* Review of Elementary Differential calculus :-

Let $y=f(x)$ be a continuous explicit function of x. Any changes in x results in a corresponding changes in y.

Let x change to $x+\delta x$ and let the corresponding change in y be $y+\delta y$. δx, δy are called the increments in x & y respectively.

$\therefore$ We have $y=f(x)$ and $y+\delta y=f(x+\delta x)$

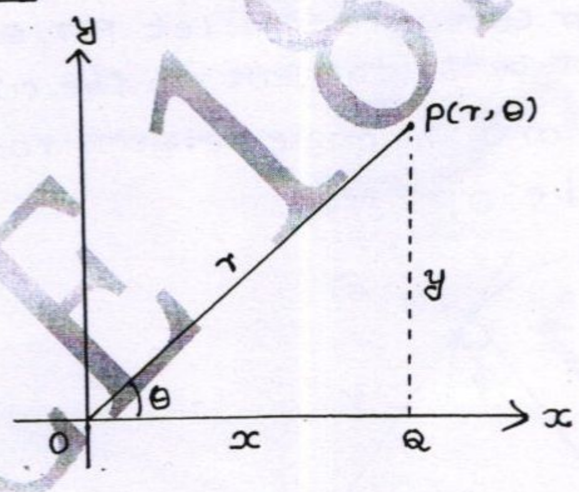
Let us consider $y+\delta y-y=f(x+\delta x)-f(x)$

$$\Rightarrow \delta y = f(x+\delta x)-f(x)$$

Hence $\lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim\limits_{\delta x\to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\}$ when exists is called as the differential coefficient or the derivative of y wrto x and is denoted by $\frac{dy}{dx}$ or $f'(x)$. or $y'(x)$

Here $\frac{d}{dx}$ is called the differential operator denoted by D.

* Polar curves:-



Consider a point P in the xoy-plane. Join the points O (origin) and P. Let r be the length of the line segment OP and θ be the angle which OP makes with the x-axis. Then (r,θ) is called Polar coordinates of the point P. and we write $P=(r,\theta)$ or $P(r,\theta)$.
In particular r is called the radial distance and θ is called the polar angle. Also, O is called the pole, the x-axis is called the initial line and the line segment OP is called the radius vector.

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Let (x, y) be the cartesian coordinates of the point P.

From the figure we have $OQ = x$, $PQ = y$. Also from the right angled triangle OQP, we have,

$$\cos\theta = \frac{OQ}{OP} = \frac{x}{r} \Rightarrow x = r\cos\theta \rightarrow (1)$$

$$\sin\theta = \frac{PQ}{OP} = \frac{y}{r} \Rightarrow y = r\sin\theta \rightarrow (2)$$

Squaring and adding (1) & (2)

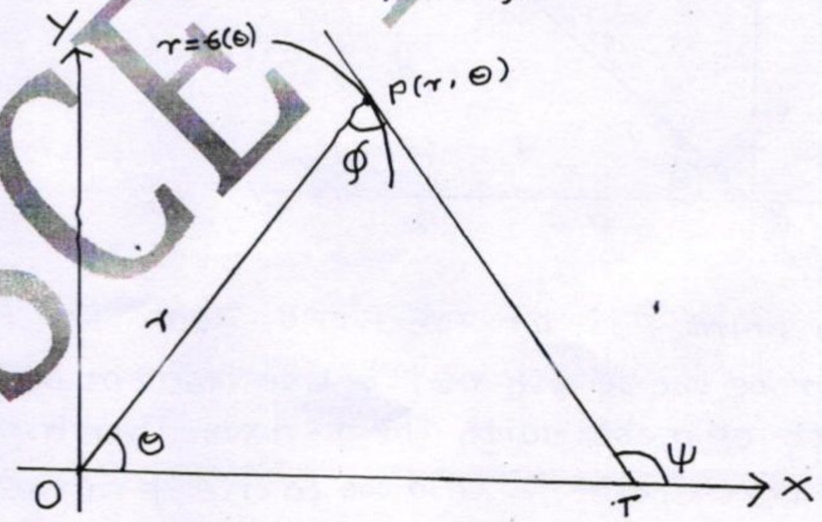
$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta \Rightarrow r = \sqrt{x^2+y^2} \quad \because \cos^2\theta+\sin^2\theta = 1$$

Also dividing (2) by (1),

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

* **Angle between radius vector and tangent :-**

Consider a polar curve $r = f(\theta)$. Let $P(r, \theta)$ be a point on the curve. Let PT be the tangent to the curve at P.
Let $P\hat{T}X = \psi$ and the angle between radius vector and tangent be ϕ. i.e $O\hat{P}T = \phi$



From the fig we have $\psi = \theta + \phi$

$\because$ An exterior angle is equal to the sum of the interior opposite angle.

$\Rightarrow \tan\psi = \tan(\theta+\phi)$

$$\Rightarrow \tan\psi = \frac{\tan\theta + \tan\phi}{1-\tan\theta\cdot\tan\phi} \longrightarrow (1)$$

Let (x, y) be the cartesian coordinates of P so that

$$x = r\cos\theta,\ y = r\sin\theta \longrightarrow (2)$$

slope of PT is given by $\tan\psi = \frac{dy}{dx}$

Given $r = f(\theta)$, noting that differentiate (2) w.r.to θ

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \& \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\text{Now } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{\frac{dr}{d\theta}\cancel{\cos\theta}\left[\tan\theta + r\cdot\frac{d\theta}{dr}\right]}{\frac{dr}{d\theta}\cancel{\cos\theta}\left[1 - r\frac{d\theta}{dr}\tan\theta\right]}$$

$$\Rightarrow \tan\psi = \frac{\tan\theta + r\cdot\frac{d\theta}{dr}}{1-\tan\theta\cdot r\frac{d\theta}{dr}} \longrightarrow (3)$$

Comparing the equations (1) & (3)

$$\Rightarrow \tan\phi = r\frac{d\theta}{dr} \text{ (or) } \cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

is the expression for angle between radius vector & tangent

* problems

- Find the angle between radius vector and tangent for the following curves

1) $r = a(1-\cos\theta)$

→ Given $r = a(1-\cos\theta)$

taking $\log_e$ on both sides

$$\Rightarrow \log_e r = \log_e[a(1-\cos\theta)]$$

$$\Rightarrow \log_e r = \log_e a + \log_e(1-\cos\theta)$$

Differentiating w.r.to θ

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}(\sin\theta)$$

$$\Rightarrow \cot\phi = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} \Rightarrow \cot\phi = \cot\theta/2$$

$$\Rightarrow \phi = \theta/2$$

2) $r^2\cos 2\theta = a^2$

→ Given $r^2\cos 2\theta = a^2$

taking $\log_e$ on both sides

$$\Rightarrow \log_e(r^2\cos 2\theta) = \log a^2$$

$$\Rightarrow 2\log_e r + \log_e\cos 2\theta = 2\log_e a$$

Differentiating w.r to θ

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta}(-2\sin 2\theta) = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} - \frac{\sin 2\theta}{\cos 2\theta} = 0$$

$$\Rightarrow \cot\phi = \tan 2\theta$$

$$\Rightarrow \cot\phi = \cot(\pi/2 - 2\theta) \quad \because \tan\theta = \cot(\pi/2-\theta)$$

$$\Rightarrow \phi = \frac{\pi}{2} - 2\theta$$

3) $r^m = a^m (\cos m\theta + \sin m\theta)$

→ Given $r^m = a^m(\cos m\theta + \sin m\theta)$

taking $\log_e$ on both sides

$$\log_e r^m = \log_e\{a^m(\cos m\theta + \sin m\theta)\}$$

$$m \log_e r = m\log_e a + \log(\cos m\theta + \sin m\theta)$$

Differentiating w.r.to θ

$$\Rightarrow \frac{m}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta + \sin m\theta}[-m\sin m\theta + m\cos m\theta]$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{[\cos m\theta - \sin m\theta]}{[\cos m\theta + \sin m\theta]} = \frac{\cos m\theta[1-\tan m\theta]}{\cos m\theta[1+\tan m\theta]}$$

$$\Rightarrow \cot\phi = \frac{1-\tan m\theta}{1+\tan m\theta}$$

$$\Rightarrow \tan\phi = \frac{1+\tan m\theta}{1-(1\times\tan m\theta)} \quad \& \tan\frac{\pi}{4} = 1$$

$$\Rightarrow \tan\phi = \frac{\tan\frac{\pi}{4} + \tan m\theta}{1-\tan\pi/4\cdot\tan m\theta}$$

$$\Rightarrow \tan\phi = \tan(\pi/4 + m\theta)$$

$$\Rightarrow \phi = \frac{\pi}{4} + m\theta$$

4) $r = a\sin^3(\theta/3)$

→ we have $r = a\sin^3(\theta/3)$

taking Logarithms on both sides

$$\log r = \log a + 3\log(\sin(\theta/3))$$

Differentiate w.r.to θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 3\cdot\frac{1}{\sin(\theta/3)}\cdot\cos(\theta/3)\cdot(1/3)$$

$\cot\phi = \cot(\theta/3)$

$\Rightarrow \boxed{\phi = \theta/3}$

5] $\frac{1}{r} = 1 + e\cos\theta$

→ We have $\frac{1}{r} = 1 + e\cos\theta$

taking Logarithms on both sides we have

$$\log\left(\frac{1}{r}\right) = \log(1 + e\cos\theta)$$

$$\Rightarrow \log 1 - \log r = \log(1 + e\cos\theta)$$

Differentiating w.r.t. θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1 + e\cos\theta}$$

$$\cot\phi = \frac{e\sin\theta}{1 + e\cos\theta}$$

$$\tan\phi = \frac{1 + e\cos\theta}{e\sin\theta}$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{1 + e\cos\theta}{e\sin\theta}\right)$$

- Find the angle between the radius vector and the tangent and also find the slope of the tangent as indicated for the following

1) $r = a(1 + \cos\theta)$ at $\theta = \pi/3$

→ Given $r = a(1 + \cos\theta)$

taking Logarithm on both sides

$$\log r = \log[a(1 + \cos\theta)]$$

$$\Rightarrow \log r = \log a + \log(1 + \cos\theta)$$

Differentiating w.r.t. θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1 + \cos\theta}(-\sin\theta)$$

$$\cot\phi = \frac{-2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2}$$

$\Rightarrow \cot\phi = -\tan\theta/2 = \tan(-\theta/2)$

$\Rightarrow \cot\phi = \cot(\pi/2+\theta/2)$ $\quad \because \tan\theta = \cot(\pi/2-\theta)$

$\Rightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$

at $\theta = \frac{\pi}{3} \Rightarrow \phi = \frac{\pi}{2}+\frac{\pi}{6} = \frac{2\pi}{3} = 120^\circ$

Also we have $\psi = \theta+\phi = \frac{\pi}{3}+\frac{2\pi}{3} = \pi = 180^\circ$

Hence, slope of the tangent is $\tan\psi = \tan\pi = 0$

2) $r\cos^2(\theta/2) = a$ at $\theta = \frac{2\pi}{3}$

→ We have $r\cos^2(\theta/2) = a$

taking Logarithms on both sides we get

$\Rightarrow \log r + 2\log\cos(\theta/2) = \log a$

Differentiate w.r.to θ

$\frac{1}{r}\frac{dr}{d\theta} + \cancel{2}\cdot\frac{1}{\cos(\theta/2)}\left[-\sin(\theta/2)\right]\cdot\frac{1}{\cancel{2}} = 0$

$\cot\phi - \tan\theta/2 = 0$

$\cot\phi = \tan\theta/2 = \cot(\pi/2-\theta/2)$

$\Rightarrow \phi = \frac{\pi}{2}-\frac{\theta}{2}$

at $\theta = \frac{2\pi}{3} \Rightarrow \phi = \frac{\pi}{2}-\frac{\pi}{3} = \frac{\pi}{6}$

Also we have $\psi = \theta+\phi = \frac{2\pi}{3}+\frac{\pi}{6} = \frac{5\pi}{6} = 150^\circ$

Hence slope of the tangent is $\tan\psi = \tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

3) $r^m \cos m\theta = a^m$ at $\theta = 0$

→ We have $r^m \cos m\theta = a^m$

taking logorithm on both sides we get

$$m \log r + \cos m\theta = m \log a$$

Differentiate w r to θ we get

$$\frac{m}{r}\frac{dr}{d\theta} + \frac{1}{\cos m\theta}(-m\sin m\theta) = 0$$

$$\cot\phi - \tan m\theta = 0$$

$$\cot\phi = \tan m\theta$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - m\theta\right)$$

$$\Rightarrow \phi = \frac{\pi}{2} - m\theta$$

at $\theta = 0 \Rightarrow \phi = \frac{\pi}{2}$

Also we have $\psi = \theta + \phi = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore$ Slope of the tangent $= \tan\psi = \tan\frac{\pi}{2} = \infty$

$\therefore$ slope will be ∞ if the tangent in 90°

4) $2a/r = 1 - \cos\theta$ at $\theta = 2\pi/3$

→ We have $\frac{2a}{r} = 1 - \cos\theta$

taking Logorithm on bothsides we get

$$\log_e(2a) - \log_e r = \log(1 - \cos\theta)$$

Differentiate w.r to θ

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}\sin\theta$$

$$-\cot\phi = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$-\cot\phi = \cot\theta/2$$

$\Rightarrow \cot(-\phi) = \cot \theta/2$

$\Rightarrow -\phi = \theta/2$

$\Rightarrow \phi = -\theta/2$

at $\theta = \frac{2\pi}{3} \Rightarrow \phi = -\pi/3$

Also $\psi = \theta + \phi = \frac{\pi}{3}$

$\Rightarrow$ Slope of the tangent $= \tan\psi = \tan\pi/3 = \sqrt{3}$

5) $r = a(1+\sin\theta)$ at $\theta = \pi/2$

$\rightarrow$ We have $r = a(1+\sin\theta)$

taking Logarithm on bothsides, we get

$\log_e r = \log_e a + \log_e(1+\sin\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$

$\Rightarrow \cot\phi = \frac{\cos\theta}{1+\sin\theta}$

at $\theta = \pi/2 \Rightarrow \cot\phi = \frac{\cos\pi/2}{1+\sin\pi/2} = 0 \Rightarrow \phi = \cot^{-1}(0)$

$\Rightarrow \phi = \pi/2$

Also $\psi = \theta + \phi = \frac{\pi}{2} + \frac{\pi}{2} \Rightarrow \psi = \pi$

$\therefore$ Slope of the tangent $= \tan\psi = \tan\pi = 0$

6) $r = e^{\theta\cot\alpha}$ at $\theta = 0$

$\rightarrow$ We have $r = e^{\theta\cot\alpha}$

taking log on both sides

$\log_e r = \theta\cot\alpha$

Differentiate w.r.to θ

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \cot\alpha$

$\Rightarrow \cot\phi = \cot\alpha$

$\Rightarrow \phi = \alpha$

$\therefore \psi = \theta + \phi \Rightarrow \psi = \alpha$ at $\theta = 0$

$\therefore$ Slope of the tangent $= \tan\psi = \tan\alpha$

7) $r\sec^2(\theta/2) = 4$ at $\theta = \pi/2$

$\rightarrow$ We have $r\sec^2(\theta/2) = 4$

$\Rightarrow \log r + 2\log\sec(\theta/2) = \log 4$

Differentiate w.r.to θ

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} + 2\frac{1}{\sec\theta/2} \cdot \sec(\theta/2)\cdot\tan(\theta/2)\cdot\frac{1}{2} = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan\theta/2$$

$\Rightarrow \cot\phi = \tan(-\theta/2)$

$\Rightarrow \cot\phi = \cot(\pi/2 + \theta/2)$ $\quad \because \tan\theta = \cot(\pi/2 - \theta)$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

at $\theta = \frac{\pi}{2} \Rightarrow \phi = \frac{3\pi}{4}$

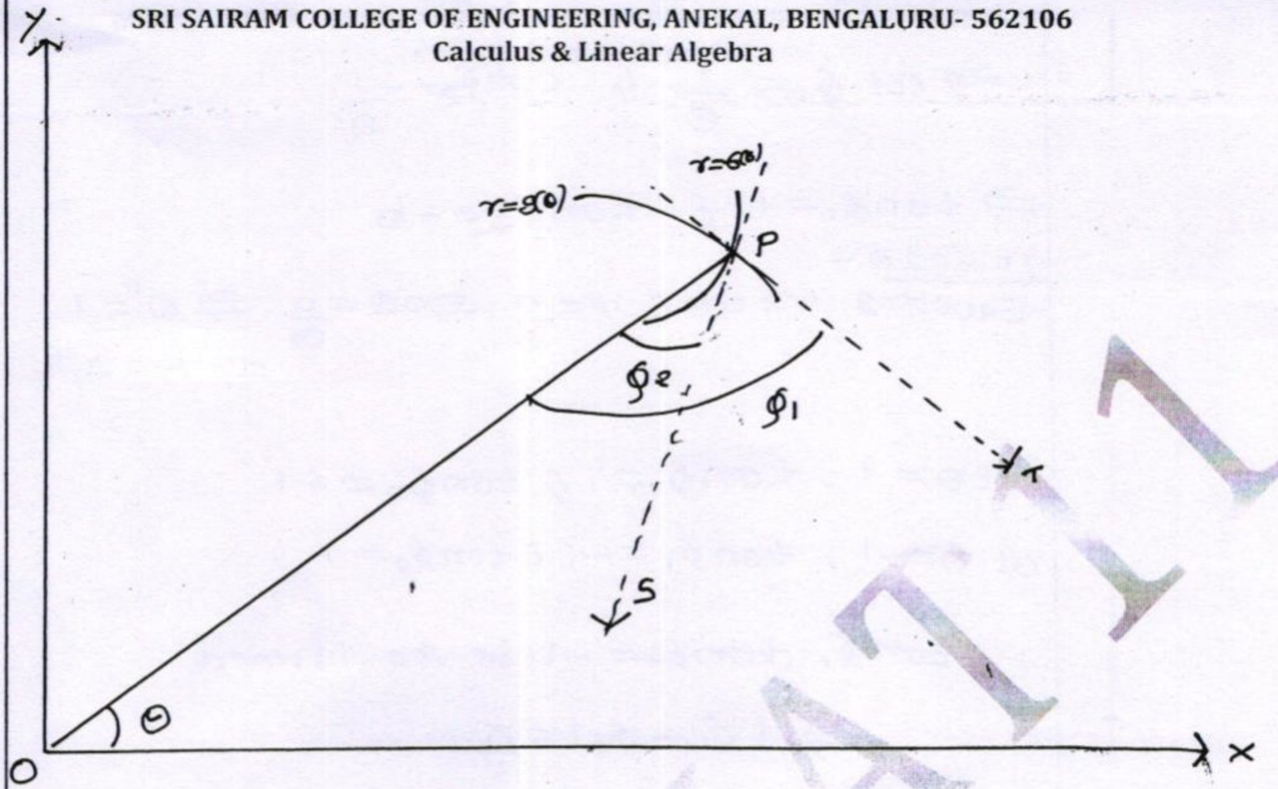
Also $\psi = \theta + \phi = \frac{\pi}{2} + \frac{3\pi}{4} \Rightarrow \psi = \frac{5\pi}{4}$

$\therefore$ Slope of tangent, $\tan\psi = \tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right)$

$= \tan\frac{\pi}{4}$

$\tan\psi = 1$

* Angle b/w two polar curves



Let $r=f(\theta)$ & $r=g(\theta)$ be any two polar curves intersecting at point P.

Let PS and PT be tangents to $r=f(\theta)$ & $r=g(\theta)$ respecti

Then angle between curves is

$$\tan|\phi_1-\phi_2| = |\tan(\phi_1-\phi_2)|$$

$$= \left|\frac{\tan\phi_1-\tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$$

Note:- Two curves are orthogonal if $|\phi_1-\phi_2| = \frac{\pi}{2}$ (or)

$$\tan\phi_1 \cdot \tan\phi_2 = -1$$

* Find the angle of intersection of following pairs of curve.

i) $r=a\theta$ and $r=\frac{a}{\theta}$

→ Given $r=a\theta$ & $r=\frac{a}{\theta} \longrightarrow (1)$

Taking Logarithms on both sides, we get

$\log r = \log(\theta a)$ and $\log r = \log\left(\frac{a}{\theta}\right)$

$\Rightarrow \log r = \log\theta + \log a$ & $\log r = \log a - \log\theta$

Differentiate w.r.to θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\theta} \quad \& \quad \frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{1}{\theta} \quad \& \quad \cot\phi_2 = -\frac{1}{\theta}$$

$$\Rightarrow \tan\phi_1 = \theta \quad \& \quad \tan\phi_2 = -\theta$$

To find θ:-

Equating $r = a\theta$ & $r = \frac{a}{\theta}$ $\Rightarrow a\theta = \frac{a}{\theta} \Rightarrow \theta^2 = 1$

$$\Rightarrow \theta = \pm 1$$

at $\theta = 1$, $\tan\phi_1 = 1$ & $\tan\phi_2 = -1$

at $\theta = -1$, $\tan\phi_1 = -1$ & $\tan\phi_2 = 1$

$\therefore \tan\phi_1 \cdot \tan\phi_2 = -1$ is satisfied.

$$\Rightarrow |\phi_1 - \phi_2| = \frac{\pi}{2}$$

2) $r = \sin\theta + \cos\theta$ & $r = 2\sin\theta$

$\rightarrow$ Given $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta \rightarrow (1)$

taking Logarithms on both sides we get

$\log r = \log(\sin\theta + \cos\theta)$ & $\log r = \log(2\sin\theta)$

Differentiate w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sin\theta + \cos\theta}\cdot(\cos\theta - \sin\theta) \quad \& \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{2\sin\theta}(2\cos\theta)$$

$$\cot\phi_1 = \frac{\cos\theta\,[1 - \tan\theta]}{\cos\theta\,[1 + \tan\theta]} \quad \& \quad \cot\phi_2 = \cot\theta$$

$$\Rightarrow \phi_2 = \theta$$

$$\tan\phi_1 = \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$\Rightarrow \tan\phi_1 = \frac{\tan\pi/4 + \tan\theta}{1 - \tan\pi/4\,\tan\theta}$$

$$\Rightarrow \tan\phi_1 = \tan\left(\frac{\pi}{4} + \theta\right)$$

$$\Rightarrow \phi_1 = \frac{\pi}{4} + \theta$$

$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \theta - \theta\right| = \frac{\pi}{4}$

$\therefore$ The angle of intersection is $\frac{\pi}{4}$

3) $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

$\rightarrow$ Given $r = a\log\theta$ & $r = \frac{a}{\log\theta} \longrightarrow ①$

taking Logarithms on both sides

$\Rightarrow \log r = \log a + \log(\log\theta)$ & $\log r = \log a - \log(\log\theta)$

$\therefore$ Differentiate w.r.to θ

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta}\cdot\frac{1}{\theta}$ & $\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta}\cdot 1/\theta$

$\Rightarrow \cot\phi_1 = \frac{1}{\theta\log\theta}$ & $\cot\phi_2 = -\frac{1}{\theta\log\theta}$

$\Rightarrow \tan\phi_1 = \theta\log\theta$ & $\tan\phi_2 = -\theta\log\theta$

$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2} = \frac{\theta\log\theta + \theta\log\theta}{1 + (\theta\log\theta)(\theta\log\theta)}$

$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2\theta\log\theta}{1 - (\theta\log\theta)^2} \longrightarrow ②$

To find θ,

Equating $r = a\log\theta$ & $r = \frac{a}{\log\theta} \Rightarrow a\log\theta = \frac{a}{\log\theta} \Rightarrow (\log\theta)^2 = 1$

$\Rightarrow \log\theta = 1 \Rightarrow \boxed{\theta = e^1 = e}$

$\therefore \tan(\phi_1 - \phi_2) = \frac{2e\log e}{1 - (e\log e)^2} = \frac{2e}{1 - e^2} \quad \because \log e = 1$

$\Rightarrow \phi_1 - \phi_2 = \tan^{-1}\left(\frac{2e}{1 - e^2}\right)$

$\therefore$ Angle of intersection is $\tan^{-1}\left(\frac{2e}{1 - e^2}\right)$

4) $r^2 \sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$

$\rightarrow r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta \rightarrow (1)$

taking $\log_e$ on both sides

$\Rightarrow 2\log r + \log(\sin 2\theta) = \log 4$ and $2\log r = \log(16\sin 2\theta)$

Differentiate w.r.to θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{1}{\sin 2\theta}(2\cos 2\theta) = 0 \text{ and } \frac{2}{r}\frac{dr}{d\theta} = \frac{32\cos 2\theta}{16\sin 2\theta}$$

$\Rightarrow \cot\phi_1 = -\cot 2\theta$ and $\cot\phi_2 = \cot 2\theta$

$\Rightarrow \cot\phi_1 = \cot(-2\theta)$ and $\cot\phi_2 = \cot 2\theta$

$\Rightarrow \phi_1 = -2\theta$ & $\phi_2 = 2\theta$

$\therefore$ The angle of intersection is

$$|\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta \longrightarrow (2)$$

Equating $r^2\sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$

$\Rightarrow 16\sin^2 2\theta = 4$

$\Rightarrow \sin^2 2\theta = \frac{1}{4}$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{12}$

Substituting $\theta = \frac{\pi}{12}$ in (2), we get

$$|\phi_1 - \phi_2| = 4\cdot\frac{\pi}{12} = \frac{\pi}{3}$$

5) $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$\rightarrow$ we have $r = a(1-\cos\theta)$ and $r = 2a\cos\theta \longrightarrow (1)$

taking logarithms on both sides we get

$\log r = \log a + \log(1-\cos\theta)$ and $\log r = \log 2a + \log\cos\theta$

Differentiate w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}(\sin\theta) \qquad \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos\theta}(-\sin\theta)$$

$$\Rightarrow \cot\phi_1 = \frac{\sin\theta}{1-\cos\theta} \quad \text{and} \quad \cot\phi_2 = -\tan\theta$$

$$\Rightarrow \cot\phi_1 = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} \quad \text{and} \quad \cot\phi_2 = \tan(-\theta)$$

$$\Rightarrow \cot\phi_1 = \cot\theta/2 \quad \text{and} \quad \cot\phi_2 = \cot(\pi/2+\theta)$$

$$\Rightarrow \phi_1 = \theta/2 \ \& \ \phi_2 = \frac{\pi}{2}+\theta$$

$$\therefore |\phi_1-\phi_2| = |\theta/2 - \pi/2 - \theta| = \frac{\pi}{2}+\frac{\theta}{2} \longrightarrow (2)$$

Equating $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$$\Rightarrow a - a\cos\theta = 2a\cos\theta$$

$$\Rightarrow 3a\cos\theta = a \Rightarrow 3\cos\theta = 1 \Rightarrow \cos\theta = 1/3 \Rightarrow \theta = \cos^{-1}(1/3)$$

$(2) \Rightarrow$

$\therefore$ Angle of intersection $= \frac{\pi}{2}+\frac{1}{2}\cos^{-1}(1/3)$

H.W

6) $r = 6\cos\theta$ and $r = 2(1+\cos\theta)$; Ans: $|\phi_1-\phi_2| = \frac{\pi}{6}$

7) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$; Ans: $|\phi_1-\phi_2| = \tan^{-1}(-3)$

* Show that the following pairs of curves intersect each other orthogonally.

1) $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$

→ We have $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta) \longrightarrow (1)$

Taking Logrithms on bothsides, We get

$\log r = \log a + \log(1+\cos\theta)$ and $\log r = \log b + \log(1-\cos\theta)$

Differentiate

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta}(-\sin\theta)$ and $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$

$\Rightarrow \cot\phi_1 = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$ and $\cot\phi_2 = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$

$\Rightarrow \cot\phi_1 = -\tan\theta/2$ and $\cot\phi_2 = \cot\theta/2$

$\Rightarrow \cot\phi_1 = \cot\left(\frac{\pi}{2}+\theta/2\right)$ and $\cot\phi_2 = \cot\theta/2$

$\Rightarrow \phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$ and $\phi_2 = \theta/2$

$\therefore$ Angle of intersection $= |\phi_1 - \phi_2|$

$$= \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right|$$

$$= \frac{\pi}{2}$$

Hence the curves intersect orthogonally

2) $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$

$\rightarrow$ We have $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta) \longrightarrow (1)$

Taking $\log_e$ on both sides

$\log r = \log(1+\sin\theta) + \log a$ & $\log r = \log a + \log(1-\sin\theta)$

Differentiate w.r.to θ, we get

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{1+\sin\theta}$ and $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{(-\cos\theta)}{1-\sin\theta}$

$\Rightarrow \cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$ and $\cot\phi_2 = -\frac{\cos\theta}{1-\sin\theta}$

$\Rightarrow \tan\phi_1 = \frac{1+\sin\theta}{\cos\theta}$ and $\tan\phi_2 = \frac{1-\sin\theta}{-\cos\theta}$

$\tan\phi_1 \cdot \tan\phi_2 = \frac{-(1+\sin\theta)(1-\sin\theta)}{\cos^2\theta} = \frac{-(1-\sin^2\theta)}{\cos^2\theta} = -\frac{\cos^2\theta}{\cos^2\theta}$

$\Rightarrow \tan\phi_1 \cdot \tan\phi_2 = -1$

Hence the curves ...

3) $r^n = a^n \cos n\theta$ & $r^n = b^n \sin n\theta$

→ We have $r^n = a^n \cos n\theta$ & $r^n = b^n \sin n\theta$

taking $\log_e$ on bothsides

$n \log r = n \log a + \log(\cos n\theta)$ and $n \log r = n \log b + \log(\sin n\theta)$

Differentiate w.r.to θ

$\Rightarrow \frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n \sin n\theta)$ & $\frac{n}{r}\frac{dr}{d\theta} = 0 + n\frac{\cos n\theta}{\sin n\theta}$

$\Rightarrow \cot\phi_1 = -\tan n\theta$ & $\cot\phi_2 = \cot n\theta$

$\Rightarrow \cot\phi_1 = \cot(\frac{\pi}{2} + n\theta)$ & $\cot\phi_2 = \cot n\theta$

$\Rightarrow \phi_1 = \frac{\pi}{2} + n\theta$ & $\phi_2 = n\theta$

$\therefore$ Angle of intersection $= |\phi_1 - \phi_2|$

$= |\frac{\pi}{2} + n\theta - n\theta|$

$= \frac{\pi}{2}$

Hence the curves intersect orthogonally

4) $r = ae^{\theta}$ and $re^{\theta} = b$

→ We have $r = ae^{\theta}$ & $re^{\theta} = b$

taking $\log_e$ on bothsides we get

$\log r = \log a + \theta$ & $\log r + \theta = \log b$

Differentiate w.r to θ

$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1$ & $\frac{1}{r}\frac{dr}{d\theta} + 1 = 0$

$\Rightarrow \cot\phi_1 = 1$ & $\cot\phi_2 = -1$

$\Rightarrow \phi_1 = \frac{\pi}{4}$ & $\phi_2 = -\frac{\pi}{4}$

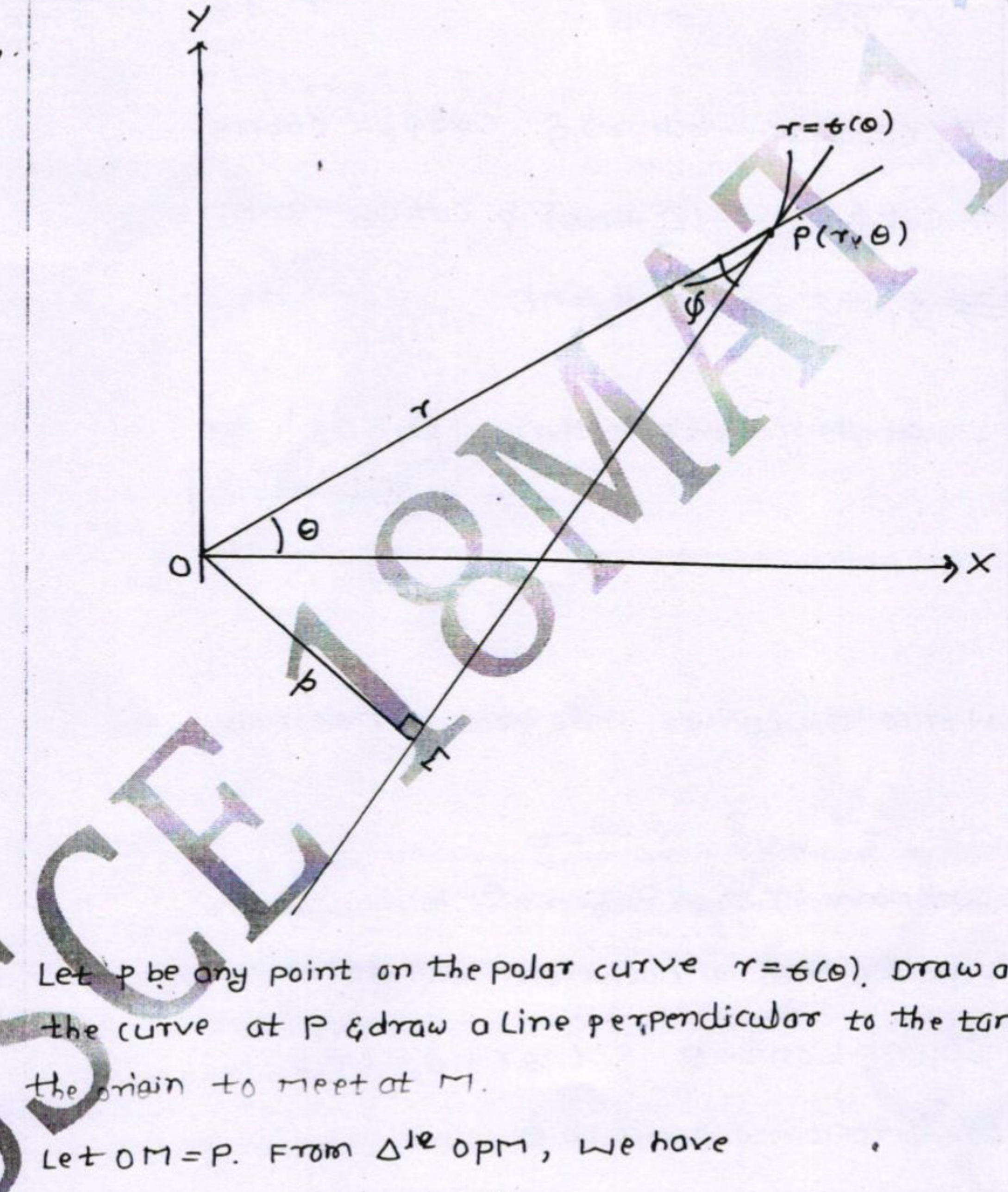
$\therefore$ Angle of intersection $= |\phi_1 - \phi_2| = |\frac{\pi}{4} + \frac{\pi}{4}| = \frac{\pi}{2}$

H.W 5) $r^2\sin 2\theta = a^2$ & $r^2\cos 2\theta = b^2$

6) $r = 4\sec^2(\theta/2)$ and $r = 3\csc^2(\theta/2)$

* Pedal equation [p-r equation]:-
(or)

- Derive an expression for Length of the perpendicular from to the tangent



Let P be any point on the polar curve $r = f(\theta)$. Draw a tangent to the curve at P & draw a Line perpendicular to the tangent from the origin to meet at M.

Let $OM = P$. From Δ^{le} OPM, we have

$$\sin\phi = \frac{OM}{OP} = \frac{P}{r}$$

$$\Rightarrow P = r\sin\phi \rightarrow (1)$$

Squaring on both sides we get

$$p^2 = r^2\sin^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \longrightarrow (2)$$

Now the relation connecting b/w p and r is called pedal (or) $p-r$ equation.

i.e> a) $p = r\sin\phi$ & b) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

* problems: Find the pedal equation of the following curves

1) $\frac{2a}{r} = (1+\cos\theta)$

→ We have $\frac{2a}{r} = (1+\cos\theta) \longrightarrow (1)$

Taking Logarithm on bothsides, We get

$\log\left(\frac{2a}{r}\right) = \log(1+\cos\theta)$

$\Rightarrow \log 2a - \log r = \log(1+\cos\theta)$

Differentiate w.r.t θ,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+\cos\theta}(-\sin\theta)$$

$$\Rightarrow -\cot\phi = -\frac{2\sin\theta/2\cdot\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi = \tan\theta/2$$

$$\Rightarrow \cot\phi = \cot\left(\frac{\pi}{2}-\theta/2\right)$$

$$\Rightarrow \phi = \frac{\pi}{2}-\theta/2$$

consider $p=r\sin\phi$ and substituting the value of ϕ

$\Rightarrow p=r\sin\left(\frac{\pi}{2}-\theta/2\right)$

$\Rightarrow p=r\cos\left(\frac{\theta}{2}\right) \longrightarrow (2)$ $\quad \because \cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$

consider equation (1) & (2) to eliminate θ

$\frac{2a}{r}=1+\cos\theta$ & $p=r\cos(\theta/2)$

$\Rightarrow \frac{2a}{r}=2\cos^2\theta/2$ & $p=r\cos\theta/2$

$\Rightarrow \frac{a}{r}=\cos^2\theta/2$ & $\frac{p}{r}=\cos\theta/2$

$\Rightarrow \frac{a}{r}=\frac{p^2}{r^2}$

$\Rightarrow p^2=ar$ is the required pedal equation

2) $r(1-\cos\theta)=2a$

$\rightarrow$ We have $r(1-\cos\theta)=2a \longrightarrow (1)$

Taking Log on both sides

$\log r+\log(1-\cos\theta)=\log 2a$

Differentiate w.r.to θ

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta}+\frac{1}{1-\cos\theta}(\sin\theta)=0$

$\Rightarrow \cot\phi+\frac{2\sin\theta/2\cdot\cos\theta/2}{2\sin^2\theta/2}=0$

$\Rightarrow \cot\phi=-\cot\theta/2=\cot(-\theta/2)$

$\Rightarrow \phi=-\theta/2$

Consider $p=r\sin\phi$ & substituting the value of ϕ

$\Rightarrow p=r\sin(-\theta/2)=-r\sin\theta/2 \longrightarrow (2)$

consider equation (1) & (2) to eliminate θ

$\Rightarrow r(1-\cos\theta) = 2a$ & $p = -r\sin\theta/2$

$\Rightarrow 2r\sin^2\theta/2 = 2a$ & $p = -r\sin\theta/2$

$\Rightarrow r\sin^2\theta/2 = a$ & $\frac{-p}{r} = \sin\theta/2$

$\Rightarrow r\left(\frac{-p}{r}\right)^2 = a$

$\Rightarrow p^2 = ar$ is the required pedal equation.

3) $r^2 = a^2\sec 2\theta$

→ We have $r^2 = a^2\sec 2\theta \rightarrow (1)$

Taking $\log_e$ on both sides

$2\log r = 2\log a + \log\sec 2\theta$

Differentiate w.r.t to θ,

$$\frac{2}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sec 2\theta}\cdot 2\sec 2\theta\cdot\tan 2\theta$$

$\Rightarrow \cot\phi = \tan 2\theta$

$\Rightarrow \cot\phi = \cot\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \phi = \frac{\pi}{2} - 2\theta$

consider $p = r\sin\phi$ & substitute ϕ

$\Rightarrow p = r\sin\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow p = r\cos 2\theta \rightarrow (2)$

consider equation (1) & (2) to eliminate θ

$\Rightarrow p = r\cos 2\theta$ & $r^2 = a^2\sec 2\theta$

$\Rightarrow \frac{r}{p} = \sec 2\theta$ & $r^2 = a^2\sec 2\theta$

$\Rightarrow r^2 = a^2\left(\frac{r}{p}\right)$

$\Rightarrow pr = a^2$ is the required pedal equation.

4) $r^n = a^n \cos n\theta$

$\rightarrow$ We have $r^n = a^n \cos n\theta \rightarrow (1)$

taking Log on both sides

$\Rightarrow n \log r = n \log a + \log \cos n\theta$

Differentiate w.r.to θ

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n \sin n\theta)$$

$\Rightarrow \cot\phi = -\tan n\theta$

$\Rightarrow \cot\phi = \cot\left(\frac{\pi}{2} + n\theta\right)$

$\Rightarrow \phi = \frac{\pi}{2} + n\theta$

consider $p = r\sin\phi$ & Substitute ϕ value

$\Rightarrow p = r\sin\left(\frac{\pi}{2} + n\theta\right)$

$\Rightarrow p = r\cos n\theta$

$\Rightarrow \frac{p}{r} = \cos n\theta \rightarrow (2)$

consider $r^n = a^n \cos n\theta$ & $\frac{p}{r} = \cos n\theta$

$$\Rightarrow r^n = a^n\left(\frac{p}{r}\right)$$

$\Rightarrow p a^n = r^{n+1}$ is the required pedal equation

5) $r^m = a^m(\cos m\theta + \sin m\theta)$

$\rightarrow$ We have $r^m = a^m(\cos m\theta + \sin m\theta)$

taking Log on both sides

$\Rightarrow m \log r = m \log a + \log(\cos m\theta + \sin m\theta)$

Differentiate w.r.to θ

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta + \sin m\theta}(-m\sin m\theta + m\cos m\theta)$$

$$\Rightarrow \cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$$\Rightarrow \cot\phi = \frac{\cos m\theta(1 - \tan m\theta)}{\cos m\theta(1 + \tan m\theta)}$$

$\Rightarrow \tan\phi = \dfrac{1+\tan m\theta}{1-\tan m\theta}$

$\Rightarrow \tan\phi = \dfrac{\tan\pi/4+\tan m\theta}{1-\tan\frac{\pi}{4}\tan m\theta}$

$\Rightarrow \tan\phi = \tan\left(\frac{\pi}{4}+m\theta\right)$

$\Rightarrow \phi = \frac{\pi}{4}+m\theta$

consider $p = r\sin\phi$ and Substitute ϕ

$\Rightarrow p = r\sin\left(\frac{\pi}{4}+m\theta\right)$

$\Rightarrow p = r\left\{\sin\frac{\pi}{4}\cos m\theta + \cos\frac{\pi}{4}\sin m\theta\right\}$

$\Rightarrow p = r\left\{\frac{1}{\sqrt{2}}\cos m\theta + \frac{1}{\sqrt{2}}\sin m\theta\right\}$

$\Rightarrow p = \frac{r}{\sqrt{2}}\{\cos m\theta + \sin m\theta\}$

$\Rightarrow \frac{\sqrt{2}\,p}{r} = \cos m\theta + \sin m\theta \longrightarrow (2)$

Equating (2) & (1)

$\Rightarrow r^m = a^m \frac{\sqrt{2}\,p}{r}$

$\Rightarrow r^{m+1} = a^m\sqrt{2}\,p$

$\Rightarrow p = \frac{r^{m+1}}{\sqrt{2}\,a^m}$ is the required pedal equation

6) $\frac{l}{r} = 1+e\cos\theta$

$\rightarrow$ We have $l/r = 1+e\cos\theta \rightarrow (1)$

taking Log on both sides

$\Rightarrow \log l - \log r = \log(1+e\cos\theta)$

Differentiate w.r.to θ

$\Rightarrow 0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+e\cos\theta}(-e\sin\theta)$

$$\Rightarrow -\cot\phi = \frac{-e\sin\theta}{1+e\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$$

& WKT $p = r\sin\phi$

$$\text{S.o.b.s} \Rightarrow p^2 = r^2\sin^2\phi \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right\} \rightarrow (2)$$

$$(1) \Rightarrow \frac{l}{r} = 1+e\cos\theta \Rightarrow e\cos\theta = \frac{l}{r}-1$$

Also $e^2\sin^2\theta = e^2(1-\cos^2\theta) = e^2 - e^2\cos^2\theta$

$$= e^2 - \left(\frac{l}{r}-1\right)^2$$

$$\text{Hence (2)} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left\{1+\frac{e^2-\left(\frac{l}{r}-1\right)^2}{l^2/r^2}\right\}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\cdot\frac{r^2}{l^2}\left[e^2-\left(\frac{l}{r}-1\right)^2\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{l^2}\left[e^2 - \frac{l^2}{r^2} + 2\cdot\frac{l}{r} - 1\right]$$

$$\Rightarrow \frac{1}{p^2} = \cancel{\frac{1}{r^2}} + \frac{e^2}{l^2} - \cancel{\frac{1}{r^2}} + \frac{2}{lr} - \frac{1}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{e^2-1}{r^2} + \frac{2}{lr}$$ is the required pedal equation

H.W

7) $r^n = a^n\operatorname{sech} n\theta$, Ans: $\frac{1}{p^2} = \frac{1}{r^2}\left\{2 - \frac{r^{2n}}{a^{2n}}\right\}$

8) $r = 2(1+\cos\theta)$, Ans: $r^3 = 4p^2$

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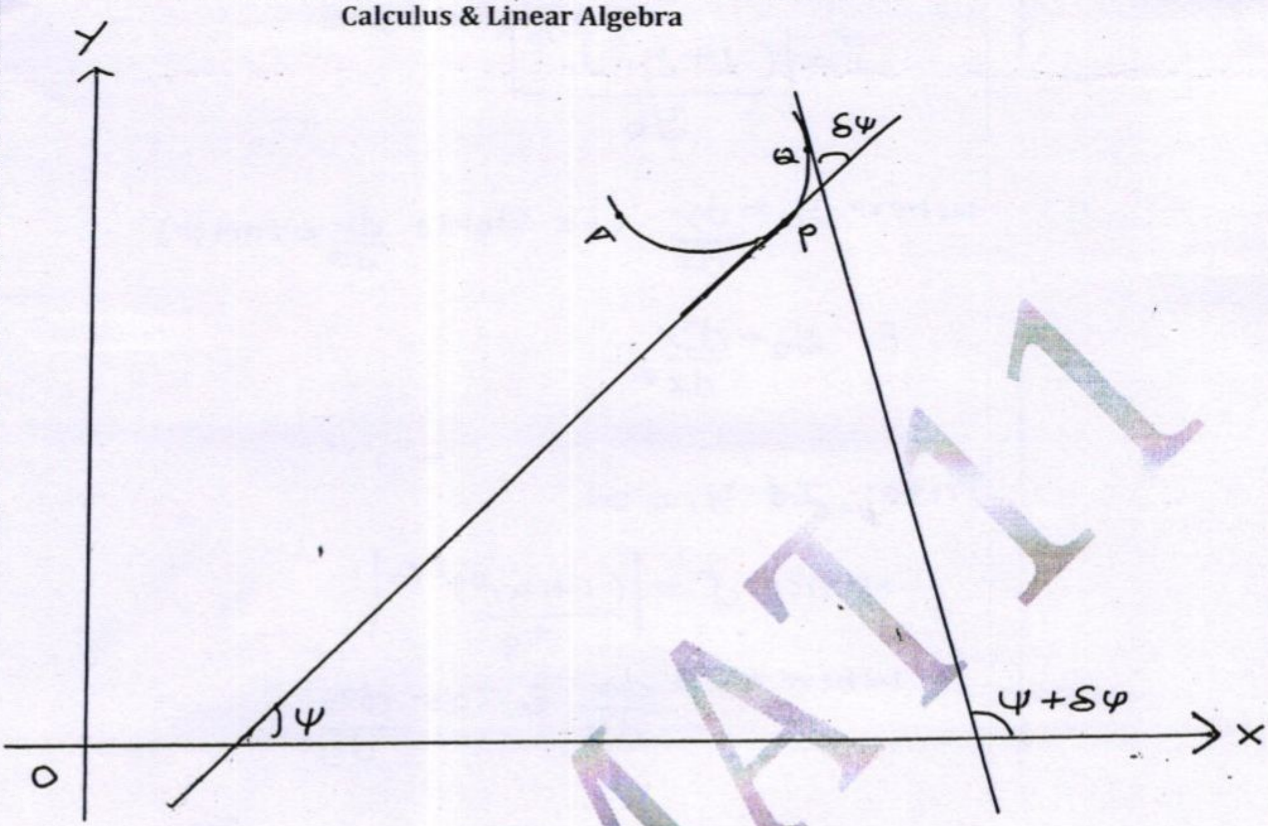

Consider a curve in the xoy plane and let A be a fixed point on it. Let P and Q be two neighbouring points on the curve such that $\widehat{AP} = s$ & $\widehat{AQ} = s+\delta s$. So that $\widehat{PQ}$

Let ψ and $\psi+\delta\psi$ respectively be the angles made by the tangents at P and Q with the x-axis.

The angles $\delta\psi$ between the tangents is called the bending of the curve which depends on δs.

$\frac{\delta\psi}{\delta s}$ is called as the mean curvature of the arc PQ. A amount of bending of the curve at P is called as the cur of the curve at P and is defined mathematically as

$$\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$

i.e curvature $= \kappa = \frac{d\psi}{ds}$ and $\kappa \neq 0$

Hence radius of curvature $= \rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$

$\Rightarrow$ The reciprocal of the curvature is called the radius of curvature

* Radius of curvature in cartesian form:-

$$\rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right|$$

where $y_1 = \frac{dy}{dx}$ (i.e slope $\frac{dy}{dx} = \tan\psi$)

& $y_2 = \frac{d^2y}{dx^2}$

Note:- If $y_1 = \infty$

Hence $\rho' = \left| \frac{(1+x_1^2)^{3/2}}{x_2} \right|$

where $x_1 = \frac{dx}{dy}$ & $x_2 = \frac{d^2x}{dy^2}$

* problems

I) Find the radius of curvature at a point (x, y) of each of t following curves:

(i) $y = a\cosh\left(\frac{x}{a}\right)$

→ Given $y = a\cosh\left(\frac{x}{a}\right) \longrightarrow (1)$ is in cartesian form

WKT Radius of curvature is given by $\rho = \left| \frac{\{1+y_1^2\}^{3/2}}{y_2} \right|$

Differentiating (1) w.r.t x

$$\Rightarrow y_1 = \frac{dy}{dx} = a\sinh\left(\frac{x}{a}\right)\cdot\frac{1}{a} = \sinh\left(\frac{x}{a}\right)$$

Differentiate again w.r.t x

$$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{1}{a}\cosh\left(\frac{x}{a}\right)$$

$$\therefore \quad \rho = \frac{\left\{1 + \left(\sinh\left(\frac{x}{a}\right)\right)^2\right\}^{3/2}}{\frac{1}{a}\cosh\left(\frac{x}{a}\right)}$$

$= a\{\cosh(\frac{x}{a})\}$ $[\because \cosh^2 x - \sinh^2 x$

$\cosh(\frac{x}{a})$

$$= \frac{a\cosh^3(\frac{x}{a})}{\cosh(\frac{x}{a})}$$

$$\rho = a\cosh^2(\frac{x}{a})$$

$$= \frac{a^2}{a}\cosh^2(\frac{x}{a})$$

$$\boxed{\rho = \frac{y^2}{a}} \quad \because \text{by (1)}$$

(ii) Astroid: $x^{2/3}+y^{2/3}=a^{2/3}$

$\rightarrow$ Given $x^{2/3}+y^{2/3}=a^{2/3} \longrightarrow (1)$ is in cartesion form

Differentiate (1) w.r.to x

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\cdot y_1 = 0$$

$$\Rightarrow \frac{2}{3}y^{-1/3}y_1 = -\frac{2}{3}x^{-1/3}$$

$$\Rightarrow y_1 = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$\Rightarrow y_1 = -x^{-1/3}y^{1/3}$$

Differentiate w.r.to x

$$\Rightarrow y_2 = -x^{-1/3}\left[\frac{1}{3}y^{-2/3}y_1\right] - \left[-\frac{1}{3}x^{-4/3}y^{1/3}\right]$$

$$= \frac{1}{3}\left[y^{1/3}x^{-4/3} - y^{-2/3}\left(-x^{-1/3}y^{1/3}\right)x^{-1/3}\right]$$

$$= \frac{1}{3}\left[y^{1/3}x^{-4/3}+x^{-2/3}y^{-1/3}\right]$$

$$= \frac{1}{3}x^{-4/3}\cdot y^{-1/3}\left[\frac{y^{1/3}}{y^{-1/3}}+\frac{x^{-2/3}}{x^{-4/3}}\right]$$

$$= \frac{1}{3}x^{-4/3}y^{-1/3}\left[y^{2/3}+x^{2/3}\right]$$

$$y_2 = \frac{1}{3}x^{-4/3}y^{-1/3}a^{2/3}$$

We have $\rho = \left| \dfrac{\{1+(y_1)^2\}^{3/2}}{y_2} \right|$

$$= \frac{\{1+(-x^{-1/3}y^{1/3})^2\}^{3/2}}{1/3\, x^{-4/3}\, y^{-1/3}\, a^{2/3}}$$

$$= \frac{3\,[1+x^{-2/3}y^{2/3}]^{3/2}}{x^{-4/3}\, y^{-1/3}\, a^{2/3}}$$

$$= \frac{3\left[1+\frac{y^{2/3}}{x^{2/3}}\right]^{3/2}}{x^{-4/3}\, y^{-1/3}\, a^{2/3}}$$

$$= \frac{3\left[\frac{x^{2/3}+y^{2/3}}{x^{2/3}}\right]^{3/2}}{x^{-4/3}\, y^{-1/3}\, a^{2/3}}$$

$$= \frac{3\left[\frac{a^{2/3}}{x}\right]^{3/2}}{x^{-4/3}\, y^{-1/3}\, a^{2/3}}$$

$$= \frac{3a}{x} \cdot a^{-2/3}\, x^{4/3}\, y^{1/3}$$

$$\boxed{\rho = 3(axy)^{1/3}}$$

(iii) $ay^2 = x^3$

$\rightarrow$ Given $ay^2 = x^3 \rightarrow (1)$ in cartesian form

Differentiate (1) w.r.t. x

$2ayy_1 = 3x^2$

$\Rightarrow y_1 = \dfrac{3x^2}{2ay}$ & by (1) $\Rightarrow y = \left(\dfrac{x^3}{a}\right)^{1/2}$

$\Rightarrow y_1 = \dfrac{3x^2}{2a\left(\frac{x^3}{a}\right)^{1/2}}$

$$\Rightarrow y_1 = \frac{3}{2}\left(\frac{x^2 \cdot x^{3/2}}{a \cdot a^{-1/2}}\right) = \frac{3}{2}\,\frac{x^{1/2}}{a^{1/2}} = \frac{3}{2}\left(\frac{x}{a}\right)^{1/2}$$

Differentiate w.r.to x

$$\Rightarrow y_2 = \frac{3}{2}\cdot\frac{1}{2}\left(\frac{x}{a}\right)^{-1/2}\cdot\frac{1}{a} = \frac{3}{4a}\left(\frac{x}{a}\right)^{-1/2}$$

We have $\rho = \left|\frac{\{1+(y_1)^2\}^{3/2}}{y_2}\right| = \frac{\left\{1+\frac{9}{4}\left(\frac{x}{a}\right)\right\}^{3/2}}{\frac{3}{4a}\left(\frac{x}{a}\right)^{-1/2}}$

$$\Rightarrow \rho = \frac{\{4a+9x\}^{3/2}}{(4a)^{3/2}}\cdot\frac{4a}{3}\cdot\left(\frac{a}{x}\right)^{-1/2}$$

$$= \frac{\{4a+9x\}^{3/2}}{8a^{3/2}}\cdot\frac{4\sqrt{ax}}{3} \qquad [\because 4^{3/2}=8]$$

$$\rho = \left\{\frac{(4a+9x)^{3/2}}{6a}\right\}\sqrt{x}$$

2) Show that the radius of curvature of the curve $y = 4\sin x - S$ at $x=\frac{\pi}{2}$ is $\frac{5\sqrt{5}}{4}$

$\rightarrow$ Given $y = 4\sin x - \sin 2x \rightarrow (1)$ is in cartesian form

Differentiate (1) w.r.to x

$$\Rightarrow y_1 = 4\cos x - 2\cos 2x \rightarrow (2)$$

at $x=\pi/2 \Rightarrow y_1 = 4\cos\frac{\pi}{2} - 2\cos\pi = 4(0) - 2(-1) = 2$

Differentiate (2) w.r.to x

$$\Rightarrow y_2 = -4\sin x + 4\sin 2x$$

at $x=\pi/2 \Rightarrow y_2 = -4\sin\frac{\pi}{2} + 4\sin\pi = 4(1)+4(0) = -4$

We have $\rho = \left|\frac{\{1+(y_1)^2\}^{3/2}}{y_2}\right|$

$$\Rightarrow \rho = \left|\frac{\{1+4\}^{3/2}}{-4}\right| = \left|\frac{5^{3/2}}{-4}\right|$$

$$\Rightarrow \rho = \frac{5\sqrt{5}}{4}$$

3) Find the radius of curvature of the curve $x^3+y^3=3axy$ at the point $P=(3a/2, 3a/2)$

→ Given $x^3+y^3=3axy \rightarrow (1)$ is in cartesian form

WKT $\rho = \left|\frac{\{1+y_1^2\}^{3/2}}{y_2}\right|$

Differentiate (1) w.r.to x

$$\Rightarrow 3x^2+3y^2y_1 = 3a(y+xy_1)$$

$$\Rightarrow x^2+y^2y_1 = ay+xy_1a$$

$$\Rightarrow y_1 = \frac{x^2-ay}{(ax-y^2)} \rightarrow (2)$$

put $(x,y) = (3a/2, 3a/2)$

$$\Rightarrow y_1 = \frac{\frac{9a^2}{4}-\frac{3a^2}{2}}{\frac{3a^2}{2}-\frac{9a^2}{4}} = \frac{\frac{3a^2}{4}}{\frac{-3a^2}{4}} \Rightarrow y_1 = -1$$

Differentiate (2) w.r.to x

$$\Rightarrow y_2 = \frac{(2x-ay_1)(ax-y^2)-(x^2-ay)(a-2yy_1)}{(ax-y^2)^2}$$

at $(x,y) = (\frac{3a}{2}, 3a/2)$

$$\Rightarrow y_2 = \frac{\left(\frac{6a}{2}+a\right)\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)-\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)\left(a+\frac{6a}{2}\right)}{\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)^2}$$

$$= \frac{4a\left[\frac{6a^2}{2}-\frac{18a^2}{4}\right]}{\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)^2} = \frac{8a\,\cancel{\left[\frac{3a^2}{2}-\frac{9a^2}{4}\right]}}{\left[\frac{3a^2}{2}-\frac{9a^2}{4}\right]^{\cancel{2}}}$$

$$= \frac{8\cancel{a}}{\frac{-3a}{4}}$$

$$= -\frac{32}{3a}$$

$$\therefore \rho = \left| \frac{\{1 + (-1)^2\}^{3/2}}{-32/3a} \right|$$

$$= \left| \frac{-3a}{32} (2)^{3/2} \right|$$

$$= \frac{3\sqrt{2}a}{16}$$

$$\rho = \frac{3a}{8\sqrt{2}}$$

4) Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a,0)$

$\Rightarrow$ Given $y^2 = \frac{a^2(a-x)}{x} \rightarrow (1)$ is in cartesian form

WKT $\rho = \left| \frac{\{1+y_1^2\}^{3/2}}{y_2} \right|$

Differentiate (1) w.r.to x

$$2yy_1 = \frac{(-a^2x) - a^2(a-x)}{x^2} = -\frac{a^3}{x^2}$$

$$\Rightarrow y_1 = -\frac{a^3}{2yx^2} \rightarrow (1)$$

at $(x,y) = (a,0) \Rightarrow y_1 = \infty \Rightarrow y_1$ does not exists

$\therefore$ use $\rho = \left| \frac{\{1+x_1^2\}^{3/2}}{x_2} \right|$

Let $x_1 = \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{y_1}$

$$\Rightarrow x_1 = -\frac{2yx^2}{a^3} \longrightarrow (3)$$

at $(x,y) = (a,0) \Rightarrow x_1 = 0$

Differentiate (3) w r to y

$$\Rightarrow x_2 = -\frac{2}{a^3}\left[x^2 + 2xyx_1\right]$$

$$= -\frac{2}{a^3} x^2 + 2yx\left(-\frac{2yx^2}{a^3}\right)$$

$$x_2 = -\frac{2x^2}{a^3}\left[1+4y^2\right]$$

at $(x,y) = (a,0) \Rightarrow x_2 = -\frac{2}{a}$

$$\therefore \rho = \left|\frac{\{1+0\}^{3/2}}{-2/a}\right| \Rightarrow \rho = \frac{a}{2}$$

5) prove that for the parabola $y^2 = 4ax$, the square of the radius of curvature at any point varies as the cube of focal distance of the point

$\rightarrow$ Given $y^2 = 4ax \longrightarrow (1)$ in cartesian form

we have $\rho = \frac{\{1+y_1^2\}^{3/2}}{y_2}$

Differentiate (1) w r to x

$$2yy = 4a$$

$$y_1 = \frac{4a}{2y} \Rightarrow y_1 = \frac{2a}{y} \longrightarrow (2)$$

Differentiate (2) w.r.to x

$$y_2 = -\frac{2a}{y^2} y_1 = -\frac{2a}{y^2}\left(\frac{2a}{y}\right) = -\frac{4 \cdot a^2}{y^3}$$

$$\therefore \rho = \left| \frac{\left\{ 1 + \frac{4a^2}{y^2} \right\}}{\frac{-4a^2}{y^3}} \right|$$

$$= \left| \frac{\frac{\{y^2+4a^2\}^{3/2}}{\cancel{y^3}}}{\frac{-4a^2}{\cancel{y^3}}} \right|$$

$$= \left| \frac{\{4ax+4a^2\}^{3/2}}{-4a^2} \right|$$

$$= \left| \frac{(4a)^{3/2}\cdot\{x+a\}^{3/2}}{-4a^2} \right|$$

$$= \left| \frac{\cancel{4}\sqrt{4}\,\cancel{a}\sqrt{a}\,\{x+a\}^{3/2}}{-\cancel{4}\,a^{\cancel{2}}} \right|$$

$$= \left| -\sqrt{\frac{4}{a}}(x+a)^{3/2} \right|$$

$$\Rightarrow \rho = \sqrt{\frac{4}{a}}\,(x+a)^{3/2}$$

6) Find the radius of curvature for the curve $x^2y = a(x^2+y^2)$ at the point $(-2a, 2a)$

$\rightarrow$ Given $x^2y = a(x^2+y^2) \longrightarrow (1)$ in cartesian form

Differentiate (1) w.r.to x

$$\Rightarrow 2xy + x^2y_1 = 2ax + 2ayy_1$$

$$\Rightarrow (x^2 - 2ay)y_1 = 2ax - 2xy$$

$$\Rightarrow y_1 = \frac{2ax - 2xy}{x^2 - 2ay} \longrightarrow (2)$$

at $(-2a, 2a) \Rightarrow y_1 = \dfrac{-4a^2 + 8a^2}{4a^2 - 4a^2} \Rightarrow y_1 = \infty$

Hence consider $\rho = \dfrac{\{1 + x_1^2\}^{3/2}}{x_2}$

Hence $x_1 = \frac{dx}{dy} = \frac{1}{y_1} = \frac{x^2-2ay}{2ax-2xy} \rightarrow (3)$

at $(-2a, 2a) \Rightarrow x_1 = 0$

Now differentiate (3) w.r.t x

$$\Rightarrow x_2 = \frac{(2ax-2xy)(2xx_1-2a) - (x^2-2ay)(2ax_1-2x_1y-2x)}{(2ax-2xy)^2}$$

at $(-2a, 2a)$

$$\Rightarrow x_2 = \frac{(-4a^2+8a^2)(0-2a) - (\cancel{4a^2-4a^2})^{0}(0-0+4a)}{(-4a^2+8a^2)^2}$$

$$= \frac{-8a^3}{16a^4}$$

$$x_2 = -\frac{1}{2a}$$

$$\therefore \rho = \left|\frac{\{1+0\}^{3/2}}{-1/2a}\right| = \left|-2a(1)^{3/2}\right|$$

$$\Rightarrow \boxed{\rho = 2a}$$

7) For the curve $y = \frac{ax}{(a+x)}$ where a is a constant prove that

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$$

$\rightarrow$ Given $y = \frac{ax}{a+x} \rightarrow (1)$ is in Cartesian form

We have $\rho = \left|\frac{\{1+y_1^2\}^{3/2}}{y_2}\right|$

Differentiate (1) w.r.to x

$$\Rightarrow y_1 = \frac{(a+x)a - ax(1)}{(a+x)^2} = \frac{a^2}{(a+x)^2} \rightarrow (2)$$

Differentiate (2) w r to x

$$\Rightarrow y_2 = \frac{(a+x)^2 \cdot 0 - 2(a+x)a^2}{(a+x)^4} = \frac{-2a^2}{(a+x)^3} \rightarrow (3)$$

$$(2) \Rightarrow y_1 = \frac{a^2x^2}{x^2(a+x)^2} = \frac{y^2}{x^2} \quad \&$$

$$(3) \Rightarrow y_2 = -2\left[\frac{a^3x^3}{ax^3(a+x)^3}\right] = \frac{-2y^3}{ax^3}$$

$$\therefore \rho = \left| \frac{\left\{1+\frac{y^4}{x^4}\right\}^{3/2}}{-\frac{2y^3}{ax^3}} \right| = \frac{ax^3}{2y^3}\left\{1+\frac{y^4}{x^4}\right\}^{3/2}$$

$$\Rightarrow \left(\frac{2\rho}{a}\right) = \frac{x^3}{y^3}\left\{1+\frac{y^4}{x^4}\right\}^{3/2}$$

take $2/3^{th}$ root on bothsides

$$\Rightarrow \left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^2}{y^2}\left\{1+\frac{y^4}{x^4}\right\}$$

$$\Rightarrow \left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^2}{y^2} + \frac{y^2}{x^2}$$

$$\Rightarrow \left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

8) prove that the radius of curvature ρ at any point (x, y) on the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ is given by $\rho = \frac{2(ax+by)^{3/2}}{ab}$

→ Given $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ is in cartesian → (1)

WKT $\rho = \left| \frac{\{1+y_1^2\}^{3/2}}{y_2} \right|$

Differentiate (1) w.r.to x

$$\Rightarrow \frac{1}{2}\left(\frac{x}{a}\right)^{-1/2} + \frac{1}{2}\left(\frac{y}{b}\right)^{-1/2} \cdot \frac{y'}{b} = 0$$

$$\Rightarrow \frac{y_1}{\cancel{2}\sqrt{yb}} = -\frac{1}{\cancel{2}\sqrt{ax}}$$

$$\Rightarrow y_1 = -\frac{\sqrt{by}}{\sqrt{ax}} \longrightarrow (2)$$

Differentiate (2) w.r.to x

$$\Rightarrow y_2 = -\frac{\left[\sqrt{ax} \cdot \frac{1}{2\sqrt{by}} \cdot b y_1 - \sqrt{by} \cdot \frac{1}{2\sqrt{ax}} \cdot a\right]}{(\sqrt{ax})^2}$$

$$= -\frac{\left[-\frac{b}{2} \frac{\sqrt{ax}}{\sqrt{by}} \cdot \frac{\sqrt{by}}{\sqrt{ax}} - \frac{a}{2} \frac{\sqrt{by}}{\sqrt{ax}}\right]}{ax}$$

$$= \frac{\frac{b}{2} + \frac{a}{2} \frac{\sqrt{by}}{\sqrt{ax}}}{ax}$$

$$= \frac{b\sqrt{ax} + a\sqrt{by}}{2(ax)^{3/2}}$$

$$= \frac{ab\left[\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}\right]}{2(ax)^{3/2}}$$

$$y_2 = \frac{ab}{2(ax)^{3/2}} \longrightarrow (3)$$ 44 $\because$ (1)

$$\rho = \left| \frac{\left[1+\left(-\sqrt{\frac{by}{ax}}\right)^2\right]^{3/2}}{\frac{ab}{2(ax)^{3/2}}} \right| = \frac{\left[1+\frac{by}{ax}\right]^{3/2}}{\frac{ab}{2(ax)^{3/2}}}$$

$$\rho = \frac{\frac{[ax+by]^{3/2}}{(ax)^{3/2}}}{\frac{ab}{2(ax)^{3/2}}}$$

$$\Rightarrow \rho = \frac{2\,[ax+by]^{3/2}}{ab}$$

9) prove that for the rectangular hyperbola $xy=c^2$, the radius of curvature at any point $P(x,y)$ is given by $\rho = \frac{r^3}{2c^2}$, where r is distance of the point P from the orig

→ Given $xy=c^2 \rightarrow (1)$ is in cartesian form
Differentiate (1) w.r.t x.

$$\Rightarrow y + xy_1 = 0 \rightarrow (2)$$

$$\Rightarrow y_1 = -\frac{y}{x}$$

Differentiate (2) w.r.t x

$$\Rightarrow y_1 + xy_2 + y_1 = 0$$

$$\Rightarrow 2y_1 + xy_2 = 0$$

$$\Rightarrow y_2 = -\frac{2y_1}{x}$$

$$\Rightarrow y_2 = \frac{2y}{x^2} \qquad \because y_1 = -y/x$$

$$\therefore \rho = \left| \frac{\{1+y_1^2\}^{3/2}}{y_2} \right|$$

$$= \left| \frac{\{1+(-y/x)^2\}^{3/2}}{\frac{2y}{x^2}} \right|$$

$$= \frac{\left\{\frac{x^2+y^2}{x^2}\right\}^{3/2}}{\frac{2y}{x^2}}$$ & $x^2+y^2=r^2$ is the rectangula hyperbola

$$= \frac{\left\{\frac{r^2}{x^2}\right\}^{3/2}}{\frac{2y}{x^2}} = \frac{\cancel{x^2}}{2y}\frac{r^3}{x^{\cancel{3}}}$$

$$\rho = \frac{r^3}{2xy} \Rightarrow \boxed{\rho = \frac{r^3}{2c^2}} \quad \because xy = c^2$$

10) In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ show that the radius of curvatu at an end point of the major axis is equal to the semi-l at rectum.

→ Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow (1)$ is in cartesion form

Differentiate (1) w.r.to x

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

$$\Rightarrow y_1 = -\frac{b^2x}{a^2y} \longrightarrow (2)$$

Differentiate (2) w.r.to x

$$\Rightarrow y_2 = \frac{-b^2(a^2y)+a^2y_1(b^2x)}{a^4y^2}$$

$$= \frac{b^2a^2(y_1x-y)}{a^4y^2}$$

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$$\Rightarrow y_2 = \frac{b^2}{a^2}\left[\frac{-b^2x^2}{a^2 y} - y\right] \Big/ y^2$$

$$= -\frac{b^2[a^2y^2+b^2x^2]}{a^4y^3}$$

$$\therefore \rho = \left|\frac{\{1+y_1^2\}^{3/2}}{y_2}\right|$$

$$= \left|\frac{-a^4y^3}{b^2[a^2y^2+b^2x^2]}\left\{1+\frac{b^4x^2}{a^4y^2}\right\}^{3/2}\right|$$

$$= \left|\frac{-a^4y^3\{a^4y^2+b^4x^2\}^{3/2}}{b^2[a^2y^2+b^2x^2]\,a^6y^3}\right|$$

$$= \left|\frac{-(a^4y^2+b^4x^2)^{3/2}}{a^2b^2(a^2y^2+b^2x^2)}\right|$$

At the end point of the major axis, we have $x=\pm a$ and $y=0$

$$\Rightarrow \rho = \left|\frac{-(a^2b^4)^{3/2}}{a^2b^2(b^2a^2)}\right| \Rightarrow \rho = \frac{b^2}{a} = \text{Semi-latus rectum}$$

* Radius of curvature in polar form: $r = f(\theta)$

$$\rho = \left| \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - r r_2} \right|$$

1) S.T radius of curvature at any point the curve cardioid $r = a(1-\cos\theta)$ varies as $\sqrt{r}$

→ Given $r = a(1-\cos\theta) \longrightarrow (1)$, Differentiate (1) w.r.t to θ

$r_1 = a\sin\theta$ & $r_2 = a\cos\theta$

$$\therefore \rho = \left| \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - r r_2} \right|$$

$$= \frac{\left(a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}{2a^2\sin^2\theta - a^2(1-\cos\theta)\cos\theta + a^2(1-\cos\theta)^2}$$

$$= \frac{\left\{a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta\right\}^{3/2}}{2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta + a^2(1+\cos^2\theta - 2\cos\theta)}$$

$$= \frac{\left\{a^2(2-2\cos\theta)\right\}^{3/2}}{3a^2 - 3a^2\cos\theta} = \frac{2\sqrt{2}a^3(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$\Rightarrow \rho = \frac{2\sqrt{2}a}{3}(1-\cos\theta)^{1/2}$$

$$= \frac{2\sqrt{2}a}{3}\left(\frac{r}{a}\right)^{1/2} \quad \because (1)$$

$$= \left(\frac{2\sqrt{2}a}{3\sqrt{a}}\right)\sqrt{r} = \left(\frac{2}{3}\sqrt{2a}\right)\sqrt{r}$$

$$\Rightarrow \rho \propto \sqrt{r}$$

2) Find the radius of curvature of the following.

(i) $r = a(1+\cos\theta)$

→ Given $r = a(1+\cos\theta)$ —(1), Differentiate (1) w r to θ

$r_1 = -a\sin\theta$ & $r_2 = -a\cos\theta$

$$\therefore \rho = \frac{\left(a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}{2a^2\sin^2\theta + a^2(1+\cos\theta)^2 + a^2(1+\cos\theta)\cos\theta}$$

$$= \frac{\{a^2(1+\cos^2\theta+2\cos\theta+\sin^2\theta)\}^{3/2}}{a^2\{2\sin^2\theta+\cos^2\theta+1+3\cos\theta+\cos^2\theta\}}$$

$$= \frac{a^3(2+2\cos\theta)^{3/2}}{a^2(3+3\cos\theta)} = \frac{(a)\,2^{3/2}(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$$

$$= \frac{2\sqrt{2}a}{3}(1+\cos\theta)^{1/2}$$

$$= \frac{2\sqrt{2}}{3}a\left(\frac{r}{a}\right)^{1/2}$$

$$\Rightarrow \rho = \frac{2}{3}\sqrt{2ar}$$

ii) $r^n = a^n \sin n\theta$

→ Given $r^n = a^n\sin n\theta$ —(1)

take Log on bothsides

$\Rightarrow n\log r = n\log a + \log\sin n\theta$

Differentiate w.r to θ

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta}(n\cos n\theta) \Rightarrow \frac{1}{r}r_1 = \cot n\theta$$

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$\Rightarrow r_1 = r\cot n\theta$

Differentiate again w.r.t θ

$$\Rightarrow r_2 = r_1 \cot n\theta + r(-n\operatorname{cosec}^2 n\theta)$$

$$= r\cot^2 n\theta - rn\operatorname{cosec}^2 n\theta$$

$$\Rightarrow r_2 = r\left[\cot^2 n\theta - n\operatorname{cosec}^2 n\theta\right]$$

$$\therefore \rho = \left|\frac{\left[r^2+r_1^2\right]^{3/2}}{r^2+2r_1^2-rr_2}\right|$$

$$= \frac{\left(r^2+r^2\cot^2 n\theta\right)^{3/2}}{\left[r^2+2r^2\cot^2 n\theta - r^2\left[\cot^2 n\theta - n\operatorname{cosec}^2 n\theta\right]\right]}$$

$$= \frac{r^3\left(1+\cot^2 n\theta\right)^{3/2}}{\left[r^2+r^2\cot^2 n\theta + n\operatorname{cosec}^2 n\theta\right]}$$

$$= \frac{r^3\left[\operatorname{cosec}^2 n\theta\right]^{3/2}}{r^2\left[\operatorname{cosec}^2 n\theta + n\operatorname{cosec}^2 n\theta\right]}$$

$$= \frac{r\left[\operatorname{cosec}^3 n\theta\right]}{(n+1)\operatorname{cosec}^2 n\theta}$$

$$= \frac{r\operatorname{cosec} n\theta}{(n+1)}$$

$$= \frac{r}{(n+1)}\cdot\frac{1}{\sin n\theta}$$

$$= \frac{r}{n+1}\left(\frac{a^n}{r^n}\right) \qquad \because r^n = a^n \sin n\theta$$

$$\rho = \frac{a^n}{r^{n-1}(n+1)}$$

3) Show that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ & $y = a(1-\cos\theta)$

→ Given $x = a(\theta+\sin\theta)$ & $y = a(1-\cos\theta)$

Let $y_1 = \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

Let $\dfrac{dx}{d\theta} = a(1+\cos\theta)$ & $\dfrac{dy}{d\theta} = a\sin\theta$

$\Rightarrow y_1 = \dfrac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)} \Rightarrow y_1 = \dfrac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} \Rightarrow y_1 = \tan\theta/2$

Let $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{d\theta}\cdot\left(\dfrac{dy}{dx}\right)\cdot\dfrac{d\theta}{dx}$

$= \dfrac{d}{d\theta}\left(\tan\theta/2\right)\cdot\dfrac{1}{a(1+\cos\theta)}$

$= \dfrac{1}{2}\sec^2\theta/2\cdot\dfrac{1}{2a\cos^2\theta/2}$

$y_2 = \dfrac{1}{4a}\sec^4\theta/2$

$\therefore \rho = \left|\dfrac{\{1+y_1^2\}^{3/2}}{y_2}\right| = \dfrac{\{1+\tan^2\theta/2\}^{3/2}}{\frac{1}{4a}\sec^4\theta/2}$

$\Rightarrow \rho = \dfrac{4a(\sec^2\theta/2)^{3/2}}{\sec^4\theta/2}$

$\Rightarrow \rho = \dfrac{4a}{\sec\theta/2}$

$\Rightarrow \rho = 4a\cos\theta/2$

4) Show that for the equiangular spiral $r = ae^{\theta\cot\alpha}$ where a and α are constants ρ/r is a constant

→ Given $r = ae^{\theta\cot\alpha} \longrightarrow (1)$

Taking Log on both sides

$$\log r = \log a + \log_e e^{\theta\cot\alpha}$$

$$\Rightarrow \log r = \log a + \theta\cot\alpha$$

Differentiate w.r.t θ

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \cot\alpha$$

$$\Rightarrow r_1 = r\cot\alpha$$

Differentiate again w.r.t θ

$$\Rightarrow r_2 = r_1\cot\alpha$$

$$\Rightarrow r_2 = r\cot^2\alpha$$

$$\therefore \rho = \left|\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}\right|$$

$$= \left|\frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha-r^2\cot^2\alpha}\right|$$

$$= \left|\frac{r^3(1+\cot^2\alpha)^{3/2}}{r^2+r^2\cot^2\alpha}\right|$$

$$= \frac{r^3(\operatorname{cosec}^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)}$$

$$\rho = \frac{r\operatorname{cosec}^3\alpha}{\operatorname{cosec}^2\alpha}$$

$$\Rightarrow \frac{\rho}{r} = \operatorname{cosec}\alpha = \text{constant}$$

5) Show that for the curve $r(1-\cos\theta) = 2a$, then ρ^2 varies as r^3

→ Given $r(1-\cos\theta) = 2a \rightarrow (1)$

taking Log on both sides

$$\Rightarrow \log r + \log(1-\cos\theta) = \log 2a$$

Differentiate w.r.t to θ

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{1-\cos\theta}\sin\theta = 0$$

$$\Rightarrow \frac{1}{r}r_1 = \frac{-2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = -\cot\theta/2$$

$$\Rightarrow r_1 = -r\cot\theta/2$$

Differentiate w.r.t to θ

$$\Rightarrow r_2 = -\left[r_1\cot\theta/2 - \frac{r}{2}\operatorname{cosec}^2\theta/2\right]$$

$$= -\left[-r\cot^2\theta/2 - \frac{r}{2}\operatorname{cosec}^2\theta/2\right]$$

$$= r\cot^2\theta/2 + \frac{r}{2}\operatorname{cosec}^2\theta/2$$

$$\therefore \rho = \left|\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}\right|$$

$$= \left|\frac{(r^2+r^2\cot^2\theta/2)^{3/2}}{r^2+2r^2\cot^2\theta/2 - \frac{r^2}{2}\operatorname{cosec}^2\theta/2 - r^2\cot^2\theta/2}\right|$$

$$= \frac{r^3(1+\cot^2\theta/2)^{3/2}}{r^2+r^2\cot^2\theta/2 - \frac{r^2}{2}\operatorname{cosec}^2\theta/2}$$

$$= \frac{r^3(\operatorname{cosec}^2\theta/2)^{3/2}}{r^2\operatorname{cosec}^2\theta/2 - \frac{r^2}{2}\operatorname{cosec}^2\theta/2} = \frac{r^3\operatorname{cosec}^3\theta/2}{\frac{r^2}{2}\operatorname{cosec}^2\theta/2}$$

$$\Rightarrow \rho = 2r\operatorname{cosec}\theta/2 \longrightarrow (2)$$

But $r(1-\cos\theta)=2a$

$$\Rightarrow r=\frac{2a}{1-\cos\theta}$$

$$\Rightarrow r=\frac{2a}{2\sin^2\theta/2} \Rightarrow \frac{r}{a}=\operatorname{cosec}^2\theta/2$$

$$\Rightarrow \operatorname{cosec}\theta/2=\sqrt{\frac{r}{a}}$$

$$① \Rightarrow \rho=2r\cdot\sqrt{\frac{r}{a}}$$

$$\Rightarrow \rho=2\frac{r^{3/2}}{a^{1/2}}$$

$$\Rightarrow \rho^2=\frac{2}{a}r^3$$

$$\Rightarrow \rho^2 \propto r^3$$

6) Find the radius of curvature of the curve $r=a\sin n\theta$ at the pole

→ Given $r=a\sin n\theta$

Differentiate w.r.to θ

$r_1=an\cos n\theta$ & $r_2=-an^2\sin n\theta$

at the pole (origin), $\theta=0$

$$\Rightarrow r_1=an\cos 0 \Rightarrow r_1=an$$

$$r_2=-an^2\sin 0 \Rightarrow r_2=0$$

$$r=a\sin 0 \Rightarrow r=0$$

$$\therefore \rho=\left|\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}\right| = \left|\frac{(0+a^2n^2)^{3/2}}{0+2a^2n^2-0}\right|$$

$$\Rightarrow \rho=\frac{a^3n^3}{2a^2n^2}$$

$$\Rightarrow \boxed{\rho=\frac{an}{2}}$$

7) show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ varies inversely as r^{n-1}

→ Given $r^n = a^n \cos n\theta \longrightarrow (1)$

take Log on bothsides

$$n \log r = n \log a + \log \cos n\theta$$

Differentiate w.r.t. θ

$$\Rightarrow \frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n \sin n\theta)$$

$$\Rightarrow r_1 = -r \tan n\theta$$

Differentiate w.r.t. θ

$$\Rightarrow r_2 = -[r_1 \tan n\theta + nr \sec^2 n\theta]$$

$$r_2 = r \tan^2 n\theta - nr \sec^2 n\theta$$

$$\therefore \rho = \left| \frac{\{r^2 + r_1^2\}^{3/2}}{r^2 + 2r_1^2 - r r_2} \right| = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{(r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta)}$$

$$= \frac{r^3 (1 + \tan^2 n\theta)^{3/2}}{r^2 (1 + \tan^2 n\theta + n \sec^2 n\theta)}$$

$$= \frac{r (\sec^2 n\theta)^{3/2}}{(n+1) \sec^2 n\theta}$$

$$\rho = \frac{r \sec n\theta}{(n+1)} \longrightarrow (2)$$

$$(1) \Rightarrow \cos n\theta = \frac{r^n}{a^n} \Rightarrow \sec n\theta = \frac{a^n}{r^n}$$

$$\Rightarrow \rho = \frac{r}{(n+1)} \frac{a^n}{r^n} = \frac{a^n}{(n+1)} \frac{1}{r^{n-1}}$$

$$\Rightarrow \rho \propto \frac{1}{r^{n-1}}$$

slope → change in x corresponding to change in y

tangent → The tangent is a Line to a plane curve at given point is the straight line that just touches the curve at that point.

pivya.R

FOCUS

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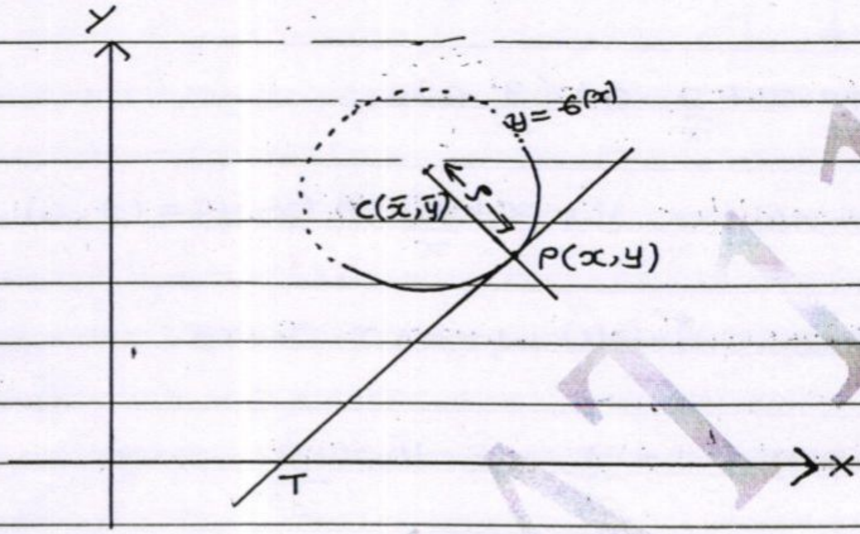
* Centre of curvature and circle of curvature



The centre of curvature of a point $P(x,y)$ on the given curve is the point $C(\bar{x},\bar{y})$ which lies inthe direction of the inward normal at P and is at distance ρ from it.

Let PT be the tangent to the curve at P

If $CP=\rho$ then $C(\bar{x},\bar{y})$ is the centre of curvature to the curve at P.

The coordinates $(\bar{x},\bar{y})$ of the centre of curvature for the cartesian curve $y=f(x)$ can be proved in the following form

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) \quad \& \quad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

The circle of curvature of the curve at $P(x,y)$ is the circle having centre at the point $C(\bar{x},\bar{y})$ and radius equal to ρ & is given by

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$$

Problems

* Find the centre of curvature and circle of curvature for the following

1) $y = x + \frac{9}{x}$ at $(3, 6)$

→ consider, $y = x + \frac{9}{x} \rightarrow (1)$ $(x, y) = (3, 6)$

Differentiat (1) w.r to x twice

$\Rightarrow y_1 = 1 - \frac{9}{x^2}$ & $y_2 = \frac{18}{x^3}$

at $(x, y) = (3, 6) \Rightarrow y_1 = 0$ & $y_2 = \frac{2}{3}$

∴ centre of curvature $C = (\bar{x}, \bar{y})$ is

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2) = 3 - \frac{0}{2/3}(1 + 0) = 3$$

$$\bar{y} = y + \frac{(1 + y_1^2)}{y_2} = 6 + \frac{(1 + 0)}{2/3} = \frac{15}{2}$$

∴ Centre of Curvature $C(\bar{x}, \bar{y}) = (3, 15/2)$

Next, we shall find ρ,

$$\rho = \left| \frac{(1 + y_1^2)^{3/2}}{y_2} \right| = \left| \frac{(1 + 0)^{3/2}}{2/3} \right| = \frac{3}{2}$$

Circle of curvature: $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$

$$\Rightarrow (x - 3)^2 + (y - 15/2)^2 = \left(\frac{3}{2}\right)^2$$

$$\Rightarrow (x-3)^2+(y-15/2)^2 = 9/4$$

2) $y^2=12x$ at $(3,6)$

$\rightarrow$ consider $y^2=12x$; $(x,y)=(3,6)$

Differentiate w.r.to x twice

$$\Rightarrow 2yy_1 = 12 \Rightarrow y_1 = \frac{12}{2y} \Rightarrow y_1 = \frac{6}{y}$$

$$\& \; y_2 = -\frac{6}{y^2} y_1$$

at $(x,y)=(3,6)$

$$\Rightarrow y_1 = \frac{6}{6} \Rightarrow y_1 = 1 \quad \& \; y_2 = -\frac{6}{36}(1) \Rightarrow y_2 = \frac{-1}{6}$$

$\therefore$ centre of curvature $C=(\bar{x},\bar{y})$ is

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = 3 - \frac{1}{(-1/6)}(1+1^2) = 3+6(2)$$

2)

$$\Rightarrow \bar{x} = 15$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 6 + \frac{(1+1)}{-1/6} = 6-12 \Rightarrow \bar{y} = -6$$

$\therefore$ centre of curvature $C(\bar{x},\bar{y}) = (15,-6)$

Next, we shall find ρ,

$$\rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right| = \left| \frac{(1+1)^{3/2}}{-1/6} \right| = 6(2\sqrt{2})$$

$\Rightarrow \rho = 12\sqrt{2}$

Circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

$\Rightarrow (x-15)^2+(y+6)^2=(12\sqrt{2})^2$

$\Rightarrow (x-15)^2+(y+6)^2=288$

3) $xy=c^2$ at (c,c)

$\rightarrow$ Given $xy=c^2$; $(x,y)=(c,c)$

Differentiate w.r.to x

$\Rightarrow y+xy_1=0$

$\Rightarrow y_1=-\dfrac{y}{x}$

& $y_2=-\dfrac{[xy_1-y]}{x^2}$

at $(x,y)=(c,c)$

$\Rightarrow y_1=-\dfrac{c}{c} \Rightarrow y_1=-1$

& $y_2=-\dfrac{[-c-c]}{c^2}=\dfrac{2c}{c^2} \Rightarrow y_2=\dfrac{2}{c}$

$\therefore$ centre of curvature $C=(\bar{x},\bar{y})$ is

$\bar{x}=x-\dfrac{y_1}{y_2}(1+y_1^2) = c+\dfrac{1}{2/c}[1+1] = 2c$

$\bar{y}=y+\dfrac{(1+y_1^2)}{y_2} = c+\dfrac{(1+1)}{2/c} = 2c$

4)

$\rightarrow$

at $\left(\frac{a}{4},\frac{a}{4}\right)$

$\therefore$ centre of curvature $(\bar{x}, \bar{y}) = (2c, 2c)$

Next we shall find ρ, $\rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right|$

$$= \left| \frac{(1+1)^{3/2}}{2/c} \right| = \frac{2\sqrt{2}c}{2}$$

$\Rightarrow \rho = \sqrt{2}c$

Circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$\Rightarrow (x-2c)^2 + (y-2c)^2 = 2c^2$

4) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a/4, a/4)$

$\rightarrow$ Given $\sqrt{x} + \sqrt{y} = \sqrt{a} \rightarrow (1)$ & $(x, y) = (a/4, a/4)$

Differentiate (1) w r to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y_1 = 0$$

$$\Rightarrow y_1 = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}}$$

$$\Rightarrow y_1 = -\frac{\sqrt{y}}{\sqrt{x}}$$

at $(x, y) = \left(\frac{a}{4}, \frac{a}{4}\right)$

$\Rightarrow y_1 = -1$

$$\& \; y_2 = \frac{\sqrt{x}\left(\frac{-1}{2\sqrt{y}} \cdot y_1\right) + \sqrt{y}\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x})^2}$$

at $\left(\frac{a}{4}, \frac{a}{4}\right)$ $\quad y_2 = \frac{\frac{1}{2} + \frac{1}{2}}{a/4}$

$\Rightarrow y_2 = \frac{4}{a}$

$\therefore$ Centre of curvature $\bar{x} = x - \frac{y_1}{y_2}[1+y_1^2]$

$= \frac{a}{4} + \frac{1}{4/a}[1+1]$

$= \frac{3a}{4}$

at (1,

$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = \frac{a}{4} + \frac{(1+1)}{4/a}$

$= \frac{3a}{4}$

Hence centre of curvature $(\bar{x}, \bar{y}) = \left(\frac{3a}{4}, \frac{3a}{4}\right)$

Next we shall find, ρ

$\rho = \left|\frac{(1+y_1^2)^{3/2}}{y_2}\right| = \left|\frac{(1+1)^{3/2}}{4/a}\right| = \frac{a}{\sqrt{2}}$

$\therefore$ Circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$\Rightarrow \left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$

5) $x^3 + y^3 = 2xy$ at $(1,1)$

$\rightarrow$ Differentiate w.r.t x

$\Rightarrow 3x^2 + 3y^2 y_1 = 2[xy_1 + y]$

$\Rightarrow y_1 = \frac{2y - 3x^2}{3y^2 - 2x} \rightarrow (1)$

at $(1,1) \Rightarrow y_1 = \frac{2-3}{3-2} \Rightarrow y_1 = -1$

Differentiate (1) w.r.to x

$$\Rightarrow y_2 = \frac{(3y^2-2x)(2y_1-6x) - (2y-3x^2)(6yy_1-2)}{(3y^2-2x)^2}$$

at $(1,1)$ $y_2 = \frac{(1)(-2-6) - (-1)(-6-2)}{1} = -16$

Hence centre of curvature, $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$$= 1 + \frac{1}{-16}(1+1)$$

$$\Rightarrow \bar{x} = \frac{7}{8}$$

$$\bar{y} = 1 + \frac{(1+1)}{-16} = \frac{7}{8}$$

$\therefore C(\bar{x}, \bar{y}) = (7/8, 7/8)$

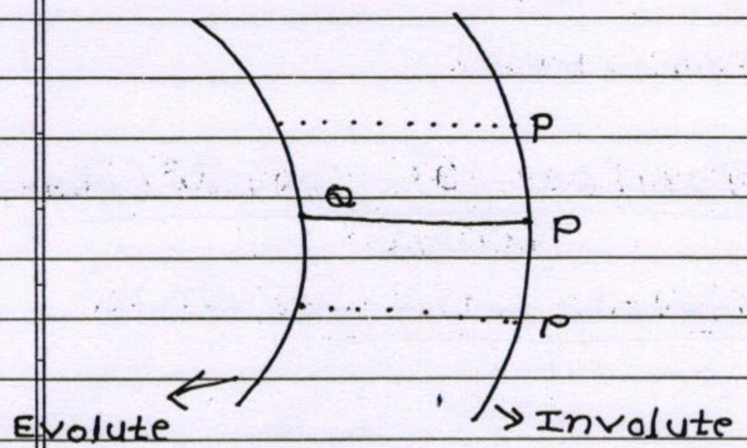
Next we shall find, ρ

$$\rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right| = \left| \frac{(1+1)^{3/2}}{-16} \right| = \frac{\sqrt{2}}{8}$$

$\therefore$ circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$\Rightarrow (x - 7/8)^2 + (y - 7/8)^2 = (\sqrt{2}/8)^2 = 1/32$$

* Evolute and Involute:-



The locus of the centre of curvature of the given curve is called the evolute of the given curve & The given curve is called the involute

Working rule:-

Step(1):- We prefer to consider the curve $y=f(x)$ in the parametric form $x=x(t)$ & $y=y(t)$

Step(2):- Compute y_1 and y_2 interms of t.

Step(3):- Compute centre of curvature $(\bar{x},\bar{y})$ for the points (x,y) in terms of t.

Step(4):- Eliminate t from $\bar{x}$ & $\bar{y}$ to get an expression of the form $F(\bar{x},\bar{y})=k$, where k being a constant

Step(5):- Taking the locus of $(\bar{x},\bar{y})$ in $F(\bar{x},\bar{y})=k$ by replacing $(\bar{x},\bar{y})$ by (x,y)

$\Rightarrow F(x,y)$ is the evolute of the involute curve $y=f(x)$

Note:-

	Name of the curve	Cartesian form	Parametric form
1	Parabola	i) $y^2 = 4ax$ ii) $x^2 = 4ay$	$x = at^2,\ y = 2at$ $x = 2at,\ y = at^2$
2.	Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t$ $y = b\sin t$
3.	Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec t$ $y = b\tan t$
4.	Rectangular Hyperbola	$xy = c^2$	$x = ct,\ y = \frac{c}{t}$
5.	Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3 t,$ $y = a\sin^3 t$

* Problems:

1) Find the evolute of the parabola $y^2 = 4ax$

$\rightarrow$ consider $y^2 = 4ax \rightarrow (1)$

Let $x = at^2$ & $y = 2at$, for parametric form

$$\Rightarrow \frac{dx}{dt} = 2at \ \& \ \frac{dy}{dt} = 2a$$

$$\text{Let } y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at}$$

$$\Rightarrow y_1 = 1/t$$

Let $y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{1}{2at} = \frac{-1}{t^2}\left(\frac{1}{2at}\right)$$

$$\Rightarrow y_2 = \frac{-1}{2at^3}$$

We have $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$$= at^2 - \frac{1/t}{(-1/2at^3)}\left(1+\frac{1}{t^2}\right)$$

$$= at^2 + 2at^2\left(1+\frac{1}{t^2}\right)$$

$$= at^2 + 2at^2 + 2a$$

$$\Rightarrow \bar{x} = 3at^2 + 2a \longrightarrow (2)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-1/2at^3}$$

$$= 2at - 2at^3\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3 - 2at$$

$$\Rightarrow \bar{y} = -2at^3 \longrightarrow (3)$$

Eliminating t from (3) & (2)

$$(2) \Rightarrow t^2 = \frac{\bar{x} - 2a}{3a}$$

$$(t^2)^{3/2} = \left(\frac{\bar{x} - 2a}{3a}\right)^{3/2}$$

$$\Rightarrow t^3 = \left(\frac{\bar{x} - 2a}{3a}\right)^{3/2}$$

$$(3) \Rightarrow \bar{y} = -2at^3 = -2a\left(\frac{\bar{x} - 2a}{3a}\right)^{3/2}$$

$$\Rightarrow (\bar{y})^2 = 4a^2\left(\frac{\bar{x} - 2a}{3a}\right)^3$$

$$\Rightarrow (\bar{y})^2 = 4a^2 \frac{(\bar{x} - 2a)^3}{27a^3}$$

$$\Rightarrow 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3$$

Now by taking the locus of $(\bar{x}, \bar{y})$, The required evolute is

$$27ay^2 = 4(x - 2a)^3$$

2) Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

→ consider the parametric equation of the ellipse
$x = a\cos t$; $y = b\sin t$

$$\Rightarrow \frac{dx}{dt} = -a\sin t \quad \& \quad \frac{dy}{dt} = b\cos t$$

Let $y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t} \Rightarrow y_1 = -\frac{b}{a}\cot t$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$\Rightarrow y_2 = \frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\cdot\frac{1}{-a\sin t} = -\frac{b}{a}\left(-\operatorname{cosec}^2 t\right)\cdot\left(-\frac{1}{a}\operatorname{cosec} t\right)$$

$$\Rightarrow y_2 = -\frac{b}{a^2}\operatorname{cosec}^3 t$$

We have $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$$= a\cos t + \frac{\frac{b}{a}\cot t}{-b/a^2 \operatorname{cosec}^3 t}\left(1+\frac{b^2}{a^2}\cot^2 t\right)$$

$$= a\cos t - a\sin^3 t\cdot\frac{\cos t}{\sin t}\left(1+\frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= a\cos t - a\sin^2 t\cos t\left(1+\frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= a\cos t - a\sin^2 t\cos t - \frac{b^2}{a}\cos^3 t$$

$$= a\cos t(1-\sin^2 t) - \frac{b^2}{a}\cos^3 t$$

$$= a\cos^3 t - \frac{b^2}{a}\cos^3 t \qquad \because \cos^2 t = 1-\sin^2 t$$

$$\bar{x} = \left(\frac{a^2-b^2}{a}\right)\cos^3 t \longrightarrow (1)$$

Let $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$

$$= b\sin t + \frac{\left(1 + \frac{b^2}{a^2}\cot^2 t\right)}{-\frac{b}{a^2}\operatorname{cosec}^3 t}$$

osect)

$$= b\sin t - \frac{a^2}{b}\sin^3 t\left(1 + \frac{b^2}{a^2}\frac{\cos^2 t}{\sin^2 t}\right)$$

$$= b\sin t - \frac{a^2}{b}\sin^3 t - b\sin t\cos^2 t$$

$$= b(1-\cos^2 t)\sin t - \frac{a^2}{b}\sin^3 t$$

$$= b\sin^3 t - \frac{a^2}{b}\sin^3 t \qquad \because 1-\cos^2 t = \sin^2 t$$

$$\bar{y} = \left(\frac{b^2-a^2}{b}\right)\sin^3 t$$

)

$$\Rightarrow \bar{y} = -\left(\frac{a^2-b^2}{b}\right)\sin^3 t. \rightarrow (2)$$

To eliminate t from (1) & (2)

Squaring & rising to the power $1/3$ on both sides of (1) & (2)

$$(\bar{x})^{2/3} = \left(\frac{a^2-b^2}{a}\right)^{2/3}\cos^2 t \;\; \& \;\; (\bar{y})^{2/3} = \left(\frac{a^2-b^2}{b}\right)^{2/3}\sin^2 t$$

$-\sin^2 t$

$$\Rightarrow (a\bar{x})^{2/3} = (a^2-b^2)^{2/3}\cos^2 t \;\; \& \;\; (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}\sin^2 t$$

$$\therefore (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}\left[\cos^2 t + \sin^2 t\right]$$

$$\Rightarrow (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

Now by taking the locus of $(\bar{x}, \bar{y})$, the required evolute is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$

3) Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

→ consider the parametric equation of the given hyperbola

$$x = a\sec t \quad \text{and} \quad y = b\tan t$$

$$\therefore \frac{dx}{dt} = a\sec t\cdot\tan t \quad \text{and} \quad \frac{dy}{dt} = b\sec^2 t$$

$$\text{Let } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t}$$

$$\Rightarrow y_1 = \frac{b}{a}\frac{1}{\cos t}\cdot\frac{\cos t}{\sin t}$$

$$\Rightarrow y_1 = \frac{b}{a}\operatorname{cosec} t$$

$$\text{Let } y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{b}{a}\operatorname{cosec} t\right)\cdot\frac{dx}{dt}$$

$$\Rightarrow y_2 = -\frac{b}{a}\operatorname{cosec} t\cot t\cdot\frac{1}{a\sec t\cdot\tan t}$$

$$= -\frac{b}{a^2}\frac{1}{\sin t}\cot t\cdot\cot t\cdot\cos t$$

$$= -\frac{b}{a^2}\cot^3 t$$

We have $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$$= a\sec t - \frac{\left(b/a\operatorname{cosec} t\right)\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2 t\right)}{-\frac{b}{a^2}\cot^3 t}$$

$$\bar{x} = a\sec t + a \cdot \frac{1}{\sin t} \cdot \frac{\sin^3 t}{\cos^3 t}\left(1 + \frac{b^2}{a^2}\operatorname{cosec}^2 t\right)$$

$$= a\sec t + a\,\frac{\sin^2 t}{\cos^3 t}\left(1 + \frac{b^2}{a^2}\operatorname{cosec}^2 t\right)$$

$$= a\sec t + \frac{a\sin^2 t}{\cos^3 t} + \frac{b^2}{a}\cdot\frac{\sin^2 t}{\cos^3 t}\cdot\operatorname{cosec}^2 t$$

$$= a\sec t + \frac{a(1-\cos^2 t)}{\cos^3 t} + \frac{b^2}{a}\cdot\frac{1}{\cos^3 t}$$

$$= a\sec t + a\sec^3 t - a\sec t + \frac{b^2}{a}\sec^3 t$$

$$= \left(a + \frac{b^2}{a}\right)\sec^3 t$$

$$\Rightarrow \bar{x} = \left(\frac{a^2+b^2}{a}\right)\sec^3 t \longrightarrow (1)$$

Let $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$

$$= b\tan t + \frac{\left(1 + b^2/a^2 \operatorname{cosec}^2 t\right)}{(-b/a^2)\cot^3 t}$$

$$= b\tan t - \frac{a^2}{b}\left(\frac{\sin^3 t}{\cos^3 t}\right)\left(1 + \frac{b^2}{a^2}\frac{1}{\sin^2 t}\right)$$

$$= b\tan t - \frac{a^2}{b}\tan^3 t - b\cdot\frac{\sin t}{\cos^3 t}$$

$$= b\tan t - \frac{a^2}{b}\tan^3 t - b\tan t\cdot\sec^2 t$$

$$= b\tan t - \frac{a^2}{b}\tan^3 t - b\tan t\,(1+\tan^2 t)$$

$$= \cancel{b\tan t} - \frac{a^2}{b}\tan^3 t - \cancel{b\tan t} - b\tan^3 t$$

$$= -\left(\frac{a^2}{b} + b\right)\tan^3 t$$

$$\bar{y} = -\left(\frac{a^2+b^2}{b}\right)\tan^3 t \longrightarrow (2)$$

To eliminate t from (1) & (2)

Squaring & rising to the power $1/3$ on bothsides of (1) & (2)

$$(\bar{x})^{2/3} = \left(\frac{a^2+b^2}{a}\right)^{2/3} \sec^2 t$$

$$\Rightarrow (\bar{x}a)^{2/3} = (a^2+b^2)^{2/3} \sec^2 t$$

$$(\bar{y})^{2/3} = \left(\frac{a^2+b^2}{b}\right)^{2/3} \tan^2 t$$

$$\Rightarrow (\bar{y}b)^{2/3} = (a^2+b^2)^{2/3} \tan^2 t$$

Hence $(\bar{x}a)^{2/3} - (\bar{y}b)^{2/3} = (a^2+b^2)^{2/3} (\sec^2 t - \tan^2 t)$

$= (a^2+b^2)^{2/3}$ $\quad \because \sec^2 t - \tan^2 t = 1$

$\therefore$ By taking the locus of $(\bar{x}, \bar{y})$, the required evolute is

$$(ax)^{2/3} - (yb)^{2/3} = (a^2+b^2)^{2/3}$$

^3t

4) Find the evolute of the curve $x^2 = 4ay$

→ consider the parametric equation of $x^2 = 4ay$,

$x = 2at \quad , \quad y = at^2$

$\frac{dx}{dt} = 2a \quad \& \quad \frac{dy}{dt} = 2at$

Let $y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2at}{2a} \Rightarrow y_1 = t$

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(t) \cdot \frac{dt}{dx} = 1 \cdot \frac{1}{2a}$

nsider

$\Rightarrow y_2 = 1/2a$

We have $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$\bar{x} = 2at - \frac{t}{1/2a}(1+t^2)$

$= 2at - 2at - 2at^3$

$\Rightarrow \bar{x} = -2at^3 \longrightarrow (1)$

$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = at^2 + \frac{(1+t^2)}{1/2a}$

tan²t)

$= at^2 + 2a + 2at^2$

$\Rightarrow \bar{y} = 3at^2 + 2a$

an²t=1

$\Rightarrow \bar{y} - 2a = 3at^2$

$\Rightarrow t^2 = \frac{\bar{y} - 2a}{3a}$

ired

taking $3/2^{th}$ power on both side

$\Rightarrow t^3 = \left(\frac{\bar{y} - 2a}{3a}\right)^{3/2} \longrightarrow (2)$

Equating (1) & (2)

$$\Rightarrow \bar{x} = -2a\left(\frac{\bar{y}-2a}{3a}\right)^{3/2}$$

S.o.b.s

$$\Rightarrow (\bar{x})^2 = 4a^2\left(\frac{\bar{y}-2a}{3a}\right)^3$$

$$\Rightarrow (\bar{x})^2 = 4a^2\,\frac{(\bar{y}-2a)^3}{27a^3}$$

$$\Rightarrow 27a(\bar{x})^2 = 4(\bar{y}-2a)^3$$

$\therefore$ By taking the locus of $(\bar{x},\bar{y})$ the required evolute is

$$27ax^2 = 4(y-2a)^3$$

5) Find the evolute of the curve $x^{2/3}+y^{2/3}=a^{2/3}$

$\rightarrow$ consider the parametric equation of the given astroid $x^{2/3}+y^{2/3}=a^{2/3}$

$x = a\cos^3 t$; $y = a\sin^3 t$

$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$, $\frac{dy}{dt} = 3a\sin^2 t\cos t$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\cos t\sin^2 t}{-3a\cos^2 t\sin t}$$

$$y_1 = -\tan t$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}(-\tan t) \cdot \frac{1}{-3a\cos^2 t \sin t}$$

$$= -\sec^2 t \cdot \frac{1}{-3a\cos^2 t\sin t}$$

$$= \frac{1}{3a\sin t\cos^4 t}$$

Let $\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$$= a\cos^3 t + \frac{\tan t}{\frac{1}{3a\sin t\cos^4 t}}(1+\tan^2 t)$$

$$= a\cos^3 t + 3a\sin t\cos^4 t \cdot \frac{\sin t}{\cos t}(\sec^2 t)$$

$$\bar{x} = a(\cos^3 t + 3\sin^2 t\cos t) \longrightarrow (1)$$

Let $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$

$$= a\sin^3 t + \frac{(1+\tan^2 t)}{\frac{1}{3a\sin t\cos^4 t}}$$

$$= a\sin^3 t + 3a\sin t\cos^4 t \cdot \sec^2 t$$

$$\bar{y} = a(\sin^3 t + 3\sin t\cos^2 t) \longrightarrow (2)$$

To eliminate t from (1) & (2)

$$\bar{x}+\bar{y} = a(\cos^3 t + \sin^3 t + 3\sin^2 t\cos t + 3\sin t\cos^2 t)$$

$$\Rightarrow \bar{x}+\bar{y} = a(\cos t + \sin t)^3 \longrightarrow (3)$$

$$\because (a+b)^3 = a^3+b^3+3ab^2+3a^2b$$

$\bar{x}-\bar{y} = a(\cos^3 t - \sin^3 t + 3\cos t\sin^2 t - 3\cos^2 t\sin t)$

$\bar{x}-\bar{y} = a(\cos t - \sin t)^3 \longrightarrow (4)$

$\because (a-b)^3 = a^3 - b^3 + 3ab^2 - 3a^2 b$

Square & rise the power of 1/3 on bothsides of (3) & (4)

$(\bar{x}+\bar{y})^{2/3} = a^{2/3}(\cos t + \sin t)^2 \rightarrow (5)$

$(\bar{x}-\bar{y})^{2/3} = a^{2/3}(\cos t - \sin t)^2 \rightarrow (6)$

Now

(5) + (6)

$\Rightarrow (\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = a^{2/3}$
$[\cos^2 t + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + \sin^2 t - \cancel{2\sin t\cos t}]$

$\Rightarrow (\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = 2a^{2/3}$ $\quad \because \sin^2 t + \cos^2 t = 1$

By taking the locous of the $(\bar{x}, \bar{y})$, the reauired evolute is

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

6) Find the evolute of the curve $xy = c^2$

→ consider the parametric equation of the form

$x = ct, \quad y = \frac{c}{t}$

$\frac{dx}{dt} = c, \quad \frac{dy}{dt} = \frac{-c}{t^2}$

$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\Rightarrow y_1 = \frac{-c/t^2}{c} \Rightarrow y_1 = -\frac{1}{t^2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\Rightarrow y_2 = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{-1}{t^2}\right)\cdot\frac{1}{c}$$

$$= \frac{2}{ct^3}$$

We have $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$$\Rightarrow \bar{x} = ct - \frac{(-1/t^2)}{(2/ct^3)}(1+1/t^4)$$

$$= ct + \frac{ct}{2}\left(1+\frac{1}{t^4}\right)$$

$$= ct + \frac{ct}{2} + \frac{c}{2t^3}$$

$$\Rightarrow \bar{x} = \frac{3ct}{2} + \frac{c}{2t^3} \rightarrow (1)$$

Let $\bar{y} = y + \frac{(1+y_1^2)}{y_2} = \frac{c}{t} + \frac{(1+1/t^4)}{2/ct^3}$

$$= \frac{c}{t} + \frac{ct^3}{2}\left(1+\frac{1}{t^4}\right)$$

$$= \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t}$$

$$\bar{y} = \frac{3c}{2t} + \frac{ct^3}{2} \longrightarrow (2)$$

To eliminate t from (1) & (2)

$$\Rightarrow \bar{x} + \bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} + \frac{3c}{2t} + \frac{ct^3}{2}$$

$$= \frac{c}{2}\left[t^3 + \frac{1}{t^3} + 3\left(t + \frac{1}{t}\right)\right]$$

$$(\bar{x} + \bar{y}) = \frac{c}{2}\left(t + \frac{1}{t}\right)^3 \longrightarrow (3)$$

$$\& \ \bar{x} - \bar{y} = \frac{3ct}{2} + \frac{c}{2t^3} - \frac{3c}{2t} - \frac{ct^3}{2}$$

$$= -\frac{c}{2}\left[t^3 - \frac{1}{t^3} - 3\left(t - \frac{1}{t}\right)\right]$$

$$(\bar{x} - \bar{y}) = -\frac{c}{2}\left(t - \frac{1}{t}\right)^3 \longrightarrow (4)$$

We shall square & rise to the power $1/3$ on both sides of (3) & (4)

$$(\bar{x} + \bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left(t + \frac{1}{t}\right)^2 \longrightarrow (5)$$

$$(\bar{x} - \bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left(t - \frac{1}{t}\right)^2 \longrightarrow (6)$$

(5) - (6) $\Rightarrow$

$$(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[t^2+\frac{1}{t^2}+2 - t^2 - \frac{1}{t^2}+2\right]$$

$$(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\cdot 4 = \left(\frac{c}{2}\right)^{2/3}\cdot 2^2$$

$$\Rightarrow (\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = (4c)^{2/3}$$

$\Rightarrow (x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3}$ is the required evolute

n both