Chapter 3: Vector Spaces and Subspaces

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Abstract

This chapter focuses on vector spaces and subspaces. Topics include vector spaces and subspaces such as the column space $C(A)$, nullspace $N(A)$, and the rest of the *four subspaces*.

1 The Nullspace of A: Solving $A\mathrm{x} = \mathbf{0}$ and $R\mathrm{x} = \mathbf{0}$

The nullspace is a subspace of matrix A (square or rectangular) that contains all of the solutions to $A\mathbf{x} = \mathbf{0}$. A readily available solution to the system is the zero vector $\mathbf{Z}$. Invertible matrices only have the zero vector as a solution while non-invertible matrices have nonzero solutions. **Each solution x belongs to the nullspace of A which is in $\mathbb{R}^n$.**

Problem 1. Describe the nullspace of the singular matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution: We produce the following from elimination:

$$A\boldsymbol{x} = 0$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Elimination produces a single line. This line is in the nullspace of A and **contains all solutions** (x_1, x_2). x_2 is a free variable. We choose $x_2 = 1$ as our special solution because all points on the line are multiples to it. From there, we get $x_1 = -2$.

$$\boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence,

$$A\boldsymbol{s} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

□

The nullspace of A consists of all combinations of the special solutions to $A\boldsymbol{x} = \mathbf{0}$

Problem 2. $x + 2y + 3z = 0$ comes from the 1×3 matrix $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Then $A\boldsymbol{x} = \mathbf{0}$ produces a plane. All vectors on the plane are perpendicular to $(1, 2, 3)$. **The plane is the nullspace of A**. There are 2 free variables y and z: Set to 0 and 1.

Solution:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

We have two free variables y and z, hence we have two special solutions. For $\boldsymbol{v}_1$, we choose $y = 1$ and $z = 0$.

$$x + 2(1) + 3(0) = 0 \qquad \rightarrow \qquad x = -2$$
$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

For $\boldsymbol{s}_2$, we choose $y = 0$ and $z = 1$.

$$x + 2(0) + 3(1) = 0 \qquad \rightarrow \qquad x = -3$$
$$\boldsymbol{s}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Checking for $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$,

$$A\boldsymbol{s}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -2 + 2 + 0 = 0$$
$$A\boldsymbol{s}_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -3 + 0 + 3 = 0$$

The vectors $\boldsymbol{s}_1, \boldsymbol{s}_2$ lie on the plane $x + 2y + 3z = 0$. All vectors on the plane are combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.
$\square$

Remark. *There are two key steps in finding the nullspace of a matrix.*

1. *Reducing A to **row echelon form** R*
2. *Finding the special solutions to $A\boldsymbol{x} = \boldsymbol{0}$*

1.1 Pivot Variables and Free Columns

Free components correspond to columns with no pivots. The special choice (1 or 0) is only for the free variables in the special solutions.

Problem 2. Find the nullspaces of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \qquad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

Solution:

Elimination with A yields

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \rightarrow U\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A is an invertible matrix, hence $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. On to B.

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first two rows do not have special solutions other than the trivial zero vector. The 3rd and 4th rows are dependent on the 1st and 2nd rows, respectively which would become zero rows when we do elimination. The nullspace is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. On to C.

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first two columns above are called the "*pivot columns*" while the last two rows are called "*free columns.*" We will have two special solutions since there are two free variables x_3 and x_4. For $\boldsymbol{s}_1$, we choose $x_3 = 1$ and $x_4 = 0$.

$$U\boldsymbol{s}_1 = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

From row 2,

$$0 + 2x_2 + 0 + 0 = 0$$

From row 1,

$$x_1 + 0 + 2 + 0 = 0$$
$$x_1 = -2$$

Hence,

$$\boxed{\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

We do the same for $\boldsymbol{s}_2$, but $x_3 = 0$ and $x_4 = 1$

$$U\boldsymbol{s}_2 = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

From row 2,

$$0 + 2x_2 + 0 + 4 = 0$$
$$x_2 = -2$$

From row 1,

$$x_1 + -4 + 0 + 4 = 0$$
$$x_1 = 0$$

Hence,

$$\boxed{\boldsymbol{s}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}}$$

□

1.2 Reduced Row Echelon Form R

We already know how to do Gaussian elimination. To get to RREF, we must produce zeros above the pivots of U and produce ones in the pivots of U by using pivot rows to eliminate upward in R and dividing the whole pivot row by its own pivot. The unique thing is that the nullspace does not change throughout the elimination process: $\mathbf{N(A) = N(U) = N(R)}$.

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$
$$R\boldsymbol{x} = \mathbf{0}$$
$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we reach R, getting the special solutions become easier. $\boldsymbol{s}_1 = (-2, 0, 1, 0)$ and $\boldsymbol{s}_2 = (0, -2, 0, 1)$

Many matrices only have one solution to $A\boldsymbol{x} = 0$, the zero vector $\boldsymbol{Z}$. The nullspaces of these matrices become $N(A) = \boldsymbol{Z}$ without any special solutions. This case stresses the notion that A's columns are **independent**. No combination of columns gives the zero vector except for the zero combination. All columns are pivots and there are no free columns. Below is another example.

Problem 3. What are the special solutions of the following matrix? $A = \begin{bmatrix} | & | & | & | & | \\ p & p & f & p & f \\ | & | & | & | & | \end{bmatrix}$

Solution:

Using elimination and doing RREF(A), we eventually reach R

$$\boldsymbol{R} = \begin{bmatrix} \mathbf{1} & 0 & a & 0 & c \\ 0 & \mathbf{1} & b & 0 & d \\ 0 & 0 & 0 & \mathbf{1} & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are only 2 free columns, therefore there are only 2 free variables. We will need $\boldsymbol{s}_1$and $\boldsymbol{s}_2$. For $\boldsymbol{s}_1$, we choose $x_3 = 1$ and $x_5 = 0$.

$$\boldsymbol{s}_1 = \begin{bmatrix} -a \\ -b \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\boldsymbol{s}_2$, we choose $x_3 = 0$ and $x_5 = 1$.

$$\boldsymbol{s}_2 = \begin{bmatrix} -c \\ -d \\ 0 \\ -e \\ 1 \end{bmatrix}$$

$\square$