a pathological function of Riemann

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The periodic mantissa function $t \mapsto t - \lfloor t \rfloor$ has at each integer value of t a jump (saltus) equal to -1, being in these points continuous from the right but not from the left. For every real value t, one has

$$0 \leqq t - \lfloor t \rfloor < 1. \tag{1}$$

Let us consider the series

$$\sum_{n=1}^{\infty} \frac{nx - \lfloor nx \rfloor}{n^2} \tag{2}$$

due to Riemann. Since by (1), all values of $x \in \mathbb{R}$ and $n \in \mathbb{Z}_+$ satisfy

$$0 \leqq \frac{nx - \lfloor nx \rfloor}{n^2} < \frac{1}{n^2}, \tag{3}$$

the series is, by Weierstrass' M-test, uniformly convergent on the whole $\mathbb{R}$ (see also the p-test). We denote by $S(x)$ the sum function of (2).

The n^{th} term of the series (2) defines a periodic function

$$x \mapsto \frac{nx - \lfloor nx \rfloor}{n^2} \tag{4}$$

with the `http://planetmath.org/PeriodicFunctions`period $\frac{1}{n}$ and having especially for $0 \leqq x < \frac{1}{n}$ the value $\frac{x}{n}$. The only points of discontinuity of this function are

$$x = \frac{m}{n} \qquad (m = 0,\, \pm 1,\, \pm 2,\, \ldots), \tag{5}$$

where it vanishes and where it is continuous from the right but not from the left; in the point (5) this function apparently has the jump $-\dfrac{1}{n^2}$.

The theorem of the entry one-sided continuity by series implies that the sum function $S(x)$ is continuous in every irrational point x, because the series (2) is uniformly convergent for every x and its terms are continuous for irrational points x.

Since the terms (4) of (2) are continuous from the right in the rational points (5), the same theorem implies that $S(x)$ is in these points continuous from the right. It can be shown that $S(x)$ is in these points discontinuous from the left having the jump equal to $-\dfrac{\pi^2}{6n^2}$.

References

[1] E. Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset III.2.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1940).