that $P \wedge \sim Q$ is false ($\because P \rightarrow Q \equiv (\sim P \vee Q)$) by demorgans. So, it can be constructed in the following manner,

(a) Assume that $P \wedge \sim Q$ is true.

(b) Based on the assumption and the information available it can be inferred that despite of having truth value true certain proposition yields the value false.

(c) Thus, this contradiction reveals that the assumption is false. Hence, $P \wedge (\sim Q)$ is false.

$\therefore$ $p \rightarrow q$ is true.

6. **Proof by Cases**

If P and Q are two propositions then the theorem of implication $P \rightarrow Q$ can be proved by establishing different cases. That is,

$$P_1 \rightarrow Q$$
$$P_2 \rightarrow Q$$
$$P_3 \rightarrow Q$$
$$\vdots$$
$$P_n \rightarrow Q$$

Where, $P = P_1 \vee P_2 \vee P_3 \vee \ldots \vee P_n$ for any proposition P_i, $1 \leq i \leq n$.

Hence, to prove $P \rightarrow Q$, the following cases i.e., $P_1 \rightarrow Q, P_2 \rightarrow Q, \ldots P_n \rightarrow Q$ must be proved.

Example

Assume 'x' be a real number. If $|x| > 4$ then prove that $x^2 > 16$ using proof by cases.

Proof

Consider a and b be two propositions wherein,

$a = x$ is a real number and $|x| > 4$

$b = x$ is a real number and $x^2 > 16$.

Inorder to prove $a \to b$, it should be proved that $a_1 \to b$ and $a_2 \to b$. So, divide the proposition 'a' into two different cases i.e., $a = a_1 \vee a_2$.

Case 1

Let a_1 be a proposition such that x is a real number, $x \leq 0$ and $|x| > 4$.

So, $|x| = -x$ and $|x| > 4$

$-x > 4$

Now, squaring on both sides,

$(-x^2) > (4)^2$

$x^2 > 16$

$\therefore$ b is true.

It can be concluded that $a_1 \to b$ is true.

Case 2

Let a_2 be a proposition such that x is a real number, $x > 0$ and $|x| > 4$.

$\Rightarrow$ $|x| = x$ and $|x| > 4$

$\Rightarrow$ $x > 4$

Now, squaring on both sides,

$\Rightarrow$ $x^2 > 16$

$\therefore$ b is true

It can concluded that $a_2 \to b$ is true. Thus from case (1) and case (2), it can be concluded that $a \to b$.

7. Exhaustive Proof

Exhaustive proof is a proof method in which few examples or cases are tested or proved. It is a special type of proof by cases method.

In this method, a single statement is divided into multiple cases and then every case is proved separately. This method is referred as an exhaustive proof, because these proofs originate by exhausting (considering) all the circumstances.

Exhaustive proof method consists of two steps. They are as follows,

Step 1: Initially consider all the cases from the given statement that is to be proved.

Step 2: Finally prove each case and conclude the statement.

Example

Prove that $(n + 1)^2 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Proof

To prove that $(n + 1)^2 \geq 3^n$, use the exhaustive proof method as follows,

Here, inequality is proved, by substituting the value of n as 1, 2, 3 and 4 in $(n + 1)^2 \geq 3^n$, if the condition is satisfied, then the statement will be proved.

For $n = 1, 2, 3, 4$, $(n + 1)^2$ can be written as,

Case (1)

If $n = 1$,

L.H.S $(n + 1)^2 = (1 + 1)^2$

$= (2)^2$

$= 4$

R.H.S $3^n = 3^1 = 3$

Case (2)

If $n = 2$,

L.H.S $(n + 1)^2 = (2 + 1)^2$

$= 3^2$

$= 9$

R.H.S $3^n = 3^2 = 9$

Case (3)

If $n = 3$,

L.H.S $(n + 1)^2 = (3 + 1)^2$

$= 4^2$

$= 16$

R.H.S $3^n = 3^3 = 27$

Case (4)

If $n = 4$,

L.H.S $(n + 1)^3 = (4 + 1)^3$

$= 5^3$

$= 125$

R.H.S $3^n = 3^4 = 81$

Since in all the four cases, $(n + 1)^2 \geq 3^n$. Hence it is proved by exhaustive proof that, $(n + 1)^2 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Q95. Find a counter example to the statement that every positive integer can be written as the sum of the squares of three integers.

Answer :

Given statement is,

Every positive integer can be written as the sum of the squares of three integers

The sum of squares of three integers

$\Rightarrow$ Integers can be expressed in the form of x^2 where $x^2 \in Z$

Consider,

$1 = 0 + 0 + 1$

$2 = 0 + 1 + 1$

$3 = 1 + 1 + 1$

$4 = 1 + 1 + 2$

It can be observed that 1 and 0 are squares of themselves

Consider,

$7 = - + - + 4$

7 cannot be expressed as sum of three squares

The only integers which are less than 7 and are squares are 0, 1 and 4

So, atleast one of the integers must be 4

If only one integer is 4,

$\Rightarrow \quad 4 + 4 > 7$

So, the sum of remaining integers must be 3

But 0 and 1 cannot make 7

Hence, any combination of three integers never gives 7

Thus, the integer which cannot be written as the sum of squares of three integers is 7

Q96. Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

Model Paper-3, Q3(b)

Answer :

A perfect cube is defined as the cube of another number which is an integer

The perfect cubes which are less than 1000 are cubes of 1 to 9

i.e., $1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343, 8^3 = 512, 9^3 = 729$

Mathematically a perfect cube is written as,

$$x^3 = y^3 + z^3, y^3, z^3 < x^3$$

Consider the first perfect cube 1

1 cannot be written as the sum of two positive small values

Similarly, 8 cannot be written as the sum of two positive small values

Consider the cube 27,

$1 + 1 = 2 \neq 27$

$1 + 8 = 9 \neq 27$

$8 + 8 = 16 \neq 27$

Consider, 64

$1 + 27 = 28 \neq 64$

$8 + 27 = 35 \neq 64$

$27 + 27 = 54 \neq 64$

Consider, 125

$1 + 64 = 65 \neq 125$

$8 + 64 = 72 \neq 125$

$27 + 64 = 91 \neq 125$

$64 + 64 = 128 \neq 125$

Consider, 216

$1 + 125 = 126 \neq 216$

$8 + 125 = 133 \neq 216$

$27 + 125 = 152 \neq 216$

$64 + 125 = 189 \neq 216$

$125 + 125 = 250 \neq 216$

Consider, 343

$1 + 216 = 217 \neq 343$

$8 + 216 = 224 \neq 343$

$27 + 216 = 243 \neq 343$

$64 + 216 = 280 \neq 343$

$125 + 216 = 341 \neq 343$

$216 + 216 = 432 \neq 343$

Consider, 512

$1 + 343 = 344 \neq 512$

$8 + 343 = 351 \neq 512$

$27 + 343 = 370 \neq 512$

$64 + 343 = 407 \neq 512$

$125 + 343 = 468 \neq 512$

$216 + 343 = 559 \neq 512$

$343 + 343 = 683 \neq 512$

Consider, 729

$1 + 512 = 513 \neq 729$

$8 + 512 = 520 \neq 729$

$27 + 512 = 539 \neq 729$

$64 + 512 = 576 \neq 729$

$125 + 512 = 637 \neq 729$

$216 + 512 = 768 \neq 729$

$343 + 512 = 855 \neq 729$

$512 + 512 = 1024 \neq 729$

It can be observed that none of the sum taken gives value more than 1000

Hence, there are no positive perfect cubes less than 1000 that are the sum of cubes of two positive integers.

Q97. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

Answer :

Given inequality is,

$$n^2 + 1 \geq 2^n$$

n is a positive integer with $1 \leq n \leq 4$

The above inequality can be proved by using the method of exhaustion. The following cases are considered

Case (i) For $n = 1$,

$$n^2 + 1 = 1^2 + 1 = 2$$

$$2^n = 2^1 = 2$$

$$\therefore \quad n^2 + 1 \geq 2^n \text{ for } n = 1$$

Case (ii) For $n = 2$,

$$n^2 + 1 = 2^2 + 1$$
$$= 4 + 1$$
$$= 5$$
$$2^n = 2^2 = 4$$
$$\Rightarrow \quad 5 \geq 4$$
$$\therefore \quad n^2 + 1 \geq 2^n \text{ for } n = 2$$

Case (iii) For $n = 3$,

$$n^2 + 1 = 3^2 + 1$$
$$= 9 + 1$$
$$= 10$$
$$2^n = 2^3$$
$$= 8$$
$$\Rightarrow \quad 10 \geq 8$$
$$\therefore \quad n^2 + 1 \geq 2^n \text{ for } n = 3$$

Case (iv) For $n = 4$,

$$n^2 + 1 = 4^2 + 1$$
$$= 16 + 1$$
$$= 17$$
$$2^n = 2^4$$
$$= 16$$
$$\Rightarrow \quad 17 \geq 16$$
$$\therefore \quad n^2 + 1 \geq 2^n \text{ for } n = 4$$

In each of these four cases,

$$n^2 + 1 \geq 2^n$$

$\therefore$ $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

Q98. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

Model Paper-5, Q2(b)

Answer :

Given equation is,

$$2x^2 + 5y^2 = 14$$

If $|x| \geq 3$

$\Rightarrow$ $2x^2 > 14$

$\Rightarrow$ $|x|$ can't take values more than 3

If $|x| \geq 2$

$5y^2 > 14$

$\Rightarrow$ $|y|$ can't take values more than 2

x can take values $-2, -1, 0, 1, 2$

y can take values $-1, 0, 1$

The possible values of $2x^2$ are obtained as,

$$2(0)^2 = 0$$
$$2(-1)^2 = 2$$
$$2(1)^2 = 2$$
$$2(-2)^2 = 8$$
$$2(2)^2 = 8$$

$\therefore$ $2x^2$ can take values 0, 2, 8

The possible values of $5y^2$ are obtained as,

$$5(-1)^2 = 5$$
$$5(0)^2 = 0$$
$$5(1)^2 = 5$$

$\therefore$ $5y^2$ can take values 0, 5

The maximum sum possible is

$$2x^2 + 5y^2 = 8 + 5 = 13 < 14$$

Hence, there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$

Q99. Prove that $\sqrt[3]{2}$ is irrational.

Answer :

Proof by Contradiction

Assume that $\sqrt[3]{2}$ is rational

From the definition of rational number there are integers p and q with $q \neq 0$ such that

$$\sqrt[3]{2} = \frac{p}{q}$$

Where p and q have no common factors

i.e., $\gcd(p, q) = 1$

Consider,

$$\sqrt[3]{2} = \frac{p}{q}$$

Cubing on both sides,

$$\left(2^{\frac{1}{3}}\right)^3 = \left(\frac{p}{q}\right)^3$$

$\Rightarrow$ $2 = \frac{p^3}{q^3}$

$\Rightarrow$ $p^3 = 2q^3$

$\Rightarrow$ p^3 is even

$\Rightarrow$ p is even

Let $p = 2k$ for some integer k

Cubing on both sides,

$p^3 = (2k)^3$

$\Rightarrow$ $p^3 = 8k^3$

Substituting $p^3 = 2q^3$ in above equation,

$2q^3 = 8k^3$

$\Rightarrow$ $q^3 = 4k^3$

$\Rightarrow$ q^3 is even

$\Rightarrow$ q is even

$\therefore$ p and q are even

$\Rightarrow$ p and q are divisible by 2

$\Rightarrow$ p and q have common factors

$\Rightarrow$ $gcd(p, q) \neq 1$

This is a contradiction

$\Rightarrow$ $\sqrt[3]{2}$ is irrational

Hence, $\sqrt[3]{2}$ is irrational

Q100. Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

Answer :

Given equation is,

$n^2 + n^3 = 100$

For $n = 1$,

$n^2 + n^3 = 1^2 + 1^3$

$= 1 + 1$

$= 2$

$\Rightarrow$ $2 \neq 100$

For $n = 2$,

$n^2 + n^3 = 2^2 + 2^3$

$= 4 + 8$

$= 12$

$\Rightarrow$ $12 \neq 100$

For $n = 3$,

$n^2 + n^3 = 3^2 + 3^3$

$= 9 + 27$

$= 36$

$\Rightarrow$ $36 \neq 100$

For $n = 4$,

$n^2 + n^3 = 4^2 + 4^3$

$= 16 + 64$

$= 80$

$\Rightarrow$ $80 \neq 100$

For $n = 5$,

$n^2 + n^3 = 5^2 + 5^3$

$= 25 + 125$

$= 150$

$\Rightarrow$ $150 > 100$

$\Rightarrow$ $n^2 + n^3 > 100 \ \forall n > 4$

Hence, there is no positive integer n such that $n^2 + n^3 = 100$.