slower divergent series

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Theorem. If

$$a_1 + a_2 + a_3 + \cdots \tag{1}$$

is a diverging series with positive , then one can always form another diverging series

$$s_1 + s_2 + s_3 + \cdots$$

with positive such that

$$\lim_{n\to\infty} \frac{s_n}{a_n} = 0. \tag{2}$$

Proof. Let $S_n = a_1 + a_2 + \cdots + a_n$ be the n^{th} partial sum of (1). Then we have

$$a_n = S_n - S_{n-1} = (\sqrt{S_n} + \sqrt{S_{n-1}})(\sqrt{S_n} - \sqrt{S_{n-1}}).$$

We set $s_1 := \sqrt{S_1}$ and

$$s_n := \frac{a_n}{\sqrt{S_n} + \sqrt{S_{n-1}}} = \sqrt{S_n} - \sqrt{S_{n-1}}$$

for $n = 2, 3, 4, \ldots$ Then the of the series

$$\sum_{n=1}^{\infty} s_n = \sqrt{S_1} + \sum_{n=1}^{\infty} (\sqrt{S_{n+1}} - \sqrt{S_n})$$

apparently are positive. This series is however divergent, because the sum of its n first is equal to $\sqrt{S_n}$ which grows without bound along with n since (1) diverges. For this reason we also get the result (2).

Remark. Niels Henrik Abel has presented a simpler example on such series $s_1 + s_2 + s_3 + \cdots$:

$$1 + \frac{a_2}{a_1 + a_2} + \frac{a_3}{a_1 + a_2 + a_3} + \frac{a_4}{a_1 + a_2 + a_3 + a_4} + \cdots$$