absolutely convergent infinite product converges

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Theorem. An http://planetmath.org/AbsoluteConvergenceOfInfiniteProductabsolutel convergent infinite product

$$\prod_{\nu=1}^{\infty}(1+c_\nu) \;=\; (1+c_1)(1+c_2)(1+c_3)\cdots \tag{1}$$

of complex numbers is convergent.

Proof. We thus assume the convergence of the http://planetmath.org/Productproduct

$$\prod_{\nu=1}^{\infty}(1+|c_\nu|) \;=\; (1+|c_1|)(1+|c_2|)(1+|c_3|)\cdots \tag{2}$$

Let ε be an arbitrary positive number. By the general convergence condition of infinite product, we have

$$|(1+|c_{n+1}|)(1+|c_{n+2}|)\cdots(1+|c_{n+p}|)-1| < \varepsilon \quad \forall\, p \in \mathbb{Z}_+$$

when $n \geqq$ certain n_ε. Then we see that

$$\begin{aligned}
|(1+c_{n+1})(1+c_{n+2})\cdots(1+c_{n+p})-1| &= |1+\sum_{\nu=n+1}^{n+p}c_\nu+\sum_{\mu,\,\nu}c_\mu c_\nu+\ldots+c_{n+1}c_{n+2}\cdots c_{n+p}-1| \\
&\leqq 1+\sum_{\nu=n+1}^{n+p}|c_\nu|+\sum_{\mu,\,\nu}|c_\mu||c_\nu|+\ldots+|c_{n+1}||c_{n+2}|\cdots|c_{n+p}|-1 \\
&= |(1+|c_{n+1}|)(1+|c_{n+2}|)\cdots(1+|c_{n+p}|)-1| < \varepsilon \qquad \forall\, p \in \mathbb{Z}_+
\end{aligned}$$

as soon as $n \geqq n_\varepsilon$. I.e., the infinite product (1) converges, by the same convergence condition.