converging alternating series not satisfying all Leibniz' conditions

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Author	pahio (2872)
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The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+(-1)^{n-1}} \;=\; \frac{1}{2}-\frac{1}{1}+\frac{1}{4}-\frac{1}{3}+\frac{1}{6}-\frac{1}{5}+-\ldots \tag{1}$$

satisfies the other requirements of Leibniz test except the monotonicity of the absolute values of the terms. The convergence may however be shown by manipulating the terms as follows.

We first multiply the numerator and the denominator of the general term by the difference $n-(-1)^{n-1}$, getting from (1)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+(-1)^{n-1}} \;=\; \frac{1}{2}+\sum_{n=2}^{\infty} \frac{n-(-1)^{n-1}}{n^2-1}(-1)^{n-1} \;=\; \frac{1}{2}+\sum_{n=2}^{\infty}\left(\frac{(-1)^{n-1}n}{n^2-1}-\frac{1}{n^2-1}\right). \tag{2}$$

One can that the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}n}{n^2-1} \tag{3}$$

satisfies all requirements of Leibniz test and thus is convergent. Since

$$0 \;<\; \frac{1}{n^2-1} \;<\; \frac{1}{n^2-\frac{1}{2}n^2} \;=\; 2\cdot\frac{1}{n^2} \quad \text{for} \quad n \geqq 2,$$

and the over-harmonic series $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges, the comparison test guarantees the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}. \tag{4}$$

Therefore the difference series of (3) and (4) and consequently, by (2), the given series (1) is convergent.