example of converging increasing sequence

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Let a be a positive real number and q an integer greater than 1. Set

$$x_1 := \sqrt[q]{a},$$

$$x_2 := \sqrt[q]{a+x_1} = \sqrt[q]{a+\sqrt[q]{a}},$$

$$x_3 := \sqrt[q]{a+x_2} = \sqrt[q]{a+\sqrt[q]{a+\sqrt[q]{a}}},$$

and generally

$$x_n := \sqrt[q]{a+x_{n-1}}. \tag{1}$$

Since $x_1 > 0$, the two first above equations imply that $x_1 < x_2$. By induction on n one can show that

$$x_1 < x_2 < x_3 < \ldots < x_n < \ldots$$

The numbers x_n are all below a finite bound M. For demonstrating this, we write the inequality $x_n < x_{n+1}$ in the form $x_n < \sqrt[q]{a+x_n}$, which implies $x_n^q < a + x_n$, i.e.

$$x_n^q - x_n - a < 0 \tag{2}$$

for all n. We study the polynomial

$$f(x) := x^q - x - a = x(x^{q-1} - 1) - 1.$$

From its latter form we see that the function f attains negative values when $0 \leqq x \leqq 1$ and that f increases monotonically and boundlessly when x increases from 1 to ∞. Because f as a polynomial function is also continuous, we infer that the equation

$$x^q - x - a = 0 \tag{3}$$

has exactly one `http://planetmath.org/Equation`root $x = M > 1$, and that f is negative for $0 < x < 1$ and positive for $x > M$. Thus we can conclude by (2) that $x_n < M$ for all values of n.

The proven facts

$$x_1 < x_2 < x_3 < \ldots < x_n < \ldots < M$$

settle, by the theorem of the http://planetmath.org/NondecreasingSequenceWithUpperBoundpa
entry, that the sequence

$$x_1,\, x_2,\, x_3,\, \ldots,\, x_n,\, \ldots$$

converges to a limit $x' \leqq M$.

Taking limits of both sides of (1) we see that $x' = \sqrt[q]{a+x'}$, i.e. $x'^q - x' - a = 0$, which means that $x' = M$, in other words: the limit of the sequence is the only M of the equation (3).

References

[1] E. Lindelöf: *Johdatus korkeampaan analyysiin.* Neljäs painos. Werner Söderström Osakeyhtiö, Porvoo ja Helsinki (1956).