THEORY OF MECHANISMS AND MACHINES

SECOND EDITION

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7

BALANCING OF INERTIA FORCES AND MOMENTS IN MACHINES

7.1 INTRODUCTION

In Chapter 4, we observed that the inertia force and the inertia moment of a body are, respectively, equal and opposite to the resultants of all the external forces and moments. Thus, when a machine has accelerating parts, the resultants of inertia forces and moments of all such accelerating parts represent the forces and the moments transmitted to the frame or foundation of the machine. By "frame", we mean only that surrounding structure of the machine which can supply the external forces and moments. If the direction and/or magnitude of these accelerations vary with time (as they generally do), the foundation is subjected to *dynamic forces* and *moments*. Besides being very harmful in reducing the life of the supports, these dynamic forces and moments set up vibrations which result in a noisy operation. So, for longer life and smoother operation of the machine, there is a need to balance these inertia forces and moments. There are two aspects of this balancing problem, namely, (i) the evaluation and analysis of inertia forces and moments, and (ii) the determination of convenient methods of balancing these quantities.

We shall first discuss the balancing of rotating machines and then consider the balancing of internal-combustion engines which have both reciprocating and rotating masses. Finally, we shall study the balancing of general mechanisms.

7.2 BALANCING OF ROTATING MASSES

If the centre of mass of a rotating machine (such as an alternator) does not lie on the axis of rotation, the inertia force is given by

$$F_i = m\omega^2 e, \tag{7.1}$$

where

m = mass of the machine,

ω = angular speed of the machine, and

e = eccentricity, i.e., the distance from the centre of mass to the axis of rotation.*

Though the magnitude of this force is constant, its direction changes continuously as the

*The term eccentricity is used here to represent a scalar quantity. However, in a problem of rotating mass, this term can also be used in the sense of a distance vector. In such a case, the direction of the resulting centrifugal force corresponds to that of the eccentricity vector.

machine rotates. Thus, even with a small value of e, a considerable dynamic force is exerted on each bearing at a high speed (since the force varies as the square of the speed). Before discussing the balancing of rotors, we shall analyze the case of discrete unbalanced masses on a rotating shaft (e.g., a disc which is eccentric with respect to the axis of rotation). The theory we shall thus develop can then be readily extended to unbalanced rotors. *Static forces*, such as weights, will be neglected in the analysis, and only the inertia forces (centrifugal forces) of unbalanced masses will be considered.

INTERNAL AND EXTERNAL BALANCING

Let a shaft carry an unbalanced mass m, with the centre of mass located at a distance e from the axis of rotation (Fig. 7.1). The shaft can be completely balanced by adding a mass m_1 at a distance e_1 from the axis of rotation, diametrically opposite to m, so that

$$m\omega^2 e = m_1\omega^2 e_1 \qquad \text{(where } \omega = \text{speed of rotation of the shaft)},$$

$$me = m_1 e_1. \tag{7.2}$$

In this case, the dynamic reactions at the bearings (say, at A and B) will be zero, i.e., $R_A = 0 = R_B$. The shaft will also be free from any dynamic bending stress. This situation is referred to as *internal balancing*.

Alternatively, the dynamic reactions R_A and R_B can be reduced to zero (see Fig. 7.2) by adding two balancing masses m_1 and m_2 at distances e_1 and e_2, respectively, from the shaft, in the

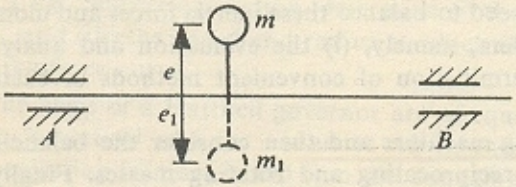


FIGURE 7.1

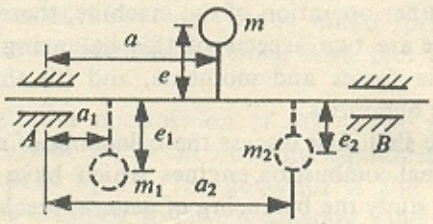


FIGURE 7.2

same axial plane as m but on the opposite side of the axis of rotation. With this arrangement, we get

$$me = m_1 e_1 + m_2 e_2, \tag{7.3}$$

$$mea = m_1 e_1 a_1 + m_2 e_2 a_2, \tag{7.4}$$

where a, a_1, and a_2 represent the distance of the masses m, m_1, and m_2, respectively, from A. This situation is referred to as *external balancing* where the entire length of the shaft is not free from dynamic bending stresses. In practice, a balancing mass can rarely be placed exactly opposite the unbalanced mass to achieve internal balancing. For this reason, the term "balancing" will generally be used to denote "external balancing".

STATIC AND DYNAMIC BALANCING

If a shaft carries a number of unbalanced masses such that the centre of mass of the system lies on the axis of rotation, the system is said to be *statically balanced*. Such a system may,

nevertheless, exert dynamic forces at the supports when it is rotated. To explain this, let us consider the statically-balanced system shown in Fig. 7.3. Two equal masses m, each with an eccentricity e, are placed on opposite sides of the axis in the same axial plane. The axial distance between them is a. The dynamic reactions at the supports will be

$$R_B = -R_A = m(\omega^2 ea/l), \tag{7.5}$$

where ω = speed of rotation, and l = distance between the supports. Since $R_A + R_B = 0$, we can refer to this situation as one in which there is no *total dynamic reaction* at the supports. In this case, the unbalanced dynamic forces result in a pure couple ($= m\omega^2 ea$).

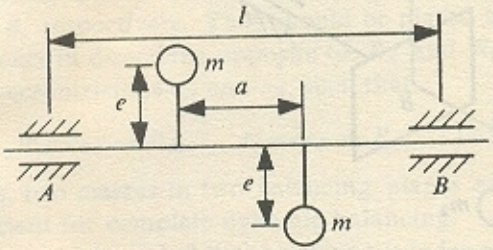


FIGURE 7.3

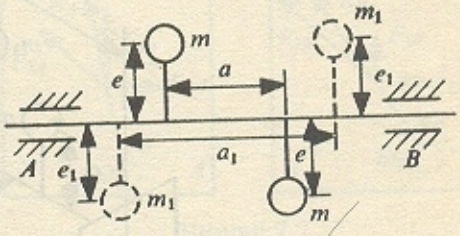


FIGURE 7.4

Now, suppose we add a pair of equal masses m_1 in the same axial plane as m, each with the same eccentricity e_1, such that they are on opposite sides of the axis and the distance between them is a_1, as shown in Fig. 7.4. For this arrangement, if the relation

$$mea = m_1 e_1 a_1 \tag{7.6}$$

is satisfied in such a manner that the couple due to the masses m_1 is opposite to that due to the masses m, then the dynamic reactions at the supports are

$$R_A = R_B = 0, \tag{7.7}$$

and the system is said to be *dynamically balanced*. It is obvious that a dynamically-balanced system is also statically balanced, but the reverse is not true. Thus, to ensure complete external balance in a rotating system, the following two conditions must be fulfilled:

(i) The centre of mass of the system should lie on the axis of rotation when the resultant of all the inertia forces during rotation will be zero. This is the condition for static balance.

(ii) The resultant couple due to all the inertia forces during rotation must be zero.

These two conditions together will give complete dynamic balance.

7.3 TWO-PLANE BALANCING

If a shaft carries a series of unbalanced masses in different axial planes, then two balancing masses in any two arbitrarily chosen planes are sufficient for complete dynamic balancing of the system. Figure 7.5 shows a shaft carrying three unbalanced masses m_1, m_2, and m_3 with eccentricities e_1, e_2, and e_3, respectively, in three different axial planes. Let A and B be the positions of two arbitrarily chosen balancing planes at a distance l from each other. In Fig. 7.6, the unbalanced masses have been replaced by their respective inertia forces F_1, F_2, and F_3 along the respective

eccentricities such that

$$F_1 = m_1\omega^2 e_1, \qquad F_2 = m_2\omega^2 e_2, \qquad F_3 = m_3\omega^2 e_3,$$

ω being the speed of rotation. Let x_1, x_2, and x_3 denote the axial distances of these forces from the plane A.

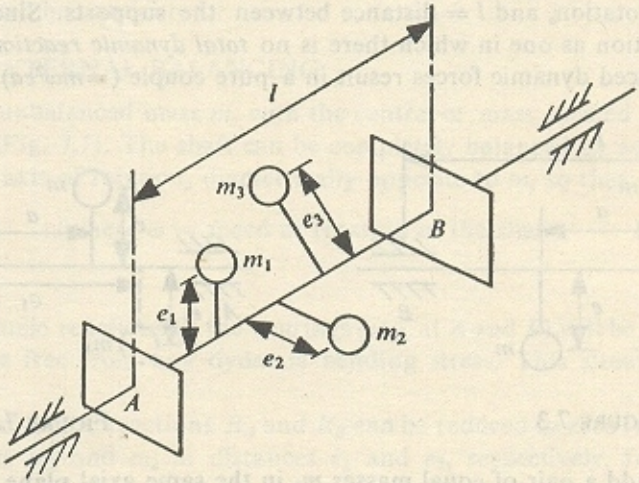


FIGURE 7.5

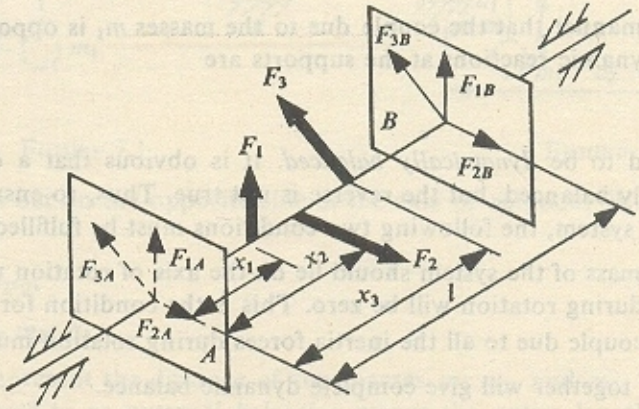


FIGURE 7.6

Now, the force F_1 can be completely replaced by two equivalent forces F_{1A} and F_{1B} in the planes A and B, respectively, acting in the direction of F_1, where

$$F_{1A} = F_1\left(1 - \frac{x_1}{l}\right), \qquad F_{1B} = F_1\frac{x_1}{l}. \tag{7.8a}$$

Similarly, F_2 and F_3 can be replaced by

$$F_{2A} = F_2\left(1 - \frac{x_2}{l}\right), \qquad F_{2B} = F_2\frac{x_2}{l}, \tag{7.8b}$$

$$F_{3A} = F_3\left(1 - \frac{x_3}{l}\right), \qquad F_{3B} = F_3\frac{x_3}{l}. \tag{7.8c}$$

Thus, the unbalanced forces F_1, F_2, and F_3 can be replaced by a set of three forces in each of the planes A and B. Let R_A be the resultant of F_{1A}, F_{2A}, and F_{3A}, and let R_B be the resultant of F_{1B}, F_{2B}, and F_{3B} (Fig. 7.7). These two forces R_A and R_B can be completely balanced by two balancing masses m_A and m_B in the planes A and B, respectively. They should be placed so as to act in directions opposite to R_A and R_B, with eccentricities e_A and e_B, such that

$$m_A\omega^2 e_A = R_A, \qquad m_B\omega^2 e_B = R_B. \tag{7.9}$$

Thus, two masses in two balancing planes are sufficient for complete dynamic balancing.

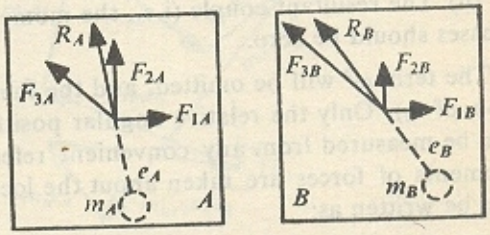


FIGURE 7.7

It should be noted that balancing is independent of the rotational speed ω as the term ω^2 is common to all the unbalanced and balancing forces. This implies that a system balanced for one speed of rotation remains balanced for any other speed. It should also be observed that the product of mass and the corresponding eccentricity appears in the expressions for all forces. Henceforth, this product me will be referred to as *unbalance*, and the term ω^2 will be omitted everywhere.

7.4 DETERMINATION OF BALANCING MASSES

If a shaft carries n unbalances in different axial planes, the two balancing masses with prescribed eccentricities in two chosen balancing planes can be determined by both graphical and analytical methods. (Actually, the product of each mass and its eccentricity is obtained.) The choice of a suitable method depends on the problem. Figure 7.8a shows n unbalances (m_1e_1, m_2e_2, m_3e_3, . . . , m_ne_n) on a shaft, in different axial planes 1, 2, 3, . . . , n, designated by angles θ_1, θ_2, θ_3, . . . , θ_n,

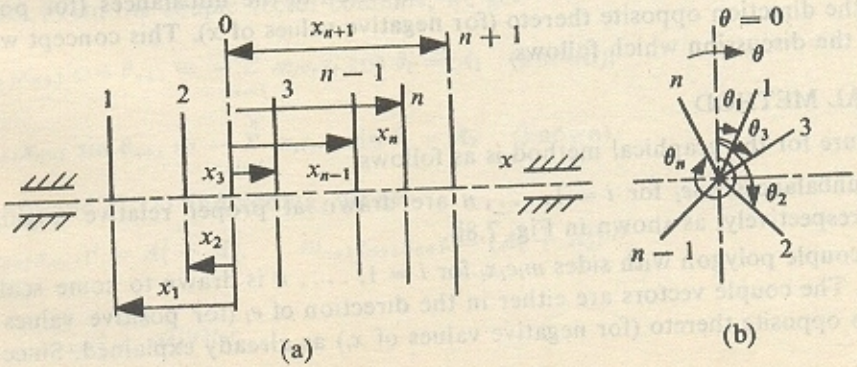


FIGURE 7.8

respectively (Fig. 7.8b), and measured conveniently in the same direction from an arbitrary reference line. The balancing planes are designated 0 and $(n + 1)$. The axial locations of planes $1, 2, 3, \ldots, n$ are denoted by the distance from one of these reference planes (say, 0), shown to be $x_1, x_2, x_3, \ldots, x_{n+1}$ in Fig. 7.8a. The axial distances will be considered positive when measured to the right. The problem is to determine m_0e_0 and $m_{n+1}e_{n+1}$ (in planes 0 and $n + 1$, respectively) along with θ_0 and θ_{n+1} (to indicate the inclination of the axial planes containing m_0e_0 and $m_{n+1}e_{n+1}$) to produce dynamic balance in the system. The following two conditions are to be satisfied:

(i) The resultant of the inertia forces of all $(n + 2)$ masses should be zero.

(ii) The resultant couple (i.e., the moment about any point) of the inertia forces of all $(n + 2)$ masses should be zero.

The term ω^2 will be omitted, and the inertia forces will be represented by m_ie_i (in the direction of e_i). Only the relative angular positions of the masses are important and the values of θ can be measured from any convenient reference line (here it is taken to be vertical). If the moments of forces are taken about the location of plane 0, the two aforementioned conditions can be written as:

(i) The vector sum of the forces is zero. Thus,

$$\sum_{i=0}^{n+1} m_ie_i = 0.$$

(ii) The vector sum of the moments is zero. Thus,

$$\sum_{i=0}^{n+1} m_ie_ix_i = 0.$$

Obviously, $x_0 = 0$, and some values of x may be negative. The couple vectors $m_ie_ix_i$ are at 90° to the corresponding force vectors.* The direction of the couple vectors will be obtained by rotating the corresponding force vectors in a clockwise direction (through 90°) for positive values of x, and in the counter-clockwise direction (through 90°) for negative values of x. Since only relative positions are important, as a convention, we shall consider all the couple vectors after rotation through 90° in the counter-clockwise direction. Thus, the couple vectors will be either in the direction of the forces, i.e., in the direction of the unbalances (for positive values of x) or in the direction opposite thereto (for negative values of x). This concept will be further clarified by the discussion which follows.

GRAPHICAL METHOD

The procedure for the graphical method is as follows:

(i) The unbalances m_ie_i for $i = 1, \ldots, n$ are drawn at proper relative angular positions $\theta_1, \ldots, \theta_n$, respectively, as shown in Fig. 7.8b.

(ii) The couple polygon with sides $m_ie_ix_i$ for $i = 1, \ldots, n$ is drawn to some scale as shown in Fig. 7.9a. The couple vectors are either in the direction of e_i (for positive values of x_i) or in the direction opposite thereto (for negative values of x_i) as already explained. Since $x_0 = 0$, the

*See Appendix B.

closing side of the polygon (shown by the dashed line in Fig. 7.9a) represents $m_{n+1}e_{n+1}x_{n+1}$. As x_{n+1} is known, we can now find $m_{n+1}e_{n+1}$. The direction of this balancing mass, θ_{n+1}, can also be determined, because the vector $m_{n+1}e_{n+1}x_{n+1}$ will be either in the direction θ_{n+1} (if x_{n+1} is positive) or in the opposite direction (if x_{n+1} is negative).

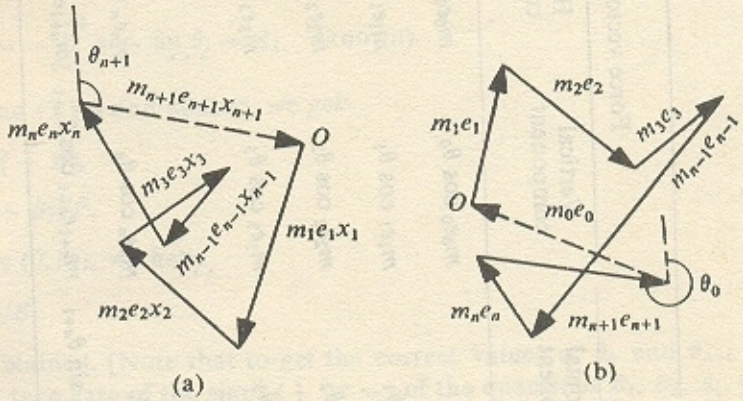


FIGURE 7.9

(iii) Now the force polygon with sides m_ie_i for $i = 1, \ldots, (n+1)$ is drawn to some scale as shown in Fig. 7.9b. The closing side represents m_0e_0 (shown by the dashed line) which gives both the magnitude of m_0e_0 and its direction θ_0.

ANALYTICAL METHOD

The foregoing problem of determining the two balancing masses can also be solved in the same way as by the graphical method but without drawing any diagrams. To do this, the forces and the couples are resolved along two mutually perpendicular directions, $\theta = 0$ and $\theta = 90°$, and their values are as shown in Table 7.1 (as before, all couple vectors are assumed to be in the direction of the corresponding force vectors for positive values of x).

For complete external balancing, the summation of each of the last four columns of the table should be zero. From the couple vector columns, we get

$$m_{n+1}e_{n+1}x_{n+1} \cos \theta_{n+1} = -\sum_{i=1}^{n} m_ie_ix_i \cos \theta_i = A_1 \quad \text{(known)}, \tag{7.10}$$

$$m_{n+1}e_{n+1}x_{n+1} \sin \theta_{n+1} = -\sum_{i=1}^{n} m_ie_ix_i \sin \theta_i = A_2 \quad \text{(known)}. \tag{7.11}$$

Squaring (7.10) and (7.11) and adding, we get

$$(m_{n+1}e_{n+1}x_{n+1})^2 = A_1^2 + A_2^2, \qquad m_{n+1}e_{n+1}x_{n+1} = (A_1^2 + A_2^2)^{1/2}.$$

Thus,

$$m_{n+1}e_{n+1} = (A_1^2 + A_2^2)^{1/2}/x_{n+1}. \tag{7.12}$$

Dividing (7.11) by (7.10), we have

$$\tan \theta_{n+1} = A_2/A_1 \tag{7.13}$$

which gives θ_{n+1}.

TABLE 7.1

Plane	Product	Distance from reference plane	Angle with reference line (vertical in this problem)	Couple vector		Force vector	
				Vertical component	Horizontal component	Vertical component	Horizontal component
0	m_0e_0	$x_0 = 0$	θ_0	0	0	$m_0e_0 \cos \theta_0$	$m_0e_0 \sin \theta_0$
1	m_1e_1	x_1	θ_1	$m_1e_1x_1 \cos \theta_1$	$m_1e_1x_1 \sin \theta_1$	$m_1e_1 \cos \theta_1$	$m_1e_1 \sin \theta_1$
2	m_2e_2	x_2	θ_2	$m_2e_2x_2 \cos \theta_2$	$m_2e_2x_2 \sin \theta_2$	$m_2e_2 \cos \theta_2$	$m_2e_2 \sin \theta_2$
3	m_3e_3	x_3	θ_3	$m_3e_3x_3 \cos \theta_3$	$m_3e_3x_3 \sin \theta_3$	$m_3e_3 \cos \theta_3$	$m_3e_3 \sin \theta_3$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	m_ne_n	x_n	θ_n	$m_ne_nx_n \cos \theta_n$	$m_ne_nx_n \sin \theta_n$	$m_ne_n \cos \theta_n$	$m_ne_n \sin \theta_n$
$n + 1$	$m_{n+1}e_{n+1}$	x_{n+1}	θ_{n+1}	$m_{n+1}e_{n+1}x_{n+1} \cos \theta_{n+1}$	$m_{n+1}e_{n+1}x_{n+1} \sin \theta_{n+1}$	$m_{n+1}e_{n+1} \cos \theta_{n+1}$	$m_{n+1}e_{n+1} \sin \theta_{n+1}$

Now, to determine m_0e_0 and θ_0, we equate the summations of each of the force vector columns of Table 7.1 to zero. Thus,

$$m_0e_0 \cos \theta_0 = -\sum_{i=1}^{n+1} m_ie_i \cos \theta_i = B_1 \quad \text{(known)}, \tag{7.14}$$

$$m_0e_0 \sin \theta_0 = -\sum_{i=1}^{n+1} m_ie_i \sin \theta_i = B_2 \quad \text{(known)}. \tag{7.15}$$

Squaring (7.14) and (7.15) and adding, we get

$$(m_0e_0)^2 = B_1^2 + B_2^2,$$

$$m_0e_0 = (B_1^2 + B_2^2)^{1/2}. \tag{7.16}$$

Dividing (7.15) by (7.14), we have

$$\tan \theta_0 = B_2/B_1. \tag{7.17}$$

Thus, θ_0 also is obtained. [Note that to get the correct values of θ_0 and θ_{n+1} from (7.17) and (7.13), we should take care of the signs (+ or −) of the quantities B_1, B_2, A_1, and A_2 since two solutions each for θ_0 and θ_{n+1} are obtained from (7.17) and (7.13), respectively.]

PROBLEM 7.1 A shaft carries four masses in parallel planes A, B, C, and D, in this order, along it. The masses at B and C are 18 kg and 12.5 kg, respectively, and each has an eccentricity of 6 cm. The masses at A and D have an eccentricity of 8 cm. The angle between the masses at B and C is 100°, and that between the masses at B and A is 190° (both angles being measured in the same direction). The axial distance between the planes A and B is 10 cm and that between B and C is 20 cm. If the shaft is in complete dynamic balance, determine

(i) the masses at A and D,

(ii) the distance between the planes C and D, and

(iii) the angular position of the mass at D.

SOLUTION

Graphical Method

As the mass at the plane A is unknown, we take the plane A as our reference for fixing the axial locations of the other planes. For locating angular positions, we take the plane B as the reference line. First, the masses are drawn at relative angular positions as shown in Fig. 7.10a, so that

$$\theta_B = 0, \qquad \theta_C = 100°, \qquad \theta_A = 190° \qquad (\theta_D \text{ is not known}).$$

With A as the reference plane for couple vectors, we have

$$x_A = 0, \qquad x_B = 10 \text{ cm}, \qquad x_C = 30 \text{ cm} \qquad (x_D \text{ is not known}),$$

$$m_B = 18 \text{ kg}, \qquad m_C = 12.5 \text{ kg} \qquad (m_A \text{ and } m_D \text{ are not known}),$$

$$e_A = 8 \text{ cm}, \qquad e_B = 6 \text{ cm}, \qquad e_C = 6 \text{ cm}, \qquad e_D = 8 \text{ cm}.$$

The couple polygon is shown in Fig. 7.10b, wherefrom by measurements we get $\theta_D = 253°$ (i.e., the angle between the masses at B and D) and $m_De_Dx_D = 2312$ kg-cm² or $m_Dx_D = 289$ kg-cm.

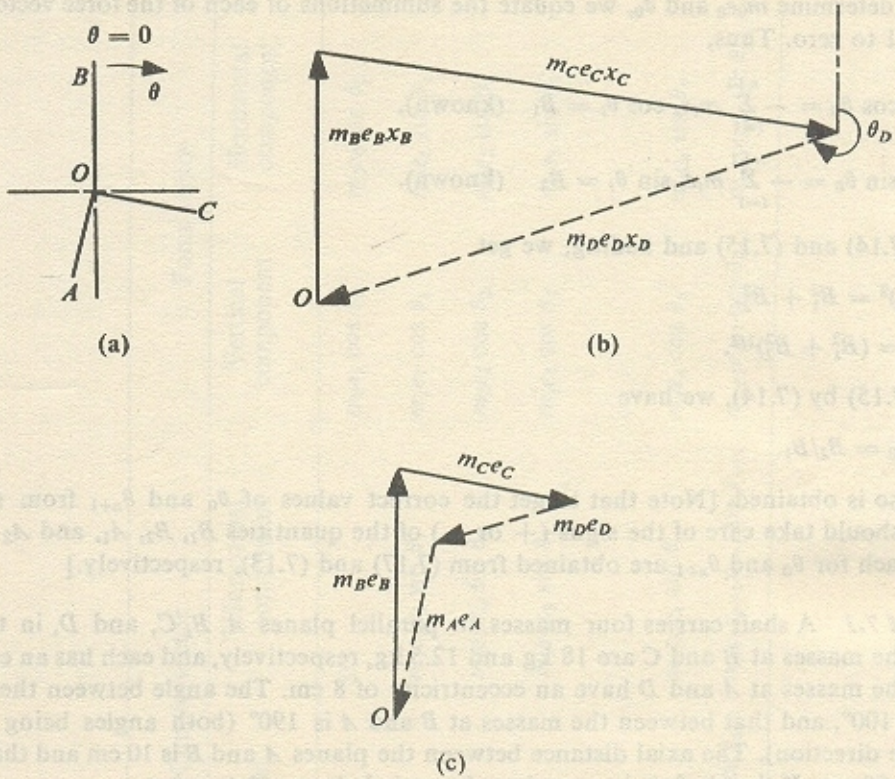


FIGURE 7.10

The force polygon is shown in Fig. 7.10c. From measurements, we get

$$m_D e_D = 63.5 \text{ kg-cm}, \qquad m_A e_A = 78 \text{ kg-cm}, \qquad m_D = 7.94 \text{ kg}, \qquad m_A = 9.75 \text{ kg}.$$

Thus, $x_D = 289/7.94 = 36.4$ cm (i.e., the distance of the plane D from the plane A). So, the distance of the plane D from the plane $C = (36.4 - 30) = 6.4$ cm.

Analytical Method

As already explained, the forces and couples are resolved in two mutually perpendicular planes (as done in Table 7.2). Summing up the last four columns of the table and equating each to zero, we get

$$m_D x_D \cos\theta_D = -86.17, \tag{a}$$

$$m_D x_D \sin\theta_D = -276.5, \tag{b}$$

$$8m_D \cos\theta_D - 7.88 m_A = -94.98, \tag{c}$$

$$8m_D \sin\theta_D - 1.39 m_A = -73.73. \tag{d}$$

From (a) and (b), $\tan\theta_D = 3.21$. Since x_D is known to be positive, both $\sin\theta_D$ and $\cos\theta_D$ are

TABLE 7.2

Plane	Mass m (kg)	Eccentricity e (cm)	Distance from reference plane A x (cm)	Angle with reference line B θ (degrees)	Couple vector		Force vector	
					Vertical component $mex \cos\theta$	Horizontal component $mex \sin\theta$	Vertical component $me \cos\theta$	Horizontal component $me \sin\theta$
A	m_A	8	0	190	0	0	$-7.88m_A$	$-1.39m_A$
B	18	6	10	0	1080	0	108	0
C	12.5	6	30	100	-390.6	2212	-13.02	73.73
D	m_D	8	x_D	θ_D	$8m_Dx_D \cos\theta_D$	$8m_Dx_D \sin\theta_D$	$8m_D \cos\theta_D$	$8m_D \sin\theta_D$

negative. So, $\theta_D = 252.7°$ (this is the angle between the masses at D and B). Thus, we get

$$\cos\theta_D = -0.2975, \qquad \sin\theta_D = -0.955.$$

Substituting these values in (c) and (d), we have

$$-2.38m_D - 7.88m_A = -94.98, \tag{e}$$

$$-7.64m_D - 1.39m_A = -73.73. \tag{f}$$

Solving (e) and (f) simultaneously, we get

$$m_A = 9.67 \text{ kg}, \qquad m_D = 7.89 \text{ kg}.$$

Then, using these values of m_D and $\cos\theta_D$ in (a), $x_D = 36.57$ cm, and the distance of D from C is $(x_D - 30) = 6.57$ cm.

7.5 BALANCING OF ROTORS

It was observed in Section 7.3 that two balancing masses in any two conveniently chosen transverse planes (balancing planes) are sufficient to produce complete dynamic balance in any rotating system. Conversely, the complete unbalance of a rotor can be represented by two unbalances in these two planes. This basic principle is utilized in various balancing machines with which rotors are balanced. In practice, it may not be feasible to add balancing masses. In such situations, balancing can be achieved by the removal of masses from the reference planes (from positions diametrically opposite to those of balancing masses). In this section, we will consider the operation of a few balancing machines in common use.

PIVOTED-CARRIAGE BALANCING MACHINE

The rotor to be balanced is mounted on half-bearings in a light but stiff carriage. This carriage is pivoted near one end and rests on a spring at the other (Fig. 7.11). Two balancing planes 1 and 2 are chosen on the rotor, and the rotor unbalance is assumed to be represented by unbalances only in these two planes. First, the rotor is mounted with plane 1 on the pivot so that the unbalance in plane 1 will have no effect on the motion of the carriage. On the rotor being rotated in this position, the carriage will oscillate due to the unbalance in plane 2. A trial mass (at a known radius) is then attached in plane 2 and the amplitude of oscillation of the carriage is noted. This procedure is repeated with the same trial mass at different angular positions. A plot of the observed amplitude against the angular position of the trial mass shows the optimum angular position for which the oscillation amplitude is minimum. The magnitude of the trial mass is varied at this position, and the exact amount of balance mass, which reduces the amplitude of oscillation to almost zero, is determined by trial and error. A similar procedure is followed to determine

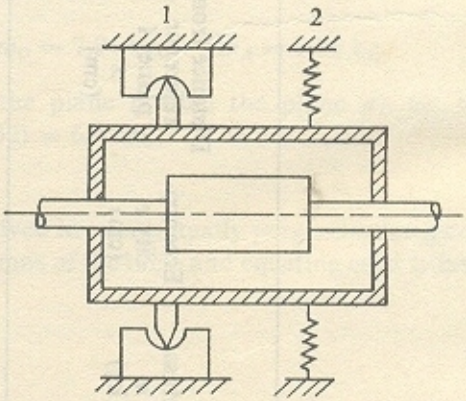


FIGURE 7.11

BALANCING OF ROTORS WITH VARIABLE MASS

There exists another class of rotors which need special attention so far as their balancing is concerned. The mass and geometrical characteristics of such a rotor are not definite and may change. A common example of this type of a rotor is the one found in many washing machines. In a washing machine, the objects to be cleaned, which form a part of the rotor, neither have a fixed quantity nor do they possess definite geometric shapes. The balancing of such a rotor running at a low speed can be achieved using special means where the requisite balancing mass is self-adjusted.*

7.6 BALANCING OF INTERNAL-COMBUSTION ENGINES (APPROXIMATE ANALYSIS)

An internal-combustion engine has both rotating and reciprocating parts. In what follows, we shall discuss the unbalanced (inertia) forces and moments resulting from the acceleration of these parts. Since steam engines (locomotives) have more or less become out of date, we shall not consider their balancing problems. Instead, we shall analyze a single-cylinder engine. In so doing, we shall first take up the unbalanced forces and moments and then discuss the proper design procedure for balancing. Subsequently, we shall consider the balancing of multicylinder engines, e.g., in-line engines, V-twin engines, and radial engines.

Before we rigorously analyze a single-cylinder engine, let us see how only the inertia forces and moments are left unbalanced and transmitted to the foundation of such an engine. For this, we shall *assume that*

(i) *the replacement of the connecting rod by two masses at its ends maintains complete dynamic equivalence*,

(ii) *the crank is so designed that the combined centre of mass of the crank and the apportioned connecting-rod mass* (m_2) *at the crank pin lies on the axis of rotation*, and

(iii) *the engine speed is constant.*

When considering multicylinder engines, we shall apply these assumptions to each constituent cylinder.

SINGLE-CYLINDER ENGINES

Let us consider a horizontal single-cylinder engine. For approximately analyzing the unbalanced forces and moments, the connecting rod is replaced by two masses m_1 and m_2 at the ends of a massless rod (as in Section 5.3). Thus, m_1 is an entirely reciprocating mass as it moves with the piston, and m_2 is subject to pure rotation along with the crank. As before, let

m_{rec} = total mass (including m_1) of the reciprocating parts (i.e., the piston and the gudgeon pin),

Q = thrust in the connecting rod acting along the direction of the connecting rod, and

$\lambda = r/l$ (where l is the length of the connecting rod and r is the crank radius).

The forces and moments acting on the mechanism and on the foundation are indicated in

*For more details on a few such mechanisms, see Macduff, J. N. and Curreri, J. R., Vibration Control, McGraw-Hill, New York, 1958.

Fig. 7.21a(i); those that act on the machine members are shown by the solid lines, and those

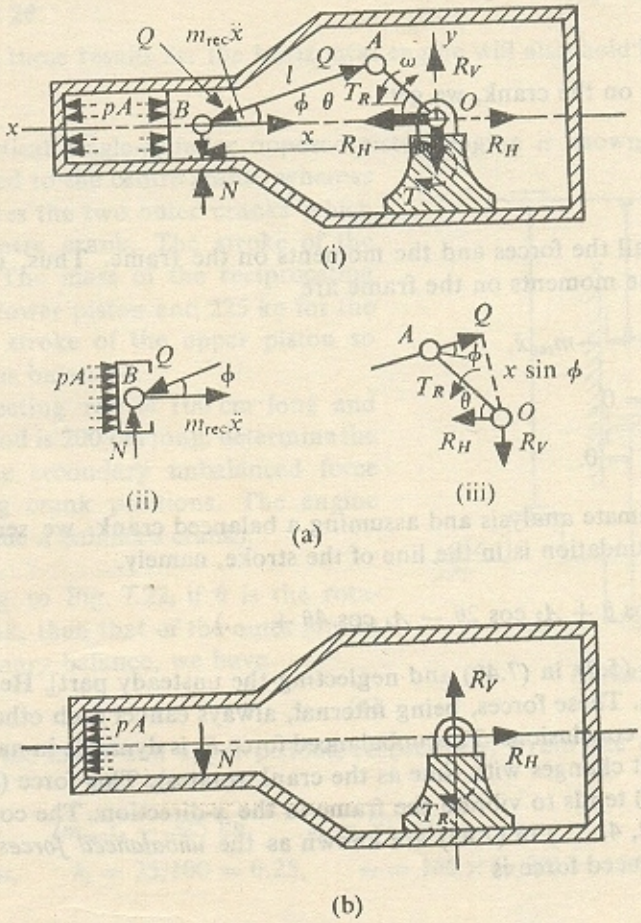


FIGURE 7.21

on the frame are shown by the dashed lines. In this figure,

pA = gas force,

T_R = load torque on the crank (the surrounding structure supplies this load torque and in turn is subjected to the reaction to it),

N = normal thrust between the cylinder and the piston, and

R_H, R_V = horizontal and vertical reactions at the crankshaft bearing.

Considering the forces at the gudgeon pin, i.e., at B [Fig. 7.21a(ii)], we get

$$Q \sin \phi = N, \tag{7.35}$$

$$Q \cos \phi = pA + m_{rec}\ddot{x}. \tag{7.36}$$

The crank rotates at a constant speed ω. Taking moments of all the forces on the crank about the point O [Fig. 7.21a(iii)], we get

$$Qx \sin \phi = T_R. \tag{7.37}$$

Balancing the forces on the crank, we get

$$R_H = Q \cos \phi, \tag{7.38}$$

$$R_V = Q \sin \phi. \tag{7.39}$$

Figure 7.21b shows all the forces and the moments on the frame. Thus, using (7.35) to (7.39), the net forces and the moments on the frame are

$$F_x = pA - R_H = -m_{rec}\ddot{x}, \tag{7.40}$$

$$F_y = R_V - N = 0, \tag{7.41}$$

$$M_z = N_x - T_R = 0. \tag{7.42}$$

So, with this approximate analysis and assuming a balanced crank, we see that the only force transmitted to the foundation is in the line of the stroke, namely,

$$F_x = m_{rec}\omega^2 r(\cos \theta + A_2 \cos 2\theta - A_4 \cos 4\theta + \ldots) \tag{7.43}$$

[substituting $\ddot{x}$ from (5.7) in (7.40) and neglecting the unsteady part]. Here, the friction forces have not been shown. These forces, being internal, always cancel each other and will in no way affect our subsequent conclusions. This unbalanced force F_x is dynamic in nature as its magnitude changes with θ (i.e., it changes with time as the crank rotates). This force (known as the longitudinal shaking force) tends to vibrate the frame in the x-direction. The coefficients of the terms $\cos j\theta$ (where $j = 1, 2, 4, \ldots$) in (7.43) are known as the *unbalanced forces of j-th order*. Thus, the first-order unbalanced force is

$$F_x^1 = m_{rec}\omega^2 r, \tag{7.44}$$

and the second-order unbalanced force is

$$F_x^2 = m_{rec}\omega^2 r A_2. \tag{7.45}$$

These are the magnitudes of the primary and secondary unbalanced forces, respectively, where the *primary unbalanced force* is

$$F_x^P = m_{rec}\omega^2 r \cos \theta, \tag{7.46}$$

and the *secondary unbalanced force* is

$$F_x^S = m_{rec}\omega^2 r A_2 \cos 2\theta. \tag{7.47}$$

Except in a very high-speed engine, forces up to the second order are of importance, those of higher order being of negligible magnitude. If the connecting-rod-to-crank-length ratio (l/r) is considerably greater than unity, then $\lambda = r/l \ll 1$, when $A_2 \approx \lambda$ (see Section 5.2). Thus, the

secondary force* can be written as

$$F_x^S = \lambda m_{rec}\omega^2 r \cos 2\theta. \tag{7.48}$$

It should be noted that these results for the horizontal engine will also hold good for a vertical engine.

PROBLEM 7.4 A vertical single-cylinder opposed-piston engine is shown in Fig. 7.22. The lower piston is connected to the centre crank, whereas the upper piston operates the two outer cranks which are at 180° to the centre crank. The stroke of the lower piston is 50 cm. The mass of the reciprocating parts is 150 kg for the lower piston and 225 kg for the upper piston. Find the stroke of the upper piston so that the primary force is balanced.

If the central connecting rod is 100 cm long and each outer connecting rod is 200 cm long, determine the maximum value of the secondary unbalanced force and the corresponding crank positions. The engine speed is 180 rpm (assume a balanced crank).

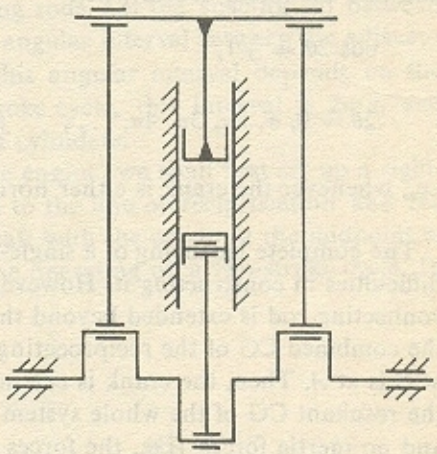


FIGURE 7.22

SOLUTION Referring to Fig. 7.22, if θ is the rotation of the central crank, then that of the outer cranks will be $\theta + \pi$. For primary balance, we have

$$F_{xl}^P + F_{xu}^P = 0, \tag{a}$$

where l and u refer to the lower and upper pistons, respectively. From the data given, we see that

$$(m_{rec})_l = 150 \text{ kg}, \qquad (m_{rec})_u = 225 \text{ kg}, \qquad s_l = 50 \text{ cm},$$

$$r_l = s_l/2 = 25 \text{ cm}, \qquad \lambda_l = 25/100 = 0.25, \qquad \omega = 180 \times (\pi/30) = 18.84 \text{ c/sec}.$$

From (7.46), we have

$$F_{xl}^P = (m_{rec})_l \omega^2 r_l \cos\theta, \qquad F_{xu}^P = (m_{rec})_u \omega^2 r_u \cos(\theta + \pi).$$

Substituting these values in equation (a), we get

$$(m_{rec})_l r_l = (m_{rec})_u r_u, \qquad r_u = (m_{rec})_l/(m_{rec})_u = 25 \times \frac{150}{225} = 16.7 \text{ cm}.$$

So, the stroke of the upper piston $s_u = 2r_u = 33.4$ cm. Thus, $\lambda_u = 16.7/200 = 0.0835$.

The secondary unbalanced force is given by

$$\begin{aligned} F_x^S &= F_{xl}^S + F_{x4}^S \\ &= \lambda_l (m_{rec})_l \omega^2 r_l \cos 2\theta + \lambda_u (m_{rec})_u \omega^2 r_u \cos 2(\theta + \pi) \\ &= \omega^2 \cos 2\theta [\lambda_l r_l (m_{rec})_l + \lambda_u r_u (m_{rec})_u]. \end{aligned}$$

*The terms "primary force" and "secondary force" are used in place of "primary unbalanced inertia force" and "secondary unbalanced inertia force", respectively.

Therefore, the maximum value of the secondary unbalanced force is

$$(F_x^S)_{\max} = \omega^2[\lambda_l r_l (m_{rec})_l + \lambda_u r_u (m_{rec})_u]$$

$$= (18.84)^2[0.25 \times .25 \times 150 + 0.0835 \times .167 \times 225]$$

$$= 4370.8 \text{ N}$$

and this maximum occurs when

$$\cos 2\theta = \pm 1,$$

$$2\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots, \qquad \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \ldots,$$

i.e., whenever the crank is either horizontal or vertical.

The complete balancing of a single-cylinder engine is extremely difficult because of the practical difficulties in constructing it. However, a scheme to achieve this is shown in Fig. 7.23. Here, the connecting rod is extended beyond the crank pin A and a suitable mass is attached at C so that the combined CG of the reciprocating masses, the extension of the connecting rod, and the mass at C is at A. Then, the crank is extended beyond O and a mass is attached at D in order to bring the resultant CG of the whole system at O. Thus, the CG of the whole system is stationary at O and no inertia forces (i.e., the forces causing unbalance) will be generated. The reader can now easily realize the serious constructional difficulties involved in incorporating the modifications we have explained.*

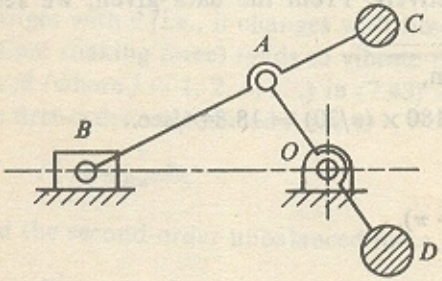


FIGURE 7.23

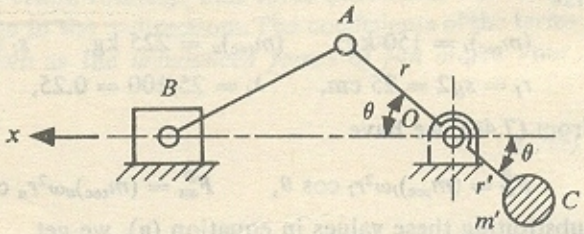


FIGURE 7.24

In view of the constructional difficulties we have outlined, attempts can be made to achieve a partial balancing following the method we shall now discuss. Let the crank be extended beyond O and a mass m' be attached at a radius r' (Fig. 7.24) so that $m'r' = m_{rec}r$. As can be seen, the component, along the line of reciprocation, of the centrifugal force of m' will always neutralize the primary unbalanced force. However, it should be noted that, in this process, an inertia force $m'r'\omega^2 \sin\theta$ is generated in the direction perpendicular to the line of reciprocation. Therefore, a compromise is made so that only a part of the original primary force is balanced

*The inertia moment still remains unbalanced (see Section 7.7).

and an unbalanced force equal to this part is created in the direction perpendicular to the line of reciprocation. Depending on the stiffness characteristics of the supporting structure, the application of the method we have given may lead to a satisfactory balancing of the engine. Thus, for a partial balancing, $m'r' = \chi m_{rec} r$, where $0 < \chi < 1$.

IN-LINE ENGINES

The cylinder centrelines of an in-line engine are parallel and lie on the same plane. In such an engine, a common crankshaft is driven by all the connecting rods. Let the spacing (a) between the adjacent centrelines be uniform. Also, assume that the angular interval between the adjacent cranks (sequenced in the order of firing) is uniform. This angular interval depends on the operating cycle and the number of cylinders. For a two-stroke cycle, this interval is $2\pi/n$, and for a four-stroke cycle, it is $4\pi/n$, where n is the number of cylinders.

To study the inertia forces and moments of an in-line engine, we shall first set up a right-handed coordinate system xyz, where the x-axis is parallel to the line of reciprocation and the z-axis is along the axis of rotation of the common crankshaft with the origin at the midpoint of the engine. Figure 7.25 shows a four-cylinder in-line engine operating on a two-stroke cycle. In

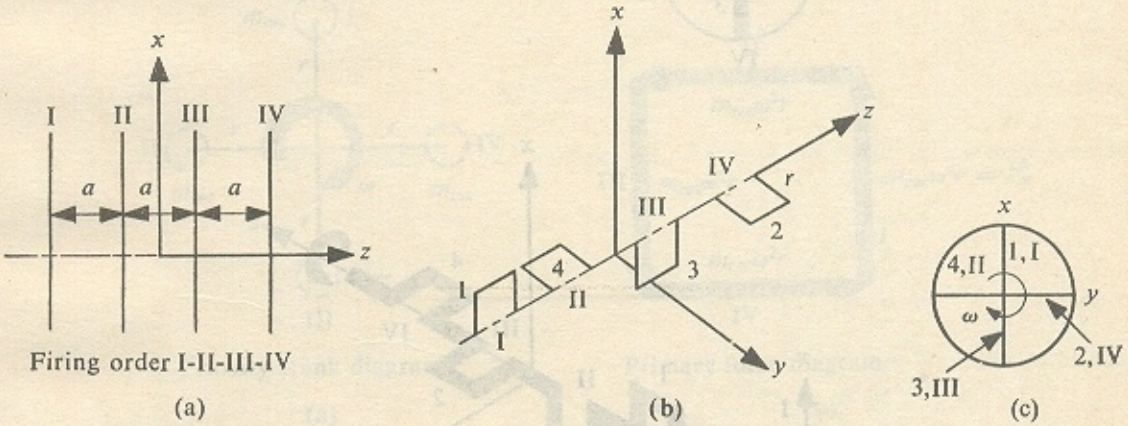


FIGURE 7.25 (r, crank radius; a, cylinder spacing; ω, angular speed)

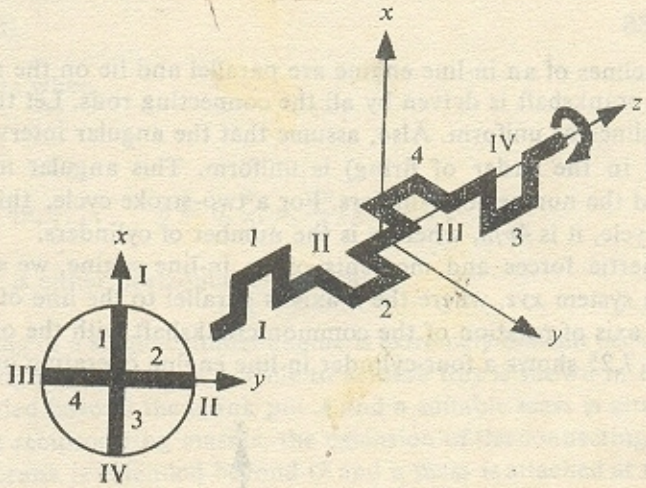
Figs. 7.25a and 7.25b, it should be noted that, starting from the left end, the cylinders are sequentially numbered as I-II-III-IV. The cranks (see Fig. 7.25c) are, however, numbered in the order of firing, i.e., as 1-4-3-2. Such a diagram is commonly known as a crank diagram. The firing order of the engine is I-II-III-IV. The firing order is always specified in terms of the cylinder numbers. The cranks, on the other hand, are always numbered in the order of firing, e.g., $1, n, n-1, \ldots, 2, 1, \ldots$ for an n-cylinder engine. Figure 7.26 shows two other crank arrangements, each with a different firing order, of the two-stroke-cycle, four-cylinder engine given in Fig. 7.25. It should be noted that the crank numbering 1-4-3-2 is not altered *in the crank diagram* when the firing order changes. This method of numbering the cranks is very convenient for expressing their angle of inclination to the x-axis (measured along the direction of crank rotation). Thus, for the K-th crank,

$$\theta_K = \theta + (K-1)(2\pi/n) \qquad \text{(for a two-stroke engine)},$$

$$\theta_K = \theta + (K - 1)(4\pi/n) \qquad \text{(for a four-stroke engine)},$$

where θ represents the angle of crank 1 (always belonging to cylinder I) from the x-axis.

In each cylinder, the only effect of the unbalance is the inertia force parallel to the x-direction.



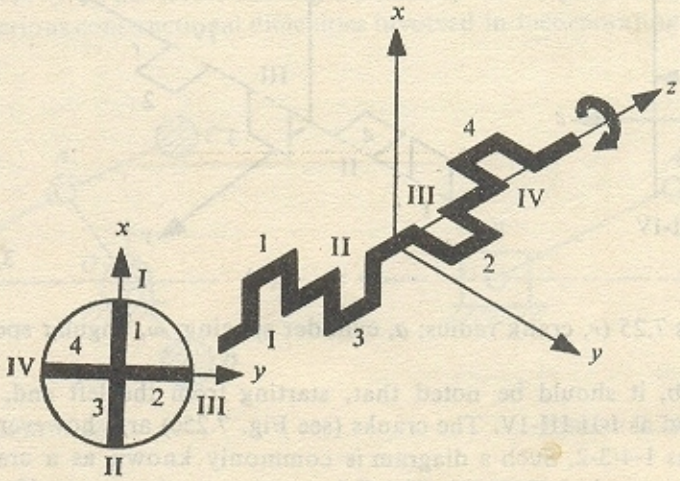
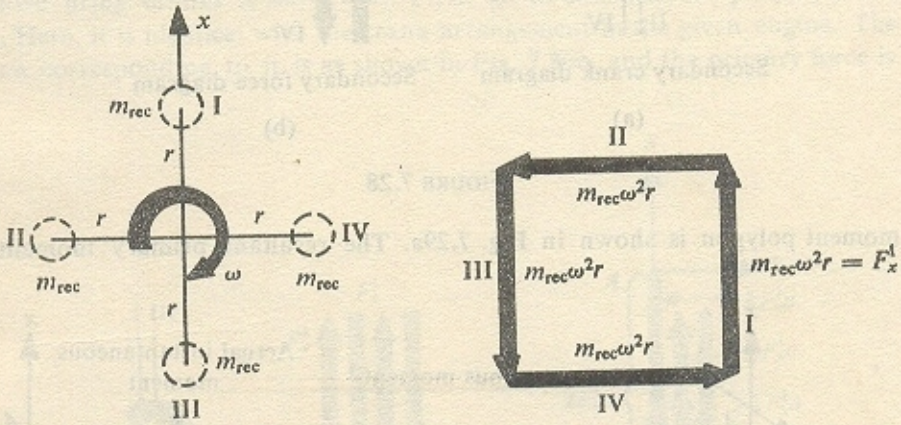


FIGURE 7.26

Thus, in the entire engine, the resultant effect consists of (i) an unbalanced force in the x-direction, and (ii) a moment about the y-axis (commonly referred to as the pitching moment). The magnitudes of these quantities and their dependence on the firing order have to be determined. As we shall subsequently see, only the pitching moment depends on the firing order. Therefore, if the firing order is suitably chosen, the state of balancing of an engine may improve. This method is known as *passive balancing*. The balancing can also be achieved by the addition of external masses. This technique is called *active balancing*. For a discussion on the methods of

active balancing, see Section 7.8. As in rotary balancing, so too in engine balancing, both the graphical and analytical procedures can be used. Of these, we shall first take up the graphical procedure.

Referring to (7.46), we find that the primary unbalanced force in each cylinder *can be thought of* as the x-component of the centrifugal force that would have developed if m_{rec} were placed at the crank pin corresponding to the cylinder. So, the resultant primary force can be determined by taking the x-component of the resultant centrifugal force vector. The maximum value of this primary force, as we can easily see, will be equal to the resultant centrifugal force. Thus, the problem reduces to that of balancing a series of discrete rotating masses. Consider the two-stroke-cycle, four-cylinder engine along with the crank diagram and other details shown in Fig. 7.25. By (hypothetically) putting m_{rec} at each crank pin (see Fig. 7.27a), the resultant centrifugal force diagram will be as in Fig. 7.27b. This force vector diagram rotates with the cranks, but as the resultant magnitude is zero, the resultant primary force also is zero.



Primary crank diagram (a) Primary force diagram (b)

FIGURE 7.27

Rewriting (7.47), we find the secondary force of each cylinder is

$$F_x^S = \frac{m_{rec}}{4}(2\omega)^2 \cdot rA_2 \cos 2\theta.$$

This, then, can be considered as the x-component of the centrifugal force developed by a mass $(A_2 m_{rec}/4)$ placed at the end of a hypothetical crank of radius r but rotating at a speed 2ω. To investigate the secondary unbalanced force of the engine, we draw the *secondary crank* diagram as done in Fig. 7.28a. Here, the secondary crank of each cylinder is so positioned that the angle it makes with the x-axis is twice the angle made by the primary crank corresponding to it. All angles are measured in the direction of rotation of the crankshaft. The secondary force diagram corresponding to Fig. 7.28a is shown in Fig. 7.28b. As we can make out from this figure, the engine is balanced so far as the secondary force is concerned.

Following the same procedure as we adopted in Section 7.4 for drawing a couple (moment)

polygon, we can draw the primary couple polygon, taking xy as the reference plane. The primary

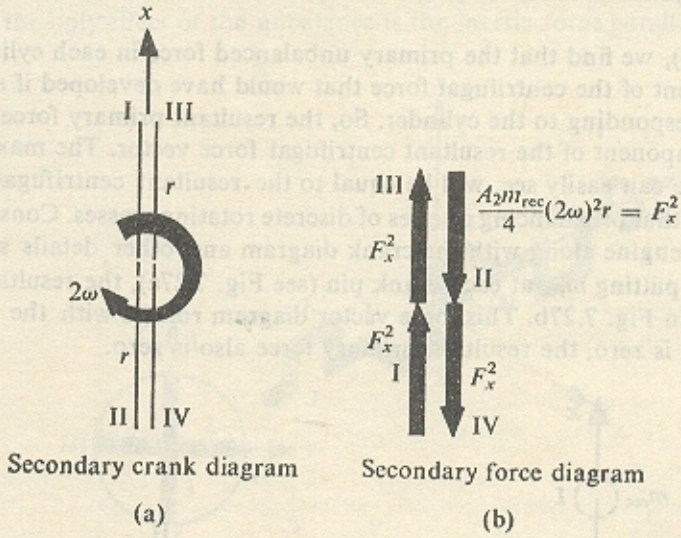


FIGURE 7.28

pitching moment polygon is shown in Fig. 7.29a. The resultant primary moment AB has a

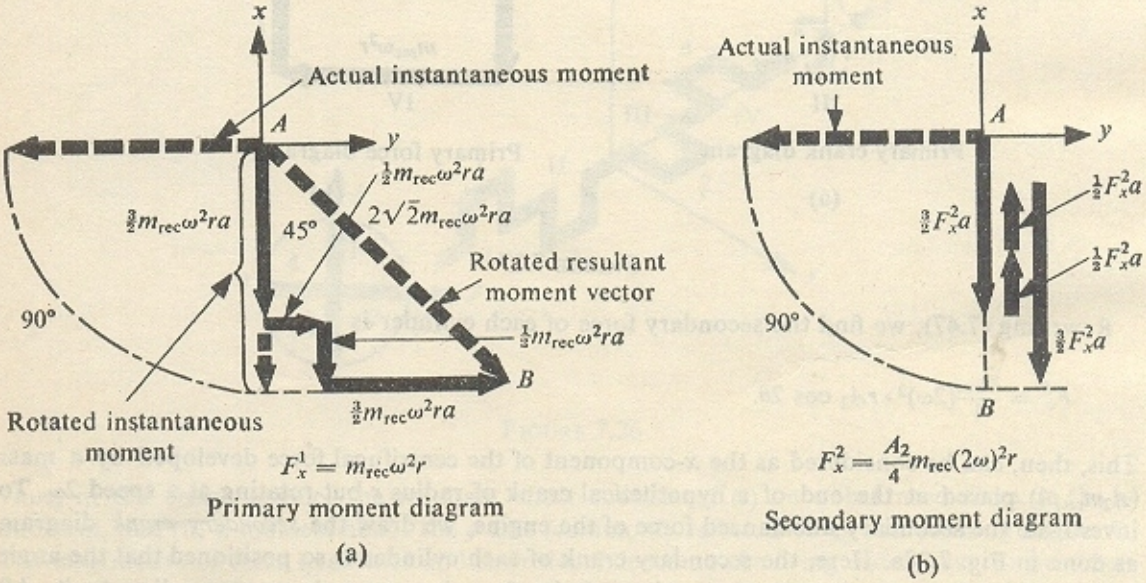


FIGURE 7.29

maximum value $2\sqrt{2}\, m_{rec}\omega^2 ra$. However, at the instant crank 1 is along the x-direction, the primary moment is given by the projection of AB along the x-axis (according to the convention

followed in this construction). The instantaneous moment then has a value $-2m_{rec}\omega^2 ra$. Since the moment polygon rotates with the cranks, the maximum primary moment, i.e., $\pm 2\sqrt{2}\, m_{rec}\omega^2 ra$, occurs when crank 1 is at 225° and 45°, respectively, with the x-axis.

Similarly, using the secondary crank diagram, the secondary moment polygon can be drawn as in Fig. 7.29b. The maximum values of the secondary moment are $\pm 2A_2 m_{rec}\omega^2 ra$ and these occur when *the secondary crank 1* is at 180° and 0°, respectively, with the x-axis. This implies that the maximum secondary moment occurs alternately with the positive and negative values when crank 1 is at 90°, 180°, 270°, and 0° with the x-axis.

PROBLEM 7.5 Investigate the state of primary and secondary balancing of a four-stroke-cycle four-cylinder engine with a firing order I-II-III-IV. What will be the change in this state when the firing order is altered to I-II-IV-III?

SOLUTION In a four-stroke-cycle four-cylinder in-line engine, the angular interval between the consecutive firing cranks is $4\pi/4\ (=\pi)$. First, let us construct the primary crank diagram (Fig. 7.30a). Here, it is identical with the crank arrangement in the given engine. The primary force polygon corresponding to it is as shown in Fig. 7.30b, and the primary force is found to

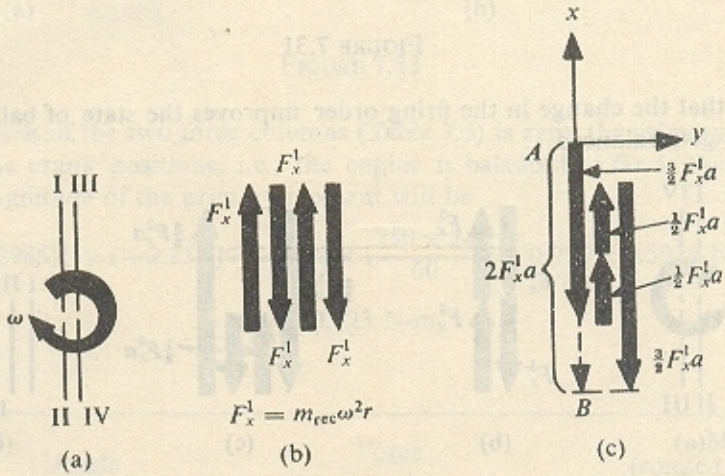


FIGURE 7.30

be balanced. The primary moment polygon (Fig. 7.30c) is drawn following the usual conventions, and the maximum values of the primary moment are found to be $\pm 2m_{rec}\omega^2 ra$. These values occur when crank 1 is at 180° and 0°, respectively.

The secondary crank diagram is drawn as in Fig. 7.31a. As we can see, the maximum values of the resulting unbalanced secondary force are $\pm 4A_2 m_{rec}\omega^2 r$ (Fig. 7.31b), and these occur when crank 1 is at 0° and 90°, respectively. After the secondary moment polygon (Fig. 7.31c) is drawn, it is found to be closed, implying the secondary moment is zero.

When the firing order is altered to I-II-IV-III, the corresponding primary crank, force, and moment diagrams would be as in Figs. 7.32a, 7.32b, and 7.32c, respectively. As we can notice, the engine has complete primary balancing. The secondary crank diagram (Fig. 7.32d) is

identical with that in Fig. 7.31a. So, the state of secondary balancing remains unchanged.

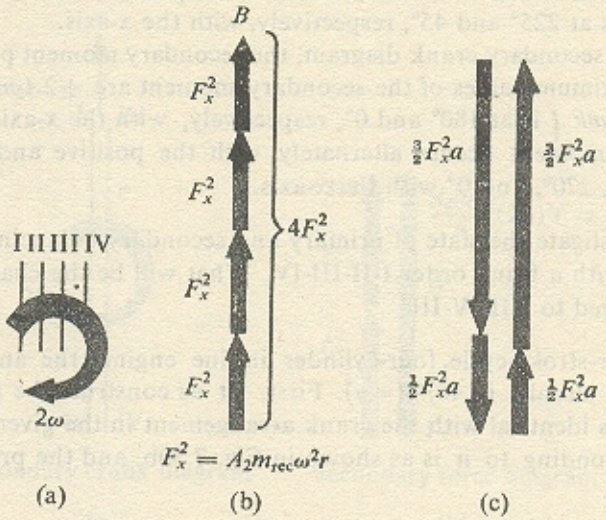


FIGURE 7.31

Thus, we find that the change in the firing order improves the state of balancing because the primary moment gets balanced.

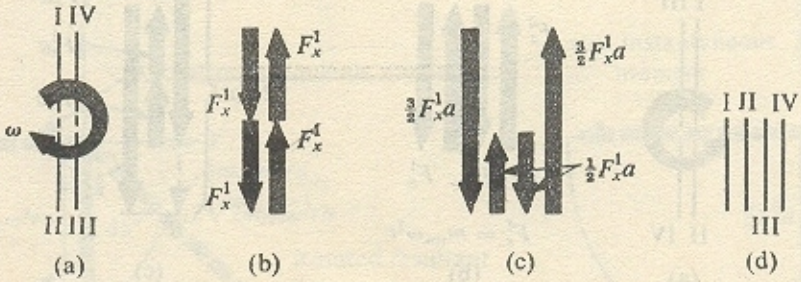


FIGURE 7.32

An analytical procedure, similar to that in Section 7.4 for rotary balancing, can also be used for an in-line engine. The problem that follows illustrates this.

PROBLEM 7.6 The firing order of a two-stroke-cycle six-cylinder diesel engine is I-V-III-VI-II-IV. The adjacent cylinder centrelines are 75 cm apart. Each cylinder has a connecting rod 120 cm long and a stroke 45 cm. The mass of the reciprocating parts in each cylinder is 200 kg. Determine the magnitudes of the primary and secondary unbalanced forces and moments when the engine runs at a constant speed of 250 rpm.

SOLUTION As mentioned at the beginning of this section, we shall assume that the cranks are so designed that the rotary unbalance has been taken care of. With the given data, the

angular interval between adjacent firing cranks $= 2\pi/6$ rad $= 60°$, $r = 45/2$ cm $= 0.225$ m, and $A_2 \approx \lambda = r/l = 0.225/1.2 = 0.188$. Then, the crank diagram and cylinder arrangement will be as shown in Fig. 7.33. In this figure, 1, 2, 3, . . . refer to the crank numbers, whereas I, II, III, . . . refer to the cylinder numbers. To study the state of primary unbalance, we shall, using the given particulars, prepare the data as in Table 7.5. The term $m_{rec}\omega^2 r$, since it is common to all force and moment expressions, has been excluded from the table.

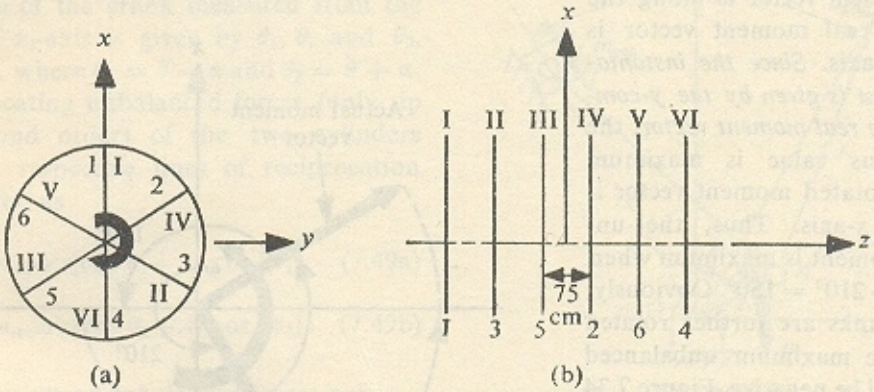


FIGURE 7.33

As the total of each of the two force columns (Table 7.5) is zero, the primary force is balanced independent of the crank positions, i.e., the engine is balanced so far as the primary force is concerned. The magnitude of the primary moment will be

$$m_{rec}\omega^2 r[(-1.2985)^2 + (-2.25)^2]^{1/2} = 200 \times (\frac{250 \times 2\pi}{60})^2 \times 0.225 \times 2.5978 \text{ N-m}$$

$$= 80{,}123 \text{ N-m.}$$

TABLE 7.5

Crank number K	Distance from origin z_K (m)	Angle θ_K (degrees)	Force: y-component $\sin\theta_K$	Force: x-component $\cos\theta_K$	Moment (rotated through 90°): y-component $z_K \sin\theta_K$	Moment (rotated through 90°): x-component $z_K \cos\theta_K$
1	−1.875	0	0	1	0	−1.875
2	0.375	60	0.866	0.5	0.325	0.1875
3	−1.125	120	0.866	−0.5	−0.97425	0.5625
4	1.875	180	0	−1	0	−1.875
5	−0.375	240	−0.866	−0.5	0.325	0.1875
6	1.125	300	−0.866	0.5	−0.97425	0.5625
		Total	0	0	−1.2985	−2.25

The inclination of this rotated moment vector with the x-axis *at the instant considered* (i.e., $\theta_1 = 0$) is given by

$$\psi = \tan^{-1}\left(\frac{\Sigma\, z_K \sin\theta_K}{\Sigma\, z_K \cos\theta_K}\right) = \tan^{-1}\left(\frac{-1.2985}{-2.25}\right) = 30^\circ \text{ or } 210^\circ.$$

As both the components of the moment are negative, $\psi = 210^\circ$ is the correct solution. When the rotated moment vector is along the x-axis, the real moment vector is along the y-axis. *Since the instantaneous moment is given by the y-component of the real moment vector*, the instantaneous value is maximum when the rotated moment vector is along the x-axis. Thus, the unbalanced moment is maximum when $\theta_1 = 360^\circ - 210^\circ = 150^\circ$. Obviously, after the cranks are further rotated by 180°, the maximum unbalanced moment will be negative. Figure 7.34 illustrates the explanation we have given.

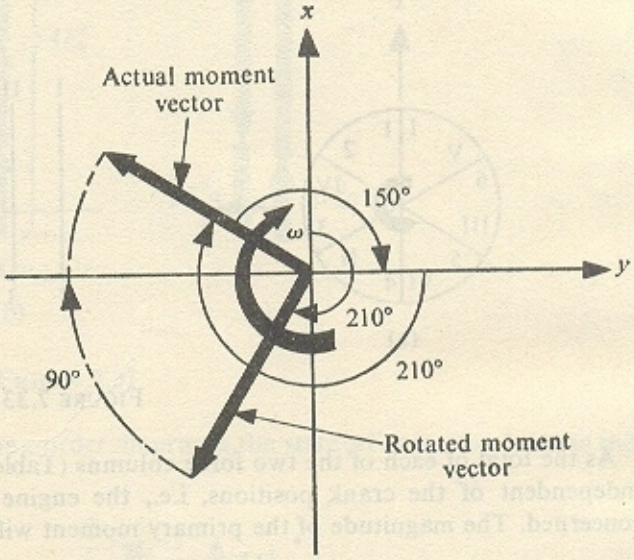


FIGURE 7.34

To investigate the state of secondary unbalance, we shall use Table 7.6. As before, the term $\frac{1}{4}A_2 m_{rec}(2\omega)^2 \cdot r$ is dropped from the table. It is obvious from this table that the engine possesses complete secondary balancing.

TABLE 7.6

Crank number K	Distance from origin z_K (m)	Secondary crank angle $2\theta_K$ (degrees)	Force: y-component $\sin 2\theta_K$	Force: x-component $\cos 2\theta_K$	Moment: y-component $z_K \sin 2\theta_K$	Moment: x-component $z_K \cos 2\theta_K$
1	−1.875	0	0	1	0	−1.875
2	0.375	120	0.866	−0.5	0.325	−0.1875
3	−1.125	240	−0.866	−0.5	0.97425	0.5625
4	1.875	0	0	1	0	1.875
5	−0.375	120	0.866	−0.5	−0.325	0.1875
6	1.125	240	−0.866	−0.5	−0.97425	−0.5625
		Total	0	0	0	0

V-TWIN ENGINES

A V-twin engine consists of two identical cylinders lying in the xy-plane such that the angle between their centrelines ox_1 and ox_2 (known as the V-angle) is 2α and it is bisected by the x-axis (Fig. 7.35). Also, it has two identical connecting rods attached to a single crank (OA). The rotation of the crank measured from the x_1-, x-, and x_2-axis is given by θ_1, θ, and θ_2, respectively, where $\theta_1 = \theta - \alpha$ and $\theta_2 = \theta + \alpha$. The reciprocating unbalanced forces (only up to the second order) of the two cylinders along their respective lines of reciprocation can be written as

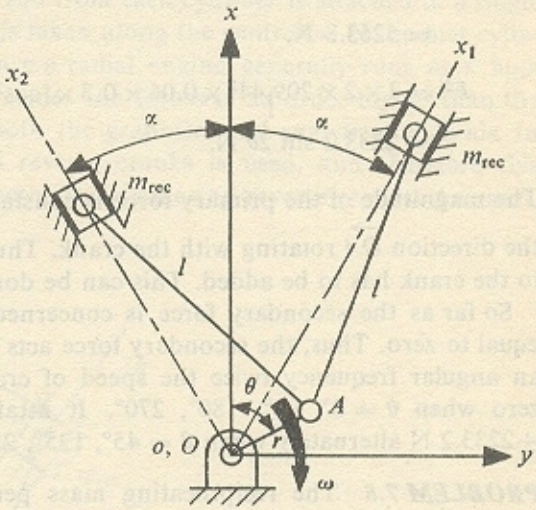


FIGURE 7.35

$$F_{x_1} = m_{rec}\omega^2 r[\cos\theta_1 + A_2\cos 2\theta_1], \quad (7.49a)$$

$$F_{x_2} = m_{rec}\omega^2 r[\cos\theta_2 + A_2\cos 2\theta_2]. \quad (7.49b)$$

Hence, the resultant unbalanced forces are

$$F_x = (F_{x_1} + F_{x_2})\cos\alpha, \qquad F_y = (F_{x_1} - F_{x_2})\sin\alpha.$$

Substituting for F_{x_1}, F_{x_2}, θ_1, and θ_2 in these expressions, we get

$$F_x = m_{rec}\omega^2 r[\cos(\theta - \alpha) + \cos(\theta + \alpha) + A_2\{\cos 2(\theta - \alpha) + \cos 2(\theta + \alpha)\}]\cos\alpha,$$

$$F_y = m_{rec}\omega^2 r[\cos(\theta - \alpha) - \cos(\theta + \alpha) + A_2\{\cos 2(\theta - \alpha) - \cos 2(\theta + \alpha)\}]\sin\alpha.$$

On simplification, we have

$$F_x = 2m_{rec}\omega^2 r\cos\alpha(\cos\alpha\cos\theta + A_2\cos 2\alpha\cos 2\theta), \quad (7.50a)$$

$$F_y = 2m_{rec}\omega^2 r\sin\alpha(\sin\alpha\sin\theta + A_2\sin 2\alpha\sin 2\theta). \quad (7.50b)$$

Separating the primary and secondary parts, we can express the resultant forces as

$$F^P \text{ (primary force)} = (F_x^{P^2} + F_y^{P^2})^{1/2} = 2m_{rec}\omega^2 r[\cos^2\theta\cos^4\alpha + \sin^2\theta\sin^4\alpha]^{1/2}, \quad (7.51a)$$

$$F^S \text{ (secondary force)} = (F_x^{S^2} + F_y^{S^2})^{1/2} = 2m_{rec}\omega^2 r A_2[\cos^2 2\theta\cos^2\alpha\cos^2 2\alpha + \sin^2 2\theta\sin^2\alpha\sin^2 2\alpha]^{1/2}. \quad (7.51b)$$

PROBLEM 7.7 Determine the maximum and minimum values of the primary and secondary forces for a 90° V-twin engine. Also, indicate the crank positions corresponding to these values. Given m_{rec} per cylinder = 2 kg, stroke = 12 cm, connecting-rod length = 20 cm, and engine speed = 2000 rpm.

SOLUTION From the given data,

$$r = 6 \text{ cm} = 0.06 \text{ m}, \qquad l = 0.2 \text{ m}, \qquad \alpha = 45°,$$

$$\omega = \frac{2000 \times 2\pi}{60} = 209.44 \text{ c/sec}, \qquad A_2 \approx \lambda = \frac{r}{l} = 0.3.$$

Substituting these values in (7.51a) and (7.51b), we get

$$F^P = 2 \times 2 \times 209.44^2 \times 0.06 \times [\cos^2 \theta \cdot \tfrac{1}{4} + \sin^2 \theta \cdot \tfrac{1}{4}]^{1/2} \text{ N}$$

$$= 5263.8 \text{ N},$$

$$F^S = 2 \times 2 \times 209.44^2 \times 0.06 \times 0.3 \times [\cos^2 2\theta \times \tfrac{1}{2} \times 0 + \sin^2 2\theta \times \tfrac{1}{2} \times 1]^{1/2} \text{ N}$$

$$= 2233.2 \sin 2\theta \text{ N}.$$

The magnitude of the primary force is constant and the force can be considered as a vector in the direction $\overrightarrow{OA}$ rotating with the crank. Thus, to balance this force, a suitable mass opposite to the crank has to be added. This can be done very easily.

So far as the secondary force is concerned, it is observed that the x-component is always equal to zero. Thus, the secondary force acts along the y-axis and fluctuates harmonically with an angular frequency twice the speed of crank rotation. The secondary unbalanced force is zero when $\theta = 0°, 90°, 180°, 270°$. It attains the maximum positive and negative values ± 2233.2 N alternately when $\theta = 45°, 135°, 225°, 315°$.

PROBLEM 7.8 The reciprocating mass per cylinder in a 60° V-twin engine is 1.2 kg. The stroke and the connecting-rod length are 10 cm and 25 cm, respectively. If the engine runs at 2000 rpm, determine the maximum and minimum values of the primary and secondary forces. Also, find out the crank positions corresponding to these values.

SOLUTION Using the given data, we have

$$r = 5 \text{ cm} = 0.05 \text{ m}, \qquad l = 0.25 \text{ m}, \qquad \alpha = 30°,$$

$$\omega = \frac{2000 \times 2\pi}{60} = 209.44 \text{ c/sec}, \qquad A_2 \approx \lambda = \frac{r}{l} = 0.2.$$

Substituting these values in (7.51a) and (7.51b), we obtain

$$F^P = 2 \times 1.2 \times 209.44^2 \times 0.05 \times (\cos^2 \theta \times 0.866^4 + \sin^2 \theta \times 0.5^4)^{1/2} \text{ N}$$

$$= 5263.8 \times (0.562 \cos^2 \theta + 0.0625 \sin^2 \theta)^{1/2} \text{ N},$$

$$F^S = 2 \times 1.2 \times 209.44^2 \times 0.05 \times 0.2(\cos^2 2\theta \times 0.866^2 \times 0.5^2 + \sin^2 2\theta \times 0.5^2 \times 0.866^2)^{1/2} \text{ N}$$

$$= 455.9 \text{ N}.$$

Thus, the secondary force is constant in magnitude and acts at an angle 2θ from the x-axis. The primary force is minimum when $\theta = 90°, 270°$ and maximum when $\theta = 0°, 180°$ as we shall now see. Rewriting F^P as

$$F^P = 5263.8 \times (0.4995 \cos^2 \theta + 0.0625)^{1/2},$$

we see it reaches the maximum magnitude when $\cos \theta = \pm 1$, i.e., $\theta = 0°, 180°$. Similarly, for $\theta = 90°, 270°$, F^P is minimum. The maximum and minimum values of F^P can be found out to be 3946.1 N and 1316 N, respectively.