proof of limit comparison test

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The main theorem we will use is the comparison test, which basically states that if $a_n > 0$, $b_n > 0$ and there is an N such that for all $n > N$, $a_n < b_n$, then if $\sum_{i=1}^{\infty} b_n$ converges so will $\sum_{i=1}^{\infty} a_n$.

Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where L can be a non negative real number or $+\infty$.

By definition, for L finite, this means that for every $\epsilon > 0$ there is a natural number n_ϵ such that for all $n > n_\epsilon$, $\left\| \frac{a_n}{b_n} - L \right\| < \epsilon$

To make matters more concrete choose $\epsilon = \frac{L}{2}$ and assume $L \neq 0$ and finite.

$0 < a_n < \frac{3L}{2} b_n$, for all $n > n_{\frac{L}{2}}$.

If $\sum_{i=1}^{\infty} b_n$ converges, so will $\sum_{i=1}^{\infty} \frac{3L}{2} b_n$ and thus by the comparison test, $\sum_{i=1}^{\infty} a_n$ will also be convergent.

For the reverse result, consider $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}$, since if L is finite so will $\frac{1}{L}$, applying the previous result we can say that if $\sum_{i=1}^{\infty} a_n$ converges so will $\sum_{i=1}^{\infty} b_n$

Consider the case $L = 0$, clearly $L = 0^+$ since both a_n and b_n are positive, this means that for all $\epsilon > 0$ there exists n_ϵ such that for all $n > n_\epsilon$, $0 < a_n < \epsilon b_n$.

Considering $\epsilon = 1$ we get the exact formulation of the comparison test, so if $\sum_{i=1}^{\infty} b_n$ converges so will $\sum_{i=1}^{\infty} a_n$.

For the case $L = +\infty$ just apply the result to $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ to conclude that if $\sum_{i=1}^{\infty} a_n$ converges so will $\sum_{i=1}^{\infty} b_n$