proof of Abel's convergence theorem

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Suppose that

$$\sum_{n=0}^{\infty} a_n = L$$

is a convergent series, and set

$$f(r) = \sum_{n=0}^{\infty} a_n r^n.$$

Convergence of the first series implies that $a_n \to 0$, and hence $f(r)$ converges for $|r| < 1$. We will show that $f(r) \to L$ as $r \to 1^-$.

Let

$$s_N = a_0 + \cdots + a_N, \quad N \in \mathbb{N},$$

denote the corresponding partial sums. Our proof relies on the following identity

$$f(r) = \sum_n a_n r^n = (1-r) \sum_n s_n r^n. \tag{1}$$

The above identity obviously works at the level of formal power series. Indeed,

$$\begin{array}{cccccccc} & a_0 & + & (a_1 + a_0)\, r & + & (a_2 + a_1 + a_0)\, r^2 & + \cdots & \\ -\,(& & & a_0\, r & + & (a_1 + a_0)\, r^2 & + \cdots &) \\ = & a_0 & + & a_1\, r & + & a_2\, r^2 & + \cdots & \end{array}$$

Since the partial sums s_n converge to L, they are bounded, and hence $\sum_n s_n r^n$ converges for $|r| < 1$. Hence for $|r| < 1$, identity (**??**) is also a genuine functional equality.

Let $\epsilon > 0$ be given. Choose an N sufficiently large so that all partial sums, s_n with $n > N$, satisfy $|s_n - L| \leq \epsilon$. Then, for all r such that $0 < r < 1$, one obtains

$$\left| \sum_{n=N+1}^{\infty} (s_n - L) r^n \right| \leq \epsilon \, \frac{r^{N+1}}{1-r}\,.$$

Note that

$$f(r) - L = (1-r) \sum_{n=0}^{N} (s_n - L) r^n + (1-r) \sum_{n=N+1}^{\infty} (s_n - L) r^n.$$

As $r \to 1^-$, the first term tends to 0. The absolute value of the second term is estimated by $\epsilon\, r^{N+1} \leq \epsilon$. Hence,

$$\limsup_{r \to 1^-} |f(r) - L| \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $f(r) \to L$ as $r \to 1^-$. QED