Chapter 4: Orthogonality

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January 2019

Abstract

This chapter focuses on the orthogonality of the four subspaces, projections, and least squares approximations.

1 Orthogonality of the Four Subspaces

Two vectors are orthogonal when their dot product is zero $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T\boldsymbol{w} = 0$. This chapter will revolve around orthogonal subspaces, orthogonal bases, and orthogonal matrices.

Definition 1.1. Orthogonal vectors have the following properties:

i. $\boldsymbol{v}^T\boldsymbol{w} = 0$

ii. $||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2 = ||\boldsymbol{v} + \boldsymbol{w}||^2 \rightarrow \boldsymbol{v}^T\boldsymbol{v} + \boldsymbol{w}^T\boldsymbol{w} = (\boldsymbol{v} + \boldsymbol{w})^T(\boldsymbol{v} + \boldsymbol{w})$

Remark. *The zero vector is orthogonal to any vector.*

Remark. *The subspaces have orthogonal properties.*

1. ***The rowspace*** $C(A^T)$ ***is perpendicular to the nullspace*** $N(A)$. *Every row of* A *is perpendicular to the solution of* $A\boldsymbol{x} = \boldsymbol{0}$.

2. ***The column space*** $C(A)$ ***is perpendicular to the left nullspaces*** $N(A^T)$. *When* $\boldsymbol{b}$ *is outside of the column space when we're trying to solve for* $A\boldsymbol{x} = \boldsymbol{b}$*, then this nullspace of* A^T *comes into its own. It contains the error* $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ *in the least-squares solution.*

Definition 1.2. Two subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ of a vector space are orthogonal if every vector $\boldsymbol{v}$ in $\boldsymbol{V}$ is perpendicular to every vector $\boldsymbol{w}$ in $\boldsymbol{W}$.

$$\boldsymbol{v}^T\boldsymbol{w} = 0 \text{ for all } \boldsymbol{v} \text{ in } \boldsymbol{V} \text{ and all } \boldsymbol{w} \text{ in } \boldsymbol{W}.$$

Theorem 1. Every vector $\boldsymbol{x}$ in the nullspace is perpendicular to every row of A, because $A\boldsymbol{x} = \mathbf{0}$. The nullspace $N(A)$ and the row space $C(A^T)$ are orthogonal subspaces of $\mathbb{R}^n$.

$$A\boldsymbol{x} = \begin{bmatrix} \text{row 1} \\ \vdots \\ \text{row m} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{gathered} C_1(\text{row}_1^T) = 0 \\ C_2(\text{row}_2^T) = 0 \\ \vdots \\ C_m(\text{row}_m^T) = 0 \end{gathered}$$

(row 1) $\cdot \boldsymbol{x}$ is zero and (row m) $\cdot$ $\boldsymbol{x}$ is also zero. Every row has a zero dot product with $\boldsymbol{x}$. Then $\boldsymbol{x}$ is perpendicular to every combination of the rows. **The whole row space $C(A^T)$ is orthogonal to $N(A)$.**

Proof. The vectors in the row space are combinations of $A^T\boldsymbol{y}$ of the rows. We take the dot product of $A^T\boldsymbol{y}$ with any $\boldsymbol{x}$ in the nullspace.

$$\boldsymbol{x} \cdot (A^T\boldsymbol{y}) = \boldsymbol{x}^T(A^T\boldsymbol{y}) = (A\boldsymbol{x})^T\boldsymbol{y} = 0^T\boldsymbol{y} = 0$$

□

Example 1. The rows of A are perpendicular to $\boldsymbol{x} = (1, 1, -1)$ in the nullspace:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+3-4 \\ 5+2-7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In this example, the column space is all of $\mathbb{R}^2$. The nullspace of A^T is the zero vector. The column space of A and the nullspace of A^T are always orthogonal subspaces.

Theorem 2. Every vector $\boldsymbol{y}$ in the nullspace of A^T is perpendicular to every column of A. The left nullspace $N(A^T)$ and the column space $C(A)$ are orthogonal in $\mathbb{R}^m$.

Proof. The nullspace of A^T is orthogonal to the row space of A^T, which is the column space of A.

$$A^T\boldsymbol{y} = \begin{bmatrix} (\text{column 1})^T \\ \vdots \\ (\text{column n})^T \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

□

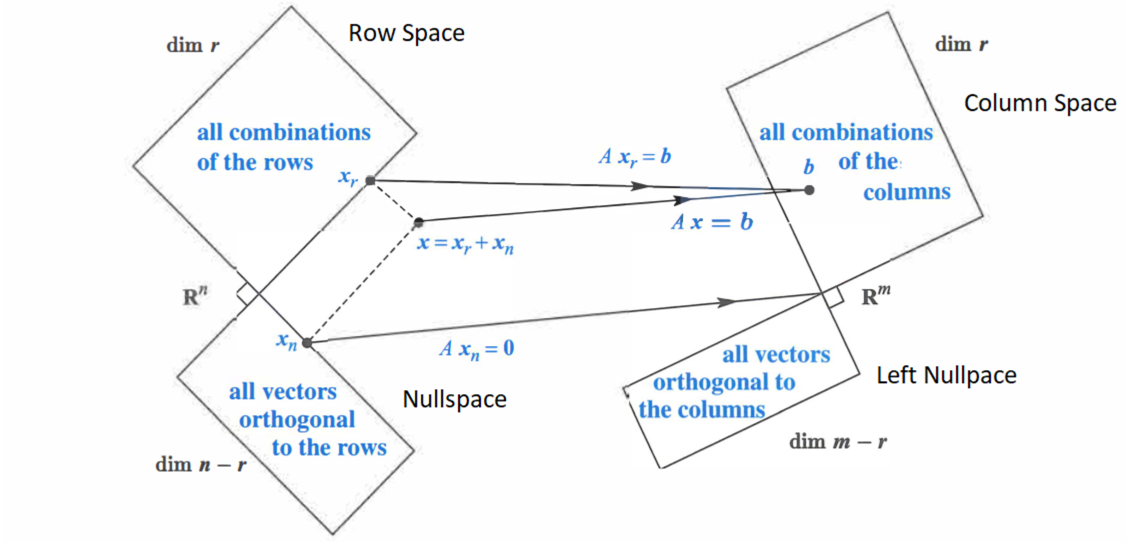

Figure 1: The Four Subspaces. There are two pairs of orthogonal subspaces.

Theorem 3. If a vector $\boldsymbol{v}$ is orthogonal to itself, then $\boldsymbol{v}$ is the zero vector.

Theorem 4. Fundamental Theorem of Linear Algebra, Part 2:
$N(A)$ is the orthogonal complement of the row space $C(A^T)$ in $\mathbb{R}^n$.
$N(A^T)$ is the orthogonal complement of the column space $C(A)$ in $\mathbb{R}^m$.

Things to note from Figure 1

1. When A multiplies to $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, it goes to $\boldsymbol{b}$ which is in the column space.

2. When A multiplies to $\boldsymbol{x}_r$, it goes to $\boldsymbol{b}$ which is also in the column space.

3. When A multiplies to $\boldsymbol{x}_n$, the nullspace component goes to $\mathbf{0}$.

1.1 Combining Bases from Subspaces

Theorem 5. Any independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$. So they are a basis.
Any n vectors that span $\mathbb{R}^n$ must be independent. So they are a basis

Theorem 6. If the n columns of A are independent, they span $\mathbb{R}^n$. So $A\boldsymbol{x} = \boldsymbol{b}$ is solvable.
If the n columns span $\mathbb{R}^n$, they are independent. So $A\boldsymbol{x} = \boldsymbol{b}$ has only one solution.

2 Problems

Problem 1.1. asd

Solution. soln □