indirect proof of identity theorem of power series

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$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \;=\; \sum_{n=0}^{\infty} b_n(z-z_0)^n \tag{1}$$

is valid in the set of points z presumed in the http://planetmath.org/IdentityTheoremOfPowerS
to be proved.

Antithesis: There are integers n such that $a_n \neq b_n$; let ν ($\geqq 0$) be least of them.

We can choose from the point set an infinite sequence $z_1,\, z_2,\, z_3,\, \ldots$ which converges to z_0 with $z_n \neq z_0$ for every n. Let z in the equation (1) belong to $\{z_1,\, z_2,\, z_3,\, \ldots\}$ and let's divide both of (1) by $(z-z_0)^\nu$ which is distinct from zero; we then have

$$\underbrace{a_\nu + a_{\nu+1}(z-z_0) + a_{\nu+2}(z-z_0)^2 + \ldots}_{f(z)} \;=\; \underbrace{b_\nu + b_{\nu+1}(z-z_0) + b_{\nu+2}(z-z_0)^2 + \ldots}_{g(z)} \tag{2}$$

Let here z to tend z_0 along the points $z_1,\, z_2,\, z_3,\, \ldots$, i.e. we take the limits $\lim_{n\to\infty} f(z_n)$ and $\lim_{n\to\infty} g(z_n)$. Because the sum of power series is always a continuous function, we see that in (2),

$$\text{left side} \longrightarrow f(z_0) = a_\nu \quad \text{and} \quad \text{right side} \longrightarrow g(z_0) = b_\nu$$

But all the time, the left and of (2) are equal, and thus also the limits. So we must have $a_\nu = b_\nu$, contrary to the antithesis. We conclude that the antithesis is wrong. This settles the proof.

Note. I learned this proof from my venerable teacher, the number-theorist Kustaa Inkeri (1908–1997).