Chapter 5: Determinants

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February 2019

Abstract

This chapter focuses on properties, and methods on computing determinants.

1 Properties of Determinants

Any square matrix, invertible or not, has a special number that contains a lot of information called the **determinant**. It tells immediately if a matrix is invertible or not since **a singular matrix has a determinant of zero**. When A is invertible, the determinant of A^{-1} is $\frac{1}{\det(A)}$. There are three common ways to compute for the determinant. These are the following:

1. Pivot formula
2. "Big" formula
3. Cofactor formula

The determinant is written in 2 ways, $\det(A)$ and $|A|$. The determinant of

$$A \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Property 1 Determinant of a square identity matrix is 1.

$$\det I = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1$$

Property 2 The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example 1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \rightarrow \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

In the earlier chapters, we know that row exchanges can be performed by multiplying matrix A with a permutation matrix P. $\det(P) = +1$ for **even** number of row exchanges and $\det(P) = -1$ for **odd** row exchanges.

Property 3 The determinant is a linear function or operator for each row. If the first row is multiplied by t, the determinant is multiplied by t.

Example 2.

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = tad - tbc = t(ad - bc) = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example 3.

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = (a + a')d - (b + b')c = (ad - bc) + (a'd - b'c) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & 'b \\ c & d \end{vmatrix}$$

Example 4.

$$\begin{aligned} \begin{vmatrix} a + a' & b + b' \\ tc & td \end{vmatrix} &= (a + a')td - (b + b')tc \\ &= t\left[ad + a'd - bc - b'c\right] \\ &= t\left[(ad - bc) + (a'd - b'c)\right] \end{aligned}$$

$$\begin{aligned} \det(A^2) &= (\det(A))^2 \\ \det(2A) &= 2^n \det(A) \end{aligned}$$

Property 4 If two rows are equal, the determinant is zero.

Example 5.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

If we perform a row exchange we still get the same matrix but because of rule 2 the determinant must change signs. We get $-D = D$, and the only way that this is consistent is when $D = 0$. A matrix with two equal rows has no inverse (matrix is singular).

Property 5 Elimination from A to U does not change the value of the determinant. Hence $\det(A) = \det(U)$.

Example 6.

$$\begin{aligned} \begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} &= a(d - lb) - b(c - la) \\ &= ad - \cancel{alb} - bc + \cancel{alb} \\ &= ad - bc \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \end{aligned}$$

Property 6 A matrix with a row of zeros has a determinant of zeros.

Property 7 The determinant of an upper triangular matrix is the product of the diagonal entries.

$$U = \begin{bmatrix} a_{11} & \dagger & \dots & \dagger \\ 0 & \ddots & \vdots & \dagger \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

$$\det U = \prod_{i=1}^{n} a_{ii}$$

To verify property 1, we simply multiply the diagonals of an identity matrix and $\det(I) = 1$. If there is a zero in the diagonal, then the determinant is zero.

Property 8 If A is a singular matrix, then $\det(A) = 0$. If A is invertible then $\det(A) \neq 0$. This is evident in singular matrices when we do elimination from A to U and we get a row of zeros and via rule 6 the determinant is zero. If A is invertible then U has pivots along its diagonal. The product of nonzero pivots gives a nonzero determinant.

$$\det(A) = \pm\det(U) = \pm(\textbf{product of the pivots})$$

The pivots of a 2×2 matrix are a and $d - (c/a)b$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (c/a)b \end{vmatrix} = ad - bc$$

The sign in $\pm\det(U)$ depends on whether the number of row exchanges is even or odd: +1 or -1 is the determinant of the permutation P that exchanges rows. With no row exchanges, $P = I$ and $\det(A) = \det(U)$.

$$PA = LU$$
$$\det(P)\det(A) = \det(L)\det(U)$$

$\det(P) = \det(I) = 1$ and L's diagonal only contains ones, hence $\det(L) = 1$.

$$\det(A) = \det(U)$$

Property 9 The determinant of AB is:

$$\det(AB) = \det(A)\det(B).$$

From this property, we can calculate the inverse.

$$AA^{-1} = I$$
$$\det(A)\det(A^{-1}) = 1$$
$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

To prove this property, suppose we have two 2×2 matrices A and B.

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad B = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

$$\begin{aligned} |A||B| &= (ad - bc)(ps - qr) \\ &= (ap + br)(cq + ds) - (aq + bs)(cp + dr) = |AB| \end{aligned}$$

Property 10 The determinant of the transpose is equal to the determinant of the original matrix.

$$\det(A^T) = \det(A)$$

Proof. Say A is invertible and we do not have any row exchanges, then $P = I$.

$$\begin{aligned} PA &= LU \\ A^T P^T &= U^T L^T \\ \det(A^T)\det(P^T) &= \det(U^T)\det(L^T) \end{aligned}$$

L^T has ones in the diagonal then its determinant is 1. Same with the determinant of P^T.

$$\det(A^T) = \det(U^T)$$

U and U^T have the same diagonals, therefore A and A^T have the same determinants. $\square$

Example 7.

$$\begin{aligned} \begin{vmatrix} a & c \\ b & d \end{vmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ ad - bc &= ad - bc \end{aligned}$$