Abel's multiplication rule for series

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Cauchy has originally presented the multiplication rule

$$\sum_{j=1}^{\infty} a_j \cdot \sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} (a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1) \tag{1}$$

for two series. His assumption was that both of the multiplicand series should be absolutely convergent. Mertens (1875) lightened the assumption requiring that both multiplicands should be convergent but at least one of them absolutely convergent (see the `http://planetmath.org/MultiplicationOfSeries`parent entry). N. H. Abel's most general form of the multiplication rule is the

Theorem. The rule (1) for multiplication of series with real or complex terms is valid as soon as all three of its series are convergent.

Proof. We consider the corresponding power series

$$\sum_{j=1}^{\infty} a_j x^j, \qquad \sum_{k=1}^{\infty} b_k x^k. \tag{2}$$

When $x = 1$, they give the series

$$\sum_{j=1}^{\infty} a_j, \quad \sum_{k=1}^{\infty} b_k$$

which we assume to converge. Thus the power series are absolutely convergent for $|x| < 1$, whence they obey the multiplication rule due to Cauchy:

$$\sum_{j=1}^{\infty} a_j x^j \cdot \sum_{k=1}^{\infty} b_k x^k = \sum_{n=1}^{\infty} (a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1) x^{n+1}. \tag{3}$$

On the other hand, the sums of the power series (2) are, as is well known, continuous functions on the interval $[0, 1]$; the same concerns the right hand side of (3), because for $x = 1$ it becomes the third series which we assume convergent. When $x \to 1-$, we infer that

$$\sum_{j=1}^{\infty} a_j x^j \to \sum_{j=1}^{\infty} a_j, \qquad \sum_{k=1}^{\infty} b_k x^k \to \sum_{k=1}^{\infty} b_k$$

and that the limit of the right hand side of (3) is the right hand side of (1). Since the equation (3) is true for $|x| < 1$, also the limits of both of (3), as $x \to 1-$, are equal. Therefore the equation (1) is in with the assumptions of the theorem.

References

[1] E. Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset III. Toinen osa.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1940).