Foliations of the 3-Sphere

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Abstract

Our goal is to prove a result of William Thurston's, namely that there are uncountably many noncobordant foliations of S^3 [10]. We begin with the definition of a foliation, but quickly find difficulty proving that a decomposition of a manifold is, in fact, a foliation. Thus, we employ the use of the Frobenius Integrability Theorem, which allows us to define foliations with smooth vector fields. Then it is possible to formulate a dual theorem which allows us to define foliations with differential forms. Differential forms allow us to easily check whether or not a decomposition is a foliation. Furthermore, certain DeRahm cohomology classes defined using forms are invariant of cobordism. We can make use of these cohomology classes to exhibit a correspondence between noncobordant foliations of S^3 and an uncountable set.

1 Introduction

In the 1930's, H. Hopf posed a question equivalent to the following:

Does there exist a 2-dimensional foliation of S^3?

G. Reeb, in the 1940's, answered Hopf's question in the affirmative by constructing smooth foliations (called Reeb components) of two solid tori; the tori were then glued along their boundaries to produce a smooth foliation of S^3 [2]. We will exhibit Reeb's construction in the following section. Following Reeb's work, a general question was posed:

Does every foliation of S^3 have a compact leaf?

In 1965, S.P. Novikov answered in the affirmative and, in fact, proved a deeper result: that every foliation of S^3 must contain a compact leaf homeomorphic to a 2-dimensional torus and further, within this torus leaf there must be a Reeb component [2]. Finally, in 1972, W. Thurston published an astonishing discovery [10]:

There are uncountably many nonequivalent smooth foliations of S^3.

Our goal is to build up to, and prove, Thurston's result. In order to so, we will develop an efficient method to define foliations. Furthermore, we will develop a notion of equivalence for foliations. Finally, we will introduce an invariant of 'equivalent' foliations - namely, the Godbillon-Vey invariant. Such an invariant will allow us to prove Thurston's theorem.

A basic knowledge of differential geometry and smooth manifolds will be assumed (cf. Spivak [7]). The reader will, however, be occasionally reminded of important definitions and notation.

2 Foliation Theory

2.1 A Quick Recollection of Smooth Manifolds

The following is adapted from Spivak [7]. A metric space M is a n-manifold if for each $p \in M$ there exists a neighborhood U of p and some integer $n \geq 0$ such that U is homeomorphic to an open set in $\mathbb{R}^n$. Note that, for our purposes, $\mathbb{R}^n$ will always have the standard euclidean metric.

If U and V are open subsets of M, then the homeomorphisms $x : U \to x(U) \subset \mathbb{R}^n$ and $y : V \to y(V) \subset \mathbb{R}^n$ are smooth-related if the maps

$$\begin{aligned} y \circ x^{-1} &: x(U \cap V) \to y(U \cap V) \\ x \circ y^{-1} &: y(U \cap V) \to x(U \cap V) \end{aligned}$$

are smooth (i.e. infinitely differentiable). If there is a collection of mutually smooth-related maps whose domains cover M, then M is called a **smooth manifold**. Furthermore, an element (x, U) in such a collection is a **coordinate system**. Oftentimes, we will refer to the homeomorphism x as the *chart* or *coordinate function*. From now on, 'manifold' will exclusively refer to 'smooth manifold'.

A map of manifolds $F : M \to N$ is **smooth** if, for every coordinate system (x, U) of M and (y, V) of N, the map $y \circ F \circ x^{-1} : x(U) \to y(V)$ is smooth.

A homeomorphism of manifolds $F : M \to N$ is a **diffeomorphism** if both F and F^{-1} are smooth. It is evident that for any coordinate system (x, U), $x : U \to x(U)$ is a diffeomorphism.

2.2 Foliations

Definition 2.1 ([5])**.** *A k-dimensional foliation, $\mathcal{F}$, of an n-dimensional manifold M is a decomposition of M into a collection of disjoint connected subsets $\{\mathcal{L}_\alpha\}_{\alpha \in A}$, called the leaves of the foliation, such that each point $p \in M$ has a coordinate system (x, U) such that for each leaf $\mathcal{L}_\alpha$, the connected components of $U \cap \mathcal{L}_\alpha$ are*

$$\left\{q \in U : x^{k+1}(q) = c^{k+1}, \ldots, x^n(q) = c^n\right\},$$

where $c^{k+1}, \ldots, c^n$ are constants.

We would like for diffeomorphisms to 'preserve' foliations. Let $F : M \to N$ be diffeomorphism of n-manifolds and let the collection $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ be a k-dimensional foliation of M. For each $p \in N$, $F^{-1}(p) \in M$ has a coordinate system (x, U) satisfying the definition of a foliation. Let $(x \circ F^{-1}, F(U))$ be a

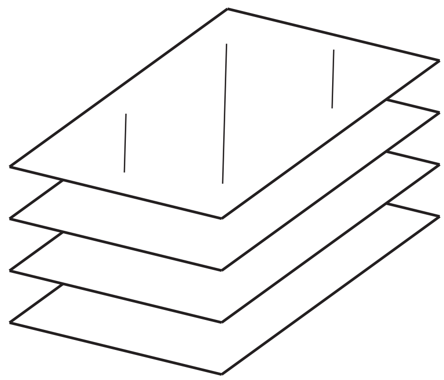

Figure 1: The trivial foliation of $\mathbb{R}^3$.

coordinate system about p. It is then evident that the collection $\{F(\mathcal{L}_\alpha)\}_{\alpha\in A}$ defines a foliation of N.

We now present a few examples of foliations.

2.2.1 The "Stack of Sheets" Foliation

We will present a trivial 2-dimensional foliation of $\mathbb{R}^3$: allow a leaf to be the set

$$\mathcal{L}_\alpha = \left\{(x, y, z) \in \mathbb{R}^3 : z = \alpha\right\},$$

where $\alpha \in \mathbb{R}$. The collection $\{\mathcal{L}_\alpha\}_{\alpha\in\mathbb{R}}$ forms the foliation, which looks like a "stack of sheets" (see Figure 1). It is obvious that this is a 2-dimensional foliation: we simply choose the coordinate system $(id, \mathbb{R}^3)$ about each point $p \in \mathbb{R}^3$.

In general, the collection of leaves

$$\left\{x \in \mathbb{R}^n : x^{k+1} = \alpha^{k+1}, \ldots, x^n = \alpha^n\right\},$$

where $\alpha^{k+1}, \ldots \alpha^n \in \mathbb{R}$ are constants, gives the trivial k-dimensional foliation of $\mathbb{R}^n$. The definition of a foliation essentially states the following: if M is a foliated manifold, there exist coordinate systems about each point $p \in M$ such that locally the leaves are "trivial" in $\mathbb{R}^n$.

2.2.2 Foliating the 3-Sphere

Although the preceding foliation is simple, a sagacious reader may sense the impending difficulty of generally proving the existence of foliations. Take, for example, the 3-Sphere, which is the 3-manifold

$$S^3 = \left\{x \in \mathbb{R}^4 : \|x\| = 1\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm. We can see the difficulty in constructing a foliation of S^3: we must work in $\mathbb{R}^4$. We will therefore find a way to circumvent work in $\mathbb{R}^4$.

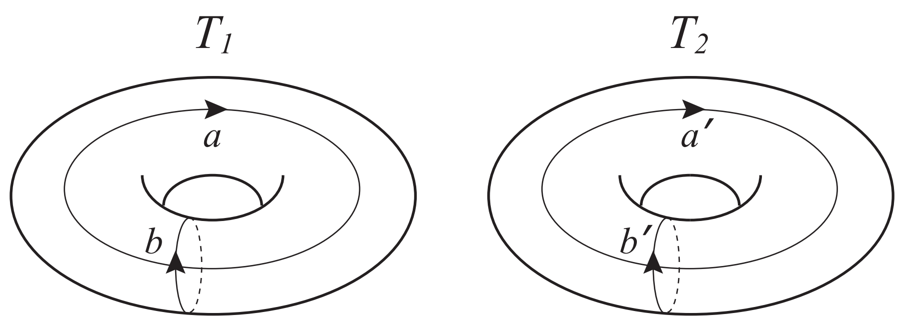

Figure 2: Loops of the form a (meridial loops) on T_1 are identified with loops of the form b' (longitudinal loops) on T_2 and loops of the form b (longitudinal loops) on T_1 are identified with loops of the form a' (meridial loops) on T_2.

2.2.3 Constructing S^3 via gluing solid tori

We can show that S^3 is diffeomorphic to two solid tori glued smoothly along their boundaries. Referring to Figure 2, our map will identify loops of the form a (loops through the longitude) on T_1 with loops of the form b' (loops through the meridian) on T_2. Similarly, loops of the form b on T_1 will be identified with loops of the form a' on T_2. The following proof of this fact is taken from Massey [6].

We will choose subsets A and B of S^3 in a clever fashion:

$$\begin{aligned} A &= \left\{x \in S^3 : \|(x^1, x^2)\|^2 \leq \|(x^3, x^4)\|^2\right\} \quad \text{and} \\ B &= \left\{x \in S^3 : \|(x^1, x^2)\|^2 \geq \|(x^3, x^4)\|^2\right\}. \end{aligned}$$

The intersection,

$$A \cap B = \left\{x \in S^3 : \|(x^1, x^2)\|^2 = \|(x^3, x^4)\|^2 = \tfrac{1}{2}\right\},$$

is diffeomorphic to the empty torus, $S^1 \times S^1 = T^2$. We then claim that A and B are each diffeomorphic to the solid torus, $D^2 \times S^1$. Note that we will use the 2-disk and 1-sphere with radius $\sqrt{\frac{1}{2}}$,

$$\begin{aligned} D^2 &= \left\{(x^1, x^2) \in \mathbb{R}^2 : \|(x^1, x^2)\|^2 \leq \tfrac{1}{2}\right\} \quad \text{and} \\ S^1 &= \left\{(x^3, x^4) \in \mathbb{R}^2 : \|(x^3, x^4)\|^2 = \tfrac{1}{2}\right\}. \end{aligned}$$

We define $f_1 : D^2 \times S^1 \to A$ and $f_2 : D^2 \times S^1 \to B$ as follows: given $(x^1, x^2) \in D^2$ and $(x^3, x^4) \in S^1$, let

$$\begin{aligned} f_1(x^1, x^2, x^3, x^4) &= \left(x^1, x^2, ax^3, ax^4\right) \quad \text{and} \\ f_2(x^1, x^2, x^3, x^4) &= \left(x^3, x^4, bx^1, bx^2\right) \end{aligned}$$

where $a = \sqrt{2 - 2\|(x^1, x^2)\|^2}$ and $b = \sqrt{2 - 2\|(x^3, x^4)\|^2}$. Given $x \in A$ or B, our corresponding inverses functions are

$$\begin{aligned} f_1^{-1}(x^1, x^2, x^3, x^4) &= \left(x^1, x^2, \tfrac{1}{c}x^3, \tfrac{1}{c}x^4\right) \quad \text{and} \\ f_2^{-1}(x^1, x^2, x^3, x^4) &= \left(\tfrac{1}{d}x^3, \tfrac{1}{d}x^4, x^1, x^2\right) \end{aligned}$$

where $c = \sqrt{2\|(x^3, x^4)\|^2}$ and $d = \sqrt{2\|(x^1, x^2)\|^2}$. It is straightforward to show that these maps define our desired diffeomorphisms.

Consider, again, the intersection $A \cap B \simeq T^2$. Both f_1^{-1} and f_2^{-1} are defined on $A \cap B$ so, given $x \in A \cap B$,

$$\begin{aligned} f_1^{-1}(x^1, x^2, x^3, x^4) &= (x^1, x^2, x^3, x^4) \quad \text{and} \\ f_2^{-1}(x^1, x^2, x^3, x^4) &= (x^3, x^4, x^1, x^2), \end{aligned}$$

which map to the boundary of $D^2 \times S^1$, i.e. $S^1 \times S^1$. Moreover, we can see that S^3 is, in fact, diffeomorphic to $A \simeq D^2 \times S^1$ and $B \simeq D^2 \times S^1$ identified along their boundaries. Furthermore, our previous representation of the gluing was correct: the maps f_1^{-1} and f_2^{-1} identify the meridian circles of one torus' boundary to corresponding parallel circles of the other's boundary, and vice-versa.

2.2.4 The Reeb Foliation

To motivate the following construction, assume that we have foliations on two solid tori. As long as these foliations each have T^2 as their "boundary leaf", we can guarantee that we will have a foliation of S^3 via our previous gluing construction. In fact, it is a theorem of Novikov's that every foliation of S^3 must contain a torus leaf [3]. Thus, our task has been reduced to foliating $D^2 \times S^1$ such that it has a torus leaf on it's boundary.

We will construct the Reeb foliation of $D^2 \times S^1$. Firstly, assume cylindrical coordinates (r, θ, z) of $\mathbb{R}^3$ where $r \in [0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be a smooth function such that $f(0) = 0$ and $\lim_{r \to 1} f(r) = +\infty$. Then let

$$\mathcal{L}_\alpha = \{(r, \theta, z) : 0 \leq r < 1 \text{ and } z = f(r) + \alpha\}$$

for $\alpha \in \mathbb{R}$ and let

$$\mathcal{L}_{T^2} = \{(r, \theta, z) : r = 1\}.$$

Our foliation is then the collection $\{\mathcal{L}_{T^2}, \{\mathcal{L}_\alpha\}_{\alpha \in \mathbb{R}}\}$ which we think of as a stack of cups contained within a cylinder (see Figure 3). If we restrict the leaves to $z \in [0, 1]$, we have a foliated bounded solid cylinder; then we glue the ends of the cylinder to obtain a foliation of the solid torus (see Figure 3), called the **Reeb foliation**. Proving that this is a foliation, however, takes longer than we would like. We will therefore develop tools to aid in this endeavor.

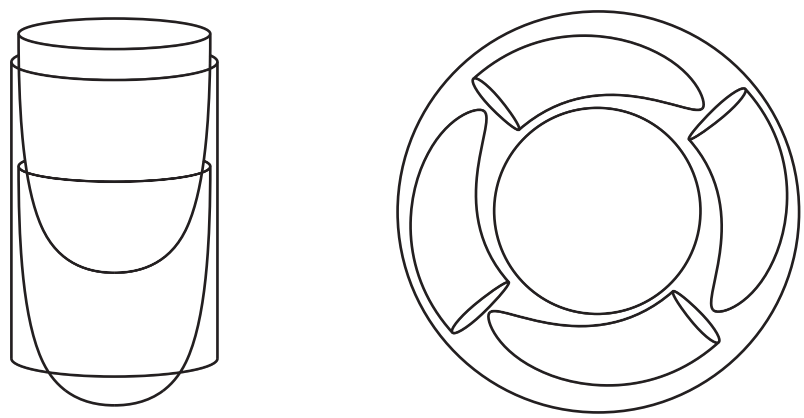

Figure 3: On the left, we have "stacked cups" with an outer cylinder. On the right, we have the Reeb foliation of the solid torus.

3 Foliations Defined by Distributions

3.1 A Recollection of Tangent Bundles and Vector Fields

Before we begin the proceeding discussion, we must recall a number of background facts (taken from Spivak [7]). For the following, let M be an n-dimensional manifold, and let (x, U) be a coordinate system about a point $p \in M$. We denote $C^\infty(M)$ to be the space of smooth functions $f : M \to \mathbb{R}$. The **tangent space** of M at p, denoted T_pM, is the set of all mappings $X_p : C^\infty(M) \to \mathbb{R}$ (called **tangent vectors** at p) such that for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

i. $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$

ii. $X_p(fg) = g(p)(X_p f) + f(p)(X_p g)$.

Moreover, T_pM is a vector space over $\mathbb{R}$.

We let the **tangent bundle** of M be the space

$$TM = \bigcup_{p \in M} T_pM,$$

which, we note, is a $2n$-dimensional manifold. Letting $\pi : TM \to M$ be the canonical projection map, we say that $X : M \to TM$ is a **smooth vector field** if X is smooth and $\pi \circ X$ is the identity map.

Furthermore, with respect to the coordinate system (x, U), we define the map

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \to \mathbb{R},$$

such that if $f \in C^\infty(M)$ and $(t^1, \dots t^n)$ are the standard coordinates on $\mathbb{R}^n$ then

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = D_i(f \circ x^{-1})(x(p)),$$

which is the partial derivative of $(f \circ x^{-1}) : \mathbb{R}^n \to \mathbb{R}$ with respect to t^i at $x(p) \in x(U) \subset \mathbb{R}^n$. We then define the map

$$\frac{\partial}{\partial x^i} : M \to TM,$$

such that if $p \in U$, then

$$\frac{\partial}{\partial x^i}(p) = \left.\frac{\partial}{\partial x^i}\right|_p.$$

We note that each $X_p \in T_pM$ is given by the form

$$X_p = \sum_{i=1}^n a_i \left.\frac{\partial}{\partial x^i}\right|_p,$$

where $a_i \in \mathbb{R}$. Similarly, each smooth vector field X is locally given by the form

$$X = \sum_{i=1}^n f_i \frac{\partial}{\partial x^i},$$

where $f_i \in C^\infty(M)$.

If $F : M \to N$ is a smooth map of manifolds, then there is an induced map $F_* : T_pM \to T_{F(p)}N$, for each $p \in M$ such that if $X_p \in T_pM$ and $f \in C^\infty$, then $(F_*X_p)(f) = X_p(f \circ F)$.

3.2 Generalizing Integral curves

We begin with an example. Consider a 1-dimensional foliation of $\mathbb{R}^2$, with leaves

$$\mathcal{L}_\alpha = \left\{(x, y) \in \mathbb{R}^2 : y = \alpha\right\},$$

for $\alpha \in \mathbb{R}$. Giving ourselves the standard coordinate system on $\mathbb{R}^2$, we can see that each leaf is the image of an *integral curve* (see Spivak [7]) of the smooth vector field

$$X = \frac{\partial}{\partial x}.$$

We can thus see the similarity between the integral curves of X and the leaves of our foliation (see Figure 4). We would like to generalize integral curves. If a single vector field defines an integral curve, we would hope that multiple vector fields would define an *integral manifold*.

In our example, notice that, for $p \in \mathbb{R}^2$, the span of the tangent vector X_p defines a 1-dimensional subspace, say Δ_p, of $T_p\mathbb{R}^2$. We can assign to each $p \in \mathbb{R}^2$ such a subspace by simply using the vector field X. This "smooth" assignment of $p \in \mathbb{R}^2$ to a subspace Δ_p is called a distribution. We can generalize as follows:

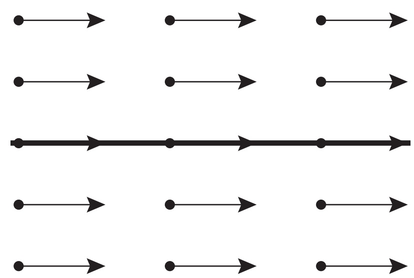

Figure 4: An integral curve of the vector field $X = \frac{\partial}{\partial x}$ in $\mathbb{R}^2$.

Definition 3.1 ([7])**.** *A* k**-dimensional distribution** Δ *on a manifold* M *is a function* $p \mapsto \Delta_p$, *where* $\Delta_p \subset T_pM$ *is a* k*-dimensional subspace of* T_pM*. We say* Δ *is smooth if there exists a neighborhood* U *for every* $p \in M$ *and* k *smooth vector fields* $X_1, \ldots, X_k$ *such that* $X_1(q), \ldots, X_k(q)$ *forms a basis for* Δ_q*, for all* $q \in U$*.*

So, for each $p \in \mathbb{R}^2$, we can think of the subspace Δ_p as being the tangent space of *the integral curve* at p. Further, these subspaces uniquely determine our integral curves. We generalize as follows:

Definition 3.2 ([7])**.** *Given an* n*-dimensional manifold* M*, let* N *be the connected imbedding of a* k*-dimensional manifold.* N *is called an* **integral manifold** *of a* k*-dimensional distribution* Δ *if for every* $p \in N$*,*

$$i_*(T_pN) = \Delta_p.$$

We note that k-dimensional integral manifolds generally do not exist for $k \geq 2$, even locally [7]. The Frobenius Integrability Theorem will, however, tell us exactly when integral manifolds do exist. Before we can prove this theorem, we must develop a greater understanding of smooth vector fields.

3.3 The Lie Derivative

It is a basic theorem of analysis that there always exist integral curves (i.e. integral manifolds of 1-dimensional distributions) on a smooth vector field (cf. Boothby [1]). We state the generalized version for manifolds.

Theorem 3.3 ([7])**.** *Let* X *be a smooth vector field on* M*, and let* $p \in M$*. Then there is an open set* V *containing* p *and an* $\epsilon > 0$*, such that there is a unique collection of diffeomorphisms* $\phi_t : V \to \phi_t(V) \subset M$ *for* $|t| < \epsilon$ *with the following properties:*

i) $\phi : (-\epsilon, \epsilon) \times V \to M$*, defined by* $\phi(t, p) = \phi_t(p)$*, is smooth.*

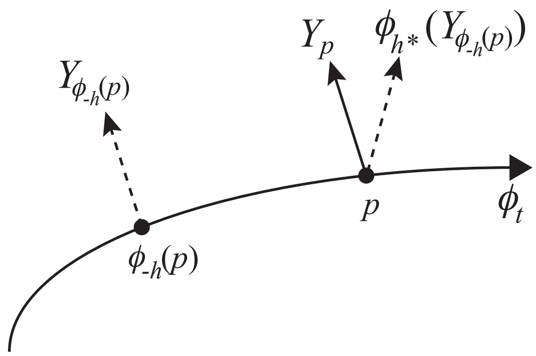

Figure 5: The Lie derivative illustrated.

ii) If $|s|, |t|, |s+t| < \epsilon$, *and* $q, \phi_t(q) \in V$, *then*

$$\phi_{s+t}(q) = \phi_s \circ \phi_t(q).$$

iii) If $q \in V$, *then* X_q *is the tangent vector at* $t = 0$ *of the curve* $t \mapsto \phi_t(q)$.

We note that each ϕ_t is an integral curve. The collection $\{\phi_t\}$ is said to be **generated** by X and for good reason, as for $q \in V$,

$$X_q f = \lim_{h \to 0} \frac{f(\phi_h(q)) - f(q)}{h}.$$

Given another vector field Y on M, we define the following:

Definition 3.4 ([7])**.** *The* **Lie derivative** *of* Y *with respect to* X *at* $p \in M$ *is*

$$[X, Y]_p = \lim_{h \to 0} \frac{1}{h} \left(Y_p - \phi_{h_*}(Y_{\phi_{-h}(p)}) \right).$$

We can picture the Lie derivative as follows (follow along with Figure 5): a point $p \in M$ is on a some curve ϕ_t which is generated by X. We find the tangent vector at p given by the vector field Y, i.e. $Y_p \in T_pM$. Then we find the tangent vector at $\phi_{-h}(p) \in M$ given by Y, i.e. $Y_{\phi_{-h}(p)} \in T_{\phi_{-h}(p)}M$. Finally, we map back to the tangent space at p via the map $\phi_{h*} : T_{\phi_{-h}(p)}M \to T_pM$. We then take the difference between Y_p and $\phi_{h_*}(Y_{\phi_{-h}(p)})$ as h goes to 0. We can see that the Lie derivative gives a vector field on M. In fact, the Lie derivative is smooth as long as X and Y are smooth.

Now, let $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ be smooth vector fields in $\mathbb{R}^2$. The integral curves of these vector fields give the coordinate lines of $\mathbb{R}^2$ (see Figure 6). Furthermore, we can again imagine the integral curves given by X and Y, but this time, in $\mathbb{R}^3$ with $z = 0$. (see Figure 6). Recalling our “stack of sheets” foliation of $\mathbb{R}^3$, we can think of giving each leaf coordinate lines (see Figure 7). Further, every 2-dimensional foliation of a 3-manifold locally looks like the

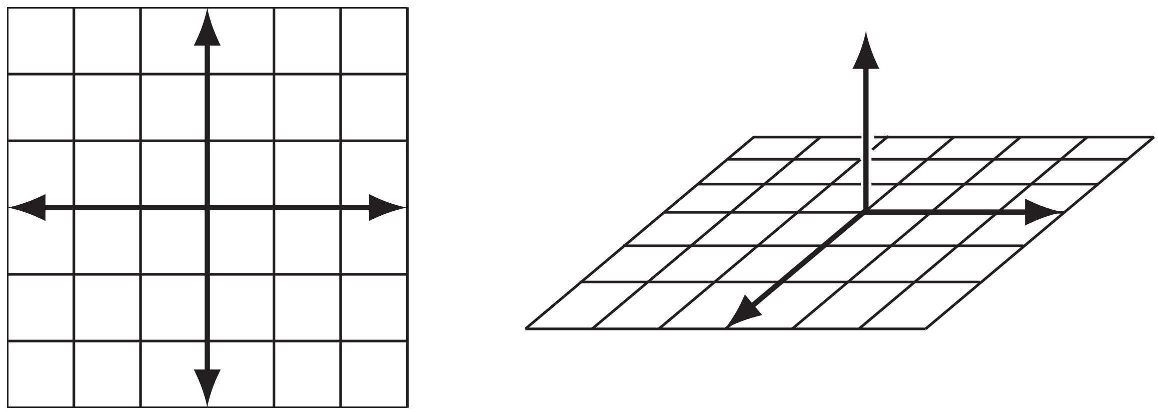

Figure 6: On the left: the coordinate lines defined by integral curves of $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ on $\mathbb{R}^2$. On the right: we imagine X and Y defining coordinate lines (in only two dimensions) on $\mathbb{R}^3$.

"stack of sheets". Thus, locally, we can give the leaves of a foliation in $\mathbb{R}^n$ coordinate lines, and then pull back the coordinate lines onto leaves of the manifold. We imagine the coordinate lines deforming to fit the shape of the leaves (see Figure 7). So, if it is possible to locally give a collection of leaves coordinate lines which, in $\mathbb{R}^2$, look like the coordinate lines given by X and Y, we will have a foliation. We will be able to generalize this result. In fact, the Lie derivative will help us determine exactly when we can find such coordinate lines.

Theorem 3.5 ([7])**.** *In $\mathbb{R}^n$, if $X_1, \ldots, X_k$ are linearly independent smooth vector fields in a neighborhood of p, and $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$, then there is a coordinate system (x, U) around p such that*

$$X_i = \frac{\partial}{\partial x^i} \quad \text{on } U, \quad i = 1, \ldots, k.$$

Before we can prove Theorem 3.5, we will need to state a few properties of the Lie derivative.

Proposition 3.6 ([7])**.** *Let X and Y be smooth vector fields on a manifold M. Then we have the following:*

1. $[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)$.
2. *If X generates $\{\phi_t\}$ and Y generates $\{\psi_t\}$. Then $[X, Y] = 0$ if and only if $\phi_t \circ \psi_s = \psi_s \circ \phi_t$ for all s, t.*

We now provide a proof for Theorem 3.5, adapted from Spivak [7].

Proof. We assume we are at $p = 0$ in $\mathbb{R}^n$ (we assert that we can use this result to prove the theorem on M). We note that there is a coordinate system (t, U)

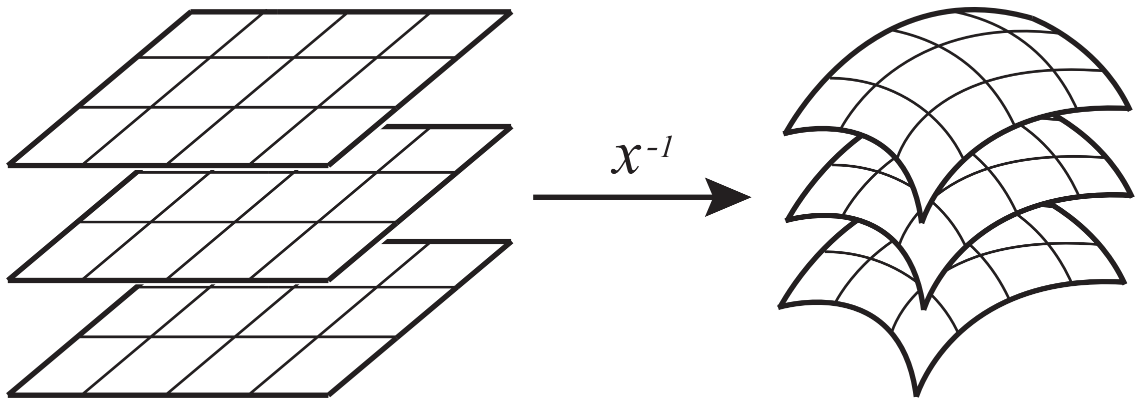

Figure 7: Coordinate lines on local leaves pulled back to coordinate lines on leaves of the manifold.

(about the origin) such that

$$X_i(0) = \left.\frac{\partial}{\partial t^i}\right|_0 \quad \alpha = 1, \ldots, k.$$

Such a coordinate system exists because if we imagine k linearly independent tangent vectors at the origin, we can simply linearly 'adjust' our first k axes so that our new axes point in the same direction as the tangent vectors (see Figure 8).

Suppose each X_i is generated by ϕ_t^i. By existence, for each $i = 1, \ldots, k$ there is an open set V_i containing $p = 0$ and an $\epsilon_i > 0$ such that $\phi_t^i : V_i \to \phi_t(V_i)$ for $|t| < \epsilon_i$, is a collection of diffeomorphisms. Let

$$V = \bigcap_{i=1}^{n} V_i$$

and let ϵ be the minimum ϵ_i. Let $(0, a^2, \ldots, a^n)$ be some element in the neighborhood V. We know that this point lies on the integral curve $\phi_t^1(0, a^2, \ldots, a^n)$. Suppose $q_1 \in V$ lies on the same integral curve, i.e.

$$q_1 = \phi_{a^1}^1(0, a^2, \ldots, a^n)$$

for $|a^1| < \epsilon$. So a^1 is the 'time' that it takes to get to q_1 from $(0, a^2, \ldots, a^n)$. We denote

$$\widetilde{V} = V \cap \left((-\epsilon, \epsilon) \times \mathbb{R}^{n-1}\right)$$

and

$$W = V \cap \left(\{0\} \times \mathbb{R}^{n-1}\right).$$

Let $a = (a^1, \ldots, a^n) \in \widetilde{V}$ and define the map $\chi_1 : \widetilde{V} \to \phi_t^1(W)$ by:

$$\chi_1(a^1, \ldots, a^n) = \phi_{a^1}^1(0, a^2, \ldots, a^n).$$

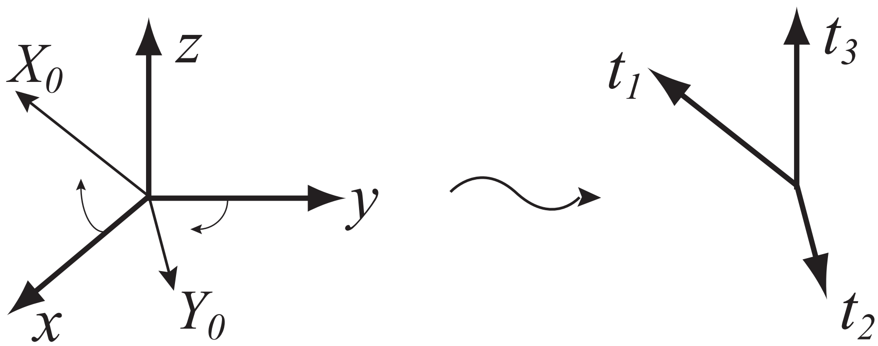

Figure 8: Given two linearly independent tangent vectors at $p = 0 \in \mathbb{R}^3$, X_0 and Y_0, we linearly adjust our axes x, y, z to get a new axes t_1, t_2, t_3 such that $X_0 = \frac{\partial}{\partial t^1}\Big|_0$ and $Y_0 = \frac{\partial}{\partial t^2}\Big|_0$.

Expanding terms, we see that:

$$\begin{aligned}
\chi_{1*}\left(\frac{\partial}{\partial t^1}\bigg|_a\right)(f) &= \frac{\partial}{\partial t^1}\bigg|_a (f \circ \chi_1) \\
&= \lim_{h\to 0}\frac{1}{h}\left(f\left(\chi_1(a^1+h, a^2, \ldots, a^n)\right) - f\left(\chi_1(a)\right)\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(f\left(\phi_{a^1+h}(0, a^2, \ldots, a^n)\right) - f\left(\chi_1(a)\right)\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(f\left(\phi_h(\chi_1(a))\right) - f\left(\chi_1(a)\right)\right) \\
&= (X_1 f)\left(\chi_1(a)\right).
\end{aligned}$$

For $j = 2, \ldots, n$, we see:

$$\begin{aligned}
\chi_{1*}\left(\frac{\partial}{\partial t^j}\bigg|_0\right)(f) &= \frac{\partial}{\partial t^j}\bigg|_0 (f \circ \chi_1) \\
&= \lim_{h\to 0}\frac{1}{h}\left(f\left(\chi_1(0, \ldots, h, \ldots, 0)\right) - f(0)\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(f(0, \ldots, h, \ldots, 0) - f(0)\right) \\
&= \frac{\partial}{\partial t^j}\bigg|_0 (f).
\end{aligned}$$

Remember that in the beginning of our proof, we made sure that

$$X_1(0) = \frac{\partial}{\partial t^1}\bigg|_0.$$

Therefore, for $j = 1, \ldots, n$,

$$\chi_{1*}\left(\left.\frac{\partial}{\partial t^j}\right|_0\right) = \left.\frac{\partial}{\partial t^j}\right|_0.$$

Therefore, at the origin, we have mapped the standard basis of $T_p\mathbb{R}^n$ to itself, i.e.

$$\chi_{1*} : T_p\mathbb{R}^n \to T_p\mathbb{R}^n$$

is an isomorphism as a map between tangent spaces at $p = 0$ (in fact, it's the identity map). By the Inverse Function Theorem, χ_1 must be a local diffeomorphism, say on a neighborhood U_1. Thus, on U_1, we can let $x = \chi_1^{-1}$ to get:

$$\frac{\partial}{\partial x^1} = X_1.$$

We've accomplished our goal for one vector field; we, can repeat this process to make

$$\chi_2(a^1, \ldots, a^n) = \phi^1_{a^1}\left(\phi^2_{a^2}(0, 0, a^3, \ldots, a^n)\right),$$

by starting at

$$(0, 0, a^3, \ldots, a^n)$$

and then moving along the integral curve defined here by ϕ^2_t we get

$$\phi^2_{a^2}(0, 0, a^3, \ldots, a^n)$$

and then moving along the integral curve defined here by ϕ^1_t we get

$$\phi^1_{a^1}\left(\phi^2_{a^2}(0, 0, a^3, \ldots, a^n)\right)$$

via the exact same process. We keep iterating until we reach the k-th step:

$$\chi(a^1, \ldots, a^n) = \phi^1_{a^1}\left(\ldots\left(\phi^k_{a^k}(0, \ldots, 0, a^{k+1}, \ldots a^n)\right)\right).$$

Through the exact same calculation as before,

$$\chi_*\left(\left.\frac{\partial}{\partial t^i}\right|_0\right) = \begin{cases} X_i(0) = \left.\frac{\partial}{\partial t^i}\right|_0 & i = 1, \ldots, k \\ \left.\frac{\partial}{\partial t^i}\right|_0 & i = k+1, \ldots n \end{cases}$$

At this point, we must use our hypothesis that $[X_i, X_j] = 0$. By Proposition 3.6, for $i = 2, \ldots, k$, we can equivalently define χ as follows:

$$\chi(a^1, \ldots, a^n) = \phi^i_{a^i}\left(\phi^k_{a^1}\left(\ldots(0, \ldots, 0, a^{k+1}, \ldots a^n)\ldots\right)\right).$$

Using the same coordinate function $x = \chi^{-1}$ we see that for all $i = 1, \ldots, k$,

$$X_i = \frac{\partial}{\partial x^i}.$$

□

3.4 The Frobenius Integrability Theorem

At this point we may turn our attention back to distributions: specifically, we would like to know the conditions necessary for a distribution to define a foliation. We introduce the following definitions and state a useful property:

Definition 3.7 ([7])**.** *Given a distribution Δ, a vector field X* **belongs to** Δ *if $X_p \in \Delta_p$ for all p. Furthermore, Δ is called* **integrable** *if $[X, Y]$ belongs to Δ whenever X and Y belong to Δ.*

Proposition 3.8 ([7])**.** *If $X_1, \ldots, X_k$ span Δ in a neighborhood U of some $p \in M$, then Δ is integrable if and only if $[X_i, X_j]$ is a linear combination of the spanning vector fields.*

Finally, we can state one of our main theorems:

Theorem 3.9 (**The Frobenius Integrability Theorem** [7])**.** *Let Δ be a smooth integrable k-dimensional distribution on M. For every $p \in M$ there is a coordinate system (x, U) such that the set*

$$\left\{q \in U : x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n\right\},$$

for $a^{k+1}, \ldots, a^n$ constants, is an integral manifold of Δ.

Furthermore, any connected integral manifold of Δ restricted to U is contained in one of these sets.

We can see that the integral manifold in the theorem looks similar to a leaf in a foliation: this is no coincidence. Recall that our intention is to show that integral manifolds are the leaves of a foliation.

Before we prove our theorem, we must state a few useful properties.

Proposition 3.10 ([7])**.** *Given M, a manifold, and $F : M \to N$, a smooth map of manifolds.*

1. *Let Y be a smooth vector field on N and F be an immersion. If $Y_{F(p)} \in F_*(T_pM)$, then there is a unique smooth vector field X on M such that*

   $$F_*(X_p) = Y_{F(p)},$$

   for each $p \in M$.

2. *Let X_1, X_2 be smooth vector fields on M and Y_1, Y_2 be smooth vector fields on N. If*

   $$F_*(X_{ip}) = Y_{iF(p)},$$

   for $i = 1, 2$, then

   $$F_*([X_1, X_2]_p) = [Y_1, Y_2]_{F(p)}.$$

We have now equipped ourself with the tools necessary to prove Theorem 3.9.

Proof. For the first part of the theorem, let's assume that we are in $\mathbb{R}^n$ (we assert that we can then translate the final result to the manifold M). As for our coordinate system, let's choose $p = 0$ and assume that $\Delta_p \subset T_p\mathbb{R}^n$ is spanned by

$$\left.\frac{\partial}{\partial t^1}\right|_p, \ldots, \left.\frac{\partial}{\partial t^k}\right|_p,$$

where $t^1, \ldots, t^n$ are coordinates for $\mathbb{R}^n$. We know we can choose such coordinates from our previous work.

We turn our attention to the projection map $\pi : \mathbb{R}^n \to \mathbb{R}^k$ which naturally projects to the first k factors. Then $\pi_* : \Delta_p \to T_p\mathbb{R}^k$ is obviously an isomorphism given our choice of coordinates (π_* restricted to Δ_p, that is). Thus, π_* restricted to Δ_p can be represented by a $k \times k$ matrix with non-zero determinant. Since π_* is continuous, there exists a neighborhood of p, say V, such that for all $q \in V$, π_* restricted to Δ_q can be represented by a $k \times k$ matrix with non-zero determinant. Thus, $\pi_* : \Delta_q \to T_q\mathbb{R}^k$ is an isomorphism. Letting

$$\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^k}$$

be smooth vector fields on $\mathbb{R}^k$, we can apply Proposition 3.10 (since π_* is injective on Δ_q) to see that there exist unique smooth vector fields $X_1, \ldots, X_k$ spanning Δ such that

$$\pi_*(X_{iq}) = \left.\frac{\partial}{\partial t^i}\right|_{\pi(q)} \qquad \text{for } i = 1, \ldots, k.$$

Therefore, by Proposition 3.10,

$$\begin{aligned} \pi_* \left[X_i, X_j\right]_q &= \left[\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right]_{\pi(q)} \\ &= 0. \end{aligned}$$

However, by assumption, $[X_i, X_j]_q \in \Delta_q$ and π_* is injective so, $[X_i, X_j] = 0$. By our hard work on Theorem 3.5, there exists a coordinate system (x, U) such that

$$X_i = \frac{\partial}{\partial x^i} \qquad \text{for } i = 1, \ldots k.$$

The sets

$$N = \{q \in U : x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n\}$$

are integral manifolds since the inclusion of their tangent spaces T_qN are spanned by

$$\left.\frac{\partial}{\partial x^1}\right|_q, \ldots, \left.\frac{\partial}{\partial x^n}\right|_q,$$

since the last $n - k$ coordinates are constant. These tangent vectors span Δ_q.

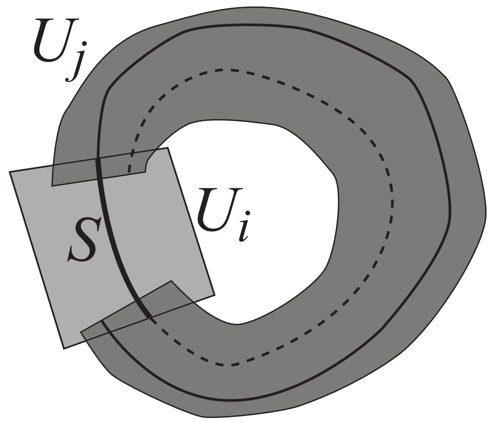

Figure 9: A slice S of U_i continuing along as a slice of U_j. Inspired by the figure on p.195 of Spivak [7].

For the second part of the theorem, if N is a connected integral manifold of Δ restricted to U, with the inclusion map $i : N \to U$, consider $(x^m \circ i)_*$ for $m = k+1, \dots n$. For any tangent vector X_q of $T_q N$ we have

$$\begin{aligned}(x^m \circ i)_*(X_q) = X_q(x^m \circ i) &= i_* X(q)(x^m) \\ &= 0,\end{aligned}$$

since $i_* X(q) \in \Delta_q$, which is spanned by

$$\left.\frac{\partial}{\partial x^i}\right|_q$$

for $i = 1, \dots, k$. Since $(x^m \circ i)_*(X_q) = 0$, N is constant for the coordinates $x^{k+1}, \dots, x^n$ which is precisely what we want. $\square$

We would like to exhibit a correspondence between integral manifolds and leaves of a foliation. Given a coordinate system (x, U) on an n-manifold M satisfying Theorem 3.9, we will denote the set

$$S = \left\{q \in U : x^{k+1}(q) = a^{k+1}, \dots, x^n(q) = a^n\right\}$$

as a *slice* of U. If we cover M with such coordinate systems, we can follow each slice along overlapping coordinate neighborhoods to obtain a leaf in M (see Figure 9). The union of these disjoint leaves will form a foliation on M. Thus, we state the following theorem:

Theorem 3.11 ([7]). *Let Δ be a smooth k-dimensional integrable distribution on M. Then M is a foliated by an integral manifold of Δ (each component is called* **maximal integral manifold of Δ***).*

4 Foliations Defined by Differential Forms

We would like to translate Frobenius' Theorem into the language of differential forms. Let $\Omega^l(M)$ be the set of all l-forms on M. Further, let $\Omega(M)$ be the direct sum of $\Omega^l(M)$, for all l. If Δ is a k-dimensional distribution on a manifold M, then $\mathcal{I}(\Delta) \subset \Omega(\Delta)$ denotes the ideal generated by the set of all l-forms ω, for all l, with the property that

$$\omega(X_1, \ldots, X_l) = 0$$

for all $X_1, \ldots, X_l$ that belong to Δ.

If we allow Δ to be locally spanned by $X_1, \ldots, X_k$, then, by duality, there exist n linearly independent 1-forms $\omega^1, \ldots, \omega^n$ such that $\omega^{k+1}, \ldots, \omega^n$ vanish on $X_1, \ldots, X_k$. Hence, $\omega^{k+1}, \ldots, \omega^n$ generate $\mathcal{I}(\Delta)$ [7].

We now state a lemma that will prove useful when translating Frobenius' Theorem.

Lemma 4.1 ([7])**.** *Given ω a 1-form, X and Y smooth vector fields on a manifold M, then $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.*

We can now state the differential form version of the **Frobenius Integrability Theorem**. The proof is adapted from Spivak [7].

Theorem 4.2 ([7])**.** *A k-dimensional distribution Δ on an n-dimensional manifold M is integrable if and only if $d(\mathcal{I}(\Delta)) \subset \mathcal{I}(\Delta)$.*

Proof. Suppose Δ is integrable. So, just as in Theorem 3.9, at each point $p \in M$ there exists a coordinate system (x, U) such that Δ is spanned by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}.$$

Thus, locally, $\mathcal{I}(\Delta)$ is generated by $n-k$ independent 1-forms

$$dx^{k+1}, \ldots, dx^n.$$

Obviously, for $\alpha = k+1, \ldots, n$,

$$d(dx^\alpha)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0$$

and so, $d(\mathcal{I}(\Delta)) \subset \mathcal{I}(\Delta)$.

For the converse, suppose that $d(\mathcal{I}(\Delta)) \subset \mathcal{I}(\Delta)$. Let $\omega^{k+1}, \ldots, \omega^n$ generate $\mathcal{I}(\Delta)$. Now, for $1 \leq i, j \leq k$ and $k+1 \leq \alpha \leq n$, we apply Lemma 4.1 to get

$$d\omega^\alpha(X_i, X_j) = X_i(\omega^\alpha(X_j)) - X_j(\omega^\alpha(X_i)) - \omega^\alpha([X_i, X_j]).$$

Since $\omega^\alpha, d\omega^\alpha \in \mathcal{I}(\Delta)$,

$$\omega^\alpha([X_i, X_j]) = 0.$$

Thus, $[X_i, X_j]$ belongs to Δ whenever X_i and X_j belong to Δ. So, Δ is integrable. □

The following corollary is obvious since, if Δ is a codimension 1 distribution on a manifold M, $\mathcal{I}(\Delta)$ is generated locally by a 1-form ω.

Corollary 4.3. *Let $\Delta = \ker(\omega)$ be a distribution with codimension 1 on a manifold M where ω is a non-vanishing 1-form. Then the following are equivalent:*

1. *Δ is integrable*
2. *There exists a 1-form η on M such that $d\omega = \omega \wedge \eta$*
3. *$\omega \wedge d\omega = 0$.*

4.1 Computational Properties of Foliations Defined by Differential Forms

We will take a look at a few examples of foliations of 3-manifolds defined by 1-forms.

4.1.1 The Reeb Foliation, Revisited.

Recall the Reeb foliation of a torus. We will now prove that it is, indeed, a foliation. The following construction is taken from Lawson [5]. Once again, consider $\mathbb{R}^3$ in cylindrical coordinates (r, θ, z). Furthermore, we restrict $\mathbb{R}^3$ such that $r \in [0, 1]$ and $z \in [0, 1]$ so that we have a solid cylinder.

We define a smooth function $\psi(r)$ such that

$$0 < \psi(r) < 1 \quad \text{for } 0 < r < 1$$

and extend it smoothly so that $\psi(r) = 0$ for $r = 0$ and $\psi(r) = 1$ for $r = 1$. We define functions $\phi_t^i : \mathbb{R}^3 \to \mathbb{R}^3$ for $i = 1, 2$ such that

$$\begin{aligned} &\phi_t^1(r, \theta, z) = (r + t(1 - \psi(r)), \theta, z + \psi(r)) \quad \text{and} \\ &\phi_t^2(r, \theta, z) = (r, \theta + t, z). \end{aligned}$$

We examine the *orbits* (cf. Boothby [1]) of these curves. For ϕ_t^1, we obtain a curve which is asymptotic to $r = 1$ and for ϕ_t^2, we obtain a circular curve about the z axis (see Figure 10).

We can see that the smooth vector fields

$$\begin{aligned} &X = (1 - \psi(r))\tfrac{\partial}{\partial r} + \psi(r)\tfrac{\partial}{\partial z} \qquad \text{and} \\ &Y = \tfrac{\partial}{\partial \theta} \end{aligned}$$

generate ϕ_t^1 and ϕ_t^2, respectively. Furthermore, the 1-form

$$\omega = \psi(r)dr + (\psi(r) - 1)dz$$

vanishes on X and Y.

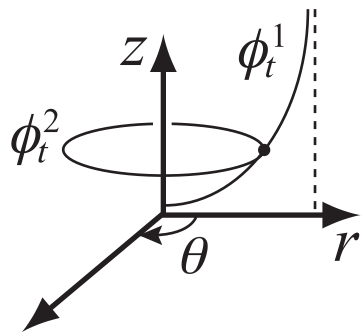

Figure 10: The orbits of ϕ^1 and ϕ^2 at some arbitrary point.

We know that ω defines a foliation on the solid cylinder if and only if $\omega \wedge d\omega = 0$. This is simple enough since

$$d\omega = \frac{\partial \psi(r)}{\partial r} dr \wedge dz,$$

and so $\omega \wedge d\omega = 0$.

Now, we glue $[0,1] \times [0, 2\pi) \times \{1\}$ to $[0,1] \times [0, 2\pi) \times \{0\}$ (i.e. the disk at the top and bottom of the cylinder, respectively) of the solid cylinder to obtain the solid torus. We assert that we obtain an induced foliation via the gluing and thus, the Reeb foliation is, indeed, a foliation of the solid torus.

4.1.2 A Foliation of $T_1\mathbb{H}$.

We examine another important foliation. The following construction is taken from Tamura [9]. Consider the Poincaré half-plane,

$$\mathbb{H} = \{(x, y) : x, y \in \mathbb{R} \text{ and } y > 0\},$$

which has the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

Recall that a **geodesic** is the shortest (with respect to a metric) path between two points in a metric space. In $\mathbb{H}$, geodesics are either [8]:

i) Euclidean semicircles centered on the x-axis.

ii) Segments of Euclidean straight lines that are perpendicular to the x-axis.

We will construct a foliation of $T_1\mathbb{H}$, the unit tangent bundle of $\mathbb{H}$. We point out that $T_1\mathbb{H}$ is diffeomorphic to $\mathbb{H} \times S^1$ (each tangent vector has distance 1, so we need only take into account the "direction" of each tangent vector).

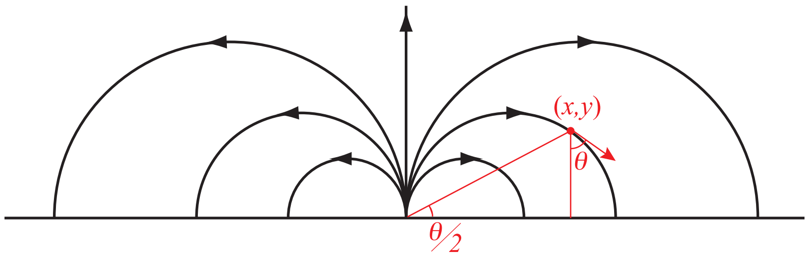

Figure 11: A family of parallel geodesics in $\mathbb{H}$. Inspired by Figure 8.4 of Tamura [9].

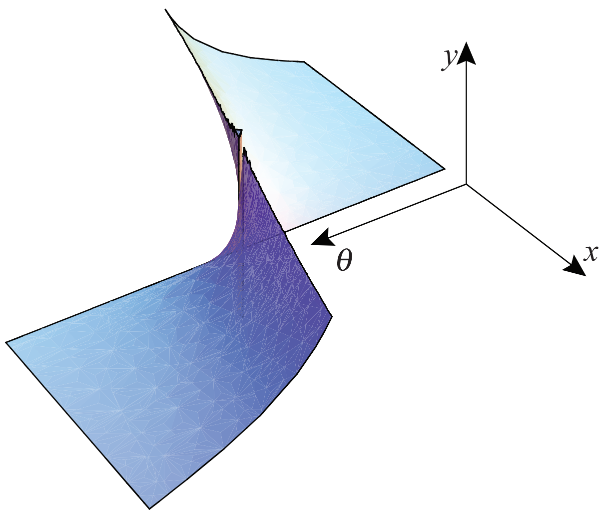

Figure 12: A leaf in the foliation of $T_1\mathbb{H}$.

If a collection of geodesics limit to the same point on the x-axis, then we call them **parallel**. Consider a family of parallel geodesics limiting to the point $(a, 0)$ (see Figure 11). We allow for a point at infinity, $\{\infty\}$, on the x-axis and thus, the collection of vertical euclidean lines pointing up is also family of parallel geodesics. Given a family of parallel geodesics, each point $(x, y) \in \mathbb{H}$ is assigned the angle θ that is obtained between the unit tangent vector at p and the vertical line between $(x, 0)$ and (x, y) (see Figure 11). Each leaf $\mathcal{L}_a$ in $T_1\mathbb{H}$ is a family of parallel geodesics limiting to a on the x-axis where each point has an associated unit tangent vector (see Figure 12). Using simple geometry, we can see that the angle formed between the line from $(a, 0)$ to (x, y) and the

x-axis is $\frac{\theta}{2}$. So, each leaf is given by

$$\mathcal{L}_a = \left\{ (x, y, \theta) \in \mathbb{H} \times S^1 : \tan \frac{\theta}{2} - \frac{y}{x-a} = 0 \right\}.$$

Note that $\mathcal{L}_\infty$, the family of vertical euclidean lines, is also given by this equation.

To prove that this is, in fact, a foliation, we let the function $f : \mathbb{H} \times S^1 \to \mathbb{R}$ be such that

$$f(x, y, \theta) = x - y \cot \frac{\theta}{2}.$$

It is obvious that $\{f^{-1}(a)\}_{a \in \mathbb{R} \cup \{\infty\}}$ is the same collection as $\{\mathcal{L}_a\}_{a \in \mathbb{R} \cup \{\infty\}}$ and thus defines the same collection of leaves. Therefore, we can see that the 1-form df will vanish on the leaves of our foliation. So, we apply the differential d to f to get the 1-form

$$df = dx - \cot \frac{\theta}{2} dy + \frac{1}{2} y \csc^2 \frac{\theta}{2} d\theta,$$

which will define the same foliation. If we let $\omega = df$, it is obvious that $\omega \wedge d\omega = 0$.

4.2 Cobordant Foliations

We would like to develop a notion of equivalence for foliations. Certainly, it is possible to change the leaves in a foliation via smooth deformations. We, however, do not want to take such foliations to be 'different'. Our intuition tells us that two foliations should be equivalent if one can be 'smoothly' deformed into the other. We can define this explicitly:

Definition 4.4 ([10]). *Let ω_0 and ω_1 be k-forms defining codimension k foliations on a manifold M. The corresponding foliations are* **cobordant** *if there exists a k-form Ω on $M \times [0, 1]$ such that Ω restricted*[1] *to $M \times i$ equals ω_i, for $i = 0, 1$.*

4.3 Thurston's Theorem

We have finally arrived at a point which allows us to state our main theorem, discovered by William Thurston:

Theorem 4.5 ([10]). *There are uncountably many noncobordant foliations of S^3.*

We have not, however, developed all of the tools necessary to actually prove this theorem. Therefore, we will take a closer look at differential forms on manifolds. We hope to find an invariant of cobordantism.

[1] We will often discuss the 'restriction' of a differential form - this is not technically correct. If we are given a k-form ω on a manifold M and we 'restrict' to $U \subset M$, we are using the pull-back of the inclusion $i : U \to M$ to get a k-form $i^* \omega$ on U.

5 DeRham Cohomology

The following section is adapted from Guillemin and Pollack [4]. We begin by examining an invariant of a manifolds: cohomology groups. We commence our exploration by pointing out that the differential d can be thought of as a function $d : \Omega^k(M) \to \Omega^{k+1}(M)$, where $\Omega^k(M)$ is the group of all k-forms on M. Furthermore, we know that $d^2 = 0$. Therefore, we can define a cochain complex, $(\Omega^k(M), d)$:

$$\Omega^0(M) \xrightarrow{d^0} \cdots \xrightarrow{d^{k-1}} \Omega^k(M) \xrightarrow{d^k} \Omega^{k+1}(M) \xrightarrow{d^{k+1}} \cdots$$

Definition 5.1. *The* k**-th DeRahm Cohomology Group** *of* M*, is defined to be*

$$H^k(M) = \ker d^k / Im\ d^{k-1}.$$

We say that a k-form ω is **closed** if $d\omega = 0$; we say that ω is **exact** if $\omega = d\theta$ for some $(k-1)$ form θ. We can see that exact implies closed, but the converse does not always hold. We can equivalently define $H^k(M)$ as the group composed of all closed k-forms on a manifold M with the following equivalence relation: given closed k-forms ω and ω', $\omega \sim \omega'$ if $\omega - \omega'$ is exact.

We can check that the pull-back of a form will also work as a map on cohomology. If $F : M \to N$ is a smooth map of manifolds such that F^* pulls back forms, we can define an induced homomorphism on cohomology:

$$F^* : H^k(N) \to H^k(M).$$

Certainly, the induced map is homomorphism (since the original map is linear). We can also check that this actually of map of cohomology groups: given a closed k-form ω on N,

$$dF^*\omega = F^*d\omega,$$

and so $F^*\omega$ is closed. Furthermore, the map is well-defined since given some exact k-form on N, $d\theta$,

$$\begin{aligned} F^*(\omega + d\theta) &= F^*\omega + F^*d\theta \\ &= F^*\omega + dF^*\omega, \end{aligned}$$

which is in the same cohomology class as $F^*\omega$ [4].

Our goal is to show that $H^3(S^3) \simeq \mathbb{R}$. Very soon, we will define an invariant of cobordism: this invariant will be a cohomology class in $H^3(M)$ for a manifold M. Thus, we will use the fact that $H^3(S^3) \simeq \mathbb{R}$ to our advantage in proving that there are uncountably many noncobordant foliations of S^3. We will prove the following generalized result:

Theorem 5.2 ([4])**.** $H^n(S^n) \simeq \mathbb{R}$.

5.1 Computing $H^1(S^1)$

As a way to familiarize ourselves with the present theory, let's run through a computation of $H^1(S^1)$. Firstly, we will cite the following lemma.

Lemma 5.3 ([4]). *Let ω be a 1-form on S^1.*

1. *ω is exact if and only if $\int_{S^1} \omega = 0$.*
2. *If ω has nonzero integral and θ is any other 1-form on S^1 then there exists a constant $a \in \mathbb{R}$ such that $\theta - a\omega$ is exact.*

We will now show that $H^1(S^1) \simeq \mathbb{R}$. Consider the 1-form on S^1

$$\omega = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

Via straightforward computation, we can see that ω is closed. However, ω is not exact: we assert that $\int_{S^1} \omega = 2\pi$; the first part of our lemma does the rest of the work. The second part asserts that, given any closed 1-form on S^1 say, θ, there exists $a \in \mathbb{R}$ such that θ and $a\omega$ are cohomologous. Furthermore, given any $a, b \in \mathbb{R}$ such that $a \neq b$,

$$\begin{aligned} \int_{S^1} a\omega - b\omega &= 2\pi(a-b) \\ &\neq 0, \end{aligned}$$

which tells us that $a\omega - b\omega$ is not exact. Therefore $a\omega$ and $b\omega$ are not in the same cohomology class and we have

$$H^1(S^1) = \{a\omega : a \in \mathbb{R}\},$$

which is isomorphic to $\mathbb{R}$.

5.2 Basic Properties

We begin with an easy fact: $H^k(M) \simeq 0$ for $k > n$, where n is the dimension of M. This is an immediate consequence of the following: there do not exist any non-vanishing k-forms on an n-manifold M for $k > n$.

We would like to prove that $H^k(M \times \mathbb{R}) \simeq H^k(M)$. We give ourselves ω, a k-form on $M \times \mathbb{R}$, where M is an n-manifold. Furthermore, let t be the standard coordinate system on $\mathbb{R}$ and (x, U) be a coordinate system on M. Let $1 \leq i_1 < \ldots < i_k \leq n$ and let $I = (i_1, \ldots, i_k)$ be a multi-index, i.e. $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. We can always represent ω as

$$\omega = \sum_I f_I(x,t) dt \wedge dx^I + \sum_J g_J(x,t) dx^J,$$

where I and J are the appropriate multi-indices and f_I and g_J are smooth functions. With this in mind, we define a linear function P which transforms a k-form into a $(k-1)$-form on $M \times \mathbb{R}$:

$$P\omega(x,t) = \sum_I \left(\int_0^t f_I(x,s)ds \right) dx^I.$$

This turns out to be quite useful.

Proposition 5.4 ([4]). *Let $\pi : M \times \mathbb{R} \to M$ be the usual projection and $i_a : M \to M \times \mathbb{R}$ be any embedding $p \mapsto (p,a)$. Then*

$$dP\omega + Pd\omega = \omega - \pi^* \circ i_a^* \omega.$$

The proof of Proposition 5.4 requires only the expanding of definitions.

Lemma 5.5 (Poincaré Lemma[4]). *The following homomorphisms*

$$i_a^* : H^k(M \times \mathbb{R}) \to H^k(M) \quad \text{and} \quad \pi^* : H^k(M) \to H^k(M \times \mathbb{R})$$

are inverses of each other.

Before we prove the Poincaré Lemma, we note an obvious (and important) consequence: that $H^k(M \times \mathbb{R})$ is isomorphic $H^k(M)$.

Proof. Begin by examining

$$\begin{aligned} \pi \circ i_a(x) &= \pi(x,a) \\ &= x. \end{aligned}$$

Since $(\pi \circ i_a) = id$, we know that $i_a^* \circ \pi^* = id^*$ on $H^k(M)$.

For the other side, we are given $\omega \in H^k(M \times \mathbb{R})$. Since ω is closed, Proposition 5.4 states that

$$dP(\omega) = \omega - \pi^* \circ i_a^*(\omega).$$

Therefore $\omega - \pi^* \circ i_a^*(\omega)$ is exact which implies ω and $\pi^* \circ i_a^*(\omega)$ are in the same cohomology class. So, $\pi^* \circ i_a^* = id^*$ on $H^k(M \times \mathbb{R})$. □

Now, we would like to prove that contractible manifolds have trivial cohomology (for $k > 0$, of course). Recall that a manifold M is contractible if the identity map is homotopic to a constant map. Thus, we would like homotopic maps to induce the same map on cohomology.

Lemma 5.6 ([4]). *If $F, G : M \to N$ are homotopic maps, then $F^* = G^*$.*

Proof. Let $H : M \times [0,1] \to N$ be a homotopy of F and G. We can suppose $H(p,0) = F(p)$ and $H(p,1) = G(p)$. We smoothly extend this map to $\widetilde{H} : M \times \mathbb{R} \to N$. Given $p \in M$, $F(p) = \widetilde{H} \circ i_0(p)$ and $G(p) = \widetilde{H} \circ i_1(p)$. Therefore,

$$F^* = i_0^* \circ \widetilde{H}^* \quad \text{and} \quad G^* = i_1^* \circ \widetilde{H}^*.$$

The induced map $i_a^* : H^*(M) \to H^*(M \times \mathbb{R})$ is an isomorphism for any $a \in \mathbb{R}$. Lemma 5.5 says that both i_0^* and i_1^* are inverses of π^* and therefore they are equal. Then we have

$$\begin{aligned} F^* &= i_0^* \circ \widetilde{H}^* \\ &= i_1^* \circ \widetilde{H}^* \\ &= G^*. \end{aligned}$$

□

Corollary 5.7 ([4])**.** *If M is contractible, then $H^k(M) \simeq 0$ for $k > 0$.*

Proof. Since M is contractible, $id : M \to M$ is homotopic to some constant map $c_q : M \to \{q\}$, for $q \in M$. Obviously, id is a diffeomorphism of $M \to M$ and so, it induces an isomorphism $H^k(M) \to H^k(M)$ by our previous work. By Lemma 5.6, c_q^* is the same isomorphism. Therefore c_q^* is an isomorphism between $H^k(\{q\})$ and $H^k(M)$. At last, since $\dim(\{q\}) = 0 < k$, we know that $H^k(\{q\}) \simeq 0$. □

5.3 Mayer-Vietoris on S^n

For the following, we let $U = S^n - \{S\}$ and $V = S^n - \{N\}$, where $S = (0, \ldots, 0, -1)$ and $N = (0, \ldots, 0, 1)$ denote the south pole and north pole, respectively.

Lemma 5.8 ([4])**.** *$U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$.*

Proof. It suffices to show that $U \cap V$ is diffeomorphic to $S^{n-1} \times (-1, 1)$. Since $(-1, 1)$ is diffeomorphic to $\mathbb{R}$, the aforementioned will be sufficient. Given the $(n+1)$-tuple $(x^1, \ldots, x^{n+1}) \in U \cap V$, define $\rho : U \cap V \to S^{n-1} \times (-1, 1)$ and it's inverse as follows:

$$\begin{aligned} \rho(x^1, \ldots, x^{n+1}) &= \left(\frac{x^1}{c}, \ldots, \frac{x^n}{c}, x^{n+1} \right) \\ \rho^{-1}(x^1, \ldots, x^{n+1}) &= \left(cx^1, \ldots, cx^n, x^{n+1} \right), \end{aligned}$$

where

$$c = \sqrt{1 - (x^{n+1})^2}.$$

Since we have removed the north and south poles, c is nonzero. It is painfully obvious that the above is a diffeomorphism. □

Lemma 5.9 (**Baby Mayer-Vietoris**[4])**.** *For $k > 1$, $H^{k-1}(U \cap V)$ and $H^k(U \cup V)$ are isomorphic.*

The following proof is adapted from Guillemin and Pollack [4].

Proof. We will define a homomorphism $f : H^{k-1}(U \cap V) \to H^k(U \cup V)$. We illustrate the path that f will take in the following diagram:

$$\begin{tikzcd}
 & \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \arrow[d, "d \oplus d"] & H^{k-1}(U \cap V) \arrow[l] \\
H^k(U \cup V) & \Omega^k(U) \oplus \Omega^k(V) \arrow[l] &
\end{tikzcd}$$

First, we find a smooth function $\rho_1 : S^n \to \mathbb{R}$ that vanishes on a neighborhood of the north pole and equals 1 in a neighborhood of the south pole. Let $\rho_2 : S^n \to \mathbb{R}$ be such that $\rho_2 = 1 - \rho_1$ (i.e., vanishes on a neighborhood of the south pole and equals 1 on a neighborhood of the north pole). We have cleverly constructed our functions so that $\rho_1 + \rho_2 = 1$ everywhere. Now given $[\omega] \in H^{k-1}(U \cap V)$, define $\theta_1 \in \Omega^{k-1}(U)$ to be $\rho_1\omega$ and $\theta_2 \in \Omega^{k-1}(V)$ on V to be $-\rho_2\omega$. In both cases, θ_i vanishes on a neighborhood of a pole on which it was originally undefined, so these are actually forms defined on all of U or V. Notice $\theta_1 - \theta_2 = \omega$ on $U \cap V$ (where θ_1 and θ_2 are restricted $U \cap V$). Define a k-form σ on $U \cup V$ by setting $\sigma = d\theta_1$ on U and $\sigma = d\theta_2$ on V. Since $d\theta_1 - d\theta_2 = d\omega = 0$ on $U \cap V$, σ is a smooth closed form on all of $U \cup V$. So $[\sigma] \in H^k(U \cup V)$. Hence, we define $f([\omega]) = [\sigma]$. It is straightforward to check that f is well-defined on cohomology.

Now we define a homomorphism $g : H^k(U \cup V) \to H^{k-1}(U \cap V)$. The path that g will take is depicted in the following diagram:

$$\begin{tikzcd}
 & \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \arrow[r] & H^{k-1}(U \cap V) \\
H^k(U \cup V) \arrow[r] & \Omega^k(U) \oplus \Omega^k(V) \arrow[u, "\hat{d} \oplus \hat{d}"] &
\end{tikzcd}$$

Note that the map $\hat{d}$ is meant to take a closed k-form to a $(k-1)$-form in the natural way, i.e. $d\eta \mapsto \eta$; accordingly, this map is not defined on the entire DeRham chain group. Let $[\sigma] \in H^k(U \cup V)$. Since U is contractible, the restriction of σ to U must be exact by Lemma 5.7, i.e. $\sigma = d\theta_1$ on U. Similarly, $\sigma = d\theta_2$ on V. Now, consider $\theta_1 - \theta_2 \in \Omega^{k-1}(U \cap V)$. Since $d\theta_1 = d\theta_2$ on $U \cap V$, $\theta_1 - \theta_2$ is closed. So we define $g([\sigma]) = [\theta_1 - \theta_2]$. It is straightforward to check that g is well-defined on cohomology.

To prove that f and g are inverses, we do simple substitution. Given a $[\sigma] \in H^k(U \cup V)$, we see that

$$f \circ g([\sigma]) \;=\; f([\theta_1 - \theta_2]).$$

Therefore, $f([\theta_1 - \theta_2]) = d(\rho_1(\theta_1 - \theta_2))$ on U and $d(-\rho_2(\theta_1 - \theta_2))$ on V. Denote these k-forms by $d\phi_1$ and $d\phi_2$, respectively, and take note that on $U \cap V$,

$$\begin{aligned} d\phi_1 - d\phi_2 &= d\theta_1 - d\theta_2 \\ &= 0. \end{aligned}$$

Let $\sigma' = \phi_1$ on U and ϕ_2 on V. We claim that this is in the same cohomology class as σ. Remember that $\sigma = d\theta_1$ on U and $d\theta_2$ on V. Moreover, on U,

$$\sigma - \sigma' = d(\theta_1 - \phi_1),$$

and on V,

$$\sigma - \sigma' = d(\theta_2 - \phi_2).$$

But, if we take a close look, we come to realize that

$$\begin{aligned}
\theta_1 - \phi_1 &= \theta_1 - (\rho_1\theta_1 - \rho_1\theta_2) \\
&= \rho_2\theta_1 + \rho_1\theta_2 \\
&= \theta_2 + (\rho_2\theta_1 - \rho_2\theta_2) \\
&= \theta_2 - \phi_2.
\end{aligned}$$

Nothing could be more welcome, for we see that the above is an element of $\Omega^{k-1}(U \cup V)$. At the north pole, an element of U, the above yields θ_1, which is defined on U. At the south pole, an element of V, the above yields θ_2, which is defined on V. We are already aware, from the construction of our maps, that the above is defined on $U \cap V$. So $\sigma - \sigma'$ is in fact $d(\theta_1 - \phi_1)$, which is certainly exact.

Conversely, given $[\omega] \in H^{k-1}(U \cap V)$ we see that

$$\begin{aligned}
g \circ f([\omega]) &= g([\sigma]) \\
&= [\theta_1 - \theta_2].
\end{aligned}$$

Since $\sigma = d(\rho_1\omega)$ on U and $\sigma = d(-\rho_2\omega)$ on V,

$$\begin{aligned}
g([\sigma]) &= [\rho_1\omega + \rho_2\omega] \\
&= [\omega].
\end{aligned}$$

So, we have constructed an isomorphism. □

5.4 $H^3(S^3) \simeq \mathbb{R}$

Now we put everything together to prove that $H^n(S^n) \simeq \mathbb{R}$. The proof is inductive and is adapted from Guillemin and Pollack [4].

Proof. Now we inductively assume the theorem holds (i.e. $H^{n-1}(S^{n-1}) = \mathbb{R}$, where $n > 1$). We know that $H^n(M \times \mathbb{R}) \simeq H^n(M)$. We let U_1 and U_2 be as before and we see the following set of equalities:

$$\begin{aligned}
H^n(S^n) &\simeq H^n(U_1 \cup U_2) \\
&\simeq H^{n-1}(U_1 \cap U_2) \\
&\simeq H^{n-1}(S^{n-1} \times \mathbb{R}) \\
&\simeq H^{n-1}(S^{n-1}) \\
&\simeq \mathbb{R}.
\end{aligned}$$

□

In the following section, we will explain the consequence of $H^3(S^3) \simeq \mathbb{R}$.

6 The Godbillon-Vey Invariant

6.1 Finding the Invariant

Recall that if ω is a 1-form defining a codimension 1 foliation on a n-manifold M then, by Corollary 4.3, there exists a 1-form η on M such that $d\omega = \omega \wedge \eta$. We then define the following:

Definition 6.1. *The cohomology class of the 3-form* $\Gamma = \eta \wedge d\eta$ *is the* **Godbillon-Vey invariant** *of the foliation defined by* ω*.*

First, we will verify that $[\Gamma] = [\eta \wedge d\eta]$ is actually in $H^3(M)$. To do so, we need only show that it is closed. We can see that

$$\begin{aligned} 0 &= d(d\omega) \\ &= d(\omega \wedge \eta) \\ &= d\omega \wedge \eta - \omega \wedge d\eta \\ &= -\omega \wedge d\eta. \end{aligned}$$

Thus, by Corollary 4.3, there exists a 1-form α on M such that

$$d\eta = \omega \wedge \alpha.$$

Via substitution,

$$\begin{aligned} d\Gamma &= d(\eta \wedge d\eta) \\ &= d\eta \wedge d\eta \\ &= \omega \wedge \alpha \wedge \omega \wedge \alpha \\ &= 0, \end{aligned}$$

since a 1-form wedged with itself is 0 via the skew-symmetric property. Hence Γ is closed [9].

This definition is well-defined with respect to our choice of η. Let ω be a 1-form on M, a 3-manifold, such that $\omega \wedge d\omega = 0$. Choose 1-forms η and η' such that

$$d\omega = \omega \wedge \eta = \omega \wedge \eta'.$$

Therefore, $\omega \wedge (\eta' - \eta) = 0$. So, there is a smooth function f on M such that $\eta' - \eta = f\omega$. Expanding terms, we get that

$$\begin{aligned} \eta' \wedge d\eta' &= (\eta + f\omega) \wedge d(\eta + f\omega) \\ &= (\eta + f\omega) \wedge (d\eta + df \wedge \omega + f d\omega) \\ &= \eta \wedge d\eta + \eta \wedge df \wedge \omega + f\omega \wedge d\eta \\ &= \eta \wedge d\eta + d(f\omega \wedge \eta). \end{aligned}$$

We see that the Godbillon-Vey invariants are in the same cohomology class since their difference is exact [9].

If we change ω to $f\omega$, where f is a non-vanishing smooth function on M, we can see that $f\omega \wedge d(f\omega) = 0$ and further, that it defines the *same foliation* [2] on M. Thus, we need to check that our Godbillon-Vey invariant is well-defined with respect to this change. Since $f\omega \wedge d(f\omega) = 0$, there exists a 1-form η'' such that $d(f\omega) = f\omega \wedge \eta''$. We can choose η'' such that

$$\eta'' = \eta - \tfrac{1}{f}df.$$

Furthermore, we see that

$$\begin{aligned} \Gamma'' &= \left(\eta - \tfrac{1}{f}df\right) \wedge d\left(\eta - \tfrac{1}{f}df\right) \\ &= \eta \wedge d\eta + d\left(\eta \wedge \tfrac{1}{f}df\right). \end{aligned}$$

Thus, $[\Gamma''] = [\Gamma]$.

We introduce another useful tool:

Definition 6.2. *Let M be a compact, boundaryless 3-manifold. Further, let ω be a 1-form inducing a foliation on M and let Γ be the associated Godbillon-Vey invariant. Then define the* **Godbillon-Vey number** *to be*

$$\Gamma[M] = \int_M \Gamma.$$

To check that the Godbillon-Vey number is well-defined, allow Γ' to be in the cohomology class of Γ, i.e. $\Gamma' = \Gamma + d\Theta$, where Θ is some 1-form on M. A quick application of Stokes' theorem yields

$$\begin{aligned} \int_M \Gamma' &= \int_M \Gamma + \int_M d\Theta \\ &= \int_M \Gamma + \int_{\partial M} \Theta \\ &= \int_M \Gamma. \end{aligned}$$

So, the Godbillon-Vey number is invariant of cohomology [9].

Heuristically, Thurston [10] points out that our Godbillon-Vey invariant can be thought of as measuring the helical wobble of a foliation's leaves (see Figure 13).

6.2 Proof of Invariance

Now we show that Godbillon-Vey number is invariant of cobordism. Let $W = M \times [0, 1]$ be a 4-manifold such that M_0 and M_1 have cobordant foliations defined by ω_0 and ω_1, respectively. By the definition of cobordism, there exists a 1-form Ω that extends ω_0 and ω_1 onto all of W. Furthermore, Ω defines a

[2] The same vector fields will be in the kernel of $f\omega$.

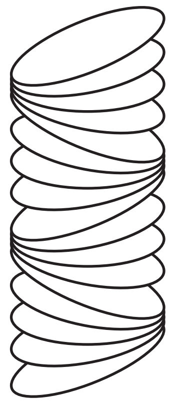

Figure 13: The leaves of a foliation “wobbling”. Inspired by the ‘helical wobble’ figure in Thurston [10].

codimension 1 foliation on W and so, $\Omega \wedge d\Omega = 0$. Let Θ be a 1-form on W such that $d\Omega = \Omega \wedge \Theta$. Denote $\Gamma = \Theta \wedge d\Theta$, the Godbillon-Vey invariant. It is obvious that Γ restricted to M_0 or M_1 is the Godbillon-Vey invariant for the foliations defined by ω_0 and ω_1, respectively. Let Γ_0 and Γ_1 be the restrictions defining Godbillon-Vey invariants on ω_0 and ω_1, respectively. Then, by Stokes' Theorem,

$$\begin{aligned}\Gamma_0[M_0] - \Gamma_1[M_1] &= \int_{M_0} \Gamma_0 - \int_{M_1} \Gamma_1 \\ &= \int_{\partial W} \Gamma \\ &= \int_W d\Gamma \\ &= 0,\end{aligned}$$

since the Godbillon-Vey invariant is closed. Therefore, cobordant foliations have the same Godbillon-Vey number [9].

6.3 Examples of the Godbillon-Vey Invariant and Godbillon-Vey Number

We will compute the Godbillon-Vey invariants and Godbillon-Vey numbers of familiar foliations.

6.3.1 Computing the Godbillon-Vey invariant of the Reeb foliation.

Recall the 1-form

$$\omega = \psi(r)dr + (\psi(r) - 1)dz,$$

which defines the Reeb foliation. We choose the 1-form η such that

$$\eta = \frac{\partial\psi(r)}{\partial r}dr + \frac{\partial\psi(r)}{\partial r}dz.$$

Via straightforward computation, we see that $d\omega = \omega \wedge \eta$. Further, another simple computation yields that the Godbillon-Vey form

$$\Gamma = \eta \wedge d\eta = 0.$$

Thus, the Godbillon-Vey number of Γ will always be 0. So, we see that the "double Reeb" foliation of S^3 will have trivial Godbillon-Vey number.

6.3.2 Computing the Godbillon-Vey invariant of the foliation on $T_1\mathbb{H}$.

We recall our 1-form

$$df = dx - \cot\frac{\theta}{2}dy + \frac{1}{2}y\csc^2\frac{\theta}{2}d\theta,$$

which defines our foliation. So as to ease the computation for the 1-form η we divide through by $y\csc^2\frac{\theta}{2}$ to get the 1-form

$$\omega = \frac{2\sin^2\frac{\theta}{2}}{y}dx - \frac{\sin\theta}{y}dy + d\theta.$$

Choose η such that

$$\eta = \frac{\sin\theta}{y}dx - \frac{\cos\theta}{y}dy.$$

Via straightforward computation, we see that our choice satisfies $d\omega = \omega \wedge \eta$. Then we define our Godbillon-Vey form to be

$$\Gamma = \eta \wedge d\eta = -\frac{1}{y^2}dx \wedge dy \wedge d\theta.$$

However, since $T_1\mathbb{H} \simeq \mathbb{H} \times S^1$ and $\mathbb{H}$ is diffeomorphic to $\mathbb{R}^2$, we see that

$$\begin{array}{rcl} H^3(T_1\mathbb{H}) & \simeq & H^3(S^1) \\ & \simeq & 0. \end{array}$$

Thus, the Godbillon-Vey invariant on $T_1\mathbb{H}$ must be the cohomology class $[0] \in H^3(T_1\mathbb{H})$.

We note that a **volume form** on an n-dimensional manifold M, is a nowhere vanishing n-form [4]. We can see that Γ satisfies the above property. In fact, the standard volume form on $\mathbb{H}$ is $\frac{1}{y^2}dx \wedge dy$ [8] and the standard volume form of S^1 is $d\theta$. Each standard volume form gives the *volume* of a bounded region within each manifold (with respect to the metric on the manifold). Therefore, we can see that the Godbillon-Vey number of Γ would produce the negative volume of a bounded region in $T_1\mathbb{H}$.

7 Thurston's Construction

We are now prepared to use the Godbillon-Vey invariant to prove Thurston's theorem, which we restate:

Theorem 7.1 ([10]). *There are uncountably many noncobordant foliations of S^3.*

We note, at this point, that the structure of the following proof is taken from Thurston [10] and the details have been largely obtained via Tamura [9].

7.1 Structure of Proof

First, we will construct a polygon in $\mathbb{H}$ with arbitrary area. We will then spin this polygon around S^1 to get a space in $T_1\mathbb{H}$ homeomorphic to (but not diffeomorphic to) a solid torus "with edges". We take our foliation of $T_1\mathbb{H}$ and restrict it to this solid torus "with edges". We will then smooth out the space by gluing in solid tori (foliated with Reeb components) at these edges to give ourselves a space diffeomorphic to a solid torus. We then glue another solid torus (with a Reeb foliation) in the proper manner to obtain a foliation of S^3. Since we saw that our Godbillon-Vey form for our foliation was, in fact, a volume form and $H^3(S^3) \simeq \mathbb{R}$, our Godbillon-Vey Invariant will not be a trivial cohomology class. Thus, the Godbillon-Vey number will be the volume of our first torus, which we chose arbitrarily. Since we can vary the volume of our solid torus continuously, we will achieve uncountably many Godbillon-Vey numbers and thus, uncountably many noncobordant foliations of S^3.

7.2 Hyperbolic Geometry

There exist functions $\tau : \mathbb{H} \to \mathbb{H}$ called **Möbius transformations**. We may represent each point $(x, y) \in \mathbb{H}$ with a complex number $z \in \mathbb{C}$ such that

$$z = x + yi.$$

The Möbius transformations are of the form

$$\tau(z) = \frac{az + b}{cz + d}, \qquad ad - bc = 1,$$

for $a, b, c, d \in \mathbb{R}$. Möbius transformations preserve distance with respect to the Poincaré Half-Plane metric and so, are **isometries**; these transformations will also map geodesics to geodesics [8].

When we consider the induced map $\tau_* : T_1\mathbb{H} \to T_1\mathbb{H}$, these maps are still isometries. Between any two geodesics there exists a *unique* orientation preserving Möbius transformation. Furthermore, if we consider our foliation of $T_1\mathbb{H}$ given by

$$\omega = \frac{2\sin^2\frac{\theta}{2}}{y}dx - \frac{\sin\theta}{y}dy + d\theta,$$

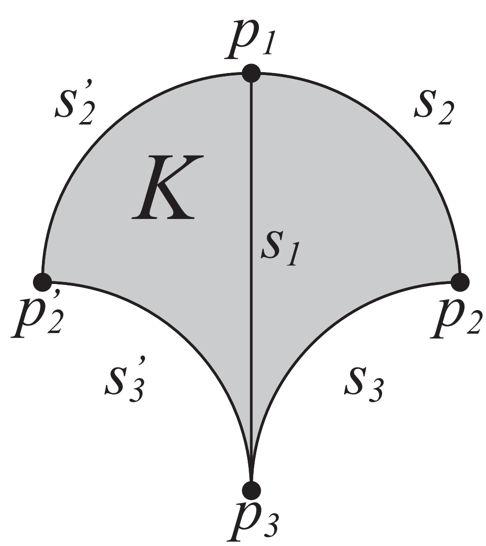

Figure 14: An example of the constructed polygon in $\mathbb{H}$.

then, given a leaf $\mathcal{L}_a$ in the foliation defined by ω,

$$\tau_*(\mathcal{L}_a) = \mathcal{L}_b,$$

where $\mathcal{L}_b$ is another leaf in the foliation (i.e. τ_* maps leaves to leaves). Thus, our foliation is preserved by Möbius transformations. Furthermore, the form ω is also preserved under these transformations, i.e.

$$(\tau_*)^*\omega = \omega.[9]$$

7.3 Construction on S^3

We begin by giving ourselves the foliation of $T_1\mathbb{H}$ given by the 1-form

$$\omega = \frac{2\sin^2\frac{\theta}{2}}{y}dx - \frac{\sin\theta}{y}dy + d\theta.$$

Remember that the Godbillon-Vey form for this foliation is the volume form for $T_1\mathbb{H}$. We will construct an arbitrary enclosed region in $T_1\mathbb{H}$. Choose vertices $p_1, \dots p_q$ and draw geodesics $s_1, \dots, s_q$, where s_i goes from p_{i-1} to p_i for $i = 2, \dots q$ and s_1 goes from p_q to p_1. We reflect each s_i with respect to s_1 to obtain reflected sides s_i' and points p_i'. Notice that s_1 reflected across itself is s_1. Hence, $s_1 = s_1'$, $p_1 = p_1'$ and $p_q = p_q'$. Let K be the resulting $(2q-2)$-gon, i.e. the region enclosed by the sides $s_2, \dots, s_q$ and $s_2', \dots, s_q'$ (refer to Figure 14).

At this point, we need to build up some notation. Choose sufficiently small $\epsilon > 0$ and let K_ϵ be the closed region K minus an open ϵ-disk (with the half-plane metric) about each vertex (see Figure 15). Let $\pi : T_1\mathbb{H} \to \mathbb{H}$ be the

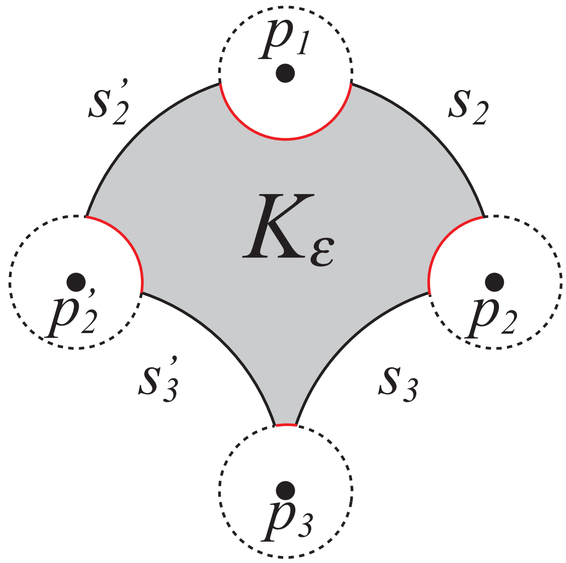

Figure 15: We remove ϵ-disks about each vertex to get the shaded region, K_ϵ.

projection map. Consider $\pi^{-1}(K_\epsilon) \subset T_1\mathbb{H}$. We can think of this as $K_\epsilon \times S^1$, i.e. a torus with edges (see Figure 16). Now, for $i = 2, \ldots, q$, let τ_i be the unique orientation preserving isometry which maps s_i to s_i'; then glue the strip $\pi^{-1}(K_\epsilon \cap s_i)$ to $\pi^{-1}(K_\epsilon \cap s_i')$ via the map τ_{i*} to obtain the quotient manifold M. We know that τ_{i*} maps leaves to leaves and so, our foliation defined by ω on K_ϵ naturally induces a foliation on M; we will still refer to this form as ω. Moreover, we can see that M is diffeomorphic to (S^2 minus q ϵ-disks) $\times S^1$, which, in turn, is diffeomorphic to (D^2 minus $(q-1)$ ϵ-disks) $\times$ S^1(refer to Figure 17). It is evident that the boundary of M is composed of q disjoint 2-dimensional tori. Our goal is to foliate M such that each boundary component is foliated by a 2-dimensional torus leaf. We will then glue in Reeb foliations.

We step back and recall our original polygon K (refer to Figure 14). Imagine that we are at some vertex, p, on our polygon K. We can modify our coordinates (x, y) in $\mathbb{H}$ to be taken with respect to p. We will use polar coordinates, (r, ϕ), where p is the center. Just as before, we will represent the unit tangent vector by the angle θ, taken as before (i.e., the angle between the tangent vector and the vertical line passing through its basepoint). We do this for each vertex on K. With this in mind, define the sets P_i and P_i':

$$\begin{aligned} P_i &= \pi^{-1}(B(p_i, 6\epsilon) \cap K_\epsilon) \qquad \text{and} \\ P_i' &= \pi^{-1}(B(p_i', 6\epsilon) \cap K_\epsilon), \end{aligned}$$

for $i = 1, \ldots, q$ (note that $P_1 = P_1'$ and $P_q = P_q'$), where $B(p, \delta)$ denotes a δ-disk about a point p. We can see that P_i and P_i' are subsets of $T_1\mathbb{H}$.[3] So

[3]The reader should note that our choice of 6ϵ is not intended to be facetious. We make

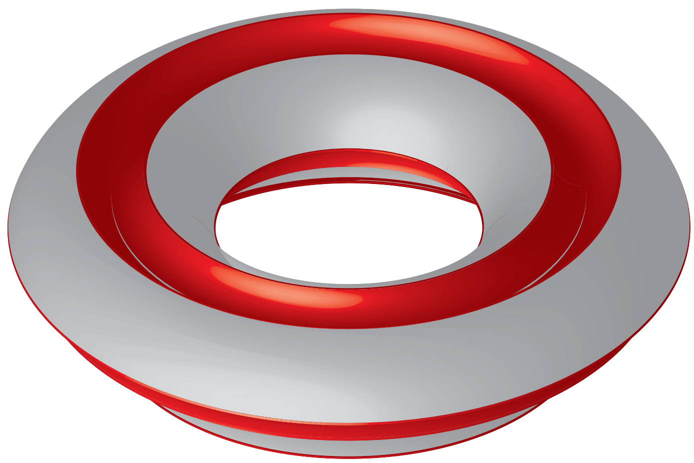

Figure 16: The space defined by $\pi^{-1}(K_\epsilon)$. The gray boundary strips are $\pi^{-1}(s\cap K_\epsilon)$ where s is a side s_i or s_i' of constructed polygon. The red boundary strips represent the $2q-2$ “troughs”.

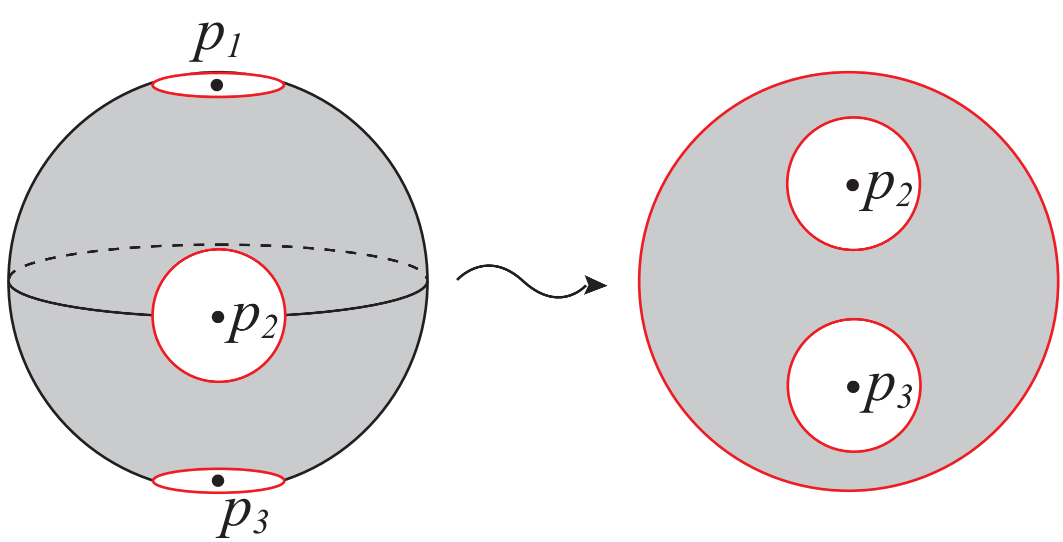

Figure 17: We take S^2 with q ϵ-surface disks removed and open up along the ϵ-disk “surrounding” p_1 to get D^2 with $(q-1)$ ϵ-disks removed.

points in P_i and P_i' will be given by polar coordinates crossed with S^1, i.e. (r, ϕ, θ), where (r, ϕ) is defined such that p_i is the center of the polar coordinate system, as discussed. Switching to polar coordinates will allow us to easily deform our foliation so that, for each component on the boundary of M, we get a 2-dimensional torus leaf.

Now, any point $(r, \phi, \theta) \in P_i$ is an element of a leaf $\mathcal{L}$ in our foliation of $T_1\mathbb{H}$. Thus, there is a unique angle $\beta \in [0, 2\pi)$ such that $(p_i, \beta) \in \mathbb{H} \times S^1$ is an element of $\mathcal{L}$. Define a function

$$\alpha_i : P_i \to [0, 2\pi)$$

such that $\alpha(r, \phi, \theta) = \beta$. This is obviously well-defined since we defined everything with unique elements. Furthermore, $\alpha_i(r, \phi, \theta) = c$ for some constant $c \in [0, 2\pi)$ for all $(r, \phi, \theta) \in P_i$ in the same leaf $\mathcal{L}$. Therefore, $d\alpha_i$ defines the same foliation as ω on P_i. We can duplicate this formulation to define the function

$$\alpha_i' : P_i' \to [0, 2\pi),$$

such that $d\alpha_i$ defines the same foliation on P_i' as ω.

We wish to show that $d\alpha_i$ and $d\alpha_i'$ are invariant under our gluing construction (i.e. $(\tau_{i*})^*$-invariant). So, given $(r, \phi, \theta) \in P_i$, we can see that $\tau_{i*}(r, \phi, \theta) \in P_i'$ since τ_{i*} preserves distance. Furthermore, keeping in mind that τ_{i*} preserves leaves on a foliation (maps leaves to leaves), define δ_i to be

$$\delta_i = \alpha_i'(\tau_{i*}(r, \phi, \theta)) - \alpha_i(r, \phi, \theta) \qquad \text{for } i = 1, \ldots, q.$$

We can take a geometric look at this equation. Let $(r, \phi, \theta) \in P_i$ which we know is on some leaf, $\mathcal{L}$, of the current foliation. The function α_i will retrieve the angle at p_i, say β_i, and then $\alpha_i' \circ \tau_{i*}$ will retrieve the angle at p_i', β_i' (see Figure 18). Thus, δ_i is the difference between these two angles. But, if we are given any other $(r_1, \phi_1, \theta_1) \in P_i$ on the same leaf, $\mathcal{L}$, then the map α_i will equal β_i, as before, since α_i is constant on a leaf of our foliation. Furthermore, since the map τ_{i*} preserves leaves, it will map $\mathcal{L}$ to the same leaf as before, and so $\alpha_i' \circ \tau_{i*}$ will be β_i', as before. Therefore, δ_i is constant on leaves which implies that $d\delta_i$ vanishes on leaves. So,

$$\begin{aligned}
\tau_i^* d\alpha_i(r, \phi, \theta) &= d\alpha_i'\left(\tau_{i*}(r, \phi, \theta)\right) \\
&= d\alpha_i(r, \phi, \theta) + d\delta_i(r, \phi, \theta) \\
&= d\alpha_i(r, \phi, \theta),
\end{aligned}$$

on any leaf.

So, not only do $d\alpha_i$ and $d\alpha_i'$ define the same foliation as ω on P_i and P_i', respectively, but they induce a foliation on the manifold obtained by gluing each $P_i \cap \pi^{-1}(s_i)$ to $P_i' \cap \pi^{-1}(s_i')$ since the leaves are preserved. Let $Q_i \subset M$ be the set $P_i \cup P_i'$ quotient-mapped into the glued manifold M. So, on Q_i, there exists a 1-form naturally induced by $d\alpha_i$ and $d\alpha_i'$, say $d\gamma_i$.

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this choice to give ourselves sufficient "room" to work with.

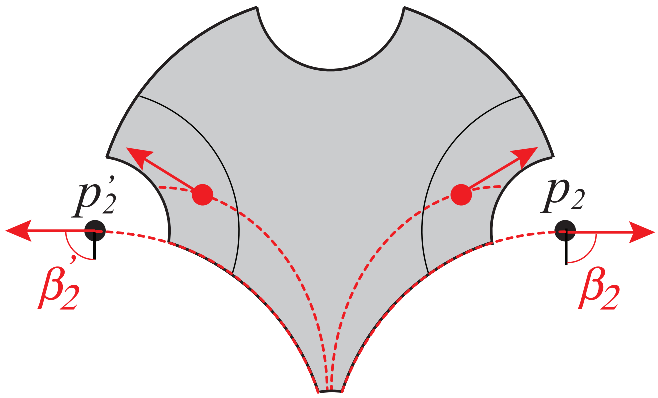

Figure 18: We will imagine $T_1\mathbb{H}$ in a planar view, i.e. as points in $\mathbb{H}$ with unit tangent vectors. The red points signify a point in P_i projected into $\mathbb{H}$ and it's unique orientation-preserving reflection in P_i'. The dashed red lines signify a family of parallel geodesics. Finally, β_i is the value given by α_i to the red point in P_i and β_i' is the value given by α_i' to the red point in P_i'.

Define a smooth, monotone decreasing function $h : [\epsilon, 6\epsilon] \to [0, 1]$ such that

$$h(r) = \begin{cases} 1 & \epsilon \le r \le 4\epsilon \\ 0 & 5\epsilon \le r \le 6\epsilon. \end{cases}$$

Further, define the 1-form ω' on M by

$$\omega' = \begin{cases} \omega & \text{on } M \setminus \bigcup_{i=1}^{q} Q_i \\ (1 - h(r))\omega + h(r) d\gamma_i & \text{on } Q_i. \end{cases}$$

It is important to notice that on $M - \cup_i^q Q_i$, we are using our standard coordinates (x, y, θ), while, on each Q_i, we are using the polar coordinates (with respect to vertices) crossed with S^1, (r, ϕ, θ). We are smoothly shifting the form from the standard coordinate system to the "polar" one. Via simple symbolic manipulation, we see that $\omega' \wedge d\omega' = 0$ and so, we still have a foliation on M.

Now that we have our nice polar coordinates, we will smoothly change our foliation so that we obtain torus leaves for the boundary components of M. Define a smooth monotone decreasing function $k : [\epsilon, 6\epsilon] \to [0, 1]$ as follows:

$$k(r) = \begin{cases} 1 & \epsilon \le r \le 2\epsilon \\ 0 & 3\epsilon \le r \le 6\epsilon. \end{cases}$$

Define the 1-form ω'' on M by

$$\omega'' = \begin{cases} \omega' & \text{on } M \setminus \bigcup_{i=1}^{q} Q_i \\ (1 - k(r))\omega' + k(r) dr & \text{on } Q_i. \end{cases}$$

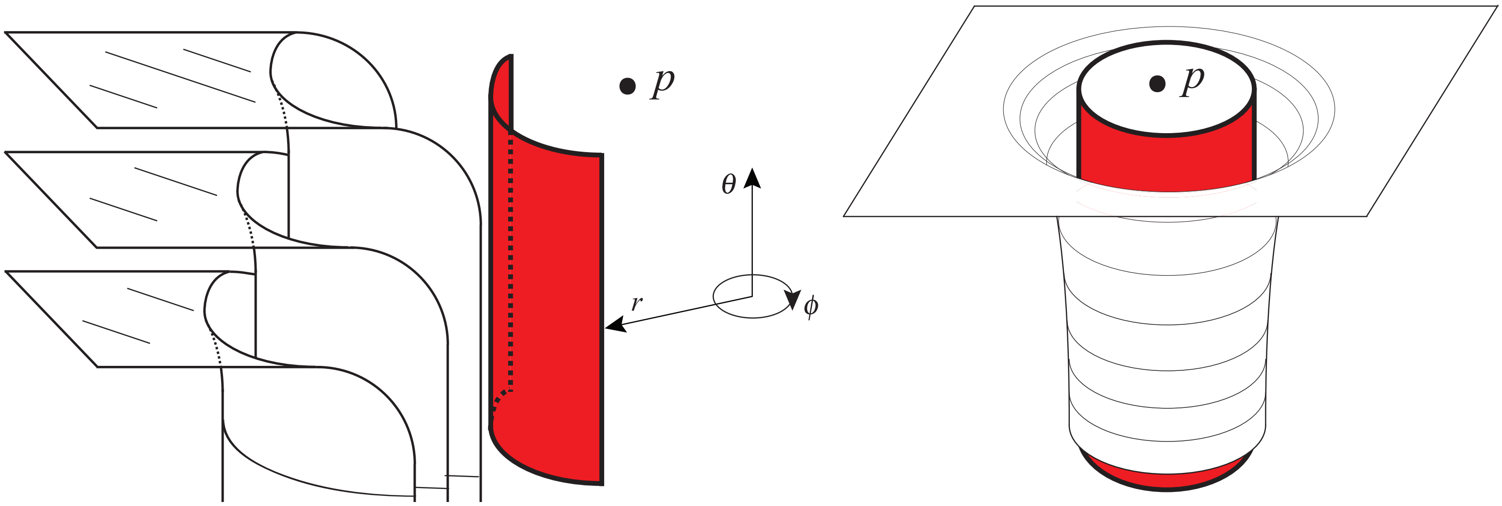

Figure 19: Our foliation smoothly deformed so as to be asymptotic to the boundary torus elements in two different perspectives. We can think about this as an "inside-out" Reeb foliation. Once again p is a point of reference: it is an element of $\pi^{-1}(p_i)$ or $\pi^{-1}(p_i')$ relative to the quotient manifold M.

Before venturing on, we point out that, when $r \in [\epsilon, 2\epsilon]$,

$$\omega'' = dr.$$

The above form defines the foliation giving leaves

$$\{r = c : \epsilon \leq c \leq 2\epsilon\} \subset M.$$

Thus, each leaf is diffeomorphic to the 2-dimensional torus. We see that our 1-form ω'' deforms our foliation by adjusting the path of each sheet so as to be asymptoptic to a torus leaf (see Figure 19). We see that the form does indeed define a folation on M since via straightforward computation, $\omega'' \wedge d\omega'' = 0$ on all of M. We may refer to Figure 20 to see the stages that our form ω'' goes through. Let Γ'' be the associated Godbillon-Vey form.[4]

7.4 Gluing in Reeb Components and Calculating the Godbillon-Vey Invariant

Our form ω'' defines the foliation that we want on M. Now, we simply glue q Reeb components inside of each of the q boundary torus leaves to obtain a foliation of S^3. To visualize the gluing, consider Figure 17: imagine spinning the disk with $(q-1)$-holes to get a solid torus with $(q-1)$ "worm holes". So,

[4]We actually must use a partition of unity to get our Godbillon-Vey form here, but it is unenlightening and downright heinus and so, the actual computation will be omitted.

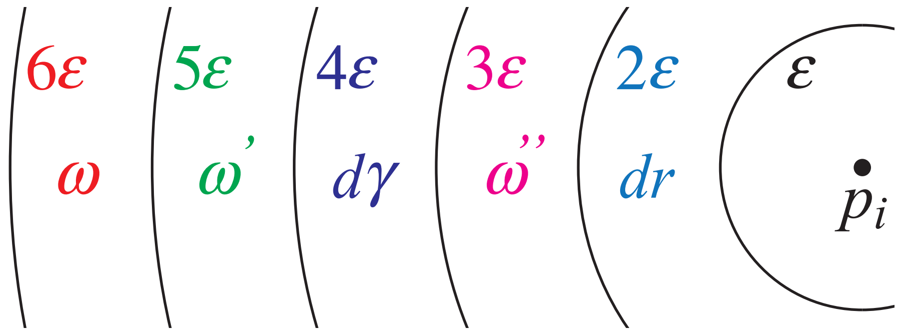

Figure 20: The transition of the final 1-form on M, ω'', is displayed as $r \in [\epsilon, 6\epsilon]$ varies on Q_i for some i.

the first $(q-1)$ Reeb components are glued into these worm holes to get a solid torus. We then glue the final Reeb component in the appropriate manner so as to obtain S^3.

We wish to compute the Godbillon-Vey number of our foliation. We know, however, that the Godbillon-Vey form (not just the cohomology class) of the Reeb foliation vanishes everywhere. Therefore, we only need to do our computation on M. Recall that Γ is the Godbillon-Vey form associated with our original foliation on M , defined by ω. Through the use of unenlightening direct (and exceedingly hideous) symbolic manipulation we get that:

$$\lim_{\epsilon \to 0} \int_M \Gamma'' = \int_M \Gamma.$$

Moreover, if we give ourselves another $\epsilon' > 0$ sufficiently small, we assert that corresponding foliation will be cobordant. Thus,

$$\Gamma''[M] = \Gamma[M].$$

So, we have proved our main result as the Godbillon-Vey number depends completely upon the area of the polygon (which affects the resulting volume of solid torus "with edges"), which we can continuously vary so as to give ourselves an uncountable set of Godbillon-Vey numbers.

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