the limit of a uniformly convergent sequence of continuous functions is continuous

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Theorem. The limit of a uniformly convergent sequence of continuous functions is continuous.

Proof. Let $f_n, f : X \to Y$, where (X, ρ) and (Y, d) are metric spaces. Suppose $f_n \to f$ uniformly and each f_n is continuous. Then given any $\epsilon > 0$, there exists N such that $n > N$ implies $d(f(x), f_n(x)) < \frac{\epsilon}{3}$ for all x. Pick an arbitrary n larger than N. Since f_n is continuous, given any point x_0, there exists $\delta > 0$ such that $0 < \rho(x, x_0) < \delta$ implies $d(f_n(x), f_n(x_0)) < \frac{\epsilon}{3}$. Therefore, given any x_0 and $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \rho(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0)) < \epsilon.$$

Therefore, f is continuous.

The theorem also generalizes to when X is an arbitrary topological space. To generalize it to X an arbitrary topological space, note that if $d(f_n(x), f(x)) < \epsilon/3$ for all x, then $x_0 \in f_n^{-1}(B_{\epsilon/3}(f_n(x_0))) \subseteq f^{-1}(B_\epsilon(f(x_0)))$, so $f^{-1}(B_\epsilon(f(x_0)))$ is a neighbourhood of x_0. Here $B_\epsilon(y)$ denote the open ball of radius ϵ, centered at y.