Cauchy criterion for convergence

Canonical name	CauchyCriterionForConvergence
Date of creation	2013-03-22 13:22:03
Last modified on	2013-03-22 13:22:03
Owner	mathwizard (128)
Last modified by	mathwizard (128)
Numerical id	14
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Entry type	Theorem
Classification	msc 40A05

A series $\sum_{i=0}^{\infty} a_i$ in a Banach space $(V, \|\cdot\|)$ is http://planetmath.org/node/2311convergent iff for every $\varepsilon > 0$ there is a number $N \in \mathbb{N}$ such that

$$\|a_{n+1} + a_{n+2} + \cdots + a_{n+p}\| < \varepsilon$$

holds for all $n > N$ and $p \geq 1$.

Proof:

First define

$$s_n := \sum_{i=0}^{n} a_i.$$

Now, since V is complete, (s_n) converges if and only if it is a Cauchy sequence, so if for every $\varepsilon > 0$ there is a number N, such that for all $n, m > N$ holds:

$$\|s_m - s_n\| < \varepsilon.$$

We can assume $m > n$ and thus set $m = n + p$. The series is iff

$$\|s_{n+p} - s_n\| = \|a_{n+1} + a_{n+2} + \cdots + a_{n+p}\| < \varepsilon.$$