Chapter 3: Vector Spaces and Subspaces

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Abstract

This chapter focuses on vector spaces and subspaces. Topics include vector spaces and subspaces such as the column space $C(A)$, nullspace $N(A)$, and the rest of the *four subspaces*.

1 The Nullspace of A: Solving $A\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{0}$

The nullspace is a subspace of matrix A (square or rectangular) that contains all of the solutions to $A\mathbf{x} = \mathbf{0}$. A readily available solution to the system is the zero vector $\mathbf{Z}$. Invertible matrices only have the zero vector as a solution while non-invertible matrices have nonzero solutions. **Each solution x belongs to the nullspace of A which is in $\mathbb{R}^n$.**

Problem 1. Describe the nullspace of the singular matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution: We produce the following from elimination:

$$A\boldsymbol{x} = 0$$
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Elimination produces a single line. This line is in the nullspace of A and **contains all solutions** (x_1, x_2). x_2 is a free variable. We choose $x_2 = 1$ as our special solution because all points on the line are multiples to it. From there, we get $x_1 = -2$.

$$\boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence,

$$A\boldsymbol{s} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

□

The nullspace of A consists of all combinations of the special solutions to $A\boldsymbol{x} = \mathbf{0}$

Problem 2. $x + 2y + 3z = 0$ comes from the 1×3 matrix $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Then $A\boldsymbol{x} = \mathbf{0}$ produces a plane. All vectors on the plane are perpendicular to $(1, 2, 3)$. **The plane is the nullspace of A**. There are 2 free variables y and z: Set to 0 and 1.

Solution:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

We have two free variables y and z, hence we have two special solutions. For $\boldsymbol{v}_1$, we choose $y = 1$ and $z = 0$.

$$x + 2(1) + 3(0) = 0 \qquad \rightarrow \qquad x = -2$$
$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

For $\boldsymbol{s}_2$, we choose $y = 0$ and $z = 1$.

$$x + 2(0) + 3(1) = 0 \qquad \rightarrow \qquad x = -3$$
$$\boldsymbol{s}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Checking for $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$,

$$A\boldsymbol{s}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -2 + 2 + 0 = 0$$
$$A\boldsymbol{s}_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = -3 + 0 + 3 = 0$$

The vectors $\boldsymbol{s}_1, \boldsymbol{s}_2$ lie on the plane $x + 2y + 3z = 0$. All vectors on the plane are combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

$\square$

Remark. *There are two key steps in finding the nullspace of a matrix.*

1. *Reducing A to **row echelon form** R*
2. *Finding the special solutions to $A\boldsymbol{x} = \boldsymbol{0}$*

1.1 Pivot Variables and Free Columns

Free components correspond to columns with no pivots. The special choice (1 or 0) is only for the free variables in the special solutions.

Problem 2. Find the nullspaces of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \qquad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

Solution:

Elimination yields

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A is an invertible matrix, hence $\boldsymbol{x}$ is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\square$