UNIT 5

GRAPHS AND TREES

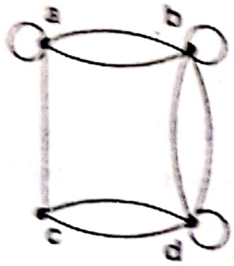
SIA GROUP

PART-A

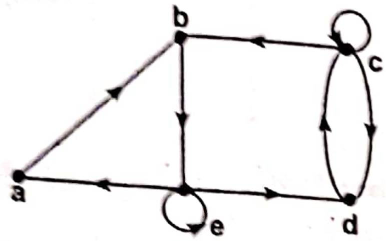
SHORT QUESTIONS WITH SOLUTIONS

Q1. Determine the type of graph

(a)



(b)



Answer :

(a) Given graph has 4 vertices *a, b, c, d*.

It has undirected edges.

The pair of edges (*a, b*), (*b, d*) and (*c, d*) have multiple edges between them and the vertices *a, b, d* have self loops

∴ The graph is a pseudograph

(b) Given graph has 5 vertices, *a, b, c, d, e*.

It has directed edges

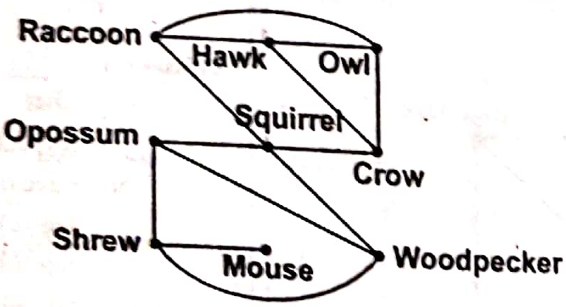
Since the direction of multiple edges between the vertices *c* and *d* is in opposite direction

∴ The graph has no multiple edges

And the vertices *c* and *e* have self loops

∴ The graph is a directed graph.

Q2. Determine the species that compete with hawks in the niche overlap graph



A Niche Overlap Graph

Answer :

The graph contains the vertices which represent species that compete for the same resources.

Here, Hawk competes with the species Raccoon, owl and crow for same resources but does not compete with squirrel for same resources.

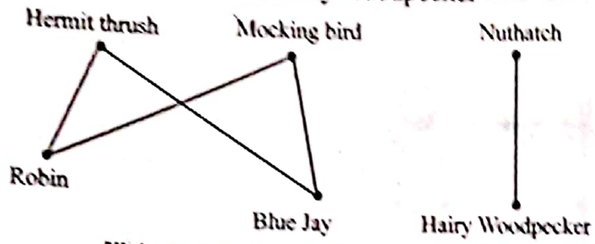
Q3. Construct a niche overlap graph for six species of birds, where the hermit thrush competes with the robin and with the blue jay, the robin also competes with the mocking bird, the mocking bird also competes with the blue jay, and the nuthatch competes with the hairy woodpecker.

Answer :

Given that,

A graph contains six species of birds which compete with each other

Hermit thrush with Robin
Hermit thrush with Blue jay
Robin with Mocking bird
Mocking bird with Blue jay
Nuthatch with Hairy woodpecker



Niche Overlap Graph [Species of Birds]

In the above graph, vertices represent species of birds and edges represent that two species of birds compete with each other.

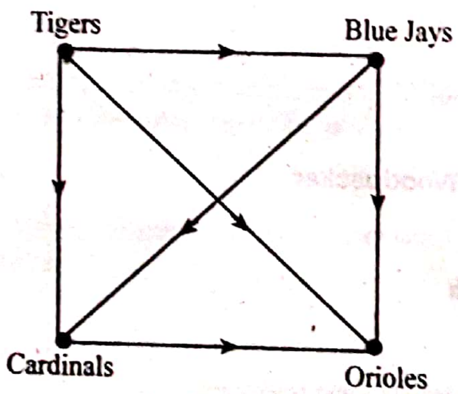
Q4. In a round-robin tournament the Tigers beat the Blue Jays, the Tigers beat the Cardinals, the Tiger beat the Orioles, the Blue Jays beat the Cardinals, the Blue Jays beat the Orioles, and the Cardinals beat the Orioles. Model this outcome with a directed graph.

Answer :

Given that,

A graph contains 4 teams

Tigers beat Blue Jays
Tigers beat Cardinals
Tigers beat Orioles
Blue Jays beat Cardinals
Blue Jays beat Orioles
Cardinals beat Orioles



Roud-Robin Tournament

In the above graph, vertices represents teams and edges represents that one team beats another team.

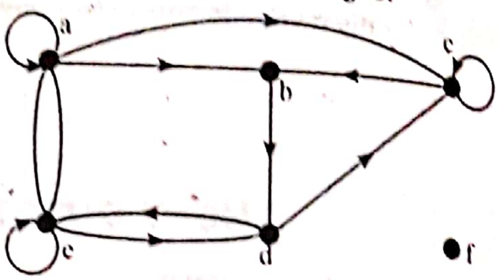
Q5. Define Underlying Undirected graph.

Answer :

Model Paper-1, Q1(

The underlying undirected graph is an undirected graph that removes the directions of edges. The edges of both the graph with directed edges and its underlying undirected graph are same.

Q6. Construct the Underlying Undirected graph for the graph with directed edges.

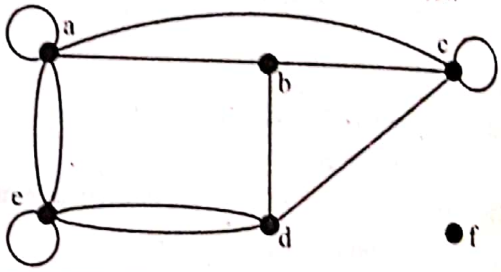


Answer :

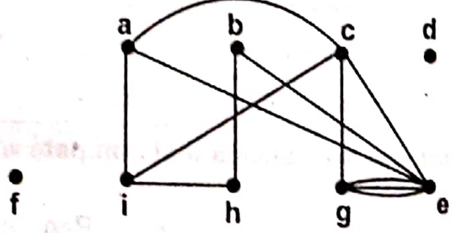
Given that,

A graph has directed edges

An underlying undirected graph obtained by removing the directions/arrows of the edges is shown below



Q7. For the undirected graph, find the sum of the degrees of the vertices and verify that it equals twice the number of edges in the graph.



Answer :

Given that,

A graph contains 9 vertices and 12 edges

The degree of vertex is the number of edges incident to it

deg $(a) = 3$
deg $(b) = 2$
deg $(c) = 4$
deg $(d) = 0$
deg $(e) = 6$
deg $(f) = 0$
deg $(g) = 4$
deg $(h) = 2$
deg $(i) = 3$

The sum of degrees is,

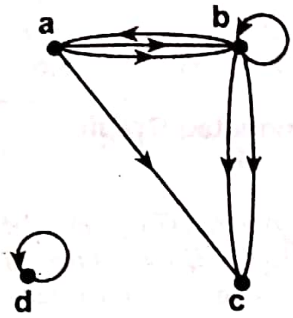
$3+2+4+0+6+0+4+2+3$

$=24$

$=2(12)$

$\therefore$ The sum of degrees is twice the number of edges

Q8. Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.



Answer :

Given that,

A graph contains 4 vertices and 8 edges i.e., 6 lines and 2 loops

The in-degree of a vertex is the number of edges that have the vertex as their terminal vertex i.e., edges incoming to a vertex

$\deg^-(a)=2$

$\deg^-(b)=3$ [$\because$ 2 edges and 1 loop]

$\deg^-(c)=2$

$\deg^-(d)=1$ [$\because$ 1 loop]

The out-degree of a vertex is the number of edges that have the vertex as their initial vertex i.e., edges outgoing from a vertex.

$\deg^+(a)=2$

$\deg^+(b)=4$ [$\because$ 3 edges and 1 loop]

$\deg^+(c)=1$

$\deg^+(d)=1$ [$\because$ 1 loop]

Q9. Define degree sequence.

Answer : **Model Paper-2, Q1(l)**

The degree sequence of a graph is the arrangement of degree of vertices in nonincreasing order i.e., from largest degree to smallest degree.

Q10. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

Answer :

Given that,

Degree sequence is 5, 2, 2, 2, 2, 1

From Handshaking theorem,

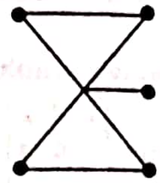
$$2m=\sum_{i=1}^{n}\deg(V_i)$$

$\Rightarrow \quad 2m=5+2+2+2+2+1$

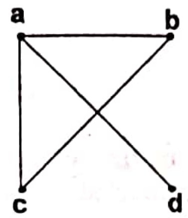
$\Rightarrow \quad 2m=14$

$\Rightarrow \quad m=7$

$\therefore$ Number of edges = 7



Q11. Use an adjacency matrix to represent the following graph.



Answer :

Given graph is,

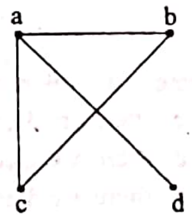


Figure: Graph G

The adjacency matrix for G is,

$$A=\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Q12. Draw the graph with the following adjacency matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Answer : **Model Paper-3, Q1(l)**

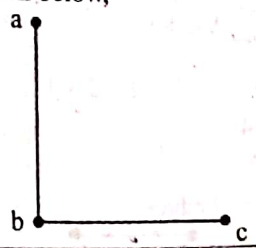
Given adjacency matrix is,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Let a, b, c represent the vertices of the graph. Then the adjacency matrix becomes,

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

The above adjacency matrix can be graphically represented as shown below,



Q13. Draw an undirected graph represented by the adjacency matrix $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

Answer : Model Paper-4, Q1(l)

Given adjacency matrix is,

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$$

The adjacency matrix i.e., $A = [a_{ij}]$ with loops and multiple edges for an undirected graph is,

$$a_{ij} = \begin{cases} m & \text{, if there are } m \text{ edges from } v_i \text{ to } v_j \\ 0 & \text{, otherwise} \end{cases}$$

From the matrix,

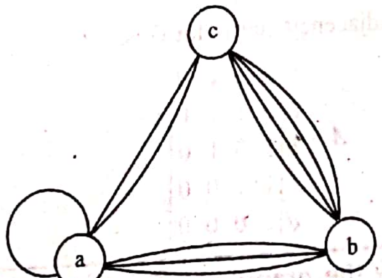
$a_{11} = 1$ i.e., there is a loop at a

$a_{12} = a_{21} = 3$ i.e., there are 3 edges from a to b

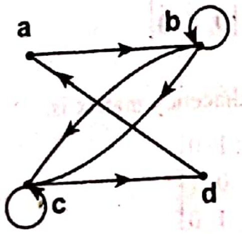
$a_{13} = a_{31} = 2$ i.e., there are 2 edges between a and c

$a_{23} = a_{32} = 4$ i.e., there are 4 edges between b and c

∴ The undirected graph is,



Q14. Find the adjacency matrix of the directed multigraph



Answer :

Given that,

A graph has 4 vertices a, b, c, dw

Here, $a_{12} = 1$ i.e., there is an edge from a to b

$a_{22} = 1$ i.e., there is a loop at b

$a_{23} = 1$ i.e., there is an edge from b to c

$a_{32} = 1$ i.e., there is an edge from c to b

$a_{33} = 1$ i.e., there is a loop at c

$a_{34} = 1$ i.e., there is an edge from c to d

$a_{41} = 1$ i.e., there is an edge from d to a

Since there are no more edges

Then other elements in matrix should be '0'

∴ The adjacency matrix is,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Q15. Define Connected Graph?

Answer : Model Paper-5, Q1(i)

An undirected graph G is said to be connected if every pair of distinct vertices in G are connected. In other words, an undirected graph G is said to be connected if there exists at least one path between every two distinct vertices in G.

Example

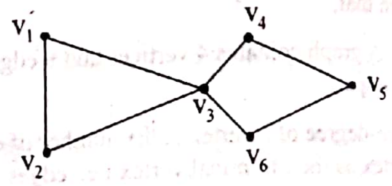


Figure : Connected Graph

Q16. What is cut-vertex?

Answer : Model Paper-1, Q1(j)

A cut vertex(also known as articulation point) is a vertex in connected graph whose deletion disconnects it into two or more sub graphs.

Q17. State Dirac's theorem?

Answer : Model Paper-2, Q1(j)

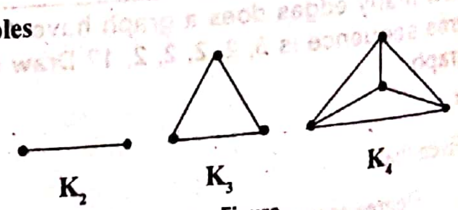
If G is simple graph with n vertices with $n \geq 3$, such that the degree of every vertex in G is atleast $n/2$, then G has Hamilton circuit.

Q18. Define planar graphs.

Answer :

A graph that can be represented by at least one plane drawing in which the edges meet only at the vertices is called a planar graph. Such a type of drawing is called a planar representation of the graph.

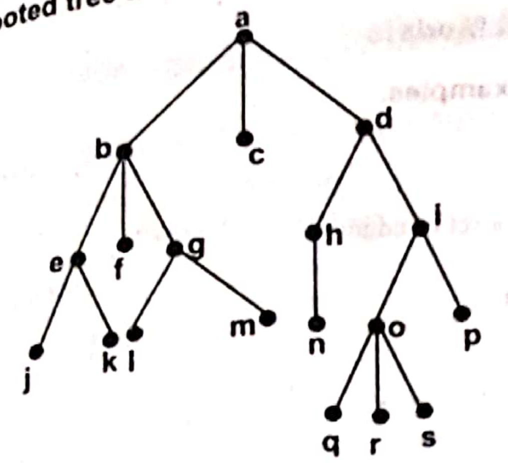
Examples



Figure

The above complete graphs K_2, K_3 and K_4 are good examples of planar graphs since they can be drawn without any crossovers

Q19. For some positive integer m, is the given rooted tree a full m-ary tree?



Answer :

The rooted tree is a full m-ary tree if every internal vertex has exactly m children

Given rooted tree has internal vertices a, b, d, e, g, h, i, o and some of these internal vertices have two children and other have three children

$\therefore$ It is not a full m-ary tree

Q20. Define infix, prefix and postfix notation.

Model Paper-3, Q1(j)

Answer :

Infix Form

In an in order traversal, if the expression produced is same as original expression with elements and operations except unary operations, then the expression is said to be in infix form.

Example

$(x + y)/(x + 3)$, $(x + (y/x)) + 3)$ and $x + (y/(x + 3))$

Infix form of all three expressions is, $x + y/x + 3$

Prefix Form

The expression obtained after traversing a rooted tree in pre order is called as prefix notation.

Example

$((x + y) \uparrow 2) + ((x - 4)/3)$

Prefix form $+ \uparrow + x\ y\ 2\ / - x\ 4\ 3$

Postfix Form

The expression obtained by traversing a binary tree in post order is called as postfix notation.

Example

$((x + y) \uparrow 2) + ((x - 4)/3)$

Postfix form $x\ y + 2 \uparrow x\ 4 - 3\ / +$

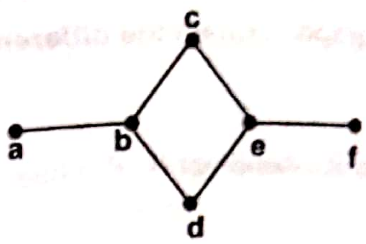
Q21. Define spaning tree.

Answer :

Model Paper-4, Q1(j)

Spanning tree of a simple graph G is defined as a subgraph containing every vertex of G. A simple graph is connected if and only if it has a spanning tree.

Q22. Draw all the spanning trees of the simple graph

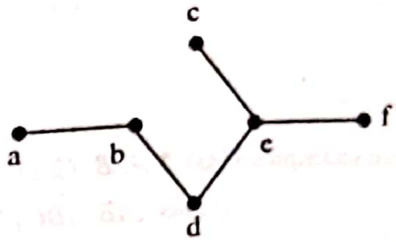


Answer :

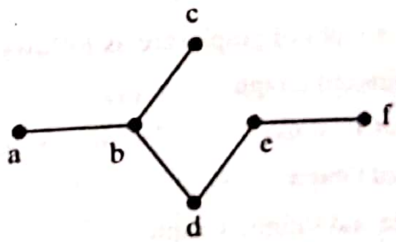
Model Paper-5, Q1(j)

The possible spanning trees are as follows,

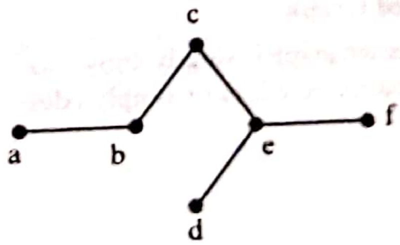
1.



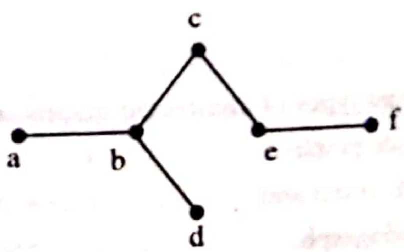
2.



3.



4.



PART-B

ESSAY QUESTIONS WITH SOLUTIONS

5.1 GRAPHS

5.1.1 Graphs and Graph Models

Q23. Define graph. Explain the different types of graphs with examples.

Answer :

Graph

A graph G is a pair of set (V, E) where V is a set of vertices and E is a set of edges.

Example

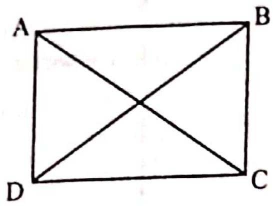


Figure (1): Graph

In the above example, $V(G)$ is A, B, C, D

$E(G)$ is $\{AB\}\{BC\}\{CD\}\{DA\}$

Types of Graphs

The various types of graphs are as follows,

1. Undirected Graph
2. Directed Graph
3. Mixed Graph
4. Finite and Infinite Graph

1. Undirected Graph

A non-directed graph is simply known as a graph. In this graph (V, E), the elements of V are called vertices and the elements of E are called undirected edges or simply edges. The set V is called a vertex set and the set E is called an edge set.

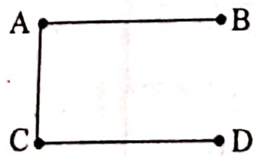


Figure (2): Undirected Graph

The different types of undirected graphs are as follows,

(i) Simple graph

(ii) Multi graph and

(iii) Pseudograph.

(i) Simple Graph

A graph without multiple edges and loops is called a simple graph.

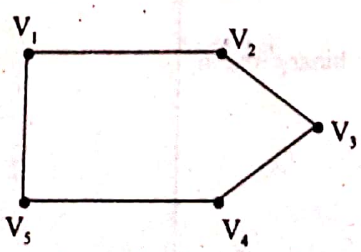


Figure (3): Simple Graph

(i) **Multigraph**

A multigraph can either be a directed or undirected graph in which each vertex have more than one edge but no loops.

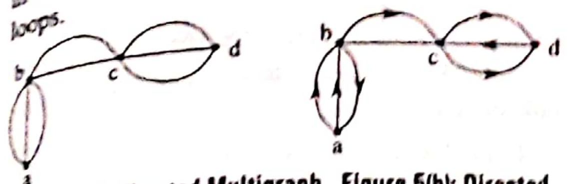


Figure5(a): Undirected Multigraph Figure 5(b): Directed Multigaph

Figure (5) : Multigraph

Example

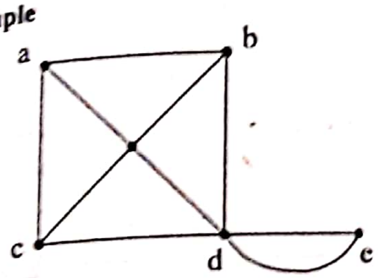


Figure (6)

The above figure shows an undirected multigraph. The vertex a has 3 edges i.e., $\{a, b\}$, $\{a, d\}$ and $\{a, c\}$. Hence, its multiplicity is 3. Similarly, the vertex d has 5 edges i.e., $\{d, e\}$, $\{d, e\}$, $\{d, b\}$, $\{d, c\}$, $\{d, a\}$. Hence, its multiplicity is 5.

Pseudographs

Pseudographs are the graphs with loops and multiple edges.

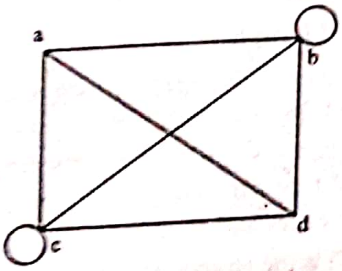


Figure (7) : Pseudograph

Directed Graph

A directed graph is also called a digraph. It consists of a ...empty set of vertices and a set of directed edges represented ...rdered pair of vertices taken from this set, namely $\{(A, ...C, D), (C, A)\}$. In a digraph (V, E), the elements of V are ...ces and elements of E are called directed edges. The set V ...led the vertex set and the set E is called the directed edge

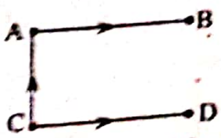


Figure (8): Directed Graph

The different types of directed graph are as follows,

(i) Simple Directed Graph and

(ii) Directed Multigraphs

(i) Simple Directed Graph

A simple directed graph is a graph without loops and multiple directed edges

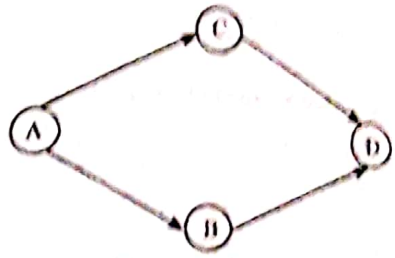


Figure (9) : Simple Directed Graph

(ii) Directed Multigraphs

Directed multigraphs are the graphs having multiple directed edges. If there are m directed edges, where each directed edge consist of an ordered pair of vertices (u, v), then (u, v) is said to be an edge of multiplicity m.

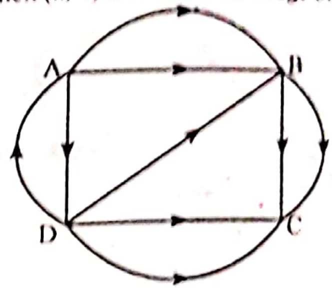


Figure (10) : Directed Multigraph

3. Mixed Graph

A combination of directed graph and undirected graph is called mixed graph.

or

A graph which consists of directed edges as well as undirected edges is called mixed graph.

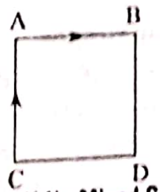


Figure (11) : Mixed Graph

In the above figure, edges (C, A) and (A, B) are directed and edges (C, D) and (B, D) are undirected.

4. Finite and Infinite Graph

A graph with finite set of vertices and as finite set of edges is called a finite graph.

A graph with infinite set of vertices and infinite set of edges is called an infinite graph.

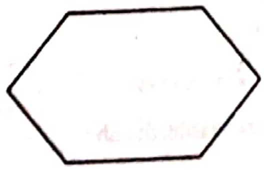


Figure (12): Finite Graph

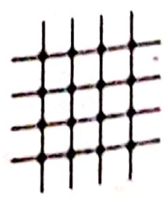


Figure (13): Infinite Graph

Q24. What are the different graph models? Explain them.

Answer :

The various graph models that can be structured using graph are as follows,

1. Niche Overlap Graphs in Ecology

Niche overlap graph is a graph used to model the structure of an ecosystem. In this graph, each vertex represents species and an undirected edge between two vertices represents that both species share some common food resources. This is a simple undirected graph with no loops and no multiple edges.

If $G(V, E)$ denotes niche overlap graph then,

$V(G)$ represents set of species and

$E(G)$ represents relationship between species who share common food resources.

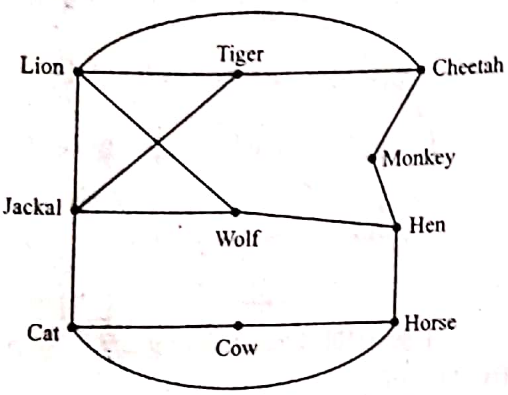


Figure (1): Niche Overlap Graph

In the above graph, lion, tiger, cheetah .... represent vertices and an edge between lion and tiger denotes they share common food resources. If there is no edge between two vertices lion and hen, then this denotes that the species lion and hen have no common food habits.

2. Acquaintanceship Graphs

Acquaintanceship graph is a graph which is used to model relation among the people. In this graph, each vertex denotes people and an edge connecting two people denotes relationship between them. This is also a simple undirected graph with no loops and no multiple edges.

If $G(V, E)$ denotes acquaintanceship graph then

$V(G)$ represents set of people and

$E(G)$ specifies relationship among peoples.

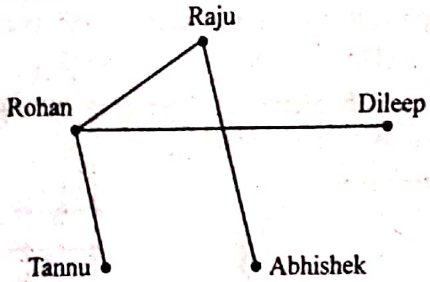


Figure (2): Acquaintanceship Graph

In the above graph, the edge between Rohan and Raju specifies that they know each other.

3. Influence Graph

Influence graph is a graph which is used to mod[el] influencing behavior among group of peoples. In this grap[h] each vertex denotes a person in a group and a directed edge fr[om] one vertex to another denotes that the first person influences th[e] next person. This is also a directed graph with no loops and w[ith] multiple directed edges.

If $G(V, E)$ denotes influence graph then,

$V(G)$ represents a person in a group and

$E(G)$ represents influencing relation between the people

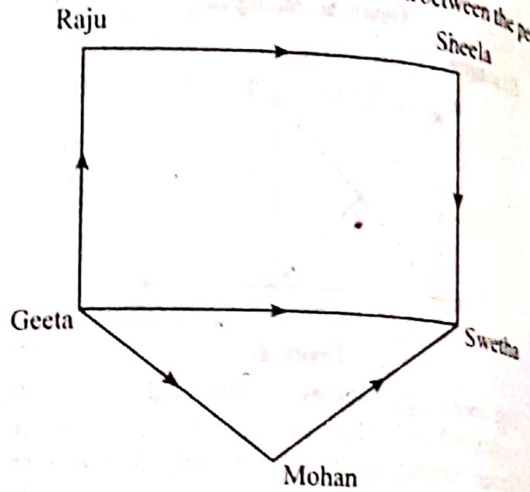


Figure (3): Influence Graph

In the above graph, edges between (Raju, Sh[eela]) denotes Raju influences Sheela. Similarly edge between (Sh[eela], Swetha) denotes Sheela influences Swetha and so on.

4. Hollywood Graph

Hollywood graph is a graph which is used to model the Hollywood world. In this graph, set of vertices denotes the actors and the movie whereas an edge between them represents that they worked together in a movie. This is a simple graph with no multiple edges and no loops.

If $G(V, E)$ is the Hollywood graph then,

$V(G)$ represents set of actors and a movie and

$E(G)$ represents that the actors worked together.

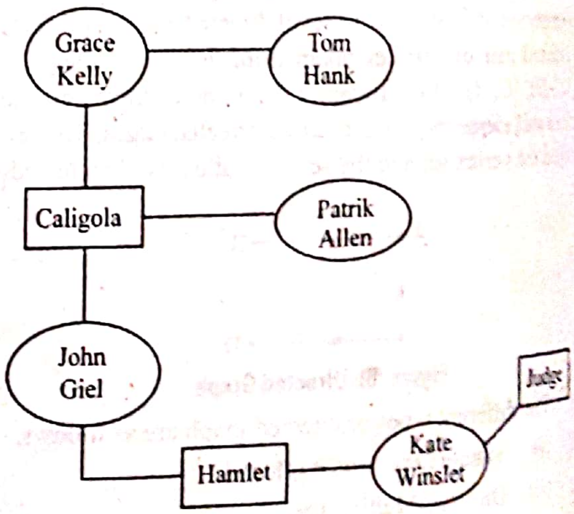


Figure (4): Hollywood Graph

In the above graph, actors are defined in ellipse and movies are defined with in square. This graph specifies that Patrik Allen and John Giel are actors who worked together in the movie "Caligola", John Giel and Kate Winslet are actors who worked together in the movie "Hamlet" etc.

5. **Round-Robin Tournment**

Round Robin tournment graph is a directed graph which is used to model a tournment in which two teams can play with each other only once. In this graph, each vertex denotes a team and an edge between two vertices denotes the team who won over the other team. This is a simple directed graph with no loops and no multiple edges.

If $G(V, E)$ is the Round-Robin tournment graph then,

$V(G)$ represents set of teams and

$E(G)$ represents set of edges where an edge {Team A, Team B} denotes that Team A won over Team B or A beats B.

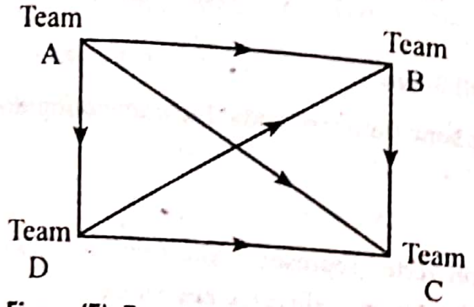


Figure (5): Round Robin Tournment Graph

In the above graph, an edge {Team A, Team B} denotes Team A won over Team B

6. **Web Graph**

Web graph is a graph which is used to model the World Wide Web. It is a simple directed graph with multiple edges and loops. In this graph each vertex denotes a web page and a directed edge between two vertex denotes that links between two web pages. That is, the first web page contain the link that directs it to the next web page.

If $G(V, E)$ is the web graph then, $V(G)$ represents set of web pages and

$E(G)$ represents sets of edges where a directed edge (A, B) denotes A is linked to B and an undirected edge $\{A, B\}$ denotes both A and B are linked together.

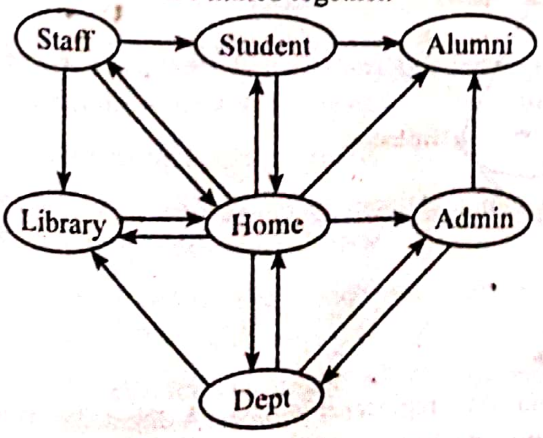


Figure (6): Library Management System

In the above graph, a directed edge (staff, library) denotes staff links to library and the edge (student, home) denotes student links to home and so on.

7. **Precedence Graph and Concurrent Processing**

Precedence graph is a graph which is used to model concurrency of statements with in a computer program. In this graph each vertex denotes a statement of computer program. A directed edge between two vertices represents that first vertex (statement) is dependent on the next vertex (statement) for execution of program. Hence, second statement cannot be executed before execution of first statement.

If $G(V, E)$ is the precedence graph, then

$V(G)$ represents set of concurrent statement and

$E(G)$ represents set of edges where an edge (A, B) denotes A is dependent on B.

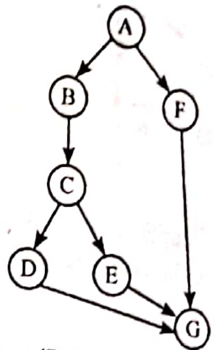


Figure (7): Precedence Graph

In the above graph, statement C cannot be executed before B and E cannot be executed before C and so on.

8. **Collaboration Graph**

Collaboration graph is a graph which is used in publication company to model the relationship among the authors who combinely publish an academic paper. In this graph, each vertex denotes author, an edge between two vertices represents that they worked together on an article/paper. This is also a simple undirected graph with no loops and no multiple edges.

If $G(V, E)$ is a collaboration graph then,

$V(G)$ represents set of authors and

$E(G)$ represents set of edges where an edge denotes authors worked together.

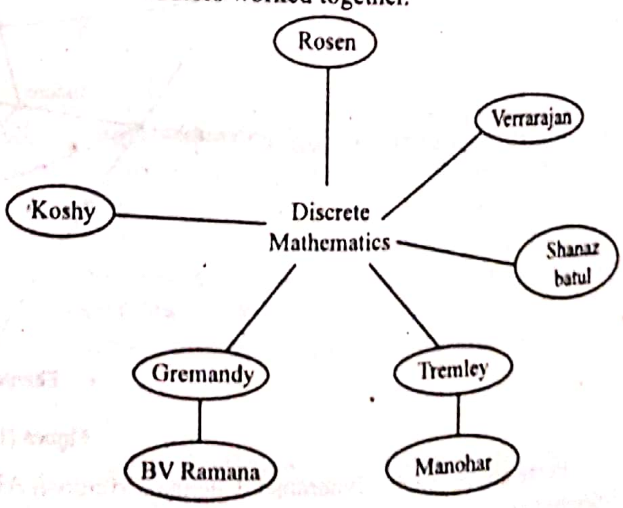


Figure (8): Subgraph of Collaboration Graph

In the above graph, Rosen, Koshy represents vertices and an edge between vertices Gremandy and B.V Ramana repres... that they have worked together in publishing the book.

9. Call Graph

Call graph is a type of graph used in telephone system to model the network of a telephone call. In this graph each ver... denotes a telephone number. A directed edge between two vertices represents a call made from first vertex to next vertex. In so... cases an undirected edge is used to represent the call made between two numbers.

If $G(V, E)$ denotes call graph then,

$V(G)$ represents set of telephone numbers (of 10 digits)

$E(G)$ represents call is made between the connecting vertices.

Example

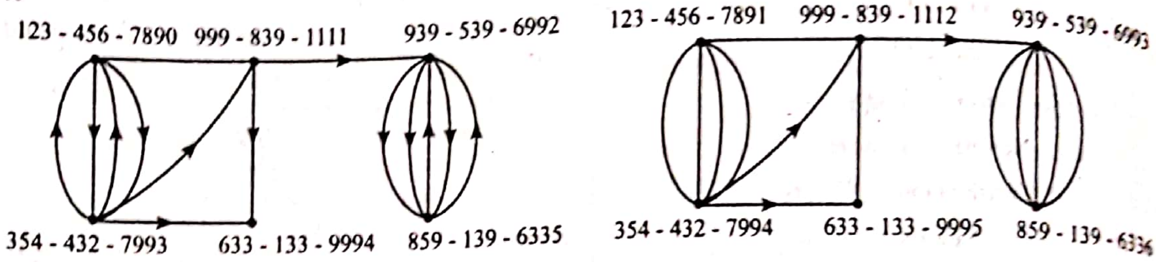


Figure (9): Directed and Undirected Call Graph

In the above example, 123 - 456 - 7890 and 354 - 432 - 7993 are telephone numbers and edge connecting them deno... calls are made between those numbers.

10. Road Maps

Road map is a graph which is used to model Road maps. In this graph vertex represents intersection of two roads an... edges represents roads. Here, a directed edge specifies that the one way road whereas an undirected edge specifies that the road i... two way road. That is, a simple directed graph with no loops and no multiple edge depicts a one way road map whereas, a mul... graph depicts both one way and two way road maps.

If $G(V, E)$ is a road map then,

$V(G)$ represents set of intersection of roads and

$E(G)$ represents set of directed edges/undirected/loops/multiple edges where directed edges denotes one way road.

Loop denotes Road loop and multiple directed edges denotes Multipath roads.

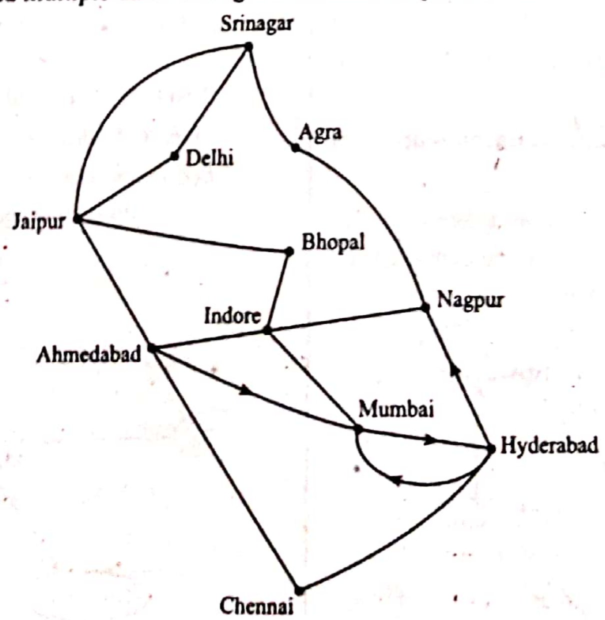


Figure (10): Road Map

In the above graph, Hyderabad, Chennai, Mumbai, Ahmedabad and so on represents vertices. A directed edge betwe... Hyderabad and Nagpur represents one-way road and an undirected edge between Chennai and Hyderabad denotes two-way roa...

Q25. Draw graph models, stating the type of graph used which determine what the vertices and edges represent, whether the edges are directed/undirected, loops are allowed. Also check whether it is a simple graph or multigraph.

Answer :

Given statement represents the airline routes,

Boston - Newark : 4 Flights

Newark - Boston : 2 Flights

Newark - Miami : 3 Flights

Miami - Newark : 2 Flights

Newark - Detroit : 1 Flight

Detroit - Newark : 2 Flights

Newark - Washington : 3 Flights

Washington - Newark : 2 Flights

Washington - Miami : 1 Flight

(a) A single edge is drawn between any two cities which contains a flight between them in either direction

Since the edges are in either direction,

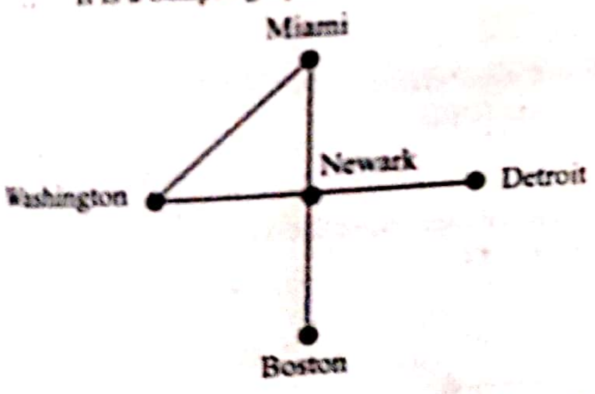
Hence, undirected edges are used.

The pair of cities either has a flight i.e., 1 edge or 0 edge between them

Thus, multiple edges are not allowed.

Hence, undirected edges, no multiple edges or no loops are used

∴ It is a Simple graph



(b) An edge is drawn between any two cities per flight between them in either direction

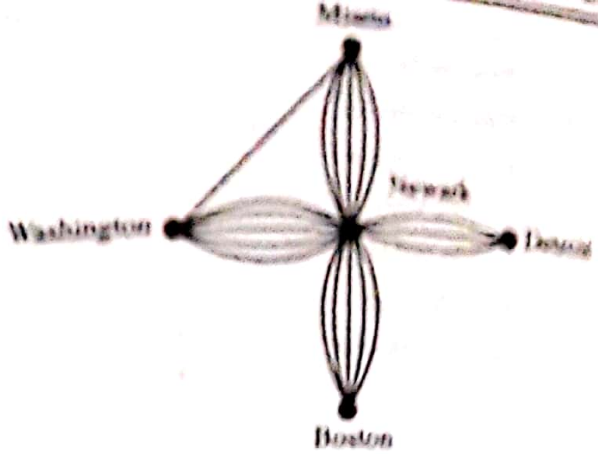
Thus, undirected edges are used.

As pair of cities may have multiple flights between them, multiple edges are allowed.

Since there are no flights between a city and itself, loops are also not allowed.

Hence, there are undirected edges, multiple edges allowed, no loops allowed.

∴ It is a Multigraph



Q26. Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that u R v if and only if there is an edge associated to (u, v) is a symmetric, reflexive relation on G.

Model Paper-1, Q10(a)

Answer :

Given that,

G is an undirected graph with a loop at every vertex

And relation $R = \{(u, v) \mid u, v \in G$, there exists an edge between u and $v\}$

Symmetric

Let $(a, b) \in R$

i.e., there is an edge between a and b in G

Since edges in G are undirected

Then there is also an edge between b and a in G

i.e., $(b, a) \in R$

Thus, when $(a, b) \in R \Rightarrow (b, a) \in R$

∴ R is symmetric

Reflexive

Let $a \in G$

i.e., a is a vertex in G

Since G contains a loop at every vertex

Then a has a loop

i.e., there is an edge between a and a

$\Rightarrow (a, a) \in R$

For every element $a \in G$, $(a, a) \in R$

∴ R is reflexive

Q27. Draw the acquaintanceship graph that represents that Tom and Patricia, Tom and Hope, Tom and Sandy, Tom and Amy, Tom and Marika, Jeff and Patricia, Jeff and Mary, Patricia and Hope, Amy and Hope, and Amy and Marika know each other, but none of the other pairs of people listed know each other.

Answer :

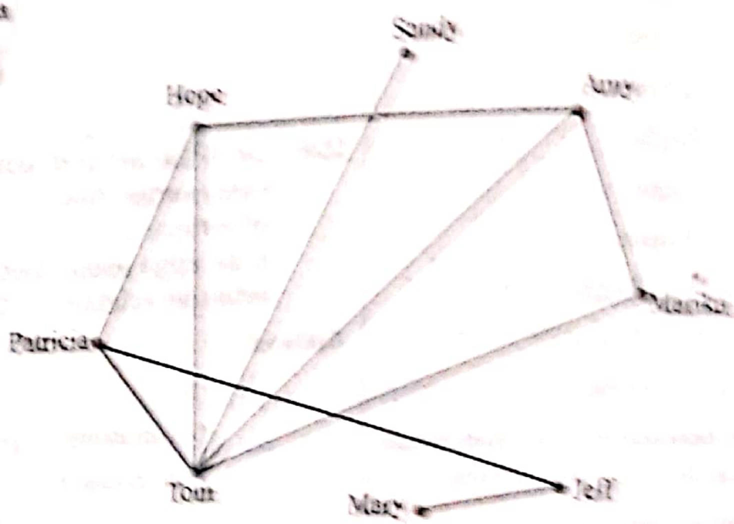
Given that,

A graph contains pairs of people who know each other

Tom and Patricia

Tom and Hope

Tom and Sandy
Tom and Amy
Tom and Marika
Jeff and Patricia
Jeff and Mary
Patricia and Hope
Amy and Hope
Amy and Marika



Acquaintanceship Graph

In the above graph, vertices represents the names of the people and edges represents that the two people know each other.

Q28. Construct an influence graph for the board members of a company if the President can influence the Director of Research and Development, the Director of Marketing, and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

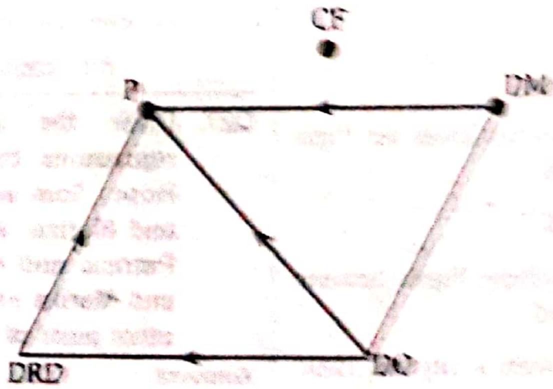
Answer :

Given that,

A graph contains 5 board members and they can influence other board members.

President (P) can influence Director of Research and Development (DRD)
President (P) can influence Director of Marketing (DM)
President (P) can influence Director of Operations (DO)
Director of Research and Development (DRD) influence Director of Operations (DO)

Also no one can influence or be influenced by Chief Financial Officer (CF)



Influence Graph

In the above graph, vertices represent board members and edges represent that board members influence other board members.

Q29. Draw the call graph for the telephone numbers 555-0011, 555-1221, 555-1333, 555-8888, 555-2222, 555-0091, and 555-1200 if there were three calls from 555-0011 to 555-8888 and two calls from 555-8888 to 555-0011, two calls from 555-2222 to 555-0091, two calls from 555-1221 to each of the other numbers, and one call from 555-1333 to each of 555-0011, 555-1221 and 555-1200.

Answer :

Given that,

A graph contains seven telephone numbers and the calls made from these numbers

Three calls from 555-0011 to 555-8888

Two calls from 555-8888 to 555-0011

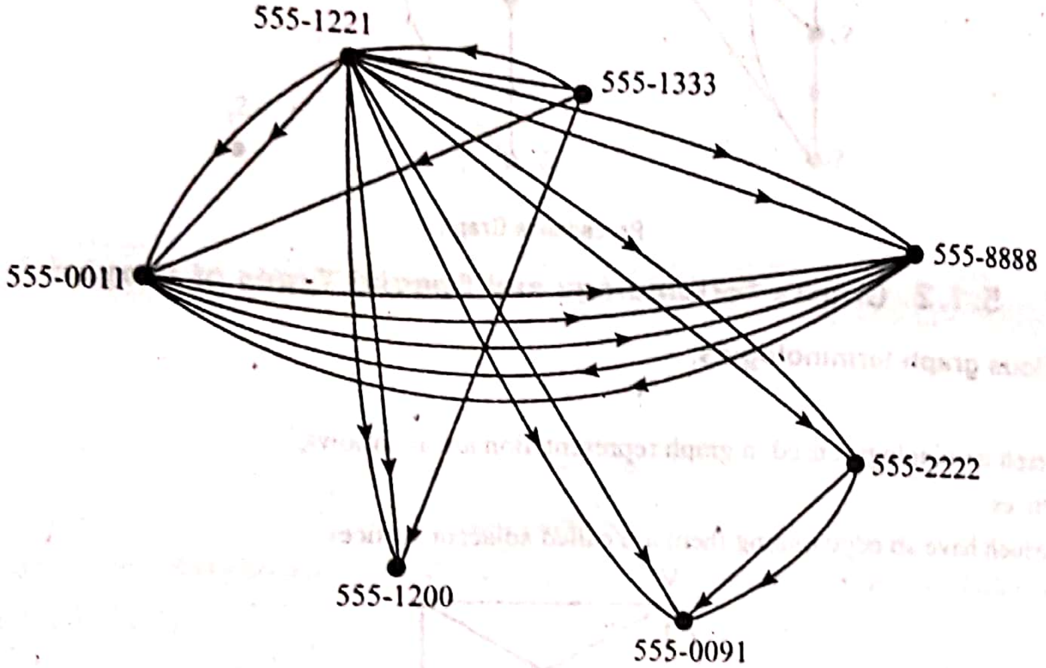
Two calls from 555-2222 to 555-0091

Two calls from 555-1221 to each of other numbers

One call from 555-1333 to 555-0011

One call from 555-1333 to 555-1221

One call from 555-1333 to 555-1200



Call Graph

Q30. Construct a precedence graph for the following program

$S_1 : x := 0$

$S_2 : x := x + 1$

$S_3 : y := 2$

$S_4 : z := y$

$S_5 : x := x + 2$

$S_6 : y := x + z$

$S_7 : z := 4$

Answer :

Given program is,

$S_1 : x := 0$

$S_2 : x := x + 1$

$S_3 : y := 2$

$S_4 : z := y$

$S_5 : x := x + 2$

$S_6 : y := x + z$

$S_7 : z := 4$

The dependence of statements on the previous statements is represented by a directed graph

Since the statements S_1, S_3 and S_7 are not dependent on any other statements

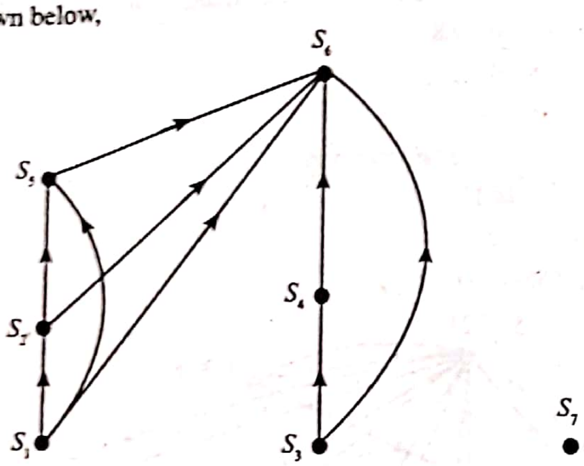
∴ These statements are executed concurrently.

Here, S_4 is adjacent from S_3

S_1 and S_5 are adjacent from S_2.

S_6 is adjacent from S_4.

The precedence graph is shown below,



Precedence Graph

5.1.2 Graphs Terminology and Special Types of Graphs

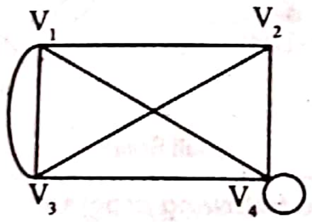
Q31. Discuss various graph terminologies.

Answer :

The various graph terminologies used in graph representation are as follows,

(i) Adjacent Vertices

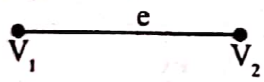
Two vertices which have an edge joining them are called adjacent vertices.



Here, V_1 and V_2 are adjacent vertices.

(ii) Incident Edge

The edge e is incident with the vertices V_1 and V_2 if it is joined with $\{V_1, V_2\}$



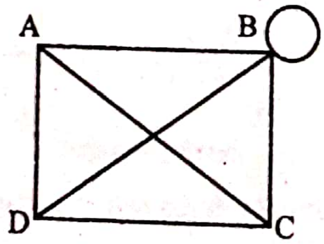
Here V_1 and V_2 are end points of edge $\{V_1, V_2\}$

(iii) Degree of Vertex

If $G(V, E)$ is an undirected graph then the number of edges incident to the vertex except that each loop at V is counted twice is said to be degree of vertex.

It is denoted by deg( ).

Example



$deg(B) = 5$

(iv) Isolated Vertex

An isolated vertex is a vertex whose degree is zero and it is not adjacent to any other vertex.

Example

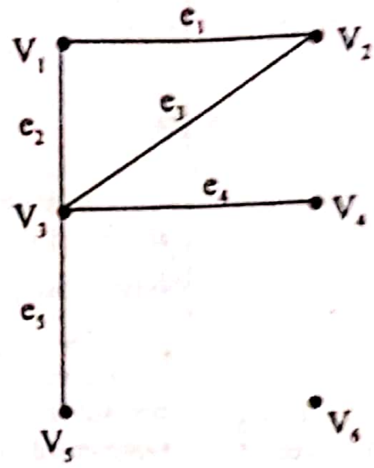


Figure : Graph(G_2)

(v) Pendant Vertex

Pendant vertex is a vertex whose degree is 1. Pendant edge is an edge incident on a pendant vertex.

Example

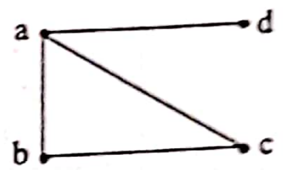


Figure : Graph G_1

In the figure, vertex d is called pendant vertex and the edge (a, d) is an edge incident on a pendent vertex.

(vi) Initial and Terminal Vertex

If (V_1, V_2) is directed edge, where V_1 is adjacent to V_2 and V_2 adjacent from V_1, then the vertex V_1 is called the initial vertex and vertex V_2 is called the terminal vertex. The initial and terminal vertices are same for the loop.

(vii) In-Degree and Out-Degree of a Vertex

In a directed graph $G(V, E)$ which have directed edges, the in-degree of a vertex is the number of edges with V as their terminal vertex and is denoted by $deg^-(V)$ and the out-degree of a vertex is the number of edges with V as their initial vertex and is denoted by $deg^+(V)$.

Q32. Explain the various special types of graphs with examples.

Answer :

(i) Complete Graph

A Complete graph is a simple graph with n vertices which contains exactly one edge between every pair of vertices. It is denoted by K_n.

The following are the examples of complete graphs.

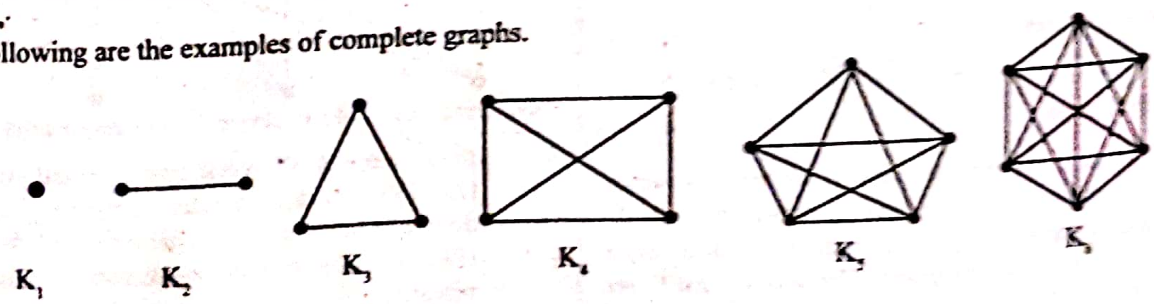


Figure (1)

(ii) Cycle Graph

A cycle graph is graph of order $n \geq 3$ which contains n vertices i.e., 1, 2, 3,.....n and {1, 2}, {2, 3},.....{(n – 1), n}, {n, 1} edges. It is denoted by C_n.

Example

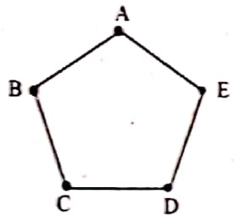


Figure (2): Cycle Graph (C_5)

The above graph is a cycle graph order 5, i.e., C_5.

(iii) Wheel Graph

A wheel graph is a graph that is obtained by joining a single new vertex to each vertex of a cycle graph of order $n \geq 3$. These graphs are represented by W_n.

Example

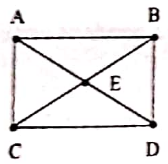


Figure (3): Wheel Graph (W_5)

The above graph is a wheel graph of order 5 i.e., W_5

(iv) n-Cubes

Cube graph is a graph of order n in which every individual vertex represents one of the 2-bit strings. It is represented as Q_n.

Number of vertices = 2^n

Number of edges = $n \times 2^{(n-1)}$

Example

Consider the following Q_3 graph

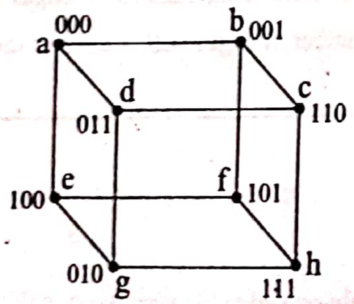


Figure (6): Q_3

Number of vertices = $2^3 = 8$

Number of edges = $3 \times 2^{(3-1)} = 3 \times 2^2 = 3 \times 4 = 12.$

(v) Bipartite Graph

If suppose there is a simple graph G wherein its vertex set V is the union of two mutually disjoint non-empty sets V_1 and V_2 such that every edge in G joins a vertex in V_1 and a vertex in V_2, then G is called a bipartite graph. The sets V_1 and V_2 are called bipartites.

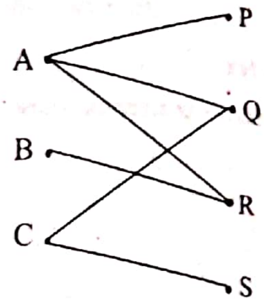


Figure (4): Bipartite Graph

The above graph shows a bipartite graph with V_1 = {A, B, C} and V_2 = {P, Q, R, S} as bipartites.

(vi) Complete Bipartite Graph

A complete bipartite graph is a graph wherein vertex set is divided into two subsets of vertices and one vertex of first subset is connected to other vertex of second subset. It is represented by $K_{m,n}$.

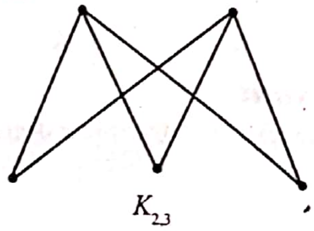


Figure (5): Complete Bipartite Graph ($K_{2,3}$)

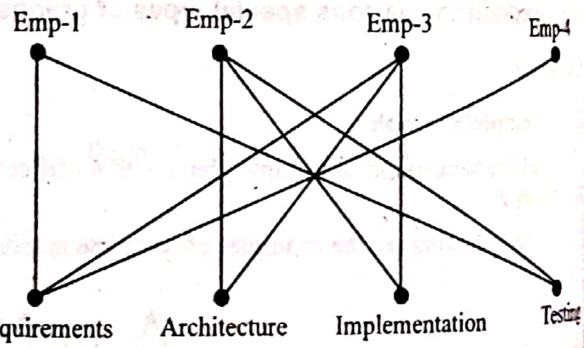
Q33. Give some brief applications of special types of graphs.

Answer : Model Paper-2, Q10(a)

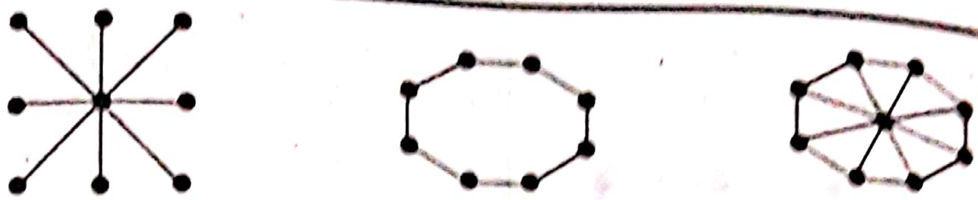
The following applications explain about special types of graphs.

1. The job assignments, each employee must be trained to do one or more jobs. Finding a match using a graph model completes assigning of jobs to employees. Maximum matching in the graph is done based on employee capabilities.

Example :



2. In a building of work space, Local Area Networks (LANS) connect various computers and peripheral devices (based on topologies) to a central control unit (CCU). LAN can be represented using a complete bipartite graph to send messages to devices through (CCU) as shown in figure (1).



(a) Star topology (Messages sent from device to device through CCU)

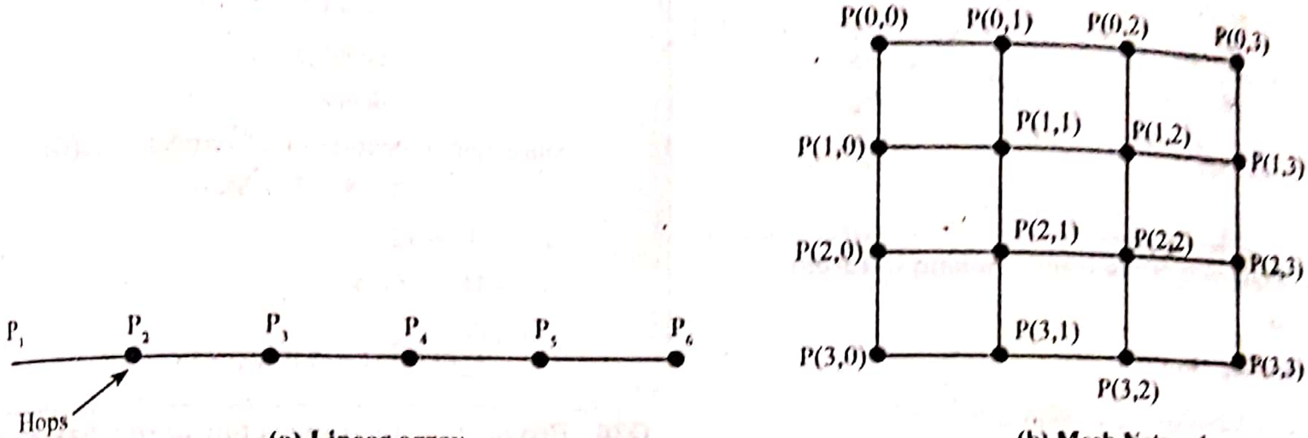
(b) Ring topology (Messages are sent around the cycle)

(c) Hybrid topology (Messages are sent are around the ring or through the central device)

Figure (1)

3. Initially, serial execution of programs occurred in computers at a time. After several computations, parallel processing came into existence which overcomes the limitations of single processor computers. It controls execution, transmission, input and output of different processors.

In parallel processing, parallel algorithms divide a problem into a number of subproblems such that it can be solved rapidly using a multiprocessor computer. In this, processors must be interconnected using the appropriate type of graph as shown in figure (2).



(a) Linear array **(b) Mesh Network**

Figure (2)

Q34. Discuss about subgraph and union of simple graphs.

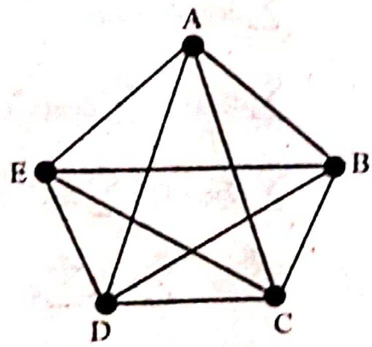
Answer :

Subgraph

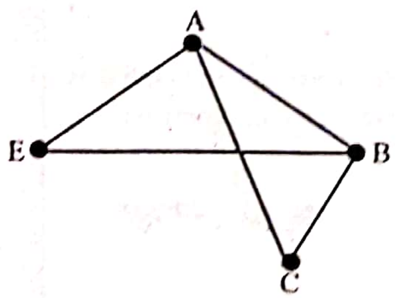
A subgraph is a graph which is obtained by removing vertices and edges without deleting the endpoints of remaining edges from the original graph.

In other words, a graph $H = (W, F)$ is a subgraph of graph $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$

A subgraph H of a graph G is called a proper subgraph of G if and only if $H \neq G$.



K_5



Subgraph of K_5

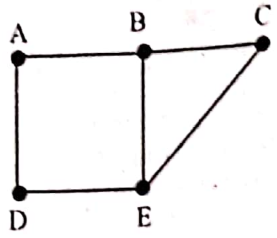
Union of Simple Graphs

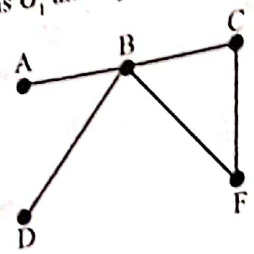
The Union of two simple graphs G_1 and G_2 is the simple graph containing a vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

It is denoted by $G_1 \cup G_2$

Example

Consider two simple graphs G_1 and G_2

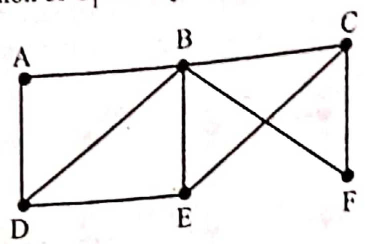




G_2

G_1

Then union of G_1 and G_2 is,



$G_1 \cup G_2$

Q35. State and prove Hand shaking theorem.

Answer :

Handshaking Theorem

The handshaking theorem states that,

If $G = (V, E)$ is an undirected graph, then

$$2|E(G)| = \sum_{u \in V} \deg(u)$$

Here, $|E(G)|$ is number of edges in the graph

$\sum_{u \in V} \deg(u)$ is the sum of the degrees of the vertices.

Proof

Let, $G(V, E)$ be an undirected graph.

Here, $|E(G)|$ = Number of edges in graph G.

$|V(G)|$ = Number of vertices in graph G.

$\Rightarrow \quad v_1, v_2, ... v_n \in V$

The sum of degrees of all the vertices in a graph G is twice the number of edges because each edge contributes 2 degrees.

Thus, $\deg(v_1) + \deg(v_2) + ... + \deg(v_n) = 2|E(G)|$

$$\Rightarrow \quad \sum_{i=1}^{n} \deg(v_i) = 2|E(G)|$$

Example

The above statement can also be proved by considering the below graph G.

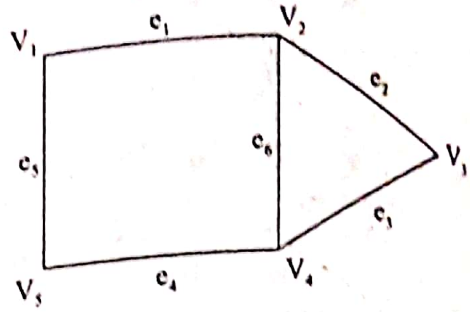


Figure: Graph

The given graph contain 5 vertices (i.e., v_1, v_2, v_3, v_4 and v_5) and 6 edges (i.e., e_1, e_2, e_3, e_4, e_5 and e_6). The degree of each vertex is,

$$\deg(V_1) = 2$$
$$\deg(V_2) = 3$$
$$\deg(V_3) = 2$$
$$\deg(V_4) = 3$$
$$\deg(V_5) = 2$$

Since sum of degrees of all vertices = $2|E(G)|$

$\Rightarrow \quad 2 + 3 + 2 + 3 + 2 = 2 \times 6$

$\Rightarrow \quad 12 = 12$

$\Rightarrow \quad$ LHS = RHS

$$\therefore \quad \sum_{i=1}^{n} \deg(V_i) = 2|E|$$

Q36. Prove that an undirected graph has an even number of vertices of odd degree.

Model Paper-3, Q10(a)

Answer :

Let, $G(V, E)$ be an undirected graph

$|V|$ = number of vertices, $|E|$ = Number of edges

$V = \{v_1, v_2, ..., v_m, v'_1, v'_2, ..., v'_n\}$ be the set of vertices where in each v_i is of odd degree and each v'_i is of even degree.

Let, $\deg(v_i) = 2p_i + 1$

$\deg(v'_i) = 2q_i$

From Handshaking theorem,

$$\sum_{v \in V} \deg(V) = 2|E|$$

$$\Rightarrow \sum_{i=1}^{m} \deg(v_i) + \sum_{i=1}^{n} \deg(v'_i) = 2|E|$$

$$\Rightarrow \sum_{i=1}^{m} (2p_i + 1) + \sum_{i=1}^{n} 2q_i = 2|E|$$

$$\Rightarrow 2\sum_{i=1}^{m} p_i + \sum_{i=1}^{m} 1 + 2\sum_{i=1}^{n} q_i = 2|E|$$

$$\Rightarrow 2\sum_{i=1}^{m} p_i + m + 2\sum_{i=1}^{n} q_i = 2|E|$$

$$\Rightarrow m = 2\left[|E| - \sum_{i=1}^{m} p_i - \sum_{i=1}^{n} q_i\right]$$

$\therefore \quad m$ is an even number.

Example

Consider the below graph.

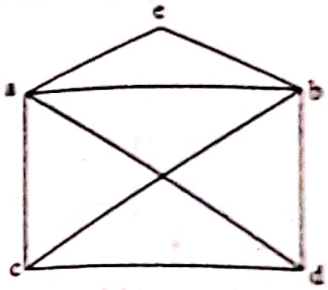


Figure: Graph (G)

The above graph contains five vertices (namely a, b, c, d and e) and 8 edges. The degrees of vertices of the graph are as follows,

$\deg(a) = (a, e), (a, b), (a, c), (a, d) = 4$

$\deg(b) = (b, e), (b, a), (b, c), (b, d) = 4$

$\deg(c) = (c, a), (c, b), (c, d) = 3$

$\deg(d) = (d, a), (d, b), (d, c) = 3$

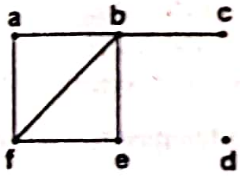
$\deg(e) = (e, a), (e, b) = 2$

Hence, number of vertices having odd degree (i.e., deg 3) are 2 (i.e., c, d)

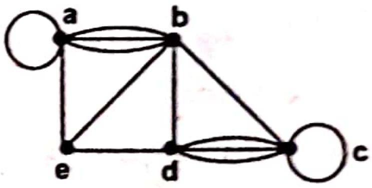
Hence, the statement is proved that in any non-directed graph, there is an even number of vertices of odd degree.

Q37. For the undirected graphs, find the number of vertices, edges and the degree of each vertex. Also identify all isolated and pendant vertices.

(a)



(b)



Answer :

(a) Given vertices are,

a, b, c, d, e, f

Number of vertices = 6

Number of edges = 6

The degree of each vertex is the number of edges incident to it.

Then, $\deg(a) = 2$

$\deg(b) = 4$

$\deg(c) = 1$

$\deg(d) = 0$

$\deg(e) = 2$

$\deg(f) = 3$

$\therefore$ Isolated vertex is d and Pendant vertex is c.

a, b, c, d, e

Number of vertices = 5

Since there are 11 lines and 2 loops

Number of edges = 13

The degree of vertex is the number of edges incident to it, a loop is counted twice

Then, $\deg(a) = 4$ lines + 2 loops = 6

$\deg(b) = 6$

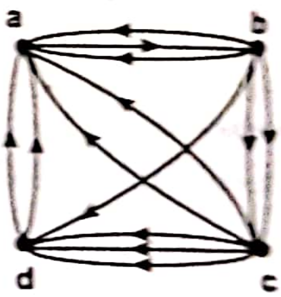
$\deg(c) = 4$ lines + 2 loops = 6

$\deg(d) = 5$

$\deg(e) = 3$

$\therefore$ There are no isolated and pendant vertices.

Q38. For the directed multigraph, determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.



Answer :

Given that,

A graph contains 5 vertices and 13 edges

The in-degrees of the vertices are,

$\deg^-(a) = 6$

$\deg^-(b) = 1$

$\deg^-(c) = 2$

$\deg^-(d) = 4$

$\deg^-(e) = 0$

The out-degrees of the vertices are,

$\deg^+(a) = 1$

$\deg^+(b) = 5$

$\deg^+(c) = 5$

$\deg^+(d) = 2$

$\deg^+(e) = 0$

The sum of in-degrees of the vertices is,

$\deg^-(a) + \deg^-(b) + \deg^-(c) + \deg^-(d) + \deg^-(e) = 6 + 1 + 2 + 4 + 0 = 13$

The sum of out-degrees of the vertices is,

$\deg^+(a) + \deg^+(b) + \deg^+(c) + \deg^+(d) + \deg^+(e) = 1 + 5 + 5 + 2 + 0 = 13$

Thus, the sum of in-degrees of the vertices and the sum of out-degrees of the vertices is equal to the number of edges.

Q39. For which values of n are these graphs bipartite.

(a) C_n

(b) W_n

Answer :

(a) Given that,

C_n is a graph

C_n is bipartite graph if and only if n is even otherwise it is not bipartite.

Example

C_4

Here, the graph contains 4 vertices.

$\therefore$ For $n = 4$, the given graph is bipartite.

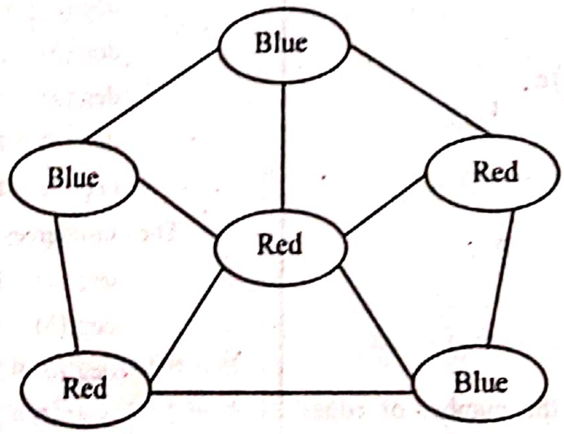
(b) Given that,

W_n is a graph

W_n is not a bipartite graph for any value of n, because the center vertex is adjacent to every vertex

i.e., For a graph to be bipartite, assignment of two different colors to each vertex is done, so that no two adjacent vertices are assigned the same color.

Example



W_5

Q40. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsibilities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development, Smith could do publicity, sales or industry relations; Chou could do planning, sales or industry relations, and Macintyre could do planning, publicity, sales or industry relations.

(a) Model the capabilities of these employees using a bipartite graph

(b) **Find an assignment of responsibilities such that each employee is assigned a responsibility.**

Answer :

Given that,

There are 5 employees : Zamora, Agraharam, Smith, Chou and Macintyre

and 6 responsibilities: Planning, Publicity, Sales, Marketing, Development and Industry relations

Also, each employee is capable of doing one or more jobs

Zamora : Planning, Sales, Marketing or Industry relations.

Agraharam : Planning or Development

Smith: Publicity, Sales or Industry relations

Chou: Planning, Sales or Industry relations

Macintyre: Planning, Publicity, Sales or Industry relations

Vertices are divided into two vertex sets

i.e., V_1 is the set of employees

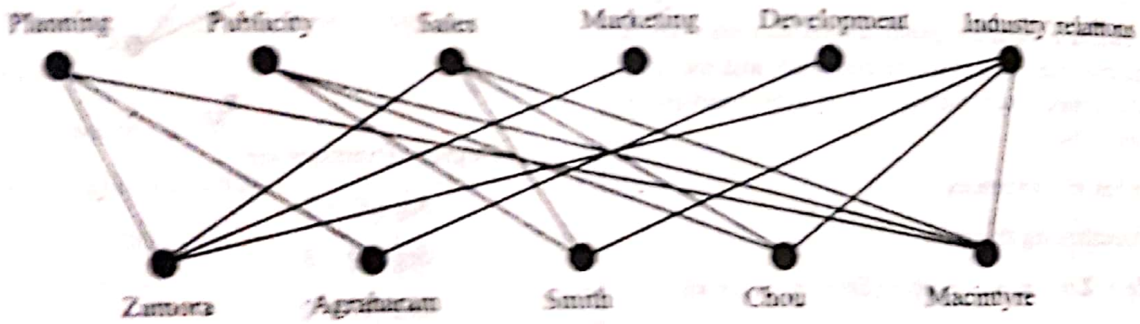
and V_2 is the set of responsibilities

V_1 {Zamora, Agraharam, Smith, Chou, Macintyre}

V_2 {Planning, Publicity, Sales, Marketing, Development, Industry relations}

E = {{Zamora, Planning}, {Zamora, Sales}, {Zamora, Marketing}, {Zamora, Industry relations}}

E = {{Zamora, Planning}, {Zamora, Sales}, {Zamora, Marketing}, {Zamora, Industry relations}, {Agraharam, Planning}, {Agraharam, Development}, {Smith, Sales}, {Smith, Publicity}, {Smith, Industry relations}, {Chou, Sales}, {Chou, Publicity}, {Chou, Industry relations}, {Macintyre, Planning}, {Macintyre, Sales}, {Macintyre, Publicity}, {Macintyre, Industry relations}}



(b) There are many possible assignments. One of the assignment is,

Zamora - Planning

Agraharam - Development

Smith - Industry relations

Chou - Sales

Macintyre - Publicity

Q41. How many vertices and how many edges do these graphs have?

(a) K_n

(b) W_n

(c) $K_{m,n}$

Answer :

(a) Given graph is,

K_n

Since K_n is a complete graph

K_n has n vertices

And there is exactly one edge between pair of distinct vertices

$\therefore$ Each vertex has degree $n-1$

From handshaking theorem,

$$2m = \Sigma(n-1) + (n-1) + ... + (n-1)$$

$$\Rightarrow \quad 2m = n(n-1)$$

$$\Rightarrow \quad m = \frac{n(n-1)}{2}$$

Thus, K_n has n vertices and $\frac{n(n-1)}{2}$ edges.

(b) Given graph is,

W_n

Here, W_n is a wheel graph, it is same as cycle graph (C_n) with an additional vertex connected to each of the n vertices

$\therefore$ W_n has $(n+1)$ vertices

It also has n extra edges incident to a new vertex

i.e., Edges in C_n + Edges connected to additional vertex

$= n + n = 2n$ edges

Hence, K_n has $(n+1)$ vertices and $2n$ edges.

(c) Given graph is,

$K_{m,n}$

It is a complete bipartite graph where its vertex set is partitioned into two subsets m and n respectively and the edge of graph should connect every vertex of first subset and every vertex of second subset

$\therefore$ It has $m+n$ vertices

From handshaking theorem,

$$2e = (\Sigma m + m + ... + m) + (\Sigma n + n + ... + n)$$

$$\Rightarrow \quad 2e = m.n + m.n$$

$$\Rightarrow \quad 2e = 2mn$$

$$\Rightarrow \quad e = mn$$

$\therefore$ $K_{m,n}$ has $m+n$ vertices and mn edges

Q42. Find the degree sequence of each of the following graphs.

(a) K_4

(b) $K_{2,3}$

(c) Q_3

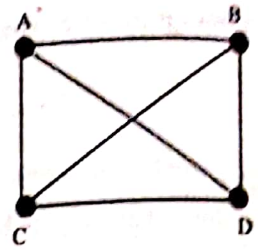
Answer :

(a) Given graph is,

K_4

It has 4 vertices

Since it is a complete graph, there exists an edge between each pair of vertices



K_4

Degree of vertices are,

deg $(A) = 3$, deg$(B) = 3$, deg$(C) = 3$ and deg$(D) = 3$

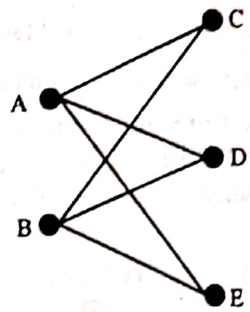
$\therefore$ Degree sequence of K_4 is 3,3,3,3.

(b) Given graph is,

$K_{2,3}$

It is a complete bipartite graph having two sets of vertices i.e., set of 2 vertices and a set of 3 vertices

Where, $S_1 = \{A, B\}$ and $S_2 = \{C, D, E\}$ and the edge of $K_{2,3}$ should connect each vertex of S_1 and each vertex of S_2



$K_{2,3}$

Degree of vertices are,

deg $(A) = 3$

deg $(B) = 3$

deg $(C) = 2$

deg $(D) = 2$

deg $(E) = 2$

$\therefore$ Degree sequence of $K_{2,3}$ is 3, 3, 2, 2, 2.

Q43. Show that if G is a bipartite simple graph with V vertices and e edges, then $e \le v^2/4$.

Answer :

Let, G be a bipartite graph having n vertices

Each vertex of G can be divided into two sets in such way that these two sets are adjacent to vertex of each other and consist of $\frac{v}{2}$ edges.

$\therefore$ The total number of edges in biparite graph are =

$$\frac{v}{2} \times \frac{v}{2} = \frac{v^2}{4}$$

Inorder to prove that they are $\frac{v^2}{4}$ maximum number of edges in a bipartite graph, consider that the first set contains vertices and the second set contains $(v-n)$ vertices.

Therefore, the total number of edges in a given graph is,

$$f(n) = n.(v - n)$$

$$= vn - n^2 \quad \ldots (1)$$

$$f'(n) = v - 2n$$

Since, maximum value of $f'(n) = 0$.

i.e., $v - 2n = 0$

$\Rightarrow \quad v = 2n$

$\Rightarrow \quad n = \frac{v}{2}$

Substituting $n = \frac{v}{2}$ in equation (1),

$$f\left(\frac{v}{n}\right) = \frac{v}{2}\left(v - \frac{v}{2}\right) = \frac{v}{2}\left(\frac{2v - v}{2}\right)$$

$$= \left(\frac{v}{2}\right).\left(\frac{v}{2}\right) = \frac{v^2}{4}$$

$\therefore$ The maximum number of edges is $\frac{v^2}{4}$ i.e., $e \leq \frac{v^2}{4}$

Q44. Prove that a simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Answer :

Model Paper-4, Q10(a)

Let $G(V, E)$ be a simple bipartite graph

Where,

V = Set of vertices

E = Set of edges that connects the vertices.

Let V_1, V_2 be the disjoint set of vertices which are connected by an edge e such that $V_1 \cap V_2 = \phi$ and $V = V_1 \cup V_2$.

Since no two adjacent vertices can have same colors, two different colors are used for coloring vertex set V_1 and vertex set V_2.

Let C_1, C_2 be two colors such that $C_1, C_2 \in C$ and $f: V \to C$ be a function containing vertices (V) and colors (C)

Where, $f(V) = C_1$, if $V \in V_1$

And $f(V) = C_2$, if $V \in V_2$

Since $V_1 \cap V_2 = \phi$, the above function is said to be well defined. Thus a graph G can be coloured using two colors i.e., $X(G) = 2$.

Conversely, if it is possible to assign one of two different colors to each vertex of the graph such that no two adjacent vertices are assigned same color.

Then, $V_1 = \{V \in f \mid f(V) = C_1\}$

$V_2 = \{V \in f \mid f(V) = C_2\}$

$\therefore$ $G(V, E)$ is bipartite.

5.1.3 Representing Graphs and Graph Isomorphism

Q45. Discuss about Adjacency Lists?

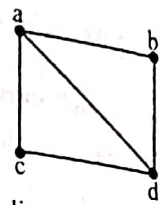
Answer :

Adjacency lists are used to represent graph with no multiple edges

Undirected Graph

Adjacency list for an undirected graph is a list of vertices that are adjacent to each vertex of the graph

Example



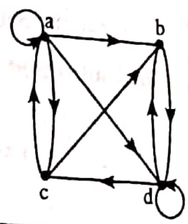
An Adjacency list for a simple graph or undirected graph is shown below

Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Directed Graph

Adjacency list for the directed graph is the listing of all the terminal vertices of edges that has the given vertex as its initial vertex

Example



Initial Vertex	Terminal Vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

Q46. List and explain different representation of graphs with example.

Answer :

Representation of Graph

Graph representation is the process of representing a graph in such a way that, it can be easily understood, manipulated and used. There are various methods for representing a graph, but matrix representation method is the most efficient method for representing graphs in computer memory. This is because matrices are easy to manipulate and can be used in studying properties of graphs.

The two types of matrices that are commonly used to represent graphs are,

1. Adjacency matrix.
2. Incidence matrix

1. Adjacency Matrix

Adjacency matrix of a graph represents the adjacency between the vertices of a graph i.e., it represents which vertices (or nodes) of a graph are adjacent to the other vertices.

Adjacency Matrix for Undirected Graph

Let $G = (V, E)$ be a simple graph (or) undirected graph and vertices of G be listed arbitrarily as $1, 2, \ldots, n$. Based on the listing of vertices, the adjacency matrix A is the $n \times n$ zero-one matrix. If i and j are adjacent, then its (i, j) th entry is 1, otherwise its (i, j) th entry is 0.

$$a_{ij} = \begin{cases} 1 & , \text{ if there is an edge from } i \text{ to } j \\ 0 & , \text{ otherwise} \end{cases}$$

Adjacency matrix of an undirected graph is symmetric i.e., $a_{ij} = a_{ji}$

For an undirected graph with loops and multiple edges, adjacency matrix is given as,

$$a_{ij} = \begin{cases} m & , \text{ if there are } m \text{ edges from } v_i \text{ to } v_j \\ 0 & , \text{ otherwise} \end{cases}$$

Adjacency Matrix for Directed Graph

Let $G = (V, E)$ be a directed graph then the adjacency matrix $A = [a_{ij}]$ for the directed graph based on the listing of vertices is,

$$a_{ij} = \begin{cases} 1 & , \text{ if there is an edge from } v_i \text{ to } v_j \\ 0 & , \text{ otherwise} \end{cases}$$

The adjacency matrix for a directed graph is not symmetric i.e., there is no edge from a_j to a_i when there is an edge from a_i to a_j. When there are multiple edges in a directed multigraph, then adjacency matrices are not zero-one matrices and is given as,

$$a_{ij} = \begin{cases} m & , \text{ if there are m edges from } v_i \text{ to } v_j \\ 0 & , \text{ otherwise} \end{cases}$$

2. Incidence Matrix

It gives the information about the incidence of vertices and edges on the graph.

Incidence Matrix for Undirected Graph

Let $G = (V, E)$ be an undirected graph where vertices are $1, 2, \ldots n$ and edges are $e_1, e_2, \ldots e_n$. The incidence matrix is the $n \times m$ matrix $M = [m_{ij}]$ and is given as,

$$m_{ij} = \begin{cases} 1 & , \text{ if edge } e_j \text{ is incident with } i \\ 0 & , \text{ otherwise} \end{cases}$$

Multiple edges and loops can also be represented using incidence matrices.

Multiple edges are represented using columns having identical entries whereas loops are represented using columns in incidence matrices having exactly one entry i.e., 1 that corresponds to the vertex incident with the loop.

Q47. When it can be said that two graphs G_1 and G_2 are isomorphic? How can it be discovered? Explain with example.

Answer :

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 is correct iff $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is known as isomorphism.

Two graphs G and G' are said to be isomorphic if there is a one-to-one correspondence between their vertices and between their edges so that the adjacency of vertices is preserved. These graphs will have same structure, only differs in the way the vertices and edges are labelled (or in the way they are represented geometrically).

When two graphs i.e., G and G' are isomorphic, we represent it as $G \equiv G'$. If a vertex A of G corresponds to the vertex $A' = f(A)$ of G' a one-to-one correspondence $f: G \to G'$, we write $A \leftrightarrow A'$. In order to show that the edge AB of G and the edge $A'B'$ of G' correspond to each other we write $\{A, B\} \leftrightarrow \{A', B'\}$

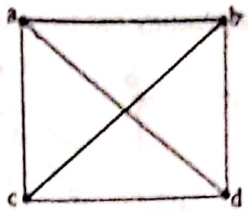


Figure: Graph K_4

The adjacency matrix for the above graph is,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(2) $K_{1,4}$

The graph of $K_{1,4}$ is shown below,

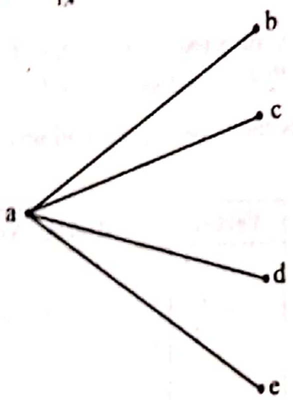


Figure: Graph $K_{1,4}$

The adjacency matrix for the above graph is,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(3) C_4

The Graph of C_4 is shown below,

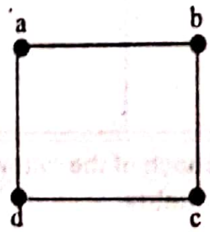


Figure: Graph C_4

The adjacency matrix for the above graph is,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(4) W_4

The graph of W_4 is shown below.

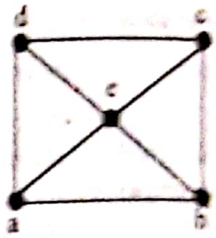


Figure: Graph W_4

The adjacency matrix for the above graph is,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q50. Draw a graph with adjacency matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Model Paper-4, Q7(a)

Answer :

Given matrix is,

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Since the adjacency matrix is a 4 × 4 zero-one matrix

∴ The graph has 4 vertices. Let the vertices be A, B, C and D.

Since the matrix is not symmetric i.e., there is no edge from a_i to a_j when there is an edge from a_j to a_i

From the adjacency matrix,

$a_{11} = 1$ i.e., there is a loop at A

$a_{12} = 1$ i.e., there is a directed edge from A to B

$a_{13} = 1$ i.e., there is a directed edge from A to C

$a_{23} = 1$ i.e., there is a directed edge from B to C

$a_{31} = 1$ i.e., there is a directed edge from C to A

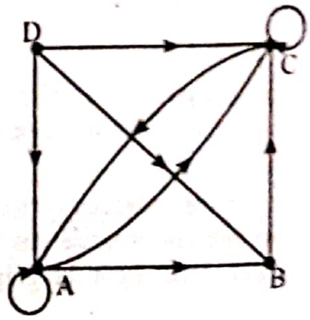
$a_{33} = 1$ i.e., there is a loop at C

$a_{41} = 1$ i.e., there is a directed edge from D to A

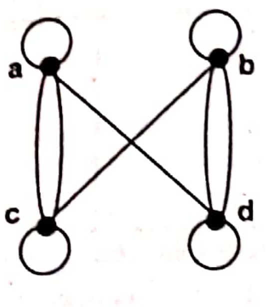
$a_{42} = 1$ i.e., there is a directed edge from D to B

$a_{43} = 1$ i.e., there is a directed edge from D to C

∴ A directed graph is shown below.



Q51. Represent the graph using an adjacency matrix



Answer :

Given that,

A graph has 4 vertices a, b, c and d

The adjacency matrix is based on the ordering of the vertices a, b, c, d

In an undirected graph,

$a_{11} = 1$ i.e., there is a loop at a

$a_{13} = a_{31}$ 2 i.e., a is connected to c by 2 edges

$a_{14} = a_{41} = 1$ i.e., a is connected to d by 1 edge

$a_{22} = 1$ i.e., there is a loop at b

$a_{23} = a_{32} = 1$ i.e., b is connected to c by 1 edge

$a_{24} = a_{42} = 2$ i.e., b is connected to d by 2 edge

$a_{33} = 1$ i.e., there is a loop at c

$a_{44} = 1$ i.e., there is a loop at d

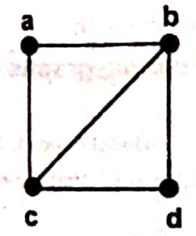
Since there are no other edges

All other elements in matrix should be 0

$\therefore$ Adjacency matrix is,

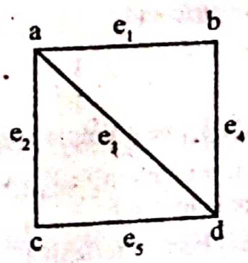
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

Q52. Use an incidence matrix to represent the graph



Answer :

Given graph is,



The above graph has 4 vertices and 5 edges

Then the incidence matrix is a 4×5 matrix

The incidence matrix for an $n \times m$ matrix is,

$$m_{ij} = \begin{cases} 1 & , \text{ if edge } e_j \text{ is incident with } i \\ 0 & , \text{ otherwise} \end{cases}$$

From the graph,

The vertex a is connected by the edges e_1, e_2 and e_3

$\therefore \quad m_{11} = 1, m_{12} = 1, m_{13} = 1$

The vertex b is connected by the edges e_1 and e_4

$\therefore \quad m_{21} = 1, m_{24} = 1$

The vertex c is connected by the edges e_2 and e_5

$\therefore \quad m_{32} = 1, m_{35} = 1$

The vertex d is connected by the edges e_3, e_4, and e_5

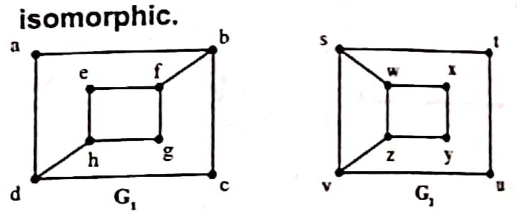
$\therefore \quad m_{43} = 1, m_{44} = 1, m_{45} = 1$

And the remaining elements in matrix are 0 because there are no other edges

$\therefore$ The incidence matrix is,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Q53. Determine whether the graphs given below are isomorphic.



Answer :

Given graphs are,

G_1 and G_2 with vertices a, b, c, d, e, f, g, h and s, t, u, v, w, x, y, z respectively.

To prove the graphs G_1 and G_2 are isomorphic following conditions are to be satisfied.

Condition 1

Check whether the vertices of graphs G_1 and G_2 are equal.

$V(G_1) = 8$ i.e., (a, b, c, d, e, f, g, h)

$V(G_2) = 8$ i.e., (s, t, u, v, w, x, y, z)

$\therefore \quad V(G_1) = V(G_2)$

Condition 1 is satisified.

Condition 2

Check whether the edges of graphs G_1 and G_2 are equal.

$E(G_1) = 10$ i.e., $\{(a, b), (b, c), (c, d), (a, d), (e, f), (f, g), (g, h), (h, e), (f, b), (h, d)\}$

$E(G_2) = 10$ i.e., $\{(s, t), (t, u), (u, v), (v, s), (w, x), (x, y), (y, z), (w, z), (s, w), (z, v)\}$

$\therefore \quad E(G_1) = E(G_2)$

Condition 2 is satisfied.

Condition 3

Check whether graphs G_1 and G_2 has similar degree of sequence.

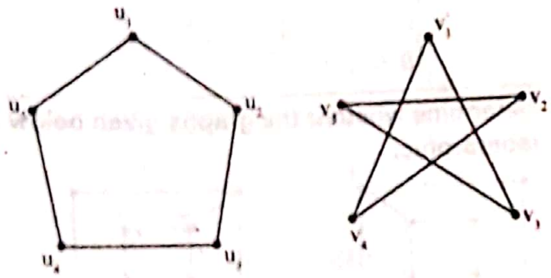
Degree of vertices of G_1	Degree of vertices of G_2
a - 2	s - 3
b - 3	t - 2
c - 2	u - 2
d - 3	v - 3
e - 2	w - 3
f - 3	x - 2
g - 2	y - 2
h - 3	z - 3

Since both the graphs contain 4 vertices of degree 3 and 4 vertices of degree 2,

Condition 3 is satisfied.

$\therefore$ The graphs G_1 and G_2 are isomorphic

Q54. Establish the isomorphism for the following graphs.



Answer :

Let the given graphs be G_1 and G_2.

To verify whether the graphs are isomorphic or not, following condition are to be satisfied.

Condition 1

Check whether both the graphs have equal number of vertices.

$$V(G_1) = 5 \text{ i.e., } (u_1, u_2, u_3, u_4, u_5)$$
$$V(G_2) = 5 \text{ i.e., } (v_1, v_2, v_3, v_4, v_5)$$
$$\therefore \quad V(G_1) = V(G_2)$$

Condition 1 is satisfied.

Condition 2

Check whether both the graphs have equal number of edges.

$$E(G_1) = 5 \text{ i.e., } \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_1)\}$$
$$E(G_2) = 5 \text{ i.e., } \{(v_1, v_3), (v_3, v_5), (v_5, v_2), (v_2, v_4), (v_4, v_1)\}$$
$$\therefore \quad E(G_1) = E(G_2)$$

Condition 2 is satisfied.

Condition 3

Check whether both the graphs have same degree of vertices.

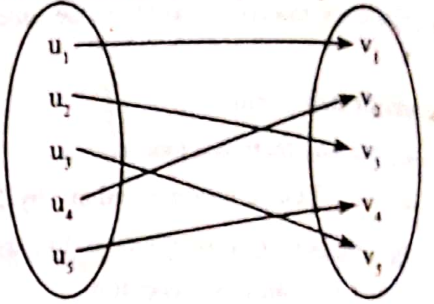
Degree of vertices of G_1	Degree of vertices of G_2
u_1 - 2	v_1 - 2
u_2 - 2	v_2 - 2
u_3 - 2	v_3 - 2
u_4 - 2	v_4 - 2
u_5 - 2	v_5 - 2

As both the graphs contain 5 vertices of degree 2

$\therefore$ Condition 3 is satisfied.

Condition 4

Verify whether, there is a one-one correspondence between the vertices of graphs G_1 and G_2.



As there is one-one correspondence in both the graphs Condition 4 is satisfied.

Condition 5

Verify that the adjacency matrix of both the graphs with respect to the corresponding vertices are equal.

Adjacency matrix of G_1 Adjacency matrix of G_2

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \\ v_1 \\ v_3 \\ v_5 \\ v_2 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_3 & v_5 & v_2 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore \quad A = B$

As the matrices are equal, condition 5 is satisfied.

Hence, the graphs G_1 and G_2 are isomorphic.

Q55. Show that isomorphism of simple graphs is an equivalence relation.

Answer : Model Paper-1, Q

Let G_1 and G_2 be two simple graphs such that G_1 and G_2 are isomorphic.

To show isomorphism is an equivalence relation on S, it must be

1. Reflexive
2. Symmetric and
3. Transitive.

1. Reflexive

Let $G = (V_1, E_1)$ be a simple graph

From identity function $f : V_1 \to V_1$ such that $f(V_1) =$ then G is isomorphic to itself

$\therefore$ Isomophism is reflexive

2. **Symmetric**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic

Then there exists one-to-one correspondence from $f: G_1 \to G_2$ and one-to-one correspondence from $f^{-1}: G_2 \to G_1$

f^{-1} is a one-to-one and onto function such that c and d are adjacent in G_2 if and only if $f^{-1}(c)$ and $f^{-1}(d)$ are adjacent in G_1

Thus, G_2 is isomorphic with G_1

$\therefore$ Isomorphism is symmetric

3. **Transitive**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic

Also $G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ be isomorphic

Since G_1 is isomorphic to G_2 then there exists a one-to-one and onto function $f: V_1 \to V_2$ such that a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2

Similarly, G_2 is isomorphic to G_3 then there exists a one-to-one and onto function

$g: V_2 \to V_3$ such that c and d are adjacent in G_2 iff $g(c)$ and $g(d)$ are adjacent in G_3

Also the functions f, g are one-one and onto

$\Rightarrow$ gof is also one-one and onto

i.e., $gof: V_1 \to V_3$ such that $gof(V_1) = g[f(V_1)]$

Thus, G_1 is isomorphic to G_3

i.e., Isomorphism is transitive

$\therefore$ Isomorphism of simple graphs is an equivalence relation.

Q56. Are the simple graphs with the following adjacency matrices ismorphic?

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Answer :

(a) Given Adjacency matrices are,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{Let, } A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$\text{and } B = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

The simple graph G corresponding with A is,

$V(G) = \{v_1, v_2, v_3\}$

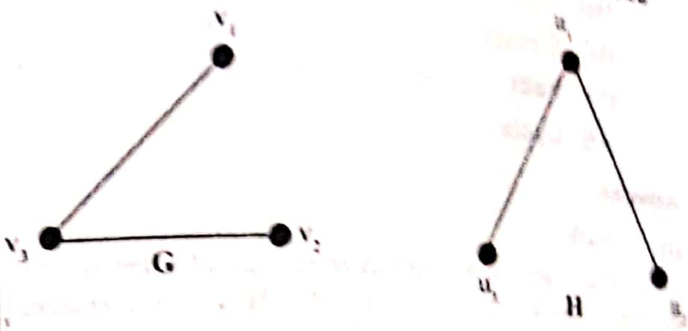
$E(G) = \{(v_1, v_3), (v_2, v_3), (v_3, v_1), (v_3, v_2)\}$

Similarly, the simple graph G' corresponding with B is,

$V(G') = \{u_1, u_2, u_3\}$

$E(G') = \{(u_1, u_2), (u_1, u_3), (u_2, u_1), (u_3, u_1)\}$

Comparing the two sets of edges $E(G)$ and $E(G')$, then the one-one and onto function $f: V(G) \to V(G')$ is defined



f is defined by,

$f(v_1) = u_3$

$f(v_2) = u_2$

$f(v_3) = u_1$

Thus, f preserves adjacency

$\therefore$ The graphs G and G' are isomorphic

(b) Given Adjacency matrices are,

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\text{Let } A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The simple graph G corresponding to A has

$V(G) = \{v_1, v_2, v_3, v_4\}$

$E(G) = \{(v_1, v_2), (v_1, v_4), (v_2, v_4), (v_3, v_4)\}$

$$\text{and } B = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The simple graph G' corresponding to B has

$V(G') = \{u_1, u_2, u_3, u_4\}$

$E(G') = \{(u_1, u_2), (u_1, u_3), (u_1, u_4), (u_2, u_4), (u_3, u_4)\}$

Since the simple graphs A and B have same number of vertices but A has 4 edges and B has 5 edges

$\therefore$ The graph G and G' represented by A and B are not isomorphic

5.1.4 Connectivity

Q57. Define the following,

(i) Walk

(ii) Open and closed walks

(iii) Trail

(iv) Circuit

(v) Path

(vi) Cycle.

Answer :

(i) Walk

If a, b are the vertices of an undirected graph $G = (V, E)$ then the walk $(a - b)$ in G is said to be a finite alternating sequence with a set of vertices and set of edges, which begins at vertex a and ends at vertex b

Example:

Consider the below graph G.

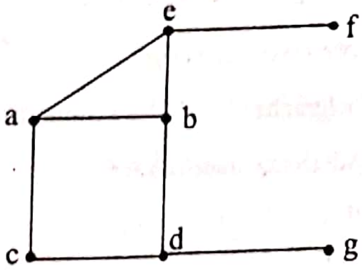


Figure: Graph (G)

The length of walk a-b i.e., n refers to the number of edges involved from vertex a to vertex b.

(ii) Open and Closed Walks

In a-b walks, if vertex a is equal to vertex b (i.e., $a = b$) and length n is greater than 1 (i.e., $n > 1$) then the walk is known as a closed walk. On the other hand, if a is not equal to b (i.e., $a \neq b$) then the walk a-b is called an open walk.

For example, the walk a-a in above graph (G) with the edges $\{a, e\}$, $\{e, b\}$ and $\{b, a\}$ of length 3 is called a closed walk and the walk a-b with the edges $\{a, e\}$ and $\{e, b\}$ of length 2 is called a open walk.

(iii) Trail

In an open walk a-b, if no edge appears more than once, then the walk is known as a-b trail.

For example, the walk a-g in above graph G containing the edges $\{a, e\}$, $\{e, b\}$, $\{b, d\}$ and $\{d, g\}$ is called an a-g trail, since none of the edges are repeated.

(iv) Circuit

A circuit is referred to a closed walk (i.e., $a = b$) in which no edge appears more than once.

For example, the walk b-b in above graph (G) with the edges $\{b, d\}$, $\{d, c\}$, $\{c, a\}$ and $\{a, b\}$ is called a b-b circuit because none of the edges are repeated.

(v) Path

A path can be defined as a trail in which no vertex appears more than once. In other words, if all the vertices in a trail are distinct then the trail is called a path.

For example, the walk a-g in above graph (G) with edges $\{a, e\}$, $\{e, b\}$, $\{b, d\}$ and $\{d, g\}$ is called a a-g path since none of the vertices are repeated.

(vi) Cycle

A cycle can be defined as a circuit in which vertex a is equal to vertex b ($a = b$) and also no vertex must appear more than once.

For example, the walk a-a in above graph containing the edges $\{a, b\}$, $\{b, d\}$, $\{d, c\}$ and $\{c, a\}$ is called an a-a cycle.

Q58. What is connectedness in a directed graphs? And also explain connected, weakly connected and strongly connected graph. Show some example graphs.

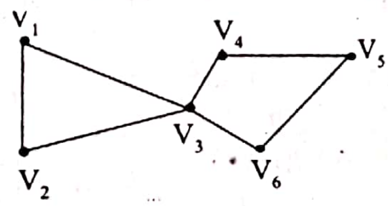
Answer :

Connectedness in a Directed Graph

The concept of connectedness in a directed graph depends upon the reachability of one vertex to another through the directed edges. This connectedness can be strong or weak. Hence reachability can be defined as the ability of one vertex to reach another in a certain path. If it is reachable then the connectedness is said to be high.

Connected Graph

A graph G is said to be connected, if every pair of distinct vertices in G are connected. In other words, a graph G is said to be connected if there exists at least one path between every two distinct vertices in G.

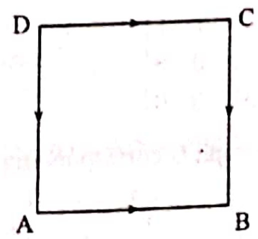


Graph (a)

Weakly Connected Graph

A graph in which connectedness of the vertices is relatively low is referred as weakly connected graph.

Consider a graph.



Graph (b)

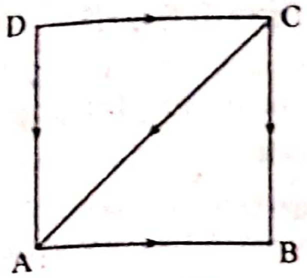
Form graph (b) consider the pair (C, D). In the pair D-C is easily reachable while C-D is not.

Strongly Connected Graph

Graph in which any two vertices will have two way connection and path to transfers in between them is called as strongly connected graph.

Example

Consider the graph.



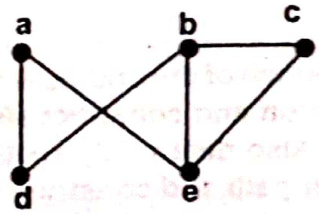
Graph (c)

The pairs for the graph (d) are (A, B) (A, C) (A, D) (B, C) (B, D) (C, D). All the pairs have two way connection between them i.e., there exist to and fro path between vertices taken as a pair.

Q59. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

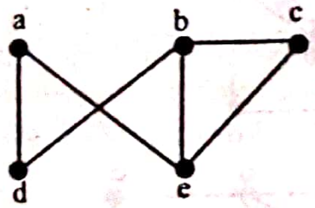
(a) a, e, b, c, b

(b) c, b, d, a, e, c



Answer :

Given graph is,



Vertices and edges of given graph are,

$V = \{a, b, c, d, e\}$

$E = \{(a, d), (a, e), (b, c), (b, d), (b, e), (c, b), (c, e), (d, a), (d, b), (e, a), (e, b), (e, c)\}$

(a) Given list of vertices are,

a, e, b, c, b

Here, $(a, e) \in E$

$(e, b) \in E$

$(b, c) \in E$

$(c, b) \in E$

Since all the sequence of edges are contained in the graph

$\therefore$ The list of vertices form a path

Since the edge (b, c) is repeated twice.

Thus, the path is not a simple path

Also, the path begins and ends at a different vertex, thus, the path is not a circuit

Since there are 4 edges in a path

$\therefore$ The length of path is 4

(b) Given list of vertices are,

c, b, d, a, e, c

Here, $(c, b) \in E$

$(b, d) \in E$

$(d, a) \in E$

$(a, e) \in E$

$(e, c) \in E$

Since all the sequence of edges are contained in the graph

$\therefore$ The list of vertices form a path

And none of edges are repeated

$\therefore$ The path is a simple path

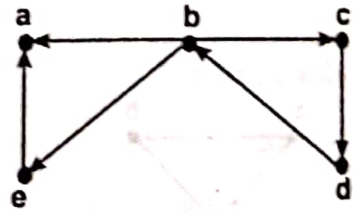
Also the path begins and ends at the same vertex

$\therefore$ The path is a circuit

Since there are 5 edges in the path

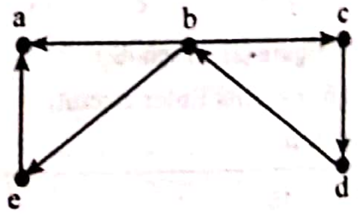
$\therefore$ The length of path is 5.

Q60. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.



Answer :

Given graph is,



The vertices and edges of the graph are,

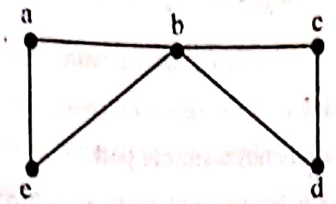
$V = \{a, b, c, d, e\}$

$E = \{(b, a), (b, c), (b, e), (c, d), (d, b), (e, a)\}$

Since there is no path from a to any other vertex, the edges (b, a) and (e, a) are directed towards a

$\therefore$ The directed graph is not strongly connected

Consider the undirected graph



Since there exists a path between every two vertices in the underlying undirected graph

∴ The directed graph is weakly connected.

5.1.5 Euler and Hamilton Paths

Q61. Define the terms with examples

(i) Euler path

(ii) Euler circuit

Answer :

(i) **Euler Path:** A Euler path in a multigraph is a path which includes each edge of the multigraph exactly once and intersects each vertex atleast once.

Example

Consider the below graph,

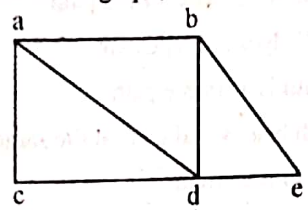


Figure (1)

In the above graph, path *a, c, d, e, b, d, a* is Euler path.

(ii) **Euler Circuits:** Consider a connected graph *G*. If there exists a circuit in *G* which contains all the edges of *G*, then that circuit is called Euler circuit.

Example

Consider the below graph (G_2)

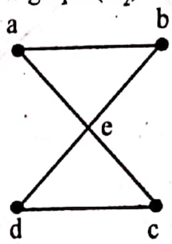


Figure (2): Graph (G_2)

The above graph contains Euler circuit,

i.e., *a, e, c, d, e, b, a*

Q62. Define the terms

(i) Hamilton path

(ii) Hamilton circuit

Answer :

Hamilton Path: A simple path in a connected graph which includes every vertex of the graph is called a hamilton path. In Hamilton path starting and ending vertex need not be same and simple path traverses every vertex exactly once.

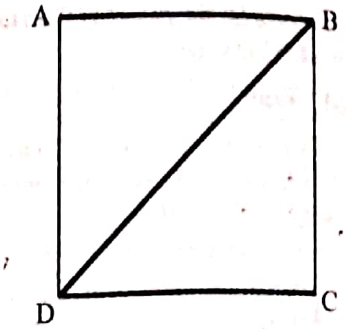


Figure (1): Graph (G)

In the above graph (*G*), the path *A-B-D-C* is Hamilton path, as it includes all the vertices of the graph.

(ii) Hamilton Circuit

A circuit in graph *G* containing every vertex of graph is known as Hamilton circuit. A Hamilton circuit traverses each vertex exactly once.

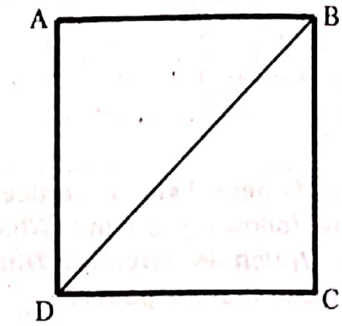
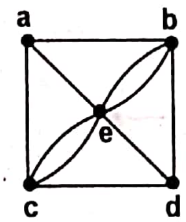


Figure (2): Graph G

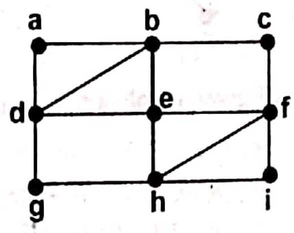
In the above graph, path of hamilton circuit is *A-B-C-D-A*.

Q63. Determine which of the undirected graphs have an Euler circuit and construct such a circuit if one exists. Also determine whether the graph has an Euler path and construct such a path if one exists.

(a)

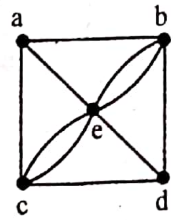


(b)



Answer :

(a) Given graph is,



The degree of vertices are,

deg(a) = 3

deg(b) = 4

deg(c) = 4

deg(d) = 3

deg(e) = 6

Since there are some vertices with odd degree

∴ The graph has no Euler circuit

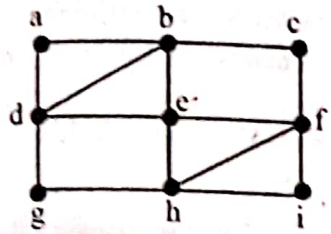
The vertices a and d have an odd degree

∴ The graph has an Euler path

One of the possible Euler path is,

$a, e, c, e, b, e, d, b, a, c, d$

(b) Given graph is,



The degree of vertices are,

deg(a) = 2

deg(b) = 4

deg(c) = 2

deg(d) = 4

deg(e) = 4

deg(f) = 4

deg(g) = 2

deg(h) = 4

deg(i) = 2

Since all the degree are even

∴ The graph has an Euler circuit

Thus, the possible Euler circuit is,

$a, b, c, f, i, h, f, e, h, g, d, e, b, d, a$

Q64. Show that a connected multigraph with atleast two vertices has an Euler circuit if and only if each of its vertices has an even degree.

Answer : **Model Paper-2, Q10(b)**

Let $G(V, E)$ be connected multigraph and graph starts with vertex 'a' and continues with an edge incident to 'a', say $\{a, b\}$ where $a, b \in V$. This edge contributes 1 to deg(a). Every time the circuit passes through a vertex, it contributes to the vertex's degree, because the circuit enters via an edge incident with this vertex and leaves via another edge of that vertex. At last the circuit stops where it started, contributing 1 to deg(a).

Now deg(a) is even since the circuit contributes 1 when it starts and ends. Thus its degree becomes 2.

The remaining vertices also have even degree. Thus it can be concluded that if a connected graph has a Euler circuit, then its vertices must be of even degree. Since there is a Euler circuit, it is a Euler graph.

Q65. Prove that a connected multi graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Answer :

If suppose a connected graph has a Euler path from a to b but not a Euler circuit, the first edge of the path contributes 1 to the degree of a. Every time the path passes through 'a', the contribution of 2 to the degree of 'a' is made. The last edge in the path contributes 1 to the degree of b. There is a contribution of 2 to 'b' whenever the path goes through 'b'.

Both a and b have odd degree. Every other vertex has an even degree because the path contributes 2 to the degree of a vertex whenever it passes through it. Therefore if it has zero or two vertices of odd degree then only the connected graph is semi-Eulerian or Euler path.

Consider the following examples.

Example (1)

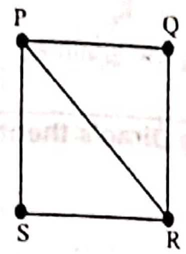


Figure: Graph G_1

The above Graph G_1 have 4 vertices named as P, Q, R and S. The Graph G_1 have exactly two vertices of odd degree, i.e, P and R. Therefore, it can be said that the Graph G_1 has an Euler path (or) semi-Eulerian that must have P and R as it end points.

Example (2)

Consider the following graph G_2.

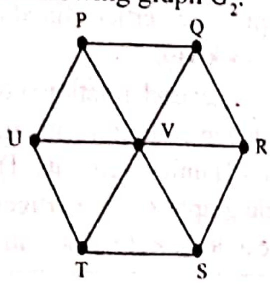


Figure: Graph G_2

The above Graph G_2 have 7 vertices named as P, Q, R, S, T, U and V. Each vertex consists of degree 3. This graph do not have an Euler path since all of its vertices have an odd degree.

Hence, a connected graph has an Euler path but not and Euler circuit if and only if it has exactly two vertices of odd degree.

Q66. Show that the complete graph k_n with n vertices has a Hamilton circuit whenever $n \geq 3$.

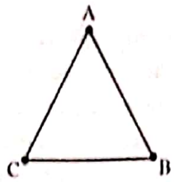
Answer : **Model Paper-3, Q10(b)**

In a complete graph k_n of n vertices there are always edges between any two vertices for all $n \geq 3$.

Hence a Hamilton circuit, i.e., a circuit traversing all the vertices of the graph exactly once can be formed in the given graph k_n.

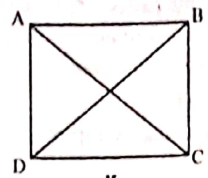
Therefore, it can be said that a complete graph with n vertices k_n has Hamilton circuit and it forms a Hamilton graph for $n \geq 3$."

Example:



K_3

In the above complete graph k_3, a path A, B, C, A forms a Hamilton circuit.



K_4

In the above complete graph k_4, a path A, B, C, D, A forms a Hamilton circuit.

Q67. State and prove Dirac's theorem?

Answer :

Statement

If G is simple graph with n vertices with $n \geq 3$, such that the degree of every vertex in G is atleast $n/2$, then G has Hamilton circuit.

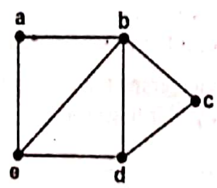
Proof

A complete graph G of n vertices has $n(n-1)/2$ edges. G consists of n edges. So, the number of edge disjoint Hamilton circuits in G cannot exceed $(n-1)/2$. There are $(n-1)/2$ edge disjoint Hamilton circuits, where n can be odd.

The subgraph of the complete graph of n vertices is a Hamilton circuit. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise.

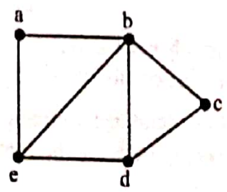
It can be observed that each rotation produces a Hamilton circuit which has no edge in common with any of the previous ones. Thus, $(n-3)/2$ new Hamilton circuits. Therefore, it can be concluded that a simple graph with n vertices $(n \geq 3)$ in which each vertex has degree at least $n/2$ has a Hamilton circuit.

Q68. Determine whether the given graph has a Hamilton circuit and find such circuit if one exists.



Answer :

Given graph is,



The graph has 5 vertices a, b, c, d, e

The degree of vertices are,

$\deg(a) = 2$

$\deg(b) = 4$

$\deg(c) = 2$

$\deg(d) = 3$

$\deg(e) = 3$

From Dirac's theorem,

The graph G has a hamilton circuit, if the degree of every vertex is atleast $\frac{n}{2}$, $n \geq 3$

Here, $n = 5$

i.e., $\frac{5}{2} = 2.5 < 3$

$\therefore$ Dirac's theorem is not satisfied

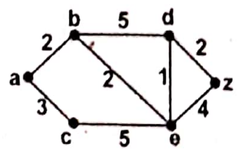
Since graph contains the cycle C_5 and it forms a Hamilton circuit

$\therefore$ The possible Hamilton circuit is the path of C_5 a, b, c, d, e, a

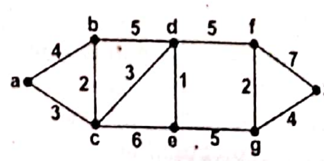
5.1.6 Shortest Path Problems

Q69. Find the length of a shortest path between a and z in the following weighted graphs

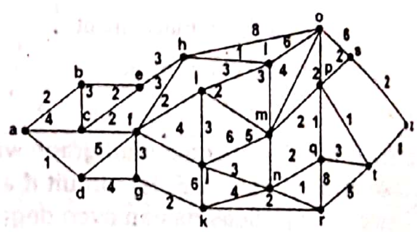
(i)



(ii)



(iii)



Answer :

(i) Given weighted graph is,

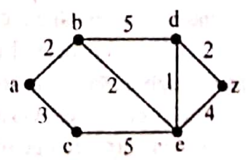


Figure -1

From the figure, the weights between the edges are given as,

$w(a, b) = 2$

$w(a, c) = 3$

$w(b, d) = 5$

$w(b, e) = 2$
$w(c, e) = 5$
$w(d, e) = 1$
$w(d, z) = 2$
$w(e, z) = 4$

Let the length of a shortest path from a to other vertex (other than a) be ∞ and the set be an empty set

i.e., $S = \phi$
$L(a) = 0$
$L(b) = \infty$
$L(c) = \infty$
$L(d) = \infty$
$L(e) = \infty$
$L(z) = \infty$

First Iteration

As $L(a)$ is the shortest length, vertex 'a' is added to the set

i.e., $S = \{a\}$
$L(b) = L(a) + w(a, b)$
$\Rightarrow L(b) = 0 + 2 = 2$
$L(c) = L(a) + w(a, c)$
$\Rightarrow L(c) = 0 + 3 = 3$
$L(d) = \infty$
$L(e) = \infty$
$L(z) = \infty$

Second Iteration

As $L(b)$ is the shortest length, vertex 'b' is added to the set

i.e., $S = \{a, b\}$
$L(c) = 3$
$L(d) = L(b) + w(b, d)$
$\Rightarrow L(d) = 2 + 5 = 7$
$L(e) = L(b) + w(b, e)$
$\Rightarrow L(e) = 2 + 2 = 4$
$L(z) = \infty$

Third Iteration

As $L(c)$ is the shortest length, vertex 'c' is added to the set

i.e., $S = \{a, b, c\}$
$L(d) = 7$
$L(e) = 4$
$L(z) = \infty$

Fourth Iteration

As $L(e)$ is the shortest length, vertex 'e' is added to the set

i.e., $S = \{a, b, c, e\}$
$L(d) = L(e) + w(d, e)$
$\Rightarrow L(d) = 4 + 1 = 5$
$L(z) = L(e) + w(e, z)$
$\Rightarrow L(z) = 4 + 4 = 8$

Fifth Iteration

As $L(d)$ is the shortest length, vertex 'd' is added to the set

i.e., $S = \{a, b, c, d, e\}$
$L(z) = L(d) + w(d, z)$
$\Rightarrow L(z) = 5 + 2 = 7$

Sixth Iteration

Finally, vertex 'z' is added to the set

i.e., $S = \{a, b, c, d, e, z\}$

The lables of the vertices in S corresponding to the iterations are as shown in table below

	a	b	c	d	e	z
Initial	0	∞	∞	∞	∞	∞
First Iteration	[0]	2	3	∞	∞	∞
Second Iteration	[0]	[2]	3	7	4	∞
Third Iteration	[0]	[2]	[3]	7	4	∞
Fourth Iteration	[0]	[2]	[3]	5	[4]	8
Fifth Iteration	[0]	[2]	[3]	[5]	[4]	7
Sixth Iteration	[0]	[2]	[3]	[5]	[4]	[7]

Table -1

$\therefore$ Length of the shortest path = 7

(ii) Given weighted graph is,

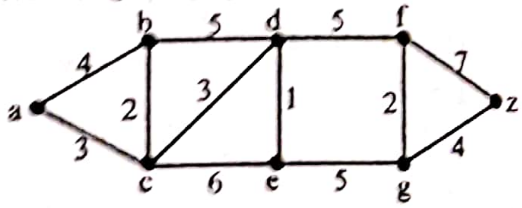


Figure -2

From the figure, the weights between the edges are given as,

$w(a, b) = 4$
$w(a, c) = 3$
$w(b, c) = 2$
$w(b, d) = 5$
$w(c, d) = 3$
$w(c, e) = 6$
$w(d, e) = 1$
$w(d, f) = 5$
$w(e, g) = 5$
$w(f, g) = 2$
$w(f, z) = 7$
$w(g, z) = 4$

Let the length of a shortest path from a to other vertex (other than a) be ∞ and the set be an empty set.

i.e., $S = \phi$
$L(a) = 0$
$L(b) = \infty$
$L(c) = \infty$
$L(d) = \infty$
$L(e) = \infty$
$L(f) = \infty$
$L(g) = \infty$
$L(z) = \infty$

First Iteration

As $L(a)$ is the shortest length, vertex 'a' is added to the set

i.e., $S = \{a\}$

$L(b) = L(a) + w(a, b)$

$\Rightarrow$ $L(b) = 0 + 4 = 4$

$L(c) = L(a) + w(a, c)$

$\Rightarrow$ $L(c) = 0 + 3 = 3$

$L(d) = \infty$

$L(e) = \infty$

$L(f) = \infty$

$L(g) = \infty$

$L(z) = \infty$

Second Iteration

As $L(c)$ is the shortest length, vertex 'c' is added to the set

i.e., $S = \{a, c\}$

$L(b) = 4$

$L(d) = L(c) + w(c, d)$

$\Rightarrow$ $L(d) = 3 + 3 = 6$

$L(e) = L(c) + w(c, e)$

$\Rightarrow$ $L(e) = 3 + 6 = 9$

$L(f) = \infty$

$L(g) = \infty$

$L(z) = \infty$

Third Iteration

As $L(b)$ is the shortest length, vertex 'b' is added to the set

i.e., $S = \{a, b, c\}$

$L(d) = 6$

$L(e) = 9$

$L(f) = \infty$

$L(g) = \infty$

$L(z) = \infty$

Fourth Iteration

As $L(d)$ is the shortest length, vertex 'd' is added to the set

i.e., $S = \{a, b, c, d\}$

$L(e) = L(d) + w(d, e)$

$\Rightarrow$ $L(e) = 6 + 1 = 7$

$L(f) = L(d) + w(d, f)$

$\Rightarrow$ $L(f) = 6 + 5 = 11$

$L(g) = \infty$

$L(z) = \infty$

Fifth Iteration

As $L(e)$ is the shortest length, vertex 'e' is added to the set

i.e., $S = \{a, b, c, d, e\}$

$L(f) = 11$

$L(g) = L(e) + w(e, g)$

$\Rightarrow$ $L(g) = 7 + 5 = 12$

$L(z) = \infty$

Sixth Iteration

As $L(f)$ is the shortest length, vertex 'f' is added to the set

i.e., $S = \{a, b, c, d, e, f\}$

$L(g) = 12$

$L(z) = L(f) + w(f, z)$

$\Rightarrow$ $L(z) = 11 + 7 = 18$

Seventh Iteration

As $L(g)$ is the shortest length, vertex 'g' is added to the set

i.e., $S = \{a, b, c, d, e, f, g\}$

$L(z) = L(g) + w(g, z)$

$\Rightarrow$ $L(z) = 12 + 4 = 16$

Eighth Iteration

Finally, vertex 'z' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, z\}$

The labels of the vertices in S corresponding to the iterations are shown in table below

	a	b	c	d	e	f	g	z
Initial	0	∞	∞	∞	∞	∞	∞	∞
First Iteration	[0]	4	3	∞	∞	∞	∞	∞
Second Iteration	[0]	4	[3]	6	9	∞	∞	∞
Third Iteration	[0]	[4]	[3]	6	9	∞	∞	∞
Fourth Iteration	[0]	[4]	[3]	[6]	7	11	∞	∞
Fifth Iteration	[0]	[4]	[3]	[6]	[7]	11	12	∞
Sixth Iteration	[0]	[4]	[3]	[6]	[7]	[11]	12	18
Seventh Iteration	[0]	[4]	[3]	[6]	[7]	[11]	[12]	16
Eighth Iteration	[0]	[4]	[3]	[6]	[7]	[11]	[12]	

Table-2

$\therefore$ Length of the shortest path = 16

(iii) Given weighted graph is,

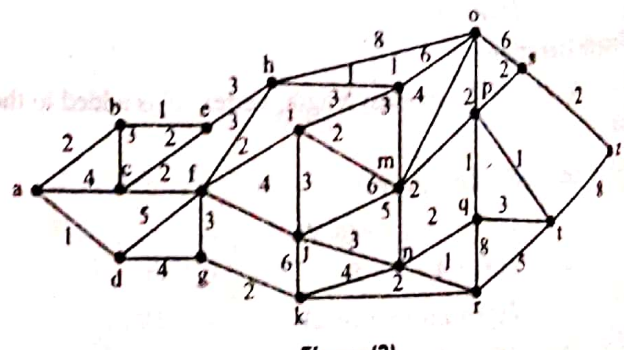


Figure (3)

From the figure, the weights between the edges are given as,

$w(a, b) = 2$
$w(a, c) = 4$
$w(a, d) = 1$
$w(b, c) = 3$
$w(b, e) = 1$
$w(c, e) = 2$
$w(c, f) = 2$
$w(d, f) = 5$
$w(d, g) = 4$
$w(e, h) = 3$
$w(f, g) = 3$
$w(f, h) = 3$
$w(f, i) = 2$
$w(f, j) = 4$
$w(g, k) = 3$
$w(h, l) = 1$
$w(h, o) = 8$
$w(i, l) = 3$
$w(i, m) = 2$
$w(j, m) = 6$
$w(j, n) = 3$
$w(k, n) = 4$
$w(k, r) = 2$
$w(l, m) = 3$
$w(l, o) = 6$
$w(m, n) = 5$
$w(m, o) = 4$
$w(m, p) = 2$
$w(n, q) = 2$
$w(n, r) = 1$
$w(o, p) = 2$
$w(o, s) = 6$
$w(p, q) = 1$
$w(p, s) = 2$
$w(p, t) = 1$
$w(q, r) = 8$
$w(q, t) = 3$
$w(r, t) = 5$
$w(s, z) = 2$
$w(t, z) = 8$

Let the length of the shortest path from a to other vertex (other than a) be ∞ and the set is an empty set

i.e., $S = \phi$

$L(a) = 0$
$L(b) = \infty$
$L(c) = \infty$
$L(d) = \infty$
$L(e) = \infty$
$L(f) = \infty$
$L(g) = \infty$
$L(h) = \infty$
$L(i) = \infty$
$L(j) = \infty$
$L(k) = \infty$
$L(l) = \infty$
$L(m) = \infty$
$L(n) = \infty$
$L(o) = \infty$
$L(p) = \infty$
$L(q) = \infty$
$L(r) = \infty$
$L(s) = \infty$
$L(t) = \infty$
$L(z) = \infty$

First Iteration

As $L(a)$ is the shortest length, vertex 'a' is added to the set

i.e., $S = \{a\}$

$L(b) = L(a) + w(a, b)$
$\Rightarrow L(b) = 0 + 2 = 2$
$L(c) = L(a) + w(a, c)$
$\Rightarrow L(c) = 0 + 4 = 4$
$L(d) = L(a) + w(a, d)$
$\Rightarrow L(d) = 0 + 1 = 1$
$L(e) = \infty$
$L(f) = \infty$
$L(g) = \infty$
$L(h) = \infty$
$L(i) = \infty$
$L(j) = \infty$
$L(k) = \infty$
$L(l) = \infty$
$L(m) = \infty$
$L(n) = \infty$
$L(o) = \infty$
$L(p) = \infty$
$L(q) = \infty$
$L(r) = \infty$
$L(s) = \infty$
$L(t) = \infty$
$L(z) = \infty$

Second Iteration

As $L(d)$ is the shortest length, vertex 'd' is added to the set

i.e., $S = \{a, d\}$

$L(b) = 2$

$L(c) = 4$

$L(e) = \infty$

$L(f) = L(d) + w(d, f)$

$\Rightarrow$ $L(f) = 1 + 5 = 6$

$L(g) = L(d) + w(d, g)$

$\Rightarrow$ $L(g) = 1 + 4 = 5$

$L(h) = \infty$

$L(i) = \infty$

$L(j) = \infty$

$L(k) = \infty$

$L(l) = \infty$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = \infty$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Third Iteration

As $L(b)$ is the shortest length, vertex 'b' is added to the set

i.e., $S = \{a, b, d\}$

$L(c) = 4$

$L(e) = L(b) + w(b, e)$

$\Rightarrow$ $L(e) = 2 + 1 = 3$

$L(f) = 6$

$L(g) = 5$

$L(h) = \infty$

$L(i) = \infty$

$L(j) = \infty$

$L(k) = \infty$

$L(l) = \infty$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = \infty$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Fourth Iteration

As $L(e)$ is the shortest length, vertex 'e' is added to the set

i.e., $S = \{a, b, d, e\}$

$L(c) = 4$

$L(f) = 6$

$L(g) = 5$

$L(h) = L(e) + w(e, h)$

$\Rightarrow$ $L(h) = 3 + 3 = 6$

$L(i) = \infty$

$L(j) = \infty$

$L(k) = \infty$

$L(l) = \infty$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = \infty$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Fifth Iteration

As $L(c)$ is the shortest length, vertex 'c' is added to the set

i.e., $S = \{a, b, c, d, e\}$

$L(f) = 6$

$L(g) = 5$

$L(h) = 6$

$L(i) = \infty$

$L(j) = \infty$

$L(k) = \infty$

$L(l) = \infty$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = \infty$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Sixth Iteration

As $L(g)$ is the shortest length, vertex 'g' is added to the set

i.e., $S = \{a, b, c, d, e, g\}$

$L(f) = 6$

$L(h) = 6$

$L(i) = \infty$

$L(j) = \infty$

$L(k) = L(g) + w(g, k)$

$\Rightarrow L(k) = 5 + 2 = 7$

$L(l) = \infty$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = \infty$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Seventh Iteration

As $L(f)$ is the shortest length, vertex 'f' is added to the set

i.e., $S = \{a, b, c, d, e, f, g\}$

$L(h) = 6$

$L(i) = L(f) + w(f, i)$

$\Rightarrow L(i) = 6 + 2 = 8$

$L(j) = L(f) + w(f, j)$

$\Rightarrow L(j) = 6 + 4 = 10$

$L(k) = 7$

$L(l) = \infty$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = \infty$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Eighth Iteration

As $L(h)$ is the shortest length, vertex 'h' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h\}$

$L(i) = 8$

$L(j) = 10$

$L(k) = 7$

$L(l) = L(h) + w(h, l)$

$\Rightarrow L(l) = 6 + 1 = 7$

$L(m) = \infty$

$L(n) = \infty$

$L(o) = L(h) + w(h, o)$

$\Rightarrow L(o) = 6 + 8 = 14$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = \infty$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Ninth Iteration

As $L(k)$ is the shortest length, vertex 'k' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, k\}$

$L(i) = 8$

$L(j) = 10$

$L(l) = 7$

$L(m) = \infty$

$L(n) = L(k) + w(k, n)$

$\Rightarrow L(n) = 7 + 4 = 11$

$L(o) = 14$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = L(k) + w(k, r)$

$\Rightarrow L(r) = 7 + 2 = 9$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Tenth Iteration

As $L(l)$ is the shortest length, vertex 'l' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, k, l\}$

$L(i) = 8$

$L(j) = 10$

$L(m) = L(l) + w(l, m)$

$\Rightarrow L(m) = 7 + 3 = 10$

$L(n) = 11$

$L(o) = L(l) + w(l, o)$

$\Rightarrow L(o) = 7 + 6 = 13$

$L(p) = \infty$

$L(q) = \infty$

$L(r) = 9$

$L(s) = \infty$

$L(t) = \infty$

$L(z) = \infty$

Eleventh Iteration

As $L(i)$ is the shortest length, vertex 'i' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, k, l\}$

$L(j) = 10$

$L(m) = 10$

$L(n) = 11$

$L(o) = 13$
$L(p) = \infty$
$L(q) = \infty$
$L(r) = 9$
$L(s) = \infty$
$L(t) = \infty$
$L(z) = \infty$

Twelfth Iteration

As $L(r)$ is the shortest length, vertex 'r' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, k, l, r\}$

$L(j) = 10$
$L(m) = 10$
$L(n) = L(r) + w(r, n)$
$\Rightarrow L(n) = 9 + 1 = 10$
$L(o) = 13$
$L(p) = \infty$
$L(q) = L(r) + w(r, q)$
$\Rightarrow L(q) = 9 + 8 = 17$
$L(s) = \infty$
$L(t) = L(r) + w(r, t)$
$\Rightarrow L(t) = 9 + 5 = 14$
$L(z) = \infty$

Thirteenth Iteration

As $L(j)$ is the shortest length, vertex 'j' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, r\}$

$L(m) = 10$
$L(n) = 10$
$L(o) = 13$
$L(p) = \infty$
$L(q) = 17$
$L(s) = \infty$
$L(t) = 14$
$L(z) = \infty$

Fourteenth Iteration

As $L(m)$ is the shortest length, vertex 'm' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, r\}$

$L(n) = 10$
$L(o) = 13$
$L(p) = L(m) + w(m, p)$
$\Rightarrow L(p) = 10 + 2 = 12$
$L(q) = 17$
$L(s) = \infty$
$L(t) = 14$
$L(z) = \infty$

Fifteenth Iteration

As $L(n)$ is the shortest length, vertex 'n' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, r\}$

$L(o) = 13$
$L(p) = 12$
$L(q) = 17$
$L(s) = \infty$
$L(t) = 14$
$L(z) = \infty$

Sixteenth Iteration

As $L(p)$ is the shortest length, vertex 'p' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, r\}$

$L(o) = 13$
$L(q) = 17$
$L(s) = L(p) + w(p, s)$
$\Rightarrow L(s) = 12 + 2 = 14$
$L(t) = L(p) + w(p, t)$
$\Rightarrow L(t) = 12 + 1 = 13$
$L(z) = \infty$

Seventeenth Iteration

As $L(o)$ is the shortest length, vertex 'o' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, r\}$

$L(q) = 17$
$L(s) = 14$
$L(t) = 13$
$L(z) = \infty$

Eighteenth Iteration

As $L(t)$ is the shortest length, vertex 't' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, r, t\}$

$L(q) = L(t) + w(q, t)$
$\Rightarrow L(q) = 13 + 3 = 16$
$L(s) = 14$
$L(z) = L(t) + w(t, z)$
$\Rightarrow L(z) = 13 + 8 = 21$

Nineteenth Iteration

As $L(s)$ is the shortest length, vertex 's' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, r, s, t\}$

$L(q) = 16$
$L(z) = L(s) + w(s, z)$
$\Rightarrow L(z) = 14 + 2 = 16$

Twentieth Iteration

As $L(q)$ is the shortest length, vertex 'q' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t\}$

$L(z) = 16$

Twenty Oneth Iteration

Finally, vertex 'z' is added to the set

i.e., $S = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, z\}$

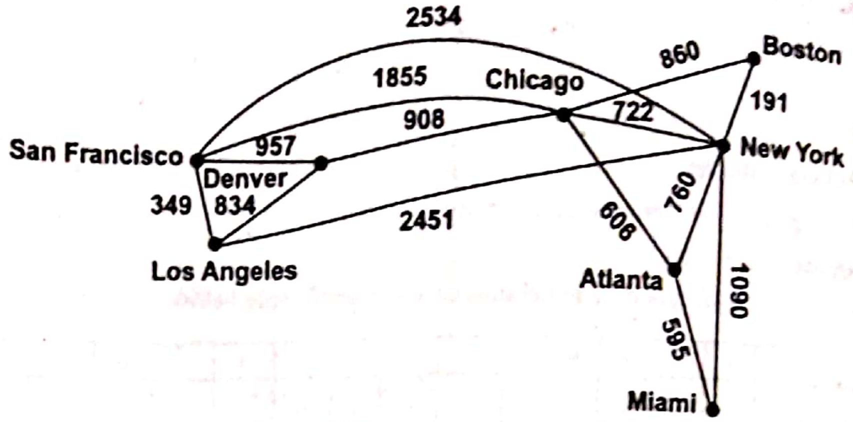
$\therefore$ Length of the shortest path = 16

The labels of the vertices in S corresponding to iterations are shown in table below

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	z
Initial	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
First Iteration	[0]	2	4	1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Second Iteration	[0]	2	4	[1]	∞	6	5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Third Iteration	[0]	[2]	4	[1]	3	6	5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Fourth Iteration	[0]	[2]	4	[1]	[3]	6	5	6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Fifth Iteration	[0]	[2]	[4]	[1]	[3]	6	5	6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Sixth Iteration	[0]	[2]	[4]	[1]	[3]	6	[5]	6	∞	∞	7	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Seventh Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	6	8	10	7	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
Eighth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	8	10	7	7	∞	∞	14	∞	∞	∞	∞	∞	∞
Ninth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	8	10	[7]	7	∞	11	14	∞	∞	9	∞	∞	∞
Tenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	8	10	[7]	[7]	10	11	13	∞	∞	9	∞	∞	∞
Eleventh Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	10	[7]	[7]	10	11	13	∞	∞	9	∞	∞	∞
Twelfth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	10	[7]	[7]	10	10	13	∞	17	[9]	∞	14	∞
Thirteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	10	10	13	∞	17	[9]	∞	14	∞
Fourteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	10	13	12	17	[9]	∞	14	∞
Fifteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	13	12	17	[9]	∞	14	∞
Sixteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	13	[12]	17	[9]	14	13	∞
Seveteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	[13]	[12]	17	[9]	14	13	∞
Eighteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	[13]	[12]	16	[9]	14	[13]	21
Nineteenth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	[13]	[12]	16	[9]	[14]	[13]	16
Twenteith Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	[13]	[12]	[16]	[9]	[14]	[13]	16
Twenty oneth Iteration	[0]	[2]	[4]	[1]	[3]	[6]	[5]	[6]	[8]	[10]	[7]	[7]	[10]	[10]	[13]	[12]	[16]	[9]	[14]	[13]	[16]

Table-3

Q70. Find the shortest path between the cities Newyork and Los Angeles in the airline system shown figure.



Answer :

Given airline system is,

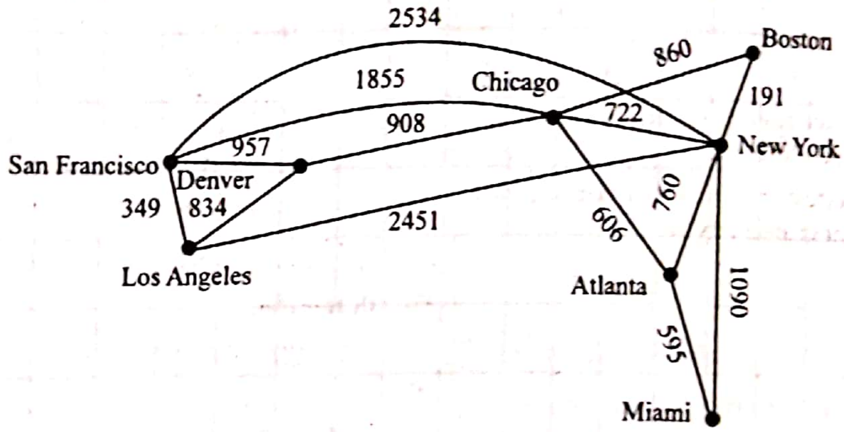


Figure (1)

Let the cities given in figure (1) can be represented as,

a - San Francisco

b - Los Angeles

c - Denver

d - Chicago

e - Boston

f - New York

g - Atlanta

h - Miami

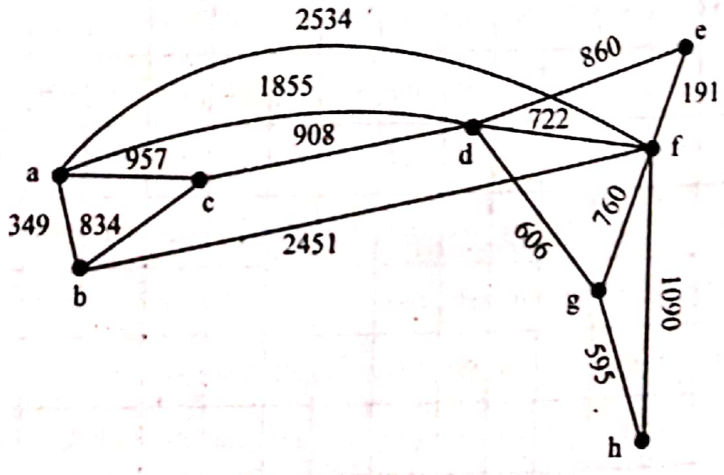


Figure (2)

From figure (2), the weights (distances) between the vertices (cities) are given as,

$w(a, b) = 349$

$w(a, c) = 957$

$w(a, d) = 1855$

$w(a, f) = 2534$

$w(b, c) = 834$

$w(b, f) = 2451$

$w(c, d) = 908$

$w(d, e) = 860$

$w(d, f) = 722$

$w(d, g) = 606$

$w(e, f) = 191$

$w(f, g) = 760$

$w(f, h) = 1090$

$w(g, h) = 595$

Here, the path to be determined is between the cities New York and Los Angeles i.e., from f to b

Let the length of a shortest path from f to other verticies (other than f) be ∞ and the set is an empty set

i.e., $S = \phi$

$L(a) = \infty$

$L(b) = \infty$

$L(c) = \infty$

$L(d) = \infty$

$L(e) = \infty$

$L(f) = 0$

$L(g) = \infty$

$L(h) = \infty$

First Iteration

As $L(f)$ is the shortest length, vertex 'f' is added to the set

i.e., $S = \{f\}$

$L(a) = L(f) + w(a, f)$

$\Rightarrow$ $L(a) = 0 + 2534 = 2534$

$L(b) = L(f) + w(b, f)$

$\Rightarrow$ $L(b) = 0 + 2451 = 2451$

$L(c) = \infty$

$L(d) = L(f) + w(d, f)$

$\Rightarrow$ $L(d) = 0 + 722 = 722$

$L(e) = L(f) + w(e, f)$

$\Rightarrow$ $L(e) = 0 + 191 = 191$

$L(g) = L(f) + w(g, f)$

$\Rightarrow$ $L(g) = 0 + 760 = 760$

$L(h) = L(f) + w(f, h)$

$\Rightarrow$ $L(h) = 0 + 1090 = 1090$

Second Iteration

As $L(e)$ is the shortest length, vertex 'e' is added to the set

i.e., $S = \{e, f\}$

$L(a) = 2534$

$L(b) = 2451$

$L(c) = \infty$

$L(d) = 722$

$L(g) = 760$

$L(h) = 1090$

Third Iteration

As $L(d)$ is the shortest length, vertex 'd' is added to the set

i.e., $S = \{d, e, f\}$

$L(a) = 2534$

$L(b) = 2451$

$L(c) = L(d) + w(c, d)$

$\Rightarrow$ $L(c) = 722 + 908 = 1630$

$L(g) = 760$

$L(h) = 1090$

Fourth Iteration

As $L(g)$ is the shortest length, vertex 'g' is added to the set

i.e., $S = \{d, e, f, g\}$

$L(a) = 2534$

$L(b) = 2451$

$L(c) = 1630$

$L(h) = 1090$

Fifth Iteration

As $L(h)$ is the shortest length, vertex 'h' is added to the set

i.e., $S = \{d, e, f, g, h\}$

$L(a) = 2534$

$L(b) = 2451$

$L(c) = 1630$

Sixth Iteration

As $L(c)$ is the shortest length, vertex 'c' is added to the set

i.e., $S = \{c, d, e, f, g, h\}$

$L(a) = 2534$

$L(b) = 2451$

Seventh Iteration

As $L(b)$ is the shortest length, vertex 'b' is added to the set

i.e., $S = \{b, c, d, e, f, g, h\}$

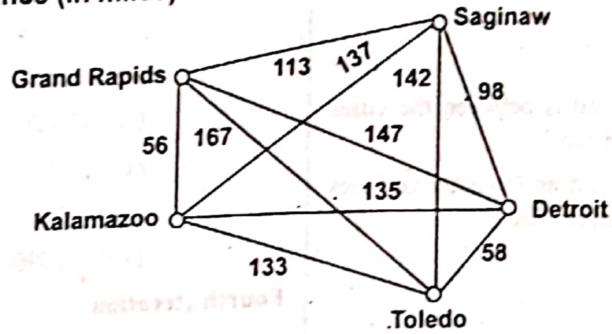
The labels of the vertices in S corresponding to the iterations are as shown in table below.

	a	b	c	d	e	f	g	h
Initial	∞	∞	∞	∞	∞	0	∞	∞
First Iteration	2534	2451	∞	722	191	[0]	760	1090
Second Iteration	2534	2451	∞	722	[191]	[0]	760	1090
Third Iteration	2534	2451	1630	[722]	[191]	[0]	760	1090
Fourth Iteration	2534	2451	1630	[722]	[191]	[0]	[760]	1090
Fifth Iteration	2534	2451	1630	[722]	[191]	[0]	[760]	[1090]
Sixth Iteration	2534	2451	[1630]	[722]	[191]	[0]	[760]	[1090]
Seventh Iteration	2534	[2451]	[1630]	[722]	[191]	[0]	[760]	[1090]

Table

∴ The shortest path is 2451 miles i.e. the length of the direct path from New York to Los Angeles.

Q71. Solve the traveling salesman problem for the given graph. In which order should salesman visit to travel the minimum total distance (in miles).



Answer :

Given weighted graph is,

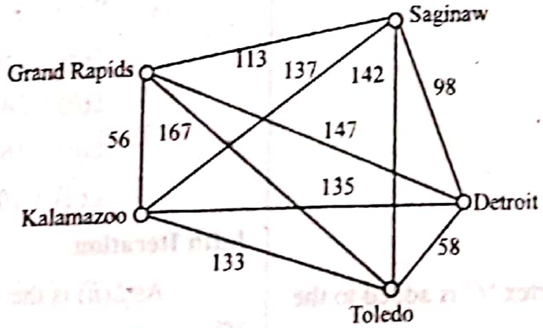


Figure (1)

Let the cities in figure (1) be represented as,

Grand Rapids - g

Saginaw - s

Kalamazoo - k

Toledo - t

Detroit - d

Then,

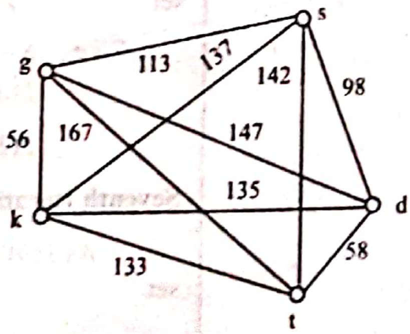
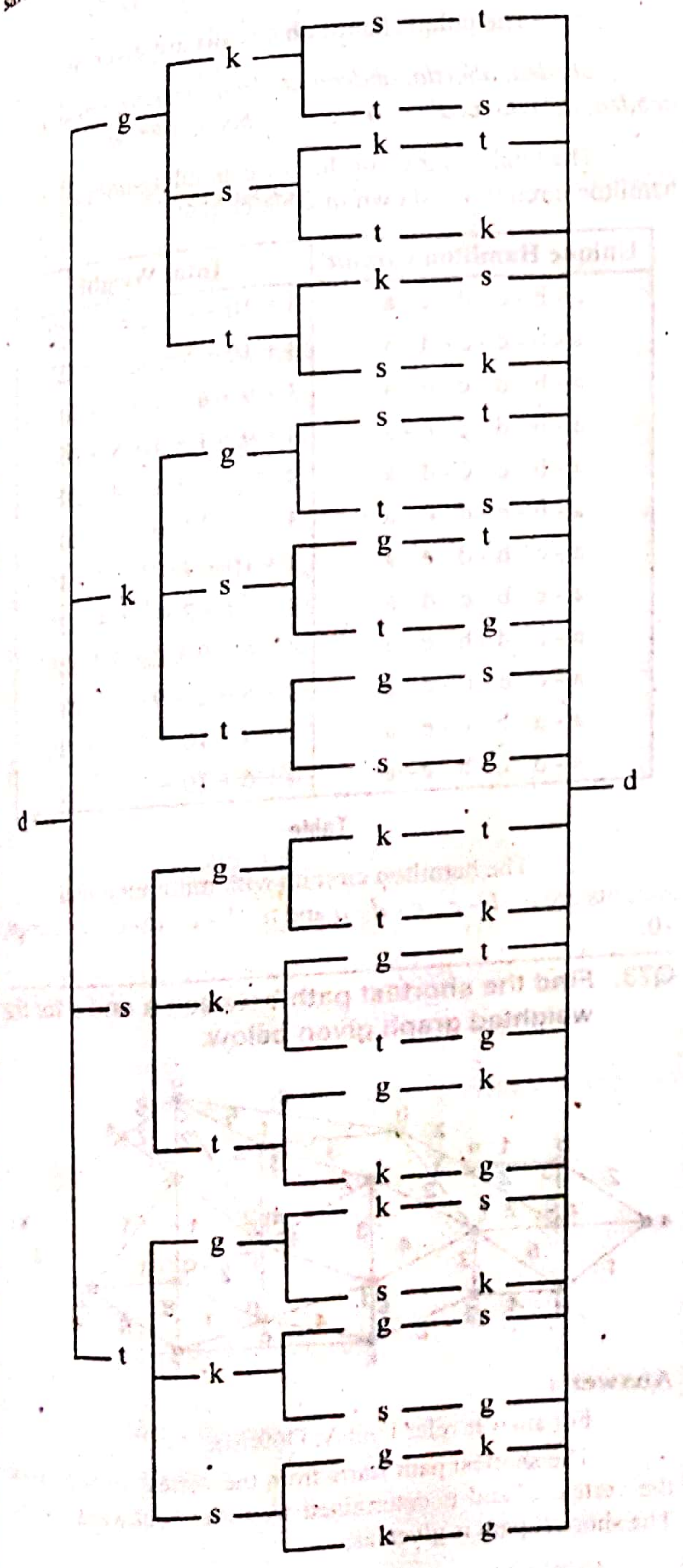


Figure (2)

Let the salesman starts from the city Detroit i.e., 'd'. Secondly he may visit any of the remaining four cities. Thirdly, he may visit any of the remaining three cities. Similarly, he can visit fourth city from remaining 2 cities and last remained city will be the fifth city. Finally, the salesman reaches the destination 'd' after visiting the five cities.

Therefore, the total number of paths in which the salesman can visit the cities are $4 \times 3 \times 2 \times 1 = 24$



Figure

Hence, the total number of paths are,

dgkstd, dgktsd, dgsktd, dgstkd, dgtksd, dgtskd, dkgstd, dkgtsd, dksgtd, dkstgd, dktsgd, dktgsd, dsgktd, dsgtkd, dskgtd,dsktgd, dstgkd, dstkgd, dtgksd, dtgskd, dtkgsd, dtksgd, dtsgkd, dtskgd.

As the traveling results in the same path in opposite direction, the unique paths in which the salesman can visit will be equal to the half of total number of paths.

i.e., Unique paths $= \frac{24}{2} = 12$

$\therefore$ The unique paths in which the salesman may travel are given as,

dtgskd, dtgksd, dtksgd, dtkgsd, dtskgd, dtsgkd, dstgkd, dstkdgd, dsktgd, dsgtkd, dgstkd, dgtskd

The total distance for the corresponding unique paths are shown in table below.

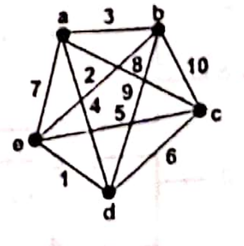
Unique Path	Total Distance (in miles)
d - t - g - s - k - d	58 + 167 + 113 + 137 + 135 = 610
d - t - g - k - s - d	58 + 167 + 56 + 137 + 98 = 516
d - t - k - s - g - d	58 + 133 + 137 + 113 + 147 = 588
d - t - k - g - s - d	58 + 133 + 56 + 113 + 98 = 458
d - t - s - k - g - d	58 + 142 + 137 + 56 + 147 = 540
d - t - s - g - k - d	58 + 142 + 113 + 56 + 135 = 504
d - s - t - g - k - d	98 + 142 + 167 + 56 + 135 = 598
d - s - t - k - g - d	98 + 142 + 133 + 56 + 147 = 576
d - s - k - t - g - d	98 + 137 + 133 + 167 + 147 = 682
d - s - g - t - k - d	98 + 113 + 167 + 133 + 135 = 646
d - g - s - t - k - d	147 + 113 + 142 + 133 + 135 = 670
d - g - t - s - k - d	147 + 167 + 142 + 137 + 135 = 728

Table

From the table, the path with minimum total distance is d - t - k - g - s - d (458 miles).

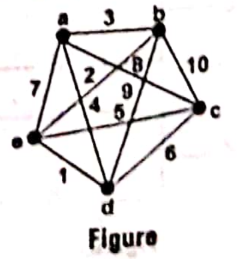
Hence, the order in which the salesman visit to travel the minimum total distance of 458 miles is Detroit - Toledo - Kalamazoo - Grand Rapids - Saginaw - Detroit.

Q72. Solve the traveling sales man problem for the given graph by finding the total weight of all Hamilton circuits and determine a circuit with minimum total weight.



Answer :

Given weighted graph is,

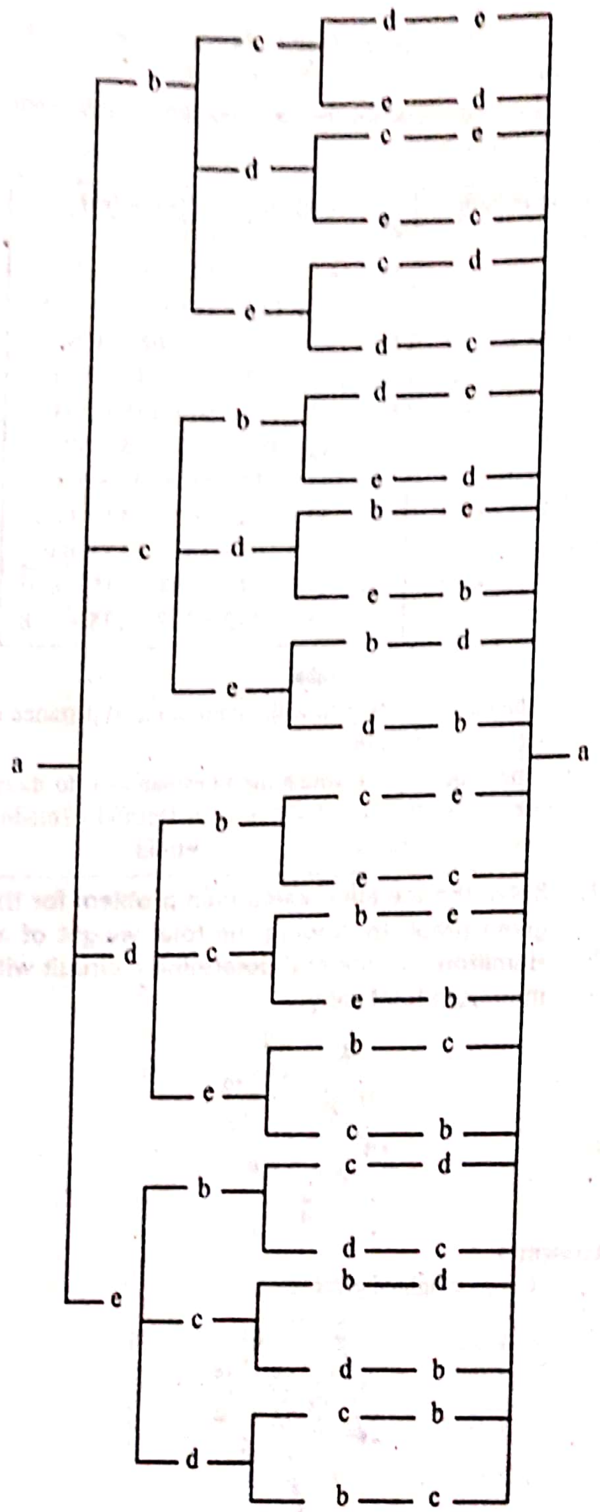


Figure

Scanned with CamScanner

Let the hamilton circuit starts from the vertex 'a'. The second vertex can be choosen from any of the remaining four vertices b, c, d, e. The third vertex can be chosen from the remaining 3 vertices. The fourth vertex can be chosen from remaining 2 vertices and the last remained vertex is choosen as fifth vertex. Finally, the hamilton circuit ends with 'a' as sixth vertex.

Therefore, the total number of possible hamilton circuits are $4 \times 3 \times 2 \times 1 = 24$



Figure

Hence, the hamilton circuits are given as,

abcdea, abceda, abdcea, abdeca, abecda, abed… abcdea, acbeda, acdbea, acdeba, acebda, abedba, abdc… abdeca, adcbea, adceca, adcbea, adceba, adceba, adeb… adecba, aebcda, aebdca, aecbda, aecdba, aedcba, aedbca

As the hamilton circuits results in the same path in opposite direction, the unique hamilton circuits will be equal to the half of total possible hamilton circuits.

i.e., Unique hamilton circuits $= \dfrac{24}{2} = 12$

$\therefore$ The unique hamilton circuits are given as,

abcdea, abceda, abdcea, abdeca, abecda, abedca, acbdea, acbeda, acdbea, acebda, adbcea, adcbea

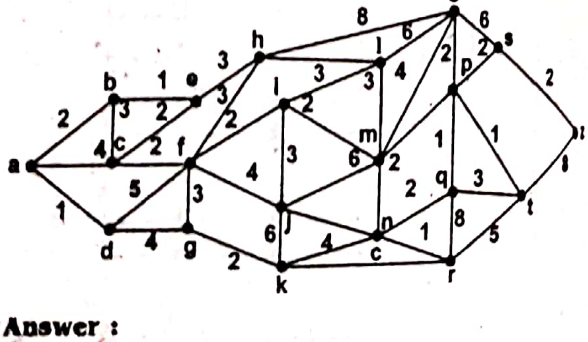
The total weights for the corresponding unique hamilton circuits are shown in table below,

Unique Hamilton Circuit	Total Weight
a - b - c - d - e - a	3 + 10 + 6 + 1 + 7 = 27
a - b - c - e - d - a	3 + 10 + 5 + 1 + 4 = 23
a - b - d - c - e - a	3 + 9 + 6 + 5 + 7 = 30
a - b - d - e - c - a	3 + 9 + 1 + 5 + 8 = 26
a - b - e - c - d - a	3 + 2 + 5 + 6 + 4 = 20
a - b - e - d - c - a	3 + 2 + 1 + 6 + 8 = 20
a - c - b - d - e - a	8 + 10 + 9 + 1 + 7 = 35
a - c - b - e - d - a	8 + 10 + 2 + 1 + 4 = 25
a - c - d - b - e - a	8 + 6 + 9 + 2 + 7 = 32
a - c - e - b - d - a	8 + 5 + 2 + 9 + 4 = 28
a - d - b - c - e - a	4 + 9 + 10 + 5 + 7 = 35
a - d - c - b - e - a	4 + 6 + 10 + 2 + 7 = 29

Table

$\therefore$ The hamilton circuits with minimum total weights are $a - b - e - c - d - a$ and $a - b - e - d - c - a$ of length 20.

Q73. Find the shortest path between a and z for the weighted graph given below.



Answer :

For answer refer Unit-V, Q69(iii).

The shortest path starts from the vertex 'a' and ends at the vertex 'z' and is determined from the backward process. The shortest path is given as,

$S.P = a \,\ldots\ldots\, z$

Twenty oneth Iteration

$L(z)$ is not calculated

Twenteith Iteration

$L(z)$ is not calculated

Nineteenth Iteration

$L(z)$ is calculated using $L(s)$ i.e., vertex 'z' is preceeded by the vertex 's'. The shortest path is given as,

$S.P = a \ldots\ldots s, z$

Eighteenth Iteration

$L(s)$ is not calculated

Seventeenth Iteration

$L(s)$ is not calculated

Sixteenth Iteration

$L(s)$ is calculated using $L(p)$ i.e., vertex 's' is preceeded by the vertex 'p'. The shortest path is given as,

$S.P = a \ldots\ldots p, s, z$

Fifteenth Iteration

$L(p)$ is not calculated

Fourteenth Iteration

$L(p)$ is calculated using $L(m)$ i.e., vertex 'p' is preceeded by vertex 'm'. The shortest path is given as,

$S.p = a \ldots\ldots m, p, s, z$

Thirteenth Iteration

$L(m)$ is not calculated

Twelfth Iteration

$L(m)$ is not calculated

Eleventh Iteration

$L(m)$ is not calculated

Tenth Iteration

$L(m)$ is calculated using $L(l)$ i.e., vertex 'm' is preceeded by vertex 'l'. The shortest path is given as,

$S.P = a \ldots l, m, p, s, z$

Ninth Iteration

$L(l)$ is not calculated

Eighth Iteration

$L(l)$ is calculated using $L(h)$ i.e., vertex 'l' is preceeded by vertex 'h'. The shortest path is given as,

$S.P = a \ldots h, l, m, p, s, z$

Seventh Iteration

$L(h)$ is not calculated

Sixth Iteration

$L(h)$ is not calculated

Fifth Iteration

$L(h)$ is not calculated

Fourth Iteration

$L(h)$ is calculated using $L(e)$ i.e., vertex 'h' is preceeded by vertex 'e'. The shortest path is given as,

$S.P = a \ldots\ldots e, h, l, m, p, s, z$

Third Iteration

$L(e)$ is calculated using $L(b)$ i.e., vertex 'e' is preceeded by vertex 'b'. The shortest path is given as,

$S.P = a \ldots b, e, h, l, m, p, s, z$

Second Iteration

$L(b)$ is not calculated

First Iteration

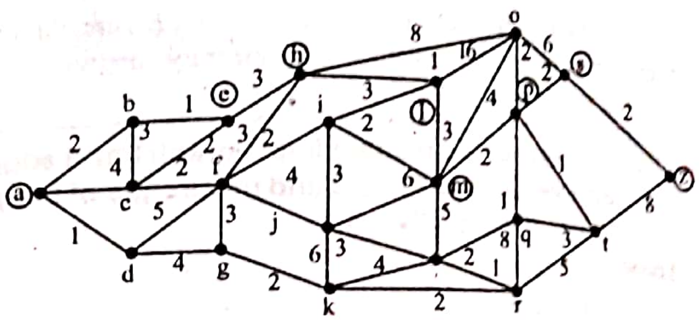
$L(b)$ is calculated using $L(a)$ i.e., vertex 'b' is preceeded by vertex 'a'. The shortest path is given as,

$S.P = a, b, e, h, l, m, p, s, z$

The length of the shortest path is verified by adding the weights of the edges

i.e:, $2 + 1 + 3 + 1 + 3 + 2 + 2 + 2 = 16$

$\therefore$ The shortest path is $a, b, e, h, l, m, p, s, z$ shown in figure below.



Figure

5.1.7 Planar Graphs

Q74. If G is a connected planar simple graph with e edges and vertices, then prove that the number of regions of G is r = e – v + 2.

Answer : Model Paper-4, Q10(b)

Given that,

G is a connected planar simple graph.

Let $G_1, G_2, G_e = G$ be subgraphs.

Let e_n be edges of G_n,

r_n be number of regions of G_n

V_n be vertices of G_n

Basis step

For $n = 1$

$r_1 = e_1 - v_1 + 2$

Induction step

If (a_{n+1}, b_{n+1}) is an edge added to G_n to obtain G_{n+1} then,

$r_{n+1} = r_n + 1$

$e_{n+1} = e_n + 1$

$V_{n+1} = V_n$

Assuming for n,

$r_n = e_n - V_n + 2$

For $n = n + 1$,

$$r_{n+1} = e_{n+1} - V_{n+1} + 2$$

Substituting the corresponding values in above equation,

$$r_{n+1} = e_n + 1 - V_n + 2$$

$$r_n = e_n - V_n + 2$$

If G is a connected planar simple graph with e edges and v vertices, where $v > 3$, then

$$e \leq 3v - 6$$

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

For a planar graph, if (u, v) edge is to be removed, it is divided using a new vertex w. This operation is called as elementary subdivision

i.e. for $(u, v) \Rightarrow \{u, w\}$ and $\{w, v\}$

If the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are obtained from same graph by a sequence of elementary subdivisions, it is called as homomorphic.

Q75. If a connected planar simple graph has e edges and vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Answer :

Given that,

A connected planar graph has e edges and vertices $v \geq 3$.

And there are no circuits of length 3.

i.e., Each region is of degree 4 or more

$\Rightarrow \quad 2e \geq 4r$ ... (1)

From Euler's formula,

$r = e - v + 2$ ... (2)

From equation (1),

$$\frac{2e}{4} \geq r$$

$\Rightarrow \quad r \leq \frac{e}{2}$ ... (3)

From equations (2) and (3),

$$e - v + 2 \leq \frac{e}{2}$$

$\Rightarrow \quad 2e - 2v + 4 \leq e$

$\Rightarrow \quad 2e - e \leq 2v - 4$

$\Rightarrow \quad e \leq 2v - 4$

$\therefore \quad e \leq 2v - 4$

Q76. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?

Answer :

Given that,

A connected planar graph has 8 vertices and degree of each vertex is 3

i.e., $\deg(v_i) = 3, i = 1, 2, \ldots 8.$

Then there are 3 edges connecting to each of 8 vertices

$\therefore$ Total number of connections to all 8 vertices is,

$3.8 = 24.$

From Handshaking theorem,

$$2e = 3.8 = 24$$

$\Rightarrow \quad e = \frac{24}{2} = 12$

$\therefore$ There are 12 edges.

From Euler's formula,

Total number of regions in a planar representation is,

$$r = e - v + 2$$

Substituting the corresponding values in above equation,

$$r = 12 - 8 + 2$$

$\therefore \quad r = 6.$

5.1.8 Graph Coloring

Q77. What is coloring problem and hence define coloring of a graph?

Answer : Model Paper-5, Q10(a)

Coloring Problem

Problems related to the coloring of maps of regions such as maps of parts of the world have generated many results in graph theory. When a map is colored, two regions with a common border are customarily assigned different colors. One way to ensure that two adjacent regions never have the same color is to use a different color for each region. However, this is insufficient and if the map has many regions it would be difficult to distinguish similar colors. Instead, a small number of colors should be used whenever possible. Consider the problem of determining the least number of colors that can be used to color a map so that adjacent regions never have the same color. The problem is known as the 4-color problem.

Each map in the plane can be represented by a graph. To set up this correspondence, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have common borders. The resulting graph is known as the dual graph of the map. The way in which dual graphs of maps are constructed, it is clear that any map in the plane has a planar dual graph.

The problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph, so that no two adjacent vertices in this graph have the same color.

Coloring of a Graph

Coloring all the vertices of the graphs with colors such that no two adjacent vertices have the same color is referred as coloring of a graph. In other words a coloring of a simple graph is the assignment of a color to each vertex of a graph so that no two adjacent vertices are assigned the same color.

Chromatic Number

The chromatic number of a graph 'G' is defined as the minimum number of different colors required for coloring the graph.

The chromatic number of a graph 'G' is denoted by $\chi(G)$.

Suppose if $\chi(G) = K$ then the graph 'G' is called 'K' chromatic.

Q78. What are chromatic numbers? Explain with an example.

Answer :

Chromatic Number

A graph G is said to be K-colorable if we can color it properly with K colors. A K-chromatic graph is a graph which can be colored properly with K colors but not less than K colors. Chromatic number of a graph is the minimum number of colors required to color a graph. It is denoted by $\chi(G)$.

Some important points are given below,

(i) A graph is 1-chromatic if it contains only isolated vertices.

(ii) A graph is at least 2-chromatic with one or more edges.

(iii) If suppose a graph G contains a subgraph G_1, then $\chi(G) \geq \chi(G_1)$.

(iv) If a graph G contains K_n as a subgraph, then $\chi(G) \geq n$.

(v) If G is a graph of n vertices then $\chi(G) \leq n$.

(vi) $\chi(K_n) = n \; \forall \; n \geq 1$

Graph Coloring

A graph G is said to be properly colored if we assign colors to its vertices in such a way that no two adjacent vertices have the same color

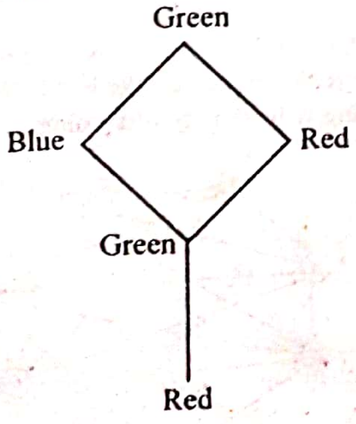


Figure (1)

Example

Let us find the chromatic number for the graph shown in figure (2).

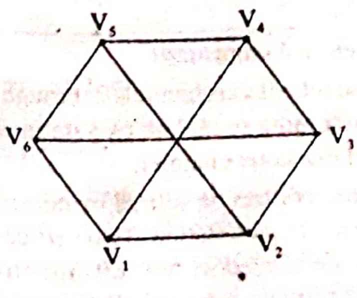


Figure (2)

This graph can be colored properly using 2 colors as shown in figure (3)

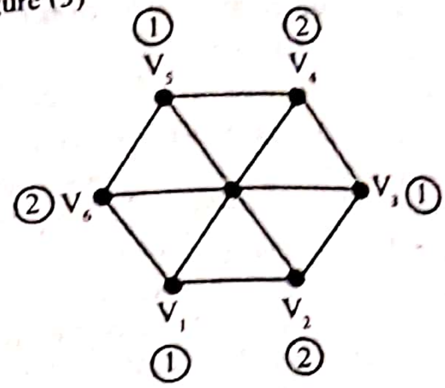


Figure (3)

In figure (3), no two adjacent vertices have the same color. Thus, the graph is colored properly. Hence, the chromatic number for the graph is 2.

In the same manner, the chromatic number of the graph shown in figure (4) is 4

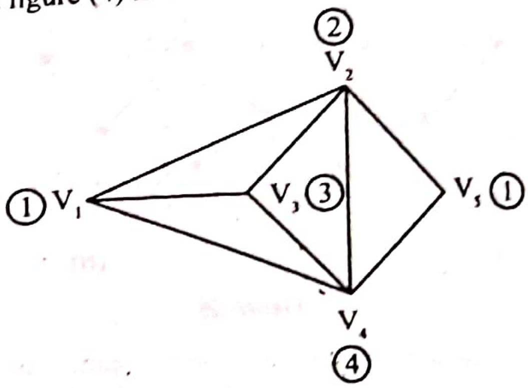


Figure (4)

Q79. State and explain the four color theorem with example.

Answer : Model Paper-1, Q11(a)

Statement

The region of a planar graph is said to be properly colored if no two adjacent regions have the same color (two regions are adjacent if they have a common edge). The minimum number of colors that are required for proper coloring of all regions of a planar graph is called 4-color problem.

The 4-color problem states that any planar graph has chromatic number 4 or less

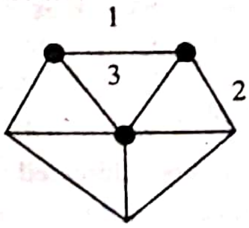


Figure (1)

Color Theorem

Every planar graph is 5 colorable i.e., the number of colors required for painting all the vertices of a planar graph is 5 or less.

Proof : (By Induction)

Consider a planar graph with 5 vertices i.e., 1, 2, 3, 4, 5

All these graphs require ≤ 5 colors. Let us assume a theorem which holds good up to $n - 1$ vertices.

Consider a planar graph with 'n' vertices we know that in a planar graph there is a vertex 'v' whose degree is ≤ 5 let us delete this vertex from the graph.

Now the graph contains $n - 1$ vertices and by hypothesis the theorem holds good up to $n - 1$ vertices, i.e., graph G' (i.e., $G - r$) requires not more than 5 colors. Let us introduce the deleted vertex v into the graph if degree of v is 1, 2, 3, 4 then we have no difficulty in assigning 5th color to vertex v. Suppose degree of v is 5 and all the 5 colors have been used in coloring the vertices adjacent to v as shown in figure 2(a).

Suppose there is a path in G' between vertices 'a' and 'd' colored alternately with colors 1 and 4 as shown in figure 2(b)

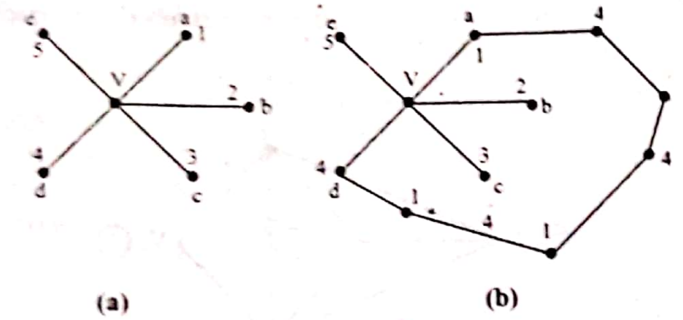


Figure (2)

A similar path between 'c' and 'e' cannot exist because these 2 paths would intersect and cause G to be non-planar. If there is no path between 'c' and 'e' then there is no way vertices 'c' and 'e' can be adjacent in which case we can give same color to vertices 'c' and 'e' which means the other color (either 3 or 5) can be given to vertex v and the graph requires still a maximum of 5 colors only.

Hence proved.

Q80. Explain about various applications of graph colorings.

Answer : Model Paper-2, Q11(a)

Applications of Graph Coloring

1. Assignment of Index Registers

During the execution of loops in the program we want to keep frequently accessed data in the C.P.U registers to speed up the process, instead in RAM. For a given loop, how many index registers are needed.

This problem can be addressed using a graph coloring model. Let each variable represent the vertex of a graph. There is an edge between two vertices, if the two variables are required to be placed in the same index register at the same time, this problem can be solved by using chromatic number of the graph. Thus, the chromatic number of the graph gives the number of index registers needed.

2. Scheduling Final Exams

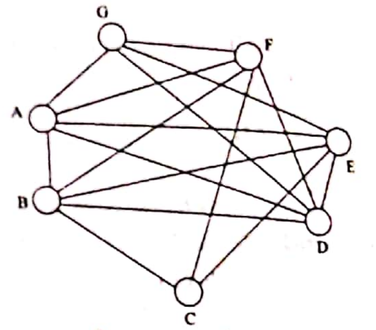
Let us consider a scheduling problem to conduct final exams at a university so that no student has two exams at the same time.

First, construct a graph model for the problem, with vertices representing courses and with an edge between two vertices if there is a common student in the course they represent (i.e., if the corresponding courses are incompatible). Each time slot for a final exam is represented by a different color. Scheduling of the exams corresponds to a coloring of the associated graph.

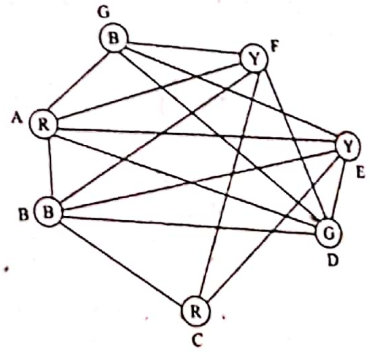
Example

Suppose there are 7 final exams to be scheduled. Let the courses are numbered A through G, such that the following pairs of courses have common students. A and G, A and F, A and E, A and D, A and B, G and F, G and E, G and D, F and B, F and C, F and D, E and B, E and C, E and D, D and B, C and B. How can we schedule?

The graph for this problem is shown below



Since the graph contains K_4 as a subgraph (with vertices A, B, D and E) its chromatic number is ≥ 4. The following figure shows a coloring with exactly four colors



How do we interpret this result? Since the chromatic number is 4, the final exams can be scheduled conflict-free using four time slots, as the following table shows

Block	1	2	3	4
Course (s)	A, C	B, G	D	E, F

3. Frequency Assignment

Suppose television channels 2 through 12 are assigned to stations in South India so that no two stations within 160 miles can operate on the same channel.

This problem can be solved by constructing a graph by assigning a vertex to each station. Two vertices are connected by an edge if they are located within 160 miles from each other. The assignment of channels corresponds to a coloring of the graph, where each color represents a different channel.

5.2 TREES

5.2.1 Introduction to Trees

Q81. Explain the following

(i) Tree

(ii) Rooted tree

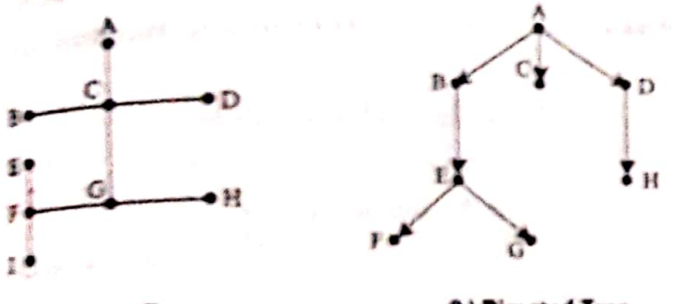
(iii) Ordered Rooted tree

Answer :

(i) Tree

A tree is a simple graph *G* where there exists a unique simple undirected path between every pair of vertices.

A tree in which there exists one designated vertex known as root is called a rooted tree. A rooted tree is a directed tree if there exists a root from which there is a directed path to each vertex. The length of the path to '*v*' from the root is called the level of a vertex '*v*'.



(a) Undirected Tree

(b) Directed Tree

Figure

(ii) Rooted Tree

G is said to be a rooted tree if it is a directed tree and if it has a unique vertex *r* called the root of the tree, which have in degree of zero (i.e., $id(r) = 0$) and all the other vertices with in degree of one (i.e., $id(v) = 1$).

Example

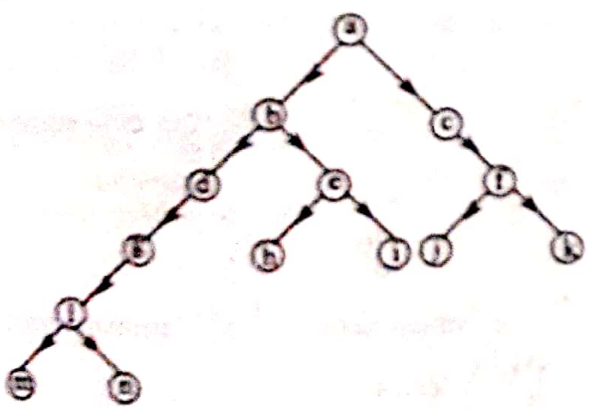


Figure (2): Rooted Tree

In the figure (2), the vertices or nodes *m*, *n*, *h*, *i*, *j*, *k* are called leaves or terminal vertices. All the other internal vertices are known as branch nodes.

The nodes *j* and *k* are called the descendants of *f*, *c* and *a* whereas the nodes *f*, *c* and *a* are called the ancestors of *j* and *k*. If two vertices have a common parent then they are referred to as siblings. For instance, *j* and *k* are called siblings since they have a common parent, which is *f*.

(iii) Ordered Rooted Tree

In a rooted tree *T* if every internal vertex is ordered from left to right then *T* is called an ordered rooted tree, which is shown in the example below.

Example

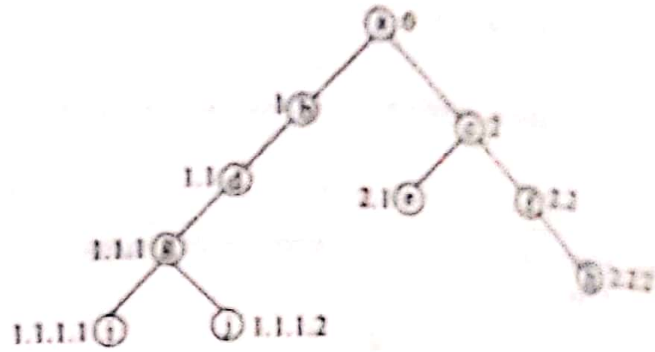
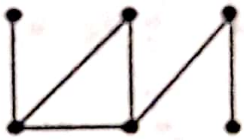


Figure (3): Ordered Rooted Tree

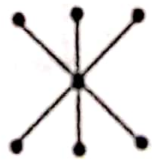
Q82. Determine which of the following graphs are trees

(a)

(b)



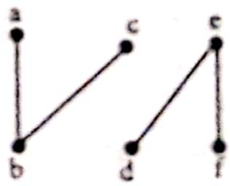
(c)



Answer :

(a) Let the vertices of given graph be *a*, *b*, *c*, *d*, *e*, *f*

i.e.,

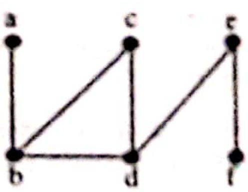


Since there is no path from *c* to *d*, the graph is not connected.

∴ The given graph is not a tree.

(b) Let the vertices of given graph be *a*, *b*, *c*, *d*, *e*, *f*

i.e.,



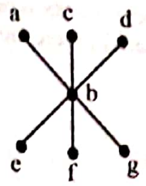
The graph contains a simple circuit

i.e., *b*, *c*, *d*, *b*

∴ The given graph is not a tree.

(c) Let the vertices of given graph be a, b, c, d, e, f, g

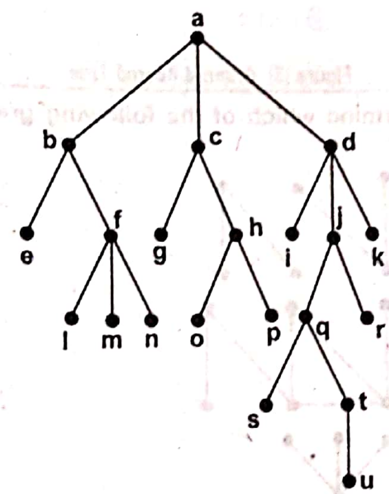
i.e.,



The graph is undirected, connected and has no simple circuit

$\therefore$ The given graph is a tree

Q83. Find the level of each vertex in the rooted tree. Also find its height



Answer :

From the given graph,

Root a is at level 0

Level 1 contains the vertices b, c, d

Level 2 contains the vertices e, f, g, h, i, j, k

Level 3 contains the vertices l, m, n, o, p, q, r

Level 4 contains the vertices s, t

Level 5 contains the vertex u

Since the largest level of any vertex is 5

$\therefore$ Rooted tree has height 5

Q84. Prove that a tree with n vertices has exactly n – 1 edges.

Answer : Model Paper-3, Q11(a)

A tree with n vertices containing exactly $n-1$ edges can be proved by using a mathematical induction. The theorem is obvious for $n = 1$, $n = 2$ and $n = 3$.

Assume that the theorem holds for all trees with fewer than k vertices. Here, k is a specified positive integer. Consider a tree T with k vertices. In T, let 'e' be an edge with end vertices u and v. Thus, deletion of e from T disconnected the graph and $T-e$ contains exactly 2 components (i.e., T_1 and T_2). These two components do not contain cycles because T does not contain any cycle.

Thus the theorem holds for these trees i.e., each of T_1 and T_2 contains one less edge than the number of vertices in it. As the total number of vertices in T_1 and T_2 is k, the total number of edges is $k-2$. But T_1 and T_2 are taken together as $T-e$. Therefore, $T-e$ contains $k-2$ edges. Consequently, T has exactly $k-1$ edges. Therefore, if the theorem is true for a tree with $n < k$ vertices, it is true for a tree with $n = k$ vertices. By induction, the theorem is true for all positive integers n.

Therefore, a tree with n vertices has $n-1$ edges.

Q85. A full m-ary tree with

(i) n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n + 1]/m$ leaves,

(ii) i internal vertices has $n = mi + 1$ vertices and $l = (m-1)i + 1$ leaves,

(iii) l leaves has $n = (ml-1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

Answer : Model Paper-4, Q11(a)

Let,

n be the number of vertices

i be the number of internal vertices

l be the number of leaves

(i) Since each vertex is either a leaf or an internal vertex

i.e., $n = l + i$ ...(1)

A full m-ary tree with i internal vertices contains vertices as,

$$n = mi + 1 \quad ...(2)$$

From equation (2),

$$n = mi + 1$$

$$\Rightarrow n - 1 = mi$$

$$\Rightarrow i = \frac{n-1}{m} \quad ...(3)$$

From equation (1),

$$l = n - i$$

$$\Rightarrow l = n - \frac{(n-1)}{m} \quad [\because \text{From equation (3)}]$$

$$\Rightarrow l = \frac{mn - n + 1}{m}$$

$$\Rightarrow l = \frac{(m-1)n + 1}{m}$$

$\therefore$ n vertices has $i = \frac{(n-1)}{m}$ internal vertices and $l = \frac{(m-1)n+1}{m}$ leaves.

(ii) From equation (2),

$$n = mi + 1$$

From equation (1),

$$l = n - i$$

$$\Rightarrow l = mi + 1 - i \quad [\because \text{From equation (2)}]$$

$$\Rightarrow l = (m-1)i + 1$$

$\therefore$ Internal vertices has $n = mi + 1$ vertices and $l = (m-1)i + 1$ leaves.

(iii)

$$l = (m-1)i + 1$$

$$\Rightarrow \quad l - 1 = (m-1)i$$

$$\Rightarrow \quad i = \frac{l-1}{m-1}$$

Substituting i value in equation (2),

$$n = m\left(\frac{l-1}{m-1}\right) + 1$$

$$\Rightarrow \quad n = \frac{ml - m + m - 1}{m-1}$$

$$\Rightarrow \quad n = \frac{ml-1}{m-1}$$

$\therefore$ l leaves has $n = \dfrac{(ml-1)}{m-1}$ vertices and $i = \dfrac{(l-1)}{(m-1)}$ internal vertices.

5.2.2 Application of Trees

Q66. Write about binary search trees.

Answer :

Binary search tree is a binary tree in which each vertex contains one right child or one left child. The keys assigned to vertices are such that the keys smaller than larger vertex key are arranged on left subtree while the keys larger than larger vertex key are arranged on right subtree.

Decision tree is a binary rooted tree in which each internal vertex corresponds to a decision with a subtree at these vertices.

Based on binary comparisons a sorting algorithm requires atleast $(\log n!)$ comparisons. To sort n elements, the average number of comparisons required are $\Omega(n\log n)$.

Q67. Discuss about Huffman coding.

Answer :

Model Paper-1, Q7(b)

Huffman coding is the one of the best and important applications of a tree in communication. For example, consider the transmitting of text. Firstly, the message should be encrypted by using 0 and 1s i.e., each of the alphabet should be represented with 0s and 1s binary code sequentially. The code given to an alphabet should be unique and each alphabet should be assigned with a code of length 5 as $2^4 < 26 < 2^5$. Then the transmitted message at receiver end is decrypted in to alphabets and then the message is recognized by the receiver. The message may consists of 'n' number of letters, from which some letters may appear many number of times while other not appear frequently. So, the most frequently occurred letters are represented with short frequencies and the less frequently occurred letters are represented with long frequencies.

Consider a message of 500 letters that is to be transmitted. The total number of bits to be sent will be 500×5. Among these 500 letters, the (assume that each letter repeated for 50 times) letters a, e, s, n, i are appeared maximum number of times and z, w appeared minimum number of times. Make the most frequently occurred letters with the size of 2. Then the length of the message will be reduced by a factor. This type of coding is called as variable length coding.

$$\Rightarrow \quad 3 \times 50 \times 5$$

$5 \rightarrow$ Each letter occurred frequently with size 5

$50 \rightarrow$ Each letter appeared 50 times

$3 \rightarrow$ Length is reduced by 3

There is a problem using the variable length coding i.e., when the sender transmits the message using various lengths, how does the receiver divides the string into the sequences corresponding to the letter.

An elegant procedure to construct an optimal tree defined by the Huffman, is as follows,

1. Arrange the data in a row in such a way that their weights are in increasing order. Each character is the leaf node of a tree.
2. Select the two nodes with small weights.
3. Make a third node by combining the weights of two small nodes.
4. Repeat the steps 2 and 3 until all the nodes on each level are combined to create a single tree.

Example to illustrate the above algorithm is as follows.

a, b, c, d, e are the nodes with weights 1, 4, 6, 7, 13 respectively then the Huffman coding for each node using the tree is

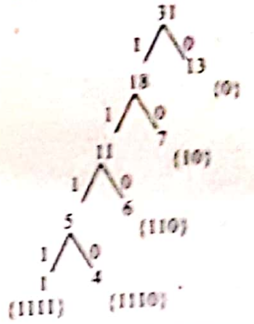
$\underbrace{1, 4}, 6, 7, 13$

$\underbrace{5, 6}, 7, 13$

$\underbrace{11, 7}, 13$

$\underbrace{18, 13}$

31



The Huffman codes are,

1 → (1111)

4 → (1110)

6 → (110)

7 → (10)

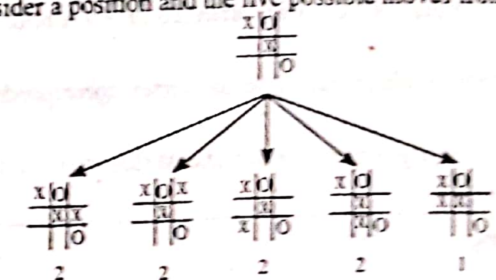
13 → (0)

Q88. Discuss about Game trees

Answer :

Game tree is the most exciting application of tree in games such as chess, nim, go, checkers, tic-tac-toe and so on. The game is explained by using the tic-tac-toe game.

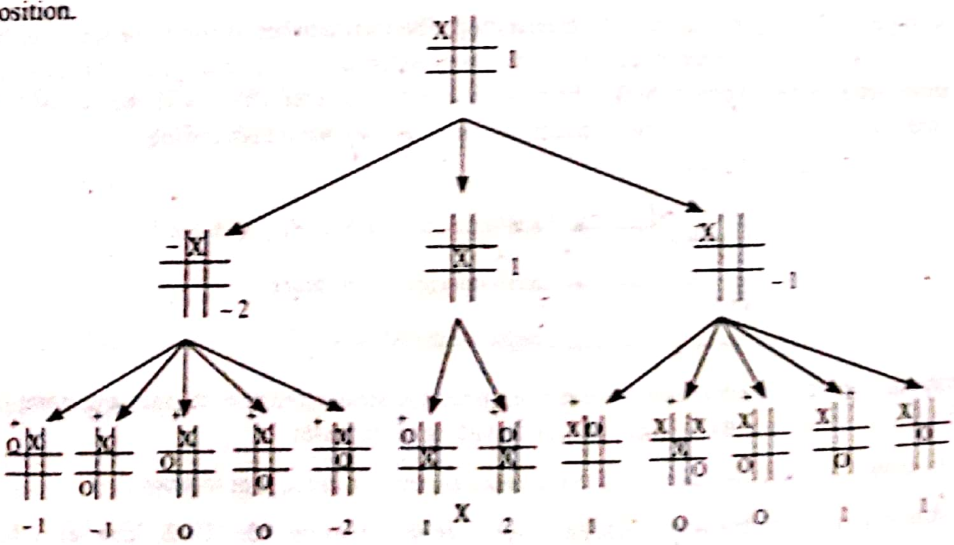
Tic-tac-toe game is a 2 player game it starts with the empty board and each of the player is assigned with a symbol like 'X' and 'O'. Each player finds the best move to win the game. The win value is used in the implementation and it is calculated as the difference between the number of rows, columns and diagonals that are left open for one player and other left for the second player. The function, say computewinvalue ( ) is used to compute and returns the winvalue. It is possible to form a tree using the possible moves from the current position. This is called as 'game tree'. For a player the move with the highest win value will be the best move. Now, consider a position and the five possible moves from that position with their winvalues.



Figure(1): Game Tree and Winvlues of Each Possible Move From a Position

The winvalues for each possible move were calculated and four possible moves have the same winvalue. The move at fourth position will lead to win for 'X' and other moves leads to the victory of 'O'. It says that the move with a small winvalue is better than the move with highest win value.

The computewinvalue( ) function is not enough to expect the result of the game it needs to revise this function. The best and efficient way to play the game is to look a head for different possible moves from the current position and significantly a better move. Let lookahead be the number of moves from a position. It is possible to form a tree using all the possible moves from the current position.



Figure(2): Game tree for tic-tac-toe Game

The game tree for a tic-tac-toe game is mentioned in above diagram. Here the '+' symbol denotes the player who starts the game first and '–' symbol denotes the opponent player. The computation for the best move starts from the root node with + symbol and the remaining are assigned with '+' or '–' depending on the computation.

Q89. Build a binary search tree for the words banana, peach, apple, pear, coconut, mango and papaya using alphabetical order.

Answer :

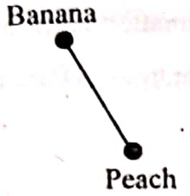
Given words are,

Banana, peach, apple, pear, coconut, mango and papaya

The word banana is the key of the root

The next word in the list is peach

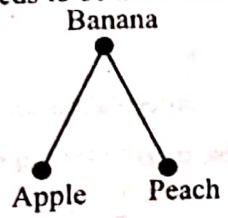
Since the word peach comes after the banana in alphabetical order, then peach needs to be the right child of banana



The next word in the list is apple.

Apple comes before banana in alphabetical order,

Then apple needs to be the left child of banana

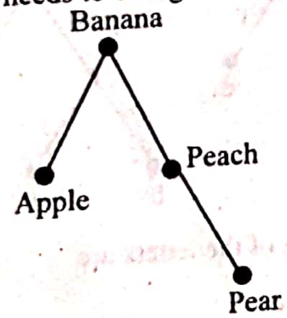


The next word in the list is pear

Pear comes after banana in alphabetical order, then pear needs to be right child of banana

Also pear occurs after peach

Thus, pear needs to be right child of peach

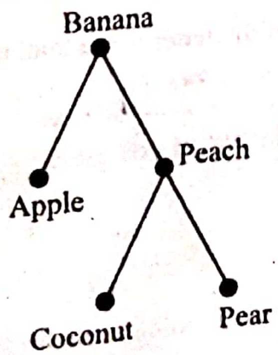


The next word in the list is coconut

Coconut comes after banana in alphabetical order, then coconut needs to be right child of banana

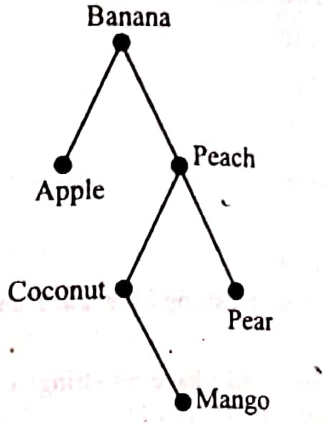
Also coconut comes before peach

Thus, coconut needs to be left child of peach



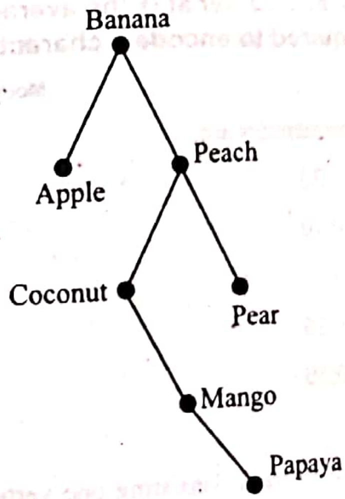
The next word in the list is mango

Mango comes after banana in alphabetical order, then mango needs to be right child of banana. Mango occurs before peach, it should be left child of peach. Again mango occurs after word coconut, thus mango needs to be right child of coconut.



The next word in the list is papaya.

Papaya comes after the word banana in alphabetical order, then papaya needs to be right child of banana. Also papaya comes before peach, it needs to be left child of peach. Again papaya comes after coconut, then papaya needs to be right child of coconut. Papaya comes after mango in alphabetical order. Thus, papaya needs to be right child of mango.



Q90. Which of these codes are prefix codes?

(a) a: 11, e:00, t: 10, s: 01

(b) a:010, e:11, t: 011, s: 1011, n: 1001, i: 10101

Model Paper-2, Q11(b)

Answer :

(a) Given codes are,

a: 11

e: 00

t: 10

s: 01

Prefix codes encode letters such that the bit string for a letter never occurs as the first part of the bit string for another letter.

Since all the given strings have length 2 and differ from each other

Thus, there are no strings of letters that start with the string of another letter

∴ The encoding forms a prefix code.

(b) Given codes are,

a: 010

e: 11

t: 011

s: 1011

n: 1001

t: 10101

The letter has bit string of length 2 and no other strings start with 11

The letters *a* and *t* have bit strings of length 3 and no other strings start with 010 or 011

The letters *s* and *n* have bit strings of length 4 and no other string start with 1011 or 1001

The letter *t* have bit string of length 5 and no other strings start with 10101

∴ The encoding forms a prefix code.

Q91. Use Huffman coding to encode these symbols with given frequencies: a: 0.20, b: 0.10, c: 0.15, d: 0.25, e: 0.30. What is the average number of bits required to encode a character?

Model Paper-5, Q11(a)

Answer :

Given frequencies are,

a: 0.20

b: 0.10

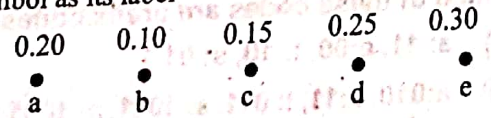
c: 0.15

d: 0.25

e: 0.30

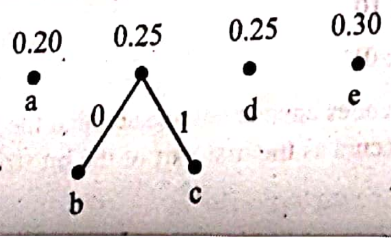
Initial Forest

The forest of trees consisting one vertex, each vertex has a symbol as its label



Step 1

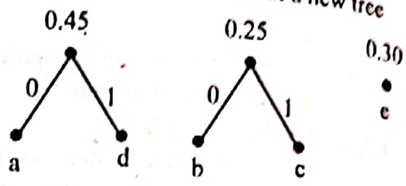
Here, *b* and *c* have the smallest labels combining two trees into a single tree by introducing a new root. The tree with larger weight is the left subtree labeled as 0 and the tree with smaller weight is the right subtree, labeled as 1



Step 2

The trees with the smallest labels are *a* and *d*

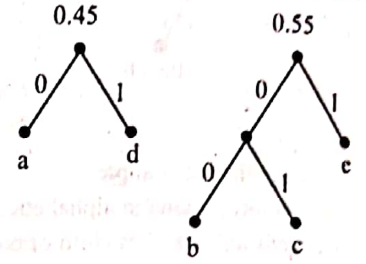
Combining these trees to form a new tree



Step 3

The trees with smallest labels are *b* + *c* and *e*

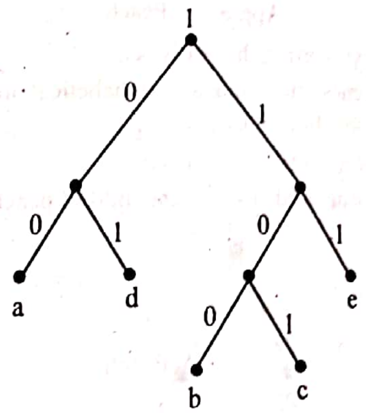
Combining these trees to form a new tree



Step 4

The trees with smallest labels are *b* + *c* + *e* and *a* + *d*

Combining these trees to form a new tree



The encoding of the letters are

a: 00

b: 100

c: 101

d: 01

e: 11

The weight of the letter is the total number of bits in the code of letter

i.e., $w_a = 2$

$w_b = 3$

$w_c = 3$

$w_d = 2$

$w_e = 2$

∴ Average number of bits is the sum of product of weight and frequencies

i.e., Average number of bits = 2(0.20) + 3(0.10) + 3(0.15) + 2(0.25) + 2(0.30)

= 0.40 + 0.3 + 0.45 + 0.5 + 0.6

= 2.25

5.2.3 Tree Traversal

Q92. Explain universal address systems.

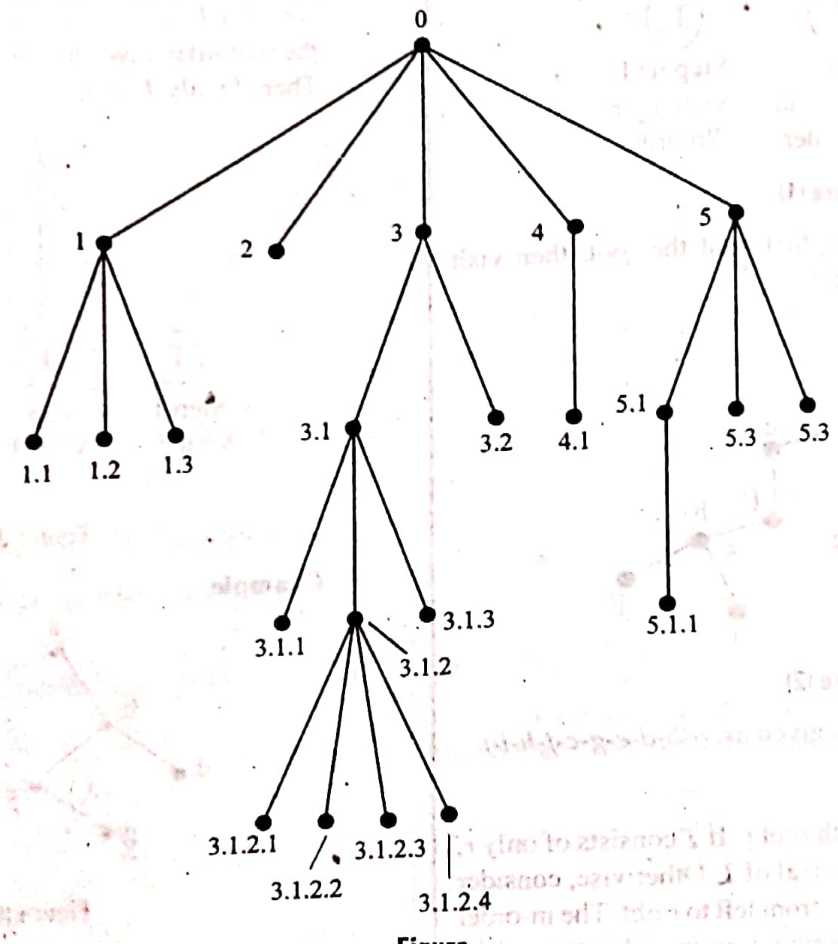
Answer :

Universal address systems

The traversal of tree is done by ordered rooted tree containing ordered children. The vertices of an ordered rooted tree is done as,

1. The root must be initially labelled with 0. The children must be labelled from left to right as 1, 2, 3, ... k.
2. For each vertex i.e. 1, 2, 3, ... k, the children are assigned as 1.1, 1.2, ... 1.k i.e. A.1, A.2, ... $A.k$.

This labeling of a tree is called as the universal address system of the ordered rooted tree.



Figure

For labelled vertex $x_1, x_2, \ldots\ldots x_n$ and $y_1, y_2, \ldots\ldots y_n$

$x_1 = y_1$

$x_2 = y_2$

$\vdots$

$x_{i-1} = y_{i-1}$; $x_i < y_i$ and $0 \leq i \leq n$.

Q93. What is meant by the tree traversal technique? Explain the different procedures used for traversing the tree.

Answer :

Tree Traversals

The process of visiting each vertex of a tree in some specific order is called Traversing the Tree.

Different procedures used for traversing the tree are,

1. Preorder traversal
2. In-order traversal and
3. Postorder traversal.

1. Preorder Traversal

If T is an ordered rooted tree with only root r, then r is the preorder traversal of T. The preorder traversal begins by visiting r. It continues by traversing T_1 in preorder, then T_2 in preorder and so on until T_n is traversed in preorder.

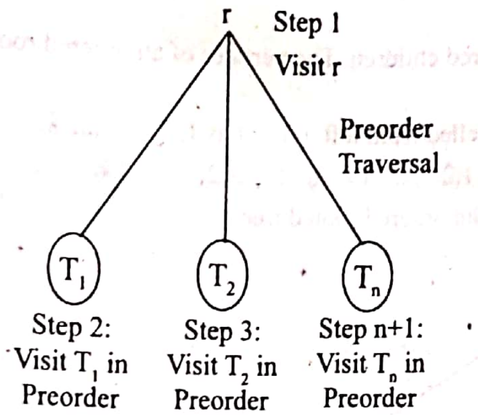


Figure (1)

In a Preorder Traversal, first visit the root, then visit subtrees from left to right.

Example

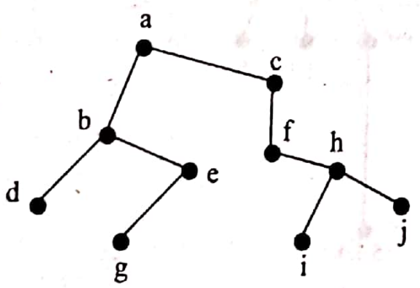


Figure (2)

The preorder traversal is given as, *a-b-d-e-g-c-f-h-i-j*.

2. In-order Traversal

Let T be a rooted tree with root r. If T consists of only r, then r is the in-order traversal of T. Otherwise, consider $T_1, T_2, \ldots, T_n$ as subtrees at r from left to right. The in-order traversal begins by traversing T_1 in in-order, then visits r. It continues by traversing T_2 in in-order and finally T_n in in-order.

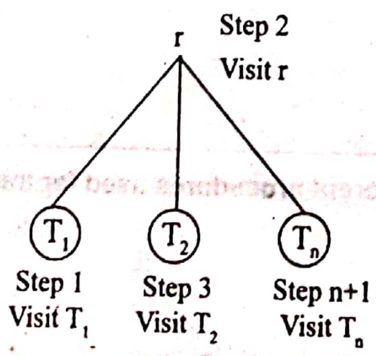


Figure (3)

Example

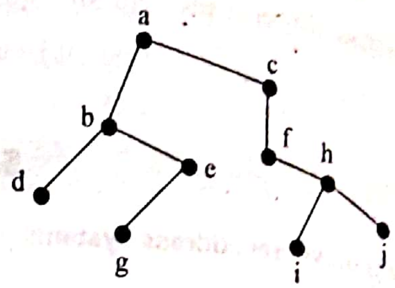


Figure (4)

The in-order traversal is given as, *d-b-g-e-a-f-i-h-j-c*

3. Postorder Traversal

Let T be an ordered rooted tree, with root r. If T consists only r, then r is the postorder traversal of T. Otherwise, consider $T_1, T_2, \ldots T_n$ as the subtrees at r from left to right, the postorder traversal begins by traversal T_1 in postorder. Then, finally T_n in postorder and ends by visiting r.

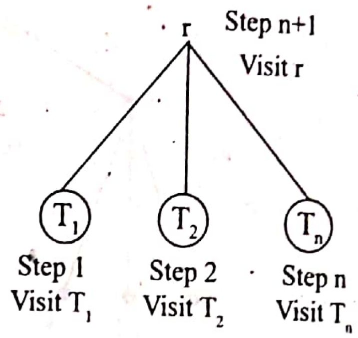


Figure (5)

Example

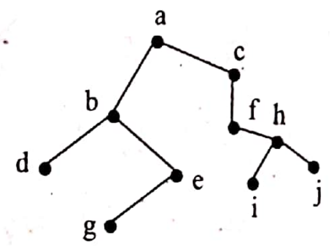


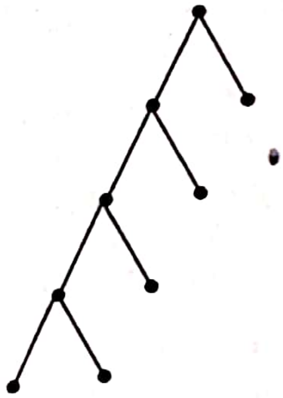
Figure (6)

The postorder traversal is given as, *d-g-e-b-i-j-k-h-f-c-a*

Q94. Construct the universal address system for the given ordered rooted tree. And use this to order its vertices using the lexicographic order of their labels

(a)

(b)

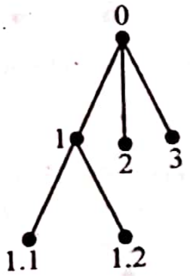


Answer :

(a) First label the root of the tree with the integer 0.

Next the children of the root are labeled as 1,2 and 3 from left to right

The two children of the vertex 1 are labeled 1.1 and 1.2 from left to right



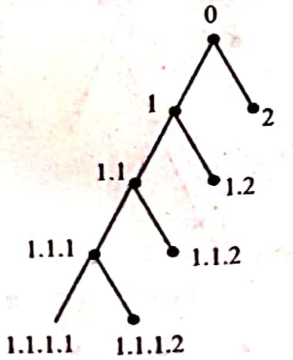
The lexicographic ordering of the labelings is,

$0 < 1 < 1.1 < 1.2 < 2 < 3$

(b) First label the root of the tree with the integer 0

The children of the root are labeled 1 and 2 from left to right

Each of the children of vertex A is labeled as $A.1$, $A.2$, $A.3$, ... from left to right.



The lexicographic ordering of the labelings is,

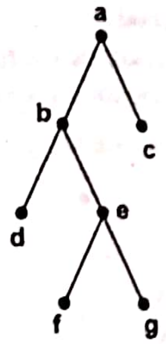
$0 < 1.1 < 1.1.1 < 1.1.1.1 < 1.1.1.2 < 1.1.2 < 1.2 < 2$

Q95. In which order are the vertices of the ordered rooted tree shown in figure below visited using

(a) Preorder traversal

(b) Inorder traversal

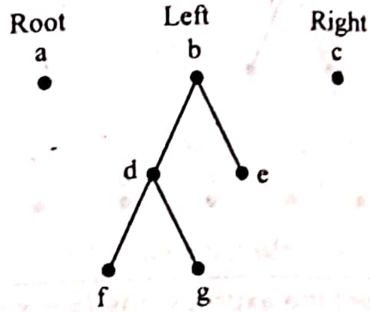
(c) Postorder traversal

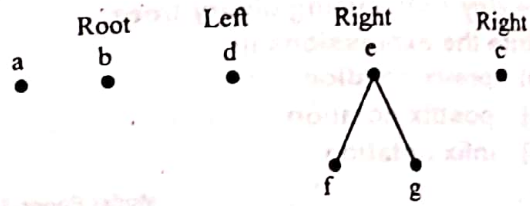


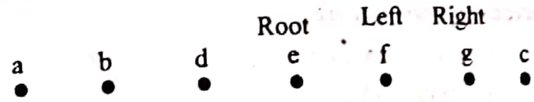
Answer :

(a) Preorder Traversal

In preorder traversal, first root of the ordered rooted tree is visited then the subtrees from left to right are visited



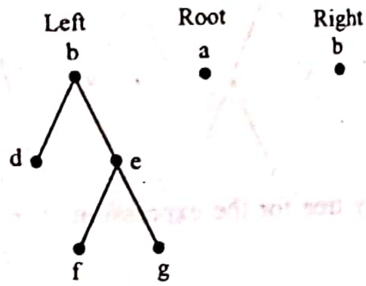


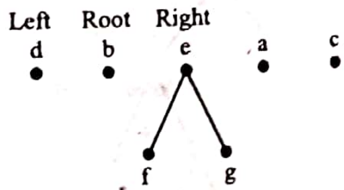


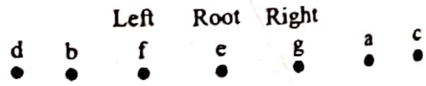
∴ The preorder traversal is a, b, d, e, f, g, c

(b) Inorder Traversal

In inorder traversal, first the left subtree is visited, then the root and finally the right subtree is visited.



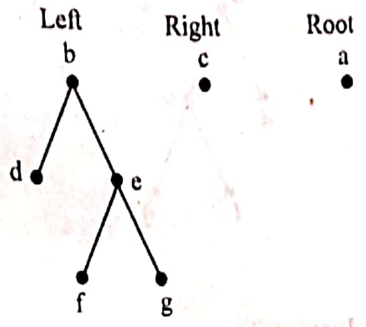


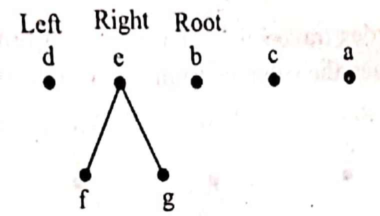


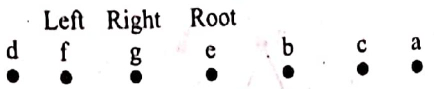
∴ The inorder traversal is d, b, f, e, g, a, c

(c) Postorder Traversal

In postorder traversal, first left subtree is visited, then right subtree and finally the root is visited







∴ The postorder traversal is *d, f, g, e, b, c, a*

Q96. Represent the expressions (x + xy) + (x/y) and x + ((xy + x)/y) using binary trees. Write the expressions in

(b) prefix notation

(c) postfix notation

(d) infix notation.

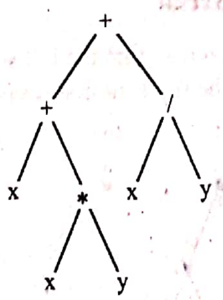
Answer : **Model Paper-3, Q11(b)**

(a) Given expressions are,

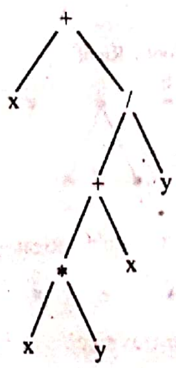
$(x + xy) + (x/y)$

$x + ((xy + x)/y)$

The binary tree for the expression $(x + xy) + (x/y)$ is shown below,

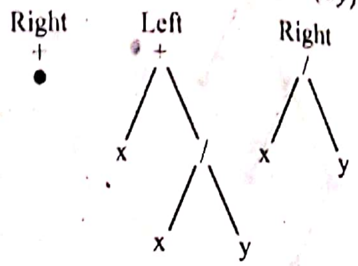


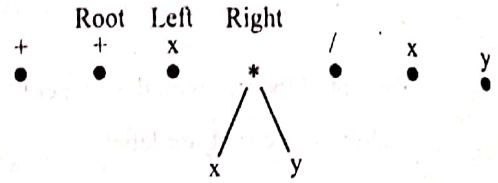
The binary tree for the expression $x + [(xy + x)/y]$ is shown below,

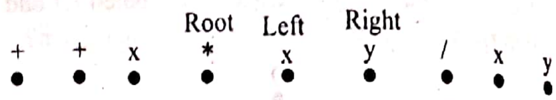


(b) The prefix of an expression is obtained by traversing the tree in preorder i.e., first root is visited, then left and right subtrees

The prefix for expression $(x + xy) + (x/y)$ is,



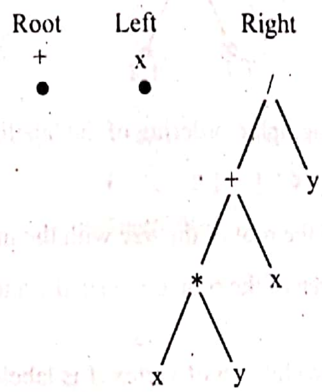


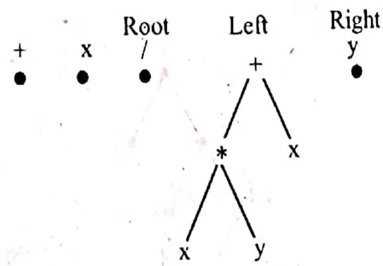


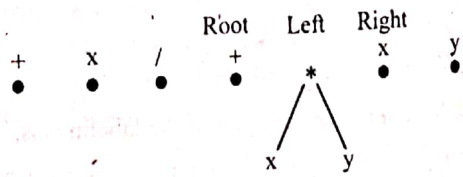
∴ The expression in prefix notation is,

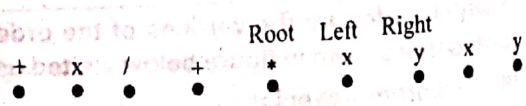
$+ + x * x y / x y$

The prefix for expression $x + [(xy + x)/y]$ is,







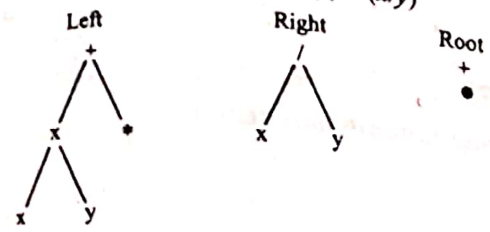


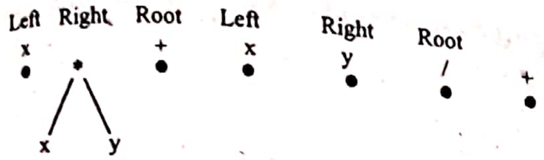
∴ The expression in prefix notation is,

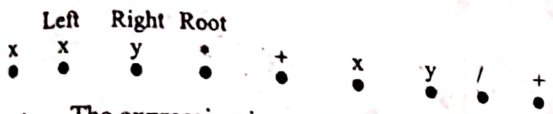
$+ x / + * x y x y$

(c) The postfix of an expression is obtained by traversing the tree in post order i.e., It begins by visiting the left subtree then right subtree and then visiting the root

Postfix for expression $(x + xy) + (x/y)$



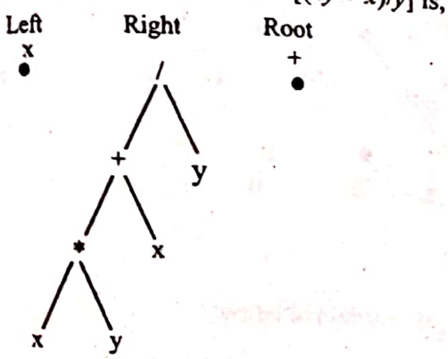


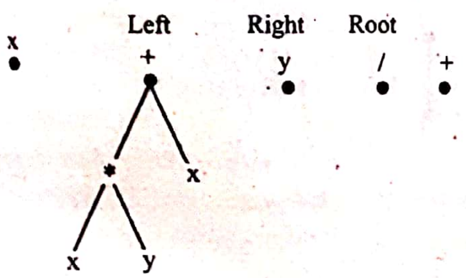


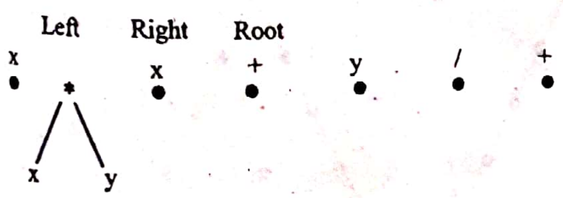
∴ The expression in postfix notation is,

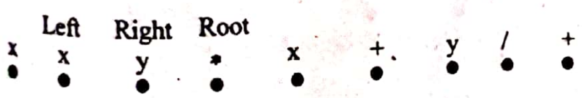
$x\,x\,y * + x\,y\,/\,+$

Postfix for expression $x + [(xy + x)/y]$ is,







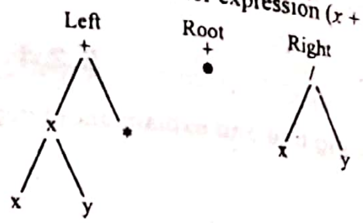


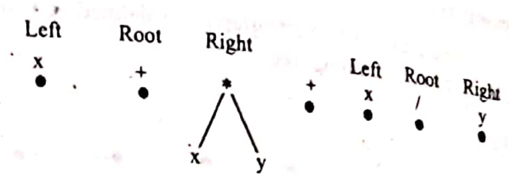
∴ The expression in postfix notation is,

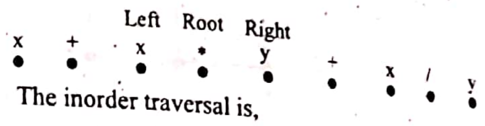
$x\,x\,y * x + y\,/\,+$

(d) The infix form of an expression is obtained by including parentheses in the inorder traversal i.e., visiting left subtree, root and then the right subtree

The infix form for expression $(x + xy) + (x/y)$ is,







The inorder traversal is,

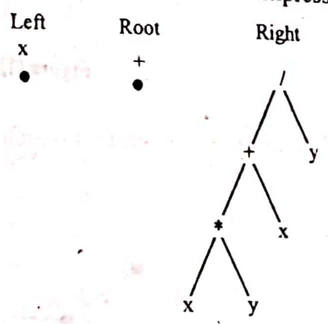
$x + x * y + x / y$

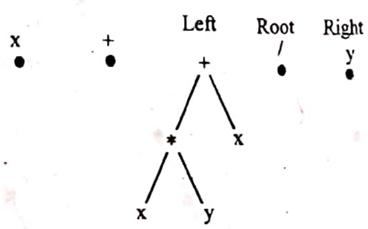
Whenever an operation is encountered, a parentheses is included in the expression of the inorder traversal

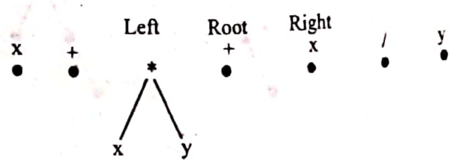
Thus, the infix notation of the expression is,

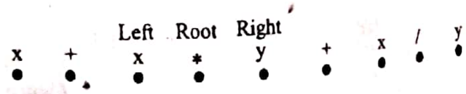
$((x + (x * y)) + (x/y))$

The infix form for the expression $((xy + x)/y)$ is,









∴ The inorder traversal is,

$$x + x * y + x / y$$

Thus, the infix notation of the expression is,

$$(x + (((x * y) + x)/y))$$

5.2.4 Spanning Trees

Q97. Define spanning tree and explain about depth-first search and breadth first search.

Answer :

Spanning tree

Spanning tree of a simple graph G is defined as a subgraph containing every vertex of G. A simple graph is connected if and only if it has a spanning tree.

Example

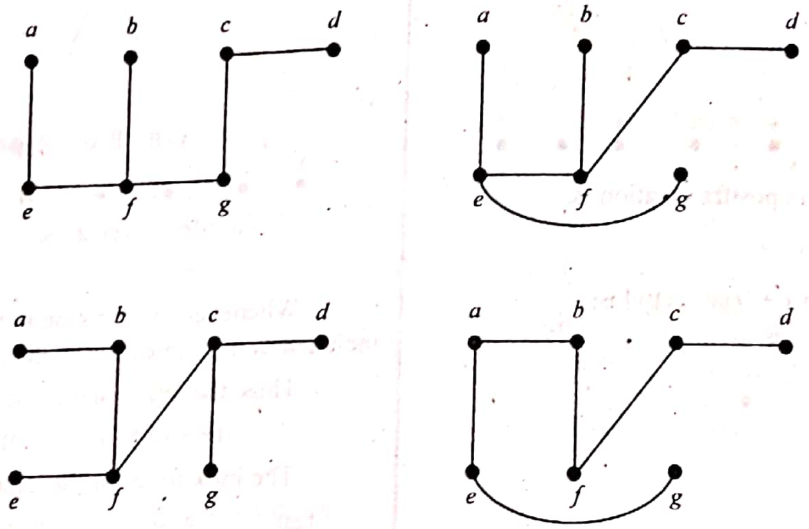


Figure (1): Spanning Trees of G

Depth-first search

The procedure to be followed for depth-first search to find a spanning tree are explained below

Example

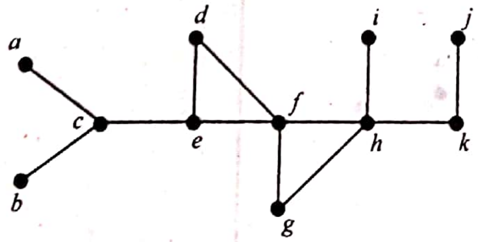


Figure (2): The Graph G

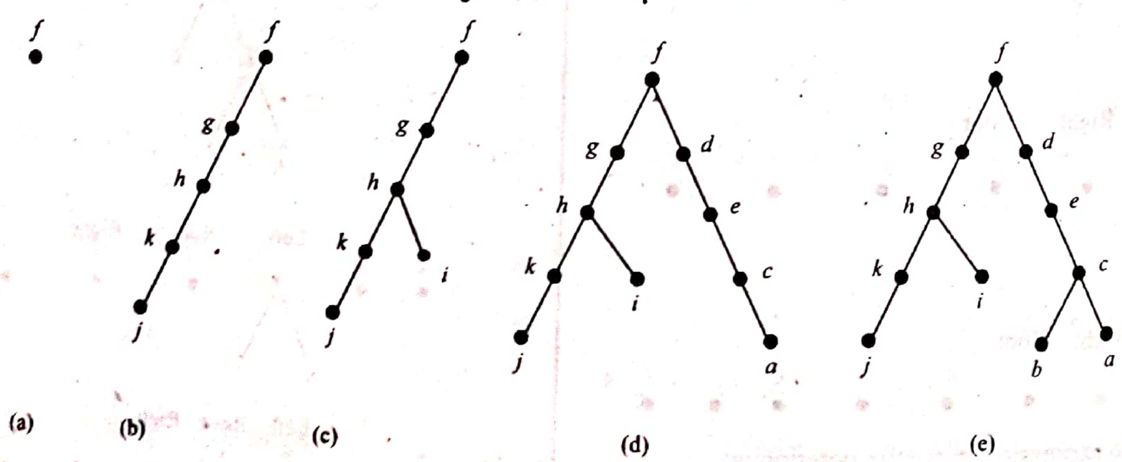


Figure (3): Depth-First Search of G

Initially, consider an arbitrary vertex f. The successive addition of edges incident with vertices and not already in the path produces a path f, g, h, k, j. Back tracking to k to find vertices that are not visited. Then backtracking to h i.e. h, i and then to f from f the path is built as f, d, e, c, a. Then backtracking to c i.e. c, b. Hence, a spanning tree is produced.

Tree edges are edges selected by depth-first search while back edges are edges that connect a vertex to an ancestor or descendent.

Breadth-first search

The procedure to find a spanning tree of a simple graph using breadth first search is explained as follows.

Example

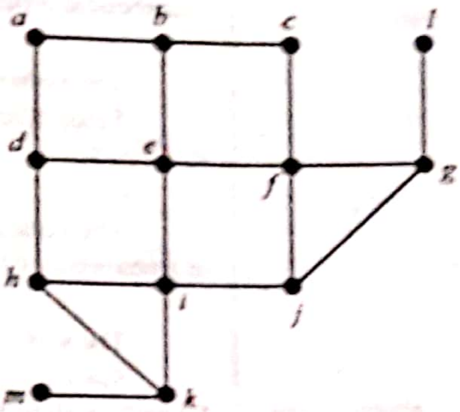


Figure (4): A Graph G

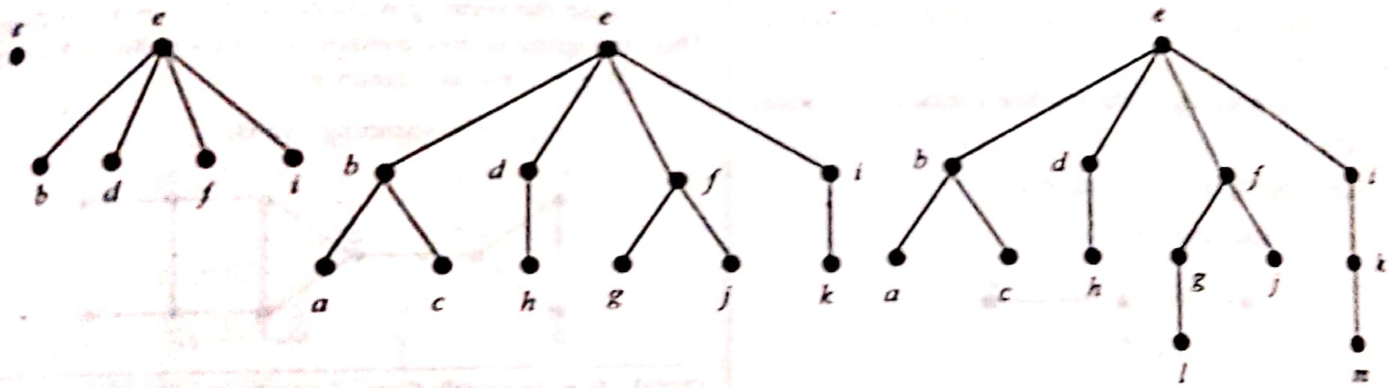


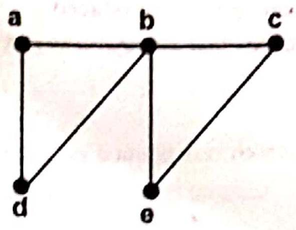
Figure (5): Breadth-first search of G

Consider a vertex e as root. The edges b, d, f and i incident with all vertices adjacent to c are added. These four vertices are at level 1 in tree. For each vertex at level 1, add the edges incident to it as long as it does not produce a simple circuit. Thus the edges b to a and c, d to h, f to j and g, and i to k are added. These new vertices are at level 2. Here, next adding the edges from these vertices to adjacent vertices (that are not already in the graph). Thus, edges g to l and k to m are added. This produces a spanning tree.

Algorithm Breadth-First Search

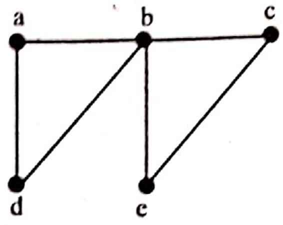
> **procedure** *BFS* (G: connected graph with vertices $v_1, v_2, \ldots\ldots v_n$)
> T := tree consisting only of vertex v_1
> L := empty list
> put v_1 in the list L of unprocessed vertices
> **while** L is not empty
> **begin**
>   remove the first vertex, v, from L
>   **for** each neighbor w of v
>   **if** w is not in L and not in T **then**
>   **begin**
>   add w to the end of the list L
>   add w and edge $\{v, w\}$ to T
>   **end**
> **end**

Q98. Find a spanning tree of the graph by removing edges in simple circuits.



Answer :

Given graph is,



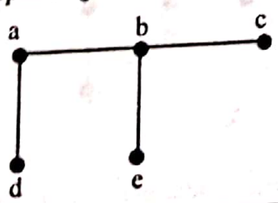
Here graph G is connected, but it contains simple circuits

The spanning tree is obtained by removing edges in simple circuits

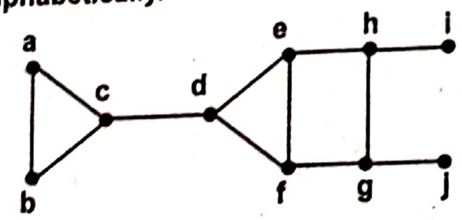
$\therefore$ The resulting spanning tree contains 5 vertices and 4 edges

Removing edges $\{b, d\}$ and $\{e, c\}$

$\therefore$ The spanning tree of G is,



Q99. Use depth-first search to produce a spanning tree for the simple graph. Choose a as the root and assume that the vertices are ordered alphabetically.



Answer : **Model Paper-4, Q11(b)**

Given that,

a is the root

Then a is the first vertex in the path

$\therefore$ Path = a

Since a is connected to both b and c, b occurs in alphabetical order, then b is added to the path.

$\therefore$ Path = a, b

Since b is connected to c, then c is added to the path

$\therefore$ Path = a, b, c

The vertex c is adjacent to d, then d is added to the path

$\therefore$ Path = a, b, c, d

Since the vertex d is connected to e and f, e occurs first in alphabetical order

$\therefore$ Path = a, b, c, d, e

Then vertex e is connected to f and h, f occurs first in alphabetical order

$\therefore$ Path = a, b, c, d, e, f

The vertex f is connected to g

Since other vertices which are adjacent to g are already in the path,

$\therefore$ Path = a, b, c, d, e, f, g

The vertex g is connected to both h and j, h occurs first in alphabetical order.

$\therefore$ Path = a, b, c, d, e, f, g, h

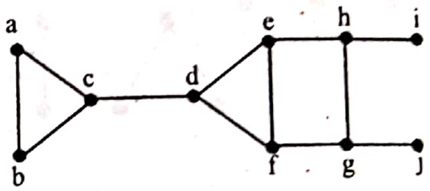
The vertex h is connected to i and g

Since g is already added in the path. Thus, i is added to the path in alphabetical order

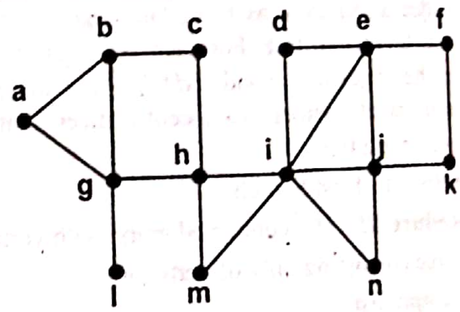
$\therefore$ Path = $a, b, c, d, e, f, g, h, i$

Also the vertex g is connected to j, j is not in the path. Thus, the spanning tree contains all edges in the path i.e., $a, b, c, d, e, f, g, h, i$ and an edge from g to j

$\therefore$ The spanning tree is,

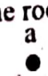


Q100. Use breadth-first search to find a spanning tree of the graph.

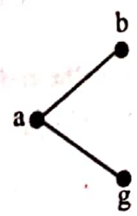


Answer :

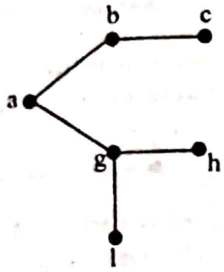
Let the vertex a be the root



The edges incident to vertex a are b and g. Thus, b and g are added

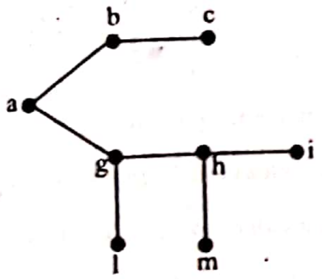


The edges incident to b is c. Thus, c is added. Also, the edges incident to g are h and l. Thus, the vertices h and l are added.



The edge incident to c is h. Since, the edge forms a simple circuit, edge $\{c, h\}$ is ignored

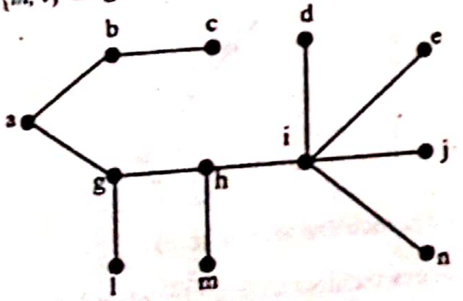
The edges incident to h are c, i, and m. Here, the edge $\{c, h\}$ is ignored as simple circuit is formed. Thus, the edges $\{h, i\}$ and $\{h, m\}$ are added.



The edges incident to i are d, m, n, j and e

Here the edge $\{i, m\}$ forms a simple circuit, it is ignored

The edge incident to m is i, it forms a simple circuit, thus edge $\{m, i\}$ is ignored.



The edges incident to d are i and e

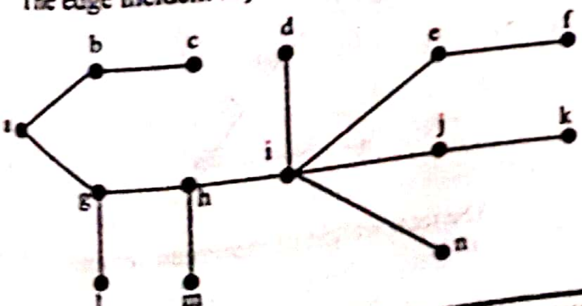
Here, an edge $\{d, e\}$ forms a simple circuit is ignored

The edges incident to e are i, j, f

Here, an edge $\{e, j\}$ forms a simple circuit, thus it is ignored

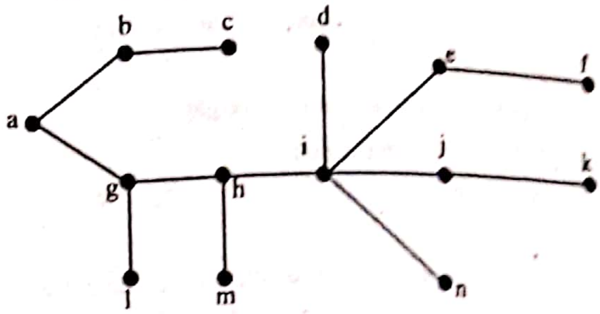
The edge incident to n is j, it forms a simple circuit, thus ignored

The edge incident to j is k, is added



The edge incident to f is k, forms a simple circuit, is ignored

∴ The spanning tree is,



5.2.5 Minimum Spanning Trees

Q101. Discuss in detail about minimum spanning trees.

Answer :

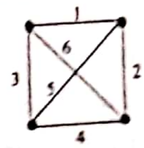
Minimum Spanning Tree

A minimum spanning tree is a spanning tree, such that the sum of the weights of its edges is at a minimum.

Prim's and Kruskal's algorithms are used to find minimum spanning tree.

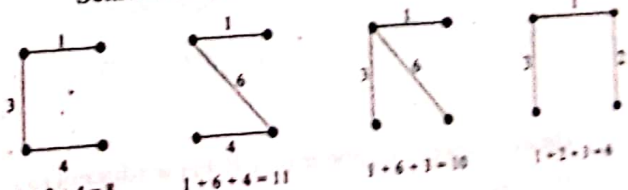
Example

Consider a graph 'G' which is as shown below,



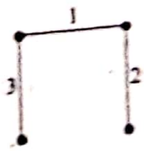
Figure(1): Graph 'G'

Some of the possible spanning trees from graph 'G' are,



Figure(2): Four Spanning Trees From Graph 'G'

From the above given spanning trees, the minimum spanning tree of graph G is shown below,



Figure(3): Minimum Spanning Tree of Graph 'G'

Kruskal's Algorithm

Kruskal's algorithm selects an edge that have minimum weight in the graph. Then add the edges individually that has minimum weight in such a way that it does not form a simple circuit with the selected edges. This process continues till the $(n - 1)$ edges have been selected.

Algorithm

Algorithm Kruskal(G)

//Input : A weighted connected graph $G = (V, E)$

//Output : T, the minimum spanning tree of G

{

Sort E in increasing order by weight

$T := \{ \}$ //empty graph

for $i = 1$ to $n - 1$

{

$e :=$ any edge in G with smallest weight that does not form a simple circuit when added to T.

$T = T \cup \{e\}$ //add e to T

}

return T.

}

Prim's Algorithm

Prim's algorithm generates minimal spanning tree by selecting smallest weight edges. Initially it selects an edge with smallest weight and puts it into the spanning tree. Then successive edges adjacent to the initial edge with minimum weight are added to the tree. However, these edges must be selected such that they do not form a circuit. The algorithm for this method is as follows.

Algorithm Prim(G)

//Input: A weighted connected graph $G = (V, E)$

//Output: T, the minimum spanning tree of G

{

$V = \{v_1\}$

$T = \{ \}$ //empty graph

for $i = 1$ to $n - 2$

{

choose a nearest neighbour v_i of V that is adjacent to v_j, $v_j \in V$ and for which the edge $e_i = (v_i, v_j)$ does not form a cycle with members of T.

$V := V \cup \{v\}$ //add v to V

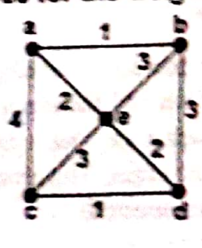
$T := T \cup \{e_i\}$ //and add (v_i, v_j) to T

}

return T.

}

Q102. Use Prim's algorithm to find a minimum spanning tree for the weighted graph.



Answer : Model Paper-3, Q11(b)

The graph contains 5 vertices

Let the vertex a be the root

The edges incident to a are (a, b), (a, c) and (a, e). The edge with the smallest weight is added i.e., (a, b) is added

The edges incident to a are (a, e) and (a, c). And the edges incident to b are (b, e) and (b, d)

Here, the edge (a, e) has smallest weight and does not form a circuit

The edge incident to a is (a, c)

The edges incident to b are (b, e) and (b, d)

The edges incident to e are (e, b), (e, c) and (e, d)

The smallest weighted edge is (e, d) and it does not form a circuit

The edge incident to a is (a, c)

The edges incident to b are (b, e) and (b, d)

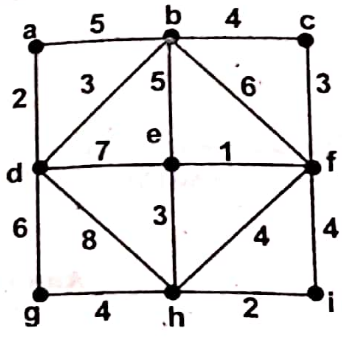
The edges incident to e are (e, b) and (e, c)

The edges incident to d are (d, b) and (d, c)

The smallest weighted edge is (d, c) and it does not form a circuit

$\therefore$ The total weight of minimum spanning tree is $1 + 2 + 2 + 1 = 6$

Q103. Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph.



Answer :

Given graph has 9 vertices and 16 vertices.

The minimum spanning tree should contain 9 vertices and 8 edges

Arranging edges in non-decreasing order of their weights is shown below

Edges	Weight
(e, f)	1
(a, d)	2
(h, i)	2
(c, f)	3
(b, d)	3
(e, h)	3
(b, c)	4
(h, f)	4
(f, i)	4
(g, h)	4
(a, b)	5
(b, e)	5
(d, g)	6
(b, f)	6
(d, e)	7
(d, h)	8

The smallest weighted edge is (*e*, *f*), then add it to the graph

The smallest weight in the graph is (*a*, *d*) and (*h*, *i*), these edges does not form a circuit, add it to graph

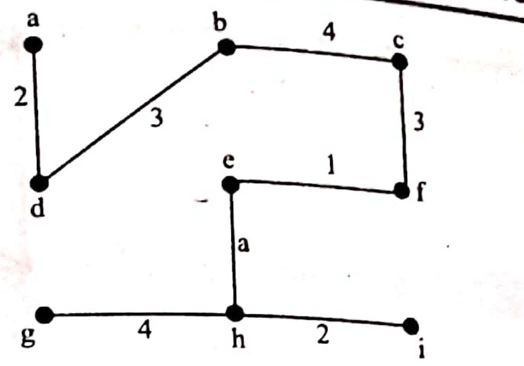
The smallest weighted edges are (*c*, *f*), (*b*, *d*) and (*e*, *h*), does not form a circuit, add these edges to graph

The next smallest edges are (*b*, *c*), (*h*, *f*), (*f*, *i*) and (*g*, *h*)

Since the edge (*f*, *i*) and (*h*, *f*) form a circuit, the other edges (*b*, *c*) and (*g*, *h*) are added to the graph

The remaining edges form a circuit, thus they are ignored.

∴ The minimum spanning tree is,



∴ Weight = 2 + 3 + 4 + 3 + 1 + 3 + 4 + 2 = 22.

EXERCISE QUESTIONS

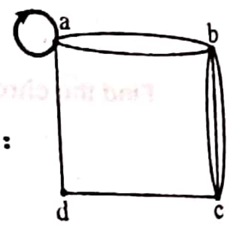
1. **Determine the degree sequence of C_4 graph.**

   Ans : 2, 2, 2, 2

2. **Draw an undirected graph represented by the given adjacency matrix.**

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Ans :



3. **How many nonisomorphic simple graphs are there with five vertices and three edges?**

   Ans : 4

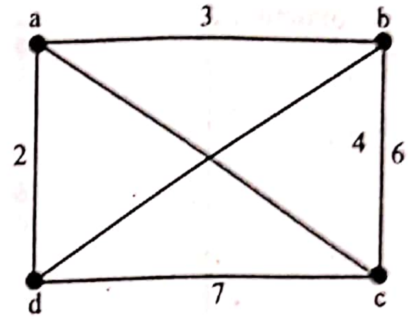
4. **Find the number of paths of length n between two different vertices in K_4 if n is, 5.**

   Ans : 61

5. **Find the Eular path but no Euler circuit for k_n.**

   Ans : n = 2

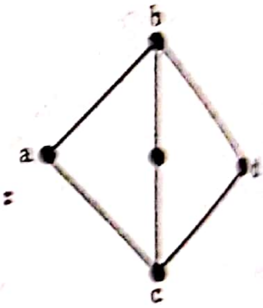
6. **Solve the travelling salesman problem for this graph by finding the total weight of all hamilton circuits and determining a circuit with minimum total weight.**



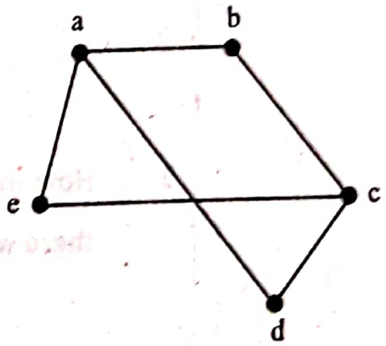
Ans : a – c – b – d –

7. **Draw the given planar graph without any crossings.**

Ans :



8. **Find the chromatic number of the given graph.**



Ans :