condition for uniform convergence of sequence of functions

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Theorem 1. *Let $f_1, f_2, \ldots$ be a sequence of real or complex functions defined on the interval $[a, b]$. The sequence converges uniformly to the* limit function *f on the interval $[a, b]$ if and only if*

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|,\ a \le x \le b\} = 0.$$

Proof. Suppose the sequence converges uniformly. By the very definition of uniform convergence, we have that for any ϵ there exist N such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2},\ a \le x \le b \quad \text{for } n > N$$

hence

$$\sup\{|f_n(x) - f(x)|,\ a \le x \le b\} < \epsilon \quad \text{for } n > N$$

Conversely, suppose the sequence does not converge uniformly. This means that there is an ϵ for which there is a sequence of increasing integers $n_i, i = 1, 2, ...$ and points x_{n_i} with the corresponding subsequence of functions f_{n_i} such that

$$|f(x_{n_i}) - f_{n_i}(x_{n_i})| > \epsilon \quad \text{for all } i = 1, 2, ...$$

therefore

$$\sup\{|f_n(x) - f(x)|,\ a \le x \le b\} > \epsilon \quad \text{for infinitely many } n.$$

Consequently, it is not the case that

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|,\ a \le x \le b\} = 0.$$

□

Theorem 2. *The uniform limit of a sequence of continuous complex or real functions f_n in the interval $[a, b]$ is continuous in $[a, b]$*

The proof is `http://planetmath.org/LimitOfAUniformlyConvergentSequenceOfContinuous`