Chapter 6: Eigenvalues and Eigenvectors

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Abstract

This chapter focuses on the properties of eigenvalues and eigenvectors, diagonalizing a matrix, systems of differential equations, symmetric matrices, and positive definite matrices.

1 Introduction

The system $A\boldsymbol{x} = \boldsymbol{b}$ is in equilibrium and steady state. Change as in time enters the picture - continuous time in a differential equation $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$ or time steps in a difference equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. Using linear algebra, eigenvalues and eigenvectors allow these types of systems to be solved beautifully.

Example 1. Say we have a matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$, calculate $A^2, A^3,$ and A^100.

$$A^2 = \begin{bmatrix} .70 & .30 \\ .30 & .55 \end{bmatrix} \quad A^3 = \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \quad A^{100} \approx \begin{bmatrix} .60 & .60 \\ .40 & .40 \end{bmatrix}$$

One way is to solve these equations using eigenvalues.

Vectors $\boldsymbol{x}$ when multiplied by A usually change direction. But there are certain exceptional vectors that maintain the same direction as $A\boldsymbol{x}$ and these are called "eigenvectors." The basic equation is

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \tag{1}$$

where the number λ is an eigenvalue of A and $\boldsymbol{x}$ is an eigenvector of A.

Example 2. What are the eigenvalues an eigenvectors of

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}?$$

$$\det(A-\lambda I) = 0$$
$$\det\left(\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$
$$\begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$$
$$(\lambda - 1)\left(\lambda - \frac{1}{2}\right) = 0$$
$$\lambda_1 = 1 \quad \lambda_2 = \frac{1}{2}$$

$A - \lambda I$ becomes a singular matrix and the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$$(A - \lambda I)\boldsymbol{x} = 0$$

Finding x_1,

$$(A - I)x = 0$$
$$\begin{bmatrix} -.2 & .3 \\ .2 & -.3 \end{bmatrix} \boldsymbol{x}_1 = 0$$
$$\boldsymbol{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$$

Finding x_2,

$$(A - \frac{1}{2}I)x = 0$$
$$\begin{bmatrix} .3 & .3 \\ .2 & .2 \end{bmatrix} \boldsymbol{x}_2 = 0$$
$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Checking,

$$A\boldsymbol{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = 1 \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \lambda_1 \boldsymbol{x}_1$$
$$A\boldsymbol{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_2 \boldsymbol{x}_2$$

And if A is multiplied n times we get $A^n\boldsymbol{x}_1 = \lambda_1^n \boldsymbol{x}_1$. Same goes for the second eigenvector. Also take note that the columns of A are a combination of the eigenvectors: $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$. The eigenvector $\boldsymbol{x}_1$ is a *steady state* because $\lambda_1 = 1$. The eigenvector $\boldsymbol{x}_2$ is a *decaying mode* that virtually disappears because $\lambda_2 = .5$. The higher the power of A, the more closely its columns approach the steady state. A is an example of a **Markov Matrix** where the sum of the entries for each column equal to one.

Definition 1.1. If A is multiplied n times, the eigenvectors stay the same and the eigenvalues are also multiplied n times.

1.1 Equation for Eigenvalues

To solve for the eigenvalues and eigenvectors, we start with Equation (1).

$$\begin{aligned} A\boldsymbol{x} &= \lambda \boldsymbol{x} \\ A\boldsymbol{x} - \lambda \boldsymbol{x} &= \mathbf{0} \\ \boxed{(A - \lambda I)\boldsymbol{x} = \mathbf{0}} \end{aligned} \tag{2}$$

The eigenvectors of $\boldsymbol{x}$ make up the nullspace of $A - \lambda I$. If we find the eigenvalue λ we can calculate for the eigenvector. To solve for the eigenvalue, we know that $A - \lambda I$ is a singular matrix. Therefore its determinant is zero.

$$\boxed{\det(A - \lambda I) = p(\lambda) = 0} \tag{3}$$

$\det(A - \lambda I) = 0$ is called the **characteristic polynomial**. Generally, the characteristic polynomial is the following:

$$p(\lambda) = (-\lambda)^n + \text{trace}(A)(-\lambda)^{n-1} + \ldots + \det(A)$$

To get the eigenvalues, we solve for λ. When A is an $n \times n$ matrix, Equation 3 has degree n. A has n eigenvalues and repeating λ are possible. Each λ leads to $\boldsymbol{x}$. **For each λ, solve $(A - \lambda I)\boldsymbol{x} = 0$ to find the eigenvector**.

Example 3. Find the eigenvalues and eigenvectors of the singular matrix: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$$\begin{gathered} \det(A - \lambda I) = 0 \\ \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \\ \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) = 0 \\ \lambda(\lambda - 5) = 0 \\ \lambda_1 = 0 \qquad \lambda_2 = 5 \end{gathered}$$

For the first eigenvector $\boldsymbol{x}_1$:

$$\begin{gathered} (A - \lambda_1 I)\boldsymbol{x}_1 = 0 \\ (A - 0I)\boldsymbol{x}_1 = 0 \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \end{gathered}$$

The nullspace solution is:

$$\boldsymbol{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

For the second eigenvector $\boldsymbol{x}_2$:

$$\begin{gathered} (A - \lambda_2 I)\boldsymbol{x}_2 = 0 \\ (A - 5I)\boldsymbol{x}_2 = 0 \\ \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \end{gathered}$$

The nullspace solution is:

$$\boldsymbol{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If A is a singular matrix, $\lambda = 0$ is an eigenvalue of A.

Summary To solve the eigenvalue problem for an $n \times n$ matrix, follow these steps:

1. **Compute the determinant of** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It's a polynomial of degree n.

2. **Find the roots of this polynomial**, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.

3. For each eigenvalue λ, **solve** $(A - \lambda I)\boldsymbol{x} = 0$ **to find the eigenvector** $\boldsymbol{x}$.

Warning! There are times when A has equal eigenvalues. Hence, there is only one line of eigenvectors. Without a full set of eigenvectors, we can't diagonalize a matrix without n independent eigenvectors.

1.2 Determinant and Trace

We cannot get the eigenvalues of A when we do elimination to U because elimination does not preserve the eigenvalues. U has its own set of eigenvalues along its diagonal (the pivots) but these aren't the eigenvalues of A. The product $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n$ and the sum $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ can be found directly from the matrix A.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{has eigenvalues } \lambda = 0, \lambda = 7 \tag{4}$$

For A, the product of the eigenvalues is $0 \cdot 7 = 0$. This agrees with the $\det(A) = 0$. The sum of the eigenvalues $0 + 7$ agrees with the sum down the diagonal of A. The sum of the diagonal entries of A is called the **trace**. These two properties are important checks to see if our calculations for the eigenvalues are correct.

Theorem 1. The product of the n eigenvalues equals the determinant of matrix A.

$$\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n = \det(A) \tag{5}$$

Proof.

Consider $\det(A - \lambda I)$,

$$\det(A - \lambda I) = p(\lambda)$$

Factorizing the characteristic equation according to its individual roots.

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

Setting $\lambda = 0$, therefore

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$$

□

Theorem 2. The sum of the n eigenvalues equals the sum of the n diagonal entries of A (the trace).

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = a_{11} + a_{22} + \ldots + a_{nn} = \mathbf{trace}(A) \tag{6}$$

Proof. Consider the general equation of the characteristic polynomial $p(\lambda)$.

$$p(\lambda) = (-\lambda)^n + \mathrm{trace}(A)(-\lambda)^{n-1} + \ldots + \det(A)$$

Another expression of the characteristic polynomial is:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

By *comparing coefficients*, we get the trace:

$$\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

□

Example 4. To see the proof above clearly, here is an example where there are two eigenvalues:

$$p(\lambda) = \lambda_2 - \mathrm{trace}(A)\lambda + \det(A)$$

$p(\lambda)$ can also be expressed with the following:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)\lambda + \lambda^2$$

By comparing coefficients, we get the trace:

$$\mathrm{trace}(A) = \lambda_1 + \lambda_2$$

Theorem 3. The eigenvalues of a triangular matrix lie along its diagonal.

Proof. Suppose A is a triangular matrix with nonzero entries.

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

Consider the determinant of $A - \lambda I$,

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} = p(\lambda)$$

Where $p(\lambda)$ is the characteristic equation. The roots of the characteristic equation are the eigenvalues. The determinant of $A - \lambda I$ is 0 by definition because it is a singular matrix.

$$\begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} = 0$$

The determinant of a triangular matrix is the product of its diagonal entries. Hence,

$$\det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda) = 0$$

Therefore, each diagonal entry in A is an eigenvalue.

$$a_{ii} = \lambda$$

$\square$

2 Diagonalizing a Matrix

Eigenvalues and eigenvectors make matrix multiplication easier. When $\boldsymbol{x}$ is an eigenvector, multiplication by A is just a multiplication by a number λ: $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Diagonalizing a matrix turns A into a diagonal matrix Λ when we use the eigenvectors properly.

Definition 2.1. Suppose the $n \times n$ matrix A has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the columns of an eigenvector matrix X. Then $X^{-1}AX$ is the **eigenvalue matrix** Λ:

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ldots & \\ & & \lambda_n \end{bmatrix} \tag{7}$$

Example 5. Diagonalize $A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$

A is a triangular matrix, therefore its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 6$. We then find its eigenvectors. Finding $\boldsymbol{x}_1$:

$$\begin{aligned}
(A - \lambda_1 I)\boldsymbol{x}_1 &= 0 \\
\left(\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \boldsymbol{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} \boldsymbol{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\boldsymbol{x}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

Finding $\boldsymbol{x}_2$,

$$\begin{aligned}
(A - \lambda_2 I)\boldsymbol{x}_2 &= 0 \\
\left(\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right) \boldsymbol{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\boldsymbol{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

Diagonalizing:

$$\begin{gathered}
X^{-1}AX = \Lambda \\
\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}
\end{gathered}$$

Theorem 4. A^n has the same eigenvectors in X and the eigenvalues raised to the power of n in Λ^n.

Proof.

$$\begin{gathered}
A^2 = X\Lambda \cancel{X^{-1}X} \Lambda X^{-1} = X\Lambda^2 X^{-1} \\
A^3 = X\Lambda \cancel{X^{-1}X} \Lambda \cancel{X^{-1}X} \Lambda X^{-1} = X^{-1}\Lambda^3 X \\
\vdots \\
\text{etc.}
\end{gathered}$$

$\square$

The matrix X must have an inverse because its columns are linearly independent. **Without n independent eigenvectors, we can't diagonalize**.

Remark. *Suppose the eigenvalues $\lambda_1, \dots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are independent. The eigenvector matrix X will be invertible.* ***Any matrix that has no repeated eigenvalues can be diagonalized.***

Remark. ***We can multiply eigenvectors by any nonzero constants.*** *$A(c\boldsymbol{x}) = \lambda(c\boldsymbol{x})$ is still true.*

Remark. *The eigenvectors in X come in the same order as the eigenvalues in Λ.*

Remark. *Matrices that have repeated eigenvalues cannot be diagonalized.*

Example 6. The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .5$ and eigenvectors of $\boldsymbol{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -.6 \end{bmatrix}$. Find $A^2, A^k,$ and A^∞.

$$A = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = X \Lambda X^{-1}$$

Computing for A^2.

$$A^2 = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

Computing for A^k.

$$A^k = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}$$

Computing for A^∞

$$A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \lim_{k \to \infty} \left(\begin{bmatrix} 1^k & 0 \\ 0 & .5^k \end{bmatrix} \right) \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

2.1 Similar Matrices: Same Eigenvalues

Similar matrices have the same Λ and different X.

Theorem 5. All matrices $A = BCB^{-1}$ are "similar". They all share the eigenvalues of C.

2.2 Fibonacci Numbers

$$0, 1, 1, 2, 3, 5, 8, 13 \dots$$

A new Fibonacci number is the sum of the two previous Fibonacci numbers in the Fibonacci sequence. The rule is: $F_{k+2} = F_{k+1} + F_k$. The obvious and slow way to get to F_{100} is by applying the rule one at a time. Linear algebra gives a faster method. The one step rule for the Fibonacci sequence is $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$.

$$\text{Let } \boldsymbol{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

The rules below are put into matrix A.

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \\ \boldsymbol{u}_{k+1} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}_k \end{aligned}$$

Every step multiplies by A. After 100 steps we reach $\boldsymbol{u}_{100} = A^{100}\boldsymbol{u}_0$.

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \boldsymbol{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

Finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Finding the eigenvector $\boldsymbol{x}_1$,

$$(A - \lambda_1 I)\boldsymbol{x}_1 = \mathbf{0}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1+\sqrt{5}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Finding the eigenvector $\boldsymbol{x}_2$,

$$(A - \lambda_2 I)\boldsymbol{x}_2 = \mathbf{0}$$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1-\sqrt{5}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{pmatrix}\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

We need to express $\boldsymbol{u}_0$ as a combination of the eigenvectors.

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\lambda_1-\lambda_2} & -\frac{b}{\lambda_1-\lambda_2}\\-\frac{1}{\lambda_1-\lambda_2} & \frac{\lambda_1}{\lambda_1-\lambda_2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}\frac{1}{\lambda_1-\lambda_2}\\-\frac{1}{\lambda_1-\lambda_2}\end{bmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}1\\-1\end{bmatrix}$$

We now have the expression:

$$\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1\\1\end{bmatrix} - \frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_2\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \boldsymbol{u}_k$$

$$\frac{1}{\lambda_1-\lambda_2}\left(\begin{bmatrix}\lambda_1\\1\end{bmatrix} - \begin{bmatrix}\lambda_2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\boldsymbol{u}_0 = \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\lambda_1-\lambda_2}$$

Then we have a generalized formula in terms of our eigenvalues:

$$\boldsymbol{u}_k = \frac{\lambda^k\boldsymbol{x}_1 - \lambda^k\boldsymbol{x}_2}{\lambda_1-\lambda_2}$$

To get 100$^{\text{th}}$ Fibonacci number, F_{100}, we use the second component of $\boldsymbol{x}_1, \boldsymbol{x}_2$

$$F_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1-\lambda_2}$$

Which is approximately,

$$F_{100} \approx 3.54224 \times 10^{20}$$

2.3 Matrix Powers A^k

$$A^k\boldsymbol{u}_0 = (X\Lambda X^{-1})\dots(X\Lambda X^{-1})\boldsymbol{u}_0 = X\Lambda^k X^{-1}\boldsymbol{u}_0$$

Fibonacci's example is a typical difference equation $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. Each step multiplies by A. The solution is $\boldsymbol{u}_{k+1} = A^k\boldsymbol{u}_0$. Diagonalizing the matrix gives a quick way to compute A^k and find $\boldsymbol{u}_k$ in three quick steps.

1. Write $\boldsymbol{u}_0$ as a combination $c_1\boldsymbol{x}_1 + \dots c_n\boldsymbol{x}_n$ of the eigenvectors. Then $\boldsymbol{c} = X^{-1}\boldsymbol{u}_0$.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by $(\lambda_i)^k$. Now we have $\Lambda^k X^{-1}\boldsymbol{u}_0$.

3. Add up the pieces $c_i(\lambda_i)^k \boldsymbol{x}_i$ to find the solution $\boldsymbol{u}_k = A^k \boldsymbol{u}_0$. This is $X\Lambda^k X^{-1}\boldsymbol{u}_0$.

The solution for $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ becomes:

$$\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k = c_1(\lambda_1)^k \boldsymbol{x}_1 + \ldots + c_n(\lambda_n)^k \boldsymbol{x}_n = X\Lambda^k X^{-1}\boldsymbol{u}_0 = X\Lambda^k \boldsymbol{c}$$
$$X\Lambda^k \boldsymbol{c} = \begin{bmatrix} | & \ldots & | \\ \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_n \\ | & \ldots & | \end{bmatrix} \begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

2.4 Nondiagonalizable Matrices

When we solve for λ and $\boldsymbol{x}$, we want to know their **multiplicity**.

Geometric Multiplicity = GM - Count the independent eigenvectors for λ. Then GM is the dimension of the nullspace of $A - \lambda I$.

Algebraic Multiplicity = AM - AM counts the repetitions of λ among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

Example 7. Find the GM and AM of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$
$$\lambda = 0$$

The eigenvalue 0 is a repeating eigenvalue. Hence its AM = 2. The eigenvalue 0 only has one eigenvector hence GM = 1.

Theorem 6. If GM = AM, the matrix A is **diagonalizable**.

Theorem 7. If GM < AM, the matrix A is **not diagonalizable**.

3 Systems of Differential Equations

We want to solve $\boldsymbol{u}$ from the following system of differential equations:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}, \quad \text{starting from the vector } \boldsymbol{u}(0) = \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix}$$

These equations are linear. If $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are solutions, so is $C\boldsymbol{u}(t) + D\boldsymbol{v}(t)$. Our job is to find n "pure exponential solutions" $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ by using $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Here λ is an eigenvalue while $\boldsymbol{x}$ is an eigenvector. First we check if $\boldsymbol{u}(t)$ is a solution.

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt} &= A\boldsymbol{u} \\ \frac{d}{dt}\left(e^{\lambda t}\boldsymbol{x}\right) &= Ae^{\lambda t}\boldsymbol{x} \\ \lambda \cancel{e^{\lambda t}}\boldsymbol{x} &= A\cancel{e^{\lambda t}}\boldsymbol{x} \\ \lambda\boldsymbol{x} &= A\boldsymbol{x} \end{aligned}$$

All components of this special solution $\boldsymbol{u} = e^{\lambda t}\boldsymbol{x}$ share the same $e^{\lambda t}$. For a certain system, the pure exponential solutions $\boldsymbol{u}_1 = e^{\lambda_1 t}\boldsymbol{x}_1$ satisfies $A\boldsymbol{u}_1 = \boldsymbol{u}_1$, $\boldsymbol{u}_2 = e^{\lambda_2 t}\boldsymbol{x}_2$ satisfies $A\boldsymbol{u}_2 = \boldsymbol{u}_2$, and so on. The solution **grows** when $\lambda > 0$. It **decays** when $\lambda < 0$. If λ is a complex number, its real part decides growth or decay. The imaginary part ω **gives oscillation** $e^{i\omega t}$ like a sine wave.

Example 8. Solve the following system of differential equations.

$$\begin{aligned} \frac{du_1}{dt} &= -u_1 + 2u_2 \\ \frac{du_2}{dt} &= u_1 - 2u_2. \end{aligned}$$

Here $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ and the initial value $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We want to solve:

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt} &= A\boldsymbol{u} \\ \frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

We need to find the eigenvalues and eigenvectors of A. We know that A is a singular matrix, hence one of its eigenvalues is $\lambda_1 = 0$. We can get the second eigenvalue from the trace which is $\lambda_2 = -3$. We then find the eigenvectors. First for λ_1,

$$
\begin{aligned}
(A - \overset{0}{\cancel{\lambda_1}} I)\boldsymbol{x}_1 &= 0 \\
\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}_1 &= 0 \\
\boldsymbol{x}_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}
$$

For the second eigenvector,

$$
\begin{aligned}
(A - \lambda_2 I)\boldsymbol{x}_2 &= 0 \\
\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \boldsymbol{x}_2 &= 0 \\
\boldsymbol{x}_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

Then the general solution becomes:

$$
\begin{aligned}
\boldsymbol{u}(t) &= c_1 e^{\lambda_1 t} \boldsymbol{x}_1 + c_2 e^{\lambda_2 t} \boldsymbol{x}_2 \\
&= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

We then find c_1 and c_2.

$$
\begin{aligned}
\boldsymbol{u}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3(0)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}
\end{aligned}
$$

Hence the solution is:

$$
\boxed{\boldsymbol{u}(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}
$$

Example 9. Solve $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. This is a vector equation for $\boldsymbol{u}$. It contains two scalar equations for the components y, z. They are "coupled together" because the matrix is not diagonal.

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$$

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

The equation above means:

$$\frac{dy}{dt} = z \quad \text{and} \quad \frac{dz}{dt} = y$$

Finding the eigenvalues,

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\boxed{\lambda_1 = 1, \quad \lambda_2 = -1}$$

Finding the first eigenvector,

$$(A - \lambda_1 I)\boldsymbol{x}_1 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\boldsymbol{x}_1 = 0$$

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finding the second eigenvector,

$$(A - \lambda_2 I)\boldsymbol{x}_2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\boldsymbol{x}_2 = 0$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution becomes:

$$\boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2 = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Finding the constants c_1, c_2,

$$\boldsymbol{u}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Therefore, the solution becomes:

$$\boxed{\boldsymbol{u}(t) = 3e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

We would like to remind ourselves that really, the final solution is expressed in terms of a diagonalization:

$$\boldsymbol{u}(t) = X\Lambda X^{-1}\boldsymbol{u}_0 = X\Lambda c = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Summary The same three steps in calculating $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$.

1. Write $\boldsymbol{u}_0$ as a combination $c_1\boldsymbol{x}_1 + \ldots c_n\boldsymbol{x}_n$ of the eigenvectors of A.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by its growth factor $e^{\lambda_i t}$.

3. The solution is the same combination of those pure solutions $e^{\lambda t}\boldsymbol{x}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \ldots + c_n e^{\lambda_n t}\boldsymbol{x}_n.$$

Warning! If a λ repeats, with only one eigenvector, another solution is needed which is $(te^{\lambda t})$.

3.1 Second Order Differential Equations

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass times the acceleration $a = y''$. The term ma balances the force F (Newton's second law). The force includes the damping $-by'$ and the elastic force $-ky$. This second order differential equation is still linear with coefficients m, b, k.

The method of solution to this equation is to substitute $y = e^{\lambda t}$. Each derivative of y

brings down a factor λ.

$$\begin{aligned}
m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky &= 0 \\
m\frac{d^2}{dt^2}e^{\lambda t} + b\frac{d}{dt}e^{\lambda t} + ke^{\lambda t} &= 0 \\
m\lambda\frac{d}{dt}e^{\lambda t} + b\lambda e^{\lambda t} + ke^{\lambda t} &= 0 \\
m\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ke^{\lambda t} &= 0 \\
(m\lambda^2 + b\lambda + k)e^{\lambda t} &= 0
\end{aligned}$$

There are two roots for λ, λ_1 and λ_2. This equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1y_1 + c_2y_2$ give the complete solution unless the λ's are the same. We want to express the problem into a vector equation for y and y'. We have:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u} \tag{8}$$

A comes from the fact that $y' = y'$ and $y'' = -by' - ky$. We then solve $\boldsymbol{u}' = A\boldsymbol{u}$ by the eigenvalues of A.

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{vmatrix} &= 0 \\
\lambda^2 + b\lambda + k &= 0
\end{aligned}$$

The roots λ_1 and λ_2 will become our eigenvalues. Now to find the eigenvectors. Finding $\boldsymbol{x}_1$:

$$\begin{aligned}
\begin{bmatrix} -\lambda_1 & 1 \\ -k & -b-\lambda_1 \end{bmatrix}\boldsymbol{x}_1 &= 0 \\
\boldsymbol{x}_1 &= \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}
\end{aligned}$$

$\boldsymbol{x}_1$ will hold if $-k + \lambda_1(-b - \lambda_1) = 0$. Finding $\boldsymbol{x}_2$.

$$\begin{aligned}
\begin{bmatrix} -\lambda_2 & 1 \\ -k & -b-\lambda_2 \end{bmatrix}\boldsymbol{x}_2 &= 0 \\
\boldsymbol{x}_2 &= \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}
\end{aligned}$$

$\boldsymbol{x}_2$ will hold if $-k + \lambda_2(-b - \lambda_2) = 0$. The general solution will then be:

$$\boldsymbol{u}(t) = c_1e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

Example 10. Motion around a circle with $y'' + y = 0$ and $y = \cos(t)$. This is our master equation with mass $m = 1$ and stiffness $k = 1$ and $d = 0$ (no damping). Substitute $y = e^{\lambda t}$ into $y'' + y = 0$ to reach $\lambda^2 + 1 = 0$. The roots are complex numbers $\lambda = \pm i$. Then half of $e^{it} + e^{-it}$ gives the solution $y = \cos(t)$. The initial values are $y(0) = 1, y'(0) = 0$. Find $\boldsymbol{u}(t)$.

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}$$

Fining the eigenvalues:

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 1 = 0$$
$$(\lambda - i)(\lambda + i) = 0$$
$$\boxed{\lambda_1 = i \qquad \lambda_2 = i}$$

Fining the eigenvector $\boldsymbol{x}_1$:

$$(A - \lambda_1 I)\boldsymbol{x}_1 = 0$$
$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Fining the eigenvector $\boldsymbol{x}_2$:

$$(A - \lambda_2 I)\boldsymbol{x}_1 = 0$$
$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The general solution is:

$$\boldsymbol{u}(t) = c_1 e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$
$$\boldsymbol{u}(t) = \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ ic_1 e^{it} - ic_2 e^{-it} \end{bmatrix}$$

Finding c_1 and c_2:

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\boldsymbol{c} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

The final solution is:

$$\boxed{\boldsymbol{u}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^{it}+e^{-it}\right) \\ \frac{1}{2}i\left(e^{it}-e^{-it}\right) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}}$$

3.2 Exponential of a Matrix

We want to write the solution $\boldsymbol{u}(t)$ in a new form $e^{At}\boldsymbol{u}(0)$. First we have to define what e^{At} means by copying the Taylor Series representation of e^x for numbers.

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots$$

Its derivative with respect to t is:

$$\frac{d}{dt}e^{At} = Ae^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \dots$$

The eigenvalues are $e^{\lambda t}$

$$e^{At}\boldsymbol{x} = \left(\sum_{n=0}^{\infty} \frac{(At)^n}{n!}\right)\boldsymbol{x} = (I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \dots)\boldsymbol{x}$$

The series always converges and its derivative is always Ae^{At}. Therefore $e^{At}\boldsymbol{u}(0)$ solves the differential equation with one quick formula - even if there is a shortage of eigenvectors. This chapter emphasizes how to find $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$ by diagonalization. Assume A does have n independent eigenvectors, so it is diagonalizable. Substitute $A = S\Lambda S^{-1}$ into the series for e^{At}.

$$\begin{aligned} e^{At} = e^{X\Lambda X^{-1}t} = \sum_{n=0}^{\infty} \frac{(X\Lambda X^{-1}t)^n}{n!} &= I + (X\Lambda X^{-1}t) + \frac{1}{2}(X\Lambda X^{-1}t)^2 + \dots \\ &= X\left[I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \dots\right]X^{-1} \end{aligned}$$

We now arrive at a diagonalized form for e^{At}:

$$\boxed{e^{At} = Xe^{\Lambda t}X^{-1}} \tag{9}$$

The solution becomes:

$$\boldsymbol{u}(t) = e^{At}\boldsymbol{u}_0 = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = \begin{bmatrix} | & \dots & | \\ \boldsymbol{x}_1 & \dots & \boldsymbol{x}_n \\ | & \dots & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Example 11. Solve $y'' - 2y' + y = 0$.

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \boldsymbol{u}$$

Finding the eigenvalues of A:

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$
$$\begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$
$$(\lambda - 1) = 0$$
$$\boxed{\lambda_{1,2} = 1}$$

Here we only have one unique eigenvalue. We then find the eigenvector $\boldsymbol{x}_1$.

$$(A - \lambda_1 I)\boldsymbol{x}_1 = \boldsymbol{0}$$
$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \boldsymbol{x}_1 = \boldsymbol{0}$$
$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence diagonalization is not possible because we do not have 2 eigenvectors. We will then compute e^{At} instead.

$$e^{At} = e^{At+It-It} = e^t e^{(A-I)t} = e^t \sum_{n=0}^{\infty} \frac{((A-I)t)^n}{n!}$$

Expanding the series:

$$\sum_{n=0}^{\infty} \frac{(((A-I)t)^n}{n!} = I + (A-I)t + \frac{1}{2}(A-I)^2t^2 + \frac{1}{6}(A-I)^3t^3 + \dots$$

When the series reaches $n = 2$ and above, the $(A - I)$ term becomes zero. Hence,

$$e^{At} = e^t(I + (A-I)t)$$

Hence,

$$\boldsymbol{u} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^t \left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t\right] \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} e^t y(0) - te^t y(0) + te^t y'(0) \\ -e^t t y(0) + e^t y'(0) + te^t y'(0) \end{bmatrix}$$

The solution to our problem becomes:

$$y(t) = e^t y(0) - te^t y(0) + te^t y'(0)$$

Example 12. Use the infinite series to find e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \frac{1}{24}(At)^4 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix}$$

Notice that when we reach $n = 5$, the signs are the same as $n = 2$,

$$= \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots & t - \frac{1}{6}t^3 + \dots \\ -t + \frac{1}{6}t^3 + \dots & 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} & \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ -\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Example 13. Solve $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We can see the eigenvalues immediately since A is a triangular matrix. These are $\lambda_1 = 1$ and $\lambda_2 = 2$. Now to find the eigenvectors. For $\boldsymbol{x}_1$:

$$(A - \lambda_1 I)\boldsymbol{x}_1 = \mathbf{0}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}$$
$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Finding $\boldsymbol{x}_2$:

$$(A - \lambda_2 I)\boldsymbol{x}_2 = \mathbf{0}$$
$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}$$
$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The solution becomes:

$$\boldsymbol{u}(t) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t + e^{2t} \\ e^{2t} \end{bmatrix}$$

4 Symmetric Matrices

Say we are given an equation $S\boldsymbol{x} = \lambda\boldsymbol{x}$ where S is a symmetric matrix ($S = S^T$). When we diagonalize $S = X^{-1}\Lambda X$ and take its transpose, we get an interesting property.]:

$$S^T = (X\Lambda X^{-1})^T = X^T\Lambda(X^{-1})^T$$

Since $S = S^T$, therefore

$$X^{-1} = X^T$$

Then,

$$X^T X = I$$

This means that the n eigenvectors in X are orthogonal.

Here are key facts about symmetric matrices:

1. A symmetric matrix has only real eigenvectors
2. The eigenvectors can be chosen orthonormal (a normalized orthogonal matrix).

Every symmetric matrix can be diagonalized. Its eigenvector matrix X becomes an orthogonal matrix Q. Orthogonal matrices make use of the property $Q^{-1} = Q^T$.

Theorem 8. First Spectral Theorem Every symmetric matrix has the factorization $S = Q\Lambda Q^T$ with real eigenvalues of Λ and orthonormal eigenvectors in the columns Q.

$$S = Q\Lambda Q^{-1} = Q\Lambda Q^T \tag{10}$$

Proof. We need to prove that a symmetric matrix does not have repeating eigenvalues. □

Proof. We need to prove that the diagonalization $Q\Lambda Q^T$ is still symmetric.

Taking the transpose:

$$(Q\Lambda Q^T)^T = Q\Lambda Q^T$$

□

Theorem 9. All eigenvalues of a real symmetric matrix are real.

Proof.

Suppose that $S\boldsymbol{x} = \lambda\boldsymbol{x}$, λ might be a complex number $a + ib$ (a and b are real). Its complex conjugate $\bar{\lambda} = a - ib$. $\boldsymbol{x}$ might be complex numbers also, $\bar{\boldsymbol{x}}$ gives the imaginary part. Then we have two equations.

$$S\boldsymbol{x} = \lambda\boldsymbol{x} \tag{11}$$
$$S\bar{\boldsymbol{x}} = \lambda\bar{\boldsymbol{x}} \tag{12}$$

Transposing Equation (12), we have,

$$\bar{\boldsymbol{x}}^T S^T = \bar{\boldsymbol{x}}^T \bar{\lambda}$$
$$\bar{\boldsymbol{x}}^T S = \bar{\boldsymbol{x}}^T \bar{\lambda} \tag{13}$$

Taking the dot product of $\bar{\boldsymbol{x}}$ with Equation (11):

$$\bar{\boldsymbol{x}} \cdot S\boldsymbol{x} = \bar{\boldsymbol{x}} \cdot \lambda\boldsymbol{x}$$
$$\bar{\boldsymbol{x}}^T S\boldsymbol{x} = \bar{\boldsymbol{x}}^T \lambda\boldsymbol{x} \tag{14}$$

Multiplying Equation (13) with $\bar{\boldsymbol{x}}$:

$$\bar{\boldsymbol{x}}^T S\boldsymbol{x} = \bar{\boldsymbol{x}}^T \bar{\lambda}\boldsymbol{x} \tag{15}$$

The right sides of Equations (14) and (15) are equal.

$$\bar{\boldsymbol{x}}^T \lambda\boldsymbol{x} = \bar{\boldsymbol{x}}^T \bar{\lambda}\boldsymbol{x}$$
$$\lambda\bar{\boldsymbol{x}}^T \boldsymbol{x} = \bar{\lambda}\bar{\boldsymbol{x}}^T \boldsymbol{x}$$

$\bar{\boldsymbol{x}}^T\boldsymbol{x}$ is just the norm $||\boldsymbol{x}||^2$ which is always positive. Hence:

$$\lambda = \bar{\lambda}$$

Therefore:

$$a + ib = a - ib$$
$$b = 0$$

λ is always real.

$\square$

Theorem 10. Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always perpendicular.

Proof.

$$S\boldsymbol{x} = \lambda_1 \boldsymbol{x} \tag{16}$$
$$S\boldsymbol{y} = \lambda_2 \boldsymbol{y} \tag{17}$$

Taking the dot product of Equation (16) with $\boldsymbol{y}$

$$S\boldsymbol{x} \cdot \boldsymbol{y} = \lambda_1 \boldsymbol{x} \cdot \boldsymbol{y}$$
$$S\boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{x}^T \lambda_1 \boldsymbol{y} \tag{18}$$

Taking the dot product of $\boldsymbol{y}$ with Equation (17)

$$\boldsymbol{x} \cdot S\boldsymbol{y} = \boldsymbol{x} \cdot \lambda_2 \boldsymbol{y}$$
$$\boldsymbol{x}^T S\boldsymbol{y} = \boldsymbol{x}^T \lambda_2 \boldsymbol{y} \tag{19}$$

The left hand sides of Equations (18) and (19) are equal. Then,

$$\boldsymbol{x}^T \lambda_1 \boldsymbol{y} = \boldsymbol{x}^T \lambda_2 \boldsymbol{y} \tag{20}$$

Since $\lambda_1 \neq \lambda_2$, for Equation (20) to hold,

$$\boxed{\boldsymbol{x}^T \boldsymbol{y} = 0}$$

The eigenvectors are perpendicular.

□

Remark. *Every symmetric matrix that are diagonalizable can be expressed by the following:*

$$S = Q\Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \ldots + \lambda_n q_n q_n^T \tag{21}$$

4.1 Complex Eigenvalues of Real Matrices

For real matrices, complex eigenvalues and eigenvectors come in conjugate pairs. Let $\lambda = a + ib$, then its complex conjugate will be $\bar{\lambda} = a - ib$:

$$\text{If } A\boldsymbol{x} = \lambda\boldsymbol{x}, \text{ then } A\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}.$$

Example 14. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Computing for the eigenvalues:

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$
$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$
$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$
$$\lambda_1 = \cos\theta + i\sin\theta \qquad \lambda_2 = \cos\theta - i\sin\theta$$

We can see that $\lambda_2 = \bar{\lambda}_1$, these are a conjugate pair. Next we find the eigenvectors:

$$(A - \lambda_1 I)\boldsymbol{x}_1 = 0$$
$$\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - (\cos\theta + i\sin\theta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\boldsymbol{x}_1 = \mathbf{0}$$
$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0}$$
$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Fining $\boldsymbol{x}_2$:

$$(A - \lambda_2 I)\boldsymbol{x}_2 = 0$$
$$\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - (\cos\theta - i\sin\theta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\boldsymbol{x}_2 = \mathbf{0}$$
$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}$$
$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Again, we see that the eigenvectors are a conjugate pair.

4.2 All Symmetric Matrices are Diagonalizable

For nonsymmetric matrices, a shortage of eigenvalues (repeating eigenvalues) cause a shortage of eigenvectors. This never happens in symmetric matrices. **There are always enough eigenvectors to diagonalize** $S = S^T$.

Every square matrix can be "triangularized" by $A = QTQ^{-1}$. If $A = S$ (symmetric), then $T = \Lambda$.

Theorem 11. Every square A factors into QTQ^{-1} where T is upper triangular and $\bar{Q}^T = Q^{-1}$. If A has real eigenvalues then Q and T can be chosen real $Q^TQ = I$.

5 Positive Definite Matrices