Unit-4

Chapter 9

Propositional Logic

9.1 Statement

[U.P.T.U. (B.Tech.) 2004, 2006]

A ***Statement*** is a declarative sentence which is true or false, but not both; or in other words a ***Statement*** is a declarative sentence which has a definite truth value.

Illustration: *Following are statements (or proposition)*:

S. No.	Declarative Sentence	Truth Value
(i)	$\{x : x^2 = 36\} = \{6, -6\}$	True
(ii)	$3 > 9$	False
(iii)	The capital of UP is Lucknow	True
(iv)	Blood is red	True
(v)	$5+4=10$	False

Following are not Statements:

(i)	How are you	(Interrogative)
(ii)	Please go from here	(Request)
(iii)	May God help you	(Wish)

9.1.1 Statement Letters or Sentence Variable

We know that symbols have great importance in Mathematics and therefore symbols will be used to represent statements. The symbols which are used to represent statements, are ***Called Statement Letters*** or ***Sentence Variables.*** To represent statement usually the letters $P, Q, R, \ldots, p, q, r, \ldots$ etc. are used. For example, if the statement 'Lucknow is the capital of U.P.' is denoted by the letter 'p', then in Mathematics it is written as follows:

p = Lucknow is the capital of U.P.

9.2 Logical Connectives or Sentence Connectives

Logical connectives or sentence connectives are the words or symbols used to combine two sentences or statements to form a compound sentence or compound statement.

Table of Logical Connectives with their Symbols

Connective Word	Name of Connective Symbol	Symbol	Rank
Not	Denial or Negation	$\sim$ or $\neg$	1
and	Conjunction	$\wedge$	2
Or	Disjunction	$\vee$	3
if...then	Conditional	$\rightarrow$ or $\Rightarrow$	4
iff or if and only if	Bi-conditional	$\leftrightarrow$ or $\Leftrightarrow$	5

Illustrations

(1) π is greater than 3 and π is less than 3.2. In symbolic language: let $p \equiv \pi$ is greater than 3, $q \equiv \pi$ is less than 3.2, so $p \wedge q$.

(2) a is equal to 4 or b is equal 4. In symbolic language : let $p \equiv a$ is equal to 4; $q \equiv b$ is equal to 4, so $p \vee q$.

(3) $(4<7) \vee (1=0)$

(4) If two lines are parallel then they do not intersect. In symbolic language : let $p \equiv$ two lines are parallel, $q \equiv$ two lines do not intersect; so $p \Rightarrow q$.

(5) A is a right angled triangle if the sum of squares of its two sides is equal to the square of the third side. In symbols : let $p \equiv A$ is a right-angled triangle, $q \equiv$ the square of two sides of a triangle is equal to the square of the third side. So $p \Leftrightarrow q$.

(6) If $\pi > 3$ then $\sim \pi < 3$

9.2.1 Use of Brackets

The use of brackets in statements (or propositions) is very important. The meaning (or explanation) of statements is included in brackets.

Illustration: The statements $(p \wedge q) \Rightarrow r$ and $p \wedge (q \Rightarrow r)$ have completely different meaning. Therefore, following rules are followed for brackets in statements:

Rule 1: If connective 'Not (*i.e.*, $\sim$)' is repeated then bracket is not required. For example: $\sim \sim p$ has the meaning as $(\sim(\sim p)$

Rule 2: If in a statement, the connectives of the same rank appear, then brackets are applied from the left.

For example:

$$p \wedge q \wedge r \wedge s \wedge t = \{(p \wedge q) \wedge s\} \wedge t$$

Rule 3: If the connectives of different ranks appear, then first of all the bracket of that connective is removed which is of lower rank.

Illustration:

$$p \vee (q \Rightarrow r) = p \vee (p \Rightarrow r)$$

Here, the rank (=4) of '$\Rightarrow$' is greater than the rank (=3) of '$\vee$'. Therefore, this bracket cannot be removed.

9.3 Kinds of Sentences

Usually the sentences are of two kinds:

(i) Simple sentence or Atomic sentence. (ii) Compound sentence or Molecular sentence.

9.3.1 Simple Sentence

A simple sentence or atomic sentence has no connective.

9.3.2 Compound Sentence

A compound sentence or molecular sentence is composed of various connectives.

A compound sentence is named on the basis of the leading connective used in it.

Illustration:

Connective	Compound Sentence
$\wedge$	Conjunctive sentence
$\vee$	Disjunctive sentence
$\rightarrow$ or $\Rightarrow$	Conditional sentence
$\leftrightarrow$ or $\Leftrightarrow$	Bi-conditional sentence
$\sim$ or $\neg$	Negative or denial sentence

(a) In conditional sentence $p \Rightarrow q$, p is called 'antecedent' or 'hypothesis' or 'premise' and q is called 'consequent or conclusion'. In language, it is read as:

(i) p means q (ii) if p then q (iii) p then q

(iv) when p then q (v) q if p (vi) q when p

(vii) p only if q (viii) necessary condition for p is q

(ix) sufficient condition for q is p.

(b) Bi conditional sentence $p \Leftrightarrow q$ is read as:

(i) If p then q and if q then p.

(ii) The necessary and sufficient condition for p is q.

(iii) p if and only if q.

(c) The sentences used to form a compound sentences are called its **components**. For example: The components of $p \wedge q$ are p and q.

Remark: The component may either be simple sentences or compound sentences.

9.4 Truth Value of A Statement

According to the definition of statement, a statement has a definite truth value which is either True or False. Consider the following examples:

(i) 15 is an odd number. (ii) 9 is a prime number.

Clearly (i) is true, therefore (i) has a definite truth value '***True***' and it is denoted by the first letter 'T' of True. Similarly, (ii) is a statement whose truth value is '**False**' and it is denoted by the first letter 'F' of False.

9.4.1 Statement Pattern or Statement Form

Statement pattern or statement form is an expression which is obtained by combining statement letters by the use of a finite number of logical connectives. For example:

If p and q be statement letters, then following are examples of statement pattern:

(i) $p \wedge q$ (ii) $p \vee q$ (iii) $p \Rightarrow q$ (iv) $p \vee q \Rightarrow \sim p$ (v) $p \vee q \Leftrightarrow q \vee p$

The statement pattern may be simple or composite, according as it is composed of one or more logical connectives. Thus statement patterns given by (i), (ii) and (iii) are simple while those given by (iv) and (v) are composite.

9.4.2 Truth Value Function

We know that every statement has definite and unique truth value which is either True or False. Hence every statement pattern defines a truth value function if its components whose range set is $\{T, F\}$.

9.4.3 Principal Connective

We know that a compound sentence is formed by the use of one or more than one logical conn ectives. The logical connective, which is used in the end in the composition of a compound sentence , is called a ***Principal Connective.***

Illustrations:

(*i*) The principal connective in $(p \wedge p) \Rightarrow r$ is '$\Rightarrow$'.

(*ii*) The principal connective in $(r \Rightarrow p) \vee (p \Leftrightarrow q)$ is '$\vee$'.

9.4.4 Open Statement

A sentence, which contains one or more variables such that when certain value are substituted for variables, becomes a statement, is called an ***Open Statement***.

Illustration: Consider $x+3=9$

If we put 6 in place of x, then this sentence becomes a true statement. Such a sentence is called an open statement.

9.4.5 Proposition

If $p, q, r \ldots$ are ***Simple Statement*** then the compound statement $P(p, q, r, s, \ldots)$ is called a proposition. The truth value of proposition P depends on the truth values of variables $p, q, r, \ldots$. If the truth value of the proposition P. A truth table is a simple way to show this relationship.

Example 1: *Find the truth table for the proposition* $(\sim p \wedge q)$.

Solution: The truth table of the proposition $(\sim p \wedge q)$ is

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

9.4.6 Well Formed Formulas

[U.P.T.U. (B.Tech.) 2005, 2006]

The statement formulas contain one or more simple statements and some connectives. If p and q be any two statements, then $p \wedge q$, $(\sim p) \wedge q$, $(p \wedge q) \vee (\sim p)$ are some statement formula derived from the statement variables p and q, where p and q called components of the statement formula.

Consider $p \wedge q$ and $q \wedge p$, where p and q are any two propositions. The truth tables of these two propositions are identical. This happen when we have any proposition in place of P and any proposition in place of Q. So we can develop the concept of a propositional variable and well formed formulas (*wff*).

Well Formed Formulas

A statement formula is a string consisting of variables, parentheses and connective symbols. A statement formula is called **well formed formulas (*wff*)** if

(i) A statement variable p standing alone is a well formed formula.

(ii) If p is a well-formed formula, then $(\sim p)$ is a well formed formula.

(iii) If p and q are well formed formula then $(p \wedge q)$, $(p \vee q)$, $(p \Rightarrow q)$ and $(p \Leftrightarrow q)$ are well formed formulas.

(iv) A string of symbols is a well formed formula if and only if it is obtained by finitely many applications of the rule (i), (ii) and (iii).

Note 1: A well formed formulas is not a proposition, but if we replace the proposition in pla... propositional variable, we get a proposition. *For example*

(i) $(\sim(p \vee q) \wedge ((\sim q) \wedge r) \Rightarrow q)$ is a wff

(ii) $(\sim p \wedge q) \Leftrightarrow q$ is a well formed formulas

Note 2: We can drop parentheses when there is no ambiguity. For the sake of convenience we used we... formed formula (*wff*)

Note 3: The final column entries of the truth table of a well formed formula gives the truth values of the formula.

Example 2: *Which of the following sentences are statements? Also state their truth value.*

(i) *Is 3 a prime number?*

(ii) $x^2 - 5x + 6 = 0$

(iii) *There will be snow in December*

(iv) *Give me ten rupees*

(v) *Ramesh is poor but honest*

Solution:

(i) It is not the statement [Interrogative]

(ii) It is not statement. Its truth value depends upon the value of x.

(iii) It is statement. It can be judged to be true or false.

(iv) It is not statement [wish]

(v) It is the statement. It can be judged to be true or false. It is true if Ramesh is poor and honest and fa... when he is poor but not honest.

Example 3: *Which of the following sentences are proposition? What are the truth values of those that... propositions?*

(i) *Kolkata is the capital of India*

(ii) *Answer this question*

(iii) *What time is it?*

(iv) $x+y = y+x$ *for every pair of real number* x *and* y

(v) $5 + 6 = 10$

(vi) $x + 3 = 3$

(vii) $x + 2 = 5$ if $x = 1$

(viii) *Do not go*

Solution:

(i)	Yes, F	(ii)	No	(iii)	No	(iv)	Yes, T
(v)	Yes, F	(vi)	Yes, F	(vii)	No	(viii)	No

Example 4: *Find out which of the following are statements and which are not.*

(i) *Where are you going?*

(ii) *Gitanjali is sick or old*

(iii) *It is raining and the sun is shining*

(iv) *Intersection of two non-empty sets is always a non-empty set*

(v) *Two individuals are always related*

(vi) *The real number x is less than 3*

(vii) $5x + 8y = 17$

(viii) *Do not pluck the flowers*

Solution:

(i) It is not a statement since it is an interrogative sentence. It can not be judged to be true or false.

(ii) It is a statement. It can be judged to be true or false. It is true if Gitanjali is sick or if Gitanjali is old or if Gitanjali is sick as well as she is old. It is false if Gitanjali is neither sick nor she is old.

(iii) It is a statement. It can be judged to be true or false. It is true if it is raining and also the sun is shining. It is false if it is not raining or if the sun is not shining or if neither it is raining nor the sun is shining.

(iv) It is a statement. It can be judged to be true or false. It is false statement because the intersection of two non-empty sets is always not recessarily a non-empty set.

Illustration: $A = \{1, 2, 7\}$ and $B = \{3, 5, 8, 11\}$

are two non-empty sets but $A \cap B = \phi$ i.e. $A \cap B$ is an empty set.

(v) It is a statement. It can be judged to be true or false. It is false statement because it is not necessary that two individuals are always related.

(vi) It is an open sentence. Its truth value depends upon x. Hence, it is not a statement.

(vii) It is an open sentence. Its truth value depends upon x and y. Hence, it is not a statement.

(viii) It is not a statement since it is an imperative sentence. It cannot be judged to be true or false.

Example 5: *Which of the following are statements? If a statement, then find its truth value.*

(i) $2 + 3 = 6$ (ii) *Listen to me, Vishu !*

(iii) *6 has three prime factors* (iv) *13 is a prime number*

Solution:

(i) '$2 + 3 = 6$' is a statement. Since $2 + 3 = 5$, so it is a false statement and hence its truth value is 'F'.

(ii) It is not statement, since it is an imperative sentence.

(iii) It is a statement. Since 6 has only two prime factors namely 2 and 3, therefore, it is a false statement. So its truth value is 'F'.

(iv) It is a statement. Since 13 is prime number so it is a true statement. Hence its truth value is 'T'.

Example 6: *State the truth values of the following statements.*

(i) *There is always a real root for any quadratic equation.*

(ii) *There is only one triangle (apart from the triangles congruent to it) with prescribed lengths for sides a, b, c with $a < b + c$, $b < c + a$ and $c < a + b$*

(iii) *Every set is a fixed set.*

(iv) *The sum of two and three is five.*

(v) *The number of ways of seating 2 persons out of n persons in two chairs is $^nP_2$.*

Solution:

(i) Since a quadratic equation may have a pair of complex roots also so the given statement is false. Hence, its truth value is 'F'.

(ii) Since all triangles with prescribed lengths for sides a, b, c with $a < b + c$, $b < a + c$ and $c < a + b$, $p = 4d$ is true. Hence, its truth value is 'T'.

(iii) Since a set may be finite or infinite so the given statement is false. Its truth value is 'F'.

(iv) The given statement is true. So, its truth value is 'T'.

(v) The given statement is true. So its truth value is 'T'.

Example 7: *Write down the truth value (T or F) of each of the following statements.*

(*i*) There are only finite number of rational numbers.

(*ii*) The quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, has always two real roots.

(*iii*) A triangle one of whose vertices lies on a circle and whose side opposite to this vertex is a diameter of the circle is a right angle triangle.

(*iv*) The Taj Mahal is in Agra.

(*v*) $\{x : x^2 = 16\} = \{4, -4\}$

(*vi*) A cow has four legs.

Solution:

(i) It is a false statement because the rational numbers are infinite. Hence, its truth value is 'F'.

(ii) It is false statement because the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, may also have roots which are not real. Hence, the truth value of this statements is 'F'.

(iii) The given statement is true because in a circle a diameter always subtends a right angle at any point on the circumference opposite to it. Hence, its truth value is 'T'.

(iv) It is a true statement. Hence, its truth value is 'T'.

(v) It is a true statement. Hence, its truth value is 'T'.

(vi) It is a true statement. Hence its truth value is 'T'.

Example 8: *Which of the following statements are true and which are false?*

(*i*) $16 + 9 = 4^2 + 3^2$ (*ii*) $9 \times 7 = 21 \times 3$ (*iii*) $\{2, 3\} \subset \{2, 4, 6\}$ (*iv*) $5 \in \{1, 3, 5\}$

Solution:

(i) True, since $25 = 25$

(ii) True, since $63 = 63$

(iii) False, since $3 \notin \{2, 4, 6\}$

(iv) True, since 5 is an element of the set $\{1, 3, 5\}$

Example 9: *What type of sentence is $5 + x = 9$? For what value of x it will become a true statement?*

Solution: The given sentence $5+x=9$ is an open statement. It becomes a true statement for $x = 4$.

Example 10: *Are the following propositions?*

(*i*) *Some roses are black.*

(*ii*) *All roses are white.*

(*iii*) *May you live long.*

(*iv*) *The girls are beautiful.*

(*v*) *Go to home.*

(*vi*) *What is your good name?*

(*vii*) *Bravo ! you have done a good work.*

(*viii*) *Children, come to me.*

(*ix*) *Is 7 a prime number?*

(*x*) *10 is a prime number.*

(*xi*) *Go to college.*

Solution:

(i) Yes, since it is a declarative sentence, whose truth value is True.

(ii) Yes, since it is a declarative sentence, whose truth value is False.

(iii) No, since it is not a declarative sentence.

(iv) (v), (vi), (vii), (viii), (ix) and (xi) are not proposition since these all are not declarative sentences.

(x) Yes, since it is a declarative sentence, whose truth value is False.

Example 11: *If p = he is poor, q = he is laborious, then write down the following statements in symbols (i.e., in the language of logic):*

(i) *He is poor and laborious.*

(ii) *He is poor but is not laborious.*

(iii) *It is false that he is poor or laborious.*

(iv) *It is false that he is not poor or is laborious.*

(v) *Neither he is poor nor he is laborious.*

(vi) *He is poor or is not poor and is laborious.*

(vii) *It is not true that he is not poor or is not laborious.*

Solution: In terms of statement letters p, q and logical connectives, the solutions are as follows:

(i) $p \wedge q$

(ii) $p \wedge \sim q$

(iii) $\sim(p \vee q)$

(iv) $\sim(\sim p \vee q)$

(v) $\sim p \wedge \sim q$

(vi) $p \vee (\sim p \wedge q)$

(vii) $\sim(\sim p \vee \sim q)$

Example 12: *Let p = "Ravi is rich" and q = "Ram is happy"*

Write the following statements in symbolic form:

(i) *Ram is poor but happy.*

(ii) *Ravi is neither rich nor happy.*

Solution:

(i) $\sim p \wedge q$

(ii) $(\sim p) \wedge (\sim q)$

Example 13: *Write the following sentences in symbols:*

(i) *When Sheela will come then I shall go to college.*

(ii) *Until Sheela will not come I shall not go to college.*

Solution: Let $p \equiv$ Sheela will come, $q \equiv$ I shall go to college.

$\therefore$ (i) $p \Rightarrow q$

(ii) $\sim p \Rightarrow \sim q$

Example 14: *Write the following in symbols:*

(i) *What so ever, I say, he refuses from that.*

(ii) *I shall reach the station in time, otherwise I shall miss the train.*

(iii) *I shall go to Delhi, but I shall not see zoo.*

(iv) *In the position when he is ill, I shall care.*

(v) *Whenever I take leave, I fall ill.*

(vi) *Until I shall not be called till then, I shall remain here.*

(vii) *Ram will go to work or will remain in the house and he will white-wash the house.*

(viii) *Not only men, but also women and children were killed.*

9.5 Truth Tables

The meaning and use of different logical connectives can be very well understood by the analysis of truth values. In symbols, true or false statements are expressed with the help of tables. These tables are called ***Truth Tables***.

9.6 Basic Logical Operations

9.6.1 Conjunction ($\wedge$)

Suppose p and q are two statements. When two statements p, q are combined by using the word 'and', then we get a new statement represented by $p \wedge q$. The resulting statement $p \wedge q$ is read as 'p and q' or 'p meet q'. The statement $p \wedge q$ so obtained is called the ***Conjunction*** of p and q. The conjunction of statements p and q *i.e.*, $p \wedge q$ is true only in the condition when p and q both are true, *i.e.*,

(i) If p is true and q is true, then $p \wedge q$ is true.

(ii) If p is true and q is false, then $p \wedge q$ is false.

(iii) If p is false and q is true, then $p \wedge q$ is false.

(iv) If p is false and q is false, then $p \wedge q$ is false.

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note: The statements, used to form the resulting statement, are called *Components*. For example, the components of resulting statement $p \wedge q$ are p and q.

9.6.2 Disjunction ($\vee$)

Suppose p and q are two statements. When two statements p, q are joined by using the word 'or', then we get a new statement represented by $p \vee q$. The resulting statement $p \vee q$ is read as 'p or q' 'p join q'. The statement $p \vee q$ so obtained is called the ***Disjunction*** of p and q. The disjunction of statements p and q *i.e.*, $p \vee q$ is false only in the condition when p and q both are false, *i.e.*,

(i) If p is true and q is true, then $p \vee q$ is true.

(ii) If p is true and q is false, then $p \vee q$ is true.

(iii) If p is false and q is true, then $p \vee q$ is true.

(iv) If p is false and q is false, then $p \vee q$ is false.

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

9.6.3 Negation ($\sim$) or ($\neg$) or Not

Let p be a statement. The negation (or denial) of the statement p is obtained by putting the word 'not' and is represented by '$\neg p$', $\sim p$ is read as 'not p'. Hence,

(i) If p is true, then $\sim p$ is false. (ii) If p is false, then $\sim p$ is true.

Therefore the truth value of $\sim p$ is also true or false.

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

Thus, the negation of a statement is a new statement which is the denial of the previous statement. But at the time of taking the negation ($\sim$) of a statement, one should be very cautious, which will be clear from the following example:

Suppose p = Ramesh is a good player of Hockey, then $\sim p$ never means 'Ramesh is a bad player of Hockey'. $\sim p$ has the following meaning:

$\sim p$ = Ramesh is not a good player of Hockey.

Note: that a negation is called a connective although it only modifies a statement. In this sense, negation is the only operator that acts on a single proposition.

Example 19: *Find the negation of the propositions*

(i) *It is cold* (ii) *Today is Sunday* (iii) *Tim Tim is poor*

Solution: The negation of the proposition are

(i) It is not cold (ii) Today is not Sunday (iii) Tim Tim is not poor

Example 20: *Write the negation of each of the following statements*

(i) $2 + 7 \leq 13$

(ii) *3 is an odd integer and 8 is an even integer*

(iii) *No nice people are dangerous*

(iv) *The weather is bad and I will not go to work*

(v) *I grow fat only if I eat too much.*

Solution:

(i) 2 + 7 is not less than or equal to 13

(ii) 3 is not an odd integer and 8 is not an even integer

(iii) Nice people are not dangerous

(iv) The weather is not bad and I will go to work

(v) I don't grow fat if I not eat too much.

9.6.4 Conditional Statement

[U.P.T.U. (B.Tech.) 2007]

Let p and q be any two statements. The statement of the type 'if p then q' is called ***Conditional Statement*** and is represented by '$p \Rightarrow q$'. It is read as 'p implies q', 'p' is called antecedent and 'q' is called ***Consequent***. To be more clear about conditional statement, consider the following example:

A father said to his son, "if you will stood first in the examination, then I shall purchase a scooter for you". If antecedent is denoted by p and consequent by q, then the given conditional statement is represented in symbols as '$p \Rightarrow q$'. Now there are following four possibilities:

(i) The son stands first in the examination and the father purchases a scooter for him, *i.e.*,

If p is true, q is true, then $p \Rightarrow q$ is true.

(ii) The son stands first in the examination and father does not purchases a scooter for him, *i.e.*,

If p is true, q is false then $p \Rightarrow q$ is false.

(iii) The son does not stands first in the examination and the father purchases a scooter for him, *i.e.*,

If p is false, q is true then $p \Rightarrow q$ is true.

(iv) The son does not stands first in the examination and the father does not purchases a scooter for him, *i.e.*,
If p is false, q is false, then $p \Rightarrow q$ is true.

Hence, we observe that ***the conditional statement is false only in condition 'when antecedent is true and consequent is false'***.

Truth table for $p \Rightarrow q$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: Some authors use the symbol '$\rightarrow$' instead of the symbol '$\Rightarrow$'. and $\leftrightarrow$ by $\Leftrightarrow$

9.6.5 Bi-Conditional ($\Leftrightarrow$)

[U.P.T.U. (B.Tech.) 2007]

Let p and q be two statements. The statement of the type '$p \Rightarrow q$ and $q \Rightarrow p$' i.e., '$p \Rightarrow q \wedge q \Rightarrow p$' is called bi-conditional statement and it is represented by '$p \Leftrightarrow q$'.

Bi-conditional statement $p \Leftrightarrow q$ is also read as follows:

(a) 'p if and only if q',

or (b) 'p then and only then when q',

or (c) 'p iff q'.

To make more clear the bi-conditional statement, in example of (iv) above of 9.6.4, if the father had promised the son that, "I shall purchase a scooter for you if and only if you will stand first in the examination". It means that the father will not purchase a scooter for him if the son does not stand first in the examination. Thus, case (ii) and (iii) both in above example are against his promise, so bi-conditional is a false statement.

Hence, we observe that a ***Bi-conditional statement is true only in those cases when its both components have the same truth values.***

Truth table for $p \Leftrightarrow q$

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: Let p and q be statement letters, then following are example of statement patterns:

(i) $p \wedge q$

(ii) $p \vee q$

(iii) $p \vee q \Rightarrow \sim p$

(iv) $p \Rightarrow q$

(v) $p \Leftrightarrow q$

(vi) $p \vee q \Leftrightarrow q \vee p$

9.7 Kinds of Conditional

If p and q are any two statements, then some other conditional, related to conditional $p \Rightarrow q$, are also used which are given in the following table.

	Conditional	Name of Kind
I.	$p \Rightarrow q$	Direct implication
II.	$q \Rightarrow p$	Converse implication
III.	$\sim p \Rightarrow \sim q$	Inverse or Opposite implication
IV.	$\sim q \Rightarrow \sim p$	Contra-positive implication

9.7.1 Contra-Positive Implication

Now, we shall discuss contra-positive implication in detail. If p and q are two statements, then implications $\sim q \rightarrow \sim p$ is called the contra-positive of implication $p \rightarrow q$, *i.e.*, $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are contra-positive of each other. For example: if in any ΔABC, $\angle B = 90^\circ$, then

$$BA^2 + BC^2 = AC^2 \qquad ...(1)$$

Its contra-positive will be as follows:

In any ΔABC, if $BA^2 + BC^2 \neq AC^2$, then $\angle B \neq 90^\circ$ ...(2)

Again above both implications (i) and (ii) are contra-positive of each other.

The following unified truth table, clearly explains the definitions of different kind of conditional.

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

9.7.2 Tautology

[U.P.T.U. (B.Tech.) 2008]

A **Tautology** is a proposition which is true for all truth values of its sub propositions or components. A tautology is also called **Logically Valid** or **Logically True**. Clearly in truth tables, all entries in the column of tautology are of 'T' only.

Example 21: *If truth set of any proposition over universal set is defined as*

$$T_{p(n)} = \{a \in u : p(a) \text{ is true}\}$$

then prove that

$$T_{p \Leftrightarrow q} = (T_p \wedge T_q) \vee (T_p^c \wedge T_q^c)$$

and

$$T_{p \rightarrow q} = (T_p^c \vee T_q)$$

[U.P.T.U. (B.Tech.) 2008]

Solution: The truth table for given proposition

$$T_{p\leftrightarrow q} = (T_p \wedge T_q) \vee T_p^c \wedge T_q^c \text{ is}$$

T_p	T_q	$T_{p\leftrightarrow q}$	$T_p \wedge T_q$	T_p^c	T_q^c	$T_p^c \wedge T_q^c$	$(T_p \wedge T_q) \vee (T_p^c \wedge T_q^c)$
T	T	T	T	F	F	F	T
T	F	F	F	F	T	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	T	T	T

Hence, from table we find

$$T_{p\leftrightarrow q} = (T_p \wedge T_q) \vee (T_p^c \wedge T_q^c)$$

The truth table for $T_{p\to q} = T_p^c \vee T_q$ is

T_p	T_q	$T_{p\to q}$	T_p^c	$T_p^c \vee T_q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence, from table we find

$$T_{p\to q} = T_p^c \vee T_q.$$

Example 22: *Show that the truth values of the following formulas are independent of their components.*

(i) $(p \wedge (p \to q)) \to q$ (ii) $(p \to q) \Leftrightarrow (\sim p \vee q)$

(iii) $((p \to q) \wedge (q \to r)) \to (p \to r)$

[U.P.T.U. (B.Tech.) 2007]

Solution: Truth value of each of these expression being "Tautology" as shown below is independent of the components.

(i) $(p \wedge (p \to q)) \to q$

p	q	$p \to q$	$p \wedge (p \to q) = A$	$A \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(ii) $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$	$(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(iii) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Example 23: (i) *Consider the conditional statement p. If the floods destroy my house or fires destroy my house, then my insurance company will pay me. Write the converse, inverse and contrapositive of the statement.* [U.P.T.U. (B.Tech.) 2003]

(ii) *There are two restaurants next to each other one has sign that says "Good food is not cheap" and other has a sign that says "cheap food is not good". Are the signs saying the same thing?* [U.P.T.U. (B.Tech.) 2003]

(iii) *Given the following statements as premises, all refering to an arbitrary meal:*

(a) *If he takes coffee, he does not drink milk.*

(b) *He eats crackers only if he drink milk.*

(c) *He does not take soup unless he eats crackers.*

(d) *At noon today, he had coffee.*

Whether he took soup at noon today? If so, what is the correct conclusion? [U.P.T.U. (B.Tech.) 2004]

Solution: (i) Let the atomic statements be

P : The floods destroy my house , q :The fires destroy my house

r : My insurance company will pay me

Then,

$$(p \vee q) \Rightarrow r$$

Its inverse is $\sim(p \vee q) \Rightarrow \sim r$ or $\sim p \wedge \sim q \Rightarrow \sim r$

Therefore, the argument will be

If the floods does not destroy my house and fires does not destroy my house, then my insurance company will not pay me. Its converse is $r \Rightarrow p \vee q$

Therefore, argument will be "If my insurance company pay me then the floods will destroy my house or fires will destroy my house.

(ii) Let P : Food is good and q : Food is cheap

Then the argument "Good food is not cheap" is written as

$$p \Rightarrow \sim q$$

and the argument "cheap food is not good" is written as

$$q \Rightarrow \sim p$$

and their truth table is

p	q	$\sim p$	$\sim q$	$p \Rightarrow \sim q$	$q \Rightarrow \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Since last two columns are same. Hence we can say that the signs are saying the same thing.

(iii) Let p: he takes coffee, q: he drinks milk, r: he eats crackers and s : he takes soup

Then, we see

(a) $p \rightarrow \sim q$ (b) $r \rightarrow q$ (c) $\sim r \rightarrow - s$

(d) $\sim r \rightarrow - s$ (e) by condition (*e*), we have p

Since implication $r \rightarrow q$ is equivalent to its contrapositive $\sim q \rightarrow \sim r$, we have the following chain of arguments.

$p \rightarrow \sim q$	a premise
$\sim q \rightarrow \sim r$	contrapositive of premise (b)
$p \rightarrow \sim r$	a conclusion of law of syllogism
$\sim r \rightarrow \sim s$	a premise
$p \rightarrow \sim s$	a conclusion by law of syllogism
p	a premise
$\sim s$	a conclusion by modusponen

Hence $\sim s$ is the conclusion, *i.e.* he did not take soup at noon day.

Example 24: *The peirce arrow $\downarrow$ (NOR) is a logical binary operation which is defined as follows*

$$p \downarrow q = \sim(p \vee q)$$

(i) *prove that* $\sim p = p \downarrow p$ (ii) *prove that* $p \wedge q = (p \downarrow p) \downarrow (q \downarrow q)$

(iii) *write* $p \to q$ *using peirce arrows only.*

[U.P.T.U. (B.Tech.) 2004]

Solution: (i)

p	$\sim p$	$p \vee p$	$\sim(p \vee p)$
T	F	T	F
F	T	F	T

We see from table $p \downarrow q = \sim(p \vee p)$

(ii) $p \wedge q = (p \downarrow p) \downarrow (q \downarrow q) = (\sim p) \downarrow (\sim q)$ [from (i)]

$= \sim(\sim p \vee \sim q)$ [by De-Morgan's law]

$= p \wedge q$

(iii) Since, we have $(p \to q) \equiv (\sim p \vee q) = \sim(p \wedge \sim q)$ by (i)

$= \sim(p \wedge (q \downarrow q))$ by (ii)

$= \sim((p \downarrow p) \downarrow (q \downarrow q) \downarrow (q \downarrow q))$

$= ((p \downarrow p) \downarrow (q \downarrow q) \downarrow (q \downarrow q) \downarrow ((p \downarrow p) \downarrow (q \downarrow q) \downarrow (q \downarrow q))$

Example 25: *The converse of a statement is given. Write the inverse and contrapositive statements "If I come early, then I can get car".* [Osmania (B.E.) Andhra 2004, 2008]

Solution: Inverse: "If I cannot get car, then I shall not come early".

Contrapositive: If I do not come early, then I can not get the car.

Example 26: *The inverse of statement is given. Write the converse and contrapositive of the statement.*

"If a man is not fisherman, then he is not swimmer".

Solution: Converse: "If he is a swimmer, then the man is a fisherman".

Contrapositive: "If he is not a swimmer, then the man is not a fisherman".

Illustration: *If* $(p \wedge q) \wedge \sim(p \vee q)$ *is a contradiction*

Hence, $p \wedge q \Rightarrow p \vee q$

Illustration: *If* $(p \to q) \wedge (q \to r) \to (p \to r)$ *is a tautology.*

Hence, $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$

Example 27: *The contrapositive of statement is given as*

"If $x < 2$ Then $x + 4 < 6$"

Write the converse and inverse

[U.P.T.U. (B.Tech.) 2009]

Solution: Converse: If $x > 2$, then $x + 4 > 6$

Inverse: If $x + 4 > 6$, then $x > 2$

Example 28: *Write the equivalent formula for* $p \wedge (q \rightarrow r) \vee (r \leftrightarrow p)$ *which does not contain bi-conditional.*

[U.P.T.U. (B.Tech.) 2009]

Solution: $p \wedge (q \rightarrow r) \wedge (r \rightarrow q) \vee (r \rightarrow p) \wedge (p \rightarrow r)$

Example 29: *Given that the value of* $p \rightarrow q$ *is true. Can you determine the value of* $\sim p \vee (p \leftrightarrow q)$?

[R.G.P.V. (B.E.) Raipur 2008; P.T.U. (B.E.) Punjab 2007]

Solution: We shall construct the truth table column for $p \rightarrow q$ and $\sim p \vee (p \rightarrow q)$

p	q	$p \rightarrow q$	$\sim p$	$p \leftrightarrow q$	$\sim p \vee (p \leftrightarrow q)$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	F	T
F	F	T	T	T	T

From the table it follows that $p \rightarrow q$ is true then the value of $\sim p \vee (p \leftrightarrow q)$ is true.

We can determine the value of $\sim p \vee (p \leftrightarrow q)$ because corresponding to each possible choice of p and q for which the value of $p \rightarrow q$ is true, the value of $\sim p \vee (p \leftrightarrow q)$ is same as T.

Example 30: *Given that the value of* $p \rightarrow q$ *is false, determine the value of* $(\sim p \vee \sim q) \rightarrow q$

[U.P.T.U. (B.Tech.) 2009; Rohtak (B.E.) 2007]

Solution: We shall construct the truth table column $p \rightarrow q$ and $(\sim p \vee \sim q) \rightarrow q$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(\sim p) \vee (\sim q)$	$(\sim p \vee \sim q) \rightarrow q$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	T	T	T
F	F	T	T	T	T	F

From the table it follow that $p \rightarrow q$ is false then the value of $(\sim p \vee \sim q) \rightarrow q$ is false.

Example 31: *Prove that* $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ *is a tautology.* **[U.P.T.U. (B.Tech.) 2003]**

Solution: For convenience, let

$$p \Leftrightarrow r = A \text{ and } (p \Leftrightarrow q) \wedge (q \Leftrightarrow r) = B$$

p	q	r	$p \Leftrightarrow q$	$q \Leftrightarrow r$	$p \Leftrightarrow r = A$ (suppose)	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) = B$ (suppose)	$B \Rightarrow A$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	T	F	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	T	F	F	F	T	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Last column shows that $B \Rightarrow A$ i.e., $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$ is a tautology.

Example 32: *Prove that each of the following statement is a tautology:*

(i) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

(ii) $[(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p)] \Rightarrow (p \Leftrightarrow q)$

(iii) $(p \Rightarrow q) \vee (r \Rightarrow p)$

(iv) $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$

[U.P.T.U. (B.Tech.) 2006]

Solution:

(i) Let the statement patterns $p \Rightarrow q, q \Rightarrow r$ and $p \Rightarrow r$ be denoted by sentence variables P, Q and R respectively.

Truth Table for $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$:

p	q	r	P $p \Rightarrow q$	Q $q \Rightarrow r$	R $p \Rightarrow r$	$P \wedge Q$	$P \wedge Q \Rightarrow R$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Last column show that $(P \wedge Q) \Rightarrow R$ i.e., $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow [p \Rightarrow r]$ is a tautology.

(ii) Let the statement patterns $\sim q \Rightarrow \sim p$, $q \Rightarrow p$ and $p \Leftrightarrow q$ be denoted by sentence variables P, Q and R, respectively.

Truth Table for $[(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p)] \Rightarrow (p \Leftrightarrow q)$

p	q	$\sim p$	$\sim q$	$P \sim q \Rightarrow \sim p$	(Q) $q \Rightarrow p$	$P \wedge Q$	(R) $p \Leftrightarrow q$	$(P \wedge Q) \Rightarrow R$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T	T

Last column shows that $(P \wedge Q) \Rightarrow R$ i.e., $[(\sim q \Rightarrow \sim p) \wedge (q \Rightarrow p)] \Rightarrow (p \Leftrightarrow q)$ is a tautology.

(iii) **Truth Table for** $(p \Rightarrow q) \vee (r \Rightarrow p)$:

p	q	r	$p \Rightarrow q$	$r \Rightarrow p$	$(p \Rightarrow q) \vee (r \Rightarrow p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

From the last column, it is clear that the given statement $(p \Rightarrow q) \vee (r \Rightarrow p)$ is a tautology.

(iv) **Truth Table for** $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$:

p	q	r	$q \wedge r$	$p \Leftrightarrow q \wedge r$	$\sim r$	$\sim p$	$\sim r \Rightarrow \sim p$	$(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$
T	T	T	T	T	F	F	T	T
T	T	F	F	F	T	F	F	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	F	F	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	F	T	T	T
F	F	F	F	T	T	T	T	T

From the last column, it is clear that the given statement $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$ is a tautology.

Example 33: *Prove that each of the following statement is a tautology:*

(i) $(p \Rightarrow q) \vee r \Leftrightarrow [(p \vee r) \Rightarrow (q \vee r)]$

(ii) $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$

[U.P.T.U. (B.Tech.) 2006]

Solution:

(i) ***Truth Table for*** $(p \Rightarrow q) \vee r \Rightarrow [(p \vee r) \Rightarrow (q \vee r)]$:

p	q	r	$p \Rightarrow q$	$(p \Rightarrow q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \Rightarrow (q \vee r)$	$(p \Rightarrow q) \vee r \Leftrightarrow [(p \vee r) \Rightarrow (q \vee r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T

From the last column, it is clear that the given statement

$$(p \Rightarrow p) \vee r \Leftrightarrow [(p \vee r) \vee (q \vee r)] \text{ is tautology.}$$

(ii) ***Truth Table for*** $(\sim q) \wedge (p \Rightarrow q) \Rightarrow (\sim p)$:

p	q	$\sim q$	$p \Rightarrow q$	$(\sim q) \wedge (p \Rightarrow q)$	$\sim p$	$(\sim q) \wedge (p \Rightarrow q) \Rightarrow (\sim p)$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

From the last column, it is clear that the given statement $(\sim q) \wedge p \Rightarrow q) \Rightarrow (\sim q)$ is a tautology.

9.7.3 Contradiction

[U.P.T.U. (B.Tech.) 2008]

A **Contradiction** is a Proposition which is always false for all truth values of its propositions or components. A contradiction is also called logically false. Clearly in truth tables, all entries in the column of contradiction are of 'F' only.

Example 34: *Prove that* $p \wedge (\sim p)$ *is a contradiction.*

Solution: Let p be a statement. Truth table for $p \wedge (\sim p)$ is:

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

Since all entries in the column of $p \wedge (\sim p)$ are of F only, it is a contradiction.

Example 35: *Prove that $p \Leftrightarrow (\sim p)$ is a contradiction.*

Solution: Truth table for $p \Leftrightarrow (\sim p)$ is:

p	$\sim p$	$p \Leftrightarrow (\sim p)$
T	F	F
F	T	F

Since all entries in the last column are of 'F' s and so it is contradiction.

Example 36: *Prove that $(p \vee q) \wedge (\sim p) \wedge (\sim q)$ is a contradiction.*

Solution: *Truth Table for* given proposition is:

p	q	$p \vee q$	$\sim p$	$\sim q$	$(p \vee q) \wedge (\sim p)$	$(p \vee q) \wedge (\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	T	F
F	F	F	T	T	F	F

Since all entries in the last column are of 'F' s and so it is contradiction.

Example 37: *Prove that each of the following statement is a contradiction:*

(i) $P = (p \vee q) \wedge (p \vee \sim q \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$

(ii) $[(p \wedge r) \vee (q \wedge \sim r)] \Leftrightarrow [(\sim p \wedge r) \vee (\sim q \wedge \sim r)]$

(iii) $(p \vee q) \wedge (\sim p) \wedge (\sim q)$

(iv) $[(p \wedge q) \Rightarrow p] \Rightarrow [q \wedge \sim q] = A$

(v) $(p \wedge q \Rightarrow q) \Rightarrow (q \wedge \sim q)$

Solution:

(i) Let $p \vee q = R$, $p \vee \sim q = S$, $\sim p \vee q = U$ and $\sim p \vee \sim q = V$.

Then construction of the truth table:

p	q	$\sim p$	$\sim q$	$R = p \vee q$	$S = p \vee \sim q$	$U = \sim p \vee q$	$V = \sim p \vee \sim q$	$P = R \wedge S \wedge U \wedge V$
T	T	F	F	T	T	T	F	F
T	F	F	T	T	T	F	T	F
F	T	T	F	T	F	T	T	F
F	F	T	T	F	T	T	T	F

From the last column of truth table it is clear that P is a contradiction because the truth value of each entry of this column is F.

(ii) Let $(p \wedge r) \vee (q \wedge \sim r) = P$ and $(\sim p \wedge r) \vee (\sim q \wedge \sim r) = Q$.

Then construction of the truth table:

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \wedge r$	$q \wedge \sim r$	P	$\sim p \wedge r$	$\sim q \wedge \sim r$	Q	$P \Leftrightarrow Q$
T	T	T	F	F	F	T	F	T	F	F	F	F
T	T	F	F	F	T	F	T	T	F	F	F	F
T	F	T	F	T	F	T	F	T	F	F	F	F
T	F	F	F	T	T	F	F	F	F	T	T	F
F	T	T	T	F	F	F	F	F	T	F	T	F
F	T	F	T	F	T	F	T	T	F	F	F	F
F	F	T	T	T	F	F	F	F	T	F	T	F
F	F	F	T	T	T	F	F	F	F	T	T	F

From the last column of the truth table it is clear that the statement $P \Leftrightarrow Q$ *i.e.,* $(p \wedge r) \vee (q \wedge \sim r) \Leftrightarrow (\sim p \wedge r) \vee (\sim q \wedge \sim r)$ is a contradiction.

(iii) On constructing of the truth table:

p	q	$\sim p$	$\sim q$	$p \vee q$	$(p \vee q) \wedge (\sim p)$	$(p \vee q) \wedge (\sim p) \wedge (\sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	F
F	F	T	T	F	F	F

Since all entries in the last column are of F' so it is a contradiction.

(iv) On constructing the truth table:

p	q	$\sim q$	$p \wedge q$	$(p \wedge q) \Rightarrow p$	$q \wedge \sim q$	A
T	T	F	T	T	F	F
T	F	T	F	T	F	F
F	T	F	F	T	F	F
F	F	T	F	T	F	F

Since all entries in the last column are of F' s and so it is a contradiction.

(v) On constructing the truth table:

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow q$	$\sim q$	$q \wedge \sim q$	$[(p \wedge q) \Rightarrow q \Rightarrow (q \wedge \sim q)]$
T	T	T	T	F	F	F
T	F	F	T	T	F	F
F	T	F	T	F	F	F
F	F	F	T	T	F	F

Since all entries in the last column are F s and so it is a contradiction.

9.7.4 Logical Equivalence

[U.P.T.U. (B.Tech.) 2008]

Two statements (or propositions) are called ***Logically Equivalent*** if the truth values of both the statements (or propositions) are always identical. In other words, two statement are called logically equivalent if when either is true the other is true and when either is false the other is false.

If two statements P and Q are logically equivalent then these are represented by '$P \equiv Q$'. If $P \equiv Q$ then $P \Rightarrow Q$ is a tautology. Consider two open statements $x = 4$ and $3x = 12$ (when $x \in N$) . Both have the same truth value T for the value 4 of x, therefore, these both open statements are logically equivalent.

Theorem: *The necessary and sufficient condition for two statements P and Q to be logically equivalent is that $P \Leftrightarrow Q$ is a tautology.* **[Rohtak (M.C.A.) 2004, 2009]**

Proof: $P \Leftrightarrow Q$ will be a tautology if its truth value is always T. The necessary and sufficient condition for the truth value of $P \Leftrightarrow Q$ to be T is that the truth values of P and Q should always be identical *i.e.*, P and Q should be logically equivalent.

Example 38: *If p and q are two statements, show that the implication $p \Rightarrow q$ and its contra-positive $(\sim q) \Rightarrow (\sim p)$ are logically equivalent.* **[U.P.T.U. (B.Tech.) 2008]**

Solution: Truth table:

p	q	$p \Rightarrow q$	$\sim p$	$\sim q$	$(\sim q) \Rightarrow (\sim p)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

The entries of third and sixth columns are identical. Hence, $p \Rightarrow q$ and $(\sim q) \Rightarrow (\sim p)$ are logically equivalent.

Example 39: *Show that* $q \Rightarrow p$ *converse of* $p \Rightarrow q$ *and its inverse* $(\sim p) \Rightarrow (\sim q)$ *are logically equivalent.*

Solution:

p	q	$q \Rightarrow p$	$\sim p$	$\sim q$	$(\sim p) \Rightarrow (\sim q)$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

Since entries of 3rd and 6th columns are identical.

$$\therefore \quad (q \Rightarrow p) \equiv \{(\sim p) \Rightarrow (\sim q)\}$$

Example 40: *Show that* $\sim (p \Rightarrow q) \equiv \{p \wedge (-q)\}$ [Kurukshetra (B.E.) 2008]

Solution: If $\sim (p \Rightarrow q)$ and $p \wedge (\sim q)$ are equivalent, then

$$\sim (p \Rightarrow q) \Leftrightarrow \{p \wedge (\sim q)\}$$

will be tautology (see 9.7.4), which is clear from the following truth table:

p	q	$\sim q$	$p \Rightarrow q$	$\sim (p \Rightarrow q)$	$p \wedge (\sim q)$	$\sim (p \Rightarrow q) \Leftrightarrow \{p \wedge (\sim q)\}$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

9.8 Algebra of Proposition

Now we shall study the Laws of Algebra of Proposition. These laws are some tautologies. Here we shall use t for tautology and f for fallacy (contradiction).

9.8.1 Idempotent Law

[U.P.T.U. (B.Tech.) 2008]

(i) $p \vee p \Leftrightarrow p$, (ii) $p \wedge p \Leftrightarrow p$.

Proof: (i) Truth table for $p \vee p \Leftrightarrow p$:

p	p	$p \vee p$	$p \vee p \Leftrightarrow p$
T	T	T	T
F	F	F	T

Since the column under $p \vee p \Leftrightarrow p$ contains only T's. *Hence it is tautology.*

(ii) Truth table for $p \wedge p \Leftrightarrow p$:

p	p	$p \wedge p$	$p \wedge p \Leftrightarrow p$
T	T	T	T
F	F	F	T

Hence, $p \wedge p \Leftrightarrow p$ is a tautology.

Note: Here the notation '$\Leftrightarrow$' can be replaced by '$\equiv$' or by '='. So the notation '=' will be used in the following laws.

9.8.2 Commutative Law

(i) $p \vee q = q \vee p$, (ii) $p \wedge q = q \wedge p$.

Proof: (i)

p	q	$p \vee q$	$q \vee p$	$p \vee q = q \vee p$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Similarly (ii) can be proved.

9.8.3 Associative Law

[U.P.T.U. (B.Tech.) 2005]

(i) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$; or $(p \vee q) \vee r = p \vee (q \vee r)$

(ii) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$; or $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

Proof: (i)

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

(ii)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	F	F	T	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T

9.8.4 Distributive Law

(i) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (ii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Proof: (i) Disjunction is distributive over conjunction. Since the columns in the following table under $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical. Hence, (i) is a equivalent.

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

(ii) Conjunction is distributive over disjunction.

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Since the columns under $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are identical. Hence (ii) is a equivalent.

9.8.5 De-Morgan's Law

(i) $\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$ or $\sim(p \vee q) = (\sim p) \wedge (\sim q)$ [U.P.T.U. (B.Tech.) 2003]

(ii) $\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$ or $\sim(p \wedge q) = (\sim p) \vee (\sim q)$ [U.P.T.U. (B.Tech.) 2003]

Proof: (i)

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$	$\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

(ii)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	F	T
F	T	F	T	T	F	F	T
F	F	F	T	T	T	T	T

9.8.6 Identity Law

[U.P.T.U. (B.Tech.) 2008]

(i) The identity element (or identity statement) for conjunction is tautology t, and is such that

$$p \wedge t = t \wedge p = p$$

(ii) The identity element (or identity statement) for disjunction is contradiction (or fallacy) f and is such that

$$p \vee f = f \vee p = p$$

Truth Table for (i)

p	t	$p \wedge t$	$t \wedge p$
T	T	T	T
F	T	F	F

Truth Table for (ii)

p	f	$p \vee f$	$f \vee p$
T	F	T	T
F	F	F	F

9.8.7 Complement Law

For every statement p there exists its negation $(\sim p)$ such that

(i) $p \vee (\sim p) = t$ (ii) $p \wedge (\sim p) = f$

9.8.8 Absorption Law

[U.P.T.U. (B.Tech.) 2008]

(i) $p \vee (p \wedge q) = p$ (ii) $p \wedge (p \vee q) = p$

p	q	$(p \wedge q)$	$(p \vee (p \wedge q))$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

From 1st and VIth column we see that

$$p \vee (p \wedge q) = p$$

(ii) Similarly as (i) part.

9.9 Truth-Table for Well Formed Formula

If we replace the propositional variables in a formula A by propositions, we get a proposition involving connectives. The table given the truth value of such proposition obtained by replacing the propositional variables by arbitrary propositions is called **the truth table of A.** If A have n propositional constants then we have 2^n possible combinations of truth values of propositions replacing the variables.

9.9.1 Null or Dominance Laws

(i) $p \wedge F = F = F \wedge p$ (ii) $p \vee T = T = T \vee p$

Proof: (i) Truth table for (i)

p	F	$p \wedge F$	$F \wedge p$
T	F	F	F
F	F	F	F

Hence $p \wedge F = F = F \wedge p$

(ii) Truth table for (ii)

p	T	$p \vee T$	$T \vee p$
T	T	T	T
F	T	T	T

Hence $p \vee T = T = T \vee p$

9.9.2 Negation Laws

(i) $p \wedge (\sim p) = F$ (ii) $p \vee (\sim p) = T$

Proof: (i) Truth Table for (i)

p	$(\sim p)$	$p \wedge (\sim p)$
T	F	F
F	T	F

Hence, from truth table we find $p \wedge (\sim p) = F$

(ii) Truth table for (ii)

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

Hence, from truth table we find $p \vee (\sim p) = T$

9.9.3 Involution Laws

[U.P.T.U. (B.Tech.) 20..]

$$\sim(\sim p) = P$$

Proof: The truth table is

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

Hence, from truth table we find

$$\sim(\sim p) = p$$

9.9.4 Conditional Rules

$$(p \Rightarrow q) = (\sim p \vee q)$$

Proof: The truth table is

p	q	$p \Rightarrow q$	$\sim p$	$(\sim p \vee q)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence, from the table we find

$$(p \Rightarrow q) = (\sim p \vee q)$$

9.9.5 Contrapositive Law

$$(p \Rightarrow q) = (\sim q) \Rightarrow (\sim p)$$

Proof: The truth table is

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$(\sim q) \Rightarrow (\sim p)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Hence, from the table we find

$$(p \Rightarrow q) = (\sim q) \Rightarrow (\sim p)$$

9.9.6 Biconditional or Equivalence Rules

(i) $(p \Leftrightarrow q) = (p \Rightarrow q) \wedge (q \Rightarrow p)$ (ii) $(p \Leftrightarrow q) = (p \wedge q) \vee ((\sim p) \wedge (\sim q))$

Proof: (i) The truth table is

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Hence, from the table we find $(p \Leftrightarrow q) = (p \Rightarrow q) \wedge (q \Rightarrow p)$

(ii) Similar table as of (i)

9.9.7 Exponential Law

$$(p \wedge q) \Rightarrow r = p \Rightarrow (q \Rightarrow r)$$

Proof: The truth table is

p	q	r	$p \wedge q$	$(p \wedge q) \Rightarrow r$	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Hence, we find from the table

$$(p \wedge q) \Rightarrow r = p \Rightarrow (q \Rightarrow r)$$

9.9.8 Absurdity Law

$$(p \Rightarrow q) \wedge (p \Rightarrow \sim q) = (\sim p)$$

Proof: The truth table is

p	q	$p \Rightarrow q$	$\sim q$	$p \Rightarrow \sim q$	$(p \Rightarrow q) \wedge (p \Rightarrow \sim q)$	$\sim p$
T	T	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Hence, we find from the table

$$(p \Rightarrow q) \wedge (p \Rightarrow \sim q) = (\sim p)$$

Example 41: *Confirm the truth value of the following:*

(i) $4 + 6 = 12 \Leftrightarrow 2 + 4 = 8$ (ii) $4 + 6 = 10 \Leftrightarrow 2 + 6 = 6$

(iii) *It is false that 'Meerut' is in U.K, then London is in India.*

Solution: Let the first statement be denoted by p and second by q, then

(i) **Statement** **Truth value**

$p \equiv 4 + 6 = 12$ F

$q \equiv 2 + 4 = 8$ F

$\therefore$ **Truth table**

p	q	$p \Leftrightarrow q$
F	F	T

Hence, the truth value of the statement $4 + 6 = 12 \Leftrightarrow 2 + 4 = 8$ *i.e.,* $p \Leftrightarrow q$ is 'true' (T).

(ii) **Statement** **Truth value**

$4 + 6 = 10$ T

$q = 2 + 4 = 6$ T

$\therefore$ **Truth table**

p	q	$p \Leftrightarrow q$
T	T	T

Hence, the truth value of the statement $4 + 6 = 10 \Leftrightarrow 2 + 4 = 6$ *i.e.,* $p \Leftrightarrow q$ is 'true' (T).

(iii) **Statement** **Truth value**

$p \equiv$ Meerut is in U.K. F

$q \equiv$ London is in India F

Thus, the statement 'Meerut' is in U.K. then London is in India' is denoted symbolically by $p \Rightarrow q$.

Hence, the statement 'it is false that Meerut is in U.K. then London is in India' is denoted symbolically by $\sim(p \Rightarrow q)$.

$\therefore$ **Truth table**

p	q	$p \Rightarrow q$	$\sim(p \Rightarrow q)$
F	F	T	F

Hence, the truth value of the given statement is 'False' (F).

Example 42: *Construct converse, inverse and contrapositive of the direct statement 'if $4x-2=10$ then $x=3$*

Solution: Let $p \equiv 4x-2=10$, $q \equiv x=3$. Thus, the statement 'if $4x-2=10$ then $x'=3$ is the direct implication $p \Rightarrow q$.

(i) **Converse:** Converse of '$p \Rightarrow q$' is '$q \Rightarrow p$', hence the converse of given statement is as follows:

" if $x=3$ *then* $4x-2=10$ "

(ii) **Inverse:** The inverse of '$p \Rightarrow q$' is '$(\sim p \Rightarrow (\sim q)$', hence the inverse of the given statement is as follows:

" If $4x-2 \neq 10$ *then* $x \neq 3$".

(iii) **Contrapositive:** The contrapositive of '$p \Rightarrow q$' is '$(\sim q) \Rightarrow (\sim p)$', hence the contrapositive of the given statement is as follows:

"If $x \neq 3$ then $4x-2 \neq 10$"

Example 43: *Write the converse, inverse and contrapositive of the following direct statements:*

(a) If $ABCD$ is a square then $ABCD$ is a rectangle.

(b) When sun rises then it is morning.

Solution: (a) Let $p \equiv ABCD$ is a square,

$q \equiv ABCD$ is a rectangle.

$\therefore$ The given statement is the direct implication '$p \Rightarrow q$'.

(i) **Converse:** Since the converse of '$p \Rightarrow q$' is '$q \Rightarrow p$' so the converse of given statement is:

'If $ABCD$ is a rectangle then $ABCD$ is a square'.

(ii) **Inverse:** Since inverse of '$p \Rightarrow q$' is '$(\sim p) \Rightarrow (\sim q)$' so the inverse of given statement is:

'If $ABCD$ is not a square then $ABCD$ is not a rectangle'.

(iii) **Contrapositive:** Since contrapositive of '$p \Rightarrow q$' is '$(\sim q) \Rightarrow (\sim p)$', so the contrapositive of the given statement is:

'If $ABCD$ is not a rectangle then $ABCD$ is not a square'.

(b) Assuming p and q as above in (a), the given statement has:

(i) **Converse:** 'When it is morning then sum rises'.

(ii) **Inverse:** 'When sun does not rise then it is not morning'.

(iii) **Contrapositive:** 'When it is not morning then sun does not rise'.

Example 44: *Let $p \equiv$ it is rainy season. $q \equiv$ the mango is delicious, construct the following statements:*

(i) $p \Rightarrow q$ (ii) its converse, (iii) its inverse, (iv) contrapositive,

Solution:

(i) If it is rainy season, then the mango is delicious.

(ii) If the mango is delicious, then it is rainy season.

(iii) If it is not rainy season, then the mango is not delicious.

(iv) If the mango is not delicious, then it is not rainy season.

(v) If it is rainy season and mango is not delicious.

Example 45: *If the truth value of $p \Leftrightarrow q$ is: (i) F, (ii) T, then find the truth value of $p \vee q$.*

Solution: We know that the truth value of $p \Leftrightarrow q$ is F, when either one of p and q has truth value F and other T.

Truth table

$p \Leftrightarrow q$	p	q	$p \vee q$
F	T	F	T
F	F	T	T

Since the entries in the column under $p \vee q$ are of only T ' s, hence the truth value of $p \vee q$ is True' (T).

(ii) The truth value of $p \Leftrightarrow q$ is T When its both components P and q has the same truth values (*i.e.*, T or F).

Truth table

$p \Leftrightarrow q$	p	q	$p \vee q$
T	T	T	T
T	F	F	F

From truth table we see that the truth value of $p \vee q$ is sometimes T and sometimes F, hence the truth value of $p \vee q$ cannot be delicious.

Example 46: *If the truth value of $p \vee q$ is F, then decide the truth value of $p \Leftrightarrow q$.*

Solution: We know that the truth value of $p \vee q$ is F, only in the condition 'when p and q both have truth values F'. Hence the truth value of $p \Leftrightarrow q$ is T (see following truth table).

$p \vee q$	p	q	$p \Leftrightarrow q$
F	F	F	T

Example 47: *If $P \Leftrightarrow Q$ is true, then determine the truth value of $P \vee (\sim Q)$.*

Solution: Truth table.

$P \Leftrightarrow Q$	P	Q	$\sim Q$	$P \vee (\sim Q)$
T	T	T	F	T
T	F	F	T	T

Hence, the truth value of $P \vee (\sim Q)$ is 'True' (T).

Example 48: *If p is a statement such that for any statement Q, the statement $P \wedge Q$ is false, then what do you conclude about the truth value of P ?*

Solution: We know that the truth value of $P \wedge Q$ is 'True' when truth values of P and Q both are 'True', and in all other case ; the truth value of $P \wedge Q$ is 'False'. Therefore, in the given condition, the truth value of P is given by the third column of the following truth table:

$P \wedge Q$	Q	P
F	F	T
F	T	F
F	F	F

Example 49: *If P is a statement such that for any statement Q the statement $P \vee Q$ is true, then what do you conclude about the truth value of P?*

Solution: The truth value of P is given by the third column of the following truth table:

$P \vee Q$	Q	P
T	T	F
T	F	T
T	T	T

Example 50: *Prepare the truth table of the statement* **[U.P.T.U. (B.Tech.) 2009]**

$$A = (p \Rightarrow q \wedge r) \vee (\sim p \wedge q).$$

or

Find a formula A that uses the variable p, q and r such that A is contradiction. **[U.P.T.U. (B.Tech.) 2009]**

Solution: Truth table for $(p \Rightarrow q \wedge r) \vee (\sim p \wedge q)$:

p	q	r	$q \wedge r$	$p \Rightarrow (q \wedge r)$	$\sim p$	$\sim p \wedge q$	$(p \Rightarrow q \wedge r) \vee (\sim p \wedge q) = A$
T	T	T	T	T	F	F	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

Example 51: *Construct a truth table for the following statement:*

(i) $(p \vee q \Rightarrow s) \Leftrightarrow (p \Leftrightarrow s) \vee (q \Rightarrow s)$

(ii) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow [p \Rightarrow r]$

[U.P.T.U. (B.Tech.) 2005]

Solution: (i) Truth table for $(p \vee q \Rightarrow s) \Leftrightarrow (p \Rightarrow s) \vee (q \Rightarrow s)$:

p	q	s	$p \vee q$	$p \vee q \Rightarrow s$	$p \Rightarrow s$	$q \Rightarrow s$	$(p \Rightarrow s) \vee (q \Rightarrow s)$	$(p \vee q \Rightarrow s) \Leftrightarrow (p \Rightarrow s) \vee (q \Rightarrow s)$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	T	F
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T	F
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

(ii) Truth table for $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow [p \Rightarrow r]$:

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow [p \Rightarrow r]$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Example 52: Prove that the following statement is a tautology.

$$(\sim B) \wedge (A \Rightarrow B) \Rightarrow (\sim A)$$

[U.P.T.U. (B.Tech.) 2005]

Solution: Truth table for given statement:

A	B	$\sim A$	$\sim B$	$A \Rightarrow B$	$\sim B \wedge (A \Rightarrow B)$	$(\sim B) \wedge (A \Rightarrow B) \Rightarrow (\sim A)$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Hence, $(\sim B) \wedge (A \Rightarrow B) \Rightarrow (\sim A)$ is a tautology.

Example 53: *If* $p \equiv$ *Ram is beautiful,* $q \equiv$ *Ram is mixable,* $r \equiv$ *His friends like Ram, then write the following statement in language:*

(a) $(p \Rightarrow q) \vee (p \Rightarrow r)$ (b) $p \Rightarrow (q \vee r)$

Examine, are the above statement equivalent?

Solution:

(a) If Ram is beautiful then either Ram is mixable or his friends like Ram.

(b) If Ram is beautiful then he is mixable or his friends like him.

Combined truth table for (a) and (b):

p	q	r	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$	$q \vee r$	$p \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T

From above truth table, we see that the entries in sixth and eighth column are identical.

Hence, the statements given by (a) and (b) above are logically equivalent.

Example 54: *Prepare the tables of the following statement:* **[U.P.T.U. (B.Tech.) 2007]**

(i) $(p \Leftrightarrow q) \wedge (r \vee q)$

(ii) $\{(p \vee q) \wedge r\} \Rightarrow q$

Solution: (i) Truth table of $(p \Leftrightarrow q) \wedge (r \vee q)$:

p	q	r	$p \Leftrightarrow q$	$r \vee q$	$(p \Leftrightarrow q) \wedge (r \vee q)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	F	F

(ii) Truth table of $\{(p \vee q) \wedge r\} \Rightarrow q$:

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$\{(p \vee q) \wedge r\} \Rightarrow q$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	F
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	T
F	F	F	F	F	T

Example 55: *Which of the following sentences are statements? What are truth values of those that are statements?*

(i) *May God bless you.*

(ii) *Do you speak English?*

(iii) $4-x = 8$

(iv) *Seven is a prime number.*

(v) *Please try to solve the problem.*

Solution: A statement (proposition)is a declarative sentence which is either true or false but not both. These two values are 'truth' and 'false' denote by symbol 1 and 0 or T and F.

(i) It is not a statement [wish]

(ii) It is not a statement.

(iii) It's not statement.

(iv) It is a statement. It's value is true.

(v) It's a command. It is not a statement.

Example 56: *Write the negation of the following:*

(i) *If she studies, she will pass in exam.*

(ii) Anil is not rich and Kanchan is poor.

(iii) *No one wants to buy my house.*

(iv) *A cow is an animal.*

(v) *If the determinant of a system of linear equations is zero then either the system has no solution or it has an infinite number of solution.*

Solution:

(i) P : If she studies., Q : She will pass in exam.

Given statement: $P \rightarrow Q = \neg P \vee Q$

Its negation: $\neg (\neg P \vee Q) = p \wedge \neg q$

i.e., If she studies, she will not pass in the exam.

(ii) P : Anil is not rich and Q : Kanchan is poor.

Given statement: $(P \wedge Q)$

Its negation: $\neg (P \wedge Q) = \neg P \vee \neg Q$ *i.e.*, Anil is rich or Kanchan is not poor.

(iii) P : x wants to buy my home.

Given statement: $\forall x (\neg p)$

Its negation: $\neg (\forall x (\neg p)) = \exists x (p)$ *i.e.*, someone wants to buy home.

(iv) P : A cow is an animal.

Its negation: $\neg P$ *i.e.*, A cow is not an animal.

(v) P : Determinant of a system of linear equation is zero.

Q : System has no solution.

R : System has an infinite No. of solution.

Given statement: $p \to (Q \vee R) = \neg P \vee (Q \vee R)$

Its negation: $\neg(\neg P \vee Q \vee R) = P \wedge \neg Q \wedge \neg R$

If determinant of a system of linear equation is zero, system has solution and not has an infinite number of solutions.

Example 57: *If P = question paper is hard.*

q = I will fail in the examination. Then translate the following sentences into symbols:

(i) *Question paper is not hard and I will fail in the examination.*

(ii) *If I will not fail in the examination, then question paper is not hard.*

(iii) *If question paper is hard, then I will fail in the examination.*

(iv) *Question paper is not hard if and only if I will fail in the examination.*

(v) *If question paper is not hard then I will fail in the examination.*

Solution: Given:

P = Question paper is hard. Q = I will fail in the examination.

(i) $\neg p \wedge q$ (ii) $\neg q \to \neg p$ (iii) $p \to q$

(iv) $\neg p \Leftrightarrow q$ (v) $\neg p \to q$

Example 58: *If $p \equiv$ it is 10 'o' clock, $q \equiv$ the train is late, then statement in words are the following resultants:*

(i) $q \vee \neg p$ (ii) $p \wedge q$ (iii) $p \wedge \neg q$

(iv) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (v) $\neg p \wedge \neg q$

Solution:

(i) $q \vee \neg p$

The train is late or it is not 10 'o' clock.

(ii) $p \wedge q$

It is 10 'o' clock and the train is late.

(iii) $p \wedge \neg q$

It is 10 'o' clock or the train is not late.

(iv) $\neg(p \wedge q) = \neg p \vee \neg q$

It is not 10 'o' clock or the train is not late.

(v) The question paper is not hard and I will not fail in the examination.

Example 59: *Determine whether the following formulate are tautology, contradiction and satisfiable:*

(i) $[p \wedge (p \Rightarrow q] \Rightarrow \neg q$ (ii) $(p \Rightarrow q) \wedge (q \Rightarrow r) \wedge (p \wedge \neg r)$ (iii) $p \wedge (q \wedge \neg p)$

Solution: Similar solution as example 14.

Example 60: *Define the term proposition (or statement). Classify the following expression are statement or not:*

(i) *May you live long.* (ii) *The population of India goes upto 100 million in year 2000.*

(iii) $x + y = 30$ (iv) *All roses are white.* (v) *7 is a prime number.*

Solution: Proposition or statement:

A statement (or proposition) is a sentence which is either true or false but not both.

(i) It is not a statement. (ii) It is a statement.

(iii) It is not a statement because it is true or false depending on the values of x and y.

(iv) It is a statement. (v) It is a statement.

Example 61: *Consider the following:*

p : *He is a carpenter.,* Q : *He is making a table.*

Write down the following statement into symbols.

(i) *He is a carpenter but is not making a table.*

(ii) *He is a carpenter or making table.*

(iii) *Neither he is a carpenter nor he is making a table.*

(iv) *It is false that he is a carpenter or making a table.*

(v) *He is not a carpenter nor he is making a table.*

Solution: Given the following statements:

P : He is a carpenter. Q : He is making a table.

(i) $p \wedge \neg Q$ (ii) $p \vee Q$ (iii) $\neg p \wedge \neg Q$

(iv) $\neg(p \vee Q)$ (v) $\neg p \rightarrow \neg Q$

Example 62: *Consider the following.*

p : You take a course in Discrete Mathematics,

q : You understand logic,

r : You get an A on the final exam.

Write in simple sentences the meaning of the following:

(i) $q \Rightarrow r$ (ii) $-p \Rightarrow -q$ (iii) $(p \wedge q) \Rightarrow r$ (iv) $(p \wedge q) \Rightarrow -r$ (v) $-(-r)$

Solution:

(i) If you understand logics then you get an A on the final exam.

(ii) If you will not take a course in Discrete Mathematics then you will not understand logic.

(iii) If you take a course in Discrete Mathematics and understand logic than you will get an A on the final exam.

(iv) If you take a course in Discrete Mathematics and you understand logic then you will not get an A on the final exam.

(v) You get an A on the final exam.

9.10.2 Elementary Products

The formula which is product (conjunction) of the variables and their negations is called an elementary product. If p and q are two given variables, then

$p \wedge \sim p, \sim p \wedge q, p \wedge (\sim p)$ are some examples of elementary product.

9.10.3 Elementary Sum

The sum of (disjunction) of variables p and their negation in a formula is called ***Elementary Sum***.

If p and q are the given variables then $p \vee (\sim p)$, $\sim p \vee q$, $\sim q \vee p \vee (\sim p)$ are some examples of elementary sum.

9.10.4 Disjunctive Normal Form

Let A be a given formula. Another formula B which is equivalent to B is called ***Disjunctive Normal Form*** of A if B is a sum of elementary products.

A disjunctive normal form of a given formula can be formed as

(i) Replace $\Rightarrow, \Leftrightarrow$ by using the logicai connectives $\wedge$, $\vee$ and $\sim$ or $\urcorner$

(ii) Use De Morgan's laws to eliminate $\sim$ before sums or products.

(iii) Apply distributive laws repeatedly and eliminate product of variables to obtain the required normal form.

Example 69: *Obtain disjunctive normal form of*

$$p \wedge (p \Leftrightarrow q)$$

Solution: $p \wedge (p \Leftrightarrow q) = p \wedge (\sim p \vee q)$

$$= \{p \wedge (\sim p)\} \vee \{p \wedge q\}$$

Example 70: *Obtain disjunctive normal form of*

$$\sim (p \vee q) \Leftrightarrow (p \wedge q)$$

Solution: Let $P = \sim (p \vee q)$ and $Q = (p \wedge q)$

$\therefore \quad \sim (p \vee q) \Leftrightarrow (p \wedge q)$

$\Rightarrow \quad P \Leftrightarrow Q$

$\Rightarrow \quad (P \wedge Q) \vee (\sim P \wedge \sim Q)$

$\Rightarrow \quad (- (p \vee q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge \sim (p \wedge q))$

$\Rightarrow \quad (- p \wedge \sim q) \wedge (p \wedge q) \vee (p \vee q) \wedge (\sim p \vee \sim q)$

[By Demorgan's laws]

$\Rightarrow \quad (\sim p \wedge \sim q \wedge p \wedge q) \vee ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge (- q))$

$\Rightarrow \quad (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge (\sim p)) \vee (q \wedge (- p)) \vee (p \wedge (- q)) \vee (q \wedge \sim q)$

This is DNF

Example 71: *Obtain disjunctive normal form of*

$$p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$$

Solution: $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim \vee r))) = p \vee (\sim p \rightarrow (-q \vee \sim r)))$

$$= p \vee (p \vee (q \vee (\sim p \vee \sim r)))$$

$$= p \vee p \vee q \vee \sim q \vee \sim r$$

$$= p \vee q \vee \sim q \vee \sim r$$

which is required disjunctive normal form.

9.10.5 Conjunctive Normal Form

Let A be a given formula, another formula B which is equivalent to A is called ***Conjunctive*** normal form if B is a product of an elementary sum.

Example 72: *Obtain conjunctive normal form:*

$$\sim (p \vee q) \leftrightarrow (p \wedge q)$$

Solution: Let $P = \sim (p \vee q)$ and $Q = (p \wedge q)$

Now, $\quad P \leftrightarrow Q$

$\Rightarrow \quad (P \rightarrow Q) \wedge (Q \rightarrow P)$

$\Rightarrow \quad \sim (p \vee q) \leftrightarrow (p \wedge q)$

$\Rightarrow \quad [\sim (p \vee q) \rightarrow (p \wedge q)] \wedge [(p \wedge q) \rightarrow \sim (p \vee q)]$

$\Rightarrow \quad [(p \vee q) \vee (p \wedge q)] \wedge [\sim (p \wedge q) \vee \sim (p \vee q)]$

$\Rightarrow \quad [(p \vee q) \vee (p \wedge q)] \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$ [By Demorgan's laws]

$\Rightarrow \quad [(p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$

which is required conjunctive normal form.

Example 73: *Obtain DNF of:*

(i) $(p \rightarrow q) \wedge (\sim p \wedge q)$ (ii) $(p \wedge (p \rightarrow q)) \rightarrow q$

Solution: (i) $p \rightarrow q$ is logical equivalent to $(\sim p \vee q)$

$\therefore$

$$(p \rightarrow q) \wedge (\sim p \wedge q)$$

$$= (\sim p \vee q) \wedge (\sim p \wedge q)$$

$$= (\sim p \vee \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$$

$$= (\sim p \wedge q) \vee (q \wedge \sim p)$$

(ii) $p \wedge (p \rightarrow q) \rightarrow q = \sim (\sim p \wedge \sim p \vee q) \vee q$

$$= \sim p \vee \sim (\sim p \vee q) \vee q$$

$$= \sim p \vee (p \wedge \sim q) \vee q$$

Example 74: *Obtain CNF and DNF of* $(p \wedge q) \wedge (q \rightarrow p)$

Solution: $(p \wedge q) \wedge (q \rightarrow p) = (\sim p \vee q) \wedge (\sim q \vee p)$

$$= [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$$

$$= (\sim p \wedge \sim q) \vee (q \wedge p) \qquad \textbf{This is DNF}$$

$$= [(\sim p \wedge \sim q) \wedge q]\, [(\sim p \wedge \sim q) \wedge p]$$

$$= (\sim p \wedge q) \wedge (b \wedge \sim q)$$

This is CNF.

9.11 Connectivity NOR AND NAND

9.11.1 NOR or joint denial (↓)

[U.P.T.U. (B.Tech.) 2005]

Let p and q be two statements.

The statement of the type $p \downarrow q$ read as ' Neither p nor q' is called joint denial or NOR statement.

Thus, $p \downarrow q = \sim (p \vee q)$

To make it more clear the joint denial statement, in the example of conditional, neither the father had promised to purchase a scooter nor the son has stood first in the examination.

It means that $p \downarrow q$ is true only when p and q both are false.

Truth Table for $p \downarrow q$

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Example 75: *Express the three connective* $\vee$, $\wedge$ *and* $\sim$ *in terms of the connective* $\downarrow$.

Solution: Truth tables for three connectives are as follows:

(i)

p	$\sim p$	$p \downarrow p$
T	F	F
F	T	T

$\therefore \quad \sim p = p \downarrow p.$

(ii)

p	q	$p \wedge q$	$p \downarrow p$	$q \downarrow q$	$(p \downarrow p) \downarrow (q \downarrow q)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	T	T	F

$\therefore$ $$p \wedge q = (p \downarrow p) \downarrow (q \downarrow q)$$

(iii)

p	q	$p \vee q$	$p \downarrow q$	$(p \downarrow q) \downarrow (p \downarrow q)$
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

$\therefore$ $$p \vee q = (p \downarrow q) \downarrow (p \downarrow q).$$

9.11.2 NAND (↑)

[U.P.T.U. (B.Tech.) 2005]

The connective NAND is denoted by symbol ↑ and is defined as "negation of AND of two statements".

If p and q are two statements then their NAND is denoted by $p \uparrow q$. Thus

$$p \uparrow q = \sim (p \wedge q)$$

Thus, $p \uparrow q$ is false only in the case when both p and q are true.

Truth table for $p \uparrow q$

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

(3) XOR (⊗)

The XOR of two statements is True if only if one of the two statements is true. In other words, if p and q are two statements then their XOR is denoted by $p \otimes q$ and is True if p is true or if q is true but not p and q both be true.

Truth Table for $p \otimes q$

p	q	$p \otimes q$
T	T	F
T	F	T
F	T	T
F	F	F

Remark: The connection $\downarrow$, $\uparrow$ and $\otimes$ are called ***Derived Connections.***

Example 76: *Construct the truth table for propositions $p \uparrow q \uparrow r$ and $p \otimes q \otimes r$.*

[Rohtak (B.E.) 2007, Kurukshetra (M.C.A.) 2005, 2009]

Solution:

p	q	r	$p \uparrow q$	$p \uparrow q \uparrow r$	$p \otimes q$	$p \otimes q \otimes r$
T	T	T	F	T	F	T
T	T	F	F	T	F	F
T	F	T	T	F	T	F
T	F	F	T	T	T	T
F	T	T	T	F	T	F
F	T	F	T	T	T	T
F	F	T	T	F	F	T
F	F	F	T	T	F	F

9.11.3 Argument

In logical mathematic, We require the process of reasoning. Given a certain set of propositions (*i.e.*, statements), we are required to derive other propositions by logical reasoning. The given set of propositions is called ***Premises*** (or ***Hypothesis***) and the proposition derived from this set is called ***Conclusion.***

OR

9.11.4 Argument

An argument is a process which yield a conclusion (*i.e.*, another proposition) from a given set of propositions, called premises. Let premises (*i.e.*, given set of propositions) be $P_1, P_2, \ldots, P_n$ and let argument yield the conclusion (*i.e.*, another proposition) q, then such an argument is denoted by

$$P_1, P_2 \cdots, P_n \vdash q$$

9.11.5 Valid Argument

An argument $p_1, p_2, \ldots, p_n \vdash q$ is called ***Valid*** if q is true whenever all its premises $p_1, p_2, \ldots, p_n$ are true. An argument is called ***Valid*** if and only if the premises implies the conclusion. Thus, the argument $p_1, p_2, \ldots, p_n \vdash q$ is said to be valid if and only if the statement

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q \text{ is tautology.}$$

9.11.6 Fallacy Argument

An argument which is not valid is said to be a fallacy or an invalid ***Argument***.

9.11.7 Representation of an Argument

An argument $p_1, p_2, \ldots, p_n \vdash q$ is written as

$$\begin{array}{c} \left.\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right\} \text{premises} \\ \hline q \text{ (Conclusion)} \end{array}$$

In the above representation premises are listed above the horizontal line and the conclusion below the horizontal line.

Example 77: *Show that the following argument is valid:*

$$\begin{array}{c} p \vee q \\ \sim p \\ \hline q \end{array}$$

Solution: Here the two premises are $p \vee q$ and $\sim p$, and the conclusion is q. The given argument will be valid if $[(p \vee q) \wedge \sim p] \rightarrow q$ is a tautology. Now constructing the truth table

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$[(p \vee q) \wedge \sim p] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Since all entries in the last column are of 'T' only, therefore $[(p \vee q) \wedge \sim p] \rightarrow q$ is a tautology. Hence the given argument is valid.

9.11.8 Law of Syllogism (or Transitive Rule)

We know that the statement $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology and therefore the argument

$$\begin{array}{ll} \left.\begin{array}{l} p \to q \\ q \to r \end{array}\right\} & \text{premises} \\ \hline p \to r & \text{(conclusion)} \end{array}$$

is a valid argument This is called *law of syllogism.*

9.11.9 Rule of Detachment

We know that the statement $[p \wedge (p \to q)] \to q$ is a tautology, and therefore the argument

$$\begin{array}{ll} \left.\begin{array}{l} p \\ p \to q \end{array}\right\} & \text{premises} \\ \hline q & \text{(conclusion)} \end{array}$$

is a valid argument. This is called ***Rule of Detachment.*** This rule is also known as ***Modus Ponens.***

The validity of a given argument can be checked by the help of a truth table and also without using the truth table.

Example 78: *Show that the argument $p,\ p \to q, q \to r \vdash r$ is valid.*

Solution:

First Method: Without using truth table.

Consider the following sequence of argument:

$$\begin{array}{ll} p & \text{(premise)} \\ p \to q & \text{(premise)} \\ \hline q & \text{(conclusion by rule of detachment)} \\ q \to r & \text{(premise)} \\ \hline r & \text{(conclusion by rule of detachment)} \end{array}$$

By the above sequence of valid arguments, it follows that the given argument is valid.

Second Method: By using truth table.

We shall construct truth table for the statement

$$[p \wedge (p \to q) \wedge (q \to r)] \to r$$

p	q	r	$p \to q$	$q \to r$	$p \wedge (p \to q)$	$p \wedge (p \to q) \wedge (q \to r) = f\,(say)$	$f \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	T

Since last column contains only T ' s, hence the given argument is valid.

Example 79: *Show that the argument*

$$p, p \to q \vdash q \text{ is valid}$$

Solution: Here p and $p \to q$ are two premises and q is the conclusion. To show that the given argument is valid, we shall show that $[p \wedge (p \to q)] \to q$ a tautology by construction truth table.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since last column contains only T 's, here the given argument is valid.

Example 80: *Test the validity of the following argument:*

If a man is a bachelor, he is worried (a premise)

If a man is worried, he dies young (a premise)

Bachelors die young (conclusion)

Solution: Let p : A man (he) is a bachelor, q : He is worried, r : He dies young.

The given argument in symbolic form can be written as

$$\begin{array}{ll} p \to q & \text{(a premise)} \\ q \to r & \text{(a premise)} \\ \hline p \to r & \text{(conclusion)} \end{array}$$

The given argument is true by law of syllogism.

Example 81: *Test the validity of the following argument: "If it rains then it will be cold. If it is cold then I shall stay at home. Since it rains therefore I shall stay at home".*

Solution: Let P : It rains, q : It will be cold; r : I shall stay at home.

The given argument in symbolic form can be written as

$$\begin{array}{ll} p \to q & \text{(a premise)} \\ q \to r & \text{(a premise)} \\ p & \text{(a premise)} \\ \hline r & \text{(conclusion)} \end{array}$$

We shall construct the truth table for the statement

$$[(p \to q) \wedge (q \to r) \wedge p] \to r.$$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r) \wedge p$	$[(p \to q) \wedge (q \to r) \wedge p] \to r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Since last column contains only T ' s, hence the given argument is valid.

Example 82: *Test the validity of the argument:*

If two sides of a triangle are equal , then the opposite angles are equal.

Two sides of a triangle are not equal.

$\therefore$ *The opposite angles are not equal*

Solution: The statement above the horizontal line are two premises. The statement below the horizontal line is the conclusion.

Let p : Two sides of a triangle are equal

q : The opposite angles of a triangle are equal.

The given argument in symbolic form can be written as

$$\begin{array}{ll} p \to q & \text{(a premise)} \\ \sim p & \text{(a premise)} \\ \hline \sim q & \text{(conclusion)} \end{array}$$

We shall construct the truth table for statement.

$$[(p \to q) \wedge \sim p] \to \sim q$$

p	q	$p \to q$	$\sim p$	$\sim q$	$(p \to q) \wedge \sim p$	$[(p \to q) \wedge \sim p] \to \sim q$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	F	T	T	T	T	T

The last column in the table shows that $[(p \to q) \wedge \sim p] \to - q$ is not a tautology. Hence the given argument is not valid.

Example 83: *State whether the argument given below is valid. If it is valid, identify the tautology or tautologies used:*

If I drive to work then I will arrive tired

I drive to work.

$\therefore$ *I will arrive tired.*

Solution: Let p : I drive to work, q : I will arrive tired.

$\therefore$ The given argument, in symbolic form, may be written as

$p \to q$ (a premise)

p (a premise)

$\therefore$ q (conclusion)

i.e., the argument is

$$p \to q, p \vdash q$$

which is valid.

Example 84: *State whether the argument given below is valid or not valid. If it is valid, identify the tautology used:*

I will become famous or I will be writer.

I will not be a writer.

$\therefore$ *I will become famous.*

Solution: Let p : I will become famous

q : I will be writer

The given argument, in symbolic form, may be written as

$$\begin{array}{ll} p \vee q & \text{(a premise)} \\ \sim q & \text{(a premise)} \\ \hline \therefore \quad p & \text{(conclusion)} \end{array}$$

The given argument i.e., $(p \vee q) \wedge (\sim q) \vdash p$ will be valid if the statement $[(p \vee q) \wedge (\sim q)] \to p$ is a tautology. Now we construct the truth table for the above statement.

p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge (\sim p)$	$[(p \vee q) \wedge (\sim q)] \to p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since last column contains only T's, *hence the given argument is valid.*

Example 85: *Test the validity of the argument: If* 8 *is even then* 2 *does not divide* 9. *Either* 7 *is not prime or* 2 *divides* 9. *But* 7 *is prime, therefore,* 8 *is odd.*

Solution: Let p : 8 is even; q : 2 divides 9; r : 7 is prime.

The given argument, in symbolic form, may be written as

$$\begin{array}{ll} p \to \sim q & \text{(a premise)} \\ \sim r \vee q & \text{(a premise)} \\ r & \text{(a premise)} \\ \hline \therefore \quad \sim p & \text{(conclusion)} \end{array}$$

The given argument will be valid if the statement $[(p \to \sim q) \wedge (\sim r \vee q) \wedge r] \to (\sim p)$ is a tautology. We construct the truth table:

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \to \sim q$	$\sim r \vee q$	$(p \to \sim q) \wedge (\sim r \vee q) \wedge r = f$ (say)	$f \to (\sim p)$
T	T	T	F	F	F	F	T	F	T
T	T	F	F	F	T	F	T	F	T
T	F	T	F	T	F	T	F	F	T
T	F	F	F	T	T	T	T	F	T
F	T	T	T	F	F	T	T	T	T
F	T	F	T	F	T	T	T	F	T
F	F	T	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T	F	T

Since last column only contains T's hence, the given argument is valid.

Example 86: *"If the labour market is perfect then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal therefore the labour market is not perfect". Test the validity of the argument.* **[R.G.P.V. (Bhopal) 2009]**

Solution: In the given case let

p_1 : "The labour market is perfect"

p_2 : "Wages fo all persons in a particular employment will be equal"

$\sim p_2$: Wages for such persons are not equal

$\sim p_1$: The labour market is not perfect.

The premise are $p_1 \Rightarrow p_2, \sim p_2$ and the conclusions is $\sim p_1$

The argument $[(p_1 \Rightarrow p_2), (\sim p_2)] \vdash \sim p_1$

valid if and only if $(p_1 \Rightarrow p_2) \wedge \sim p_2 \Rightarrow \sim p_1$ is a tautology.

We construct the truth tables as below:

p_1	p_2	$\sim p_1$	$\sim p_2$	$p_1 \Rightarrow p_2$	$(p_1 \Rightarrow p_2) \wedge \sim p_2$	$[(p_1 \Rightarrow p_2) \wedge \sim p_2] \Rightarrow \sim p_1$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

It follows that $p_1 \Rightarrow p_2 \wedge \sim p_2 \Rightarrow \sim p_1$ is tautology.

Hence, the argument is valid.

Note: Hence the conclusion is given by the symbol $\Rightarrow$. But in some example it is given by $\rightarrow$ which has same meaning.

Example 87: *Test the validity of the following argument.*

"If Ashok wins then Ram will be happy. If Kamal wins Raju will be happy. Either Ashok will win or Kamal will win. However if Ashok win, Raju will not be happy and if Kamal wins Ram will not be happy. So Ram will be happy if and only if Raju is not happy". **[Kurukshetra (B.E.) 2008]**

Solution: Here let

p_1 : Ashok wins $\quad$ p_2 : Ram is happy $\quad$ p_3 : Kamal wins $\quad$ p_4 : Raju is happy.

The premises are $p_5 : p_1 \Rightarrow p_2$, $p_6 : p_3 \Rightarrow p_4$, $p_7 : p_1 \vee p_3$, $p_8 : p_1 \Rightarrow \sim p_4$, $p_9 : p_3 \Rightarrow \sim p_2$.

The conclusion is $p_1 : p_2 \Leftrightarrow \sim p_4$

The above argument is valid $p_1 : p_2 \Leftrightarrow \sim p_4$ *is a tautology so we construct the truth table.*

1	2	3	4	5	6	7	8	9	10	11	12	13
P_1	P_2	P_3	P_4	$P_1 \Rightarrow P_2$	$P_3 \Rightarrow P_4$	$P_1 \vee P_3$	$-P_4$ or $-P_2$	$P_1 \Rightarrow -P_4$	$P_3 \Rightarrow -P_2$	$P_2 \Leftrightarrow -P_4$	$P_2 \Rightarrow P_2 \wedge P_3 \Rightarrow P_4 \wedge (P_1 \wedge P_2) \wedge P_1 \Rightarrow -P_4 \wedge {}'_3 \Rightarrow P_2$	(12) $\Rightarrow$ (11)
T	T	T	T	T	T	T	F	F	F	F	F	T
T	T	F	T	T	T	T	F	F	T	F	F	T
T	F	T	F	F	F	T	T	T	T	F	F	T
T	F	F	F	F	T	T	T	T	T	F	F	T
F	T	T	T	T	T	T	F	T	F	F	F	T
F	T	F	T	T	T	F	F	T	T	F	F	T
F	F	T	F	T	F	T	T	T	F	F	F	T
F	F	F	F	T	T	F	T	T	T	F	F	T

Since the given statement is a tautology. Hence, the argument is valid.

Example 88: *Show that the following argument is not valid:*

$$p$$

$$\neg q \vee r$$

$$\neg p \Rightarrow q$$

$$\overline{}$$

$$r$$

[Rohtak (B.E.) 2007]

Solution: Let $f = [\{p \wedge (\neg q \vee r)\} \wedge (\neg p \Rightarrow q)] \Rightarrow r$.

Construct the truth table for f

p	q	r	$\neg p$	$\neg q$	$\neg q \vee r$	$\neg p \Rightarrow q$	$p \wedge (\neg q \vee r) \wedge (\neg p \Rightarrow q)$	f
T	T	T	F	F	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	F	T	T	T	T	T
F	T	T	T	F	T	T	F	T
T	F	F	F	T	T	T	T	F
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	F	T
F	F	F	T	T	T	F	F	T

Since f column does not contain all the truth values T. Thus it is not a tautology. Hence, the given argument is not valid.

Example 89: *Check the validity of each of the following arguments:*

(a)
$$\begin{array}{l} p \wedge q \\ \neg p \Rightarrow q \\ \hline \neg q \end{array}$$

(b)
$$\begin{array}{l} r \\ p \Rightarrow \neg q \\ q \Rightarrow r \\ \hline p \end{array}$$

(c)
$$\begin{array}{l} p \wedge r \\ (p \Rightarrow q) \Rightarrow (r \Rightarrow s) \\ \hline s \end{array}$$

(d)
$$\begin{array}{l} [(p \Rightarrow q) \Rightarrow r] \Rightarrow s \\ r \\ \hline s \end{array}$$

Solution: (a) Let $f = [(p \wedge q) \wedge (\neg p \Rightarrow q)] \Rightarrow \neg q$

Now, construct the truth table for f.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \Rightarrow q$	$(p \vee q) \wedge (\neg p \Rightarrow \neg q)$	f
T	T	F	F	T	T	T	F
T	F	F	T	F	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	F	T

Since f column does not contain all T. Hence, the given argument is not valid.

(b) Let $f = [\{r \wedge (p \Rightarrow \neg q)\} \wedge (q \Rightarrow r)] \Rightarrow p$

Construct the truth table for f

p	q	r	$\rceil q$	$p \Leftrightarrow \rceil q$	$q \Rightarrow r$	$r \wedge (p \Leftrightarrow \rceil q) \wedge (q \Rightarrow r)$	f
T	T	T	F	F	T	F	T
T	T	F	F	F	F	F	T
T	F	T	T	T	T	F	T
F	T	T	F	T	T	T	T
T	F	F	T	T	T	T	F
F	T	F	F	T	F	F	T
F	F	T	T	F	T	F	T
F	F	F	T	F	T	F	T

Since f column contain all the truth values T. Thus, it is a tautology and hence the given argument is valid.

(c) Let $f = [(p \wedge r) \wedge [(p \Rightarrow q) \Rightarrow (r \Rightarrow s)]] \Rightarrow s$

Construct the truth table

p	q	r	s	$p \wedge r$	$p \Rightarrow q$	$r \Rightarrow s$	$(p \Rightarrow q) \Rightarrow (r \Rightarrow s)$	$(p \wedge r) \wedge [(p \Rightarrow q) \Rightarrow (r \Rightarrow s)]$	f
T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	F	F	F	T
T	T	F	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T	T	T
F	T	T	T	F	T	T	T	F	T
T	T	F	F	F	T	T	T	F	T
T	F	F	T	F	F	T	T	F	T
F	F	T	T	F	T	T	T	F	T
T	F	T	F	T	F	T	F	T	F
F	T	T	F	F	T	F	F	F	T
F	T	F	T	F	T	T	T	F	T
T	F	F	F	F	F	T	T	F	T
F	T	F	F	F	T	T	T	F	T
F	F	T	F	F	T	F	F	F	T
F	F	F	T	F	T	T	T	F	T
F	F	F	F	F	T	T	T	F	T

Since f column does not contain throughout T. Hence the given argument is invalid.

(d) $F = [[[(p \Rightarrow q) \Rightarrow r] \Rightarrow s] \wedge r] \Rightarrow r$

Construct the truth table for f

p	q	r	s	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow r$	$((p \Rightarrow q) \Rightarrow r) \Rightarrow s$	$[((p \Rightarrow q) \Rightarrow r) \Rightarrow s] \wedge r$	f
T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	F	F	T
T	T	F	T	T	F	T	F	T
T	F	T	T	F	T	T	T	T
F	T	T	T	T	F	T	T	T
T	T	F	F	T	T	T	F	T
T	F	F	T	F	T	T	F	T
F	F	T	T	T	T	T	T	T
T	F	T	F	F	T	F	F	T
F	T	T	F	T	T	F	F	T
F	T	F	T	T	F	T	F	T
T	F	F	F	F	T	F	F	T
F	T	F	F	T	F	T	F	T
F	F	T	F	T	T	F	F	T
F	F	F	T	T	F	F	F	T
F	F	F	F	T	F	T	F	T

The f column contains throughout T. Hence, the given statement is valid.

Example 90: *Check the validity of the following arguments. Show with the use of symbolic notation.*

$x^2 = y^2$ only if $x = y$

$x = y$

$x^2 = y^2$ **[U.P.T.U. (B.Tech.) 2005]**

Solution: Let $p \equiv x^2 - y^2$, $q \equiv x = y$, then

$p \Rightarrow q$

q

---

p

Let $f = [(p \Rightarrow q) \wedge q] \Rightarrow p$.

Construct the truth table for f

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge q$	f
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Since f column does not contain T throughout so it is not a tautology. Hence the given argument is invalid.

9.12 Law of Duality

In this section, we consider only those statements which contain the connectives $\wedge$, $\vee$ and $\sim$only. Two statements p and q are said to be dual of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F and F by T.

Example 91: *Find the dual of the following:*

(a) $(p \vee q) \wedge r$ (b) $(p \wedge q) \vee T$ (c) $\sim(p \vee q) \wedge (p \vee \sim(p \wedge s)$

Solution: (a) Interchanging $\vee$ and $\wedge$, we have, dual as $(p \wedge q) \vee r$

(b) Dual is $(p \vee q) \wedge F$ (c) Dual is $\sim (p \wedge q) \vee (p \wedge \sim (p \vee s))$

9.12.1 Principle of Duality

It states that if any two statements are equal then their dual are also equal.

Example 92: *Prove the following:*

(a) $\sim(p \wedge q) = \sim p \vee \sim q$ (b) $\sim(p \vee q) = \sim p \wedge \sim q$

Solution: We shall prove (a) result then by principle of duality (b) already proved. To prove (a), consider the following table:

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, it follows that $\sim(p \wedge q) = \sim p \vee \sim q$.

Examples 93: *Write the dual of the following*

(i) $(p \wedge T) \wedge (F \vee p') = F$ (ii) $p \wedge (p' \wedge q) = p \wedge q'$

(iii) $(p \vee q) \wedge r = p \wedge (p \wedge r)$ (iv) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Solution: The dual are

(i) $(p \vee F) \vee (T \wedge p') = T$ (ii) $p \vee (p' \vee q) = p \vee q'$

(iii) $(p \wedge q) \vee r = p \vee (q \vee r)$ (iv) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Answers

1. (a) $(p \wedge q) \vee r$ (b) $(p \vee q) \wedge F$

   (c) $\sim (p \wedge q) \vee (p \wedge \sim (p \vee s)$

4. (a) valid (b) valid (c) valid

5. valid

6. $p \rightarrow q = (\sim p \downarrow q) \downarrow (\sim p \downarrow q)$

7. True

9.13 Predicate or Propositional Function or Open Sentence

In the previous articles we have discussed the simple statements and the logical techniques to combine simple statements into compound statements. We cannot apply those techniques to the arguments of the following forms:

All human are mortal. Newton is human. Therefore, Newton is mortal.

The validity of such type of argument depends upon the inner logical structure of simple statements it contains.

The second premise of the above argument is a *singular proposition*; it states that the individual Newton has the characteristic of being a human. We say 'Newton' the *subject* term and 'human' the *predicate* term. The individuals are not necessarily persons, but may be any *things,* such as planets, stars, cities, animals etc. of which the characteristics may be predicated. Similarly the characteristic can be designated by not only adjectives but also by nouns, pronouns or verb.

In symbolic notation, we shall use small letters to denote the individuals such as the individual 'Newton' will be denoted by the small letter 'n' [the first letter of the name], and is called *individual constant.* The characteristic will be denoted by the capital letter such as 'mortal' by M and 'human' by H. The symbolic notation for 'Newton is mortal' will be used as $M(n)$ and for 'Newton is human' as $H(n)$. Some other examples of singular propositions with the same characteristic are 'Brijpal is human'. 'Lakshay is human' and their symbolic formulation are respectively '$H(b)$', '$H(l)$'. Let us use the symbolic representations $H(x)$ to denote the pattern common to all such singular propositions. The small letter 'x' is called '*individual variable*', and takes values from the set of individual constants. When an individual constant is substituted for x, a singular proposition, is produced, such as $H(n)$, $H(b)$, $H(l)$.

The singular propositions $H(n)$, $H(b)$, $H(l)$ are either true or false *i.e.*, they have their truth values but three is no truth value (true or false) or $H(x)$ since $H(x)$ is not a proposition. The expressions of the type $H(x)$ are called ***'Propositional Functions'*** or ***'Open Sentences'***. Thus propositional functions are expressions that contain individual variables and become propositions when their individual constants are replaced by individual variables.

Definition: Let A be a given set. An expression denoted by $P(x)$ is called a *propositional function* or, simply, an *open sentence* or, a *predicate* on A if $P(a)$ is true or false for each $a \in A$ *i.e.* $P(a)$ has a truth value for each $a \in A$. In other words, $P(x)$ is called a *propositional function* or, *an open sentence* if $P(x)$ becomes a statement whenever any element $a \in A$ is substituted for the variable x.

Illustration: *Let* $P(x)$ *be* '$x + 4 < 9$',

then $P(x)$ *is a propositional function on the set of natural numbers* **N.**

Clearly $P(x)$ *is true for* $n = 1, 2, 3, 4$ *and false for* $x = 5, 6, 7, \ldots$ *Hence* $P(x)$ *becomes a statement whenever any element* $a \in \mathbf{N}$ *is substituted for* x.

Illustration: *Let* $P(x)$ *be* '$x + 3 > 6$', *then* $P(x)$ *is a propositional function on the set of natural numbers* **N**.

Illustration: *Let* $P(x)$ *be* '$x + 3 > 6$', *then* $P(x)$ *is not a propositional function on the set of complex numbers* **C**, *because inequalities are not defined on* **C**.

9.13.1 Truth Set Definition

Let $P(x)$ be a propositional function and D be its domain. The set of elements $d \in D$ with the property that $P(d)$ is true is called the ***Truth Set*** $T(P)$ of $P(x)$. Symbolically:

$$T(P) = \{x : x \in D, P(x) \text{ is true}\} \text{ or } T(P) = \{x : P(x)\}.$$

Illustration: *If* '$x + 3 > 6$' *be a propositional function defined on* N, *then find the truth set of* $P(x)$.

We have $P(x) \equiv$ '$x + 3 > 6$', $D \equiv \mathbf{N}$.

$\therefore \quad T(P) = \{x : x + 3 > 6, x \in \mathbf{N}\} = \{4, 5, 6, \ldots\}$.

Illustration: *If* $P(x)$ *be* '$x + 2 < 3$' *and* $D \equiv \mathbf{N}$, *then find the truth set.*

$$T(P) = \{x : x + 2 < 3, x \in \mathbf{N}\} = \phi.$$

Illustration: *If* $P(x)$ *be* '$x < 5$' *and* $D \equiv \{5, 6, 7, 8, \ldots\}$, *then*

$$T(P) = \{x : x < 5, x \in D\} = \phi.$$

Illustration: *If* $P(x)$ *be* '$x > 5$' *and* $D \equiv \mathbf{N}$, *then*

$$T(P) = \{x : x > 5, n \in \mathbf{N}\} = \{6, 7, 8, \ldots\}.$$

Illustration: *If* $P(x)$ *be* '$x > 2$' *and* $D \equiv \{3, 4, 5, \ldots\}$, *then*

$T(P) = \{3, 6, 5, \ldots\} = D$ *i.e.*, $P(x)$ *is true for every* $x \in D$.

9.14 Quantifiers

The restrictions namely 'for every' and 'for some' are called ***Quantifiers.***

There are two types of quantifiers.

9.14.1 Universal Quantifier

Definition: *The symbol $\forall$ which is read as **'For Every'** or **'For All'** is called the universal quantifier.*

Let $P(x)$ be a propositional function defined on the set D. If 'for every $x \in D$, $P(x)$ is a true statement' then by the use of universal quantifier ($\forall$) it is written as

$$(\forall\ x \in D)\, P(x) \text{ or } \forall\ x\ P(x) \text{ or } \forall\ x,\ P(x). \qquad ...(1)$$

We see that the truth set of $P(x)$ is the entire set D,

i.e.,
$$T(P) = \{x : x \in D,\ P(x)\} = D \qquad ...(2)$$

Illustration: The statement $(\forall\ n \in \mathbf{N})\,(n + 2 > 1)$ is true since its truth set

$$T(P) = \{n : n \in \mathbf{N},\ n + 2 > 1\} = \{1, 2, 3, 4, \ldots\} = \mathbf{N}.$$

Illustration: *The statement $(\forall\ n \in \mathbf{N})\,(n + 1 > 5)$ is false since its truth set*

$$T(P) = \{5, 6, 7, \ldots\} \neq \mathbf{N}.$$

Illustration: *The statement $(\forall\ x \in \mathbf{R})\,(x^2 > -1)$ where $\mathbf{R}$ is the set of real numbers, is true since its truth set*

$$T(P) = \{x : x \in \mathbf{R},\ x^2 > -1\} = \mathbf{R}$$

Illustration: *Let D be the set of all men, then the statement 'All men are mortal' is written as*

$$(\forall\ x \in D) \qquad (x \text{ is Mortal}).$$

Illustration: *Let $\{A_t\}_{t \in T}$ be a family of sets, where T is the index set. The intersection of family of sets denoted by $\bigcap_{t \in T} A_t$ may also be written as (by the use of $\forall$) follows*

$$\bigcap_{t \in T} A_t = \{x : \forall\ t \in T,\ x \in A_t\}$$

Note: The truth set $T(P)$ is also called '**Universe of discourse**'.

9.14.2 Existential Quantifiers

Definition: *The symbol $\exists$ which is read as **'There Exists'** or **'For Some'** or **'For at Least One'** is called the **Existential Quantifier.***

Let $P(x)$ be a propositional function defined on the set D. If 'there exists an $x \in D$, such that $P(x)$ is a true statement' or 'for some $x \in D$, $P(x)$ is a true statement' or 'for at least one $x \in D$, $P(x)$ is a true statement' then it is written as

$$(\exists\, x \in D)\, P(x) \quad \text{or} \quad \exists\, x\, P(x). \qquad ...(1)$$

Hence, we find that the truth set (or universe of discourse) is not the empty set, *i.e.*,

$$T(P) = \{x : x \in D,\, P(x)\} \neq \phi.$$

Therefore, we may state that

(i) if $T(P) \neq \phi$ then $\exists\, x\, P(x)$ is true,

and (ii) if $T(P) = \phi$ then $\exists\, x\, P(x)$ is false.

Illustration: *The statement* $(\exists\, n \in \mathbf{N})\,(n + 7 < 6)$ *is false since*

$$T(P) = \{n : n + 7 < 6\} = \phi.$$

Illustration: *The statement* $(\exists\, x \in \mathbf{R})\,(x^2 - 1 = 0)$, $\mathbf{R}$ *is the set of real numbers, is true since*

$$T(P) = \{x : x^2 - 1 = 0\} = \{1, -1\} \neq \phi.$$

We see the variable x can take only two real values $1, -1$ for which $x^2 - 1 = 0$ is true.

Illustration: *Let* $\{A_t\}_{t \in T}$ *be a family of sets. The union of family of sets denoted by* $\bigcup_{t \in T} A_t$ *may also be written as (by the use of* $\exists$*) follows:*

$$\bigcup_{t \in T} A_t = \{x : \exists\, t \in T,\, x \in A_t\}.$$

9.15 Negation of A Quantifier

Let us consider the proposition. 'All Indians are honest'. If M denotes the set of Indians, then it can be is written as

$$(\forall\, x \in M) \quad (x \text{ is honest}).$$

This statement will become false, if we say that 'there is an Indian who is not honest' or in symbolic form

$$(\exists\, x \in M)\ (x \text{ is not-honest}).$$

Therefore, the negation of the statement 'All Indians are honest' will be 'there is an Indian who is not honest' or in symbolic form $(\exists\, x \in M)\,(x \text{ is not honest})$.

Thus the negation of $\forall\, x\, P(x)$ is $\exists\, x\,[\sim P(x)]$; where $P(x)$ denotes 'x' is honest'.

Hence, we see that the negation of a statement with 'universal quantifier' is a statement with 'existential quantifier'.

Again we consider the proposition. 'Some Indian are honest'.

If M denotes the set of Indians, then this statement in symbolic form can be written as

$$(\exists\, x \in M)\,(x \text{ is honest}).$$

This statement will become false, if we say that 'No India is honest' or in symbolic form

$$(\forall\, x \in M)\,(x \text{ is not honest}).$$

Therefore, the negation of the statement 'some Indians are honest' will be 'No Indian is honest'. In symbolic form, the negation of $(\exists\, x \in M)\, P(x)$ is $(\forall\, x \in M)(\sim P(x))$ where $P(x)$ denotes 'x is honest'.

Hence, we see that the negation of a statement with 'existential quantifier' is a statement with 'universal quantifier'.

9.16 De-Morgan Laws

If $P(x)$ is a propositional function defined on the domain D, then negation of $(\forall\, x \in D)\, P(x)$ is $(\exists\, x \in D) \sim P(x)$, which can be written as

$$\sim (\forall\, x \in D)\, P(x) \equiv (\exists\, x \in D) \sim P(x) \qquad \ldots(1)$$

Similarly $$\sim (\exists\, x \in D)\, P(x) \equiv (\forall\, x \in D) \sim P(x) \qquad \ldots(2)$$

(1) and (2) are called De-Morgan Laws.

Note: The laws which hold for propositions also hold for propositional functions, for example:

(i) $\sim (P(x) \vee Q(x)) \equiv \sim P(x) \wedge \sim Q(x)$.

(ii) $\sim (P(x) \wedge Q(x)) \equiv \sim P(x) \vee \sim Q(x)$ etc.

Example 94: *Use quantifiers to say that $\sqrt{3}$ is not a rational number.* **[U.P.T.U. (B.Tech.) 2008]**

Solution: $A(x)$: x is prime, $B(x)$: $\sqrt{x}$, x is prime and $C(x)$: x is rational

$\forall\, x,\, x \in A\,(x),\, B(x) \to \sim C(x)$ *i.e.*, square root of every prime number is not rational

Example 95: *Let $M(x)$ be "x is mammal". Let $A(x)$ be "x is an animal". Let $W(x)$ be "x is warm blooded". Translate into a formula; Every mammal is warm, blooded. Translate into English.*

$(\exists\, x)\,(A(x) \wedge (\sim M(x)))$ **[U.P.T.U. (B.Tech.) 2008]**

Solution: Formula for "every mammal is warn blooded" is $\forall\, x\,(M(x) \to W(x))$

$(\exists\, x)\,(A(x) \wedge (\sim M(x))$

Translated into English is there are some animals there are not mammals.

Example 96: *The proposition:*

(i) *There is a dog without a toil can be written as ($\exists$ a dog) (the dog without toil).*

(ii) *There is an integer between 2and 8 inclusive may be written as ($\exists$ an integer) (the integer is between 2and 8).*

Example 97: *Nagate the proposition "All integers are greater then 8".*

[Nagpur (B.E.) 2005; Pune (B.E.) 2004, 2007; M.K.U. (B.E.) 2008; Rohtak (M.C.A.) 2009]

Solution: The given proposition can be written as ($\forall$ integer x) $(x > 8)$

The negation is ($\exists$ an integer x) $(x \le 8)$ *i.e.*, There is an integer less than or equals to 8

Remark: In negation a proposition "for all" becomes "there is" and "there is" becomes "for all".

The symbol $\forall$ becomes $\exists$ and $\exists$ becomes $\forall$.

Example 98: Translate the following statements given in English into equivalent statements of Propositional/Predicate Calculus, after introducing appropriate symbolism.

(i) Some pet dogs are dangerous
(ii) Some physicists are not good in chemistry
(iii) Americans will stop driving big cars only if there are comfortable small cars.
(iv) Not all birds can fly
(v) Himadari, who is a doctor, is a good sports man also
(vi) Sum of two positive integers is greater than either of integers.
(vii) Some cats are black but all buffaloes are black
(viii) Some patients like all doctors
(ix) Some people have six fingers on one hand
(x) Some people help every body
(xi) Some mangoes are green
(xii) Some real numbers are integers
(xiii) Some mathematicians are not good in computer science
(xiv) All lions have tails
(xv) All cows are not white
(xvi) All frogs are brown
(xvii) All lions are dangerous animals
(xviii) All integers are either odd integers or even integers
(xix) All cows have four legs
(xx) All fish except shark are kind to children
(xxi) All men are mortal
(xxii) Any integer is either positive or negative
(xxiii) Every integer is also a real number
(xxiv) Nine plus ten equal nineteen
(xxv) Let A subset of B. Express this fact in predicate calculus.
(xxvi) Some irrational numbers x and y, the number x^y is a rational numbers
(xxvii) Couple's only surviving child is male

Solution: (i) Some Pets dogs are dangerous

Let $P(x)$: x is pet dog and $D(x)$: x is dangerous

∴ In predicate calculus we have

$$x\, P(x) \rightarrow D(x)$$

(ii) Some physicists are not good in chemistry

Let $P(x)$: x is a physicist and $C(x)$: x is good in chemistry

$$\exists\, x\, P(x) \rightarrow C(x)$$

(iii) Americans will stop driving big cars only if there are comfortable small cars.

Let $A(x)$: x is American will stop driving big cars and $C(x)$: x is comfortable s

In predicate calculus, we have

$$C(x) \to A(x)$$

(iv) Not all birds can fly.

Let $B(x)$: x is bird and $F(x)$: x is fly

In predicate calculus, we have

$$\sim (\forall\, B(x)) \to F(x)$$

(v) Himadari, who is a doctor, is a good sports person also

Let x : denote Himadari, $D(x)$: is doctor and $S(x)$: x is sport person

In predicate calculus, we have

$$(\exists\, x)\, D(x) \wedge S(x)$$

(vi) Sum of two positive integers is greater than either of the integers

Let $I(x)$: x is positive integer, $GT\,(x, y)$: x is greater then y and $Su\,(x, y)$: sum of x

Thus, in predicate calculus, we have

$$(\forall x)\,(\forall y)\,(I(x)\, I(y))\,(GT\ (Su\ (x, y)\, GTSu\ (x, y), y)))$$

(vii) Some cats are black but all buffaloes are black

Let $C(x)$: x is a cat, $B(x)$: x is a black, and $BF(x)$: x is buffalo

$\therefore$ $\exists\, x \in C(x) \Rightarrow B(x)$

and $\forall x\ BF(x) \Rightarrow B(x)$

(viii) Some patients like all doctors

Let $P(x)$: x is patient, and $D(y)$: y is doctor, (x, y): x likes by y

$\therefore$ $x\ P(x)\ y$

(ix) Some people have six fingers on one hand

Let $P(x)$: x is people, $S(x)$: x is a people having six fingers on one hand

Therefore in predicate calculus, we have

$$\exists\, x \in P(x) \to S(x)$$

(x) Some people help everybody

Let $P(x)$: x is a people, $HB(y)$: y is human being, and $H(x, y)$: x helps y

$\therefore$ The given statement can be expressed as

$$\exists\, x \in P(x) \to H(x, y)\ \forall\ x \in (P(x) \wedge HB(y))$$

or $\exists\, x \in P(x) \to\ \forall\ y \in (P(x) \wedge HB(y)) \to H(x, y)$

(xi) Some mangoes are green

Let $M(x)$: x are mangoes and $G(x)$: x are green

$\therefore$ In predicate calculus, we have

$$\exists\, x\ (M(x) \to G(x))$$

(xii) Some real numbers are integers

Let $R(y)$: y is real number and $I(y)$: y is an integer

∴ In predicate calculus, we have

$$\exists y \in R(y) \rightarrow I(y)$$

(xiii) Some mathematicians are not good in computer science

Let $M(x)$: x is a mathematician and $G(x)$: x is good in computer science

∴ The given statement can be expressed in predicate calculus as

$$\exists x \in M(x) \rightarrow \sim G(x)$$

(xiv) All lions have tails

Let $L(x)$: x is lions and $T(x)$: x is tails

In predicate calculus, we have

$$(\forall x)(L(x) \rightarrow T(x))$$

(xv) All cows are not white

Let $C(x)$: x is cow and $W(x)$: x is white

∴ In predicate calculus, we have

$$\forall x\, C(x) \rightarrow \sim W(x)$$

(xvi) All frogs are brown

Let $F(x)$: x is frogs, $B(x)$: x is an brown frog

∴ In predicate calculus, we have

$$\forall x \in F(x) \rightarrow B(x)$$

(xvii) All lions are dangerous animals

Let $L(x)$: x is a lion, $D(x)$: x is a dangerous animal

∴ In predicate calculus, we have

$$\forall x \in L(x) \rightarrow D(x)$$

(xviii) All integers are either odd integers or even integers

Let $I(x)$: x are integers, $O(x)$: x is odd, $E(x)$: x is even

∴ In predicate calculus, we have

$$\forall x\, I(x) \rightarrow \sim (O(x) \vee E(x))$$

(xix) All cows have four legs

Let $C(x)$: x is cow, $F(x)$: x has four legs

∴ In predicate calculus, we have

$$\forall x\, (C(x) \rightarrow F(x))$$

(xx) All fish except shark are kind to children

Let $F(x)$: x is a fish, $S(x)$: x is a shark, $K(x)$: x is kind to child

∴ In predicate calculus, we have

$$\forall x\, (F(x) \neq S(x)) \rightarrow K(x)$$

(xxi) All men are mortal

Let $H(x)$: x is man, $M(x)$: x is mortal

$\therefore$ In predicate calculus, we have

$$(\forall x)(C(x) \rightarrow M(x))$$

(xxii) Any integer is either positive or negative

Let $I(x)$: x is an integer, $P(x)$: x is positive integer

$\therefore$ In predicate calculus, we have

$$\forall x \in I(x) \rightarrow \{P(x) \vee N(x)\}$$

(xxiii) Every integer is also a real number

Let $I(x)$: x is an integer

$R(x)$: x is real number

$\therefore$ In predicate calculus, we have

$$\exists x \in I(x) \rightarrow R(x)$$

(xxiv) Nine plus ten equal nineteen

Let $N(x)$: x is nine, $T(x)$: x is ten, $C(x, y)$: sum of x and y

$NT(x)$: is nineteen

$\therefore$ In predicate calculus, we have $xy\,(N(x)T(y)\,\text{sum}\,(x, y)\,NT(x)$

(xxv) Let A be subset of a set B. Express this fact in predicate calculus.

Let $A(x)$: x is an element of A, $B(x)$: x is an element of B

$\therefore$ In predicate calculus, we have

$$\forall x \in A(x) \wedge B(x)$$

(xxvi) Let $I(x)$: x is an irrational number, $Q(x)$: x is an rational number, $EX\,P(x, y)$: x^y

$\therefore$ In predicate calculus, we have

$$(\exists x, y)[I(x) \wedge I(y) \rightarrow Q\,(EX\,P(x, y)]$$

(xxvii) Couple's only surviving child is a male

Let $SC(x)$: x is couple's surviving child, $M(x)$: x is male

$\therefore$ In predicate calculus, we have

$$\exists x \in SC(x) \rightarrow M(x)$$

Example 99: *Let $Q(x) \equiv x$ is a rational number, $R(x) \equiv x$ is a real number*

$$E(x, y) \equiv \text{'}x = y\text{'},\ G(x, y) \equiv \text{'}x > y\text{'}.$$

Translate the following sentences into symbols:

(i) *π is a real number.*

(ii) *e is a real number.*

(iii) *4/5 is a rational number.*

(iv) *$\sqrt{3}$ is an irrational number.*

(v) *Every rational number is a real number.*

(vi) *Some real numbers are rational.*

(vii) *The square of every real number is not nega*

Solution: (i) $R(\pi)$,
(ii) $R(e)$,
(iii) $Q(4/5)$,
(iv) $\sim Q(\sqrt{3})$,
(v) $\forall x\,[Q(x) \Rightarrow R(x)]$,
(vi) $\exists x\,[R(x) \wedge Q(x)]$,
(vii) $\forall x\, R(x) \Rightarrow \sim G(0, x^2)]$.

Note that $G(0, x^2)$ means $0 > x^2$ *i.e.,* x^2 is negative and therefore $\sim G(0, x^2)$ means x^2 is not negative.

Example 100: *Negate each of the following statements:*

(i) $\forall x\,(|x| = x)$;
(ii) $\forall x\,(x + 1 > x)$,
(iii) $\forall x\,(x \neq 1, x \neq 2)$,
(iv) $\exists x\,(x^2 < 0)$,
(v) $\forall x\,(x \neq 0) \Rightarrow (x^2 > 0)$,
(vi) $\exists x\,|x| = 0)$,
(vii) $\exists x\,(x^2 = x)$,
(viii) $\exists x\,(x + 2 = x)$,
(ix) $\exists x\,(x^2 = 1$ *and* $x^2 - 2x + 3 = 0)$,
(x) *If there is a will there is a way,*
(xi) *All men are mortal.*
(xii) *All integer are greater than zero.*

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Take the universal set as set of real numbers from (i) to (ix).

Solution: (i) $\exists x \sim (|x| = x)$ or $\exists x\,(|x| \neq x)$.

(ii) $\exists x \sim (x + 1 > x)$ or $\exists x\,(x + 1 \leq x)$.

(iii) $\exists x\,((x-1)(x-2) = 0)$ or $\exists x\,(x^2 - 3x + 2 = 0)$.

(iv) $\forall x \sim (x^2 < 0)$ or $\forall x\,(x^2 \geq 0)$.

(v) $\exists x\,[(x \neq 0) \wedge (x^2 < 0)]$.

(vi) $\forall x \sim (|x| = 0)$ or $\forall x\,(|x| \neq 0)$.

(vii) $\forall x \sim (x^2 = x)$ or $\forall x\,(x^2 \neq x)$.

(viii) $\forall x\,(x + 2 \neq x)$.

(ix) $\forall x \sim (x^2 = 1$ and $x^2 - 2x + 3 = 0)$.

or $\forall x\,(x^2 \neq 1$ or $x^2 - 2x + 3 \neq 0)$.

(x) We know that $\sim (p \Rightarrow q) \equiv p \wedge \sim q$. Thus required negation is:
There is a will and there is no way.

(xi) Some men are not mortal.

(xii) Let $A(x)$ denote "x is an integer"
and $(B)x$ denoted "x is greater than 0"
Then given proposition can be written as $(\exists x \in A(x)(-B(x)))$

Example 101: *Let* $D = \{1, 2, 3, 4, 5, 6 \ldots\}$, *negate the following statements:*

(i) $(\forall x \in D)(x + 4 \geq 8)$,
(ii) $(\forall x \in D)(x + 2 < 9)$,
(iii) $(\exists x \in D)(x + 5 = 9)$,
(iv) $(\exists x \in D)(x + 1 > 6)$,
(v) $(\exists x \in D)(x^2 = 6)$,
(vi) $(\forall x \in D)(x^2 > 25)$,

Solution: (i) $(\exists x \in D)(x + 4 < 8)$,
(ii) $(\exists x \in D)(x + 2 \geq 9)$,
(iii) $(\forall x \in D)(x + 5 \neq 9)$,
(iv) $(\forall x \in D)(x + 1 \leq 6)$,
(v) $(\forall x \in D)(x^2 \neq 6)$,
(vi) $(\exists x \in D)(x^2 \leq 25)$,

Example 102: *Negate the statement 'He is poor and laborious'.*

Solution: We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Hence the negation of the given statement is:

'It is false that he is poor and laborious'. ≡ 'He is not poor or he is not laborious'.

Example 103: *Negate the statement 'It is daylight and all the people have arisen'.*

Solution: We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Hence the negation of the given statement is:

'It is false that it is daylight and all the people have arisen'.

≡ 'It not daylight or it is false that all the people have arisen'.

≡ 'It is not daylight or some one has not arisen'.

Example 104: *Negate the statements:*

(i) $\exists\, x\, P(x) \vee \forall\, y\, Q(y)$.   (ii) $\forall\, x\, P(x) \wedge \exists\, y\, Q(y)$.

Solution: (i) We know that $\sim(p \vee q) = \sim p \wedge \sim q$.

$\therefore \quad \sim[\exists\, x\, P(x) \vee \forall\, y\, Q(y)] \equiv \sim\exists\, x\, P(x) \wedge \sim\forall\, y\, Q(y) \equiv \forall\, x \sim P(x) \wedge \exists\, y \sim Q(y)$.

$\because \quad \sim\exists\, x\, P(x) \equiv \forall\, x \sim P(x)$ and $\sim\forall\, y\, Q(y) \equiv \exists\, y \sim Q(y)$

(ii) We know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

$\therefore \quad \sim[\forall\, x\, P(x) \wedge \exists\, y\, Q(y)] \equiv \sim\forall\, x\, P(x) \vee \sim\exists\, y\, Q(y) \equiv \exists\, x \sim P(x) \vee \forall\, y \sim Q(y)$.

Example 105: *Write the three properties of equivalence relation in symbolic notations.*

Solution: Let $R(x, y)$ denote that x has a relation R with y, then the three properties for R to be equivalence relation are:

(i) ***Reflexivity:*** $\forall\, x\, R(x, x)$.

(ii) ***Symmetric:*** $\forall\, x, y\, [R(x, y) \Rightarrow R(y, x)]$.

(iii) ***Transitivity:*** $\forall\, x, y, z\, [R(x, y) \wedge R(y, z) \Rightarrow R(x, y)]$.

Example 106: *Negate the statement* $\exists\, x\, \forall\, y\, [P(x) \vee \sim Q(y)]$.

Solution: $\sim\exists\, x\, \forall\, y\, [P(x) \vee Q(y)] \equiv \forall\, x\, \exists\, y\, [\sim P(x) \wedge Q(y)]$.

Example 107: *Negate the statement* $\forall\, x\, \exists\, y\, [P(x, y) \Rightarrow Q(y)]$.

Solution: $\sim\forall\, x\, \exists\, y\, [P(x, y) \Rightarrow Q(y)] \equiv \exists\, x\, \forall\, y\, [P(x, y) \wedge -Q(y)]$ $[\because \quad \sim(p \Rightarrow q) \equiv p \wedge \sim q]$.

Example 108: *Write the following statement into symbols*

(using quantifier and the symbol '<' for 'less than'):

(i) *A number x, is less than 7 and greater than 5.*

(ii) *For a given number x there is a greater number y.*

(iii) *For two given numbers x and y, there is a number z such that the difference of x and y is less than the product of x^2 and z.*

(iv) *The numbers x, y, z are such that x + y is greater than xz.*

Solution:

(i) $\exists x [x < 7 \wedge 5 < x]$.

(ii) $\forall x \exists y (x < y)$.

(iii) $\forall x \forall y \exists z (|x - y| < x^2 z)$.

(iv) $(\exists x)(\exists y)(\exists z)(xz < x + y)$.

Example 109: *Write the following sentences into symbols:*

(i) *The square of any rational number is not 2.*

(ii) *Two non-parallel coplanar straight lines have a common point.*

(iii) *If there is no prize, then a person does not purchase a ticket.*

Solution: (i) Let $Q(x) = x$ is a rational number, $E(x, y) \equiv$ '$x = y$'.

The given sentence in symbols, using quantifiers, is

$$\exists x [Q(x) \wedge E(x^2, 2)].$$

(ii) Let $P(x) = x$ is a point, $L(x) = x$ is a straight line, $I(x, y) =$ the point x is on y, $D(x, y) = x, y$ are non-parallel and coplanar.

The given sentence in symbols is

$$\forall mn [L(m) \wedge L(n) \wedge D(m n) \Rightarrow \exists x \{P(x) \wedge I(x, m) \wedge I(x, n)\}].$$

(iii) Let $P(x) =$ a prize, $T(x) =$ a ticket, $M(x) =$ a person, $A(x, y) = x$ purchases y.

The given sentence is symbols is:

$$\{\sim \exists x P(x)\} \Rightarrow \forall x \forall y \{M(x) \wedge T(y) \Rightarrow \sim A(x, y)\}.$$

Example 110: *'If the product of two rational numbers is zero, then at least one factor is zero'. Write this statement into symbols.*

Solution: Let $P(x) \equiv$ product of rational numbers is x and $F(x, y) \equiv x$ in a factor of y.

The given sentence is symbols is

$$\forall x [P(x) \wedge (x = 0) \Rightarrow \exists y [F(x, y) \Rightarrow (y = 0)\}].$$

Example 111: *Show that the negation of $(\forall x)(P(x))$ is $(\exists x)(\sim P(x))$.*

Or

If the projection of $P(x)$ on D is a propositional function, then show that

$$\sim (\forall x \in D : P(x)) \equiv (\exists x \in D : \sim P(x)).$$

Solution: Here following two cases arise:

Case (i): *The truth value of $(\forall x \in D : P(x))$ is T. In this case $P(x)$ is satisfied for all $x \in D$. Therefore, there is an $x \in D$ for which $\sim P(x)$ is not satisfied.*

So the truth value of $(\exists x \in D : \sim P(x))$ is F.

Case (ii): *The truth value of $(\forall x \in D : P(x))$ is F. In this case $P(x)$ is not satisfied for some $x \in D$, therefore, $\sim P(x)$ is satisfied for some $x \in D$.*

So the truth value of $(\exists x \in D : \sim P(x))$ is T. Hence

$$\sim (\forall x \in D : P(x)) \equiv (\exists x \in D : \sim P(x)).$$

Example 112: *Write the following predicate into symbolic language:*

(i) *For every real number there is a greater real number.*

(ii) *Every irrational number is a real number.*

(iii) *The number divisible by an even number is even.*

(iv) *Every teacher of a college is learned.*

(v) *All students are not wise.*

(vi) *Some people like to listen only instrumental music.*

(vii) *If the product of a finite number of numbers is zero, then at least one factor is zero.*

(viii) *The function $f(x)$ is continuous at $x = a$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(a)| < \varepsilon \quad \text{where} \quad |x - a| < \delta.$$

Solution: (i) Let $R(x) \equiv x$ is a real number, $G(x, y) \equiv x > y$.

Then the given statement in symbols is

$$\forall\, x\,[R(x) \Rightarrow \exists\, y\,[\{R(y) \wedge G(y, x)\}].$$

(ii) Let $I(x) \equiv x$ is irrational number, $R(x) \equiv x$ is real number.

Then $\quad \forall\, x\,[I(x) \Rightarrow R(x)]$.

(iii) $R(x) \equiv x$ is a number, $E(x) \equiv x$ is an even number

$D(x, y) \equiv y$ is divisible by x.

Then $\quad \forall\, x, y\,[E(x) \wedge R(y)] \wedge D(x, y) \Rightarrow E(y)]$.

(iv) $T(x) \equiv x$ is a teacher of college, $L(x) \equiv x$ is learned.

$\therefore \quad \forall\, x\,[T(x) \Rightarrow L(x)]$.

(v) $S(x) \equiv x$ is student and $W(x) \equiv x$ is wise.

$\therefore \quad \sim \forall\, x\,[S(x) \wedge W(x)]$.

(vi) $P(x) \equiv x$ is people, $I(x) \equiv x$ is instrumental music, $H(x, y) \equiv x$ likes to listen y.

$\therefore \quad \exists\, x\,[P(x) \wedge H(x, y) \Rightarrow Q(y)]$.

(vii) $P(x) \equiv$ Product of finite number of members

$F(x, y) \equiv x$ is a factor of y.

$\therefore \quad \forall\, x\,[P(x) \wedge (x = 0) \Rightarrow \exists\, y\,\{F(y, x) \Rightarrow (y = 0)\}]$.

(viii) $F(x) \equiv$ The function f is continuous at x.

$\therefore \quad [\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, x\,\{|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\}] \Rightarrow f(a)$.

Example 113: *Write the following predicate into symbolic form:*

(i) *There are real numbers which are greater than all real numbers.*

(ii) *Some ladies lawyer are such that they are also homely ladies.*

(iii) *Some ladies are lawyer and also homely ladies.*

(iv) *Every student is a member of N.C.C. or N.S.S., but some students are not players.*

(v) *If a and b are non-zero integers and p is a prime number such that $p \mid ab$ then $p \mid a$ or $p \mid b$.*

Solution: (i) Let $R(x) \equiv x$ is real number

$G(x, y) \equiv x > y.$

Then the given predicate in symbolic form is

$$\forall x[R(x) \Rightarrow \exists y\{R(y) \wedge G(x, y)\}].$$

(ii) $W(x) \equiv x$ is lady, $L(x) \equiv x$ is lawyer, $H(x) \equiv x$ is homely lady.

Then $\exists x[W(x) \wedge L(x) \wedge H(x)].$

(iii) Same as (ii).

(iv) $S(x) \equiv x$ is student. $N(x) \equiv x$ is a member of N.C.C. $M(x) \equiv x$ is a member of N.S.S. $P(x) \equiv x$ is a player.

Then $\forall x[S(x) \Rightarrow \{N(x) \wedge M(x)\}] \wedge \exists y[S(y) \wedge \sim P(y)].$

(v) $I(x) \equiv x$ is a non-zero integer, $P(x) \equiv x$ is prime number, $D(x, y) \equiv x$ is divisible y.

Then $\forall a, b, p[I(a) \wedge I(b) \wedge P(p) \wedge D(p, ab)] \Rightarrow [D(p, a) \vee D(p, b)].$

Example 114: *Write the following predicates into symbolic form:*

(i) *All are not mortal.* (ii) *All are immortal.*

(iii) *For all real numbers x, y, z addition is an associative operation.*

Solution: Let x denote any person and $M(x) \equiv x$ is Mortal. Then

(i) $\sim \forall x[M(x)].$ (ii) $\forall x \Rightarrow [\sim M(x)].$

(iii) Let x, y, z be real numbers. Then $\forall x, y, z[(x+y)+z = x+(y+z)].$

Example 115: *Write the following predicate into symbols and also write its negative in symbols:*

"Every rational number is a real number".

Solution: Let $Q(x) \equiv x$ is rational number, $R(x) \equiv x$ is real number, and $P(x) \equiv Q(x) \Rightarrow R(x)$.

Then symbolic form of given predicate is $\forall x\, P(x)$.

Negative: $\exists x[\sim P(x)]$ i.e., $\exists x[\sim Q(x) \Rightarrow R(x)].$

Exercise

1. Show that negative of $\exists x(\sim P(x))$ is $\forall x(P(x))$.

   [**Hint:** $\sim \exists x(\sim P(x)) = \forall x(\sim\sim P(x)) = \forall x\, P(x)$].
2. Write the empty set ϕ and universal set U by the use of quantifier.
3. Show that $\sim \forall x\, P(x) \equiv \exists x(\sim P(x))$.
4. Define quantifiers, universal quantifiers and existential quantifiers by giving an example.
5. Define existential and universal quantifiers.
6. Which are existential and universal quantifiers? Write three-three statements using each.
7. Explain the following terms and also give examples to explain them:

   (a) Quantifier. (b) Universal quantifier.

   (c) Existential quantifier. (d) Negation of a quantifier.