

Chapter 6: Eigenvalues and Eigenvectors

Val Anthony Balagon

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Abstract

This chapter focuses on the properties of eigenvalues and eigenvectors, diagonalizing a matrix, systems of differential equations, symmetric matrices, and positive definite matrices.

1 Introduction

The system $A\mathbf{x} = \mathbf{b}$ is in equilibrium and steady state. Change as in time enters the picture - continuous time in a differential equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ or time steps in a difference equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$. Using linear algebra, eigenvalues and eigenvectors allow these types of systems to be solved beautifully.

Example 1. Say we have a matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$, calculate A^2, A^3 , and A^{100} .

$$A^2 = \begin{bmatrix} .70 & .30 \\ .30 & .55 \end{bmatrix} \quad A^3 = \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \quad A^{100} \approx \begin{bmatrix} .60 & .60 \\ .40 & .40 \end{bmatrix}$$

One way is to solve these equations using eigenvalues.

Vectors \mathbf{x} when multiplied by A usually change direction. But there are certain exceptional vectors that maintain the same direction as $A\mathbf{x}$ and these are called “eigenvectors.” The basic equation is

$$A\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

where the number λ is an eigenvalue of A and \mathbf{x} is an eigenvector of A .

Example 2. What are the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}?$$

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{bmatrix}.8 & .3 \\ .2 & .7\end{bmatrix} - \begin{bmatrix}\lambda & 0 \\ 0 & \lambda\end{bmatrix}\right) &= 0 \\
\begin{vmatrix}.8 - \lambda & .3 \\ .2 & .7 - \lambda\end{vmatrix} &= 0 \\
\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} &= 0 \\
(\lambda - 1)\left(\lambda - \frac{1}{2}\right) &= 0 \\
\lambda_1 = 1 \quad \lambda_2 = \frac{1}{2}
\end{aligned}$$

$A - \lambda I$ becomes a singular matrix and the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$$(A - \lambda I)\mathbf{x} = 0$$

Finding x_1 ,

$$\begin{aligned}
(A - I)x &= 0 \\
\begin{bmatrix}-.2 & .3 \\ .2 & -.3\end{bmatrix}\mathbf{x}_1 &= 0 \\
\mathbf{x}_1 &= \begin{bmatrix}.6 \\ .4\end{bmatrix}
\end{aligned}$$

Finding x_2 ,

$$\begin{aligned}
(A - \frac{1}{2}I)x &= 0 \\
\begin{bmatrix}.3 & .3 \\ .2 & .2\end{bmatrix}\mathbf{x}_2 &= 0 \\
\mathbf{x}_2 &= \begin{bmatrix}1 \\ -1\end{bmatrix}
\end{aligned}$$

Checking,

$$\begin{aligned}
A\mathbf{x}_1 &= \begin{bmatrix}.8 & .3 \\ .2 & .7\end{bmatrix} \begin{bmatrix}.6 \\ .4\end{bmatrix} = 1 \begin{bmatrix}.6 \\ .4\end{bmatrix} = \lambda_1 \mathbf{x}_1 \\
A\mathbf{x}_2 &= \begin{bmatrix}.8 & .3 \\ .2 & .7\end{bmatrix} \begin{bmatrix}1 \\ -1\end{bmatrix} = \frac{1}{2} \begin{bmatrix}1 \\ -1\end{bmatrix} = \lambda_2 \mathbf{x}_2
\end{aligned}$$

And if A is multiplied n times we get $A^n \mathbf{x}_1 = \lambda_1^n \mathbf{x}_1$. Same goes for the second eigenvector. Also take note that the columns of A are a combination of the eigenvectors: $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2$. The eigenvector \mathbf{x}_1 is a *steady state* because $\lambda_1 = 1$. The eigenvector \mathbf{x}_2 is a *decaying mode* that virtually disappears because $\lambda_2 = .5$. The higher the power of A , the more closely its columns approach the steady state. A is an example of a **Markov Matrix** where the sum of the entries for each column equal to one.

Definition 1.1. If A is multiplied n times, the eigenvectors stay the same and the eigenvalues are also multiplied n times.

1.1 Equation for Eigenvalues

To solve for the eigenvalues and eigenvectors, we start with Equation (1).

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ \boxed{(A - \lambda I)\mathbf{x} = \mathbf{0}} \end{aligned} \tag{2}$$

The eigenvectors of \mathbf{x} make up the nullspace of $A - \lambda I$. If we find the eigenvalue λ we can calculate for the eigenvector. To solve for the eigenvalue, we know that $A - \lambda I$ is a singular matrix. Therefore its determinant is zero.

$$\boxed{\det(A - \lambda I) = p(\lambda) = 0} \tag{3}$$

$\det(A - \lambda I) = 0$ is called the **characteristic polynomial**. Generally, the characteristic polynomial is the following:

$$p(\lambda) = (-\lambda)^n + \text{trace}(A)(-\lambda)^{n-1} + \dots + \det(A)$$

To get the eigenvalues, we solve for λ . When A is an $n \times n$ matrix, Equation 3 has degree n . A has n eigenvalues and repeating λ are possible. Each λ leads to \mathbf{x} . **For each λ , solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find the eigenvector.**

Example 3. Find the eigenvalues and eigenvectors of the singular matrix: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) &= 0 \\ \lambda(\lambda - 5) &= 0 \\ \lambda_1 = 0 & \quad \lambda_2 = 5 \end{aligned}$$

For the first eigenvector \mathbf{x}_1 :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ (A - 0I)\mathbf{x}_1 &= \mathbf{0} \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \mathbf{0} \end{aligned}$$

The nullspace solution is:

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

For the second eigenvector \mathbf{x}_2 :

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= 0 \\ (A - 5I)\mathbf{x}_2 &= 0 \\ \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \end{aligned}$$

The nullspace solution is:

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If A is a singular matrix, $\lambda = 0$ is an eigenvalue of A .

Summary To solve the eigenvalue problem for an $n \times n$ matrix, follow these steps:

1. **Compute the determinant of $A - \lambda I$.** With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It's a polynomial of degree n .
2. **Find the roots of this polynomial**, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A . They make $A - \lambda I$ singular.
3. For each eigenvalue λ , **solve $(A - \lambda I)\mathbf{x} = 0$ to find the eigenvector \mathbf{x} .**

Warning! There are times when A has equal eigenvalues. Hence, there is only one line of eigenvectors. Without a full set of eigenvectors, we can't diagonalize a matrix without n independent eigenvectors.

1.2 Determinant and Trace

We cannot get the eigenvalues of A when we do elimination to U because elimination does not preserve the eigenvalues. U has its own set of eigenvalues along its diagonal (the pivots) but these aren't the eigenvalues of A . The product $\lambda_1 \times \lambda_2 \times \dots \times \lambda_n$ and the sum $\lambda_1 + \lambda_2 + \dots + \lambda_n$ can be found directly from the matrix A .

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{has eigenvalues } \lambda = 0, \lambda = 7 \quad (4)$$

For A , the product of the eigenvalues is $0 \cdot 7 = 0$. This agrees with the $\det(A) = 0$. The sum of the eigenvalues $0 + 7$ agrees with the sum down the diagonal of A . The sum of the

diagonal entries of A is called the **trace**. These two properties are important checks to see if our calculations for the eigenvalues are correct.

Theorem 1. The product of the n eigenvalues equals the determinant of matrix A .

$$\lambda_1 \times \lambda_2 \times \dots \times \lambda_n = \det(A) \quad (5)$$

Proof.

Consider $\det(A - \lambda I)$,

$$\det(A - \lambda I) = p(\lambda)$$

Factorizing the characteristic equation according to its individual roots.

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

Setting $\lambda = 0$, therefore

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

□

Theorem 2. The sum of the n eigenvalues equals the sum of the n diagonal entries of A (the trace).

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = a_{11} + a_{22} + \dots + a_{nn} = \text{trace}(A) \quad (6)$$

Proof. Consider the general equation of the characteristic polynomial $p(\lambda)$.

$$p(\lambda) = (-\lambda)^n + \text{trace}(A)(-\lambda)^{n-1} + \dots + \det(A)$$

Another expression of the characteristic polynomial is:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

By *comparing coefficients*, we get the trace:

$$\text{trace}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

□

Example 4. To see the proof above clearly, here is an example where there are two eigenvalues:

$$p(\lambda) = \lambda^2 - \text{trace}(A)\lambda + \det(A)$$

$p(\lambda)$ can also be expressed with the following:

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)\lambda + \lambda^2$$

By comparing coefficients, we get the trace:

$$\text{trace}(A) = \lambda_1 + \lambda_2$$

Theorem 3. The eigenvalues of a triangular matrix lie along its diagonal.

Proof. Suppose A is a triangular matrix with nonzero entries.

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

Consider the determinant of $A - \lambda I$,

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} = p(\lambda)$$

Where $p(\lambda)$ is the characteristic equation. The roots of the characteristic equation are the eigenvalues. The determinant of $A - \lambda I$ is 0 by definition because it is a singular matrix.

$$\begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} = 0$$

The determinant of a triangular matrix is the product of its diagonal entries. Hence,

$$\det(A - \lambda I) = \prod_{i=1}^n (a_{ii} - \lambda) = 0$$

Therefore, each diagonal entry in A is an eigenvalue.

$$a_{ii} = \lambda$$

□

2 Diagonalizing a Matrix

Eigenvalues and eigenvectors make matrix multiplication easier. When \mathbf{x} is an eigenvector, multiplication by A is just a multiplication by a number λ : $A\mathbf{x} = \lambda\mathbf{x}$. Diagonalizing a matrix turns A into a diagonal matrix Λ when we use the eigenvectors properly.

Definition 2.1. Suppose the $n \times n$ matrix A has n linearly independent eigenvectors x_1, \dots, x_n . Put them into the columns of an eigenvector matrix X . Then $X^{-1}AX$ is the **eigenvalue matrix** Λ :

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{bmatrix} \tag{7}$$

Example 5. Diagonalize $A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$

A is a triangular matrix, therefore its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 6$. We then find its eigenvectors. Finding \mathbf{x}_1 :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= 0 \\ \left(\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix} \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finding \mathbf{x}_2 ,

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= 0 \\ \left(\begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right) \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Diagonalizing:

$$\begin{aligned} X^{-1}AX &= \Lambda \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

Theorem 4. A^n has the same eigenvectors in X and the eigenvalues raised to the power of n in Λ^n .

Proof.

$$\begin{aligned} A^2 &= X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^2 X^{-1} \\ A^3 &= X\Lambda X^{-1}X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^3 X^{-1} \\ &\vdots \\ &\text{etc.} \end{aligned}$$

□

The matrix X must have an inverse because its columns are linearly independent. **Without n independent eigenvectors, we can't diagonalize.**

Remark. Suppose the eigenvalues $\lambda_1, \dots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ are independent. The eigenvector matrix X will be invertible. **Any matrix that has no repeated eigenvalues can be diagonalized.**

Remark. We can multiply eigenvectors by any nonzero constants. $A(c\mathbf{x}) = \lambda(c\mathbf{x})$ is still true.

Remark. The eigenvectors in X come in the same order as the eigenvalues in Λ .

Remark. Matrices that have repeated eigenvalues cannot be diagonalized.

Example 6. The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = .5$ and eigenvectors of $\mathbf{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -.6 \end{bmatrix}$. Find A^2, A^k , and A^∞ .

$$A = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = X\Lambda X^{-1}$$

Computing for A^2 .

$$A^2 = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

Computing for A^k .

$$A^k = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}$$

Computing for A^∞

$$A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \lim_{k \rightarrow \infty} \left(\begin{bmatrix} 1^k & 0 \\ 0 & .5^k \end{bmatrix} \right) \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -.1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

2.1 Similar Matrices: Same Eigenvalues

Similar matrices have the same Λ and different X .

Theorem 5. All matrices $A = BCB^{-1}$ are “similar”. They all share the eigenvalues of C .

2.2 Fibonacci Numbers

0, 1, 1, 2, 3, 5, 8, 13, ...

A new Fibonacci number is the sum of the two previous Fibonacci numbers in the Fibonacci sequence. The rule is: $F_{k+2} = F_{k+1} + F_k$. The obvious and slow way to get to F_{100} is by applying the rule one at a time. Linear algebra gives a faster method. The one step rule for the Fibonacci sequence is $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

$$\text{Let } \mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

The rules below are put into matrix A .

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \\ \mathbf{u}_{k+1} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}_k \end{aligned}$$

Every step multiplies by A . After 100 steps we reach $\mathbf{u}_{100} = A^{100}\mathbf{u}_0$.

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

Finding the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 \\ \lambda_1 &= \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \end{aligned}$$

Finding the eigenvector \mathbf{x}_1 ,

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1+\sqrt{5}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \mathbf{x}_1 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \end{aligned}$$

Finding the eigenvector \mathbf{x}_2 ,

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= \mathbf{0} \\ \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1-\sqrt{5}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{bmatrix} \mathbf{x}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} &= \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \end{aligned}$$

We need to express \mathbf{u}_0 as a combination of the eigenvectors.

$$\begin{aligned}
c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} & -\frac{1}{\lambda_1 - \lambda_2} \\ -\frac{1}{\lambda_1 - \lambda_2} & \frac{1}{\lambda_1 - \lambda_2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \\ -\frac{1}{\lambda_1 - \lambda_2} \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

We now have the expression:

$$\begin{aligned}
\frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u}_k \\
\frac{1}{\lambda_1 - \lambda_2} \left(\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{u}_0 &= \frac{\mathbf{x}_1 - \mathbf{x}_2}{\lambda_1 - \lambda_2}
\end{aligned}$$

Then we have a generalized formula in terms of our eigenvalues:

$$\mathbf{u}_k = \frac{\lambda^k \mathbf{x}_1 - \lambda^k \mathbf{x}_2}{\lambda_1 - \lambda_2}$$

To get 100th Fibonacci number, F_{100} , we use the second component of $\mathbf{x}_1, \mathbf{x}_2$

$$F_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}$$

Which is approximately,

$$F_{100} \approx 3.54224 \times 10^{20}$$

2.3 Matrix Powers A^k

$$A^k \mathbf{u}_0 = (X \Lambda X^{-1}) \dots (X \Lambda X^{-1}) \mathbf{u}_0 = X \Lambda^k X^{-1} \mathbf{u}_0$$

Fibonacci's example is a typical difference equation $\mathbf{u}_{k+1} = A \mathbf{u}_k$. Each step multiplies by A . The solution is $\mathbf{u}_{k+1} = A^k \mathbf{u}_0$. Diagonalizing the matrix gives a quick way to compute A^k and find \mathbf{u}_k in three quick steps.

1. Write \mathbf{u}_0 as a combination $c_1 \mathbf{x}_1 + \dots c_n \mathbf{x}_n$ of the eigenvectors. Then $\mathbf{c} = X^{-1} \mathbf{u}_0$.
2. Multiply each eigenvector \mathbf{x}_i by $(\lambda_i)^k$. Now we have $\Lambda^k X^{-1} \mathbf{u}_0$.

3. Add up the pieces $c_i(\lambda_i)^k \mathbf{x}_i$ to find the solution $\mathbf{u}_k = A^k \mathbf{u}_0$. This is $X\Lambda^k X^{-1} \mathbf{u}_0$.

The solution for $\mathbf{u}_{k+1} = A\mathbf{u}_k$ becomes:

$$\mathbf{u}_{k+1} = A\mathbf{u}_k = c_1(\lambda_1)^k \mathbf{x}_1 + \dots + c_n(\lambda_n)^k \mathbf{x}_n = X\Lambda^k X^{-1} \mathbf{u}_0 = X\Lambda^k \mathbf{c}$$

$$X\Lambda^k \mathbf{c} = \begin{bmatrix} | & \dots & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & \dots & | \end{bmatrix} \begin{bmatrix} (\lambda_1)^k & & \\ & \ddots & \\ & & (\lambda_n)^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

2.4 Nondiagonalizable Matrices

When we solve for λ and \mathbf{x} , we want to know their **multiplicity**.

Geometric Multiplicity = GM - Count the independent eigenvectors for λ . Then GM is the dimension of the nullspace of $A - \lambda I$.

Algebraic Multiplicity = AM - AM counts the repetitions of λ among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

Example 7. Find the GM and AM of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

$$\lambda = 0$$

The eigenvalue 0 is a repeating eigenvalue. Hence its AM = 2. The eigenvalue 0 only has one eigenvector hence GM = 1.

Theorem 6. If GM = AM, the matrix A is **diagonalizable**.

Theorem 7. If GM < AM, the matrix A is **not diagonalizable**.

3 Systems of Differential Equations

We want to solve \mathbf{u} from the following system of differential equations:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \quad \text{starting from the vector } \mathbf{u}(0) = \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix}$$

These equations are linear. If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions, so is $C\mathbf{u}(t) + D\mathbf{v}(t)$. Our job is to find n “pure exponential solutions” $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$ by using $A\mathbf{x} = \lambda\mathbf{x}$. Here λ is an eigenvalue while \mathbf{x} is an eigenvector. First we check if $\mathbf{u}(t)$ is a solution.

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= A\mathbf{u} \\ \frac{d}{dt}(e^{\lambda t}\mathbf{x}) &= Ae^{\lambda t}\mathbf{x} \\ \lambda e^{\lambda t}\mathbf{x} &= Ae^{\lambda t}\mathbf{x} \\ \lambda\mathbf{x} &= A\mathbf{x} \end{aligned}$$

All components of this special solution $\mathbf{u} = e^{\lambda t}\mathbf{x}$ share the same $e^{\lambda t}$. For a certain system, the pure exponential solutions $\mathbf{u}_1 = e^{\lambda_1 t}\mathbf{x}_1$ satisfies $A\mathbf{u}_1 = \lambda_1\mathbf{u}_1$, $\mathbf{u}_2 = e^{\lambda_2 t}\mathbf{x}_2$ satisfies $A\mathbf{u}_2 = \lambda_2\mathbf{u}_2$, and so on. The solution **grows** when $\lambda > 0$. It **decays** when $\lambda < 0$. If λ is a complex number, its real part decides growth or decay. The imaginary part ω **gives oscillation** $e^{i\omega t}$ like a sine wave.

Example 8. Solve the following system of differential equations.

$$\begin{aligned} \frac{du_1}{dt} &= -u_1 + 2u_2 \\ \frac{du_2}{dt} &= u_1 - 2u_2. \end{aligned}$$

Here $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ and the initial value $\mathbf{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We want to solve:

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= A\mathbf{u} \\ \frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

We need to find the eigenvalues and eigenvectors of A . We know that A is a singular matrix, hence one of its eigenvalues is $\lambda_1 = 0$. We can get the second eigenvalue from the trace which is $\lambda_2 = -3$. We then find the eigenvectors. First for λ_1 ,

$$\begin{aligned}(A - \cancel{\lambda_1}^0 I)\mathbf{x}_1 &= 0 \\ \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \mathbf{x}_1 &= 0 \\ \mathbf{x}_1 &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}\end{aligned}$$

For the second eigenvector,

$$\begin{aligned}(A - \lambda_2 I)\mathbf{x}_2 &= 0 \\ \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 &= 0 \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

Then the general solution becomes:

$$\begin{aligned}\mathbf{u}(t) &= c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 \\ &= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\end{aligned}$$

We then find c_1 and c_2 .

$$\begin{aligned}\mathbf{u}(0) &= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3(0)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}\end{aligned}$$

Hence the solution is:

$$\boxed{\mathbf{u}(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Example 9. Solve $\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}$ starting from $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. This is a vector equation for \mathbf{u} . It contains two scalar equations for the components y, z . They are “coupled together” because the matrix is not diagonal.

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

$$\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

The equation above means:

$$\frac{dy}{dt} = z \quad \text{and} \quad \frac{dz}{dt} = y$$

Finding the eigenvalues,

$$\det(A - \lambda I) = 0$$

$$\det \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\boxed{\lambda_1 = 1, \quad \lambda_2 = -1}$$

Finding the first eigenvector,

$$(A - \lambda_1 I)\mathbf{x}_1 = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}_1 = 0$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finding the second eigenvector,

$$(A - \lambda_2 I)\mathbf{x}_2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 = 0$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution becomes:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Finding the constants c_1, c_2 ,

$$\begin{aligned}\mathbf{u}(0) &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}\end{aligned}$$

Therefore, the solution becomes:

$$\boxed{\mathbf{u}(t) = 3e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

We would like to remind ourselves that really, the final solution is expressed in terms of a diagonalization:

$$\mathbf{u}(t) = X\Lambda X^{-1}\mathbf{u}_0 = X\Lambda c = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Summary The same three steps in calculating $\mathbf{u}_{k+1} = A\mathbf{u}_k$ now solve $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$.

1. Write \mathbf{u}_0 as a combination $c_1\mathbf{x}_1 + \dots c_n\mathbf{x}_n$ of the eigenvectors of A .
2. Multiply each eigenvector \mathbf{x}_i by its growth factor $e^{\lambda_i t}$.
3. The solution is the same combination of those pure solutions $e^{\lambda_i t}\mathbf{x}_i$:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} \quad \mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + \dots + c_n e^{\lambda_n t} \mathbf{x}_n.$$

Warning! If a λ repeats, with only one eigenvector, another solution is needed which is $(te^{\lambda t})$.

3.1 Second Order Differential Equations

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass times the acceleration $a = y''$. The term ma balances the force F (Newton's second law). The force includes the damping $-by'$ and the elastic force $-ky$. This second order differential equation is still linear with coefficients m, b, k .

The method of solution to this equation is to substitute $y = e^{\lambda t}$. Each derivative of y

brings down a factor λ .

$$\begin{aligned}
m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky &= 0 \\
m \frac{d^2}{dt^2} e^{\lambda t} + b \frac{d}{dt} e^{\lambda t} + k e^{\lambda t} &= 0 \\
m \lambda \frac{d}{dt} e^{\lambda t} + b \lambda e^{\lambda t} + k e^{\lambda t} &= 0 \\
m \lambda^2 e^{\lambda t} + b \lambda e^{\lambda t} + k e^{\lambda t} &= 0 \\
(m \lambda^2 + b \lambda + k) e^{\lambda t} &= 0
\end{aligned}$$

There are two roots for λ , λ_1 and λ_2 . This equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1 y_1 + c_2 y_2$ give the complete solution unless the λ 's are the same. We want to express the problem into a vector equation for y and y' . We have:

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A \mathbf{u} \quad (8)$$

A comes from the fact that $y' = y'$ and $y'' = -by' - ky$. We then solve $\mathbf{u}' = A \mathbf{u}$ by the eigenvalues of A .

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{vmatrix} &= 0 \\
\lambda^2 + b\lambda + k &= 0
\end{aligned}$$

The roots λ_1 and λ_2 will become our eigenvalues. Now to find the eigenvectors. Finding \mathbf{x}_1 :

$$\begin{aligned}
\begin{bmatrix} -\lambda_1 & 1 \\ -k & -b - \lambda_1 \end{bmatrix} \mathbf{x}_1 &= 0 \\
\mathbf{x}_1 &= \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}
\end{aligned}$$

\mathbf{x}_1 will hold if $-k + \lambda_1(-b - \lambda_1) = 0$. Finding \mathbf{x}_2 .

$$\begin{aligned}
\begin{bmatrix} -\lambda_2 & 1 \\ -k & -b - \lambda_2 \end{bmatrix} \mathbf{x}_2 &= 0 \\
\mathbf{x}_2 &= \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}
\end{aligned}$$

\mathbf{x}_2 will hold if $-k + \lambda_2(-b - \lambda_2) = 0$. The general solution will then be:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

Example 10. Motion around a circle with $y'' + y = 0$ and $y = \cos(t)$. This is our master equation with mass $m = 1$ and stiffness $k = 1$ and $d = 0$ (no damping). Substitute $y = e^{\lambda t}$ into $y'' + y = 0$ to reach $\lambda^2 + 1 = 0$. The roots are complex numbers $\lambda = \pm i$. Then half of $e^{it} + e^{-it}$ gives the solution $y = \cos(t)$. The initial values are $y(0) = 1, y'(0) = 0$. Find $\mathbf{u}(t)$.

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\mathbf{u}$$

Finding the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} &= 0 \\ \lambda^2 + 1 &= 0 \\ (\lambda - i)(\lambda + i) &= 0 \\ \boxed{\lambda_1 = i \quad \lambda_2 = -i} \end{aligned}$$

Finding the eigenvector \mathbf{x}_1 :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= 0 \\ \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ i \end{bmatrix} \end{aligned}$$

Finding the eigenvector \mathbf{x}_2 :

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_1 &= 0 \\ \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \mathbf{x} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ -i \end{bmatrix} \end{aligned}$$

The general solution is:

$$\begin{aligned} \mathbf{u}(t) &= c_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ \mathbf{u}(t) &= \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ ic_1 e^{it} - ic_2 e^{-it} \end{bmatrix} \end{aligned}$$

Finding c_1 and c_2 :

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{c} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

The final solution is:

$$\mathbf{u}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^{it} + e^{-it}) \\ \frac{1}{2}i(e^{it} - e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

3.2 Exponential of a Matrix

We want to write the solution $\mathbf{u}(t)$ in a new form $e^{At}\mathbf{u}(0)$. First we have to define what e^{At} means by copying the Taylor Series representation of e^x for numbers.

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots$$

Its derivative with respect to t is:

$$\frac{d}{dt}e^{At} = Ae^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \dots$$

The eigenvalues are $e^{\lambda t}$

$$e^{At}\mathbf{x} = \left(\sum_{n=0}^{\infty} \frac{(At)^n}{n!} \right) \mathbf{x} = (I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots)\mathbf{x} = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \dots)\mathbf{x}$$

The series always converges and its derivative is always Ae^{At} . Therefore $e^{At}\mathbf{u}(0)$ solves the differential equation with one quick formula - even if there is a shortage of eigenvectors. This chapter emphasizes how to find $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$ by diagonalization. Assume A does have n independent eigenvectors, so it is diagonalizable. Substitute $A = S\Lambda S^{-1}$ into the series for e^{At} .

$$\begin{aligned} e^{At} &= e^{X\Lambda X^{-1}t} = \sum_{n=0}^{\infty} \frac{(X\Lambda X^{-1}t)^n}{n!} = I + (X\Lambda X^{-1}t) + \frac{1}{2}(X\Lambda X^{-1}t)^2 + \dots \\ &= X \left[I + \Lambda t + \frac{1}{2}(\Lambda t)^2 + \dots \right] X^{-1} \end{aligned}$$

We now arrive at a diagonalized form for e^{At} :

$$\boxed{e^{At} = X e^{\Lambda t} X^{-1}} \tag{9}$$

The solution becomes:

$$\mathbf{u}(t) = e^{At}\mathbf{u}_0 = X e^{\Lambda t} X^{-1}\mathbf{u}(0) = \begin{bmatrix} | & \dots & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_n \\ | & \dots & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Example 11. Solve $y'' - 2y' + y = 0$.

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{u}$$

Finding the eigenvalues of A :

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ -1 & 2 - \lambda \end{vmatrix} &= 0 \\ (\lambda - 1) &= 0 \\ \boxed{\lambda_{1,2} = 1} \end{aligned}$$

Here we only have one unique eigenvalue. We then find the eigenvector \mathbf{x}_1 .

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}_1 &= \mathbf{0} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Hence diagonalization is not possible because we do not have 2 eigenvectors. We will then compute e^{At} instead.

$$e^{At} = e^{At+It-It} = e^t e^{(A-I)t} = e^t \sum_{n=0}^{\infty} \frac{((A-I)t)^n}{n!}$$

Expanding the series:

$$\sum_{n=0}^{\infty} \frac{((A-I)t)^n}{n!} = I + (A-I)t + \frac{1}{2}(A-I)^2 t^2 + \frac{1}{6}(A-I)^3 t^3 + \dots$$

When the series reaches $n = 2$ and above, the $(A - I)$ term becomes zero. Hence,

$$e^{At} = e^t(I + (A - I)t)$$

Hence,

$$\mathbf{u} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^t \left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \right] \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} e^t y(0) - te^t y(0) + te^t y'(0) \\ -e^t t y(0) + e^t y'(0) + te^t y'(0) \end{bmatrix}$$

The solution to our problem becomes:

$$y(t) = e^t y(0) - te^t y(0) + te^t y'(0)$$

Example 12. Use the infinite series to find e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$\begin{aligned} e^{At} &= I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \frac{1}{24}(At)^4 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 0 & -t^3 \\ t^3 & 0 \end{bmatrix} + \frac{1}{24} \begin{bmatrix} t^4 & 0 \\ 0 & t^4 \end{bmatrix} \end{aligned}$$

Notice that when we reach $n = 5$, the signs are the same as $n = 2$,

$$\begin{aligned} &= \begin{bmatrix} 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots & t - \frac{1}{6}t^3 + \dots \\ -t + \frac{1}{6}t^3 + \dots & 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \dots \end{bmatrix} \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} & \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ -\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \end{bmatrix} \\ e^{At} &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Example 13. Solve $\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{u}$ starting from $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We can see the eigenvalues immediately since A is a triangular matrix. These are $\lambda_1 = 1$ and $\lambda_2 = 2$. Now to find the eigenvectors. For \mathbf{x}_1 :

$$\begin{aligned} (A - \lambda_1 I)\mathbf{x}_1 &= \mathbf{0} \\ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x}_1 &= \mathbf{0} \\ \mathbf{x}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finding \mathbf{x}_2 :

$$\begin{aligned} (A - \lambda_2 I)\mathbf{x}_2 &= \mathbf{0} \\ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_2 &= \mathbf{0} \\ \mathbf{x}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution becomes:

$$\mathbf{u}(t) = X e^{At} X^{-1} \mathbf{u}(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t + e^{2t} \\ e^{2t} \end{bmatrix}$$

4 Symmetric Matrices

5 Positive Definite Matrices