absolute convergence implies uniform convergence

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Theorem 1. *Let T be a topological space, f be a continuous function from T to $[0, \infty)$, and let $\{f_k\}_{k=0}^{\infty}$ be a sequence of continuous functions from T to $[0, \infty)$ such that, for all $x \in T$, the sum $\sum_{k=0}^{\infty} f_k(x)$ converges to $f(x)$. Then the convergence of this sum is uniform on compact subsets of T.*

Proof. Let X be a compact subset of T and let ϵ be a positive real number. We will construct an open cover of X. Because the series is assumed to converge pointwise, for every $x \in X$, there exists an integer n_x such that $\sum_{k=n_x}^{\infty} f_k(x) < \epsilon/3$. By continuity, there exists an open neighborhood N_1 of x such that $|f(x) - f(y)| < \epsilon/3$ when $y \in N_1$ and an open neighborhood N_2 of x such that $|\sum_{k=0}^{n_x} f_k(x) - \sum_{k=0}^{n} f_k(y)| < \epsilon/3$ when $y \in N_2$. Let N_x be the intersection of N_1 and N_2. Then, for every $y \in N$, we have

$$f(y) - \sum_{k=0}^{n_x} f_k(y) < |f(y) - f(x)| + \left| f(x) - \sum_{k=0}^{n_x} f_k(x) \right| + \left| \sum_{k=0}^{n_x} f_k(x) - \sum_{k=0}^{n_x} f_k(y) \right| < \epsilon.$$

In this way, we associate to every point x an neighborhood N_x and an integer n_x. Since X is compact, there will exist a finite number of points $x_1, \ldots x_m$ such that $X \subseteq N_{x_1} \cup \cdots \cup N_{x_m}$. Let n be the greatest of $n_{x_1}, \ldots, n_{x_m}$. Then we have $f(y) - \sum_{k=0}^{n} f_k(y) < \epsilon$ for all $y \in X$, so, the functions f_k being positive, $f(y) - \sum_{k=0}^{h} f_k(y) < \epsilon$ for all $h \geq n$, which means that the sum converges uniformly. □

Note: This result can also be deduced from Dini's theorem, since the partial sums of positive functions are monotonically increasing.