proof of Abel's limit theorem

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Without loss of generality we may assume $r = 1$, because otherwise we can set $a'_n := a_r^n$, so that $\sum a'_n x^n$ has radius 1 and $\sum a'$ is convergent if and only if $\sum a_n r^n$ is. We now have to show that the function $f(x)$ generated by $\sum a_n x^n$ (with $r = 1$)is continuous from below at $x = 1$ if it is defined there. Let $s := \sum a_n$. We have to show that

$$\lim_{x \to 1^-} f(x) = s.$$

If $|x| < 1$ we have:

$$\begin{aligned} s - f(x) &= \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (1 - x^n) a_n \\ &= (1 - x) \sum_{n=1}^{\infty} (x^{n-1} + x^{n-2} + \cdots + x + 1) a_n \\ &= (1 - x) \sum_{n=0}^{\infty} (s - s_n) x^n \end{aligned}$$

with $s_n := \sum_{i=0}^{n} a_i$. Now, since $s - s_n \to 0$ as $n \to \infty$ we can choose an N for every $\varepsilon > 0$ such that $|s - s_n| < \frac{\varepsilon}{2}$ for all $m > N$. So for every $0 < x < 1$ we have:

$$\begin{aligned} |s - f(x)| &< (1 - x) \sum_{n=0}^{m} |r_n| x^n + \frac{\varepsilon}{2} (1 - x) \sum_{n=m+1}^{\infty} x^n \\ &< (1 - x) \sum_{n=0}^{m} |r_n| + \frac{\varepsilon}{2}. \end{aligned}$$

This is smaller than ε for all $x < 1$ sufficiently close to 1, which proves

$$\lim_{x \to r^-} \sum a_n x^n = \sum a_n r^n = \sum \lim_{x \to r^-} a_n x^n.$$