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uncountable sums of positive numbers

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The notion of sum of a series can be generalized to sums of nonnegative real numbers over arbitrary index sets.

let I be a set and let c be a mapping from I to the nonnegative real numbers. Then we may define the sum as follows:

$$\sum_{i \in I} c_i = \sup_{\substack{s \subset I \\ \#s < \infty}} \sum_{i \in s} c_i$$

In words, we are taking the supremum over all sums over finite subsets of the index set. This agrees with the usual notion of sum when our set is countably infinite, but generalizes this notion to uncountable index sets.

An important fact about this generalization is that the sum can only be finite if the number of elements $i \in I$ such that $c_i > 0$ is countable. To demonstrate this fact, define the sets s_n (where n is a nonnegative integer) as follows:

$$s_0 = \{i \in I \mid c_i \geq 1\}$$

when $n > 0$,

$$s_n = \{i \in I \mid 1/n > c_i \geq 1/(n+1)\}$$

If any of these sets is infinite, then the sum will diverge so, for the sum to be finite, all these sets must be finite. However, if these sets are all finite, then their union is countable. In other words, the number of indices for which $c_i > 0$ will be countable.

This notion finds use in places such as non-separable Hilbert spaces. For instance, given a vector in such a space and a complete orthonormal set, one can express the norm of the vector as the sum of the squares of its components using this definition even when the orthonormal set is uncountably infinite.

This discussion can also be phrased in terms of Lesbegue integration with respect to counting measure. For this point of view, please see the entry support of integrable function with respect to counting measure is countable.