# proof that e is not a natural number

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Here, we are going to show that the natural log base $e$ is not a natural number by showing a sharper result: that $e$ is between 2 and 3.

**Proposition.** $2 < e < 3$.

*Proof.* There are several infinite series representations of $e$. In this proof, we will use the most common one, the Taylor expansion of $e$:

$$\sum_{i=0}^{\infty} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots . \tag{1}$$

We chop up the Taylor expansion of $e$ into two parts: the first part $a$ consists of the sum of the first two terms, and the second part $b$ consists of the sum of the rest, or $e - a$. The proof of the proposition now lies in the estimation of $a$ and $b$.

**Step 1: e>2.** First, $a = \frac{1}{0!} + \frac{1}{1!} = 1 + 1 = 2$. Next, $b > 0$, being a sum of the terms in (1), all of which are positive (note also that $b$ must be bounded because (1) is a convergent series). Therefore, $e = a + b = 2 + b > 2 + 0 = 2$.

**Step 2: e<3.** This step is the same as showing that $b = e - a = e - 2 < 3 - 2 = 1$. With this in mind, let us compare term by term of the series (2) representing $b$ and another series (3):

$$\frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots \tag{2}$$

and

$$\frac{1}{2^{2-1}} + \frac{1}{2^{3-1}} + \cdots + \frac{1}{2^{n-1}} + \cdots . \tag{3}$$

It is well-known that the second series (a geometric series) sums to 1. Because both series are convergent, the term-by-term comparisons make sense. Except for the first term, where $\frac{1}{2!} = \frac{1}{2} = \frac{1}{2^{2-1}}$, we have $\frac{1}{n!} < \frac{1}{2^{n-1}}$ for all other terms. The inequality $\frac{1}{n!} < \frac{1}{2^{n-1}}$, for $n$ a positive number can be translated into the basic inequality $n! > 2^{n-1}$, the proof of which, based on mathematical induction, can be found `http://planetmath.org/AnExampleOfMathematicalInduction`here.

Because the term comparisons show

- that the terms from (2) $\leq$ the corresponding terms from (3), and
- that at least one term from (2) $<$ than the corresponding term from (3),

we conclude that (2) $<$ (3), or that $b < 1$. This concludes the proof. $\square$