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MODULE-1

Fourier series

- **Introduction:-**

In many Engineering problems in the study of periodic phenomena. It is necessary to express a function in a series of sines and cosines within a desired range of values of the variable. Such a series is known as Fourier series.

- **Periodic function:-**

A real valued function $f(x)$ is said to be periodic of period T. if $f(x+nT) = f(x), T>0$.

Ex:-

1) Both $\sin x$ & $\cos x$ are periodic with period $2n\pi$

(i) $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \ldots$

(ii) $\cos x = \cos(x+2\pi) = \cos(x+4\pi) = \ldots$

2) $\tan x$ is periodic with period $(2n+1)\pi$

3) constant is periodic with any positive period of π

i.e) If $f(x) = k$

then $f(x+n\pi) = k$

* Dirichlet's conditions :-

A function $f(x)$ defined in the interval $c \le x \le c+2\pi$ can be expanded as an infinite trignometric series of the form

(i) $f(x)$ is single-valued and finite in $(c, c+2\pi)$

(ii) $f(x)$ is continuous with finite number of finite discontinuities in $(c, c+2\pi)$

(iii) $f(x)$ has no (or) finite number of maxima (or) minima in $(c, c+2\pi)$

* Definition of fourier series :-

Fourier series for the function $f(x)$ in the interval $c < x < c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx$$

The above formula is called Eulers formula, Where a_0, a_n & b_n are called Fourier coefficients of $f(x)$.

* Even and odd functions :-

A function $f(x)$ is said to be even if $f(-x) = f(x)$

Ex:- x^2, $\cos x$, $\sin^2 x$, etc...

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

Ex:- x, x^3, $\sin x$, $\tan^3 x$, etc...

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* Fourier series expansion over the interval $(-\pi, \pi)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad \text{for } n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad \text{for } n = 1, 2, 3 \ldots.$$

If $f(x)$ is an odd function in $(-\pi, \pi)$
then $a_0 = 0$ & $a_n = 0$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \text{ for } n=1,2,3\ldots$$

If $f(x)$ is an even function in $(-\pi,\pi)$ then $b_n=0$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad \&$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \text{ for } n=1,2,3\ldots$$

- Problems:-

1 Obtain the Fourier expansion of the function $f(x)=x$ over the interval $(-\pi,\pi)$. Deduce that $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

→ Given $f(x)=x$ & $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \to (1)$

Now $f(-x) = -x = -f(x)$

Hence the given function $f(x)$ is odd function in the interval $(-\pi,\pi)$

$\therefore a_0=0$ & $a_n=0$

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

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$$= \frac{2}{\pi}\left[x\frac{(-\cos nx)}{n} - 1\frac{(-\sin nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \left[-\frac{\pi \cos n\pi}{n} + \frac{\sin n\pi}{n^2} \right] - \left[\frac{0 \cos 0}{n} + \frac{\sin 0}{n^2} \right] \right\}$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n} \right]$$

$$\Rightarrow b_n = (-1)^{n+1}\frac{2}{n}$$

$$\therefore ① \Rightarrow f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$\Rightarrow x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

$$\Rightarrow x = 2\left\{ \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right\} \rightarrow ②$$

is the required Fourier expansion.

Further put $x = \frac{\pi}{2}$ in ②

$$\Rightarrow \frac{\pi}{2} = 2\left\{ \sin\frac{\pi}{2} - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) - \dots \right\}$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{0}{2} + \frac{1}{3}(-1) - \frac{0}{4} + \frac{1}{5} - \dots$$

$$\Rightarrow \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

2. Find the Fourier series for the function $f(x) = x^2$ in the interval $-\pi \le x \le \pi$.

Deduce the following

(i) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (ii) $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

(iii) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

→ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

Given $f(x) = x^2$

Now $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ the given function $f(x)$ is even function

$\therefore$ $b_n = 0$ in the interval $(-\pi, \pi)$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi} \left.\frac{x^3}{3}\right|_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^3}{3} - 0\right)$

$\Rightarrow a_0 = \frac{2\pi^2}{3}$

Now $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$

$$= \frac{2}{\pi}\left\{ x^2 \frac{\sin nx}{n} - 2x\frac{(-\cos nx)}{n^2} + 2(1)\frac{(-\sin nx)}{n^3} \right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3} \right] - \left[\frac{0 \sin 0}{n} + \frac{0 \cos 0}{n^2} - \frac{2\sin 0}{n^3} \right] \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{\pi^2(0)}{n} + \frac{2\pi(-1)^n}{n^2} - \frac{2(0)}{n^3} \right\} = \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2} \right)$$

$$\Rightarrow a_n = \frac{4(-1)^n}{n^2}$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{2}\left(\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + 0$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \longrightarrow (2)$$

is the required fourier exponsion

(i) put $x = \pi$ in (2)

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos n\pi$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} \qquad \text{but } (-1)^{2n} = 1$$

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$$\Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}} \longrightarrow (3)$$

(ii) Put $x=0$ in (2)

$$\Rightarrow 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0$$

$$\Rightarrow \frac{\pi^2}{3} = -4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}} \longrightarrow (4)$$

(iii) Adding (3) & (4)

$$\Rightarrow \frac{\pi^2}{6} + \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} \left[1 + (-1)^{n+1}\right]$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{2}{1^2} + \frac{0}{2^2} + \frac{2}{3^2} + \frac{0}{4^2} + \cdots$$

$$\frac{\pi^2}{4} = 2\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}$$

3 Find the Fourier series for the function $f(x)=|x|$ in $(-\pi,\pi)$. Hence deduce that $\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$\rightarrow$ WKT $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx \longrightarrow (1)$

Given $f(x)=|x|$

Now $f(-x)=|-x|=|x|=f(x)$ is an even function

Hence $b_n=0$ in the interval $(-\pi,\pi)$.

Now $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$=\frac{2}{\pi}\int_0^{\pi} x\,dx \quad [\because f(x)=|x|=x \text{ for } x>0]$

$=\frac{2}{\pi}\left.\frac{x^2}{2}\right|_0^{\pi}=\frac{2}{\pi}\left(\frac{\pi^2}{2}-0\right)$

$\Rightarrow a_0=\pi$

Now $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

$=\frac{2}{\pi}\left[x\frac{\sin nx}{n}-1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$

$=\frac{2}{\pi}\left\{\left[\frac{\pi\sin n\pi}{n}+\frac{\cos n\pi}{n^2}\right]-\left[\frac{0\sin 0}{n}+\frac{\cos 0}{n^2}\right]\right\}$

$=\frac{2}{\pi}\left[0+\frac{(-1)^n}{n^2}-0-\frac{1}{n^2}\right]$

$$\Rightarrow a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{2}(\pi) + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx + 0$$

$$\Rightarrow |x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2}\cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left\{-\frac{2}{1^2}\cos x + \frac{0}{2^2}\cos 2x - \frac{2}{3^2}\cos 3x + \frac{0}{4^2}\cos 4x - \dots\right\}$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left\{-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \dots\right\}$$

$$= \frac{\pi}{2} + \frac{2}{\pi}(-2)\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right\}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \longrightarrow (2)$$

is the required fourier expansion

put $x = 0$ in (2)

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 0}{(2n-1)^2}$$

$$\Rightarrow -\frac{\pi}{2} = -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{\pi}{2} = +\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

4. Expand $f(x)$ given by $f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$

where $k > 0$ is a constant, in a Fourier series for $-\pi < x < \pi$. Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

Given $f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$

Now $f(-x) = \begin{cases} -k & \text{for } -\pi < (-x) < 0 \\ k & \text{for } 0 < (-x) < \pi \end{cases} = -f(x)$

$\therefore$ Given function $f(x)$ is odd function

Hence $a_0 = 0$, $a_n = 0$ in the interval $(-\pi, \pi)$

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} k \sin nx\, dx = \frac{2k}{\pi}\int_0^{\pi}\sin nx\, dx$$

$$= \frac{2k}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2k}{\pi}\left[-\frac{\cos n\pi}{n} + \frac{\cos 0}{n}\right]$$

$$= \frac{2k}{\pi n}\left(-(-1)^n + 1\right)$$

$$b_n = \frac{2k}{n\pi}\left[1-(-1)^n\right]$$

$$\Rightarrow b_n = \frac{2k}{n\pi}(0) \quad \text{if } n \text{ is even}$$

$$\Rightarrow b_n = 0$$

(or)

$$b_n = \frac{2k}{n\pi}(2) \quad \text{if } n \text{ is odd}$$

$$\Rightarrow b_n = \frac{4k}{n\pi}$$

$$\therefore b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4k}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore (1) \Rightarrow f(x) = 0 + 0 + \sum_{n=1,3,5\ldots}^{\infty} \frac{4k}{n\pi}\sin nx \quad \left\{ \because n \text{ is odd} \right.$$

$$f(x) = \frac{4k}{\pi}\left\{ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right\} \longrightarrow (2)$$

is the required fourier expansion.

put $x = \pi/2$ in (2)

$$\cancel{k} = \frac{4\cancel{k}}{\pi}\left\{ \sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots \right\}$$

$$\Rightarrow \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

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5. Obtain the Fourier series expansion of $f(x) = \sin ax$ where a is not an integer over the interval $(-\pi, \pi)$

→ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = \sin ax$

Now $f(-x) = \sin a(-x) = -\sin ax = -f(x)$

$\therefore f(x)$ is an odd function

Hence $a_0 = 0$, $a_n = 0$ in the interval $(-\pi, \pi)$

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} \frac{[\cos(a-n)x - \cos(a+n)x]}{2} dx$

$= \frac{1}{\pi}\left[\frac{\sin(a-n)x}{(a-n)} - \frac{\sin(a+n)x}{(a+n)}\right]_0^{\pi}$

$= \frac{1}{\pi}\left\{\left[\frac{\sin(a-n)\pi}{(a-n)} - \frac{\sin(a+n)\pi}{(a+n)}\right] - 0\right\}$

$= \frac{1}{\pi}\left\{\frac{1}{a-n}[\sin a\pi \cos n\pi - \cos a\pi \cancel{\sin n\pi}^{0}] - \frac{1}{a+n}[\sin a\pi \cos n\pi + \cos a\pi \cancel{\sin n\pi}^{0}]\right\}$

$= \frac{1}{\pi}\left\{\frac{1}{a-n} - \frac{1}{a+n}\right\} \sin a\pi \cos n\pi$

$$= \frac{1}{\pi}\left\{\frac{a+n-a+n}{(a+n)(a-n)}\right\} \sin a\pi (-1)^n$$

$$\Rightarrow b_n = \frac{2n(-1)^n \sin a\pi}{\pi(a^2-n^2)}$$

$$\therefore (1) \Rightarrow f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2n(-1)^n \sin a\pi \sin nx}{\pi(a^2-n^2)}$$

$$\Rightarrow \sin ax = \frac{2}{\pi}\sin a\pi \sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2-n^2} \sin nx$$

is the required fourier expansion

6 Expand $f(x) = \sqrt{1-\cos x}$ in a Fourier series over the interval $(-\pi, \pi)$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = \sqrt{1-\cos x}$

Now $f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$

$\therefore f(x)$ is an even function.

Hence $b_n = 0$ in the interval $(-\pi, \pi)$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\,dx$$

But $1-\cos x = 2\sin^2 x/2$

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$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2\sin^2\left(\frac{x}{2}\right)}\, dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \sin(x/2)\, dx$$

put $\frac{x}{2} = t \Rightarrow \frac{dx}{2} = dt \Rightarrow dx = 2dt$

at $x = 0 \Rightarrow t = 0$
at $x = \pi \Rightarrow t = \frac{\pi}{2}$

$$\therefore a_0 = \frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \sin t\,(2dt) = \frac{4\sqrt{2}}{\pi}\int_0^{\pi/2} \sin t\, dt$$

$$= \frac{4\sqrt{2}}{\pi}\Big[-\cos t\Big]_0^{\pi/2}$$

$$= \frac{4\sqrt{2}}{\pi}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$$\Rightarrow a_0 = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{1-\cos x}\,\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2\sin^2\frac{x}{2}}\,\cos nx\, dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\pi} \sin\frac{x}{2}\,\cos nx\, dx$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^\pi \frac{1}{2}\left[\sin(n+1/2)x - \sin(n-1/2)x\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos(n+1/2)x}{(n+1/2)} - \left(-\frac{\cos(n-1/2)x}{(n-1/2)}\right)\right]_0^\pi$$

$$= \frac{\sqrt{2}}{\pi}\left\{\left[\cancel{-\frac{\cos(n+1/2)\pi}{(n+1/2)}}^{\,0} + \cancel{\frac{\cos(n-1/2)\pi}{(n-1/2)}}^{\,0}\right] - \left[-\frac{\cos 0}{(n+1/2)} + \frac{\cos 0}{(n-1/2)}\right]\right\}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{n+1/2} - \frac{1}{n-1/2}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{n-1/2-n-1/2}{(n+1/2)(n-1/2)}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{-1}{n^2-1/4}\right) = -\frac{\sqrt{2}}{\pi}\frac{1}{\frac{(4n^2-1)}{4}}$$

$$\therefore a_n = -\frac{4\sqrt{2}}{\pi}\frac{1}{(4n^2-1)}$$

$$\therefore ① \Rightarrow f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty} -\frac{4\sqrt{2}}{\pi}\frac{1}{(4n^2-1)}\cos nx + 0$$

$$\Rightarrow \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi}\left[1 - 2\sum_{n=1}^{\infty}\frac{\cos nx}{(4n^2-1)}\right]$$

7 Obtain the Fourier series in $(-\pi, \pi)$ for $f(x) = x\cos x$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = x\cos x$

Now $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$

$\therefore$ given $f(x)$ is an odd function

Hence $a_0 = 0$, $a_n = 0$ in the interval $(-\pi, \pi)$

$$\text{Now } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x[\sin nx \cdot \cos x] \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \frac{1}{2}[\sin(n+1)x + \sin(n-1)x] \, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n+1)x \, dx + \int_0^{\pi} x\sin(n-1)x \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - \left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} + \left[x\left(-\frac{\cos(n-1)x}{n-1}\right) - \left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\pi\cos(n+1)\pi}{n+1} + \cancel{\frac{\sin(n+1)\pi}{(n+1)^2}}^{\,0} - \frac{\pi\cos(n-1)\pi}{(n-1)} + \cancel{\frac{\sin(n-1)\pi}{(n-1)^2}}^{\,0}\right] - \left[\frac{-0\cos 0}{n+1} + \frac{\sin 0}{(n+1)^2} - \frac{0\cos 0}{n-1} + \frac{\sin 0}{(n-1)^2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi(-1)^{n+1}}{n+1} - \frac{\pi(-1)^{n-1}}{n-1}\right\}$$

$$= \frac{\pi}{\pi}(-1)^n\left\{\frac{-(-1)}{n+1} - \frac{(-1)^{-1}}{n-1}\right\}$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{n-1+n+1}{(n+1)(n-1)}\right]$$

$$\Rightarrow b_n = \frac{(-1)^n 2n}{n^2-1} \quad (n \neq 1)$$

To find b_n for $n=1$:-

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\cos x\,dx$$

Put $n=1$

$$\Rightarrow b_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\, 2\sin x\cos x\,dx$$

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$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi\cos 2\pi}{2} + \cancel{\frac{\sin 2\pi}{4}}^{0}\right) - (0+0)\right]$$

$$= -\frac{\pi}{\pi}\left(\frac{1}{2}\right)$$

$$\Rightarrow b_1 = -1/2$$

$$\therefore (1) \Rightarrow f(x) = 0 + 0 + b_1\sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$x\cos x = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1}\sin nx$$

is the required Fourier expansion

8. Obtain the Fourier series in $(-\pi,\pi)$ for $f(x) = x\sin x$

WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = x\sin x$

Now $f(-x) = (-x)\sin(-x) = -x(-\sin x) = x\sin x = f(x)$

$\therefore$ $f(x)$ is even function.

Hence $b_n = 0$ in the interval $(-\pi,\pi)$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

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$$= \frac{2}{\pi}\int_0^{\pi} x\sin x \, dx$$

$$= \frac{2}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[(-\pi\cos\pi + \sin\pi) - (-0\cos 0 + \sin 0)\Big]$$

$$= \frac{2}{\pi}\Big[-\pi(-1) + 0 + 0 - 0\Big]$$

$$\Rightarrow a_0 = 2$$

Now $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\left[\frac{1}{2}\Big(\sin(1+n)x + \sin(1-n)x\Big)\right]dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(1+n)x\,dx + \int_0^{\pi} x\sin(1-n)x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos(1+n)x}{1+n}\right) - 1\left(-\frac{\sin(1+n)x}{(1+n)^2}\right)\right]_0^{\pi} + \left[x\left(-\frac{\cos(1-n)x}{1-n}\right) - 1\left(-\frac{\sin(1-n)x}{(1-n)^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\pi\cos(1+n)\pi}{(1+n)} + \frac{\sin(1+n)\pi}{(1+n)^2} - \frac{\pi\cos(1-n)\pi}{(1-n)}\right.\right.$$

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$$+\frac{\sin(1-n)\pi}{(1-n)^2}\Big] - \Big[\frac{0\cos 0}{1+n} + \frac{\sin 0}{(1+n)^2} - \frac{0\cos 0}{1-n} + \frac{\sin 0}{(1-n)^2}\Big]\Big\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi(-1)^{n+1}}{1+n} + 0 - \frac{\pi(-1)^{n-1}}{1-n} + 0\right\}$$

$$= \frac{\pi(-1)^n}{\pi}\left\{\frac{-(-1)}{1+n} - \frac{(-1)^{-1}}{1-n}\right\}$$

$$= (-1)^n\left[\frac{1}{1+n} + \frac{1}{1-n}\right] = (-1)^n\left[\frac{1-n+1+n}{(1+n)(1-n)}\right]$$

$$= \frac{2(-1)^n}{1^2-n^2} = \frac{2(-1)^n}{-(n^2-1^2)} = \frac{(-1)2(-1)^n}{n^2-1^2}$$

$$\Rightarrow a_n = \frac{2(-1)^{n+1}}{n^2-1^2} \quad (n \neq 1)$$

To find a_n at $n=1$:-

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx$$

put $n=1$

$$\Rightarrow a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{2^2}\right)\right]_0^\pi$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[\left(\frac{-\pi\cos 2\pi}{2} + \frac{\sin 2\pi}{4}\right) - (0+0)\right]$$

$$\Rightarrow a_1 = \frac{1}{\pi}\left[-\frac{\pi(1)}{2} + 0\right]$$

$$\Rightarrow a_1 = -1/2$$

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + 0$$

$$x\sin x = \frac{1}{2}(2) + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$$

$$\Rightarrow x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx \longrightarrow (2)$$

is the required Fourier expansion

9. Expand $f(x) = |\cos x|$ in a fourier series over $(-\pi, \pi)$

$\longrightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = |\cos x|$

Now $f(-x) = |-\cos x| = |\cos x| = f(x)$

$\therefore$ Given $f(x)$ is even function

Hence $b_n = 0$ in the interval $(-\pi, \pi)$

$$\text{Now } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right]$$

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$$= \frac{2}{\pi}\left[\sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{\pi}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\pi - \sin\frac{\pi}{2}\right) \right]$$

$$= \frac{2}{\pi}\left[1 - 0 - 0 + 1 \right]$$

$$\Rightarrow a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \cos x \cos nx\, dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx\, dx \right\}$$

$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \frac{1}{2}\left[\cos(n+1)x + \cos(n-1)x\right] dx - \int_{\pi/2}^{\pi} \frac{1}{2}\left[\cos(n+1)x + \cos(n-1)x\right] dx \right\}$$

$$= \frac{2}{\pi}\left(\frac{1}{2}\right)\left\{ \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)}\right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)\pi/2}{(n+1)} + \frac{\sin(n-1)\pi/2}{(n-1)} - 0 - 0\right] - \right.$$

$$\left.\left[\cancel{\frac{\sin(n+1)\pi}{(n+1)}}^{0} + \cancel{\frac{\sin(n-1)\pi}{(n-1)}}^{0} - \frac{\sin(n+1)\pi/2}{(n+1)} - \frac{\sin(n-1)\pi/2}{(n-1)}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\sin(n+1)\pi/2}{(n+1)} + \frac{2\sin(n-1)\pi/2}{(n-1)}\right\}$$

$$= \frac{2}{\pi}\left[\frac{\cos(n\pi/2)}{n+1} + \frac{-\cos(n\pi/2)}{n-1}\right]$$

$$= \frac{2\cos\left(\frac{n\pi}{2}\right)}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\left[\frac{\cancel{n}-1-\cancel{n}-1}{(n+1)(n-1)}\right]$$

$$a_n = \frac{-4\cos(n\pi/2)}{\pi(n^2-1)} \quad \text{if } n \text{ is even } \& \text{ if } n \neq 1$$

$$\therefore a_n = \begin{cases} \dfrac{-4\cos(n\pi/2)}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Now to find a_1 :-

$$\text{Consider } a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x \cos nx \, dx - \int_{\pi/2}^{\pi} \cos x \cos nx \, dx\right\}$$

Put $n=1$

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$$\Rightarrow a_1 = \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x \cos x\, dx - \int_{\pi/2}^{\pi} \cos x \cos x\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos^2 x\, dx - \int_{\pi/2}^{\pi} \cos^2 x\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx - \int_{\pi/2}^{\pi} \left(\frac{1+\cos 2x}{2}\right) dx\right\}$$

$$= \frac{2}{\pi}\left(\frac{1}{2}\right)\left\{\int_0^{\pi/2} (1+\cos 2x)\, dx - \int_{\pi/2}^{\pi} (1+\cos 2x)\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi}{2} + \frac{1}{2}\sin\left(\frac{2\pi}{2}\right) - 0 - 0\right] - \left[\pi + \frac{\sin 2\pi}{2} - \frac{\pi}{2} - \frac{1}{2}\sin\left(\frac{2\pi}{2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + 0 - \pi + 0 + \frac{\pi}{2} + 0\right]$$

$$\Rightarrow a_1 = 0$$

$$\therefore ① \Rightarrow f(x) = \frac{1}{2}\left(\frac{4}{\pi}\right) + a_1 \cos x + \sum_{n=2,4,6\cdots}^{\infty} a_n \cos nx + 0$$

$$|\cos x| = \frac{2}{\pi} + 0 + \sum_{n=2,4,6\ldots}^{\infty} \frac{-4\cos(n\pi/2)}{\pi(n^2-1)} \cdot \cos nx$$

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6\ldots}^{\infty} \frac{\cos(n\pi/2)}{(n^2-1)} \cos nx$$

is the required Fourier expansion.

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10. Obtain the Fourier expansion of the function $f(x)$ defined by

$$f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$\rightarrow$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Now $f(-x) = \begin{cases} 1+\frac{2(-x)}{\pi}, & -\pi \le (-x) \le 0 \\ 1-\frac{2(-x)}{\pi}, & 0 \le (-x) \le \pi \end{cases}$

$$= \begin{cases} 1-\frac{2x}{\pi}, & 0 \le x \le \pi \\ 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases} = f(x)$$

$\therefore$ given function $f(x)$ is even

Hence $b_n = 0$ in the interval $(-\pi, \pi)$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi}\right]$$

$\Rightarrow a_0 = 0$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi}\cos nx\,dx - \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{n} - \frac{2}{\pi}\left(\frac{x\sin nx}{n} - 1\left(\frac{-\cos nx}{n^2}\right)\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\sin n\pi}{n} - \frac{2}{\pi}\left(\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right)\right] - \left[\frac{\sin 0}{n} - \frac{2}{\pi}\left(\frac{0\sin 0}{n} + \frac{\cos 0}{n^2}\right)\right]\right.$$

$$= \frac{4}{\pi^2}\left[0 - 0 - \frac{(-1)^n}{n^2} - 0 + 0 + \frac{1}{n^2}\right]$$

$$a_n = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$\therefore ① \Rightarrow f(x) = 0 + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]\cos nx + 0$$

$$\Rightarrow f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos nx \longrightarrow ②$$

is the required Fourier expansion.

put $x=0$ in ②

$$f(0) = 1 = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos 0$$

$$1 = \frac{4}{\pi^2}\left\{\frac{2}{1^2}+\frac{0}{2^2}+\frac{2}{3^2}+\frac{0}{4^2}+\frac{2}{5^2}+\dots\right\}$$

$$1 = \frac{4\times 2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots \quad (\text{or}) \quad \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

11. Find the Fourier series expansion for

$$f(x) = \begin{cases} \pi + x \; ; & -\pi \le x \le 0 \\ \pi - x \; ; & 0 \le x \le \pi \end{cases}, \text{ Hence deduce } \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = \begin{cases} \pi + x \; ; & -\pi \le x \le 0 \\ \pi - x \; ; & 0 \le x \le \pi \end{cases}$

Now $f(-x) = \begin{cases} \pi + (-x) \; ; & -\pi \le (-x) \le 0 \\ \pi - (-x) \; ; & 0 \le (-x) \le \pi \end{cases}$

$$= \begin{cases} \pi - x \; ; & 0 \le x \le \pi \\ \pi + x & -\pi \le x \le 0 \end{cases} = f(x)$$

Hence the given $f(x)$ is even function

$\therefore b_n = 0$ in the interval $(-\pi, \pi)$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$\Rightarrow a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[(\pi-\pi)\frac{\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right] - \left[(\pi - 0)\frac{\sin 0}{n} - \frac{\cos 0}{n^2}\right]\right\}$$

$$a_n = \frac{2}{\pi}\left[0 - \frac{(-1)^n}{n^2} - 0 + \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]$$

$$\Rightarrow a_n = \begin{cases} \dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty}\frac{4}{\pi n^2}\cos nx + 0$$

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos nx}{n^2} \longrightarrow (2)$$

is the required fourier expansion

Put $x=0$ in ②

$$\Rightarrow f(0)=\pi-0=\frac{\pi}{2}+\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos 0}{n^2}$$

$$\Rightarrow \pi-\frac{\pi}{2}=\frac{4}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right\}$$

$$\Rightarrow \boxed{\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots \quad \text{(or)} \quad \frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

12. Expand $f(x)=e^{-ax}$ as a Fourier series over the interval $(-\pi,\pi)$. Hence deduce the Fourier expansions of e^{ax}, $\cosh ax$ and $\sinh ax$ over the interval $(-\pi,\pi)$.
Also prove the following
(i) $\dfrac{\pi}{\sinh\pi}=2\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2+1}$ (ii) $\operatorname{cosech}\pi=\dfrac{2}{\pi}\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2+1}$

$\rightarrow$ WKT $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \longrightarrow$ ①

Given $f(x)=e^{-ax}$ is neither even nor odd
then

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\,dx=\frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{-a} - \frac{e^{a\pi}}{-a}\right] = \frac{-1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right]$$

$$a_0 = \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

$$\Rightarrow a_0 = \frac{2}{a\pi}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right] = \frac{2}{a\pi}\sinh a\pi$$

Now

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx$$

WKT $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\left[-a\cos nx + n\sin nx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{e^{-a\pi}}{a^2+n^2}\left(-a\cos n\pi + n\sin n\pi\right)\right] - \left[\frac{e^{a\pi}}{a^2+n^2}\left(-a\cos n(-\pi) + n\sin n(-\pi)\right)\right]\right\}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}\left(-a(-1)^n + n(0)\right) - e^{a\pi}\left(-a(-1)^n + n(0)\right)\right]$$

$$= \frac{-a(-1)^n}{\pi(a^2+n^2)}\left[e^{-a\pi} - e^{a\pi}\right]$$

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$$\Rightarrow a_n = \frac{a(-1)^n}{\pi(a^2+n^2)} \; \frac{2[e^{a\pi}-e^{-a\pi}]}{2}$$

$$\Rightarrow a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx \, dx$$

WKT $\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}[-a\sin nx - n\cos nx]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left\{\left[e^{-a\pi}(-a\sin n\pi - n\cos n\pi)\right] - \left[e^{a\pi}(-a\sin n(-\pi) - n\cos n(-\pi))\right]\right\}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{-a\pi}(-a(0)-n(-1)^n) - e^{a\pi}(-a(0)-n(-1)^n\right.$$

$$= \frac{-n(-1)^n}{\pi(a^2+n^2)}\left[e^{-a\pi}-e^{a\pi}\right]$$

$$= \frac{2n(-1)^n}{\pi(a^2+n^2)} \; \frac{[e^{a\pi}-e^{-a\pi}]}{2}$$

$$\Rightarrow b_n = \frac{2n(-1)^n}{\pi(a^2+n^2)} \sinh a\pi$$

$$\therefore ① \Rightarrow f(x) = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n}{\pi(a^2+n^2)} \sinh a\pi \cos nx + \sum_{n=1}^{\infty} \frac{2n(-1)^n}{\pi(a^2+n^2)} \sinh a\pi \sin nx$$

$$\Rightarrow \boxed{e^{-ax} = \frac{\sinh a\pi}{a\pi}\left[1 + 2a^2\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}\cos nx + 2a\sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2+n^2}\sin nx\right]} \longrightarrow ②$$

is the required Fourier series expansion for e^{-ax}

Replace a by $-a$ in ②

$$e^{ax} = \frac{\sinh(-a)\pi}{-a\pi}\left[1 + 2(-a)^2\sum_{n=1}^{\infty} \frac{(-1)^n}{(-a)^2+n^2}\cos nx + 2(-a)\sum_{n=1}^{\infty} \frac{n(-1)^n}{(-a)^2+n^2}\sin nx\right]$$

$$\boxed{e^{ax} = \frac{\sinh a\pi}{a\pi}\left[1 + 2a^2\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}\cos nx - 2a\sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2+n^2}\sin nx\right]} \longrightarrow ③$$

is the required Fourier series expression for e^{ax}

Consider

$$\cosh ax = \frac{1}{2}\left[e^{ax} + e^{-ax}\right]$$

$$= \frac{\sinh a\pi}{2a\pi}\left[2 + 4a^2\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}\cos nx\right] \quad \because \text{add } ② \,\&\, ③$$

$$\Rightarrow \boxed{\cosh ax = \frac{\sinh a\pi}{a\pi}\left[1 + 2a^2\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}\cos nx\right]} \longrightarrow ④$$

consider

$$\sinh ax = \frac{1}{2}\left[e^{ax} - e^{-ax}\right]$$

$$= \frac{\sinh a\pi}{2a\pi}\left[0+0-4a\sum_{n=1}^{\infty}\frac{n(-1)^n \sin nx}{a^2+n^2}\right]$$

$$\boxed{\sinh ax = \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}\sin nx}{a^2+n^2}} \rightarrow (5)$$

(4) & (5) are the Fourier expansions for cosh ax & sinh ax.

(i) Now put $x=0$ & $a=1$ in (2)

$$\Rightarrow e^0 = \frac{\sinh\pi}{\pi}\left[1+\sum_{n=1}^{\infty}\frac{2(-1)^n \cos 0}{1+n^2}+\sum_{n=1}^{\infty}\frac{2n(-1)^n \sin 0}{1+n^2}\right]$$

$$\Rightarrow 1 = \frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}+0\right]$$

$$\Rightarrow \frac{\pi}{\sinh\pi} = 1+2\left\{\frac{-1}{2}+\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}\right\}$$

$$= 1-1+2\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\Rightarrow \boxed{\frac{\pi}{\sinh\pi} = 2\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}}$$

(or)

(ii) $$\boxed{\operatorname{cosech}\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}}$$

13. Find the Fourier Series for the function $f(x)=x+x^2$ over the interval $-\pi \le x \le \pi$. Hence deduce

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots$

(ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = x + x^2$ is neither even nor odd then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} + \frac{\pi^3}{3}\right] - \left[\frac{(-\pi)^2}{2} + \frac{(-\pi)^3}{3}\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$\Rightarrow a_0 = \frac{2\pi^2}{3}$$

Now

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + (0+2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi+\pi^2)\frac{\sin n\pi}{n} + (1+2\pi)\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3}\right] - \left[(-\pi+(-\pi)^2)\frac{\sin n(-\pi)}{n} + (1-2\pi)\frac{\cos n(-\pi)}{n^2} - \frac{2\sin n(-\pi)}{n^3}\right]\right\}$$

But $\sin n\pi = 0 = \sin n(-\pi) = -\sin n\pi$

$$\therefore a_n = \frac{1}{\pi}\left[(1+2\pi)\frac{\cos n\pi}{n^2} - (1-2\pi)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{\cos n\pi}{\pi n^2}\left[1+2\pi-1+2\pi\right]$$

$$a_n = \frac{4\pi(-1)^n}{\pi n^2}$$

$$\Rightarrow a_n = \frac{4(-1)^n}{n^2}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

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$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+(0+2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right)+(1+2\pi)\left(\cancel{\frac{\sin n\pi}{n^2}}\right)^{0}+\frac{2\cos n\pi}{n^3}\right]\right.$$

$$\left.-\left[(-\pi+(-\pi)^2)\left(-\frac{\cos n(-\pi)}{n}\right)+(1-2\pi)\cancel{\frac{\sin n(-\pi)}{n^2}}^{0}+\frac{2\cos n(-\pi)}{n^3}\right]\right\}$$

$$=\frac{1}{\pi}\left\{-\frac{\pi\cos n\pi}{n}-\cancel{\frac{\pi^2\cos n\pi}{n}}+\cancel{\frac{2\cos n\pi}{n^3}}-\frac{\pi\cos n\pi}{n}+\cancel{\frac{\pi^2\cos n\pi}{n}}-\cancel{\frac{2\cos n\pi}{n^3}}\right\}$$

$$=\frac{1}{\pi}\left(-\frac{2\pi\cos n\pi}{n}\right)$$

$$b_n=\frac{-2(-1)^n}{n}=\frac{2(-1)(-1)^n}{n}$$

$$\Rightarrow b_n=\frac{2(-1)^{n+1}}{n}$$

$$\therefore ① \Rightarrow f(x)=\frac{1}{2}\frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

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$$\therefore x+x^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \longrightarrow (2)$$

is the required Fourier expansion

put $x=0$ in (2)

$$\Rightarrow 0+0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 0+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin 0$$

$$\Rightarrow 0=\frac{\pi^2}{3}+4\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\dots\right]$$

$$\Rightarrow \frac{\pi^2}{3}=-4\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\dots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots} \longrightarrow (3)$$

put $x=\pi$ in (2)

$$\Rightarrow \pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cancel{\sin n\pi}^{\,0}$$

$$\Rightarrow \pi+\pi^2-\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}+0, \quad \text{But } (-1)^{2n}=1$$

$$\Rightarrow \pi+\frac{2\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2} \longrightarrow (4)$$

Put $x=-\pi$ in (2)

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$$\Rightarrow -\pi+\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n(-\pi)+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cancel{\sin n(\pi)}^{0}$$

$$\Rightarrow -\pi+\pi^2-\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}+0$$

$$\Rightarrow -\pi+\frac{2\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2} \longrightarrow (5)$$

Add (4) + (5) $\Rightarrow$

$$\pi+\frac{2\pi^2}{3}-\pi+\frac{2\pi^2}{3}=8\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{4\pi^2}{3}=8\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots} \longrightarrow (6)$$

Add (3) & (6)

$$\frac{\pi^2}{12}+\frac{\pi^2}{6}=\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right)+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\frac{\pi^2}{4}=\frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\cdots$$

$$\frac{\pi^2}{4}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\Rightarrow \boxed{\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}$$

14 Find a Fourier series in $(-\pi,\pi)$ to represent $f(x) = x - x^2$. Hence deduce

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow$ ①

Given $f(x) = x - x^2$ is neither even nor odd then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] - \left[\frac{(-\pi)^2}{2} - \frac{(-\pi)^3}{3}\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi^3}{3}\right)$$

$$\Rightarrow a_0 = -\frac{2\pi^2}{3}$$

Now

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

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$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (0-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-\pi^2)\cancel{\frac{\sin n\pi}{n}}^{0} + (1-2\pi)\frac{\cos n\pi}{n^2} + \cancel{\frac{2\sin n\pi}{n^3}}^{0}\right] - \left[(-\pi-(-\pi)^2)\cancel{\frac{\sin n(-\pi)}{n}}^{0} + (1+2\pi)\left(\frac{\cos n(-\pi)}{n^2}\right) + \cancel{\frac{2\sin n(-\pi)}{n^3}}^{0}\right]\right\}$$

$$a_n = \frac{1}{\pi}\left[(1-2\pi)\frac{\cos n\pi}{n^2} - (1+2\pi)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{\cos n\pi}{\pi n^2}\left[1-2\pi-1-2\pi\right]$$

$$= \frac{(-1)^n(-4\pi)}{\pi n^2} = \frac{4(-1)(-1)^n}{n^2}$$

$$\Rightarrow a_n = \frac{4(-1)^{n+1}}{n^2}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) - (0-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-\pi^2)\left(-\frac{\cos n\pi}{n}\right) + (1-2\pi)\cancel{\frac{\sin n\pi}{n^2}}^{0} + \frac{2\cos n\pi}{n^3}\right] - \right.$$

$$\left.\left[(-\pi-(-\pi)^2)\left(-\frac{\cos n(-\pi)}{n}\right) + (1+2\pi)\cancel{\frac{\sin n(-\pi)}{n^2}}^{0} + \frac{2\cos n(-\pi)}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \cancel{\frac{\pi^2\cos n\pi}{n}} + \cancel{\frac{2\cos n\pi}{n^3}} - \frac{\pi\cos n\pi}{n} - \cancel{\frac{\pi^2\cos n\pi}{n}} - \cancel{\frac{2\cos n\pi}{n^3}}\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right]$$

$$= \frac{(-1)2(-1)^n}{n}$$

$$\Rightarrow b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{2}\left(-\frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

$$\therefore x - x^2 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \longrightarrow (2)$$

is the required Fourier expansion.

put $x=0$ in ②

$$\Rightarrow 0+0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos 0 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cancelto{0}{\sin 0}$$

$$\Rightarrow 0 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$\Rightarrow \frac{\pi^2}{3} = 4\left\{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right\}$$

$$\Rightarrow \boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots} \longrightarrow ③$$

put $x=\pi$ in ②

$$\Rightarrow \pi - \pi^2 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cancelto{0}{\sin n\pi}$$

$$\Rightarrow \pi - \pi^2 + \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}(-1)^n + 0$$

$$\Rightarrow \pi - \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^2}$$

$$\pi - \frac{2\pi^2}{3} = 4\left\{-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \dots\right\} \longrightarrow ④$$

put $x=-\pi$ in ②

$$\Rightarrow -\pi - (-\pi)^2 = -\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n(-\pi) + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cancelto{0}{\sin n(-\pi)}$$

$$\Rightarrow -\pi - \pi^2 + \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}(-1)^n + 0$$

$$\Rightarrow -\pi - \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^2}$$

$$-\pi - \frac{2\pi^2}{3} = 4\left\{-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \dots\right\} \longrightarrow ⑤$$

Add ④ & ⑤

$$\pi - \frac{2\pi^2}{3} - \pi - \frac{2\pi^2}{3} = 8\left\{-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right\}$$

$$-\frac{4\pi^2}{3} = -8\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right\}$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots} \longrightarrow ⑥$$

Adding ③ & ⑥

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

15 Expand $f(x) = |\sin x|$ as a Fourier series in $(-\pi, \pi)$

$$\rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ①$$

Given $f(x) = |\sin x|$

Now $f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$

$\therefore f(x)$ is even function.

Hence $b_n = 0$ in the interval $(-\pi, \pi)$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x|\, dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \sin x\, dx + \int_{\pi/2}^{\pi} -\sin x\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\cos x\right]_0^{\pi/2} - \left[-\cos x\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\cos\frac{\pi}{2} + \cos 0\right] + \left[\cos\pi - \cos\frac{\pi}{2}\right]\right\}$$

$$= \frac{2}{\pi}\left\{0 + 1 + (-1) - 0\right\}$$

$$\Rightarrow a_0 = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \sin x \cos nx\, dx + \int_{\pi/2}^{\pi} -\sin x \cos nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right) dx - \right.$$

$$\left.\int_{\pi/2}^{\pi} \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right) dx\right\}$$

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$$= \frac{2}{\pi}\cdot\frac{1}{2}\left\{\left[-\frac{\cos(n+1)x}{(n+1)} - \left(-\frac{\cos(n-1)x}{(n-1)}\right)\right]_0^{\pi/2} - \left[-\frac{\cos(n+1)x}{(n+1)} - \left(-\frac{\cos(n-1)x}{(n-1)}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\cos(n+1)\pi/2}{(n+1)} + \frac{\cos(n-1)\pi/2}{(n-1)}\right] - \left[-\frac{\cos 0}{(n+1)} + \frac{\cos 0}{(n-1)}\right]\right\} - \frac{1}{\pi}\left\{\left[-\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{n-1}\right] - \left[-\frac{\cos(n+1)\pi/2}{(n+1)} + \frac{\cos(n-1)\pi/2}{(n-1)}\right]\right\}$$

$$a_n = \frac{1}{\pi}\left\{-\left(-\frac{\sin(n\pi/2)}{n+1}\right) + \frac{\sin(n\pi/2)}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} + \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{(n-1)} - \left(-\frac{\sin(n\pi/2)}{n+1}\right) + \frac{\sin(n\pi/2)}{n-1}\right\}$$

$$= \frac{1}{\pi}\left\{2\,\frac{\sin(n\pi/2)}{n+1} + \frac{2\sin(n\pi/2)}{n-1} + \left(\frac{1}{n+1} - \frac{1}{n-1}\right) + (-1)^n\left(\frac{(-1)}{n+1} - \frac{(-1)^{-1}}{n-1}\right)\right\}$$

$$= \frac{1}{\pi}\left\{2\sin(n\pi/2)\left(\frac{1}{n+1} + \frac{1}{n-1}\right) + \left(\frac{-2}{n^2-1^2}\right) + (-1)^n\left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right\}$$

$$= \frac{1}{\pi}\left\{2\sin(n\pi/2)\left(\frac{2n}{n^2-1^2}\right) - \frac{2}{(n^2-1^2)} + (-1)^n\left(\frac{2}{n^2-1^2}\right)\right\}$$

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$$a_n = \frac{1}{\pi(n^2-1^2)}\left[4n\sin\left(\frac{n\pi}{2}\right) - 2(1-(-1)^n)\right], \quad n \neq 1$$

To find a_n at $n=1$:-

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} \sin x \sin nx\, dx - \int_{\pi/2}^{\pi} \sin x \sin nx\, dx\right\}$$

put $n=1$ $\Rightarrow$ $a_1 = \frac{2}{\pi}\left\{\int_0^{\pi/2} \sin x \sin x\, dx - \int_{\pi/2}^{\pi} \sin x \sin x\, dx\right\}$

$$\Rightarrow a_1 = \frac{2}{\pi}\left\{\int_0^{\pi/2} \sin^2 x\, dx - \int_{\pi/2}^{\pi} \sin^2 x\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right) dx - \int_{\pi/2}^{\pi}\left(\frac{1-\cos 2x}{2}\right) dx\right\}$$

$$= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\Big|_0^{\pi/2} - x - \frac{\sin 2x}{2}\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi}{2} - \frac{1}{2}\sin 2\pi/2\right] - 0 - \left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(\frac{\pi}{2} - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)\right)\right]\right\}$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2} - 0 - 0 - \pi + 0 + \frac{\pi}{2} - 0\right]$$

$$\Rightarrow a_1 = 0$$

$\therefore$ ① $\Rightarrow f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_n$

$$|\sin x| = 0 + 0 + \sum_{n=2}^{\infty} \frac{1}{\pi(n^2-1^2)}\left[4n\sin\left(\frac{n\pi}{2}\right) - 2\left(1-(-1)^n\right)\right]\cos nx + 0$$

$$\Rightarrow |\sin x| = \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1}{(n^2-1^2)}\left[2n\sin\left(\frac{n\pi}{2}\right) - (1-(-1)^n)\right]\cos nx$$

is the required fourier expansion.

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* Fourier series expansion over the interval $(0, 2\pi)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad n = 1, 2, 3 \ldots$$

* Problems:-

1. obtain the Fourier series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = \frac{\pi - x}{2}$ in the interval $(0, 2\pi)$

Now

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\,dx$$

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$$= \frac{1}{2\pi}\int_0^{2\pi} \pi - x\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[2\pi^2 - \frac{4\pi^2}{2}\right] - 0\right\}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - 2\pi^2\right]$$

$$\Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left\{(\pi - x)\frac{\sin nx}{n} - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[(\pi - 2\pi)\frac{\sin n2\pi}{n} - \frac{\cos n2\pi}{n^2}\right] - \left[(\pi - 0)\frac{\sin 0}{n} - \frac{\cos 0}{n^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[0 - 1/n^2 - 0 + 1/n^2\right] \qquad \{\because \cos 2n\pi = 1 = \cos 0\}$$

$$\Rightarrow a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\sin nx\,dx$$

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$$= \frac{1}{2\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[(\pi-2\pi)\left(-\frac{\cos 2n\pi}{n}\right) - \frac{\sin 2n\pi}{n^2}\right] - \left[(\pi-0)\left(-\frac{\cos 0}{n}\right) - \frac{\sin 0}{n^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[\frac{-\pi\cos 2n\pi}{n} + \frac{2\pi\cos 2n\pi}{n} - 0 + \frac{\pi\cos 0}{n} - 0 + 0\right]$$

$$= \frac{1}{2\pi}\left[-\pi/n + 2\pi/n + \pi/n\right] \quad \because \cos 2n\pi = 1$$

$$\Rightarrow b_n = \frac{1}{2\pi}\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow b_n = 1/n$$

$$\therefore (1) \Rightarrow f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

$$\Rightarrow \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \longrightarrow (2)$$

is the required Fourier expansion

Now put $x = \pi/2$ in (2)

$$\Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n}$$

$$\Rightarrow \frac{\pi}{4} = \sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin\frac{4\pi}{2}$$

$$= 1 + \frac{1}{2}(0) + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}(1) + \dots$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

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2 If $f(x) = x(2\pi - x)$ in $0 \le x \le 2\pi$. Show that $f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \dots\right]$ is a fourier series. Hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = x(2\pi - x)$ in $(0, 2\pi)$

$\Rightarrow f(x) = (2\pi x - x^2)$

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\,dx = \frac{1}{\pi}\left[2\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left\{\left[\pi(2\pi)^2 - \frac{(2\pi)^3}{3}\right] - 0\right\}$

$= \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right] = \frac{1}{\pi}\left[\frac{4\pi^3}{3}\right]$

$\Rightarrow a_0 = \frac{4\pi^2}{3}$

Now $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\cos nx\,dx$

$= \frac{1}{\pi}\left[(2\pi x - x^2)\frac{\sin nx}{n} - (2\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) - (0 - 2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$

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$$= \frac{1}{\pi}\left\{\left[(4\pi^2-4\pi^2)\frac{\cancel{\sin 2n\pi}}{n} + (2\pi-4\pi)\frac{\cos 2n\pi}{n^2} - \frac{2\cancel{\sin 2n\pi}}{n^3}\right]\right.$$

$$\left. - \left[(0-0)\frac{\cancel{\sin 0}}{n} + (2\pi-0)\frac{\cos 0}{n^2} - \frac{2\cancel{\sin 0}}{n^3}\right]\right\}$$

$$a_n = \frac{1}{\pi}\left[0 - 2\pi\left(\frac{1}{n^2}\right) - 0 - 0 - 2\pi\frac{1}{n^2} - 0\right]$$

$$= \frac{1}{\pi}\left(-\frac{4\pi}{n^2}\right)$$

$$\Rightarrow a_n = -\frac{4}{n^2}$$

Now $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (2\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (0-2)\left(-\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[(4\pi^2-4\pi^2)\left(-\frac{\cos 2n\pi}{n}\right) + (2\pi-4\pi)\left(\frac{\cancel{\sin 2n\pi}}{n^2}\right) + \frac{2\cos 2n\pi}{n^3}\right]\right.$$

$$\left. - \left[(0-0)\left(-\frac{\cos 0}{n}\right) + (2\pi-0)\frac{\cancel{\sin 0}}{n^2} + \frac{2\cos 0}{n^3}\right]\right\}$$

$$b_n = \frac{1}{\pi}\left[0 + 0 + 2\left(\frac{1}{n^3}\right) - 0 - 0 - 2\left(\frac{1}{n^3}\right)\right] = \frac{1}{\pi}(0)$$

$$\Rightarrow b_n = 0$$

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$$\therefore ① \Rightarrow f(x) = \frac{1}{2}\left(\frac{4\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\right)\cos nx + 0$$

$$x(2\pi - x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

$$\Rightarrow x(2\pi - x) = \frac{2\pi^2}{3} - 4\left\{\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right\} \longrightarrow ②$$

is the required Fourier expansion

Put $x = 0$ in ②

$$\Rightarrow 0 = \frac{2\pi^2}{3} - 4\left\{\frac{\cos 0}{1^2} + \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} + \cdots\right\}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right\}$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \longrightarrow ③$$

Put $x = \pi$ in ②

$$\Rightarrow \pi(2\pi - \pi) = \frac{2\pi^2}{3} - 4\left\{\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \cdots\right\}$$

$$2\pi^2 - \pi^2 - \frac{2\pi^2}{3} = -4\left\{-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right\}$$

$$\frac{\pi^2}{3} = 4\left\{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right\}$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \longrightarrow ④$$

Add ③ & ④

$$\frac{\pi^2}{6}+\frac{\pi^2}{12} = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)+\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots\right)$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots}$$

3. Express $f(x)=(\pi-x)^2$ as a Fourier series of period 2π in the interval $0<x<2\pi$.
Hence deduce that $1+\frac{1}{2^2}+\frac{1}{3^2}+\dots =$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = (\pi-x)^2$

Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2 dx = \frac{1}{\pi}\int_0^{2\pi}[\pi^2+x^2-2\pi x]\,dx$$

$$= \frac{1}{\pi}\left[\pi^2 x+\frac{x^3}{3}-2\pi\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[2\pi^3+\frac{8\pi^3}{3}-4\pi^3\right]-0\right\}$$

$$a_0 = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right)$$

$$\Rightarrow a_0 = \frac{2}{3}\pi^2$$

Now $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2 \cos nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2 \frac{\sin nx}{n} - 2(\pi-x)(-1)\left(-\frac{\cos nx}{n^2}\right) + (-2)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-2\pi)^2 \frac{\cancel{\sin 2n\pi}^{0}}{n} - 2(\pi-2\pi)\frac{\cos 2n\pi}{n^2} - 2\frac{\cancel{\sin 2n\pi}^{0}}{n^3}\right]\right.$$

$$\left. - \left[(\pi-0)^2 \frac{\cancel{\sin 0}^{0}}{n} - 2(\pi-0)\frac{\cos 0}{n^2} - 2\frac{\cancel{\sin 0}^{0}}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left[0 + 2\pi\frac{1}{n^2} - 0 - 0 + 2\pi\frac{1}{n^2} + 0\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right)$$

$$\Rightarrow a_n = \frac{4}{n^2}$$

$$\text{Now } b_n = \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right) + (-2)(-1)\left(+\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi-2\pi)^2\left(-\frac{\cos 2n\pi}{n}\right) - 2(\pi-2\pi)\frac{\sin 2n\pi}{n^2} + 2\frac{\cos 2n\pi}{n^3}\right]\right.$$

$$\left. - \left[(\pi-0)^2\left(-\frac{\cos 0}{n}\right) - 2(\pi-0)\frac{\sin 0}{n^2} + 2\frac{\cos 0}{n^3}\right]\right\}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-(-\pi)^2\frac{1}{n} - 0 + \frac{2}{n^3} + \frac{\pi^2}{n} + 0 - \frac{2}{n^3}\right]$$

$\Rightarrow bn = \frac{1}{\pi}[0-0]$

$\Rightarrow bn = 0$

$\therefore ① \Rightarrow f(x) = \frac{1}{2}\left(\frac{2}{3}\pi^2\right) + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + 0$

$\Rightarrow (\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \longrightarrow ②$

is the required Fourier expression.

put $x=0$ in ②

$\Rightarrow (\pi-0)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos 0}{n^2}$

$\pi^2 - \frac{\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right\}$

$\frac{2\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right\}$

$\Rightarrow \boxed{\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots}$

4 Expand $f(x) = \sqrt{1-\cos x}$, $0 < x < 2\pi$ in Fourier series. Hence evaluate $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots\dots =$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ①$

Given $f(x) = \sqrt{1-\cos x}$

Let $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

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$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{1-\cos x}\, dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2\sin^2\frac{x}{2}}\, dx$$

$$a_0 = \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \sin\frac{x}{2}\, dx$$

Put $x/_2 = t \Rightarrow \frac{dx}{2} = dt \Rightarrow dx = 2dt$

at $x=0 \Rightarrow t=0$ & $x=2\pi \Rightarrow t=\pi$

$$\therefore a_0 = \frac{\sqrt{2}}{\pi}\int_0^{\pi} \sin t\,(2dt) = \frac{2\sqrt{2}}{\pi}\left[-\cos t\right]_0^{\pi}$$

$$= \frac{2\sqrt{2}}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2\sqrt{2}}{\pi}\left[-(-1)+1\right]$$

$$\Rightarrow a_0 = \frac{4\sqrt{2}}{\pi}$$

Let $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{1-\cos x}\,\cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2\sin^2 x/_2}\,\cos nx\, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \sin\frac{x}{2}\cos nx\, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \left[\frac{1}{2}\left(\sin(n+1/_2)x - \sin(n-1/_2)x\right)\right] dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{\cos(n+1/2)x}{(n+1/2)} - \left(-\frac{\cos(n-1/2)x}{(n-1/2)} \right) \right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left\{ \left[-\frac{\cos(n+1/2)2\pi}{(n+1/2)} + \frac{\cos(n-1/2)2\pi}{(n-1/2)} \right] - \left[-\frac{\cos 0}{(n+1/2)} + \frac{\cos 0}{(n-1/2)} \right] \right\}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-(-1)}{n+1/2} + \frac{(-1)}{n-1/2} + \frac{1}{n+1/2} - \frac{1}{n-1/2} \right] = \frac{2\sqrt{2}}{2\pi}\left[\frac{1}{n+1/2} - \frac{1}{n-1/2} \right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{n-1/2-n-1/2}{(n+1/2)(n-1/2)} \right]$$

$$a_n = \frac{\sqrt{2}}{\pi}\left[\frac{-1}{n^2-\frac{1}{4}} \right]$$

$$\Rightarrow a_n = \frac{-4\sqrt{2}}{\pi}\left[\frac{1}{4n^2-1} \right]$$

Let $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{1-\cos x}\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2\sin^2 x/2}\,\sin nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \sin x/2 \cdot \sin nx\,dx$$

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$$= \frac{\sqrt{2}}{\pi} \int_0^{2\pi} \frac{1}{2} [\cos(n-1/2)x - \cos(n+1/2)x] \, dx$$

$$= \frac{\sqrt{2}}{2\pi} \left[\frac{\sin(n-1/2)x}{(n-1/2)} - \frac{\sin(n+1/2)x}{(n+1/2)} \right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi} \left[\frac{\sin(n-1/2)2\pi}{(n-1/2)} - \frac{\sin(n+1/2)2\pi}{(n+1/2)} - \frac{\sin 0}{(n-1/2)} + \frac{\sin 0}{(n+1/2)} \right]$$

$$= \frac{\sqrt{2}}{2\pi} [0-0-0+0]$$

$$\Rightarrow b_n = 0$$

$$\therefore ① \Rightarrow f(x) = \frac{1}{2}\left(\frac{4\sqrt{2}}{\pi}\right) + \sum_{n=1}^{\infty} \left(-\frac{4\sqrt{2}}{\pi} \frac{1}{(4n^2-1)}\right) \cos nx + 0$$

$$\Rightarrow \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi}\left[1 - 2\sum_{n=1}^{\infty} \frac{\cos nx}{4n^2-1}\right] \longrightarrow ②$$

is required Fourier expansion

put $x=0$ in ②

$$\Rightarrow \sqrt{1-\cos 0} = \frac{2\sqrt{2}}{\pi}\left[1 - 2\sum_{n=1}^{\infty} \frac{\cos 0}{4n^2-1}\right]$$

$$\Rightarrow 0 = \frac{2\sqrt{2}}{\pi}\left[1 - 2\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\right]$$

$$\Rightarrow 0 = 1 - 2\left\{\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots\right\}$$

$$\Rightarrow 1 = 2\left\{\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right\}$$

$$\Rightarrow \boxed{\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots}$$

5. An alternating current after passing through a rectifier is given by

$$I = \begin{cases} I_0 \sin x, & 0 < x \leq \pi \\ 0, & \pi < x < 2\pi \end{cases}, \text{ where } I_0 \text{ is a constant}$$

Develop I in a fourier series in $(0, 2\pi)$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = I = \begin{cases} I_0 \sin x, & 0 < x \leq \pi \\ 0, & \pi < x < 2\pi \end{cases}$

Let $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$= \frac{1}{\pi} \left\{ \int_0^{\pi} I_0 \sin x\, dx + \cancel{\int_{\pi}^{2\pi} 0\, dx}^{\,0} \right\}$

$= \frac{I_0}{\pi} \int_0^{\pi} \sin x\, dx = \frac{I_0}{\pi} \left(-\cos x \right)_0^{\pi} = \frac{I_0}{\pi} \left[-\cos \pi + \cos 0 \right]$

$= \frac{I_0}{\pi} \left[-(-1) + 1 \right]$

$\Rightarrow a_0 = \frac{2 I_0}{\pi}$

Let $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$

$= \frac{1}{\pi} \left\{ \int_0^{\pi} I_0 \sin x \cos nx\, dx + \cancel{\int_{\pi}^{2\pi} 0 \cdot \cos nx\, dx}^{\,0} \right\}$

$= \frac{I_0}{\pi} \int_0^{\pi} \sin x \cos nx\, dx$

$$= \frac{I_0}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{I_0}{2\pi}\left\{ \frac{-\cos(n+1)x}{n+1} - \left(\frac{-\cos(n-1)x}{n-1}\right) \right\}_0^{\pi}$$

$$= \frac{I_0}{2\pi}\left\{\left[\frac{-\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right] - \left[\frac{-\cos 0}{n+1} + \frac{\cos 0}{n-1}\right]\right\}$$

$$= \frac{I_0}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{I_0}{2\pi}\left[(-1)^n\left(-\frac{(-1)}{n+1} + \frac{(-1)^{-1}}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{I_0}{2\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{I_0}{2\pi}\left[(-1)^n + 1\right]\left(\frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$= \frac{I_0}{2\pi}\left[(-1)^n + 1\right]\left(\frac{n-1-n-1}{(n+1)(n-1)}\right) = -\frac{2I_0}{2\pi}\frac{\left[(-1)^n+1\right]}{n^2-1^2}$$

$$\Rightarrow a_n = \frac{-I_0}{\pi(n^2-1^2)}\left[(-1)^n + 1\right], \quad n \neq 1$$

To find a_n at $n=1$:-

$$\text{Let } a_n = \frac{I_0}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

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Put $n=1 \Rightarrow a_1 = \frac{I_0}{\pi}\int_0^{\pi} \sin x \cos x \, dx$

$$= \frac{I_0}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin(1+1)x - \sin(1-1)x\right] dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} [\sin 2x - \cancel{\sin 0 x}^{\,0}] \, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \sin 2x \, dx = \frac{I_0}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{I_0}{4\pi}\left[-\cos 2\pi + \cos 0\right] = \frac{I_0}{4\pi}\left[-1+1\right]$$

$\Rightarrow a_1 = 0$

Let $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} I_0 \sin x \sin nx \, dx + \cancel{\int_{\pi}^{2\pi} 0(\sin nx) \, dx}^{\,0}\right\}$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

$$= \frac{I_0}{\pi}\int_0^{\pi} \frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{I_0}{2\pi}\left\{\int_0^{\pi} \cos(n-1)x \, dx - \int_0^{\pi} \cos(n+1)x \, dx\right\}$$

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$$= \frac{I_0}{2\pi}\left\{ \frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)} \right\}_0^{\pi}$$

$$= \frac{I_0}{2\pi}\left\{ \left[\frac{\sin(n-1)\pi}{(n-1)} - \frac{\sin(n+1)\pi}{(n+1)}\right] - \left[\frac{\sin 0}{(n-1)} - \frac{\sin 0}{(n+1)}\right] \right\}$$

$$= \frac{I_0}{2\pi}[0-0]$$

$\Rightarrow b_n = 0$ at $n \neq 1$

<u>To find b_n at $n=1$:-</u>

$$b_n = \frac{I_0}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

put $n=1 \Rightarrow b_1 = \frac{I_0}{\pi}\int_0^{\pi} \sin x \sin x \, dx$

$$= \frac{I_0}{\pi}\int_0^{\pi} \frac{1}{2}\left[\cos(1-1)x - \cos(1+1)x\right] dx$$

$$\Rightarrow b_1 = \frac{I_0}{2\pi}\int_0^{\pi}[\cos 0x - \cos 2x]\,dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi}[1-\cos 2x]\,dx$$

$$= \frac{I_0}{2\pi}\left\{ x - \frac{\sin 2x}{2} \right\}_0^{\pi}$$

$$= \frac{I_0}{2\pi}\left\{ \left[\pi - \frac{\cancel{\sin 2\pi}}{2}\right] - \left[0 - \frac{\sin 0}{2}\right] \right\}$$

$$= \frac{I_0}{2\pi}[\pi - 0]$$

$$\Rightarrow b_1 = \frac{I_0}{2}$$

$$\therefore ① \Rightarrow f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$\Rightarrow I = f(x) = \left(\frac{2I_0}{\pi}\right)\cdot\frac{1}{2} + 0\cdot\cos x + \sum_{n=2}^{\infty}\left(\frac{-I_0\,[1+(-1)^n]}{\pi(n^2-1)}\right)\cos nx + \frac{I_0}{2}\sin x + \sum_{n=2}^{\infty} 0\cdot \sin nx$$

$$\Rightarrow I = \frac{I_0}{\pi} - \frac{I_0}{\pi}\sum_{n=2}^{\infty}\frac{[1+(-1)^n]}{n^2-1}\cos nx + \frac{I_0}{2}\sin x \longrightarrow ②$$

is the required Fourier series

6. obtain the Fourier expansion of $f(x) = x\sin x$ in the interval $(0, 2\pi)$. Deduce the following
(i) $\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$ (ii) $\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty}\frac{\cos(n\pi/2)}{n^2-1}$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ①$

Given $f(x) = x\sin x$ in $(0, 2\pi)$

Let $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$

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$\therefore y(x_0+h) = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$= \frac{1}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{2\pi}$

$y(0+0.2) = 1+\frac{1}{6}[0.02+2(0.0206)+2(0.0206)+0.02]$

$= \frac{1}{\pi}\left[(2\pi(-\cos 2\pi)+\sin 2\pi) - (0+\sin 0)\right]$

$y(0.2) = 1.0207$ is the required solution

$= \frac{1}{\pi}[2\pi(-1)+0-0]$

5) Given $\frac{dy}{dx} = 3x+\frac{y}{2}$, $y(0)=1$ compute $y(0.2)$ by taking $h=0.2$ using Fourth order Runge-kutta method. Also find the analytical solution.

$a_0 = -2$

Let $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$\rightarrow$ Given $x_0=0$, $y_0=1$, we need to find $y(0.2) \Rightarrow x_1=0.2$

$h = x_1 - x_0 = 0.2 - 0 \Rightarrow h=0.2$

$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$

Let $\frac{dy}{dx} = 3x+\frac{y}{2} \Rightarrow f(x,y) = 3x+\frac{y}{2}$

$= \frac{1}{\pi}\int_0^{2\pi} x\cdot\frac{1}{2}[\sin(1+n)x+\sin(1-n)x]\,dx$

$k_1 = h f(x_0,y_0) \Rightarrow k_1 = 0.2 f(0,1)$

$\Rightarrow k_1 = 0.2\left[3(0)+\frac{1}{2}\right]$

$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\sin(1+n)x\,dx + \int_0^{2\pi} x\sin(1-n)x\,dx\right\}$

$k_1 = 0.1$

$= \frac{1}{2\pi}\left\{\left[x\left(-\frac{\cos(1+n)x}{(1+n)}\right)-(1)\left(-\frac{\sin(1+n)x}{(1+n)^2}\right)\right]+\right.$

$k_2 = h f\left[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right] \Rightarrow k_2 = 0.2 f\left[0+\frac{0.2}{2}, 1+\frac{0.1}{2}\right]$

$\Rightarrow k_2 = 0.2 f[0.1, 1.05]$

$\left.\left[x\left(\frac{\cos(1-n)x}{(1-n)}\right)-(1)\left(-\frac{\sin(1-n)x}{(1-n)^2}\right)\right]\right\}_0^{2\pi}$

$= 0.2\left[3\times 0.1+\frac{1.05}{2}\right]$

$= 0.165$

$= \frac{1}{2\pi}\left\{\left(\left[-2\pi\frac{\cos(1+n)2\pi}{(1+n)}+\frac{\sin(1+n)2\pi}{(1+n)^2}\right]-\right.\right.$

$k_3 = h f\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right] \Rightarrow k_3 = 0.2 f\left[0+\frac{0.2}{2}, 1+\frac{0.165}{2}\right]$

$\left.\left[-\frac{0\cos 0}{(1+n)}+\frac{\sin 0}{(1+n)^2}\right]\right)+\left(\left[-2\pi\frac{\cos(1-n)2\pi}{(1-n)}+\frac{\sin(1-n)2\pi}{(1-n)^2}\right]\right.$

$\Rightarrow k_3 = 0.2 f[0.1, 1.0825]$

$= 0.2\left[3\times 0.1+\frac{1.0825}{2}\right]$

$\left.\left.-\left[\frac{0\cos 0}{(1-n)}+\frac{\sin 0}{(1-n)^2}\right]\right)\right\}$

$$=\frac{1}{2\pi}\left[-2\pi\cdot\frac{1}{n+1}+0+0-0-2\pi\frac{1}{1-n}+0+0-0\right]$$

$$=\frac{2\pi}{2\pi}\left[-\frac{1}{n+1}-\left(\frac{-1}{n-1}\right)\right]$$

$$=\frac{-1}{n+1}+\frac{1}{n-1}=\frac{-n+1+n+1}{(n+1)(n-1)}.$$

$$\Rightarrow a_n=\frac{2}{n^2-1^2} \text{ for } n\neq 1.$$

<u>To find a_n at $n=1$:-</u>

Consider $a_n=\frac{1}{2\pi}\int_0^{2\pi}x\left[\sin(1+n)x+\sin(1-n)x\right]dx$

put $n=1$

$$\Rightarrow a_1=\frac{1}{2\pi}\int_0^{2\pi}x\left[\sin 2x+\sin 0x\right]dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi}x\sin 2x\,dx$$

$$=\frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-(1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\left(-2\pi\frac{\cos 4\pi}{2}+\cancel{\frac{\sin 4\pi}{4}}^{0}\right)-\left(\frac{0\cdot\cos 0}{2}+\frac{\sin 0}{4}\right)\right]$$

$$=\frac{-2\pi}{2\pi}\left(\frac{1}{2}\right)$$

$$\Rightarrow a_1=-1/2$$

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Consider $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cdot\frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\cos(n-1)x\,dx - \int_0^{2\pi} x\cos(n+1)x\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[x\frac{\sin(n-1)x}{(n-1)} - 1\cdot\left(-\frac{\cos(n-1)x}{(n-1)^2}\right)\right] - \left[\frac{x\sin(n+1)x}{(n+1)} - 1\left(-\frac{\cos(n+1)x}{(n+1)^2}\right)\right]\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[\cancel{\frac{2\pi\sin(n-1)2\pi}{(n-1)}}^{0} + \frac{\cos(n-1)2\pi}{(n-1)^2} - \cancel{\frac{2\pi\sin(n+1)2\pi}{(n+1)}}^{0} - \frac{\cos(n+1)2\pi}{(n+1)^2}\right] - \left[0 + \frac{\cos 0}{(n-1)^2} - 0 - \frac{\cos 0}{(n+1)^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[0 + \frac{1}{(n-1)^2} - 0 - \frac{1}{(n+1)^2} - 0 - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$\Rightarrow b_n = 0$ for $n \neq 1$

To find b_n at $n=1$:-

Consider $b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$

Put $n=1 \Rightarrow b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin x\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2\pi}\int_0^{2\pi} (x - x\cos 2x).dx$

$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \left(x\frac{\sin 2x}{2} - 1\left(-\frac{\cos 2x}{4}\right)\right)\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[\left(\frac{4\pi^2}{2} - \frac{2\pi \sin 2\pi}{2} - \frac{\cos 4\pi}{4}\right) - \left(0 - 0 - \frac{\cos 0}{4}\right)\right]$

$= \frac{1}{2\pi}\left[2\pi^2 - 0 - \frac{1}{4} + \frac{1}{4}\right]$

$\Rightarrow b_1 = \pi$

$\therefore$ ① $\Rightarrow f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$

$x\sin x = \frac{1}{2}(-2) + \left(-\frac{1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x + 0$

$\Rightarrow x\sin x = -1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1} + \pi \sin x \longrightarrow$ ②

is the required Fourier expansion

(i) put $x=0$ in (2)

$$\Rightarrow 0 = -1 - \frac{\cos 0}{2} + 2\sum_{n=2}^{\infty} \frac{\cos 0}{n^2-1} + \pi . \sin 0$$

$$= -1 - \frac{1}{2} + 2\left\{\sum_{n=2}^{\infty} \frac{1}{n^2-1}\right\} + 0$$

$$= -\frac{3}{2} + 2\left\{\sum_{n=2}^{\infty} \frac{1}{n^2-1}\right\}$$

$$\Rightarrow \frac{3}{2} = 2\left\{\sum_{n=2}^{\infty} \frac{1}{n^2-1}\right\}$$

$$\Rightarrow \boxed{\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2-1}}$$

(ii) put $x=\pi/2$ in (2)

$$\Rightarrow \frac{\pi}{2} \cdot \sin\frac{\pi}{2} = -1 - \frac{1}{2}\cancel{\cos\frac{\pi}{2}}^{0} + 2\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1} + \pi . \sin\frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}(1) = -1 + 2\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1} + \pi(1)$$

$$\Rightarrow \frac{\pi}{2} - \pi = -1 + 2\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1}$$

$$\Rightarrow -\frac{\pi}{2} = -1 + 2\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1}$$

$$\Rightarrow \boxed{\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1}}$$

7 Obtain the Fourier Series expansion of $f(x)=x\cos x$ for $0<x<2\pi$

→ WKT $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \longrightarrow (1)$

Given $f(x)=x\cos x$ in $(0,2\pi)$

Let $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$

$$=\frac{1}{\pi}\int_0^{2\pi}x\cos x\,dx=\frac{1}{\pi}\Big[x\sin x-(1)(-\cos x)\Big]_0^{2\pi}$$

$$=\frac{1}{\pi}\Big\{\Big[2\pi\cancelto{0}{\sin 2\pi}+\cos 2\pi\Big]-\Big[0\sin 0+\cos 0\Big]\Big\}$$

$$=\frac{1}{\pi}[1-1]$$

$\Rightarrow a_0=0$

Let $a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$

$$=\frac{1}{\pi}\int_0^{2\pi}x\cos x\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}x\cdot\frac{1}{2}\left[\cos(1-n)x+\cos(1+n)x\right]dx$$

$$=\frac{1}{2\pi}\left\{\int_0^{2\pi}x\cos(1-n)x\,dx+\int_0^{2\pi}x\cos(1+n)x\,dx\right\}$$

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$$= \frac{1}{2\pi}\left\{\left[x\frac{\sin(1-n)x}{(1-n)} - (1)\left(-\frac{\cos(1-n)x}{(1-n)^2}\right)\right] + \left[x\frac{\sin(1+n)x}{(1+n)} - (1)\left(-\frac{\cos(1+n)x}{(1+n)^2}\right)\right]\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{2\pi\sin(1-n)2\pi}{(1-n)} + \frac{\cos(1-n)2\pi}{(1-n)^2} + \frac{2\pi\sin(1+n)2\pi}{(1+n)} + \frac{\cos(1+n)2\pi}{(1+n)^2}\right] - \left[\frac{0\cdot\sin 0}{(1-n)} + \frac{\cos 0}{(1-n)^2} + \frac{0\sin 0}{(1+n)} + \frac{\cos 0}{(1+n)^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[0 + \frac{1}{(1-n)^2} + 0 + \frac{1}{(1+n)^2} - 0 - \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2}\right]$$

$\Rightarrow a_n = 0$ for $n \neq 1$

<u>To find a_n at $n=1$:-</u>

consider $a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos x\cos nx\,dx$

put $n=1 \Rightarrow a_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x\cos x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x + x\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + x\frac{\sin 2x}{2} - (1)\left(-\frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

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$$=\frac{1}{2\pi}\left\{\left[\frac{4\pi^2}{2}+\frac{2\pi\cdot\sin 4\pi}{2}+\frac{\cos 4\pi}{4}\right]-\left[0+\frac{0\sin 0}{2}+\frac{\cos 0}{4}\right]\right.$$

$$=\frac{1}{2\pi}\left[2\pi^2+0+\frac{1}{4}-0-0-\frac{1}{4}\right]=\frac{1}{2\pi}(2\pi^2)$$

$$\Rightarrow a_1=\pi$$

Let $b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$=\frac{1}{\pi}\int_0^{2\pi} x\cos x\sin nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x\cdot\frac{1}{2}\left[\sin(1+n)x-\sin(1-n)x\right]dx$$

$$=\frac{1}{2\pi}\left\{\int_0^{2\pi} x\sin(1+n)x\,dx-\int_0^{2\pi} x\sin(1-n)x\,dx\right\}$$

$$=\frac{1}{2\pi}\left\{\left[x\left(-\frac{\cos(1+n)x}{(1+n)}\right)-(1)\left(-\frac{\sin(1+n)x}{(1+n)^2}\right)\right]-\right.$$

$$\left.\left[x\left(-\frac{\cos(1-n)x}{(1-n)}\right)-(1)\left(-\frac{\sin(1-n)x}{(1-n)^2}\right)\right]\right\}_0^{2\pi}$$

$$=\frac{1}{2\pi}\left\{\left[-\frac{2\pi\cos(1+n)2\pi}{1+n}+\frac{\sin(n+1)2\pi}{(1+n)^2}+\frac{2\pi\cos(1-n)2\pi}{(1-n)}\right.\right.$$

$$\left.\left.-\frac{\sin(1-n)2\pi}{(1-n)^2}\right]-\left[\frac{0\cdot\cos 0}{1+n}+\frac{\sin 0}{(1+n)^2}+\frac{0\cos 0}{1-n}-\frac{\sin 0}{(1-n)^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left[-2\pi\frac{1}{1+n} + 0 + 2\pi\frac{1}{1-n} - 0 - 0 - 0 - 0 + 0\right]$$

$$= \frac{2\pi}{2\pi}\left[\frac{1}{1-n} - \frac{1}{1+n}\right] = \left[\frac{1+n-1+n}{1^2-n^2}\right]$$

$$\Rightarrow b_n = \frac{2n}{1-n^2} \quad \text{for } n \neq 1$$

To find b_n at $n=1$:-

Consider $b_n = \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(1+n)x - \sin(1-n)x]\,dx$

put $n = 1 \Rightarrow$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin 2x - \cancel{\sin 0x}^{\,0}]\,dx$$

$$\Rightarrow b_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{-2\pi\cos 4\pi}{2} + \frac{\sin 4\pi}{4}\right) - \left(\frac{0\cos 0}{2} + \frac{\sin 0}{4}\right)\right]$$

$$= \frac{1}{2\pi}\left[-2\pi\cdot\frac{1}{2} + 0 - 0 - 0\right]$$

$$\Rightarrow b_1 = -1/2$$

$$\therefore \text{①} \Rightarrow f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$x\cos x = 0 + \pi\cos x + 0 + (-1/2)\sin x + \sum_{n=2}^{\infty}\frac{2n}{1-n^2}\sin nx$$

$$\Rightarrow x\cos x = \pi\cos x - \frac{\sin x}{2} + 2\sum_{n=2}^{\infty}\frac{n\sin nx}{1-n^2} \longrightarrow ②$$

is the Fourier expansion.

8. Obtain the Fourier series for $f(x) = e^{-ax}$ in the interval $(0, 2\pi)$. Hence deduce for e^{-x} & e^{x}

→ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

Given $f(x) = e^{-ax}$ in $(0, 2\pi)$

Let $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-ax}\,dx$

$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{e^{-2a\pi}}{-a} - \frac{e^0}{-a}\right]$

$\Rightarrow a_0 = \frac{1}{a\pi}\left[1 - e^{-2a\pi}\right]$

Let $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^{-ax}\cos nx\,dx$

$= \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\left[-a\cos nx + n\sin nx\right]\right]_0^{2\pi}$

$\left(\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right)$

$= \frac{1}{\pi}\left\{\left[\frac{e^{-2a\pi}}{a^2+n^2}(-a\cos 2n\pi + n\sin 2n\pi)\right] - \left[\frac{e^0}{a^2+n^2}(-a\cos 0 + n\sin 0)\right]\right.$

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$$= \frac{1}{\pi(a^2+n^2)} \left[e^{-2a\pi}(-a(1)+n(0)) - 1(-a(1)+n(0)) \right]$$

$$a_n = \frac{a}{\pi(a^2+n^2)} [1 - e^{-2a\pi}]$$

$$\text{Let } b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-ax} \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-ax}}{(-a)^2+n^2} (-a \sin nx - n \cos nx) \right]_0^{2\pi}$$

$$\left(\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right)$$

$$= \frac{1}{\pi(a^2+n^2)} \left\{ \left[e^{-2a\pi}(-a \sin 2n\pi - n \cos 2n\pi) \right] - \left[e^0(-a \sin 0 - n \cos 0) \right] \right\}$$

$$b_n = \frac{1}{\pi(a^2+n^2)} \left\{ \left[e^{-2a\pi}(-a(0) - n(1)) \right] - \left[1(-a(0) - n(1)) \right] \right\}$$

$$b_n = \frac{n}{\pi(a^2+n^2)} [1 - e^{-2a\pi}]$$

$$① \Rightarrow f(x) = \frac{[1 - e^{-2a\pi}]}{2\pi a} + \sum_{n=1}^{\infty} \frac{a[1 - e^{-2a\pi}]}{\pi(a^2+n^2)} \cos nx + \sum_{n=1}^{\infty} \frac{n[1 - e^{-2a\pi}]}{\pi(a^2+n^2)} \sin nx$$

$$\Rightarrow e^{-ax} = \frac{[1-e^{-2a\pi}]}{\pi}\left\{\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a\cos nx + n\sin nx}{a^2+n^2}\right\} \longrightarrow (2)$$

is the required Fourier expansion

put $a=1$ in (2)

$$\boxed{\Rightarrow e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx + n\sin nx}{1+n^2}\right\}}$$

Put $a=-1$ in (2)

$$\boxed{\Rightarrow e^{x} = \frac{1-e^{2\pi}}{\pi}\left\{\frac{-1}{2} + \sum_{n=1}^{\infty} \frac{n\sin nx - \cos nx}{1+n^2}\right\}}$$

9. Find the Fourier expansion of the function $f(x)$ defined by
$$f(x) = \begin{cases} x, & \text{for } 0 \le x < \pi \\ 2\pi - x, & \text{for } \pi < x \le 2\pi \end{cases}$$
Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$\rightarrow$ WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$

Given $f(x) = \begin{cases} x, & \text{for } 0 \le x \le \pi \\ 2\pi - x, & \text{for } \pi < x \le 2\pi \end{cases}$ in $(0, 2\pi)$

Let $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

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$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{x^2}{2}\Big|_0^{\pi} + 2\pi x - \frac{x^2}{2}\Big|_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} - 0\right] + \left[\left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} + \frac{4\pi^2}{2} - \frac{3\pi^2}{2}\right\} = \frac{1}{\pi}\left(\frac{2\pi^2}{2}\right)$$

$$\Rightarrow a_0 = \pi$$

Let $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\frac{\sin nx}{n} - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2}\right] - \left[\frac{0.\sin 0}{n} + \frac{\cos 0}{n^2}\right] + \left[\frac{(2\pi - 2\pi)\sin 2n\pi}{n} - \frac{\cos 2n\pi}{n^2}\right] - \left[\frac{(2\pi - \pi)\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2} + 0 - \frac{1}{n^2} - 0 + \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\text{Let } b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \left[-\frac{0\cdot\cos 0}{n} + \frac{\sin 0}{n^2}\right] + \left[\frac{-(2\pi - 2\pi)\cos 2n\pi}{n} - \frac{\sin 2n\pi}{n^2}\right] - \left[-(2\pi - \pi)\left(\frac{\cos n\pi}{n}\right) - \frac{\sin n\pi}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi(-1)^n}{n} + 0 + 0 - 0 - 0 - 0 + \frac{\pi(-1)^n}{n} + 0\right\}$$

$$\Rightarrow b_n = 0$$

$$\therefore ① \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx + 0$$

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$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos nx \longrightarrow (2)$$

is the required Fourier expansion

put $x=0$ in (2) & Given $f(x) = \begin{cases} x \\ 2\pi - x \end{cases}$ becomes 0

$$f(0) = 1/2\,[LHL + RHL] = 1/2\,[f(0) + f(2\pi)] = 1/2\,[0 + (2\pi - 2\pi)]$$

i.e $f(0) = 0$

$$(2) \Rightarrow 0 = f(0) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos 0$$

$$\Rightarrow -\frac{\pi}{2} = \frac{2}{\pi}\left\{-\frac{2}{1^2} - \frac{0}{2^2} - \frac{2}{3^2} - \frac{0}{4^2} - \frac{2}{5^2} - \dots\right\}$$

$$\Rightarrow -\frac{\pi^2}{4} = -2\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right\}$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

* Fourier Series expansion over the interval $(-L, L)$:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$

Where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx, \quad n = 1, 2 \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx, \quad n = 1, 2 \ldots$$

If $f(x)$ is an <u>odd function</u> in $(-L, L)$
then $a_0 = 0$ & $a_n = 0$

$$\therefore b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx, \quad n = 1, 2, 3 \ldots$$

If $f(x)$ is an <u>even function</u> in $(-L, L)$
then $b_n = 0$

$$\therefore a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx, \quad n = 1, 2, 3 \ldots$$

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* Problems:-

1 obtain the Fourier Series expansion of $f(x)=x^3$ in the interval $(-2,2)$

→ Given $f(x)=x^3$ in the interval $(-2,2)$, where $L=2$

Now $f(-x)=(-x)^3=-x^3=-f(x)$

$\therefore f(x)$ is an odd function.

Hence $a_0=0$ & $a_n=0$, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}\right)x+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}\right)x$

Let $b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx \quad \rightarrow ①$

Now at $L=2 \Rightarrow b_n=\frac{2}{2}\int_0^2 x^3\sin\left(\frac{n\pi}{2}\right)x\,dx$

$$b_n=\left[x^3\left(\frac{-\cos\left(\frac{n\pi}{2}\right)x}{(n\pi/2)}\right)-3x^2\left(\frac{-\sin\left(\frac{n\pi}{2}\right)x}{(n\pi/2)^2}\right)+6x\left(\frac{\cos\left(\frac{n\pi}{2}\right)x}{(n\pi/2)^3}\right)-6\left(\frac{\sin\left(\frac{n\pi}{2}\right)x}{(n\pi/2)^4}\right)\right]_0^2$$

$$=\left\{\left[2^3\left(\frac{-\cos\left(\frac{n\pi}{2}\right)\cdot 2}{(n\pi/2)}\right)+3\cdot 2^2\left(\frac{\sin\left(\frac{n\pi}{2}\right)\cdot 2}{(n\pi/2)^2}\right)+6\cdot 2\left(\frac{\cos\left(\frac{n\pi}{2}\right)2}{(n\pi/2)^3}\right)-6\left(\frac{\sin\left(\frac{n\pi}{2}\right)\cdot 2}{(n\pi/2)^4}\right)\right]-\right.$$

$$\left[0\frac{\cos 0}{(n\pi/2)} + 0\frac{\sin 0}{(n\pi/2)^2} - 0\frac{\cos 0}{(n\pi/2)^3} + 6\frac{\sin 0}{(n\pi/2)^4} \right] \Bigg\}$$

$$= -8\frac{(-1)^n}{\left(\frac{n\pi}{2}\right)} + 12\cdot\frac{0}{\left(\frac{n\pi}{2}\right)^2} + 12\frac{(-1)^n}{\left(\frac{n\pi}{2}\right)^3} - 6\cdot\frac{0}{\left(\frac{n\pi}{2}\right)^4}$$

$$= \frac{-16}{n\pi}(-1)^n + 0 + \frac{96}{n^3\pi^3}(-1)^n - 0$$

$$b_n = \frac{16}{n^3\pi^3}(-1)^n\left[6 - n^2\pi^2\right]$$

$$\therefore ① \Rightarrow f(x) = \frac{1}{2}(0) + 0 + \sum_{n=1}^{\infty}\frac{16}{n^3\pi^3}(-1)^n\left[6-n^2\pi^2\right]\sin\left(\frac{n\pi}{2}\right)x$$

$$\Rightarrow x^3 = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left[6-n^2\pi^2\right]\sin\left(\frac{n\pi}{2}\right)x$$

is the required Fourier expansion

2 Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$

→ Given $f(x) = e^{-x}$ in the interval $(-l, l)$ is neither odd nor even.

$$\text{WKT } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}\right)x$$

$$\text{Let } a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

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$$\Rightarrow a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{-1}\right]_{-l}^{l} = \frac{1}{l}\left[-e^{-l}-(-e^{l})\right]$$

$$= \frac{1}{l}\left[e^{l}-e^{-l}\right] = \frac{2}{l}\left(\frac{e^{l}-e^{-l}}{2}\right)$$

$$\Rightarrow a_0 = \frac{2\sinh l}{l}$$

$$\text{Let } a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$$

$$\Rightarrow a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{(-1)^2+\left(\frac{n\pi}{l}\right)^2}\left(-1\cos\left(\frac{n\pi}{l}\right)x + \frac{n\pi}{l}\sin\left(\frac{n\pi}{l}\right)x\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left\{\left[\frac{e^{-l}}{1^2+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\left(\frac{n\pi}{l}\right)l + \frac{n\pi}{l}\cancel{\sin\left(\frac{n\pi}{l}\right)l}^{\,0}\right)\right]\right.$$

$$\left. - \left[\frac{e^{l}}{1^2+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\left(\frac{n\pi}{l}\right)(-l) + \frac{n\pi}{l}\cancel{\sin\left(\frac{n\pi}{l}\right)(-l)}^{\,0}\right)\right]\right\}$$

$$= \frac{1}{l}\left[\frac{e^{-l}}{\frac{l^2+n^2\pi^2}{l^2}}\left(-(-1)^n\right) - \frac{e^{l}}{\frac{l^2+n^2\pi^2}{l^2}}\left(-(-1)^n\right)\right]$$

$$= \frac{l^2(-1)^n}{l(l^2+n^2\pi^2)}\left[e^{l}-e^{-l}\right]$$

$$= \frac{l(-1)^n}{l^2+n^2\pi^2}\, 2\cdot\frac{\left[e^{l}-e^{-l}\right]}{2}$$

$$\Rightarrow a_n = \frac{2l(-1)^n \sinh l}{l^2+n^2\pi^2}$$

Let $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \sin\left(\frac{n\pi}{L}\right)x\, dx$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\left(\frac{n\pi}{l}\right)x\, dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{(-1)^2+\left(\frac{n\pi}{l}\right)^2}\left(-1\sin\left(\frac{n\pi}{l}\right)x - \frac{n\pi}{l}\cos\left(\frac{n\pi}{l}\right)x\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left\{\left[\frac{e^{-l}}{1+\frac{n^2\pi^2}{l^2}}\left(-\cancel{\sin\left(\frac{n\pi}{l}\right)l}^{\,0} - \frac{n\pi}{l}\cos\left(\frac{n\pi}{l}\right)l\right)\right] - \left[\frac{e^{l}}{1+\frac{n^2\pi^2}{l^2}}\left(-\cancel{\sin\left(\frac{n\pi}{l}\right)(-l)}^{\,0} - \frac{n\pi}{l}\cos\left(\frac{n\pi}{l}\right)(-l)\right)\right]\right\}$$

$$= \frac{1}{l}\left(\frac{1}{\frac{l^2+n^2\pi^2}{l^2}}\right)\left[e^{-l}\left(-\frac{n\pi}{l}(-1)^n\right)+e^{l}\,\frac{n\pi}{l}(-1)^n\right]$$

$$= \frac{l^2}{l}\left(\frac{n\pi}{l}\right)\frac{(-1)^n}{l^2+n^2\pi^2}\left[e^{l}-e^{-l}\right]$$

$$= \frac{n\pi(-1)^n}{l^2+n^2\pi^2}\cdot 2\cdot\frac{\left[e^{l}-e^{-l}\right]}{2}$$

$$\Rightarrow b_n = \frac{2n\pi(-1)^n \sinh l}{l^2+n^2\pi^2}$$

$$\therefore ① \Rightarrow f(x) = \frac{2\sinh l}{l}\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{2l(-1)^n \sinh l}{l^2+n^2\pi^2} \cos\left(\frac{n\pi}{l}\right)x$$

$$+ \sum_{n=1}^{\infty} \frac{2n\pi(-1)^n \sinh l}{l^2+n^2\pi^2} \sin\left(\frac{n\pi}{l}\right)x$$

$$\Rightarrow e^{-x} = \sinh l\left[\frac{1}{l} + 2l\sum_{n=1}^{\infty} \frac{(-1)^n}{l^2+n^2\pi^2} \cos\left(\frac{n\pi}{l}\right)x + 2\pi\sum_{n=1}^{\infty} \frac{n(-1)^n}{l^2+n^2\pi^2} \cdot \sin\left(\frac{n\pi}{l}\right)x\right]$$

is the required Fourier expansion.

3. Express e^x as Fourier series in $[-1, 1]$

$\rightarrow$ Given $f(x) = e^x$ is neither odd nor even in $(-1, 1)$ where $l = 1$

$$\text{WKT } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$

$$\text{Let } a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} e^x\,dx = e^x\Big|_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

$$\Rightarrow a_0 = e - \frac{1}{e}$$

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$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} e^{x}\cos\left(\frac{n\pi}{1}\right)x\,dx$$

$$= \left[\frac{e^{x}}{1^2+(n\pi)^2}\left[1\cos n\pi x + n\pi\sin n\pi x\right]\right]_{-1}^{1}$$

$$= \left\{\left[\frac{e^{1}}{1+n^2\pi^2}\left[\cos n\pi + n\pi \cancel{\sin n\pi}^{\,0}\right]\right] - \right.$$

$$\left.\left[\frac{e^{-1}}{1+n^2\pi^2}\left[1\cos n\pi(-1) + n\pi\cancel{\sin n\pi(-1)}^{\,0}\right]\right]\right\}$$

$$= \frac{1}{1+n^2\pi^2}\left[e(-1)^n - e^{-1}(-1)^n\right]$$

$$\Rightarrow a_n = \frac{(-1)^n}{1+n^2\pi^2}\left[e - \frac{1}{e}\right]$$

Let $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{1}{1}\int_{-1}^{1} e^{x}\sin\left(\frac{n\pi}{1}\right)x\,dx$$

$$= \left[\frac{e^{x}}{1^2+(n\pi)^2}\left[1\sin n\pi x - n\pi\cos n\pi x\right]\right]_{-1}^{1}$$

$$= \left\{\left[\frac{e^{1}}{1^2+n^2\pi^2}\left[\cancel{\sin n\pi}^{\,0} - n\pi\cos n\pi\right]\right] - \right.$$

$$\left[\frac{e^{-1}}{1+n^2\pi^2}\left[1\,\cancel{\sin n\pi}^{\,0}(-1) - n\pi\cos n\pi(-1)\right]\right]\Bigg\}$$

$$\Rightarrow b_n = \frac{1}{1+n^2\pi^2}\left[e(-n\pi(-1)^n) - e^{-1}(-n\pi(-1)^n)\right]$$

$$= \frac{n\pi(-1)^n}{1+n^2\pi^2}\left[\frac{1}{e} - e\right]$$

$$b_n = -\frac{n\pi(-1)^n}{1+n^2\pi^2}\left[e - \frac{1}{e}\right]$$

$$\therefore ① \Rightarrow f(x) = \frac{1}{2}\left[e - \frac{1}{e}\right] + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\left[e - \frac{1}{e}\right]\cos\left(\frac{n\pi}{1}\right)x$$

$$+ \sum_{n=1}^{\infty}\frac{-n\pi(-1)^n}{1+n^2\pi^2}\left[e - \frac{1}{e}\right]\sin\left(\frac{n\pi}{1}\right)x$$

$$\Rightarrow e^x = \left(e - \frac{1}{e}\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\left[\cos n\pi x - n\pi\sin n\pi x\right]\right]$$

is the required Fourier expansion

4 Find the Fourier series that represents the function $f(x) = 1 + \sin x$ in the interval $-1 < x < 1$

→ Given $f(x) = 1 + \sin x$ is neither odd nor even in the interval $(-1, 1)$

$$\text{WKT } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow ①$$

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Let $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

$= \frac{1}{1}\int_{-1}^{1} (1+\sin x)\,dx$

$= x - \cos x \Big|_{-1}^{1}$

$\Rightarrow a_0 = (1-\cos 1) - (-1-\cos(-1))$
$= 1 - \cos 1 + 1 + \cos 1$

$\Rightarrow a_0 = 2$

Let $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$= \frac{1}{1}\int_{-1}^{1} (1+\sin x)\cos\left(\frac{n\pi}{1}\right)x\,dx$

$= \int_{-1}^{1} (\cos n\pi x + \sin x \cos n\pi x)\,dx$

$= \int_{-1}^{1} \left[\cos n\pi x + \frac{1}{2}\left(\sin(1+n\pi)x + \sin(1-n\pi)x\right)\right] dx$

$= \left[\frac{\sin n\pi x}{n\pi} + \frac{1}{2}\left\{-\frac{\cos(1+n\pi)x}{(1+n\pi)} - \frac{\cos(1-n\pi)x}{(1-n\pi)}\right\}\right]_{-1}^{1}$

$= \left[\cancel{\frac{\sin n\pi}{n\pi}}^{\,0} - \frac{1}{2}\left(\frac{\cos(1+n\pi)}{(1+n\pi)} + \frac{\cos(1-n\pi)}{(1-n\pi)}\right)\right] -$

$\left[\cancel{\frac{\sin n\pi(-1)}{n\pi}}^{\,0} - \frac{1}{2}\left(\frac{\cos(1+n\pi)(-1)}{(1+n\pi)} + \frac{\cos(1-n\pi)(-1)}{(1-n\pi)}\right)\right]$

$$= \left[0 - \frac{1}{2}\left(\frac{\cos(1+n\pi)}{1+n\pi} + \frac{\cos(1-n\pi)}{1-n\pi} \right) \right.$$

$$\left. - 0 + \frac{1}{2}\left(\frac{\cos(1+n\pi)}{1+n\pi} + \frac{\cos(1-n\pi)}{1-n\pi} \right) \right]$$

$$\Rightarrow a_n = 0$$

Let $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}\right) x \, dx$

$$= \frac{1}{1} \int_{-1}^{1} (1+\sin x) \sin\left(\frac{n\pi}{1}\right) x \, dx$$

$$= \int_{-1}^{1} (\sin n\pi x + \sin x \sin n\pi x) \, dx$$

$$= \int_{-1}^{1} \left(\sin n\pi x + \frac{1}{2}\left[\cos(n\pi - 1)x - \cos(n\pi+1)x \right] \right) dx$$

$$= \left\{ -\frac{\cos n\pi x}{n\pi} + \frac{1}{2}\left[\frac{\sin(n\pi-1)x}{(n\pi-1)} - \frac{\sin(n\pi+1)x}{(n\pi+1)} \right] \right\}_{-1}^{1}$$

$$= \left\{ \left[-\frac{\cos n\pi}{n\pi} + \frac{1}{2}\left(\frac{\sin(n\pi-1)}{(n\pi-1)} - \frac{\sin(n\pi+1)}{(n\pi+1)} \right) \right] - \right.$$

$$\left. \left[-\frac{\cos n\pi(-1)}{n\pi} + \frac{1}{2}\left(\frac{\sin(n\pi-1)(-1)}{(n\pi-1)} - \frac{\sin(n\pi+1)(-1)}{(n\pi+1)} \right) \right] \right\}$$

$$= \left[-\frac{\cos n\pi}{n\pi} + \frac{1}{2}\left[\frac{\sin(n\pi-1)}{(n\pi-1)} - \frac{\sin(n\pi+1)}{(n\pi+1)} \right] \right] -$$

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$$\left[\frac{-\cos n\pi}{n\pi} + \frac{1}{2}\left(-\frac{\sin(n\pi-1)}{(n\pi-1)} + \frac{\sin(n\pi+1)}{(n\pi+1)}\right)\right]$$

$$\Rightarrow b_n = \cancel{\frac{-\cos n\pi}{n\pi}} + \frac{1}{2}\left[\frac{\sin(n\pi-1)}{(n\pi-1)} - \frac{\sin(n\pi+1)}{(n\pi+1)}\right] + \cancel{\frac{\cos n\pi}{n\pi}}$$

$$+ \frac{1}{2}\left[\frac{\sin(n\pi-1)}{(n\pi-1)} - \frac{\sin(n\pi+1)}{(n\pi+1)}\right]$$

$$\Rightarrow b_n = \frac{2}{2}\left[\frac{\sin(n\pi-1)}{(n\pi-1)} - \frac{\sin(n\pi+1)}{(n\pi+1)}\right]$$

$$\Rightarrow b_n = \frac{-(\cos n\pi)\sin 1}{(n\pi-1)} - \frac{(\cos n\pi)\sin 1}{n\pi+1}$$

$$= -(\cos n\pi)\sin 1\left[\frac{1}{n\pi-1} + \frac{1}{n\pi+1}\right]$$

$$= (-1)(-1)^n \sin 1\left[\frac{n\pi+1+n\pi-1}{n^2\pi^2-1^2}\right]$$

$$b_n = \frac{(-1)^{n+1}\sin 1}{n^2\pi^2-1}[2n\pi]$$

$$\therefore ① \Rightarrow f(x) = \frac{2}{2} + 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin 1(2n\pi)}{n^2\pi^2-1}\sin\left(\frac{n\pi}{1}\right)x$$

$$1+\sin x = 1 + 2\pi(\sin 1)\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{n^2\pi^2-1}\sin n\pi x$$

is the required Fourier expansion

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5. Expand the function $f(x) = \begin{cases} 1+2x & \text{for } -3 < x \le 0 \\ 1-2x & \text{for } 0 \le x < 3 \end{cases}$

as a Fourier series & deduce that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

→ Given $f(x) = \begin{cases} 1+2x, & -3 < x \le 0 \\ 1-2x, & 0 \le x < 3 \end{cases}$ in the interval $(-3, 3)$

Let $f(-x) = \begin{cases} 1+2(-x), & -3 < -x \le 0 \\ 1-2(-x) & 0 \le -x < 3 \end{cases}$

$$\Rightarrow f(-x) = \begin{cases} 1-2x & 0 \le x < 3 \\ 1+2x, & -3 < x \le 0 \end{cases} = f(x)$$

$\therefore f(-x) = f(x)$ is an even function.

Where $L = 3$, Hence $b_n = 0$

WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \rightarrow ①$

Let $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{3}\int_0^3 (1-2x)\,dx$

$$= \frac{2}{3}\left[x - \frac{2x^2}{2}\right]_0^3 = \frac{2}{3}\left[(3-9) - (0-0)\right]$$

$$= \frac{2}{3}(-6)$$

$$\Rightarrow a_0 = -4$$

Let $a_n = \frac{2}{L}\int^L a_n \cos\left(\frac{n\pi}{L}\right)x\,dx$

$$a_n = \frac{2}{3}\int_0^3 (1-2x)\cos\left(\frac{n\pi}{3}\right)x\,dx$$

$$= \frac{2}{3}\left[(1-2x)\left(\frac{\sin\left(\frac{n\pi}{3}\right)x}{(n\pi/3)}\right) - (0-2)\left(\frac{-\cos\left(\frac{n\pi}{3}\right)x}{(n\pi/3)^2}\right)\right]_0^3$$

$$= \frac{2}{3}\left\{\left[(1-6)\left(\frac{\cancel{\sin\left(\frac{n\pi}{3}\right)3}}{(n\pi/3)}\right) - 2\left(\frac{\cos\left(\frac{n\pi}{3}\right)3}{(n\pi/3)^2}\right)\right] - \left[(1-0)\left(\frac{\cancel{\sin 0}}{(n\pi/3)}\right) - 2\left(\frac{\cos 0}{(n\pi/3)^2}\right)\right]\right\}$$

$$= \frac{2}{3}\left[\frac{-2(-1)^n}{(n\pi/3)^2} - \frac{(-2)1}{(n\pi/3)^2}\right]$$

$$= \frac{2\times(-2)}{3\left(\frac{n\pi}{3}\right)^2}\left[(-1)^n - 1\right]$$

$$= \frac{-4\times 9}{3n^2\pi^2}\left[(-1)^n - 1\right] = \frac{-12}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\Rightarrow a_n = \frac{12}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\therefore f(x) = \frac{-4}{2} + \sum_{n=1}^{\infty}\frac{12}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi}{3}\right)x + 0$$

$$\Rightarrow f(x) = -2 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos\left(\frac{n\pi}{3}\right)x \longrightarrow (2)$$

is the required Fourier expansion

Put $x=0$ in ② & given $f(x)=\begin{cases}1+2x, & -3<x\le 0\\ 1-2x, & 0\le x<3\end{cases}$

$$\Rightarrow f(0)=1$$

$$\Rightarrow 1=-2+\frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos 0$$

$$\Rightarrow 1+2=\frac{12}{\pi^2}\left\{\frac{2}{1^2}+\frac{0}{2^2}+\frac{2}{3^2}+\cdots\right\}$$

$$\Rightarrow \frac{3\pi^2}{12}=2\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right\}$$

$$\Rightarrow \boxed{\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

6. Find the Fourier series represention the function

$$f(x)=\begin{cases}0 & \text{in } -2<x<0\\ a & \text{in } 0<x<2\end{cases}$$

→ Given $f(x)$ neither odd nor even function in $(-2,2)$.

WKT $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}\right)x+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}\right)x \longrightarrow ①$

where $L=2$

Let $a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx$

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$$a_0 = \frac{1}{2}\left\{\cancel{\int_{-2}^{0} 0 \cdot dx}^{\,0} + \int_{0}^{2} a\, dx\right\}$$

$$= \frac{1}{2}(ax)\Big|_0^2 \quad = \frac{1}{2}[2a - 0 \cdot a]$$

$$\Rightarrow a_0 = a$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\, dx$$

$$= \frac{1}{2}\left\{\cancel{\int_{-2}^{0} 0 \cdot \cos\left(\frac{n\pi}{2}\right)x\, dx}^{\,0} + \int_{0}^{2} a \cdot \cos\left(\frac{n\pi}{2}\right)x\, dx\right\}$$

$$= \frac{a}{2}\left[\frac{\sin\left(\frac{n\pi}{2}\right)x}{(n\pi/2)}\right]_0^2$$

$$= \frac{a}{2\left(\frac{n\pi}{2}\right)}\left[\sin\left(\frac{n\pi}{2}\right) \cdot 2 - \sin 0\right]$$

$$= \frac{a}{n\pi}[0 - 0]$$

$$\Rightarrow a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\, dx$$

$$= \frac{1}{2}\left\{\cancel{\int_{-2}^{0} 0 \cdot \sin\left(\frac{n\pi}{2}\right)x\, dx}^{\,0} + \int_{0}^{2} a \cdot \sin\left(\frac{n\pi}{2}\right)x\, dx\right\}$$

$$= \frac{a}{2}\left[\frac{-\cos\left(\frac{n\pi}{2}\right)x}{(n\pi/2)}\right]_0^2$$

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$$= \frac{-a}{2\cdot\left(\frac{n\pi}{2}\right)}\left[\cos\left(\frac{n\pi}{2}\right)\cdot 2 - \cos 0\right]$$

$$= \frac{-a}{n\pi}\left[(-1)^n - 1\right]$$

$$\Rightarrow b_n = \frac{a}{n\pi}\left[1 - (-1)^n\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{a}{2} + 0 + \sum_{n=1}^{\infty} \frac{a}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi}{2}\right)x$$

$$\Rightarrow f(x) = \frac{a}{2} + \frac{a}{\pi}\sum_{n=1}^{\infty} \frac{\left[1-(-1)^n\right]}{n} \sin\left(\frac{n\pi}{2}\right)x.$$

* Fourier series expansion over the interval $(0, 2L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$

Where $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx, \text{ for } n=1,2,3\ldots$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx, \text{ for } n=1,2,3\ldots$$

* Problems :-

1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2$.

→ Given $f(x) = e^{-x}$ in the interval $(0,2) = (0,2L)$

$\Rightarrow 2L = 2 \Rightarrow L = 1$

WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$

Let $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$

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$$= \frac{1}{1}\int_0^2 e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^2 = -e^{-2} - (-e^0)$$

$$= -e^{-2} + 1 = 1 - \frac{1}{e^2}$$

$$\Rightarrow a_0 = \frac{e^2 - 1}{e^2}$$

$$\text{Let } a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{1}{1}\int_0^2 e^{-x}\cos\left(\frac{n\pi}{1}\right)x\,dx$$

$$= \frac{e^{-x}}{(-1)^2 + (n\pi)^2}\left[-1\cos n\pi x + n\pi \sin n\pi x\right]\Big|_0^2$$

$$= \frac{1}{1+n^2\pi^2}\left\{\left[e^{-2}(-\cos 2n\pi + \cancel{n\pi \sin 2n\pi}^{\,0})\right] - \left[e^0(-\cos 0 + \cancel{n\pi \sin 0}^{\,0})\right]\right\}$$

$$= \frac{1}{1+n^2\pi^2}\left[e^{-2}(-1) - (-1)\right]$$

$$\Rightarrow a_n = \frac{1}{1+n^2\pi^2}\left[1 - \frac{1}{e^2}\right]$$

$$\Rightarrow a_n = \frac{[e^2 - 1]}{e^2(1+n^2\pi^2)}$$

$$\text{Let } b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

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$$= \frac{1}{1}\int_0^2 e^{-x} \sin\left(\frac{n\pi}{1}\right)x\,dx$$

$$= \frac{e^{-x}}{(-1)^2+(n\pi)^2}\left[-1\sin n\pi x - n\pi\cos n\pi x\right]\Big|_0^2$$

$$= \frac{1}{1^2+n^2\pi^2}\left\{\left[e^{-2}(-1\cancel{\sin 2n\pi}^{0} - n\pi\cos 2n\pi)\right] - \left[e^{0}(-1\cancel{\sin 0}^{0} - n\pi\cos 0)\right]\right\}$$

$$= \frac{1}{1+n^2\pi^2}\left[-n\pi(e^{-2}) - (-n\pi(1))\right]$$

$$= \frac{n\pi}{1+n^2\pi^2}\left[1-e^{-2}\right] = \frac{n\pi}{1+n^2\pi^2}\left[1-\frac{1}{e^2}\right]$$

$$\Rightarrow b_n = \frac{n\pi\left[e^2-1\right]}{e^2(1+n^2\pi^2)}$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{2}\frac{(e^2-1)}{e^2} + \sum_{n=1}^{\infty}\frac{(e^2-1)}{e^2(1+n^2\pi^2)}\cos\left(\frac{n\pi}{1}\right)x + \sum_{n=1}^{\infty}\frac{n\pi(e^2-1)}{e^2(1+n^2\pi^2)}\sin\left(\frac{n\pi}{1}\right)x$$

$$\Rightarrow e^{-x} = \left(\frac{e^2-1}{e^2}\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos n\pi x + n\pi\sin n\pi x}{1+n^2\pi^2}\right]$$

is the required Fourier expansion.

2. Find the Fourier series of the periodic function defined by $f(x) = 2x - x^2$ in the interval $0 < x < 3$

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$\rightarrow$ Given $f(x) = 2x - x^2$ in the interval $(0, 3) = (0, 2L)$

$$\Rightarrow 2L = 3 \Rightarrow L = 3/2$$

WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \rightarrow (1)$

Let $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[\left(9 - \frac{27}{3}\right) - 0\right]$$

$$\Rightarrow a_0 = 0$$

Let $a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)\cos\frac{n\pi}{3/2}x\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi}{3}\right)x\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(\frac{\cancelto{0}{\sin\left(\frac{2n\pi}{3}\right)x}}{\left(\frac{2n\pi}{3}\right)}\right) - (2 - 2x)\left(\frac{-\cos\left(\frac{2n\pi}{3}\right)x}{\left(\frac{2n\pi}{3}\right)^2}\right) + (0 - 2)\left(\frac{-\cancelto{0}{\sin\left(\frac{2n\pi}{3}\right)x}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$\because \sin 2n\pi =$

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$$= \frac{2}{3}\left\{ \frac{(2-2x)\cos\left(\frac{2n\pi}{3}\right)x}{\left(\frac{2n\pi}{3}\right)^2} \right\}_0^3$$

$$= \frac{2/3}{\frac{4n^2\pi^2}{9}} \left\{ \left[(2-6)\cos\left(\frac{2n\pi}{3}\right)\cdot 3\right] - \left[(2-0)\cos 0\right] \right\}$$

$$= \frac{3}{2n^2\pi^2}\left[-4(1) - 2(1)\right]$$

$$= \frac{3(-6)}{2n^2\pi^2}$$

$$\Rightarrow a_n = -9/n^2\pi^2$$

$$\text{Let } b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\left(\frac{n\pi}{3/2}\right)x\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi}{3}\right)x\,dx$$

$$= \frac{2}{3}\left\{ (2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi}{3}\right)x}{(2n\pi/3)}\right) - (2-2x)\cancelto{0}{\left(\frac{-\sin\left(\frac{2\pi n}{3}\right)x}{(2n\pi/3)^2}\right)} + (0-2)\left(\frac{\cos\left(\frac{2n\pi}{3}\right)x}{(2n\pi/3)^3}\right) \right\}_0^3$$

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$$= \frac{2}{3}\left\{\left[-\frac{(6-9)\cos\left(\frac{2n\pi}{3}\right)3}{(2n\pi/3)} - \frac{2\cos\left(\frac{2n\pi}{3}\right)3}{(2n\pi/3)^3}\right] - \left[\frac{-0\cos 0}{(2n\pi/3)} - \frac{2\cos 0}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}$$

$$= \frac{2}{3}\left[3\frac{1}{\left(\frac{2n\pi}{3}\right)} - 2\cdot\frac{1}{\left(\frac{2n\pi}{3}\right)^3} + 0 + 2\cdot\frac{1}{\left(\frac{2n\pi}{3}\right)^3}\right]$$

$$= \frac{2}{3}\cdot\frac{9}{2n\pi}$$

$$\Rightarrow b_n = \frac{3}{n\pi}$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{2}\cdot 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\right)\cos\left(\frac{n\pi}{3/2}\right)x + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\left(\frac{n\pi}{3/2}\right)x$$

$$\Rightarrow (2x-x^2) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi}{3}\right)x + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi}{3}\right)x$$

is the required Fourier expansion.

3 Obtain the Fourier series of the Saw-tooth function $f(t) = Et/T$ for $0<t<T$ that $f(t+T) = f(t)\ \forall t>0$

$\rightarrow$ Given $f(t) = Et/T$ in the interval $(0, T) = (0, 2L)$

Where $2L = T \Rightarrow L = \frac{T}{2}$

WKT $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)t + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)t \rightarrow x$

Let $a_0 = \frac{1}{L}\int_0^{2L} f(t)\,dt$

$$= \frac{1}{T/2}\int_0^{T} \frac{E}{T} t\,dt = \frac{2E}{T^2}\int_0^{T} t\,dt$$

$$= \frac{2E}{T^2} \left.\frac{t^2}{2}\right|_0^{T} = \frac{2E}{T^2}\left[\frac{T^2}{2} - \frac{0}{2}\right]$$

$$= \frac{2E}{T^2}\left(\frac{T^2}{2}\right)$$

$\Rightarrow a_0 = E$

Let

$a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi}{L}\right)t\,dt$

$$= \frac{1}{T/2}\int_0^{T} \frac{E}{T} t \cos\left(\frac{n\pi}{T/2}\right)t\,dt$$

$$= \frac{2E}{T^2}\int_0^{T} t\cos\left(\frac{2n\pi}{T}\right)t\,dt$$

$$= \frac{2E}{T^2}\left[t\,\frac{\cancel{\sin\left(\frac{2n\pi}{T}\right)t}^{\,0}}{\left(\frac{2n\pi}{T}\right)} - (1)\left(\frac{-\cos\left(\frac{2n\pi}{T}\right)t}{\left(\frac{2n\pi}{T}\right)^2}\right)\right]_0^{T}$$

$\because \sin 2n\pi = 0$

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$$= \frac{2E}{T^2}\left[\frac{1}{\left(\frac{2n\pi}{T}\right)^2}\right]\left\{\cos\left(\frac{2n\pi}{T}\right)T - \cos 0\right\}$$

$$= \frac{2E}{T^2}\,\frac{T^2}{4n^2\pi^2}\left[1-1\right]$$

$$\Rightarrow a_n = 0$$

Let $b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi}{L}\right)t\,dt$

$$= \frac{1}{T/2}\int_0^{T} \frac{E}{T}t\,\sin\left(\frac{n\pi}{T/2}\right)t\,dt$$

$$= \frac{2E}{T^2}\int_0^{T} t\sin\left(\frac{2n\pi}{T}\right)t\,dt$$

$$= \frac{2E}{T^2}\left\{t\left(\frac{-\cos\left(\frac{2n\pi}{T}\right)t}{\left(\frac{2n\pi}{T}\right)}\right) - (1)\left(\frac{-\cancelto{0}{\sin\left(\frac{2n\pi}{T}\right)t}}{\left(\frac{2n\pi}{T}\right)^2}\right)\right\}$$

$\because \sin 2n\pi = 0$

$$= \frac{2E}{T^2}\,\frac{1}{\frac{2n\pi}{T}}\left\{-T\cos\left(\frac{2n\pi}{T}\right)T - \left(-0\cdot\cos 0\right)\right\}$$

$$= \frac{2ET}{2n\pi T^2}\left[-T(1)+0\right]$$

$$\Rightarrow b_n = -E/n\pi$$

$$\therefore (1) \Rightarrow f(t) = E/2 + 0 + \sum_{n=1}^{\infty}\left(\frac{-E}{n\pi}\right)\sin\left(\frac{n\pi}{T/2}\right)t$$

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$$\Rightarrow \frac{Et}{T} = \frac{E}{2} - \frac{E}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{2n\pi}{T}\right)t$$

is the required Fourier series

4 Find the Fourier series of $f(x)$ with period $2l$ by $f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l<x<2l \end{cases}$

Hence deduce that (i) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

(ii) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

$\rightarrow$ Given $f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l<x<2l \end{cases}$ in the interval

$(0, 2l) = (0, 2L)$

$$\Rightarrow 2L = 2l \Rightarrow L = l$$

WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow (1)$

Let $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$

$$= \frac{1}{l}\left\{\int_0^{l} (l-x)\,dx + \cancel{\int_l^{2l} 0 \cdot dx}\right\}$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l$$

$$= \frac{1}{l}\left[l^2 - \frac{l^2}{2}\right] = \frac{1}{l}\left[\frac{l^2}{2}\right]$$

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$$\Rightarrow a_0 = l/2$$

Let $a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{1}{l}\left\{\int_0^l (l-x)\cos\left(\frac{n\pi}{l}\right)x\,dx + \int_l^{2l} 0\cdot\cos\left(\frac{n\pi}{l}\right)x\,dx\right\}$$

$$= \frac{1}{l}\left[(l-x)\frac{\sin\left(\frac{n\pi}{l}\right)x}{(n\pi/l)} - (0-1)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{(n\pi/l)^2}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\frac{1}{(n\pi/l)^2}\right]\left\{\left[-\cos\left(\frac{n\pi}{l}\right)\cdot l - \left(-\cos 0\right)\right]\right\}$$

$$= \frac{l^2}{l(n^2\pi^2)}\left[-(-1)^n - (-1)\right]$$

$$\Rightarrow a_n = \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]$$

Let $b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{1}{l}\left\{\int_0^l (l-x)\sin\left(\frac{n\pi}{l}\right)x\,dx + \int_l^{2l} 0\cdot\sin\left(\frac{n\pi}{l}\right)x\,dx\right\}$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{(n\pi/l)}\right) - (0-1)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{1}{l}\left(\frac{1}{(n\pi/l)}\right)\left[-(l-l)\cos\left(\frac{n\pi}{l}\right)l - \left(-(l-0)\cos 0\right)\right]$$

$$= \frac{l}{n\pi l}\left[0 + l(1)\right]$$

$$\Rightarrow b_n = \frac{l}{n\pi}$$

① $\Rightarrow$

$$\therefore f(x) = \frac{1}{2}\left(\frac{l}{2}\right) + \sum_{n=1}^{\infty} \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi}{l}\right)x +$$

$$\sum_{n=1}^{\infty} \frac{l}{n\pi} \sin\left(\frac{n\pi}{l}\right)x$$

$$\Rightarrow f(x) = \frac{l}{4} + \frac{l}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos\left(\frac{n\pi}{l}\right)x + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}\right)x \longrightarrow ②$$

is the required Fourier expansion.

(i) put $x = 0$, in ②

at $x = 0$, $f(0) = \frac{1}{2}\left[f(0) + f(2l)\right]$

$$= \frac{1}{2}\left[l + 0\right]$$

$$\Rightarrow f(0) = \frac{l}{2}$$

$$\therefore ② \Rightarrow f(0) = \frac{l}{4} + \frac{l}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos 0 + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin 0$$

$$\Rightarrow \frac{l}{2} = \frac{l}{4} + \frac{l}{\pi^2}\left[\frac{2}{1^2} + \frac{0}{2^2} + \frac{2}{3^2} + \frac{0}{4^2} + \frac{2}{5^2} + \dots\right]$$

$$\Rightarrow \frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$$\Rightarrow \frac{l}{4} = \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots}$$

(ii) put $x = l/2$, $f(l/2) = l - l/2 = l/2$

$$\Rightarrow f(l/2) = \frac{l}{4} + \frac{l}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos\left(\frac{n\pi}{l}\right)\cdot\frac{l}{2} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{l}\right)\cdot\frac{l}{2}$$

$$\Rightarrow \frac{l}{2} = \frac{l}{4} + \frac{l}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cancelto{0}{\cos\left(\frac{n\pi}{2}\right)} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{l}{2} - \frac{l}{4} = \frac{l}{\pi}\left[1\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{4\pi}{2}\right) + \dots\right]$$

$$\Rightarrow \frac{l}{4} = \frac{l}{\pi}\left[1\cdot(1) + \frac{1}{2}(0) + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}(1) + \dots\right]$$

$$\Rightarrow \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}$$

5. If $f(x) = \begin{cases} 2-x & \text{in } 0 \le x \le 4 \\ x-6 & \text{in } 4 \le x \le 8 \end{cases}$ Express $f(x)$ as a Fourier series & hence deduce $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

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$\rightarrow$ Given $f(x) = \begin{cases} 2-x & \text{in } 0 \le x \le 4 \\ x-6 & \text{in } 4 \le x \le 8 \end{cases}$ in $(0,8) = (0, 2L)$

$\Rightarrow 2L = 8 \Rightarrow L = 4$

WKT $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \rightarrow (1)$

Let $a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$

$$= \frac{1}{4}\left[\int_0^4 (2-x)\,dx + \int_4^8 (x-6)\,dx\right]$$

$$= \frac{1}{4}\left\{\left[2x - \frac{x^2}{2}\right]_0^4 + \left[\frac{x^2}{2} - 6x\right]_4^8\right\}$$

$$= \frac{1}{4}\left\{\left[\left(8 - \frac{16}{2}\right) - (0-0)\right] + \left[\left(\frac{64}{2} - 48\right) - \left(\frac{16}{2} - 24\right)\right]\right\}$$

$$= \frac{1}{4}\left[0 - 0 + (-16) - (-16)\right]$$

$\Rightarrow a_0 = 0$

Let $a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{1}{4}\left\{\int_0^4 (2-x)\cos\left(\frac{n\pi}{4}\right)x\,dx + \int_4^8 (x-6)\cos\left(\frac{n\pi}{4}\right)x\,dx\right\}$$

$$= \frac{1}{4}\left\{\left[(2-x)\cancelto{0}{\left(\frac{\sin\left(\frac{n\pi}{4}\right)x}{\left(\frac{n\pi}{4}\right)}\right)} - (0-1)\left(\frac{-\cos\left(\frac{n\pi}{4}\right)x}{\left(\frac{n\pi}{4}\right)^2}\right)\right]_0^4 + \right.$$

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$$\left[(x-6)\left(\cancel{\frac{\sin\left(\frac{n\pi}{4}\right)x}{\left(\frac{n\pi}{4}\right)}}^{0}\right)-(1-0)\left(-\frac{\cos\left(\frac{n\pi}{4}\right)x}{\left(\frac{n\pi}{4}\right)^2}\right)\right]_4^8\Bigg\}$$

$$=\frac{1}{4}\cdot\frac{1}{\left(\frac{n\pi}{4}\right)^2}\left\{\left[-\cos\left(\frac{n\pi}{4}\right)4-\left(-\cos 0\right)\right]+\left[\cos\left(\frac{n\pi}{4}\right)8-\cos\left(\frac{n\pi}{4}\right)4\right]\right\}$$

$$=\frac{16}{4n^2\pi^2}\left[-(-1)^n+1+1-(-1)^n\right]$$

$$=\frac{4}{n^2\pi^2}\left[2-2(-1)^n\right]=\frac{4\times 2}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\Rightarrow a_n=\frac{8}{n^2\pi^2}\left[1-(-1)^n\right]$$

Let $b_n=\frac{1}{L}\int_0^{2L}f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$

$$=\frac{1}{4}\left\{\int_0^4(2-x)\sin\left(\frac{n\pi}{4}\right)x\,dx+\int_4^8(x-6)\sin\left(\frac{n\pi}{4}\right)x\,dx\right\}$$

$$=\frac{1}{4}\left\{\left[(2-x)\left(\frac{-\cos\left(\frac{n\pi}{4}\right)x}{(n\pi/4)}\right)-(0-1)\left(\cancel{\frac{-\sin\left(\frac{n\pi}{4}\right)x}{(n\pi/4)^2}}^{0}\right)\right]_0^4\right.$$

$$\left.+\left[(x-6)\left(\frac{-\cos(n\pi/4)x}{(n\pi/4)}\right)-(1-0)\left(\cancel{\frac{-\sin(n\pi/4)x}{(n\pi/4)^2}}^{0}\right)\right]_4^8\right\}$$

$$= \frac{1}{4}\frac{1}{\left(\frac{n\pi}{4}\right)}\left\{\left[-(2-4)\cos\left(\frac{n\pi}{4}\right)4 - \left(-(2-0)\cos 0\right)\right] + \left[(8-6)\cos\left(\frac{n\pi}{4}\right)8 - \left(-(4-6)\cos\left(\frac{n\pi}{4}\right)4\right)\right]\right\}$$

$$= \frac{4}{4n\pi}\left[2(-1)^n + 2(1) - 2(1) - 2(-1)^n\right]$$

$$\Rightarrow b_n = 0$$

$$\therefore f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi}{4}\right)x + 0$$

$$\Rightarrow f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi}{4}\right)x \rightarrow ②$$

is the required Fourier expansion

put $x=0$ in ②

Now $f(0) = \frac{1}{2}[LHL+RHL] = \frac{1}{2}[f(0)+f(8)]$

$$\Rightarrow f(0) = \frac{1}{2}[2+2] = 2$$

$$\therefore ② \Rightarrow f(0) = 2 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]\cos 0}{n^2}$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{2}{1^2} + \frac{0}{2^2} + \frac{2}{3^2} + \frac{0}{4^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{4} = 2\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

* Half-Range Fourier Series :-

When $f(x)$ is defined in $(0, \pi)$ or $(0, L)$ then we represent $f(x)$ as a sine or cosine series only. These series are said to be half range series.

• Half range cosine series :-

1) If $f(x)$ is defined in $0<x<\pi$ then the Half range Fourier cosine series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \text{ for } n=1,2,3\ldots$$

2) If $f(x)$ is defined in $0<x<L$ then the Half range Fourier cosine series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x$$

Where $a_0 = \frac{2}{L}\int_0^{L} f(x)\,dx$

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx, \text{ for } n=1,2,3\ldots$$

- Half range Sine series :-

1) If $f(x)$ is defined in $0<x<\pi$ then the Half range Fourier Sine series is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$, for $n=1,2,3\ldots$

2) If $f(x)$ is defined in $0<x<L$ then the Half range Fourier Sine series is given by,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x$$

Where $b_n=\frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$, for $n=1,2,3\ldots$

* Problems:-

1. Find the half range cosine series of $f(x)=(x-1)^2$ in $0<x<1$

→ Given $f(x)=(x-1)^2$ in the interval $(0,1)$, where $L=1$
From Fourier cosine series we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}\right)x \longrightarrow (1)$$

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Let $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$= \frac{2}{1}\int_0^1 (x-1)^2 dx = 2\left[\frac{(x-1)^3}{3}\right]_0^1$

$= \frac{2}{3}\left[(1-1)^3 - (0-1)^3\right]$

$\Rightarrow a_0 = \frac{2}{3}$

Let $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$= \frac{2}{1}\int_0^1 (x-1)^2 \cos\left(\frac{n\pi}{1}\right)x\,dx$

$= 2\left[\cancel{(x-1)^2\frac{\sin n\pi x}{n\pi}}^{0} - 2(x-1)\left(-\frac{\cos n\pi x}{(n\pi)^2}\right) - 2(1-0)\cancel{\left(-\frac{\sin n\pi x}{(n\pi)^3}\right)}^{0}\right]_0^1$

$= 2\left[2(x-1)\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$

$= \frac{4}{n^2\pi^2}\left[(1-1)\cos n\pi - (0-1)\cos 0\right]$

$= \frac{4}{n^2\pi^2}\left[0+1\right]$

$\Rightarrow a_n = 4/\pi^2 n^2$

$\therefore (1) \Rightarrow f(x) = \frac{1}{2}\left(\frac{2}{3}\right) + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\cos\left(\frac{n\pi}{1}\right)x$

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$$\Rightarrow (x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2}$$

is the required Fourier expansion

• 2 Find the cosine half range series of $f(x) = x\sin x$ in $0<x<\pi$. Deduce that

(i) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots = \frac{\pi-2}{4}$

(ii) $1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \dots = \frac{\pi}{2}$

→ Given $f(x) = x\sin x$ in $(0,\pi)$

From Fourier cosine series we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

To find a_0 & a_n

• [Refer problem no. 8 in page no. 11]

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx.$$

$$\Rightarrow x\sin x = \frac{1}{2}(2) + \left(-\frac{1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$$

$$\Rightarrow x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx \longrightarrow (2)$$

is the required Fourier expansion

Now put $x = \pi/2$ in (2)

(i)

$$\Rightarrow \frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cancel{\cos\frac{\pi}{2}}^{0} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}(1) = 1 - 0 + 2\left\{ \frac{-1}{3}\cos\frac{2\pi}{2} + \frac{1}{8}\cancel{\cos\frac{3\pi}{2}}^{0} + \left(\frac{-1}{15}\right)\cos\frac{4\pi}{2} + \frac{1}{24}\cancel{\cos\frac{5\pi}{2}}^{0} + \dots \right\}$$

$$\frac{\pi}{2} - 1 = 2\left\{ -\frac{1}{3}(-1) + 0 - \frac{1}{15}(1) + 0 - \frac{1}{35}(-1) + \dots \right\}$$

$$\frac{\pi - 2}{2} = 2\left\{ \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots \right\}$$

$$\Rightarrow \boxed{\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots} \longrightarrow (3)$$

(ii) $(3) \Rightarrow \frac{1}{2}\left[\frac{\pi}{2} - \frac{2}{2}\right] = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots$

$$\Rightarrow \frac{\pi}{2} - 1 = \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \dots$$

$$\Rightarrow \boxed{\frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \dots}$$

• 3 Show that $\frac{L}{2} - x = \frac{L}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi}{L}\right)x,\ 0 < x < L$

→ From the given problem we can say that it is half range Fourier sine series

Given $f(x) = \frac{L}{2} - x$ in the interval $(0, L)$

∴ Half range Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow (1)$$

$$\text{Let } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{L}\int_0^L \left(\frac{L}{2}-x\right)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{L}\left[\left(\frac{L}{2}-x\right)\left(\frac{-\cos\left(\frac{n\pi}{L}\right)x}{(n\pi/L)}\right) - (0-1)\left(\frac{-\sin\left(\frac{n\pi}{L}\right)x}{(n\pi/L)}\right)\right]_0^L$$

$$= \frac{2}{L}\left[-\left(\frac{L}{2}-x\right)\frac{\cos\left(\frac{n\pi}{L}\right)x}{(n\pi/L)}\right]_0^L$$

$$= \frac{2}{L}\cdot\frac{1}{\frac{n\pi}{L}}\left[-\left(\frac{L}{2}-L\right)\cos\left(\frac{n\pi}{L}\right)\cdot L - \left(-\left(\frac{L}{2}-0\right)\cos 0\right)\right]$$

$$= \frac{2}{n\pi}\left[-\left(-\frac{L}{2}\right)(-1)^n + \frac{L}{2}(1)\right]$$

$$= \frac{2}{n\pi}\left(\frac{L}{2}\right)\left[(-1)^n+1\right]$$

$$\Rightarrow b_n = \frac{L}{n\pi}\left[(-1)^n+1\right]$$

$$\therefore (1) \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{L}{n\pi}\left[(-1)^n+1\right]\sin\left(\frac{n\pi}{L}\right)x$$

$$\Rightarrow \frac{L}{2} - x = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[(-1)^n+1\right]\sin\left(\frac{n\pi}{L}\right)x$$

But $(-1)^n + 1 = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

$$\therefore \quad \frac{L}{2} - x = \frac{L}{\pi} \sum_{n=2,4,6\ldots}^{\infty} \frac{2}{n} \sin\left(\frac{n\pi}{L}\right) x$$

put $n = 2n$

$$\Rightarrow \frac{L}{2} - x = \frac{L}{\pi} \sum_{2n=2,4,6\ldots}^{\infty} \frac{2}{2n} \sin\left(\frac{2n\pi}{L}\right) x$$

$$\Rightarrow \boxed{\frac{L}{2} - x = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{2n\pi}{L}\right) x}$$

Hence proved

4. Find the half range Fourier cosine series for the function

$$f(x) = \begin{cases} kx & , 0 \le x \le l/2 \\ (l-x)k & , l/2 < x \le l \end{cases} \quad \text{Where } k \text{ is constant}$$

$\rightarrow$ consider $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right) x \longrightarrow ①$

Where $L = l$

Let $a_0 = \frac{2}{L} \int_0^L f(x)\,dx = \frac{2}{l} \left\{ \int_0^{l/2} kx\,dx + \int_{l/2}^{l} (l-x)k\,dx \right\}$

$$= \frac{2}{l} \left\{ k\left(\frac{x^2}{2}\right)_0^{l/2} + k\left[lx - \frac{x^2}{2}\right]_{l/2}^{l} \right\}$$

$$= \frac{2}{l} \left\{ k\left[\frac{l^2/4}{2} - 0\right] + k\left[\left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} - \frac{l^2/4}{2}\right)\right] \right\}$$

$$= \frac{1}{l}\left[K\frac{l^2}{8} + \left(\frac{l - l^2}{2}\right) - \left(\frac{8l - l^2}{8}\right)\right]$$

$$= \frac{1}{8l}\left[Kl^2 + 4\pi(4l - l^2) - \pi(8l - l^2)\right]$$

$$= \frac{1}{8l}\left[Kl^2 + \pi(16l - 4l^2 - 8l + l^2)\right]$$

$$= \frac{1}{8l}\left[Kl^2 + \pi(8l - 3l^2)\right]$$

$$= \frac{l}{8l}\left[Kl + \pi(8 - 3l)\right]$$

$$\Rightarrow a_0 = \frac{1}{8}\Big[$$

$$a_0 = \frac{2K}{l}\left[\frac{l^2}{8} + \frac{l^2}{2} - \frac{l^2}{2} + \frac{l^2}{8}\right]$$

$$= \frac{2K}{l}\left[\frac{2l^2}{8}\right]$$

$$\Rightarrow a_0 = \frac{lK}{2}$$

Consider $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{2}{l}\left[\int_0^{l/2} Kx\cos\left(\frac{n\pi}{l}\right)x\,dx + \int_{l/2}^{l} K(l - x)\cos\left(\frac{n\pi}{l}\right)x\,dx\right]$$

$$= \frac{2K}{l}\left\{\left[\frac{x\sin\left(\frac{n\pi}{l}\right)x}{\frac{n\pi}{l}} - 1\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2} + \left[\frac{(l - x)\sin\left(\frac{n\pi}{l}\right)x}{\frac{n\pi}{l}}\right.\right.$$

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$$- (0-1)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\Bigg]_{l/2}^{l}\Bigg\}$$

$$= \frac{2k}{l}\left\{\left[\left(\frac{\frac{l}{2}\sin\left(\frac{n\pi}{l}\right)\frac{l}{2}}{\frac{n\pi}{l}} + \frac{\cos\left(\frac{n\pi}{l}\right)\frac{l}{2}}{\frac{n^2\pi^2}{l^2}}\right) - \left(\frac{\cancel{0\cdot\sin 0}}{\frac{n\pi}{l}} + \frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right)\right]\right.$$

$$+ \left[\left(\frac{\cancel{(l-l)\sin\left(\frac{n\pi}{l}\right)l}}{\frac{n\pi}{l}} - \frac{\cos\left(\frac{n\pi}{l}\right)l}{\frac{n^2\pi^2}{l^2}}\right) - \left(\frac{\left(l-\frac{l}{2}\right)\sin\left(\frac{n\pi}{l}\right)\frac{l}{2}}{\frac{n\pi}{l}}\right.\right.$$

$$\left.\left.\left. - \frac{\cos\left(\frac{n\pi}{l}\right)\frac{l}{2}}{\frac{n^2\pi^2}{l^2}}\right)\right]\right\}$$

$$= \frac{2k}{l}\left[\frac{\cancel{\frac{l}{2}\sin\left(\frac{n\pi}{2}\right)}}{\frac{n\pi}{l}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2}{l}} - \frac{1}{\frac{n^2\pi^2}{l^2}} - \frac{\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right.$$

$$\left. \cancel{-\frac{l}{2}\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2k}{l}\frac{1}{\frac{n^2\pi^2}{l^2}}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$\Rightarrow a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$\therefore ① \Rightarrow f(x) = \frac{kl}{2}\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty}\frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos$$

$$\Rightarrow f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos\left(\frac{n\pi}{l}\right)x$$

is the required Fourier expansion

5. Find the half range cosine series of $f(x) = x$ in $0 < x < 2$.

→ Fourier cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x \longrightarrow (1)$$

Given $f(x) = x$ in $(0, 2) = (0, L)$

$$\Rightarrow \boxed{L = 2}$$

consider $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{2}\int_0^2 x\,dx$

$$\Rightarrow a_0 = \left.\frac{x^2}{2}\right|_0^2 = \frac{1}{2}[4 - 0]$$

$$\Rightarrow \boxed{a_0 = 2}$$

Consider $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi}{2}\right)x\,dx$$

$$= \left[\frac{x\sin\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)} - 1\left(\frac{-\cos\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)^2}\right)\right]_0^2$$

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$$= \left[\frac{\cancel{2\sin\left(\frac{n\pi}{2}\right)(2)}}{\frac{n\pi}{2}} + \frac{\cos\left(\frac{n\pi}{2}\right)2}{\frac{n^2\pi^2}{4}} \right] - \left[\frac{\cancel{0\sin 0}}{\frac{n\pi}{2}} + \frac{\cos 0}{\frac{n^2\pi^2}{4}} \right]$$

$$= \frac{1}{\frac{n^2\pi^2}{4}} \left[(-1)^n - 1 \right]$$

$$\Rightarrow a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore ① \Rightarrow f(x) = 2\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi}{2}\right)x$$

$$\Rightarrow x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2}\cos\left(\frac{n\pi}{2}\right)x$$

is the required Fourier series

6. Find the half range cosine series of $f(x) = 1 - \frac{x}{l}$ in (0,

→ consider Fourier cosine series in $(0, L)$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x \longrightarrow ①$$

Given $f(x) = 1 - \frac{x}{l}$ in $(0, l) = (0, L)$

$$\Rightarrow \boxed{L = l}$$

consider $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$$= \frac{2}{l}\int_0^l \left(1-\frac{x}{l}\right) dx = \frac{2}{l}\left[x - \frac{x^2}{2l}\right]_0^l$$

$$= \frac{2}{l}\left\{\left[l - \frac{l^2}{2l}\right] - [0-0]\right\}$$

$$\Rightarrow a_0 = \frac{2}{l}\left[l - \frac{l}{2}\right] = \frac{2}{l}\left[\frac{l}{2}\right]$$

$$\Rightarrow \boxed{a_0 = 1}$$

consider $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$= \frac{2}{l}\int_0^l \left(1-\frac{x}{l}\right)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\left[\left(1-\frac{x}{l}\right)\frac{\sin\left(\frac{n\pi}{l}\right)x}{\frac{n\pi}{l}} - \left(0-\frac{1}{l}\right)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left\{\left[\left(1-\frac{l}{l}\right)\frac{\cancelto{0}{\sin\left(\frac{n\pi}{l}\right)l}}{\frac{n\pi}{l}} - \frac{1}{l}\frac{\cos\left(\frac{n\pi}{l}\right)l}{\frac{n^2\pi^2}{l^2}}\right] - \left[(1-0)\frac{\cancelto{0}{\sin 0}}{\frac{n\pi}{l}} - \frac{1}{l}\frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right]\right\}$$

$$= \frac{2}{l}\left[-\frac{1}{l}\frac{\cos n\pi}{\frac{n^2\pi^2}{l^2}} + \frac{1}{l}\frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{2}{l}\left(\frac{1}{l}\right)\frac{1}{\frac{n^2\pi^2}{l^2}}\left[-(-1)^n + 1\right]$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2}\left[-(-1)^n+1\right]$$

$$\therefore ① \Rightarrow f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[-(-1)^n+1\right]\cos\left(\frac{n\pi}{l}\right)x$$

$$\Rightarrow 1-\frac{x}{l} = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\left[-(-1)^n+1\right]}{n^2}\cos\left(\frac{n\pi}{l}\right)x$$

is the required Fourier expansion.

7 obtain the half range cosine series of $f(x)$ defined by $f(x) = lx - x^2$ $x \in (0, l)$

→ consider the Fourier cosine series in $(0, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x \longrightarrow ①$$

Given $f(x) = lx - x^2$ in $(0, l) = (0, L)$

$$\Rightarrow L = l$$

consider $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{l}\int_0^l (lx - x^2)\,dx$

$$= \frac{2}{l}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right)_0^l = \frac{2}{l}\left\{\left(\frac{l^3}{2} - \frac{l^3}{3}\right) - (0-0)\right\}$$

$$= \frac{2l^3}{l}\left(\frac{1}{2} - \frac{1}{3}\right) = 2l^2\left(\frac{1}{6}\right)$$

$$\Rightarrow \boxed{a_0 = \frac{l^2}{3}}$$

Consider $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx$

$$\Rightarrow a_n = \frac{2}{l}\int_0^l (lx-x^2)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{2}{l}\left[\frac{(lx-x^2)\sin\left(\frac{n\pi}{l}\right)x}{\frac{n\pi}{l}} - (l-2x)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right) + (0-2)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left\{\left[\frac{(l^2-l^2)\cancel{\sin\left(\frac{n\pi}{l}\right)l}^{0}}{\left(\frac{n\pi}{l}\right)} + \frac{(l-2l)\cos\left(\frac{n\pi}{l}\right)l}{\frac{n^2\pi^2}{l^2}} + \frac{2\cancel{\sin\left(\frac{n\pi}{l}\right)l}^{0}}{\frac{n^3\pi^3}{l^3}}\right] - \left[\frac{(0-0)\cancel{\sin 0}^{0}}{\frac{n\pi}{l}} + \frac{(l-0)\cos 0}{\frac{n^2\pi^2}{l^2}} + \frac{2\cancel{\sin 0}^{0}}{\frac{n^3\pi^3}{l^3}}\right]\right\}$$

$$= \frac{2}{l}\left[\frac{-l\cos n\pi}{\frac{n^2\pi^2}{l^2}} - \frac{l\cos 0}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= -\frac{2l}{l} \cdot \frac{l^2}{n^2\pi^2}\left[(-1)^n+1\right]$$

$$\Rightarrow a_n = \frac{-2l^2}{n^2\pi^2}\left[(-1)^n+1\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{1}{2}\left(\frac{l^2}{3}\right) + \sum_{n=1}^{\infty} \frac{-2l^2}{n^2\pi^2}\left[(-1)^n+1\right]\cos\left(\frac{n\pi}{l}\right)x$$

$$\Rightarrow (lx - x^2) = \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n+1\right]}{n^2}\cos\left(\frac{n\pi}{l}\right)x$$

is the required Fourier series

• 8. Expand $f(x) = e^{-x}$ as a Half range Fourier sine series in the interval $(0, 1)$

$\rightarrow$

consider the Fourier sine series in the interval $(0, L)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow (1)$$

Given $f(x) = e^{-x}$ in $(0, 1) = (0, L)$

$$\Rightarrow L = 1$$

consider

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{1}\int_0^1 e^{-x}\sin n\pi x\,dx$$

$$= 2\left[\frac{e^{-x}}{(-1)^2+n^2\pi^2}\left(-1\sin n\pi x - n\pi\cos n\pi x\right)\right]_0^1$$

$$= \frac{2}{1+n^2\pi^2}\left[e^{-1}\left(-1\cancel{\sin n\pi(1)}^{0} - n\pi\cos n\pi(1)\right) - e^{0}\left(-\cancel{\sin 0}^{0} - n\pi\cos 0\right)\right]$$

$$= \frac{2}{n^2\pi^2+1}\left[\frac{1}{e}\left(-n\pi\cos n\pi\right) + n\pi\cos 0\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[\frac{-1}{e}(-1)^n + 1\right]$$

$$b_n = \frac{2n\pi}{1+n^2\pi^2}\left[\frac{e-(-1)^n}{e}\right]$$

$$\therefore ① \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}\left[\frac{e-(-1)^n}{e}\right]\sin n\pi x$$

$$e^{-x} = \frac{2\pi}{e}\sum_{n=1}^{\infty} \frac{n\left[e-(-1)^n\right]}{1+n^2\pi^2}\sin n\pi x$$

is the required Fourier sine series

- 9. Find the half range sine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$

→ Consider the half range Fourier sine series in $(0, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow ①$$

Given $f(x) = x(\pi - x)$ in $(0, \pi)$

consider $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

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$$= \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (0-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[-\cancel{(\pi^2-\pi^2)\frac{\cos n\pi}{n}} + \cancel{(\pi - 2\pi)\frac{\sin n\pi}{n^2}} - \frac{2\cos n\pi}{n^3}\right] - \left[-\cancel{(0-0)\frac{\cos 0}{n}} + \cancel{(\pi - 0)\frac{\sin 0}{n^2}} - \frac{2\cos 0}{n^3}\right]\right\}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}(-1)^n + \frac{2}{n^3}(1)\right]$$

$$b_n = \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

$$\therefore (1) \Rightarrow x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left[1 - (-1)^n\right]\sin nx$$

is the required Fourier series.

10. Obtain the half range sine series of the function

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < 1/2 \\ x - \frac{3}{4} & 1/2 < x < 1 \end{cases}$$

→ Consider Half range Fourier sine series in $(0, L)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow (1)$$

Given $f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < 1/2 \\ x - 3/4, & 1/2 < x < 1 \end{cases}$ in $(0,1) = (0,L)$ $\Rightarrow \boxed{L=1}$

Consider $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right)x \, dx$

$$= \frac{2}{1}\int_0^1 f(x) \sin\left(\frac{n\pi}{1}\right)x \, dx$$

$$= 2\left\{\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x \, dx + \int_{1/2}^{1}\left(x - 3/4\right)\sin n\pi x \, dx\right\}$$

$$= 2\left\{\left[\left(\frac{1}{4} - x\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - (0-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^{1/2}\right.$$

$$\left. + \left[(x - 3/4)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1-0)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\left[-\left(\frac{1}{4} - \frac{1}{2}\right)\frac{\cos\frac{n\pi}{2}}{n\pi} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right] - \left[-\left(\frac{1}{4} - 0\right)\frac{\cos 0}{n\pi}\right.\right.$$

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$$-\cancel{\frac{\sin 0}{n^2\pi^2}}^{0}\Bigg] + \left[-\left(1-\frac{3}{4}\right)\frac{\cos n\pi}{n\pi} + \cancel{\frac{\sin n\pi}{n^2\pi^2}}^{0}\right] -$$

$$\left[-\left(\frac{1}{2}-\frac{3}{4}\right)\frac{\cos\frac{n\pi}{2}}{n\pi} + \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$= 2\left[\frac{1}{4}\cancel{\frac{\cos\frac{n\pi}{2}}{n\pi}} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{1}{4}\frac{\cos 0}{n\pi} - \frac{1}{4}\frac{\cos n\pi}{n\pi}\right.$$

$$\left. -\frac{1}{4}\cancel{\frac{\cos\frac{n\pi}{2}}{n\pi}} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$= 2\left[\frac{1}{4n\pi}(1) - \frac{1}{4n\pi}(-1)^n - \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow b_n = 2\left[\frac{1}{4n\pi}\left(1-(-1)^n\right)\right] - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore b_n = \frac{1}{2n\pi}\left[1-(-1)^n\right] - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$\therefore$ ① $\Rightarrow$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{1}{2n\pi}\left[1-(-1)^n\right] - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\sin n\pi x$$

is the required fourier series

11. If $f(x) = \begin{cases} x & \text{in } 0<x<\pi/2 \\ \pi - x & \text{in } \pi/2<x<\pi \end{cases}$ Show that

(i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \dots\right]$

→ Here $f(x)$ is defined in $(0,\pi)$, then we need to find the Fourier sine half range series & cosine half range series

(i) The sine half range series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

consider $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (0-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\left(-\frac{\pi}{2}\frac{\cos \frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right) - \left(\cancel{-0\frac{\cos 0}{n}} + \cancel{\frac{\sin 0}{n^2}}\right)\right] + \left[\left(\cancel{-(\pi-\pi)\frac{\cos n\pi}{n}} - \cancel{\frac{\sin n\pi}{n^2}}\right) - \left(-\left(\pi - \frac{\pi}{2}\right)\frac{\cos\frac{n\pi}{2}}{n} - \frac{\sin \frac{n\pi}{2}}{n^2}\right)\right]\right\}$$

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$$= \frac{2}{\pi}\left[\cancel{-\frac{\pi}{2n} \cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \cancel{\frac{\pi}{2n}\cos\frac{n\pi}{2}} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right]$$

$$\Rightarrow b_n = \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$\therefore ① \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\cdot\sin nx$$

$$= \frac{4}{\pi}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\sin x + \frac{1}{2^2}\overset{0}{\sin\frac{2\pi}{2}}\sin 2x + \frac{1}{3^2}\sin\frac{3\pi}{2}\sin 3x + \dots\right]$$

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \dots\right]$$

(ii) The cosine half range series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow ②$$

consider $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right\}$

$$= \frac{2}{\pi}\left\{\frac{x^2}{2}\Big|_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{2}\left(\frac{\pi^2}{4} - 0\right) + \left[\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{1}{2}\frac{\pi^2}{4}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{2}{\pi}\left(\frac{2\pi^2}{8}\right)$$

$$\Rightarrow a_0 = \frac{\pi}{2}$$

consider $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi}^{\pi/2}(\pi - x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\frac{\sin nx}{n} - (0-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\left(\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right) - \left(\cancel{\frac{0\cdot\sin 0}{n}} + \frac{\cos 0}{n^2}\right)\right] + \right.$$

$$\left.\left[\left(\cancel{\frac{(\pi-\pi)}{n}\sin n\pi} - \frac{\cos n\pi}{n^2}\right) - \left(\left(\pi - \frac{\pi}{2}\right)\frac{1}{n}\sin\frac{n\pi}{2} - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right)\right]\right\}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \cancel{\frac{\pi}{2n}\sin\frac{n\pi}{2}} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi n^2}\left[-1 - (-1)^n + 2\cos\frac{n\pi}{2}\right] = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - (1 + (-1)^n)\right]$$

But $1 + (-1)^n = \begin{cases} 2 & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$

$$\therefore a_n = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 2\right] = \frac{2(-2)}{\pi n^2}\left[1 - \cos\frac{n\pi}{2}\right] \text{ if } n \text{ is even}$$

But $1 - \cos\frac{n\pi}{2} = \begin{cases} 1 - (-1) = 2 & \text{where } n = 2, 6, 10 \ldots \\ 1 - (1) = 0 & \text{where } n = 4, 8, 12 \ldots \end{cases}$

$\Rightarrow a_n = \frac{-4}{\pi n^2}(2) \Rightarrow a_n = \frac{-8}{\pi n^2}$ if $n = 2, 6, 10, \ldots$

$\therefore ② \Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=2,6,\ldots}^{\infty} \frac{-8}{\pi n^2} \cos nx$

$\Rightarrow f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left\{\frac{\cos 2x}{2^2} + \frac{\cos 6x}{6^2} + \frac{\cos 10x}{10^2} + \ldots\right\}$

$\Rightarrow f(x) = \frac{\pi}{4} - \frac{8}{\pi}\cdot\left(\frac{1}{2^2}\right)\left\{\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right\}$

$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right]$

• 12. Find the Half range sine series of $f(x) = lx - x^2$, $x \in (0, l)$

$\rightarrow$ The Fourier Half range sine series in the in $(0, L)$ is

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow ①$ Where $L = l$

Consider $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right)x\, dx$

$= \frac{2}{l}\int_0^l (lx - x^2) \sin\left(\frac{n\pi}{l}\right)x\, dx$

$= \frac{2}{l}\left[(lx-x^2)\left(\frac{-\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)}\right) - (l-2x)\left(\frac{-\sin\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^2}\right) - (0-2)\left(\frac{\cos\left(\frac{n\pi}{l}\right)x}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$

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$$= \frac{2}{l}\left\{\left[\frac{-(l^2-l^2)\cancel{\cos\left(\frac{n\pi}{l}\right)l}}{\frac{n\pi}{l}} + \frac{(l-2l)\cancel{\sin\left(\frac{n\pi}{l}\right)l}}{\left(\frac{n\pi}{l}\right)^2} + \frac{2\cos\left(\frac{n\pi}{l}\right)l}{\left(\frac{n\pi}{l}\right)^3}\right] - \right.$$

$$\left.\left[\frac{-(0-0)\cancel{\cos 0}}{\frac{n\pi}{l}} + \frac{(l-0)\cancel{\sin 0}}{\left(\frac{n\pi}{l}\right)^2} + \frac{2\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]\right\}$$

$$= \frac{2}{l}\left(\frac{2l^3}{n^3\pi^3}\right)\left[\cos n\pi - \cos 0\right]$$

$$b_n = \frac{4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$\therefore (1) \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]\sin\left(\frac{n\pi}{l}\right)x$$

$$\Rightarrow lx - x^2 = \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^3}\sin\left(\frac{n\pi}{l}\right)x$$

* Practical Harmonic Analysis :-

Harmonic analysis is the process of finding the constant term and the first few cosine and sine terms numerically

(i) The Fourier series of period 2π of a function $y=f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

Where $a_0 = \frac{2}{N} \sum y$

$$a_n = \frac{2}{N} \sum y \cos nx$$

$$b_n = \frac{2}{N} \sum y \sin nx$$

(ii) The Fourier series of period $2L$ of a function $y=f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right)x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\right)x \longrightarrow (2)$$

$$\left.\begin{aligned} \text{where } a_0 &= \frac{2}{N} \sum y \\ a_n &= \frac{2}{N} \sum y \cos\left(\frac{n\pi}{L}\right)x \\ b_n &= \frac{2}{N} \sum y \sin\left(\frac{n\pi}{L}\right)x \end{aligned}\right\} \longrightarrow (3)$$

put $\frac{\pi x}{L} = \theta$ in (3)

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$\times^{ly}$ & $\div$ by $(1-in\pi)$

$$c_n = \frac{-1}{2}\frac{(1-in\pi)}{(1-in\pi)(1+in\pi)}\left[e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right]$$

$$= \frac{-1}{2}\frac{(1-in\pi)}{1^2-(in\pi)^2}\left[e^{-1}(-1)^n - e(-1)^n\right]$$

$$= \frac{-1}{2}\frac{(1-in\pi)(-1)^n}{1^2 - i^2n^2\pi^2}\left(e^{-1} - e^{1}\right)$$

$$= \frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\left(\frac{e^{1}-e^{-1}}{2}\right)$$

$$\Rightarrow c_n = \frac{\sinh 1\,(-1)^n(1-in\pi)}{1+n^2\pi^2}$$

$$\therefore (1) \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} \frac{\sinh 1\,(-1)^n(1-in\pi)}{1+n^2\pi^2}\,e^{in\pi x}$$

is the required complex Fourier expansion

$\therefore a_0 = \frac{2}{N} \sum y = \frac{2}{8}(8) \Rightarrow a_0 = 2$

$a_1 = \frac{2}{N} \sum y \cos x = \frac{2}{8}(3.4142) \Rightarrow a_1 = 0.85355$

$b_1 = \frac{2}{N} \sum y \sin x = \frac{2}{8}(0) \Rightarrow b_1 = 0$

$\therefore (1) \Rightarrow y = \frac{2}{2} + 0.85355 \cos x + 0 \cdot \sin x$

$\Rightarrow y = 1 + 0.85355 \cos x$

2. Obtain the Fourier series of y upto the second harmonics for the following values

x°	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	-1.5	0	2.8	3.4

$\rightarrow$ Here the interval of x is $0 < x \leq 360$ i.e) $0 < x \leq 2\pi$

Where $N = 8$

$\therefore$ The Fourier series of y up to second harmonic is

$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x \longrightarrow (1)$

x°	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
45	4.0	0.7071	0.7071	0	1	2.8284	2.8284	0	4.0
90	3.8	0	1	-1	0	0	3.8	-3.8	0
135	2.4	-0.7071	0.7071	0	-1	-1.69704	1.69704	0	-2.4
180	2.0	-1	0	1	0	-2.0	0	2.0	0
225	-1.5	-0.7071	-0.7071	0	1	1.06065	1.06065	0	-1.5
270	0	0	-1	-1	0	0	0	0	0
315	2.8	0.7071	-0.7071	0	-1	1.97988	-1.97988	0	-2.8
360	3.4	1	0	1	0	3.4	0	3.4	0

$\sum y = 16.9$ $\quad \sum y\cos x = 5.57189$ $\quad \sum y \sin x = 7.40621$ $\quad \sum y\cos 2x = 1.6$ $\quad \sum y \sin 2x = -2.7$

$$\Rightarrow a_0 = \frac{2}{N}\sum y$$

$$\Rightarrow a_n = \frac{2}{N}\sum y\cos n\theta$$

$$\Rightarrow b_n = \frac{2}{N}\sum y\sin n\theta$$

- Problems:-

1 Determine the constant term and the first cosine and sine terms of the Fourier series expansion of y from the following data.

x°	0	45	90	135	180	225	270	315
y	2	3/2	1	1/2	0	1/2	1	3/2

→ Here the interval of x is $0 \le x < 360$, i.e) $0 \le x < 2\pi$ where $N=8$, ∴ The Fourier series of y upto First harmonic is $y = \frac{a_0}{2} + a_1\cos x + b_1 \sin x$ ⟶ (1)

x°	y	cos x	sin x	y cos x	y sin x
0	2	1	0	2.0	0
45	3/2	0.7071	0.7071	1.06065	1.06065
90	1	0	1	0	1.0
135	1/2	-0.7071	0.7071	-0.35355	0.35355
180	0	-1	0	0	0
225	1/2	-0.7071	-0.7071	-0.35355	-0.35355
270	1	0	-1	0	-1
315	3/2	0.7071	-0.7071	1.06065	-1.06065

$\Sigma y = 8$ $\quad$ $\Sigma y\cos x = 3.4142, \Sigma y \sin x = 0$

θ°	T	$\cos\theta$	$\sin\theta$	$T\cos\theta$	$T\sin\theta$
0	0	1	0	0	0
30	2.7	0.866	0.5	2.3382	1.35
60	5.2	0.5	0.866	2.6	4.5032
90	7.0	0	1	0	7.0
120	8.1	-0.5	0.866	-4.05	7.0146
150	8.3	-0.866	0.5	-7.1878	4.15
180	7.9	-1	0	-7.9	0
210	6.8	-0.866	-0.5	-5.8888	-3.4
240	5.5	-0.5	-0.866	-2.75	-4.763
270	4.1	0	-1	0	-4.1
300	2.6	0.5	-0.866	1.3	-2.2516
330	1.2	0.866	-0.5	1.0392	-0.6

$\sum T = 59.4$ $\quad \sum T\cos\theta = -20.4992$ $\quad \sum T\sin\theta = 8.9032$

$$a_0 = \frac{2}{N}\sum T = \frac{2}{12}(59.4) \Rightarrow a_0 = 9.9$$

$$a_1 = \frac{2}{N}\sum T\cos\theta = \frac{2}{12}(-20.4992) \Rightarrow a_1 = -3.4165$$

$$b_1 = \frac{2}{N}\sum T\sin\theta = \frac{2}{12}(8.9032) \Rightarrow b_1 = 1.4839$$

$$\therefore (1) \Rightarrow T = \frac{9.9}{2} - 3.4165\cos\theta + 1.4839\sin\theta$$

$$\Rightarrow T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta$$

4. Given the following table

x°	0°	60°	120°	180°	240°	300°
y	7.9	7.2	3.6	0.5	0.9	6.8

obtain the Fourier series neglecting terms higher than first harmonics

$$a_0 = \frac{2}{N}\sum y = \frac{2}{8}(16.9) \Rightarrow a_0 = 4.225$$

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{8}(5.57189) \Rightarrow a_1 = 1.393$$

$$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{8}(7.40621) \Rightarrow b_1 = 1.8516$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{8}(1.6) \Rightarrow a_2 = 0.4$$

$$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{8}(-2.7) \Rightarrow b_2 = -0.675$$

$$\therefore (1) \Rightarrow y = \frac{4.225}{2} + (1.393\cos x + 1.8516\sin x) + (0.4\cos 2x - 0.675\sin 2x)$$

$$\Rightarrow y = 2.1125 + (1.393\cos x + 1.8516\sin x) + (0.4\cos 2x - 0.675\sin 2x)$$

3. The turning moment T on the crank shaft of a steam engine for the crank angle θ is given as follows.

θ°	0	30	60	90	120	150	180	210	240	270	300	330
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

Express T as a Fourier series up to first harmonics

→ Here the interval of θ is $0 \le \theta < 360$ i.e) $0 \le \theta < 2\pi$
Where $N = 12$

∴ The Fourier series up to first harmonic is

$$T = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta \longrightarrow (1)$$

→ Here the interval of x is $0 \leq x \leq 2\pi$,
the values of y at $x=0$ & $x=2\pi$ must be same by the periodic property $f(x+2\pi)=f(x)$
So we must neglect any one value either 0 (or) 2π

∴ The values of x interms of degrees is 0, 60, 120, 180, 240, 300
Where $N = 6$
∴ The Fourier series up to third harmonic is

$$y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + (a_3\cos 3x + b_3\sin 3x) \longrightarrow (1)$$

x°	y	$\cos x$	$y\cos x$	$\cos 2x$	$y\cos 2x$	$\cos 3x$	$y\cos 3x$
0	1.98	1	1.98	1	1.98	1	1.98
60	1.3	0.5	0.65	-0.5	-0.65	-1	-1.3
120	1.05	-0.5	-0.525	-0.5	-0.525	1	1.05
180	1.3	-1	-1.3	1	1.3	-1	-1.3
240	-0.88	-0.5	0.44	-0.5	0.44	1	-0.88
300	-0.25	0.5	-0.125	-0.5	0.125	-1	0.25

$\Sigma y = 4.5$ $\quad \Sigma y\cos x = 1.12$ $\quad \Sigma y\cos 2x = 2.67$ $\quad \Sigma y\cos 3x = -0.2$

$\sin x$	$y\sin x$	$\sin 2x$	$y\sin 2x$	$\sin 3x$	$y\sin 3x$
0	0	0	0	0	0
0.866	1.1258	0.866	1.1258	0	0
0.866	0.909	-0.866	-0.909	0	0
0	0	0	0	0	0
-0.866	0.762	0.866	-0.762	0	0
-0.866	0.2165	-0.866	0.2165	0	0
				0	

$\Sigma y\sin x = 3.0133$ $\quad \Sigma y\sin 2x = -0.3287$ $\quad \Sigma y\sin 3x = 0$

→ Here the interval of x is $0 \le x < 360°$ i.e) $0 \le x < 2\pi$

Where $N=6$,

∴ The Fourier series up to first Harmonic is given by

$$y = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x \longrightarrow (1)$$

$x°$	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	7.9	1	7.9	0	0
60	7.2	0.5	3.6	0.866	6.2352
120	3.6	-0.5	-1.8	0.866	3.1176
180	0.5	-1	-0.5	0	0
240	0.9	-0.5	-0.45	-0.866	-0.7794
300	6.8	0.5	3.4	-0.866	-5.8888

$\sum y = 26.9$ $\sum y \cos x = 12.15$ $\sum y \sin x = 2.6846$

$$a_0 = \frac{2}{N} \sum y = \frac{2}{6}(26.9) \Rightarrow a_0 = 8.9667$$

$$a_1 = \frac{2}{N} \sum y \cos x = \frac{2}{6}(12.15) \Rightarrow a_1 = 4.05$$

$$b_1 = \frac{2}{N} \sum y \sin x = \frac{2}{6}(2.6846) \Rightarrow b_1 = 0.8949$$

$$\therefore (1) \Rightarrow y = \frac{8.9667}{2} + 4.05 \cos x + 0.8949 \sin x$$

$$\Rightarrow y = 4.48335 + 4.05 \cos x + 0.8949 \sin x$$

5. Express y as a Fourier series upto the third harmonics given

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

$\therefore$ The values of x interms of degrees is 0, 60, 120, 180, 240, 300, where $N = 6$.

The Fourier series of first two Harmonics is

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) \longrightarrow ①$$

x^o	y	$\cos x$	$y\cos x$	$\cos 2x$	$y\cos 2x$	$\sin x$	$y\sin x$	$\sin 2x$	$y\sin 2x$
0	1	1	1	1	1	0	0	0	0
60	1.4	0.5	0.7	−0.5	−0.7	0.866	1.2124	0.866	1.2124
120	1.9	−0.5	−0.95	−0.5	−0.95	0.866	1.6454	−0.866	−1.6454
180	1.7	−1	−1.7	1	1.7	0	0	0	0
240	1.5	−0.5	−0.75	−0.5	−0.75	−0.866	−1.299	0.866	1.299
300	1.2	0.5	0.6	−0.5	−0.6	−0.866	−1.0392	−0.866	−1.0392

$\Sigma y = 8.7$ $\Sigma y\cos x = -1.1$ $\Sigma y\cos 2x = -0.3$ $\Sigma y\sin x = 0.5196$ $\Sigma y\sin 2x = -0.1732$

$$a_1 = \frac{2}{N} \Sigma y\cos x = \frac{2}{6}(-1.1) \Rightarrow a_1 = -0.367$$

$$a_2 = \frac{2}{N} \Sigma y\cos 2x = \frac{2}{6}(-0.3) \Rightarrow a_2 = -0.1$$

$$b_1 = \frac{2}{N} \Sigma y\sin x = \frac{2}{6}(0.5196) \Rightarrow b_1 = 0.1732$$

$$b_2 = \frac{2}{N} \Sigma y\sin 2x = \frac{2}{6}(-0.1732) \Rightarrow b_2 = -0.0577$$

$$a_0 = \frac{2}{N} \Sigma y = \frac{2}{6}(8.7) = 2.9$$

$$① \Rightarrow f(x) = \frac{2.9}{2} + (-0.367\cos x + 0.1732\sin x) + (-0.1\cos 2x - 0.0577\sin 2x)$$

7. Obtain the constant term and coefficients of the first cosine and sine terms in the Fourier expansion of y from the table

$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(4.5) \Rightarrow a_0 = 1.5$

$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(1.12) \Rightarrow a_1 = 0.373$

$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(2.67) \Rightarrow a_2 = 0.89$

$a_3 = \frac{2}{N}\sum y\cos 3x = \frac{2}{6}(-0.2) \Rightarrow a_3 = -0.066$

$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(3.0133) \Rightarrow b_1 = 1.0044$

$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{6}(-0.3287) \Rightarrow b_2 = -0.1096$

$b_3 = \frac{2}{N}\sum y\sin 3x = \frac{2}{6}(0) \Rightarrow b_3 = 0$

$\therefore (1) \Rightarrow y = \frac{1.5}{2} + (0.373\cos x + 1.0044\sin x) + (0.89\cos 2x - 0.1096\sin 2x) + (-0.066\cos 3x + 0\cdot\sin x)$

$\Rightarrow y = 0.75 + (0.373\cos x + 1.0044\sin x) + (0.89\cos 2x - 0.1096\sin 2x) - 0.066\cos 3x$

6. Compute the first two harmonics of the Fourier series of $f(x)$ given the following table

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

→ Here the interval of x is $0 \le x \le 2\pi$

The values of y at $x=0$ & $x=2\pi$ must be same by the periodic property $f(x+2\pi) = f(x)$

So we must neglect any one value either 0 (or) 2π

$b_1 = \frac{2}{N}\sum y\sin\theta = \frac{2}{6}(-3.464) \Rightarrow b_1 = -1.155$

$\therefore (2) \Rightarrow y = \frac{41.67}{2} - 8.333\cos\theta - 1.1555\sin\theta$

at $\theta = \frac{\pi x}{3}$

$$\Rightarrow y = 20.835 - 8.333\cos\left(\frac{\pi x}{3}\right) - 1.1555\sin\left(\frac{\pi x}{3}\right)$$

8. Analyses Harmonically the data given below and express 'y' in Fourier series up to the second harmonics

x°	0	60	120	180	240	300	360
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

→ Refer problem No. 6, page No. 76

9. Obtain the constant term, first two coefficient of cosine terms a_1, a_2 and first two coefficient of sine terms b_1, b_2 in the Fourier series expansion of the function y, given by the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

→ The values of x at 0, 1, 2, 3, 4, 5 are given $(N=6)$
Hence the interval of x is arbitrary.
i.e $0 \le x < 6$ (or) $0 \le x < 2L \Rightarrow 2L = 6$ or $L = 3$

The Fourier series of constant term & first two harmonic is

$$y = \frac{a_0}{2} + a_1\cos\frac{\pi x}{L} + b_1\sin\frac{\pi x}{L} + a_2\cos 2\left(\frac{\pi x}{L}\right) + b_2\sin 2\left(\frac{\pi x}{L}\right)$$

x	0	1	2	3	4	5
y	9	18	24	28	26	20.

→ The values of x at 0, 1, 2, 3, 4, 5 are given ($N=6$)
Hence the interval of x is arbitrary
i.e $0 \le x < 6$ (or) $0 \le x < 2L \Rightarrow 2L = 6$
$\Rightarrow \boxed{L=3}$

The Fourier series of first harmonic of period $2L$ is

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L}$$

at $L=3$ $\Rightarrow y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} \longrightarrow (1)$

put $\frac{\pi x}{3} = \theta \Rightarrow y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta \longrightarrow (2)$

x	$\theta = \frac{\pi x}{3}$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1	9	0	0
1	60	18	0.5	9	0.866	15.588
2	120	24	-0.5	-12	0.866	20.784
3	180	28	-1	-28	0	0
4	240	26	-0.5	-13	-0.866	-22.516
5	300	20	0.5	10	-0.866	-17.32

$\Sigma y = 125$ $\quad \Sigma y\cos\theta = -25$ $\quad \Sigma y\sin\theta = -3.464$

$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(125) \Rightarrow a_0 = 41.67$

$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-25) \Rightarrow a_1 = -8.333$

10. Express y as a Fourier series up to the 2^{nd} Harmonics given the following values

x	0	1	2	3	4	5
y	4	8	15	7	6	2

$\rightarrow$ The values of x at $0, 1, 2, 3, 4, 5$ are given $(N=6)$

Hence the interval of x is arbitrary

i.e) $0 \le x < 6$ (or) $0 \le x < 2L \Rightarrow 2L = 6$ or $L = 3$

$\therefore$ Fourier series up to second Harmonic is

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L} + a_2 \cos 2\frac{\pi x}{L} + b_2 \sin 2\left(\frac{\pi x}{L}\right)$$

$$\text{at } L=3 \Rightarrow y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) + a_2 \cos 2\left(\frac{\pi x}{3}\right) + b_2 \sin 2\left(\frac{\pi x}{3}\right) \rightarrow (1)$$

$$\text{put } \theta = \frac{\pi x}{3} \Rightarrow y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta \rightarrow (2)$$

x	$\theta = \frac{\pi x}{3}$	y	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$	$\sin\theta$	$y\sin\theta$	$\sin 2\theta$	$y\sin 2\theta$
0	0	4	1	4	1	4	0	0	0	0
1	60	8	0.5	4	-0.5	-4	0.866	6.928	0.866	6.928
2	120	15	-0.5	-7.5	-0.5	-7.5	0.866	12.99	-0.866	-12.99
3	180	7	-1	-7	1	7	0	0	0	0
4	240	6	-0.5	-3	-0.5	-3	-0.866	-5.196	0.866	5.196
5	300	2	0.5	1	-0.5	-1	-0.866	-1.732	-0.866	-1.732

$\Sigma y = 42$ $\quad \Sigma y\cos\theta = -8.5$ $\quad \Sigma y\cos 2\theta = -4.5$ $\quad \Sigma y\sin\theta = 12.99$ $\quad \Sigma y\sin 2\theta = -2.598$

$$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(42) \Rightarrow a_0 = 14$$

at $L=3 \Rightarrow y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3} + a_2 \cos 2\left(\frac{\pi x}{3}\right) + b_2 \sin 2\left(\frac{\pi x}{3}\right)$

Replace $\theta = \frac{\pi x}{3}$

$$\Rightarrow y = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2 \sin 2\theta \longrightarrow (1)$$

x	$\theta=\frac{\pi x}{3}$	y	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$	$\sin\theta$	$y\sin\theta$	$\sin 2\theta$	$y\sin 2\theta$
0	0	9	1	9	1	9	0	0	0	0
1	60	18	0.5	9	-0.5	-9	0.866	15.588	0.866	15.588
2	120	24	-0.5	-12	-0.5	-12	0.866	20.784	-0.866	-20.784
3	180	28	-1	-28	1	28	0	0	0	0
4	240	26	-0.5	-13	-0.5	-13	-0.866	-22.516	0.866	22.516
5	300	20	0.5	10	-0.5	-10	-0.866	-17.32	-0.866	-17.32

$\Sigma y = 125$ $\quad \Sigma y\cos\theta = -25$ $\quad \Sigma y\cos 2\theta = -7$ $\quad \Sigma y\sin\theta = -3.464$ $\quad \Sigma y \sin 2\theta = 0$

$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(125) \Rightarrow a_0 = 41.67$

$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-25) \Rightarrow a_1 = -8.333$

$a_2 = \frac{2}{N}\Sigma y\cos 2\theta = \frac{2}{6}(-7) \Rightarrow a_2 = -2.333$

$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(-3.464) \Rightarrow b_1 = -1.155$

$b_2 = \frac{2}{N}\Sigma y\sin 2\theta = \frac{2}{6}(0) \Rightarrow b_2 = 0$

constant term is $\frac{a_0}{2} = 20.835$

coefficient of the first two cosine terms $\rightarrow a_1 = -8.333$ & $a_2 = -2.333$
coefficient of the first two sine terms $\rightarrow b_1 = -1.155$ & $b_2 = 0$

put $\theta = \frac{\pi x}{6}$

$$\Rightarrow y = \frac{a_0}{2} + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta \longrightarrow (2)$$

x	$\theta=\frac{\pi x}{6}$	y	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$	$\cos 3\theta$	$y\cos 3\theta$
0	0	4	1	4	1	4	1	4
1	30	8	0.866	6.928	0.5	4	0	0
2	60	15	0.5	7.5	-0.5	-7.5	-1	-15
3	90	7	0	0	-1	-7	0	0
4	120	6	-0.5	-3	-0.5	-3	1	6
5	150	2	-0.866	-1.732	0.5	1	0	0

$\sum y = 42$ $\quad \sum y\cos\theta = 13.696$ $\quad \sum y\cos 2\theta = -8.5$ $\quad \sum y\cos 3\theta = -5$

$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(42) \Rightarrow a_0 = 14$

$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{2}{6}[13.696] \Rightarrow a_1 = 4.565$

$a_2 = \frac{2}{N}\sum y\cos 2\theta = \frac{2}{6}[-8.5] \Rightarrow a_2 = -2.833$

$a_3 = \frac{2}{N}\sum y\cos 3\theta = \frac{2}{6}(-5) \Rightarrow a_3 = -1.667$

The constant term is $\frac{a_0}{2} = 7$

The first three coefficients of cosine series is $a_1 = 4.565$

$a_2 = -2.833$

$a_3 = -1.667$

12. The following values of y and x are given. Find the Fourier series of y upto second harmonics

$a_1 = \frac{2}{N} \sum y\cos\theta = \frac{2}{6}(-8.5) \Rightarrow a_1 = -2.833$

$a_2 = \frac{2}{N} \sum y\cos 2\theta = \frac{2}{6}(-4.5) \Rightarrow a_2 = -1.5$

$b_1 = \frac{2}{N} \sum y\sin\theta = \frac{2}{6}(12.99) \Rightarrow b_1 = 4.33$

$b_2 = \frac{2}{N} \sum y\sin 2\theta = \frac{2}{6}(-2.598) \Rightarrow b_2 = -0.866$

$\therefore$ (2) $\Rightarrow y = \frac{14}{2} - 2.833\cos\theta + 4.33\sin\theta - 1.5\cos 2\theta - 0.866\sin 2\theta$

at $\theta = \frac{\pi x}{3}$

$\Rightarrow y = 7 - 2.833\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3} - 1.5\cos 2\left(\frac{\pi x}{3}\right) - 0.866\sin 2\left(\frac{\pi x}{3}\right)$

11 Obtain the constant term and the first three coefficients in the Fourier cosine series for y using the following table.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

$\rightarrow$ Here the interval of x is $0 \le x < 6$ & $N = 6$
The coefficients Fourier cosine series [Half range cosine series in $(0, L)$] is given by

$$y = \frac{a_0}{2} + a_1\cos\frac{\pi x}{L} + a_2\cos 2\left(\frac{\pi x}{L}\right) + a_3\cos 3\left(\frac{\pi x}{L}\right)$$

Where $y = f(x)$ is in $(0, 6) = (0, L) \Rightarrow L = 6$

$$\therefore y = \frac{a_0}{2} + a_1\cos\frac{\pi x}{6} + a_2\cos 2\left(\frac{\pi x}{6}\right) + a_3\cos 3\left(\frac{\pi x}{6}\right) \rightarrow (1)$$

$\therefore a_0 = \frac{2}{N}\sum y = \frac{2}{6}(128.9) \Rightarrow a_0 = 42.967$

$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{2}{6}(-24.65) \Rightarrow a_1 = -8.217$

$a_2 = \frac{2}{N}\sum y\cos 2\theta = \frac{2}{6}(-9.25) \Rightarrow a_2 = -3.083$

$b_1 = \frac{2}{N}\sum y\sin\theta = \frac{2}{6}(-5.9754) \Rightarrow b_1 = -1.9918$

$b_2 = \frac{2}{N}\sum y\sin 2\theta = \frac{2}{6}(-0.6062) \Rightarrow b_2 = -0.202$

$\therefore (2) \Rightarrow y = \frac{42.967}{2} - 8.217\cos\theta - 1.9918\sin\theta - 3.083\cos 2\theta - 0.202\sin 2\theta$

at $\theta = \frac{\pi x}{6}$

$\Rightarrow y = 21.4835 - \left(8.217\cos\frac{\pi x}{6} + 1.9918\sin\frac{\pi x}{6}\right) - \left(3.083\cos\frac{\pi x}{3} + 0.202\sin\frac{\pi x}{3}\right)$

13. The following table gives the variation of a periodic current A over a period

t(sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
A(amp)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that there is a constant part of 0.75 amps in the current A & also obtain the amplitude of the first harmonic (OR)

[Show that there is a direct current part of 0.75 amps in the variable current and obtain the amplitude of the first harmonic.]

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22	9.0

→ The values of y at $x=0$ & $x=12$ are same.
Hence the interval of x is $0 \le x \le 12 \Rightarrow 0 \le x \le 2L$

$\therefore 2L = 12$

$\Rightarrow L = 6$

So we have to neglect 0 (or) 12

The Fourier series expansion over $(0, 2L)$. up to second Harmonic is

$$y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + b_1 \sin\left(\frac{\pi x}{L}\right) + a_2 \cos 2\left(\frac{\pi x}{L}\right) + b_2 \sin 2\left(\frac{\pi x}{L}\right)$$

at $L=6$.

$$\Rightarrow y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{6}\right) + b_1 \sin\left(\frac{\pi x}{6}\right) + a_2 \cos\left(\frac{2\pi x}{6}\right) + b_2 \sin\left(\frac{2\pi x}{6}\right) \rightarrow (1)$$

put $\theta = \frac{\pi x}{6}$

$$\Rightarrow y = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) \longrightarrow (2)$$

x	$\theta = \frac{\pi x}{6}$	y	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$	$\sin\theta$	$y\sin\theta$	$\sin 2\theta$	$y\sin 2\theta$
0	0	9.0	1	9	1	9.0	0	0	0	0
2	60	18.2	0.5	9.1	-0.5	-9.1	0.866	15.7612	0.866	15.7612
4	120	24.4	-0.5	-12.2	-0.5	-12.2	0.866	21.1304	-0.866	-21.1304
6	180	27.8	-1	-27.8	1	27.8	0	0	0	0
8	240	27.5	0.5	-13.75	-0.5	-13.75	-0.866	-23.815	0.866	23.815
10	300	22	0.5	11.0	-0.5	-11.0	-0.866	-19.052	-0.866	-19.052

$\Sigma y = 128.9$ $\Sigma y\cos\theta = -24.65$ $\Sigma y\cos 2\theta = -9.25$ $\Sigma y\sin\theta = -5.9754$ $\Sigma y\sin 2\theta = -0.6062$

at $\theta = \left(\frac{2\pi}{T}\right)t$,

$$② \Rightarrow A = 0.75 + 0.3733\cos\left(\frac{2\pi}{T}\right)t + (1.0046)\sin\left(\frac{2\pi}{T}\right)t$$

This shows that A has a constant part 0.75 in it

Now the amplitude of first harmonic is

$$\sqrt{a_1^2+b_1^2} = \sqrt{(0.3733)^2+(1.0046)^2} = 1.0717$$

14. Express 'y' in a Fourier series up to first harmonics using the tabular values of x & y. Given below.

x	0	30	60	90	120	150	180	210	240	270	300	330
y	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.00

→ Here the interval of x is $0 \le x < 360$ i.e) $0 \le x < 2\pi$
where $N=12$

The Fourier series up to first harmonics is

$$y = \frac{a_0}{2} + a_1\cos x + b_1\sin x \longrightarrow ①$$

→ Here, $A = A(t)$ is periodic with period T and is defined over the Time interval $(0, T)$
Here $t=0$ & $t=T$ are the same, so we should neglect any one. $\Rightarrow 0 \le t \le 2L \Rightarrow 2L = T \Rightarrow L = T/2$
First we shout convert $A = f(t)$ to the period 2π by putting $\theta = \left(\frac{2\pi}{T}\right)t$

When $t=0 \Rightarrow \theta = 0$
$t = T \Rightarrow \theta = 2\pi$

∴ The Fourier series of first harmonic in $(0, 2\pi)$ is

$$A = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta \longrightarrow (1)$$

t	$\theta = \frac{2\pi t}{T}$	A	$\cos\theta$	$A\cos\theta$	$\sin\theta$	$A\sin\theta$
0	0	1.98	1	1.98	0	0
$T/6$	60	1.30	$1/2$	0.65	$\sqrt{3}/2$	1.1258
$T/3$	120	1.05	$-1/2$	-0.525	$\sqrt{3}/2$	0.9093
$T/2$	180	1.30	-1	-1.30	0	0
$2T/3$	240	-0.88	$-1/2$	0.44	$-\sqrt{3}/2$	0.7621
$5T/6$	300	-0.25	$1/2$	-0.125	$-\sqrt{3}/2$	0.2165

$\Sigma A = 4.5$ $\quad \Sigma A\cos\theta = 1.12$ $\quad \Sigma A\sin\theta = 3.0137$

$$a_0 = \frac{2}{N}\Sigma A = \frac{2}{6}(4.5) = 1.5$$

$$a_1 = \frac{2}{N}\Sigma A\cos\theta = \frac{2}{6}(1.12) \Rightarrow a_1 = 0.3733$$

$$b_1 = \frac{2}{N}\Sigma A\sin\theta = \frac{2}{6}(3.0137) \Rightarrow b_1 = 1.0046$$

Question bank problems :-

1. Find a Fourier series to represent $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

→ Given $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$ in $(-\pi, \pi)$ is neither odd nor even

consider $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow ①$

consider $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\cancel{\int_{-\pi}^{0} 0\cdot dx}^{\,0} + \int_{0}^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\,\frac{x^3}{3}\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^3}{3} - 0\right)$$

$$\Rightarrow a_0 = \frac{\pi^2}{3}$$

consider $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left\{\cancel{\int_{-\pi}^{0} 0\cdot\cos nx\,dx}^{\,0} + \int_{0}^{\pi} x^2 \cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\cancel{\pi^2\frac{\sin n\pi}{n}}^{\,0} + \frac{2\pi\cos n\pi}{n^2} - \cancel{\frac{2\sin n\pi}{n^3}}^{\,0}\right] - \left[\cancel{0\cdot\frac{\sin 0}{n} + 0\frac{\cos 0}{n^2} - \frac{2\sin 0}{n^3}}^{\,0}\right.\right.$$

x	y	$\cos x$	$y\cos x$	$\sin x$	$y\sin x$
0	1.8	1	1.8	0	0
30	1.1	0.866	0.9526	0.5	0.55
60	0.3	0.5	0.15	0.866	0.2598
90	0.16	0	0	1	0.16
120	0.5	−0.5	−0.25	0.866	0.433
150	1.3	−0.866	−1.1258	0.5	0.65
180	2.16	−1	−2.16	0	0
210	1.25	−0.866	−1.0825	−0.5	−0.625
240	1.3	−0.5	−0.65	−0.866	−1.1258
270	1.52	0	0	−1	−1.52
300	1.76	0.5	0.88	−0.866	−1.524
330	2.00	0.866	1.7321	−0.5	−1

$\sum y = 15.15 \quad \sum y\cos x = 0.2464 \quad \sum y\sin x = -3.742$

$$a_0 = \frac{2}{N}\sum y = \frac{2}{12}(15.15) \Rightarrow a_0 = 2.525$$

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{12}(0.2464) \Rightarrow a_1 = 0.04106$$

$$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{12}(-3.742) \Rightarrow b_1 = -0.6237$$

$\therefore (1) \Rightarrow y = 1.2625 + 0.04106\cos x - 0.6237\sin x$

2. obtain the Fourier series of $f(x)$ defined by

$$f(x)=\begin{cases}-\pi, & x\in[-\pi,0] \\ x, & x\in[0,\pi]\end{cases}$$

$\rightarrow$ Given $f(x)=\begin{cases}-\pi, & x\in[-\pi,0] \\ x, & x\in[0,\pi]\end{cases}$ in $(-\pi,\pi)$ is neither odd nor even.

Consider $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx \rightarrow (1)$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{\pi}\left\{\int_{-\pi}^{0}(-\pi)dx+\int_{0}^{\pi}x\,dx\right\}$$

$$=-\pi x\Big|_{-\pi}^{0}+\frac{x^2}{2}\Big|_{0}^{\pi}$$

$$=\left[-\pi(0)-(-\pi)(-\pi)\right]+\left[\frac{\pi^2}{2}-0\right]$$

$$=-\pi^2+\frac{\pi^2}{2}$$

$$\Rightarrow a_0=-\frac{\pi^2}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\pi\cos nx\,dx+\int_{0}^{\pi}x\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left\{-\pi\frac{\sin nx}{n}\Big|_{-\pi}^{0}+\left[x\frac{\sin nx}{n}-1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$\Rightarrow a_n = \frac{2(-1)^n}{n^2}$$

consider $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} \cancel{0\cdot\sin nx\,dx}^{\,0} + \int_{0}^{\pi} x^2 \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[-\pi^2\frac{\cos n\pi}{n} + \cancel{2\pi\frac{\sin n\pi}{n^2}}^{\,0} + \frac{2\cos n\pi}{n^3}\right] - \left[\cancel{-0\frac{\cos 0}{n}}^{\,0} + \cancel{0\frac{\sin 0}{n^2}}^{\,0} + \frac{2\cos 0}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} + \frac{2}{n^3}\right\}$$

$$b_n = \frac{1}{\pi n^3}\left[2(-1)^n + 2 - n^2\pi^2(-1)^n\right]$$

$$\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1}{\pi n^3}\left[2(-1)^n + 2 - n^2\pi^2(-1)^n\right]\sin nx \;//$$

$$\therefore f(x) = \frac{-\pi^2}{4} + \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{[1-2(-1)^n]}{n} \sin nx$$

3. Expand $f(x) = x\sin x$ as a Fourier series in the interval $(-\pi, \pi)$ and deduce that $\frac{\pi-2}{4} = \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \ldots$

$\rightarrow$ [Refer problem No.8, page No.11]

$$\therefore x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx \longrightarrow (1)$$

put $x = \frac{\pi}{2}$ in (1)

$$\Rightarrow \frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cancelto{0}{\cos\frac{\pi}{2}} + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}(1) = 1 - 0 + 2\left[\frac{-1}{3}\cos\frac{2\pi}{2} + \frac{1}{8}\cos\frac{3\pi}{2} - \frac{1}{15}\cos\frac{4\pi}{2} + \ldots\right]$$

$$\Rightarrow \frac{\pi}{2} - 1 = 2\left[\frac{-1}{3}(-1) + \frac{1}{8}(0) - \frac{1}{15}(1) + \ldots\right]$$

$$\Rightarrow \frac{\pi-2}{2} = 2\left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \ldots\right]$$

$$\Rightarrow \frac{\pi-2}{4} = \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \ldots$$

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$$= \frac{1}{\pi}\left\{\left[-\pi\left(\frac{\cancel{\sin n(0)}}{n} - \frac{\cancel{\sin n(-\pi)}}{n}\right)\right] + \left[\left(\frac{\cancel{\pi \sin n\pi}}{n} + \frac{\cos n\pi}{n^2}\right) - \left(\frac{\cancel{0 \sin 0}}{n} + \frac{\cos 0}{n^2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi \sin nx\,dx + \int_{0}^{\pi} x \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} + \left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\pi\left(\frac{\cos 0}{n} - \frac{\cos n(-\pi)}{n}\right) + \left[\left(-\frac{\pi \cos n\pi}{n} + \frac{\cancel{\sin n\pi}}{n^2}\right) - \left(\frac{\cancel{-0 \cos 0}}{n} + \frac{\sin 0}{n^2}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

$$= \frac{\pi}{\pi n}\left[1 - 2(-1)^n\right]$$

$$\Rightarrow b_n = \frac{1 - 2(-1)^n}{n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \longrightarrow (3)$$

Adding (2) & (3)

$$\Rightarrow \frac{\pi^2}{6} + \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n^2}$$

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{0}{2^2} + \frac{2}{3^2} + \frac{0}{4^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \ (\text{or})\ \pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

5. Find the complex form of Fourier series $f(x) = \cos ax$ in $-\pi < x < \pi$

$\rightarrow$ Given $f(x) = \cos ax$ in $(-\pi, \pi)$

consider $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} \cos ax\, dx$$

4. Find the half range cosine series for $f(x)=(x-1)^2$ in the interval $0<x<1$. Hence show that
$$\pi^2 = 8\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)$$

→ [Refer problem No.1, Page No.57]

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2} \rightarrow (1)$$

put $x=0$ in (1) $\Rightarrow$

$$(0-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 0}{n^2}$$

$$1-\frac{1}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2}{3} = \frac{4}{\pi^2}\sum \frac{1}{n^2} \quad \Rightarrow \frac{2\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \rightarrow (2)$$

put $x=1$ in (1)

$$\Rightarrow (1-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}$$

$$\Rightarrow 0-\frac{1}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow \frac{1}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)(-1)^n}{n^2}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2+a^2}\left(-in\cos ax + a\sin ax\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(-n^2+a^2)}\left\{\left[e^{-in\pi}\left(-in\cos a\pi + a\sin a\pi\right)\right] - \left[e^{in\pi}\left(-in\cos a(-\pi) + a\sin a(-\pi)\right)\right]\right\}$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[(-1)^n\left(-in\cos a\pi + a\sin a\pi\right) - (-1)^n\left(-in\cos a\pi - a\sin a\pi\right)\right]$$

$$= \frac{(-1)^n}{2\pi(a^2-n^2)}\left[-in\cancel{\cos a\pi} + a\sin a\pi + in\cancel{\cos a\pi} + a\sin a\pi\right]$$

$$= \frac{\cancel{2}a\sin a\pi\,(-1)^n}{\cancel{2}\pi(a^2-n^2)}$$

$$\Rightarrow c_n = \frac{a\sin a\pi\,(-1)^n}{\pi(a^2-n^2)}$$

$$\therefore f(x) = \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2-n^2}e^{inx} \quad //$$