Index and Inversions

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1. Introduction

This paper examines Foata's work [1] on proving a bijection between the index and inversions permutation statistics. The definitions are identified and presented in order of need. Each definition is illustrated with an example. By the end of this paper, we shall have found a one-to-one correspondence and given an example of this at work.

Note that ALL results, proofs and definitions come from Foata's paper [1].

2. Words: Index and Inversion Number

Definition 1. *Let $\mathcal{X}$ denote a set of ordered elements. A word $f = x_1 x_2 \cdots x_n$ is a finite string of elements such that each $x_i \in \mathcal{X}$. $\mathcal{X}^*$*

Note that each element need not be distinct. Repeated elements, such as the word $f = 1111$, are allowed under this definition. We would now like to define a set of words.

Definition 2. *Let $\mathcal{X}^*$ be the set of all possible words $f = x_1 x_2 \cdots x_n$, where $n \geq 0$.*

Once again, notice that this definition allows for words of *any* size, i.e. $f = 1\ 2\ 3$, $g = 4\ 3\ 5\ 6$ and $h = \emptyset$. $\mathcal{X}^*$ is therefore all encompassing.

We would like to provide define the permutation statistics upon which this entire paper is based:

Definition 3. *The inversion number of f, $S(f) = \#\{j \mid 1 \leq j < k \leq n, x_j > x_k\}$, and the index of f, $T(f) = \sum_{i=0}^{n} \{i \mid x_i > x_{i+1}\}$.*

To illustrate these permutation statistics, we provide an example.

Example 4. We are given $f = 6\ 3\ 4\ 7\ 9\ 1$. We do the following to determine the inversion number.

$$\begin{aligned}
x_1 = &\ 6 \ > x_2 = 3, x_3 = 4, x_6 = 1 \\
x_2 = &\ 3 \ > x_6 = 1 \\
x_3 = &\ 4 \ > x_6 = 1 \\
x_4 = &\ 7 \ > x_6 = 1 \\
x_5 = &\ 9 \ > x_6 = 1 \\
x_6 = &\ 1 \ > x_6 = 1
\end{aligned}$$

Therefore, $S(f) = 7$.

To find the index of f, we mark the locations where $x_i > x_{i+1}$ to get

$$6 \mid 3\ 4\ 7\ 9 \mid 1$$

which indicates that $T(f) = 1 + 5 = 6$.

3. Sets of Words

We would now like to work within this framework to construct a permutation that will prove the bijection between the index and number of inversions.

Definition 5. *Let Y, $Z \subseteq \mathcal{X}$. Y^* is the set of all possible words such that each element $x_i \in Y$. Y^*Z is the set of all possible words in Y^* where the final element, $x_n \in Z$.*

Similarly, we can see that ZY^* begins with the element $x_0 \in Z$, followed by a word in Y^*. We can demonstrate the above notation with an example.

Example 6. Let $\mathcal{X} = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 3\}$ and $Z = \{4, 5\}$. One can see that $Y, Z \subseteq \mathcal{X}$. Furthermore,

$$\begin{aligned} f &= \ 1\ 2\ 3\ 2\ 1 \ \in Y^*, \\ g &= \ 1\ 2\ 3\ 2\ 4 \ \in Y^*Z, \\ h &= \ 4\ 1\ 2\ 3\ 2 \ \in ZY^*. \end{aligned}$$

Definition 7. *Let* $L_x = \{y \in \mathcal{X} \mid y \leq x\}$ *and* $R_x = \{z \in \mathcal{X} \mid z > x\}$.

Example 8. Let $\mathcal{X} = \{1, 3, 4, 6, 7, 9\}$ and $x = 4$. $L_x = \{1, 3, 4\}$ and $R_x = \{6, 7, 9\}$.

Definition 9. $l_x f = \#\{i \mid 1 \leq i \leq n, x_i \leq x\}$ *and* $r_x f = \#\{j \mid 1 \leq j \leq n, x_i > x\}$.

Example 10. Let $f = 6\ 3\ 4\ 7\ 9\ 1$, $\mathcal{X} = \{1, 3, 4, 6, 7, 9\}$ and $x = 4$. We then have that

$$\begin{aligned} l_x f &= \ |\{1, 3, 4\}| \ = 3, \\ r_x f &= \ |\{6, 7, 9\}| \ = 3. \end{aligned}$$

Definition 11. *Given* $f = x_1 x_2 \cdots x_n \in \mathcal{X}^*$, *we let*

$$\begin{aligned} f^\pi &= \quad f & \text{if } n = 0, 1 \\ &= x_n x_1 x_2 \cdots x_{n-1} & \text{for } n > 1. \end{aligned}$$

Example 12. Let $f = 6\ 3\ 4\ 7\ 9\ 1$. We see that $f^\pi = 1\ 6\ 3\ 4\ 7\ 9$.

4. The Good Stuff

Now that we have established the tools that we will be working with, we can proceed onto the core of the proof that the index and number of inversions are one-to-one.

Let $f = x_1 x_2 \cdots x_n \in \mathcal{X}^* L_x$, for some x. Let $(r_1, r_2, \ldots, r_s)$ equal the integer i, where $1 \leq i \leq n$ and $x_i \in L_x$. We know this sequence of r_i's is non-empty since at the least our last element is such that $x_n \in L_x$. Set $r_0 = 0$ and for $p = 1, 2, \ldots, s$

$$f_p = x_{r_{p-1}+1} x_{r_{p-1}+2} \ldots x_{r_p}.$$

Remark 13. If we have $f \in \mathcal{X}^* R_x$ the above formula does not change. However, if $f \in L_x \mathcal{X}^*$ or $R_x \mathcal{X}^*$ then set $r_{s+1} = n + 1$ and for $p = 1, 2, \ldots, s$

$$f_p = x_{r_p} x_{r_p+1} \ldots x_{r_{p+1}}.$$

Example 14. Let $f = 6\ 3\ 4\ 7\ 9\ 1$, $\mathcal{X} = \{1, 3, 4, 6, 7, 9\}$, $x = 4$ ($\Rightarrow L_x = \{1, 3, 4\}$ and $R_x = \{6, 7, 9\}$). We see that $f \in \mathcal{X}^* L_x$ and so, $r_0 = 0, r_1 = 2, r_2 = 3, r_3 = 6$. We apply our above equation for f_p to get

$$\begin{aligned} f_1 &= \ x_{r_0+1} x_{r_1} \\ &= \ x_1 x_2 \\ &= \ 6\ 3 \\ f_2 &= \ x_{r_2} \\ &= \ x_3 \\ &= \ 4 \\ f_3 &= \ x_{r_2+1} x_{r_2+2} x_{r_3} \\ &= \ x_4 x_5 x_6 \\ &= \ 7\ 9\ 1. \end{aligned}$$

Remark 15. We see that $f = f_1\, f_2\, f_3$ from above. This is called a factorization of f.

For $f \in \mathcal{X}^* L_x$, $f_1\, f_2 \cdots f_s$ is the unique factorization of f s.t. each $f_p \in R_x^* L_x$. Furthermore, if we execute the f_p method for

$$\begin{aligned} f \in &\ \mathcal{X}^* R_x,\ L_x \mathcal{X}^*,\ R_x \mathcal{X}^* \\ f_p \in &\ L_x^* R_x,\ L_x R_x^*,\ R_x L_x^* \ \text{ respectively.} \end{aligned}$$

We now denote $\gamma_x(f) = f_1^\pi\, f_s^\pi \cdots f_s^\pi$.

Example 16. Use f, x from previous example to get

$$\begin{array}{ll} f_1^\pi & 3\ 6 \\ f_2^\pi & 4 \\ f_3^\pi & 1\ 7\ 9 \end{array} \quad \Rightarrow \gamma_x(f) = 3\ 6\ 4\ 1\ 7\ 9.$$

We can clearly see that γ_x is a permutation of $f \in \mathcal{X}^*$. We are simply rearranging the elements in f. We say $\alpha(f) = \alpha(f')$, when f and f' are rearrangements of the same elements. Therefore, we have that $\alpha(f) = \alpha(\gamma_x(f))$. Now, we introduce a lemma that displays some useful properties. Note that fx, is a word f with the element x appended at the end of the word.

Lemma 17. *For $f = x_1\, x_2 \cdots x_n \in \mathcal{X}^*$ and $x \in \mathcal{X}$, we have that*

$$\begin{aligned} S(fx) = &\ S(f) + r_x f && (1)\\ S(\gamma_x(f)) = &\ S(f) - r_x f \ \text{ if } f \in \mathcal{X}^* L_x && (2)\\ S(\gamma_x(f)) = &\ S(f) + l_x f \ \text{ if } f \in \mathcal{X}^* R_x && (3)\\ T(fx) = &\ T(f) \qquad \text{ if } f \in \mathcal{X}^* L_x && (4)\\ T(fx) = &\ T(f) + n \quad \text{ if } f \in \mathcal{X}^* R_x. && (5) \end{aligned}$$

Proof.

(1) When we add an element to the end of a word, the elements that are greater than it, i.e. each of the elements $x_i \in R_x$, are such that $x_i > x$. Therefore we are increasing the number of inversions by $r_x f$, the size of the set R_x.

(2) If $f \in \mathcal{X}^* L_x$, then by applying our γ_x permutation, we are having a modified factorization $f_1^\pi\, f_2^\pi \cdots f_s^\pi$, where each $f_p \in L_x R_x^*$. In this case, when we apply the f_p^π operation we are taking the smallest element at the end and moving it to the front of the word. Therefore, we are "losing" $r_x f_p$ inversions. When we put all the f_p^π elements together, we have "lost" $r_x f_1 + \ldots + r_x f_s = r_x f$ inversions.

(3) Same argument as (2).

(4) The last element of $f \in \mathcal{X}^* L_x$ is an element $x_n \leq x$. Therefore, when we append x to the word, the index value does not change.

(5) The last element of $f \in \mathcal{X}^* R_x$ is an element $x_n > x$. Therefore, when we append x to the word, the index value n is added to the previous index $T(f)$. [1] □

5. The Main Theorem

We are now ready to prove that $S(f)$ and $T(f)$ are one-to-one, (this implies they have the same generating function). First, we denote

$$\begin{aligned} \Phi(f) = &\ f && n \leq 1 \\ \Phi(fx) = &\ \gamma_x(\Phi(f))x && n > 2. \end{aligned}$$

We let $\Phi_n(f)$ be the map for words of length n. We are now able to display our main theorem. We note without proof that $\Phi(f)$ is a permutation that rearranges the elements of f.

Theorem 18. *Given $f \in \mathcal{X}^*$, the map $\Phi\colon \mathcal{X}^* \to \mathcal{X}^*$ is such that $S(\Phi(f)) = T(f)$.*

Proof. We will inductively assume that $S(\Phi_{n-1}(f)) = T(f)$, for a word of length $n-1$. Now, we are asked to prove this for $S(\Phi_n(fx))$. We know from our definition of the Φ map that

$$\begin{aligned} S(\Phi_n(fx)) = &\ S(\gamma_x(\Phi_n(f)x) \\ = &\ S(\gamma_x(\Phi_n(f) + r_x\gamma_x(\Phi(f)) && \text{from (3)} \\ = &\ S(\gamma_x(\Phi_n(f)) + r_x f && \text{since } \Phi \text{ just rearranges } f. \end{aligned}$$

We now break up the rest of the proof into two cases.

(i) $f \in \mathcal{X}^* L_x$. It follows that

$$\begin{aligned} S(\gamma_x(\Phi_{n-1}(f))) = &\ S(\Phi_{n-1}(f)) - r_x\Phi_{n-1}(f) && \text{from (2)} \\ = &\ S(\Phi_{n-1}(f)) - r_x f && \text{since } \Phi \text{ just rearranges } f. \end{aligned}$$

And so, we get that

$$\begin{aligned} S(\Phi_n(fx)) = &\ S(\Phi_{n-1}(f)) + r_x f - r_x f && \text{combining the final steps above} \\ = &\ S(\Phi_{n-1}(f)) \\ = &\ T(f) && \text{from our inductive step} \\ = &\ T(fx) && \text{since } x_n \in f \in \mathcal{X}^* L_x,\ x \geq x_n. \end{aligned}$$

Thus, we have shown our theorem is true for this case.

(ii) $f \in \mathcal{X}^* R_x$. It follows that

$$\begin{aligned} S(\gamma_x(\Phi_{n-1}(f))) = &\ S(\Phi_{n-1}(f)) + l_x\Phi_{n-1}(f) && \text{from (3)} \\ = &\ S(\Phi_{n-1}(f)) + l_x f && \text{since } \Phi \text{ just rearranges } f. \end{aligned}$$

And so, we get that

$$\begin{aligned} S(\Phi_n(fx)) = &\ S(\Phi_{n-1}(f)) + r_x f + l_x f && \text{combining the final steps above} \\ = &\ S(\Phi_{n-1}(f)) + (n-1) \\ = &\ T(f) + (n-1) && \text{from our inductive step} \\ = &\ T(fx) && \text{since } x_n \in f \in \mathcal{X}^* R_x, x < x_n. \end{aligned}$$

(Proof derived from [1]) □

We will display an example of how this works.

Example 19. Given $f = 6\ 3\ 4\ 7\ 9\ 1$, we want to find $\Phi(f)$. Our recurrence fans out to the following:

$$\begin{aligned} &\gamma_1(\gamma_9(\gamma_7(\gamma_4(\gamma_3(6)3)4)7)9)1 \\ &\to \gamma_1(\gamma_9(\gamma_7(\gamma_4(6\ 3)4)7)9)1 \\ &\to \gamma_1(\gamma_9(\gamma_7(3\ 6\ 4)7)9)1 \\ &\to \gamma_1(\gamma_9(3\ 6\ 4\ 7)9)1 \\ &\to \gamma_1(3\ 6\ 4\ 7\ 9)1 \\ &\to 3\ 6\ 4\ 7\ 9\ 1 \end{aligned}$$

We can see that $S(3\ 6\ 4\ 7\ 9\ 1) = 6$ and $T(6\ 3\ 4\ 7\ 9\ 1) = 6$, which is exactly what we want!

6. Conclusion

We have given a general overview of Foata's work [1] and in doing so, have provided an intuition as to why the Φ map works.

Bibliography

[1] D. Foata. On the netto inversion number of a sequence. *Proceedings of the American Mathematical Society*, 19:236–240, 1968.