sequence determining convergence of series

Canonical name	SequenceDeterminingConvergenceOfSeries
Date of creation	2013-03-22 19:06:54
Last modified on	2013-03-22 19:06:54
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	6
Author	pahio (2872)
Entry type	Definition
Classification	msc 40A05
Related topic	LimitComparisonTest

Theorem. Let $a_1+a_2+\ldots$ be any series of real a_n. If the positive numbers $r_1,\,r_2,\,\ldots$ are such that

$$\lim_{n\to\infty}\frac{a_n}{r_n} \;=\; L\;\neq\;0, \tag{1}$$

then the series converges simultaneously with the series $r_1+r_2+\ldots$

Proof. In the case that the limit (1) is positive, the supposition implies that there is an integer n_0 such that

$$0.5L \;<\; \frac{a_n}{r_n} \;<\; 1.5L \quad \text{for } n \geqq n_0. \tag{2}$$

Therefore

$$0 \;<\; 0.5Lr_n \;<\; a_n \;<\; 1.5Lr_n \quad \text{for all } n \geqq n_0,$$

and since the series $\sum_{n=1}^{\infty}0.5Lr_n$ and $\sum_{n=1}^{\infty}1.5Lr_n$ converge simultaneously with the series $r_1+r_2+\ldots$, the comparison test guarantees that the same concerns the given series $a_1+a_2+\ldots$

The case where (1) is negative, whence we have

$$\lim_{n\to\infty}\frac{-a_n}{r_n} \;=\; -L>0,$$

may be handled as above.

Note. For the case $L=0$, see the limit comparison test.