proof of Tauber's convergence theorem

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Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

be a complex power series, convergent in the open disk $|z| < 1$. We suppose that

1. $na_n \to 0$ as $n \to \infty$, and that
2. $f(r)$ converges to some finite L as $r \to 1^-$;

and wish to show that $\sum_n a_n$ converges to the same L as well.

Let $s_n = a_0 + \cdots + a_n$, where $n = 0, 1, \ldots$, denote the partial sums of the series in question. The enabling idea in Tauber's convergence result (as well as other Tauberian theorems) is the existence of a correspondence in the evolution of the s_n as $n \to \infty$, and the evolution of $f(r)$ as $r \to 1^-$. Indeed we shall show that

$$\left| s_n - f\left(\frac{n-1}{n}\right) \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{1}$$

The desired result then follows in an obvious fashion.

For every real $0 < r < 1$ we have

$$s_n = f(r) + \sum_{k=0}^{n} a_k (1 - r^k) - \sum_{k=n+1}^{\infty} a_k\, r^k.$$

Setting

$$\epsilon_n = \sup_{k>n} |ka_k|,$$

and noting that

$$1 - r^k = (1 - r)(1 + r + \cdots + r^{k-1}) < k(1 - r),$$

we have that

$$|s_n - f(r)| \le (1 - r) \sum_{k=0}^{n} ka_k + \frac{\epsilon_n}{n} \sum_{k=n+1}^{\infty} r^k.$$

Setting $r = 1 - 1/n$ in the above inequality we get

$$|s_n - f(1 - 1/n)| \le \mu_n + \epsilon_n (1 - 1/n)^{n+1},$$

where

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n} |ka_k|$$

are the Cesàro means of the sequence $|ka_k|, \ k = 0, 1, \ldots$ Since the latter sequence converges to zero, so do the means μ_n, and the suprema ϵ_n. Finally, Euler's formula for e gives

$$\lim_{n \to \infty} (1 - 1/n)^n = e^{-1}.$$

The validity of (**??**) follows immediately. QED