Chapter 4: Orthogonality

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Abstract

This chapter focuses on the orthogonality of the four subspaces, projections, and least squares approximations.

Two vectors are orthogonal when their dot product is zero $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = 0$. This chapter will revolve around orthogonal subspaces, orthogonal bases, and orthogonal matrices.

Definition 0.1. Orthogonal vectors have the following properties:

i. $\boldsymbol{v}^T \boldsymbol{w} = 0$

ii. $||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2 = ||\boldsymbol{v} + \boldsymbol{w}||^2$

Remark. *The subspaces have orthogonal properties.*

1. ***The rowspace*** $C(A^T)$ ***is perpendicular to the nullspace*** $N(A)$. *Every row of* A *is perpendicular to the solution of* $A\boldsymbol{x} = \boldsymbol{0}$.

2. ***The column space*** $C(A)$ ***is perpendicular to the left nullspaces*** $N(A^T)$. *When* $\boldsymbol{b}$ *is outside of the column space when we're trying to solve for* $A\boldsymbol{x} = \boldsymbol{b}$, *then this nullspace of* A^T *comes into its own. It contains the error* $\boldsymbol{e} = \boldsymbol{b} - A\boldsymbol{x}$ *in the least-squares solution.*

Definition 0.2. Two subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ of a vector space are orthogonal if every vector $\boldsymbol{v}$ in $\boldsymbol{V}$ is perpendicular to every vector $\boldsymbol{w}$ in $\boldsymbol{W}$.

$$\boldsymbol{v}^T \boldsymbol{w} = 0 \text{for all } \boldsymbol{v} \text{ in } \boldsymbol{V} \text{ and all } \boldsymbol{w} \text{ in } \boldsymbol{W}.$$

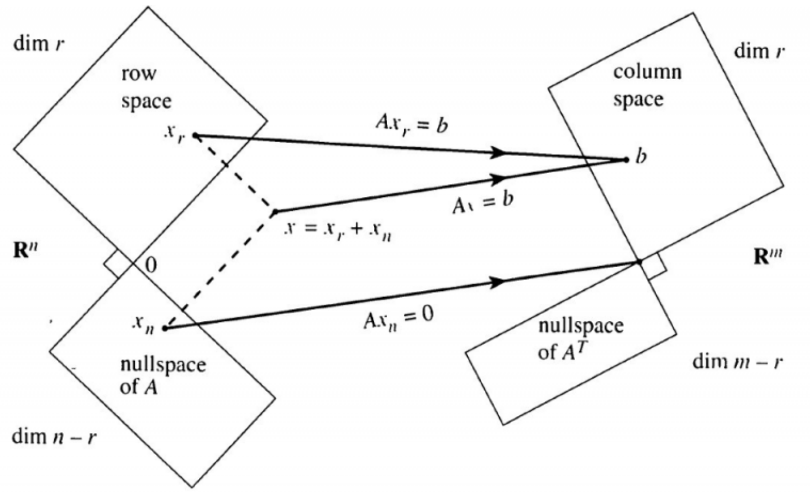

Figure 1: The Four Subspaces. There are two pairs of orthogonal subspaces.

Remark. *Every vector $\boldsymbol{x}$ in the nullspace is perpendicular to every row of A, because $A\boldsymbol{x} = \boldsymbol{0}$. The nullspace $N(A)$ and the row space $C(A^T)$ are orthogonal subspaces of $\mathbb{R}^n$.*

$$A\boldsymbol{x} = \begin{bmatrix} \textit{row 1} \\ \vdots \\ \textit{row } m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

(row 1) $\cdot\boldsymbol{x}$ is zero and (row m) $\cdot\, \boldsymbol{x}$ is also zero. Every row has a zero dot product with $\boldsymbol{x}$. Then $\boldsymbol{x}$ is perpendicular to every combination of the rows. **The whole row space $C(A^T)$ is orthogonal to $N(A)$.**