example of summation by parts

Canonical name	ExampleOfSummationByParts
Date of creation	2013-03-22 17:27:56
Last modified on	2013-03-22 17:27:56
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	8
Author	pahio (2872)
Entry type	Example
Classification	msc 40A05
Related topic	ExampleOfTelescopingSum
Related topic	SineIntegralInInfinity
Related topic	ExampleOfSolvingTheHeatEquation

Proposition. The series $\sum_{n=1}^{\infty}\frac{\sin n\varphi}{n}$ and $\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n}$ converge for every complex value φ which is not an even multiple of π.

Proof. Let ε be an arbitrary positive number. One uses the

$$\sin\varphi+\sin 2\varphi+\ldots+\sin n\varphi = \frac{\sin(n+\frac{1}{2})\varphi-\sin\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}}, \qquad (1)$$

$$\cos\varphi+\cos 2\varphi+\ldots+\cos n\varphi = \frac{-\cos(n+\frac{1}{2})\varphi+\cos\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}}, \qquad (2)$$

proved in the entry "`http://planetmath.org/ExampleOfTelescopingSum`example of telescoping sum". These give the

$$|\sin\varphi+\sin 2\varphi+\ldots+\sin n\varphi| \leqq \frac{2}{2|\sin\frac{\varphi}{2}|} \;:=\; K_\varphi,$$

$$|\cos\varphi+\cos 2\varphi+\ldots+\cos n\varphi| \leqq \frac{2}{2|\sin\frac{\varphi}{2}|} \;:=\; K_\varphi$$

for any $n = 1,\,2,\,3,\,\ldots$. We want to apply to the series $\sum_{n=1}^{\infty}\frac{\cos n\varphi}{n}$ the `http://planetmath.org/CauchyCriterionForConvergence`Cauchy general convergence criterion for series. Let us use here the short notation

$$\cos N\varphi+\cos(N+1)\varphi+\ldots+\cos(N+p)\varphi := S_{N,N+p} \quad (p=0,\,1,\,2,\,\ldots).$$

Then, utilizing Abel's summation by parts, we obtain

$$\left|\sum_{n=N}^{N+P}\frac{\cos n\varphi}{n}\right| = \left|\sum_{p=0}^{P}\frac{1}{N+p}\cos(N+p)\varphi\right| = \left|\sum_{p=0}^{P-1}\left(\frac{1}{N+p}-\frac{1}{N+p+1}\right)S_{N,N+p}+\frac{1}{N+P}S_{N,N+P}\right| \leqq$$

$$\leqq \sum_{p=0}^{P-1}\left(\frac{1}{N+p}-\frac{1}{N+p+1}\right)|S_{N,N+P}|+\frac{1}{N+P}|S_{N,N+P}| <$$

$$< \sum_{p=0}^{P-1}\left(\frac{1}{N+p}-\frac{1}{N+p+1}\right)\cdot 2K_\varphi+\frac{1}{N+P}\cdot 2K_\varphi \;=\; \frac{1}{N}\cdot 2K_\varphi;$$

the last form is gotten by `http://planetmath.org/TelescopingSum`telescoping the preceding sum and before that by using the identity

$$S_{N,N+p} = [\cos\varphi+\cos 2\varphi+\ldots+\cos(N+p)\varphi]-[\cos\varphi+\cos 2\varphi+\ldots+\cos(N-1)\varphi].$$

Thus we see that

$$\left|\sum_{n=N}^{N+P} \frac{\cos n\varphi}{n}\right| < \frac{2K_\varphi}{N} < \varepsilon$$

for all natural numbers P as soon as $N > \frac{2K_\varphi}{\varepsilon}$. According to the Cauchy criterion, the latter series is convergent for the mentioned values of φ. The former series is handled similarly.