ratio test of d'Alembert

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A lighter version of the ratio test is the

Ratio test of d'Alembert. Let $a_1+a_2+\dots$ be a series with positive terms.

1°. If there exists a number q such that $0<q<1$ and

$$\frac{a_{n+1}}{a_n} \le q \quad \text{for all } n \ge n_0, \tag{1}$$

then the series converges.

2°. If there exists a number n_0 such that

$$\frac{a_{n+1}}{a_n} \ge 1 \quad \text{for all } n \ge n_0, \tag{2}$$

then the series diverges.

Proof. 1°. By the condition (1), we have $a_{n+1} \le a_n q$; thus we get the estimations

$$\begin{aligned} a_{n_0+1} &\le a_{n_0} q, \\ a_{n_0+2} &\le a_{n_0+1} q \le a_{n_0} q^2, \\ &\dots \quad \dots \quad \dots \\ a_{n_0+p} &\le a_{n_0+p-1} q \le \dots \le a_{n_0} q^p, \\ &\dots \quad \dots \quad \dots \end{aligned}$$

Because $a_{n_0}q + a_{n_0}q^2 + \dots + a_{n_0}q^p + \dots$ is a convergent geometric series, those inequalities and the comparison test imply that the series

$$a_{n_0+1}+a_{n_0+2}+\dots+a_{n_0+p}+\dots$$

and as well the whole series $a_1+a_2+\dots$ is convergent.

2°. The condition (2) yields

$$a_{n_0+1} \ge a_{n_0}, \quad a_{n_0+2} \ge a_{n_0+1} \ge a_{n_0}, \quad \dots$$

and since a_{n_0} is positive, the limit of a_n as n tends to infinity cannot be 0. Hence the given series does not fulfil the necessary condition of convergence.

Example. If the variable x in the power series

$$\sum_{n=0}^{\infty} n!x^n$$

is distinct from zero, we have

$$\frac{|(n+1)!x^{n+1}|}{|n!x^n|} \;=\; (n+1)|x| \;\geq\; 1 \quad \text{for all} \;\; n \geq n_0.$$

Then the series does not `http://planetmath.org/AbsoluteConvergence`converge absolutely. The known theorem of Abel says that the series diverges for all $x \neq 0$. It means that the radius of convergence is 0.

References

[1] Л. Д. Кудрявцев: *Математический анализ. I том.* Издательство "Высшая школа". Москва (1970).