

# Chapter 6: Eigenvalues and Eigenvectors

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## Abstract

This chapter focuses on the properties of eigenvalues and eigenvectors, diagonalizing a matrix, systems of differential equations, symmetric matrices, and positive definite matrices.

## 1 Introduction

The system  $A\mathbf{x} = \mathbf{b}$  is in equilibrium and steady state. Change as in time enters the picture - continuous time in a differential equation  $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$  or time steps in a difference equation  $\mathbf{u}_{k+1} = A\mathbf{u}_k$ . Using linear algebra, eigenvalues and eigenvectors allow these types of systems to be solved beautifully.

**Example 1.** Say we have a matrix  $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ , calculate  $A^2, A^3$ , and  $A^{100}$ .

$$A^2 = \begin{bmatrix} .70 & .30 \\ .30 & .55 \end{bmatrix} \quad A^3 = \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \quad A^{100} \approx \begin{bmatrix} .60 & .60 \\ .40 & .40 \end{bmatrix}$$

One way is to solve these equations using eigenvalues.

Vectors  $\mathbf{x}$  when multiplied by  $A$  usually change direction. But there are certain exceptional vectors that maintain the same direction as  $A\mathbf{x}$  and these are called “eigenvectors.” The basic equation is

$$A\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

where the number  $\lambda$  is an eigenvalue of  $A$  and  $\mathbf{x}$  is an eigenvector of  $A$ .

**Example 2.** What are the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}?$$

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix}.8 & .3 \\ .2 & .7\end{bmatrix} - \begin{bmatrix}\lambda & 0 \\ 0 & \lambda\end{bmatrix}\right) &= 0 \\
\begin{vmatrix}.8 - \lambda & .3 \\ .2 & .7 - \lambda\end{vmatrix} &= 0 \\
\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} &= 0 \\
(\lambda - 1)\left(\lambda - \frac{1}{2}\right) &= 0 \\
\lambda_1 = 1 \quad \lambda_2 = \frac{1}{2}
\end{aligned}$$

$A - \lambda I$  becomes a singular matrix and the eigenvectors  $\mathbf{v}_1, \mathbf{v}_2$  are in the nullspaces of  $A - I$  and  $A - \frac{1}{2}I$ .

$$(A - \lambda I)\mathbf{x} = 0$$

Finding  $x_1$ ,

$$\begin{aligned}
(A - I)x &= 0 \\
\begin{bmatrix}-.2 & .3 \\ .2 & -.3\end{bmatrix}\mathbf{x}_1 &= 0 \\
\mathbf{x}_1 &= \begin{bmatrix}.6 \\ .4\end{bmatrix}
\end{aligned}$$

Finding  $x_2$ ,

$$\begin{aligned}
(A - \frac{1}{2}I)x &= 0 \\
\begin{bmatrix}.3 & .3 \\ .2 & .2\end{bmatrix}\mathbf{x}_2 &= 0 \\
\mathbf{x}_2 &= \begin{bmatrix}1 \\ -1\end{bmatrix}
\end{aligned}$$

Checking,

$$\begin{aligned}
A\mathbf{x}_1 &= \begin{bmatrix}.8 & .3 \\ .2 & .7\end{bmatrix} \begin{bmatrix}.6 \\ .4\end{bmatrix} = 1 \begin{bmatrix}.6 \\ .4\end{bmatrix} = \lambda_1 \mathbf{x}_1 \\
A\mathbf{x}_2 &= \begin{bmatrix}.8 & .3 \\ .2 & .7\end{bmatrix} \begin{bmatrix}1 \\ -1\end{bmatrix} = \frac{1}{2} \begin{bmatrix}1 \\ -1\end{bmatrix} = \lambda_2 \mathbf{x}_2
\end{aligned}$$

And if  $A$  is multiplied  $n$  times we get  $A^n \mathbf{x}_1 = \lambda_1^n \mathbf{x}_1$ . Same goes for the second eigenvector. Also take note that the columns of  $A$  are a combination of the eigenvectors:  $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$ . The eigenvector  $\mathbf{x}_1$  is a *steady state* because  $\lambda_1 = 1$ . The eigenvector  $\mathbf{x}_2$  is a *decaying mode* that virtually disappears because  $\lambda_2 = .5$ . The higher the power of  $A$ , the more closely its columns approach the steady state.  $A$  is an example of a **Markov Matrix** where the sum of the entries for each column equal to one.

**Definition 1.1.** If  $A$  is multiplied  $n$  times, the eigenvectors stay the same and the eigenvalues are also multiplied  $n$  times.

## 1.1 Equation for Eigenvalues

To solve for the eigenvalues and eigenvectors, we start with Equation (1).

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ \boxed{(A - \lambda I)\mathbf{x} = \mathbf{0}} \end{aligned} \tag{2}$$

The eigenvectors of  $\mathbf{x}$  make up the nullspace of  $A - \lambda I$ . If we find the eigenvalue  $\lambda$  we can calculate for the eigenvector. To solve for the eigenvalue, we know that  $A - \lambda I$  is a singular matrix. Therefore its determinant is zero.

$$\boxed{\det(A - \lambda I) = p(\lambda) = 0} \tag{3}$$

$\det(A - \lambda I) = 0$  is called the **characteristic polynomial**. Generally, the characteristic polynomial is the following:

$$p(\lambda) = (-\lambda)^n + \text{trace}(A)(-\lambda)^{n-1} + \dots + \det(A)$$

To get the eigenvalues, we solve for  $\lambda$ . When  $A$  is an  $n \times n$  matrix, Equation 3 has degree  $n$ .  $A$  has  $n$  eigenvalues and repeating  $\lambda$  are possible. Each  $\lambda$  leads to  $\mathbf{x}$ . **For each  $\lambda$ , solve  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  to find the eigenvector.**

**Example 3.** Find the eigenvalues and eigenvectors of the singular matrix:  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ .

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) &= 0 \\ \lambda(\lambda - 5) &= 0 \\ \lambda_1 = 0 &\quad \lambda_2 = 5 \end{aligned}$$

For the first eigenvector  $\mathbf{x}_1$ :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ (A - 0I)\mathbf{x}_1 &= \mathbf{0} \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \mathbf{0} \end{aligned}$$

The nullspace solution is:

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

For the second eigenvector  $\mathbf{x}_2$ :

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= 0 \\ (A - 5I)\mathbf{x}_2 &= 0 \\ \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \end{aligned}$$

The nullspace solution is:

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If  $A$  is a singular matrix,  $\lambda = 0$  is an eigenvalue of  $A$ .

**Summary** To solve the eigenvalue problem for an  $n \times n$  matrix, follow these steps:

1. **Compute the determinant of  $A - \lambda I$ .** With  $\lambda$  subtracted along the diagonal, this determinant starts with  $\lambda^n$  or  $-\lambda^n$ . It's a polynomial of degree  $n$ .
2. **Find the roots of this polynomial**, by solving  $\det(A - \lambda I) = 0$ . The  $n$  roots are the  $n$  eigenvalues of  $A$ . They make  $A - \lambda I$  singular.
3. For each eigenvalue  $\lambda$ , **solve  $(A - \lambda I)\mathbf{x} = 0$  to find the eigenvector  $\mathbf{x}$ .**

**Warning!** There are times when  $A$  has equal eigenvalues. Hence, there is only one line of eigenvectors. Without a full set of eigenvectors, we can't diagonalize a matrix without  $n$  independent eigenvectors.

## 1.2 Determinant and Trace

We cannot get the eigenvalues of  $A$  when we do elimination to  $U$  because elimination does not preserve the eigenvalues.  $U$  has its own set of eigenvalues along its diagonal (the pivots) but these aren't the eigenvalues of  $A$ . The product  $\lambda_1 \times \lambda_2 \times \dots \times \lambda_n$  and the sum  $\lambda_1 + \lambda_2 + \dots + \lambda_n$  can be found directly from the matrix  $A$ .

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{has eigenvalues } \lambda = 0, \lambda = 7 \tag{4}$$

For  $A$ , the product of the eigenvalues is  $0 \cdot 7 = 0$ . This agrees with the  $\det(A) = 0$ . The sum of the eigenvalues  $0 + 7$  agrees with the sum down the diagonal of  $A$ . The sum of the

diagonal entries of  $A$  is called the **trace**. These two properties are important checks to see if our calculations for the eigenvalues are correct.

**Theorem 1.** The product of the  $n$  eigenvalues equals the determinant of matrix  $A$ .

$$\lambda_1 \times \lambda_2 \times \dots \times \lambda_n = \det(A) \quad (5)$$

*Proof.*

Consider  $\det(A - \lambda I)$ ,

$$\det(A - \lambda I) = p(\lambda)$$

Factorizing the characteristic equation according to its individual roots.

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

Setting  $\lambda = 0$ , therefore

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

□

**Theorem 2.** The sum of the  $n$  eigenvalues equals the sum of the  $n$  diagonal entries of  $A$  (the trace).

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = a_{11} + a_{22} + \dots + a_{nn} = \mathbf{trace}(A) \quad (6)$$

*Proof.* Consider the general equation of the characteristic polynomial  $p(\lambda)$ .

$$p(\lambda) = (-\lambda)^n + \text{trace}(A)(-\lambda)^{n-1} + \dots + \det(A)$$

Another expression of the characteristic polynomial is:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

By *comparing coefficients*, we get the trace:

$$\text{trace}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

□

**Example 4.** To see the proof above clearly, here is an example where there are two eigenvalues:

$$p(\lambda) = \lambda^2 - \text{trace}(A)\lambda + \det(A)$$

$p(\lambda)$  can also be expressed with the following:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)\lambda + \lambda^2$$

By comparing coefficients, we get the trace:

$$\text{trace}(A) = \lambda_1 + \lambda_2$$

**Theorem 3.** The eigenvalues of a triangular matrix lie along its diagonal.

*Proof.* Suppose  $A$  is a triangular matrix with nonzero entries.

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

Consider the determinant of  $A - \lambda I$ ,

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} = p(\lambda)$$

Where  $p(\lambda)$  is the characteristic equation. The roots of the characteristic equation are the eigenvalues. The determinant of  $A - \lambda I$  is 0 by definition because it is a singular matrix.

$$\begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} = 0$$

The determinant of a triangular matrix is the product of its diagonal entries. Hence,

$$\det(A - \lambda I) = \prod_{i=1}^n (a_{ii} - \lambda) = 0$$

Therefore, each diagonal entry in  $A$  is an eigenvalue.

$$a_{ii} = \lambda$$

□

## 2 Diagonalizing a Matrix

Eigenvalues and eigenvectors make matrix multiplication easier. When  $\mathbf{x}$  is an eigenvector, multiplication by  $A$  is just a multiplication by a number  $\lambda$ :  $A\mathbf{x} = \lambda\mathbf{x}$ . Diagonalizing a matrix turns  $A$  into a diagonal matrix  $\Lambda$  when we use the eigenvectors properly.

**Definition 2.1.** Suppose the  $n \times n$  matrix  $A$  has  $n$  linearly independent eigenvectors  $x_1, \dots, x_n$ . Put them into the columns of an eigenvector matrix  $X$ . Then  $X^{-1}AX$  is the **eigenvalue matrix**  $\Lambda$ :

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{bmatrix} \tag{7}$$

**Example 5.** Diagonalize  $A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$

$A$  is a triangular matrix, therefore its eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = 6$ . We then find its eigenvectors. Finding  $\mathbf{x}_1$ :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= 0 \\ \left( \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finding  $\mathbf{x}_2$ ,

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= 0 \\ \left( \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right) \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Diagonalizing:

$$\begin{aligned} X^{-1}AX &= \Lambda \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

**Theorem 4.**  $A^n$  has the same eigenvectors in  $X$  and the eigenvalues raised to the power of  $n$  in  $\Lambda^n$ .

*Proof.*

$$\begin{aligned} A^2 &= X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^2 X^{-1} \\ A^3 &= X\Lambda X^{-1}X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^3 X^{-1} \\ &\vdots \\ &\text{etc.} \end{aligned}$$

□

The matrix  $X$  must have an inverse because its columns are linearly independent. **Without  $n$  independent eigenvectors, we can't diagonalize.**

**Remark.** Suppose the eigenvalues  $\lambda_1, \dots, \lambda_n$  are all different. Then it is automatic that the eigenvectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are independent. The eigenvector matrix  $X$  will be invertible. **Any matrix that has no repeated eigenvalues can be diagonalized.**

**Remark.** We can multiply eigenvectors by any nonzero constants.  $A(c\mathbf{x}) = \lambda(c\mathbf{x})$  is still true.

**Remark.** The eigenvectors in  $X$  come in the same order as the eigenvalues in  $\Lambda$ .

**Remark.** Matrices that have repeated eigenvalues cannot be diagonalized.

**Example 6.** The Markov matrix  $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$  has eigenvalues  $\lambda_1 = 1$  and  $\lambda_2 = .5$  and eigenvectors of  $\mathbf{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$  and  $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -.6 \end{bmatrix}$ . Find  $A^2, A^k$ , and  $A^\infty$ .

$$A = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = X\Lambda X^{-1}$$

Computing for  $A^2$ .

$$A^2 = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

Computing for  $A^k$ .

$$A^k = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}$$

Computing for  $A^\infty$

$$A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \lim_{k \rightarrow \infty} \left( \begin{bmatrix} 1^k & 0 \\ 0 & .5^k \end{bmatrix} \right) \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

## 2.1 Similar Matrices: Same Eigenvalues

Similar matrices have the same  $\Lambda$  and different  $X$ .

**Theorem 5.** All matrices  $A = BCB^{-1}$  are “similar”. They all share the eigenvalues of  $C$ .

## 2.2 Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, ...

A new Fibonacci number is the sum of the two previous Fibonacci numbers in the Fibonacci sequence. The rule is:  $F_{k+2} = F_{k+1} + F_k$ . The obvious and slow way to get to  $F_{100}$  is by applying the rule one at a time. Linear algebra gives a faster method. The one step rule for the Fibonacci sequence is  $\mathbf{u}_{k+1} = A\mathbf{u}_k$ .



$$\text{Let } \mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

The rules below are put into matrix  $A$ .

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \\ \mathbf{u}_{k+1} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}_k \end{aligned}$$

Every step multiplies by  $A$ . After 100 steps we reach  $\mathbf{u}_{100} = A^{100}\mathbf{u}_0$ .

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

Finding the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 \\ \lambda_1 &= \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \end{aligned}$$

Finding the eigenvector  $\mathbf{x}_1$ ,

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1+\sqrt{5}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \end{aligned}$$

Finding the eigenvector  $\mathbf{x}_2$ ,

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= \mathbf{0} \\ \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1-\sqrt{5}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{bmatrix} \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} &= \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \end{aligned}$$

We need to express  $\mathbf{u}_0$  as a combination of the eigenvectors.

$$\begin{aligned}
c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} & -\frac{1}{\lambda_1 - \lambda_2} \\ -\frac{1}{\lambda_1 - \lambda_2} & \frac{1}{\lambda_1 - \lambda_2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \\ -\frac{1}{\lambda_1 - \lambda_2} \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

We now have the expression:

$$\begin{aligned}
\frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u}_k \\
\frac{1}{\lambda_1 - \lambda_2} \left( \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{u}_0 &= \frac{\mathbf{x}_1 - \mathbf{x}_2}{\lambda_1 - \lambda_2}
\end{aligned}$$

Then we have a generalized formula in terms of our eigenvalues:

$$\mathbf{u}_k = \frac{\lambda^k \mathbf{x}_1 - \lambda^k \mathbf{x}_2}{\lambda_1 - \lambda_2}$$

To get 100<sup>th</sup> Fibonacci number,  $F_{100}$ , we use the second component of  $\mathbf{x}_1, \mathbf{x}_2$

$$F_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}$$

Which is approximately,

$$F_{100} \approx 3.54224 \times 10^{20}$$

## 2.3 Matrix Powers $A^k$

$$A^k \mathbf{u}_0 = (X \Lambda X^{-1}) \dots (X \Lambda X^{-1}) \mathbf{u}_0 = X \Lambda^k X^{-1} \mathbf{u}_0$$

Fibonacci's example is a typical difference equation  $\mathbf{u}_{k+1} = A \mathbf{u}_k$ . Each step multiplies by  $A$ . The solution is  $\mathbf{u}_{k+1} = A^k \mathbf{u}_0$ . Diagonalizing the matrix gives a quick way to compute  $A^k$  and find  $\mathbf{u}_k$  in three quick steps.

1. Write  $\mathbf{u}_0$  as a combination  $c_1 \mathbf{x}_1 + \dots c_n \mathbf{x}_n$  of the eigenvectors. Then  $\mathbf{c} = X^{-1} \mathbf{u}_0$ .
2. Multiply each eigenvector  $\mathbf{x}_i$  by  $(\lambda_i)^k$ . Now we have  $\Lambda^k X^{-1} \mathbf{u}_0$ .

3. Add up the pieces  $c_i(\lambda_i)^k \mathbf{x}_i$  to find the solution  $\mathbf{u}_k = A^k \mathbf{u}_0$ . This is  $X\Lambda^k X^{-1} \mathbf{u}_0$ .

The solution for  $\mathbf{u}_{k+1} = A\mathbf{u}_k$  becomes:

$$\mathbf{u}_{k+1} = A\mathbf{u}_k = c_1(\lambda_1)^k \mathbf{x}_1 + \dots + c_n(\lambda_n)^k \mathbf{x}_n = X\Lambda^k X^{-1} \mathbf{u}_0 = X\Lambda^k \mathbf{c}$$

$$X\Lambda^k \mathbf{c} = \begin{bmatrix} | & \dots & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & \dots & | \end{bmatrix} \begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

## 2.4 Nondiagonalizable Matrices

When we solve for  $\lambda$  and  $\mathbf{x}$ , we want to know their **multiplicity**.

**Geometric Multiplicity = GM** - Count the independent eigenvectors for  $\lambda$ . Then GM is the dimension of the nullspace of  $A - \lambda I$ .

**Algebraic Multiplicity = AM** - AM counts the repetitions of  $\lambda$  among the eigenvalues. Look at the  $n$  roots of  $\det(A - \lambda I) = 0$ .

**Example 7.** Find the GM and AM of  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

$$\lambda = 0$$

The eigenvalue 0 is a repeating eigenvalue. Hence its AM = 2. The eigenvalue 0 only has one eigenvector hence GM = 1.

**Theorem 6.** If GM = AM, the matrix  $A$  is **diagonalizable**.

**Theorem 7.** If GM < AM, the matrix  $A$  is **not diagonalizable**.

### 3 Systems of Differential Equations

We want to solve  $\mathbf{u}$  from the following system of differential equations:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \quad \text{starting from the vector } \mathbf{u}(0) = \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix}$$

These equations are linear. If  $\mathbf{u}(t)$  and  $\mathbf{v}(t)$  are solutions, so is  $C\mathbf{u}(t) + D\mathbf{v}(t)$ . Our job is to find  $n$  “pure exponential solutions”  $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$  by using  $A\mathbf{x} = \lambda\mathbf{x}$ . Here  $\lambda$  is an eigenvalue while  $\mathbf{x}$  is an eigenvector. First we check if  $\mathbf{u}(t)$  is a solution.

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= A\mathbf{u} \\ \frac{d}{dt}(e^{\lambda t}\mathbf{x}) &= Ae^{\lambda t}\mathbf{x} \\ \lambda e^{\lambda t}\mathbf{x} &= Ae^{\lambda t}\mathbf{x} \\ \lambda\mathbf{x} &= A\mathbf{x} \end{aligned}$$

All components of this special solution  $\mathbf{u} = e^{\lambda t}\mathbf{x}$  share the same  $e^{\lambda t}$ . For a certain system, the pure exponential solutions  $\mathbf{u}_1 = e^{\lambda_1 t}\mathbf{x}_1$  satisfies  $A\mathbf{u}_1 = \mathbf{u}_1$ ,  $\mathbf{u}_2 = e^{\lambda_2 t}\mathbf{x}_2$  satisfies  $A\mathbf{u}_2 = \mathbf{u}_2$ , and so on. The solution **grows** when  $\lambda > 0$ . It **decays** when  $\lambda < 0$ . If  $\lambda$  is a complex number, its real part decides growth or decay. The imaginary part  $\omega$  **gives oscillation**  $e^{i\omega t}$  like a sine wave.

**Example 8.** Solve the following system of differential equations.

$$\begin{aligned} \frac{du_1}{dt} &= -u_1 + 2u_2 \\ \frac{du_2}{dt} &= u_1 - 2u_2. \end{aligned}$$

Here  $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$  and the initial value  $\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

We want to solve:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= A\mathbf{u} \\ \frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

We need to find the eigenvalues and eigenvectors of  $A$ . We know that  $A$  is a singular matrix, hence one of its eigenvalues is  $\lambda_1 = 0$ . We can get the second eigenvalue from the trace which is  $\lambda_2 = -3$ . We then find the eigenvectors. First for  $\lambda_1$ ,

$$\begin{aligned}(A - \cancel{\lambda_1}^0 I)\mathbf{x}_1 &= 0 \\ \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \mathbf{x}_1 &= 0 \\ \mathbf{x}_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}\end{aligned}$$

For the second eigenvector,

$$\begin{aligned}(A - \lambda_2 I)\mathbf{x}_2 &= 0 \\ \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 &= 0 \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

Then the general solution becomes:

$$\begin{aligned}\mathbf{u}(t) &= c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 \\ &= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

We then find  $c_1$  and  $c_2$ .

$$\begin{aligned}\mathbf{u}(0) &= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3(0)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}\end{aligned}$$

Hence the solution is:

$$\boxed{\mathbf{u}(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

**Example 9.** Solve  $\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}$  starting from  $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ . This is a vector equation for  $\mathbf{u}$ . It contains two scalar equations for the components  $y, z$ . They are “coupled together” because the matrix is not diagonal.

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

$$\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

The equation above means:

$$\frac{dy}{dt} = z \quad \text{and} \quad \frac{dz}{dt} = y$$

Finding the eigenvalues,

$$\det(A - \lambda I) = 0$$

$$\det \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\boxed{\lambda_1 = 1, \quad \lambda_2 = -1}$$

Finding the first eigenvector,

$$(A - \lambda_1 I)\mathbf{x}_1 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}_1 = 0$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finding the second eigenvector,

$$(A - \lambda_2 I)\mathbf{x}_2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 = 0$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution becomes:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Finding the constants  $c_1, c_2$ ,

$$\begin{aligned}\mathbf{u}(0) &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}\end{aligned}$$

Therefore, the solution becomes:

$$\boxed{\mathbf{u}(t) = 3e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

We would like to remind ourselves that really, the final solution is expressed in terms of a diagonalization:

$$\mathbf{u}(t) = X\Lambda X^{-1}\mathbf{u}_0 = X\Lambda c = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

**Summary** The same three steps in calculating  $\mathbf{u}_{k+1} = A\mathbf{u}_k$  now solve  $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ .

1. Write  $\mathbf{u}_0$  as a combination  $c_1\mathbf{x}_1 + \dots c_n\mathbf{x}_n$  of the eigenvectors of  $A$ .
2. Multiply each eigenvector  $\mathbf{x}_i$  by its growth factor  $e^{\lambda_i t}$ .
3. The solution is the same combination of those pure solutions  $e^{\lambda_i t}\mathbf{x}_i$ :

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} \quad \mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + \dots + c_n e^{\lambda_n t} \mathbf{x}_n.$$

**Warning!** If a  $\lambda$  repeats, with only one eigenvector, another solution is needed which is  $(te^{\lambda t})$ .

### 3.1 Second Order Differential Equations

The most important equation in mechanics is  $my'' + by' + ky = 0$ . The first term is the mass times the acceleration  $a = y''$ . The term  $ma$  balances the force  $F$  (Newton's second law). The force includes the damping  $-by'$  and the elastic force  $-ky$ . This second order differential equation is still linear with coefficients  $m, b, k$ .

The method of solution to this equation is to substitute  $y = e^{\lambda t}$ . Each derivative of  $y$

brings down a factor  $\lambda$ .

$$\begin{aligned}
m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky &= 0 \\
m \frac{d^2}{dt^2} e^{\lambda t} + b \frac{d}{dt} e^{\lambda t} + k e^{\lambda t} &= 0 \\
m \lambda \frac{d}{dt} e^{\lambda t} + b \lambda e^{\lambda t} + k e^{\lambda t} &= 0 \\
m \lambda^2 e^{\lambda t} + b \lambda e^{\lambda t} + k e^{\lambda t} &= 0 \\
(m \lambda^2 + b \lambda + k) e^{\lambda t} &= 0
\end{aligned}$$

There are two roots for  $\lambda$ ,  $\lambda_1$  and  $\lambda_2$ . This equation for  $y$  has two pure solutions  $y_1 = e^{\lambda_1 t}$  and  $y_2 = e^{\lambda_2 t}$ . Their combinations  $c_1 y_1 + c_2 y_2$  give the complete solution unless the  $\lambda$ 's are the same. We want to express the problem into a vector equation for  $y$  and  $y'$ . We have:

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A \mathbf{u} \quad (8)$$

$A$  comes from the fact that  $y' = y'$  and  $y'' = -by' - ky$ . We then solve  $\mathbf{u}' = A\mathbf{u}$  by the eigenvalues of  $A$ .

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{vmatrix} &= 0 \\
\lambda^2 + b\lambda + k &= 0
\end{aligned}$$

The roots  $\lambda_1$  and  $\lambda_2$  will become our eigenvalues. Now to find the eigenvectors. Finding  $\mathbf{x}_1$ :

$$\begin{aligned}
\begin{bmatrix} -\lambda_1 & 1 \\ -k & -b - \lambda_1 \end{bmatrix} \mathbf{x}_1 &= 0 \\
\mathbf{x}_1 &= \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}
\end{aligned}$$

$\mathbf{x}_1$  will hold if  $-k + \lambda_1(-b - \lambda_1) = 0$ . Finding  $\mathbf{x}_2$ .

$$\begin{aligned}
\begin{bmatrix} -\lambda_2 & 1 \\ -k & -b - \lambda_2 \end{bmatrix} \mathbf{x}_2 &= 0 \\
\mathbf{x}_2 &= \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}
\end{aligned}$$

$\mathbf{x}_2$  will hold if  $-k + \lambda_2(-b - \lambda_2) = 0$ . The general solution will then be:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$



**Example 10. Motion around a circle with  $y'' + y = 0$  and  $y = \cos(t)$ .** This is our master equation with mass  $m = 1$  and stiffness  $k = 1$  and  $d = 0$  (no damping). Substitute  $y = e^{\lambda t}$  into  $y'' + y = 0$  to reach  $\lambda^2 + 1 = 0$ . The roots are complex numbers  $\lambda = \pm i$ . Then half of  $e^{it} + e^{-it}$  gives the solution  $y = \cos(t)$ . The initial values are  $y(0) = 1, y'(0) = 0$ . Find  $\mathbf{u}(t)$ .

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\mathbf{u}$$

Finding the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 1 &= 0 \\ (\lambda - i)(\lambda + i) &= 0 \\ \boxed{\lambda_1 = i \quad \lambda_2 = -i} \end{aligned}$$

Finding the eigenvector  $\mathbf{x}_1$ :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= 0 \\ \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ i \end{bmatrix} \end{aligned}$$

Finding the eigenvector  $\mathbf{x}_2$ :

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_1 &= 0 \\ \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ -i \end{bmatrix} \end{aligned}$$

The general solution is:

$$\begin{aligned} \mathbf{u}(t) &= c_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ \mathbf{u}(t) &= \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ ic_1 e^{it} - ic_2 e^{-it} \end{bmatrix} \end{aligned}$$

Finding  $c_1$  and  $c_2$ :

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{c} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

The final solution is:

$$\mathbf{u}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^{it} + e^{-it}) \\ \frac{1}{2}i(e^{it} - e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

### 3.2 Exponential of a Matrix

We want to write the solution  $\mathbf{u}(t)$  in a new form  $e^{At}\mathbf{u}(0)$ . First we have to define what  $e^{At}$  means by copying the Taylor Series representation of  $e^x$  for numbers.

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots$$

Its derivative with respect to  $t$  is:

$$\frac{d}{dt}e^{At} = Ae^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \dots$$

The eigenvalues are  $e^{\lambda t}$

$$e^{At}\mathbf{x} = \left( \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \right) \mathbf{x} = (I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots)\mathbf{x} = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \dots)\mathbf{x}$$

The series always converges and its derivative is always  $Ae^{At}$ . Therefore  $e^{At}\mathbf{u}(0)$  solves the differential equation with one quick formula - even if there is a shortage of eigenvectors. This chapter emphasizes how to find  $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$  by diagonalization. Assume  $A$  does have  $n$  independent eigenvectors, so it is diagonalizable. Substitute  $A = S\Lambda S^{-1}$  into the series for  $e^{At}$ .

$$\begin{aligned} e^{At} &= e^{X\Lambda X^{-1}t} = \sum_{n=0}^{\infty} \frac{(X\Lambda X^{-1}t)^n}{n!} = I + (X\Lambda X^{-1}t) + \frac{1}{2}(X\Lambda X^{-1}t)^2 + \dots \\ &= X \left[ I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \dots \right] X^{-1} \end{aligned}$$

We now arrive at a diagonalized form for  $e^{At}$ :

$$\boxed{e^{At} = X e^{\Lambda t} X^{-1}} \tag{9}$$

The solution becomes:

$$\mathbf{u}(t) = e^{At}\mathbf{u}_0 = X e^{\Lambda t} X^{-1}\mathbf{u}(0) = \begin{bmatrix} | & \dots & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & \dots & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

**Example 11.** Solve  $y'' - 2y' + y = 0$ .

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{u}$$

Finding the eigenvalues of  $A$ :

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ -1 & 2 - \lambda \end{vmatrix} &= 0 \\ (\lambda - 1) &= 0 \\ \boxed{\lambda_{1,2} = 1} \end{aligned}$$

Here we only have one unique eigenvalue. We then find the eigenvector  $\mathbf{x}_1$ .

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}_1 &= \mathbf{0} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Hence diagonalization is not possible because we do not have 2 eigenvectors. We will then compute  $e^{At}$  instead.

$$e^{At} = e^{At+It-It} = e^t e^{(A-I)t} = e^t \sum_{n=0}^{\infty} \frac{((A-I)t)^n}{n!}$$

Expanding the series:

$$\sum_{n=0}^{\infty} \frac{(((A-I)t))^n}{n!} = I + (A-I)t + \frac{1}{2}(A-I)^2 t^2 + \frac{1}{6}(A-I)^3 t^3 + \dots$$

When the series reaches  $n = 2$  and above, the  $(A - I)$  term becomes zero. Hence,

$$e^{At} = e^t(I + (A - I)t)$$

Hence,

$$\mathbf{u} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^t \left[ I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \right] \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} e^t y(0) - te^t y(0) + te^t y'(0) \\ -e^t t y(0) + e^t y'(0) + te^t y'(0) \end{bmatrix}$$

The solution to our problem becomes:

$$y(t) = e^t y(0) - te^t y(0) + te^t y'(0)$$

**Example 12.** Use the infinite series to find  $e^{At}$  for  $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ .

$$\begin{aligned} e^{At} &= I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \frac{1}{24}(At)^4 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{24} \begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} \end{aligned}$$

Notice that when we reach  $n = 5$ , the signs are the same as  $n = 2$ ,

$$\begin{aligned} &= \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots & t - \frac{1}{6}t^3 + \dots \\ -t + \frac{1}{6}t^3 + \dots & 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots \end{bmatrix} \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} & \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \end{bmatrix} \\ e^{At} &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

**Example 13.** Solve  $\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{u}$  starting from  $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

We can see the eigenvalues immediately since  $A$  is a triangular matrix. These are  $\lambda_1 = 1$  and  $\lambda_2 = 2$ . Now to find the eigenvectors. For  $\mathbf{x}_1$ :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}_1 &= \mathbf{0} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finding  $\mathbf{x}_2$ :

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= \mathbf{0} \\ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_2 &= \mathbf{0} \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution becomes:

$$\mathbf{u}(t) = X e^{At} X^{-1} \mathbf{u}(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t + e^{2t} \\ e^{2t} \end{bmatrix}$$

## 4 Symmetric Matrices

Say we are given an equation  $S\mathbf{x} = \lambda\mathbf{x}$  where  $S$  is a symmetric matrix ( $S = S^T$ ). When we diagonalize  $S = X^{-1}\Lambda X$  and take its transpose, we get an interesting property.]:

$$S^T = (X\Lambda X^{-1})^T = X^T\Lambda(X^{-1})^T$$

Since  $S = S^T$ , therefore

$$X^{-1} = X^T$$

Then,

$$X^T X = I$$

This means that the  $n$  eigenvectors in  $X$  are orthogonal.

Here are key facts about symmetric matrices:

1. A symmetric matrix has only real eigenvectors
2. The eigenvectors can be chosen orthonormal (a normalized orthogonal matrix).

**Every symmetric matrix can be diagonalized.** Its eigenvector matrix  $X$  becomes an orthogonal matrix  $Q$ . Orthogonal matrices make use of the property  $Q^{-1} = Q^T$ .

**Theorem 8. First Spectral Theorem** Every symmetric matrix has the factorization  $S = Q\Lambda Q^T$  with real eigenvalues of  $\Lambda$  and orthonormal eigenvectors in the columns  $Q$ .

$$S = Q\Lambda Q^{-1} = Q\Lambda Q^T \tag{10}$$

*Proof.* We need to prove that a symmetric matrix does not have repeating eigenvalues.  $\square$

*Proof.* We need to prove that the diagonalization  $Q\Lambda Q^T$  is still symmetric.

Taking the transpose:

$$(Q\Lambda Q^T)^T = Q\Lambda Q^T$$

$\square$

**Theorem 9.** All eigenvalues of a real symmetric matrix are real.

*Proof.*

Suppose that  $S\mathbf{x} = \lambda\mathbf{x}$ ,  $\lambda$  might be a complex number  $a + ib$  ( $a$  and  $b$  are real). Its complex conjugate  $\bar{\lambda} = a - ib$ .  $\mathbf{x}$  might be complex numbers also,  $\bar{\mathbf{x}}$  gives the imaginary part. Then we have two equations.

$$S\mathbf{x} = \lambda\mathbf{x} \quad (11)$$

$$S\bar{\mathbf{x}} = \lambda\bar{\mathbf{x}} \quad (12)$$

Transposing Equation (12), we have,

$$\begin{aligned} \bar{\mathbf{x}}^T S^T &= \bar{\mathbf{x}}^T \bar{\lambda} \\ \bar{\mathbf{x}}^T S &= \bar{\mathbf{x}}^T \bar{\lambda} \end{aligned} \quad (13)$$

Taking the dot product of  $\bar{\mathbf{x}}$  with Equation (11):

$$\begin{aligned} \bar{\mathbf{x}} \cdot S\mathbf{x} &= \bar{\mathbf{x}} \cdot \lambda\mathbf{x} \\ \bar{\mathbf{x}}^T S\mathbf{x} &= \bar{\mathbf{x}}^T \lambda\mathbf{x} \end{aligned} \quad (14)$$

Multiplying Equation (13) with  $\bar{\mathbf{x}}$ :

$$\bar{\mathbf{x}}^T S\mathbf{x} = \bar{\mathbf{x}}^T \bar{\lambda}\mathbf{x} \quad (15)$$

The right sides of Equations (14) and (15) are equal.

$$\begin{aligned} \bar{\mathbf{x}}^T \lambda\mathbf{x} &= \bar{\mathbf{x}}^T \bar{\lambda}\mathbf{x} \\ \lambda\bar{\mathbf{x}}^T \mathbf{x} &= \bar{\lambda}\bar{\mathbf{x}}^T \mathbf{x} \end{aligned}$$

$\bar{\mathbf{x}}^T \mathbf{x}$  is just the norm  $||\mathbf{x}||^2$  which is always positive. Hence:

$$\lambda = \bar{\lambda}$$

Therefore:

$$\begin{aligned} a + ib &= a - ib \\ b &= 0 \end{aligned}$$

$\lambda$  is always real.

□

**Theorem 10.** Eigenvectors of a real symmetric matrix (when they correspond to different  $\lambda$ 's) are always perpendicular.

*Proof.*

$$S\mathbf{x} = \lambda_1\mathbf{x} \quad (16)$$

$$S\mathbf{y} = \lambda_2\mathbf{y} \quad (17)$$

Taking the dot product of Equation (16) with  $\mathbf{y}$

$$\begin{aligned} S\mathbf{x} \cdot \mathbf{y} &= \lambda_1\mathbf{x} \cdot \mathbf{y} \\ S\mathbf{x}^T\mathbf{y} &= \mathbf{x}^T\lambda_1\mathbf{y} \end{aligned} \quad (18)$$

Taking the dot product of  $\mathbf{y}$  with Equation (17)

$$\begin{aligned} \mathbf{x} \cdot S\mathbf{y} &= \mathbf{x} \cdot \lambda_2\mathbf{y} \\ \mathbf{x}^T S\mathbf{y} &= \mathbf{x}^T\lambda_2\mathbf{y} \end{aligned} \quad (19)$$

The left hand sides of Equations (18) and (19) are equal. Then,

$$\mathbf{x}^T\lambda_1\mathbf{y} = \mathbf{x}^T\lambda_2\mathbf{y} \quad (20)$$

Since  $\lambda_1 \neq \lambda_2$ , for Equation (20) to hold,

$$\boxed{\mathbf{x}^T\mathbf{y} = 0}$$

The eigenvectors are perpendicular.

□

**Remark.** Every symmetric matrix that are diagonalizable can be expressed by the following:

$$S = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots + \lambda_n q_n q_n^T \quad (21)$$

## 4.1 Complex Eigenvalues of Real Matrices

For real matrices, complex eigenvalues and eigenvectors come in conjugate pairs. Let  $\lambda = a + ib$ , then its complex conjugate will be  $\bar{\lambda} = a - ib$ :

$$\text{If } A\mathbf{x} = \lambda\mathbf{x}, \text{ then } A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}.$$

**Example 14.** Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Computing for the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} \cos \theta - \lambda & -\sin \theta \\ \sin \theta & \cos \theta - \lambda \end{vmatrix} &= 0 \\
(\cos \theta - \lambda)^2 + \sin^2 \theta &= 0 \\
\lambda^2 - 2\lambda \cos \theta + 1 &= 0 \\
\lambda_1 = \cos \theta + i \sin \theta & \quad \lambda_2 = \cos \theta - i \sin \theta
\end{aligned}$$

We can see that  $\lambda_2 = \bar{\lambda}_1$ , these are a conjugate pair. Next we find the eigenvectors:

$$\begin{aligned}
(A - \lambda_1 I) \mathbf{x}_1 &= 0 \\
\left( \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} - (\cos \theta + i \sin \theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{x}_1 &= \mathbf{0} \\
\begin{bmatrix} -i \sin \theta & -\sin \theta \\ \sin \theta & -i \sin \theta \end{bmatrix} \mathbf{x}_1 &= \mathbf{0} \\
\mathbf{x}_1 &= \begin{bmatrix} 1 \\ -i \end{bmatrix}
\end{aligned}$$

Finding  $\mathbf{x}_2$ :

$$\begin{aligned}
(A - \lambda_2 I) \mathbf{x}_2 &= 0 \\
\left( \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} - (\cos \theta - i \sin \theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{x}_2 &= \mathbf{0} \\
\begin{bmatrix} i \sin \theta & -\sin \theta \\ \sin \theta & i \sin \theta \end{bmatrix} \mathbf{x}_2 &= \mathbf{0} \\
\mathbf{x}_2 &= \begin{bmatrix} 1 \\ i \end{bmatrix}
\end{aligned}$$

Again, we see that the eigenvectors are a conjugate pair.

## 4.2 All Symmetric Matrices are Diagonalizable

For nonsymmetric matrices, a shortage of eigenvalues (repeating eigenvalues) cause a shortage of eigenvectors. This never happens in symmetric matrices. **There are always enough eigenvectors to diagonalize  $S = S^T$ .**

Every square matrix can be “triangularized” by  $A = QTQ^{-1}$ . If  $A = S$  (symmetric), then  $T = \Lambda$ .

**Theorem 11.** Every square  $A$  factors into  $QTQ^{-1}$  where  $T$  is upper triangular and  $\bar{Q}^T = Q^{-1}$ . If  $A$  has real eigenvalues then  $Q$  and  $T$  can be chosen real  $Q^T Q = I$ .

## 5 Positive Definite Matrices