point preventing uniform convergence

Canonical name	PointPreventingUniformConvergence
Date of creation	2013-03-22 17:27:18
Last modified on	2013-03-22 17:27:18
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	6
Author	pahio (2872)
Entry type	Theorem
Classification	msc 40A30
Related topic	NotUniformlyContinuousFunction
Related topic	LimitFunctionOfSequence

Theorem. If the sequence $f_1,\, f_2,\, f_3,\, \ldots$ of real functions converges at each point of the interval $[a,\, b]$ but does not converge uniformly on this interval, then there exists at least one point x_0 of the interval such that the function sequence converges uniformly on no closed sub-interval of $[a,\, b]$ containing x_0.

Proof. Let the limit function of the sequence on the interval $[a,\, b]$ be f. According the entry uniform convergence on union interval, the sequence can not converge uniformly to f both half-intervals $[a,\, \frac{a+b}{2}]$ and $[\frac{a+b}{2},\, b]$, since otherwise it would do it on the union $[a,\, b]$. Denote by $[a_1,\, b_1]$ the first (fom left) of those half-intervals on which the convergence is not uniform. We have $[a,\, b] \supset [a_1,\, b_1]$. Then the interval $[a_1,\, b_1]$ is halved and chosen its half-interval $[a_2,\, b_2]$ on which the convergence is not uniform. We can continue similarly arbitrarily far and obtain a unique endless sequence

$$[a,\, b] \supset [a_1,\, b_1] \supset [a_2,\, b_2] \supset \ldots$$

of nested intervals on which the convergence of the function sequence is not uniform, and besides the length of the intervals tend to zero:

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\frac{b-a}{2^n} = 0.$$

The nested interval theorem thus gives a unique real number x_0 belonging to each of the intervals $[a,\, b]$ and $[a_n,\, b_n]$. Then $\lim_{n\to\infty} a_n = x_0 = \lim_{n\to\infty} b_n$. Let us choose α and β such that $a \leqq \alpha \leqq x_0 \leqq \beta \leqq b$. There exist the integers n_1 and n_2 such that

$$|a_n - x_0| = x_0 - a_n \leqq x_0 - \alpha \quad \text{when } \; n > n_1$$

$$|b_n - x_0| = b_n - x_0 \leqq \beta - x_0 \quad \text{when } \; n > n_2.$$

Therefore

$$\alpha \leqq a_n \leqq x_0 \leqq b_n \leqq \beta \quad \text{when } \; n > \max\{n_1,\, n_2\}.$$

This means that $f_n \to f$ not uniformly on $[a_n,\, b_n] \subset [\alpha,\, \beta]$, whence the function sequence does not converge uniformly on the arbitrarily chosen subinterval $[\alpha,\, \beta]$ of $[a,\, b]$ containing x_0.