convergence of integrals

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Similarly as one speaks of convergence of series, one can speak of *convergence of integrals*, especially of Riemann integrals

$$\int_I f(t)\,dt.$$

This integral is *convergent*, if it exists, and otherwise *divergent*. One can also speak of *absolute convergence of integrals*.

Example. Study the convergence of the integral

$$\int_1^2 \frac{dx}{(\ln x)^c} \tag{1}$$

where c is a real constant.

According to the logarithm series, we may write for $1 < x < b$, where b is sufficiently close to 1, the estimations

$$\ln(x-1) \;=\; x-1+O((x-1)^2) \;=\; (x-1)[1+O(x-1)] \begin{cases} \leq 2(x-1), \\ \geq \frac{1}{2}(x-1). \end{cases}$$

Let $1 < a < b$.

1°. For $c > 1$:

$$\int_a^b \frac{dx}{(\ln x)^c} \geqq \int_a^b \frac{dx}{2^c(x-1)^c} \;=\; -\frac{1}{2^c} \Big/_{\!x=a}^{b} \frac{1}{(c-1)(x-1)^{c-1}}$$

$$= \frac{1}{2^c(c-1)}\left[\frac{1}{(a-1)^{c-1}} - \frac{1}{(b-1)^{c-1}}\right] \longrightarrow \infty \quad \text{as} \quad a \to 1+$$

2°. For $c = 1$:

$$\int_a^b \frac{dx}{\ln x} \geqq \int_a^b \frac{dx}{2(x-1)} \;=\; \frac{1}{2} \Big/_{\!a}^{b} \ln(x-1)$$

$$= \frac{1}{2}\left[\ln(b-1) - \ln(a-1)\right] \longrightarrow \infty \quad \text{as} \quad a \to 1+$$

3°. For $c < 1$:

$$0 \;<\; \int_a^b \frac{dx}{(\ln x)^c} \leqq \int_a^b \frac{2^c\,dx}{(x-1)^c} \;=\; 2^c \Big/_{\!x=a}^{b} \frac{x^{1-c}}{1-c}$$

$$= \frac{2^c}{1-c}\left[(b-1)^{1-c} - (a-1)^{1-c}\right] \longrightarrow \frac{2^c}{1-c}(b-1)^{1-c} \quad \text{as} \quad a \to 1+$$

Consequently, the integral $\int_a^b \frac{dx}{(\ln x)^c}$, and thus also (1), converges if and only if $c < 1$.