Induction.

Many mathematical statements assert that a Property is true for all +ve integers.

Examples of such statements are that for every Positive integer n: $n! < n^3$, $n^3 - n$ is divisible by 3; and the sum of the first n Positive integers is $n(n+1)/2$.

Proofs using mathematical Induction has two Parts. First, they show that the stmts holds for the Positive integer 1. Second, they show that if the stmt holds for a Positive integer then it must also hold for the next larger integer

Introduction.

Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder. We know 2 things:

1. We can reach the first rung of the ladder.
2. If we can reach a Particular rung of the ladder, then we can reach the next rung.

But can we conclude that we are able to reach every rung of this infinite ladder? The answer is Yes, something we can Verify using an important Proof technique called mathematical Induction.

Mathematical Induction

In general, mathematical induction can be used to Prove statements that assert that $P(n)$ is true for all Positive integers n, where $P(n)$ is a Propositional function.

A Proof by mathematical induction has two Parts, a basic step, where we show that $P(1)$ is true and an inductive step, where we show that for all Positive integers K, if $P(K)$ is true, then $P(K+1)$ is true.

The assumption that $P(K)$ is true is called * inductive hypothesis.

Example 1: Show that if n is a +ve integer, then

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Soln: Let $P(n)$ be the Proposition that the sum of the first n Positive integers is $n(n+1)/2$.

We must do two things to Prove that $P(n)$ is true for $n=1,2,3,\ldots$ Namely, we must show that $P(1)$ is true & the conditional statement $P(K)$ implies $P(K+1)$ is true for $k=1,2,3,\ldots$

Basis step: $P(1)$ is True, because $1 = \frac{1(1+1)}{2}$

Inductive step:

For the inductive hypothesis we assume that P(K) holds for an arbitrary Positive integer K. That is, we assume that

Assumption $1+2+\cdots+K = \frac{K(K+1)}{2}$

Under this assumption, it must be shown that P(K+1) is true, namely that

Prove $1+2+\cdots+K+(K+1) = \frac{(K+1)[(K+1)+1]}{2}$

$= \frac{(K+1)(K+2)}{2}$

is also true, when we add K+1 to both sides of the Equation in P(K), we Obtain

L.H.S $1+2+\cdots+K+(K+1) = \frac{K(K+1)}{2}+(K+1)$

$= \frac{K(K+1)+2(K+1)}{2}$

$= \frac{(K+1)(K+2)}{2}$

$\therefore$ L.H.S = R.H.S.

This last Equation shows that P(K+1) is true under the assumption that P(K) is true.

This Completes the inductive step.

We have Completed the basis step & the inductive step, so by mathematical induction we know that P(n) is true for all Positive integer n.

That is, we have Proven that

$1+2+\cdots+n = n(n+1)/2$ for all +ve integers n

Example 2.

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Soln: The sum of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$1 = 1$, $1+3 = 4$, $1+3+5 = 9$,

$1+3+5+7 = 16$, $1+3+5+7+9 = 25$.

From these values it is reasonable to conjecture that the sum of the first n positive odd integers is n^2, that is $1+3+5+\dots+(2n-1) = n^2$.

Let $p(n)$ denote the proposition that the sum of the first n odd +ve integers is n^2.

Our conjecture is that $P(n)$ is true for all +ve integers. To use mathematical induction to prove this conjecture, we must first complete the basis step that is, we must show that $p(1)$ is true. Then we must carry out inductive step, that is we must show that $P(k+1)$ is true when $P(k)$ is assumed to be true.

Basis step: $P(1)$ states that the sum of the first one odd positive integer is 1^2.

This is true because the sum of the first odd +ve integer is 1.

The basis step is complete.

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Inductive Step:- To Complete the inductive Step we must show that the Proposition $P(K) \rightarrow P(K+1)$ is true for every Positive integer k.

To do this, we first assume the inductive hypothesis. The inductive hypothesis is the statement that $P(K)$ is true, that is,

$$1+3+5+\cdots+(2K-1)=K^2 \quad -①$$

[Note that the kth odd Positive integer is $(2K-1)$, because this integer is obtained by adding 2 a total of $K-1$ times to 1].

To show that $\forall K\,(P(K) \rightarrow P(K+1))$ is true, we must show that if $P(K)$ is true, then $P(K+1)$ is true. Note that $P(K+1)$ is the Stmt that

$$1+3+5+\cdots+(2K-1)+(2K+1)=(K+1)^2$$

$2(K+1)-1 = 2K+2-1 = 2K+1$

So, assuming that $P(K)$ is true, it follows that

L.H.S:

$$1+3+5+\cdots+(2K-1)+(2K+1)$$

$$= [1+3+\cdots+(2K-1)]+(2K+1)$$

$$= K^2+(2K+1)$$ Substitute Eqn ① Value

$$= K^2+2K+1$$

$$= (K+1)^2$$

$\therefore$ L.H.S = R.H.S

This Shows that $P(K+1)$ follows from $P(K)$.

Note that we used inductive hypothesis $P(K)$ in the second Equality to replace the sum of the first K odd Positive integers by K^2.

Example

3 Sums of Geometric Progressions.

Use mathematical induction to Prove this formula for the Sum of a finite number of terms of a geometric Progression.

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1} \quad \text{when } r \neq 1.$$

Where n is a nonnegative integer.

Soln: To Prove this formula using mathematical induction, let $P(n)$ be the stmt that the sum of the first $n+1$ terms of a geometric Progression in this formula is correct.

Basis Step: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r-1} = \frac{ar - a}{r-1} = \frac{a(r-1)}{r-1} = a.$$

Inductive step

The inductive hypothesis is the stmt that $P(k)$ is true, where k is a nonnegative integer.

That is, $P(k)$ is the stmt that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r-1}.$$

To Compute the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true.

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Now for $n = k+1$

$a + ar + ar^2 + \dots + ar^k + ar^{k+1} =$

$(a + ar + ar^2 + \dots + ar^k) + ar^{k+1}$ — (2)

Substitute Eqn (1) Value in Eqn (2)

$$= \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$

$$\therefore a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{(k+1)+1} - a}{r-1}$$

Thus $P(k+1)$ is true

We have Completed the basis step & inductive step, So by mathematical induction, $P(n)$ is true for all non negative integers n.

This Shows that the formula for the Sum of the terms of a geometric Series is correct.

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4. Use mathematical induction to show that

$$1+2+2^2+\cdots+2^n = 2^{n+1}-1.$$

for all nonnegative integers n.

Soln Let $P(n)$ be the Proposition that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for integers n.

Basis Step: $P(0)$ is true, because

$$2^0 = 1 = 2^1 - 1$$
$$1 = 1$$

This completes the basis step.

Inductive Step. For the inductive hypothesis, we assume that $P(k)$ is true. That is we assume that

$$1+2+2^2+\cdots+2^k = 2^{k+1}-1. \quad \text{— (1)}$$

To carryout the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$\underbrace{1+2+2^2+\cdots+2^k}+2^{k+1} = 2^{(k+1)+1}-1 \quad \text{— (2)}$$
$$= 2^{k+2}-1$$

Substituting Eqn (1) value in Eqn (2)

L.H.S. $2^{k+1}-1+2^{k+1}$

$= 2.2^{k+1}-1$ (In multiplication when bases are same Powers Can be added)

$= 2^{k+2}-1$

We have Completed inductive Step.

Because we have Computed the basis step & inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n.

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II Proving Inequalities

Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a particular positive integer

Example (1) Use mathematical induction to prove the inequality $n<2^n$ for all positive integers n

Soln: Let $P(n)$ be the proposition that $n<2^n$ (1)

Basis step: $P(1)$ is true, because $1<2^1=2$
$1<2$

This completes the basis step

Inductive step

for $n=k$, the eqn (1) becomes
$k<2^k$ - (2)

for $n=k+1$
$k+1<2^{k+1}$

That is we need to show that if $k<2^k$, then $k+1<2^{k+1}$. To show that this conditional statement is true for the positive integer k, we first add 1 to both sides of $k<2^k$

$$k+1<2^k+1$$
$$<2^k+2^k \quad (\because 1<2^k)$$
$$<2\cdot 2^k$$
$$k+1<2^{k+1}$$

This shows that $P(k+1)$ is true, namely $k+1<2^{k+1}$, based on assumption that $P(k)$ is true. The induction step is complete

since we have done basis & induction step we show that $n<2^n$

2. Use mathematical induction to Prove that $2^n < n!$ for every Positive integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2$ and 3)

Soln:- Let $P(n)$ be the Proposition that $2^n < n!$

Basis step: To Prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

Inductive Step:

For the inductive step, we assume that $P(K)$ is true for the Positive integer K with $K \geq 4$. That is, we assume that $2^K < K!$ for the Positive integer K with $K \geq 4$.

We must Show that under this hypothesis, $P(K+1)$ is also true.

That is, we must Show that if $2^K < K!$ for the Positive integer K where $K \geq 4$, then

$$2^{K+1} < (K+1)!$$

Take L.H.S

$2^{K+1} = 2 \cdot 2^K$ by definition of Exponent

$< 2 \cdot K!$ by inductive hypothesis $K \geq 4$.

$< (K+1)K!$ because $2 < K+1$. $(2<5)$ $(2<6)$ $(2<7)$

$= (K+1)!$

by definition of factorial function

This Shows that $P(K+1)$ is true when $P(K)$ is true.
This Completes the inductive step of the Proof.

11 (6) Divisibility Results

Use mathematical Induction to Prove that

$$2 - 2\cdot 7 + 2\cdot 7^2 \cdots + 2(-7)^n = (1-(-7)^{n+1})/4$$

whenever n is a non negative integer

Soln: Let $p(n)$ be $2 - 2\cdot 7 + 2\cdot 7^2 + \cdots + 2(-7)^n = \frac{(1-(-7)^{n+1})}{4}$ — (1)

Basis Step:

For $n=0$

$$L.H.S = 2\cdot(-7)^0$$
$$= 2\cdot(1)$$
$$= 2$$

$$R.H.S : \frac{1-(-7)^{0+1}}{4} = \frac{1-(-7)}{4} = \frac{8}{4} = 2$$

$\therefore$ L.H.S = R.H.S

$\therefore p(0)$ is true.

Inductive Step:

for $n=k$, the Eqn (1) becomes

$$2 - 2\cdot 7 + 2\cdot(7)^2 + \cdots 2(-7)^k = \frac{(1-(-7)^{k+1})}{4} \quad (2)$$

$\therefore P(k)$ is true

For $n = k+1$

$$2 - 2\cdot 7 + 2\cdot(7)^2 + \cdots + 2(-7)^k + 2(-7)^{k+1} \quad (3)$$

Substitute Eqn (2) value in eqn (3)

$$= \frac{(1-(-7)^{k+1})}{4} + 2\cdot(-7)^{k+1}$$

$$= \frac{1-(-7)^{k+1} + 8(-7)^{k+1}}{4}$$

$$= \frac{1+7^{k+1} + 8\cdot -7^{k+1}}{4}$$

$$= \frac{1 + 7^{k+1} - 56^{k+1}}{4}$$

$$= \frac{1 + 7 \cdot (-7)^{k+1}}{4}$$

$$= \frac{1 - (-7)(-7)^{k+1}}{4}$$

$$= \frac{1 - (-7)^{k+2}}{4}$$

$$\therefore \; 2 - 2 \cdot 7 + 2 \cdot (7)^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{1 - (-7)^{(k+1)+1}}{4}$$

$\therefore$ $p(k+1)$ is true

By mathematical induction
$P(n)$ is true $\forall$ non-negative integers n.

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III Proving Divisibility Results

Mathematical induction can be used to prove divisibility results about integers.

Example.

Use mathematical induction to prove that n^3-n is divisible by 3 whenever n is a positive integer.

Soln To construct the proof, let $P(n)$ denote the proposition. "n^3-n is divisible by 3".

Basis step :- The statement $P(1)$ is true because $1^3-1=0$ is divisible by 3.

This completes the basis step

Inductive step :- For the inductive hypothesis we assume that $P(K)$ is true; that is, we assume that K^3-K is divisible by 3.

To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(K+1)$, the statement that $(K+1)^3-(K+1)$ is divisible by 3, is also true.

That is, we must show that $(K+1)^3-(K+1)$ is divisible by 3. Note that

$$(K+1)^3-(K+1) = (K^3+1+3K^2+3K)-(K+1)$$

$$\because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$$

$$= \cancel{3(K^2+K)} \quad K^3+\cancel{1}+3K^2+3K-K-\cancel{1}$$

$$= 3(K^2+K)+(K^3-K)$$

Because both terms in this sum are divisible by 3 (the second by inductive hypothesis, & first because it is

3 times an integer), it follows that
$(k+1)^3 - (k+1)$ is also divisible by 3.
This completes the inductive step.

Because we have completed both the basis step & inductive step, by the Principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

IV Proving Results about sets.

Mathematical induction can be used to prove many results about sets.

Example:
Use mathematical induction to prove the following generalization of one of De Morgan's law.

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

whenever $A_1, A_2 \ldots A_n$ are subsets of a universal set U and $n \geq 2$.

Soln: Let $P(n)$ be the identity for n sets.

Basis step: The statement $P(2)$ asserts that
$\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$. This is one of De Morgan's law.

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Inductive Step:

The inductive hypothesis is the statement that $P(K)$ is true, where K is an integer with $K \geq 2$. That is, it is the statement that

$$\overline{\bigcap_{j=1}^{K} A_j} = \bigcup_{j=1}^{K} \overline{A_j}.$$

Whenever $A_1, A_2 \ldots A_K$ are subsets of the universal set U. To carry out the inductive step, we need to show that this assumption implies that $P(K+1)$ is true.

That is, we need to show that if this equality holds for every collection of K subsets of U, then it must also hold for every collection of $K+1$ subsets of U.

Suppose that $A_1 A_2, \ldots A_K, A_{K+1}$ are subsets of U. When inductive hypothesis is assumed to hold, it follows that

$$\overline{\bigcap_{j=1}^{K+1} A_j} = \overline{\left(\bigcap_{j=1}^{K} A_j\right) \cap A_{K+1}} \quad \text{by definition of intersection}$$

$$= \overline{\left(\bigcap_{j=1}^{K} A_j\right)} \cup \overline{A_{K+1}} \quad \text{by De Morgan's Law.}$$

$$= \left(\bigcup_{j=1}^{K} \overline{A_j}\right) \cup \overline{A_{K+1}} \quad \text{by inductive hypothesis}$$

$$= \bigcup_{j=1}^{K+1} \overline{A_j}.$$

This completes inductive step.

By mathematical induction we know that $P(n)$ is true whenever n is a positive integer, $n \geq 2$.