Chapter 5: Determinants

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Abstract

This chapter focuses on properties, and methods on computing determinants.

1 Properties of Determinants

Any square matrix, invertible or not, has a special number that contains a lot of information called the **determinant**. It tells immediately if a matrix is invertible or not since **a singular matrix has a determinant of zero**. When A is invertible, the determinant of A^{-1} is $\frac{1}{\det(A)}$. There are three common ways to compute for the determinant. These are the following:

1. Pivot formula
2. "Big" formula
3. Cofactor formula

The determinant is written in 2 ways, $\det(A)$ and $|A|$. The determinant of

$$A \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Property 1 Determinant of a square identity matrix is 1.

$$\det I = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1$$

Property 2 The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example 1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \rightarrow \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

In the earlier chapters, we know that row exchanges can be performed by multiplying matrix A with a permutation matrix P. $\det(P) = +1$ for **even** number of row exchanges and $\det(P) = -1$ for **odd** row exchanges.

Property 3 The determinant is a linear function or operator for each row. If the first row is multiplied by t, the determinant is multiplied by t.

Example 2.

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = tad - tbc = t(ad - bc) = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example 3.

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = (a + a')d - (b + b')c = (ad - bc) + (a'd - b'c) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & 'b \\ c & d \end{vmatrix}$$

Example 4.

$$\begin{aligned} \begin{vmatrix} a + a' & b + b' \\ tc & td \end{vmatrix} &= (a + a')td - (b + b')tc \\ &= t\left[ad + a'd - bc - b'c\right] \\ &= t\left[(ad - bc) + (a'd - b'c)\right] \end{aligned}$$

$$\begin{aligned} \det(A^2) &= (\det(A))^2 \\ \det(2A) &= 2^n \det(A) \end{aligned}$$

Property 4 If two rows are equal, the determinant is zero.

Example 5.

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

If we perform a row exchange we still get the same matrix but because of rule 2 the determinant must change signs. We get $-D = D$, and the only way that this is consistent is when $D = 0$. A matrix with two equal rows has no inverse (matrix is singular).

Property 5 Elimination from A to U does not change the value of the determinant. Hence $\det(A) = \det(U)$.

Example 6.

$$\begin{aligned}
\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} &= a(d-lb) - b(c-la) \\
&= ad - \cancel{alb} - bc + \cancel{alb} \\
&= ad - bc \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix}
\end{aligned}$$

Property 6 A matrix with a row of zeros has a determinant of zeros.

Property 7 The determinant of an upper triangular matrix is the product of the diagonal entries.

$$U = \begin{bmatrix} a_{11} & \dagger & \dots & \dagger \\ 0 & \ddots & \vdots & \dagger \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

$$\det U = \prod_{i=1}^{n} a_{ii}$$

To verify property 1, we simply multiply the diagonals of an identity matrix and $\det(I) = 1$. If there is a zero in the diagonal, then the determinant is zero.

Property 8 If A is a singular matrix, then $\det(A) = 0$. If A is invertible then $\det(A) \neq 0$. This is evident in singular matrices when we do elimination from A to U and we get a row of zeros and via rule 6 the determinant is zero. If A is invertible then U has pivots along its diagonal. The product of nonzero pivots gives a nonzero determinant.

$$\det(A) = \pm\det(U) = \pm(\textbf{product of the pivots})$$

The pivots of a 2×2 matrix are a and $d - (c/a)b$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d-(c/a)b \end{vmatrix} = ad - bc$$

The sign in $\pm\det(U)$ depends on whether the number of row exchanges is even or odd: +1 or -1 is the determinant of the permutation P that exchanges rows. With no row exchanges, $P = I$ and $\det(A) = \det(U)$.

$$PA = LU$$
$$\det(P)\det(A) = \det(L)\det(U)$$

$\det(P) = \det(I) = 1$ and L's diagonal only contains ones, hence $\det(L) = 1$.

$$\det(A) = \det(U)$$

Property 9 The determinant of AB is:

$$\det(AB) = \det(A)\det(B).$$

From this property, we can calculate the inverse.

$$AA^{-1} = I$$
$$\det(A)\det(A^{-1}) = 1$$
$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

To prove this property, suppose we have two 2×2 matrices A and B.

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad B = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$$

$$\begin{aligned} |A||B| &= (ad - bc)(ps - qr) \\ &= (ap + br)(cq + ds) - (aq + bs)(cp + dr) = |AB| \end{aligned}$$

Property 10 The determinant of the transpose is equal to the determinant of the original matrix.

$$\det(A^T) = \det(A)$$

Proof. Say A is invertible and we do not have any row exchanges, then $P = I$.

$$PA = LU$$
$$A^T P^T = U^T L^T$$
$$\det(A^T)\det(P^T) = \det(U^T)\det(L^T)$$

L^T has ones in the diagonal then its determinant is 1. Same with the determinant of P^T.

$$\det(A^T) = \det(U^T)$$

U and U^T have the same diagonals, therefore A and A^T have the same determinants. $\square$

Example 7.

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
$$ad - bc = ad - bc$$

2 Inverses, Cramer's Rule, and Volumes

2.1 Inverse

For the 2×2 case, the formula for the inverse of A is

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For the $n \times n$ case, we have

$$A^{-1} = \frac{1}{\det(A)} C^T$$

2.2 Cramer's Rule

To solve $A\boldsymbol{x} = \boldsymbol{b}$, we use $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. According to the last section, this formula is equivalent to

$$\boldsymbol{x} = \frac{1}{\det(A)} C^T \boldsymbol{b}.$$

Definition 2.1. If $\det(A) \neq 0$, $A\boldsymbol{x} = \boldsymbol{b}$ is solvable using determinants:

$$\boldsymbol{x}_1 = \frac{\det(B_1)}{\det(A)} \quad \boldsymbol{x}_2 = \frac{\det(B_2)}{\det(A)} \quad \ldots \boldsymbol{x}_n = \frac{\det(B_n)}{\det(A)} \tag{1}$$

Where B_i is

$$B_i = A \text{ with column } i \text{ replaced by } \boldsymbol{b}$$

Cramer's rule is an explicit formula for the solution $\boldsymbol{x}$ but it involves way too many operations.

Example 8. Solve for x_1, x_2:

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The determinant of A is:

$$\det(A) = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 3 \cdot 6 - 4 \cdot 5 = -2$$

Solving for x_1:

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{\begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix}}{\det(A)} = \frac{2 \cdot 6 - 4 \cdot 4}{-2} = \frac{-4}{-2} = 2$$

Solving for x_2:

$$x_2 = \frac{\det(B_2)}{\det(A)} = \frac{\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}}{\det(A)} = \frac{3 \cdot 4 - 2 \cdot 5}{-2} = \frac{2}{-2} = -1$$

Hence,

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

2.3 Cross Product

Definition 2.2. The cross product of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is a vector

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}. \tag{2}$$

Using the Levi-Civita symbol:

$$\boldsymbol{u} \times \boldsymbol{v} = \sum_i \sum_j \sum_k \epsilon_{ijk} u_j v_k$$

This vector $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. The cross product $\boldsymbol{v} \times \boldsymbol{u}$ is $-(\boldsymbol{u} \times \boldsymbol{v})$.

Property 1 $\boldsymbol{v} \times \boldsymbol{u}$ reverses rows 2 and 3 in the determinant so it equals -($\boldsymbol{u} \times \boldsymbol{v}$).

Property 2 The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{u}$.

Proof.

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$$

□

Property 3 The cross product of any vector with itself is $\boldsymbol{u} \times \boldsymbol{u} = 0$. This is because the determinant has two equal rows. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, their cross product is zero. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, their dot product is zero.

$$||\boldsymbol{u} \times \boldsymbol{v}|| = ||\boldsymbol{u}||||\boldsymbol{v}||| \sin\theta| \qquad ||\boldsymbol{u} \cdot \boldsymbol{v}|| = ||\boldsymbol{u}||||\boldsymbol{v}||| \cos\theta|$$

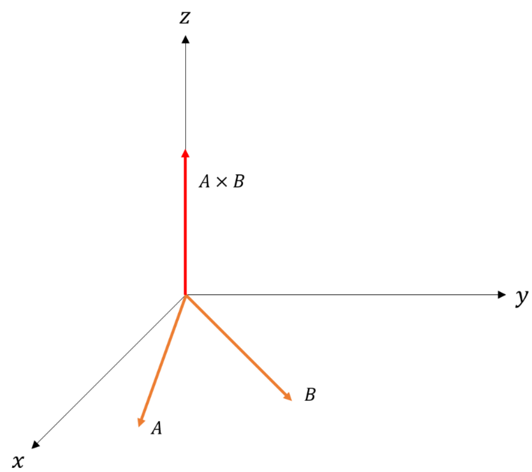

Figure 1: Geometry of the cross product.

Property 4 The length of $\boldsymbol{u} \times \boldsymbol{u}$ equals the area of the parallelogram with sides $\boldsymbol{u}$ and $\boldsymbol{v}$.

Definition 2.3. The cross product is a vector with length $||\boldsymbol{u}||||\boldsymbol{v}||| \sin\theta|$. Its direction is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. It points up or down by the right hand rule.

Example 9. Find $\boldsymbol{u} \times \boldsymbol{v}$ and $||\boldsymbol{u} \times \boldsymbol{v}||$ of $\boldsymbol{u} = (1, 1, 1)$ and $\boldsymbol{v} = (1, 1, 2)$.

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \boldsymbol{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j}$$

$$||\boldsymbol{u} \times \boldsymbol{v}|| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

2.4 Scalar Triple Product

The triple product is

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}.$$

It is a scalar because the answer is a number. The scalar triple product is a determinant - it gives the volume of a $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box.

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

When $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are on the same plane,

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = 0.$$