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properties of superexponentiation

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In this entry, we list some basic properties of the superexponetial function $f : \mathbb{N}^2 \to \mathbb{N}$, defined recursively by

$$f(m, 0) = m, \qquad f(m, n+1) = m^{f(m,n)}.$$

Furthermore, we set $f(0, n) := 0$ for all n.

Given m, the values of f are

$$m, m^m, m^{m^m}, \cdots, m^{\cdot^{\cdot^{m}}}, \cdots,$$

where the evaluation of these values start from the top, for example: $3^{3^3} = 3^{81}$.

Proposition 1. *Suppose $x, y, z \in \mathbb{N}$ (including 0), and for all except the first assertion, $x > 1$.*

1. *$x \leq f(x, y)$.*
2. *$f(x, y)$ is increasing in both arguments.*
3. *$2f(x, y) \leq f(x, y + 1)$.*
4. *$f(x, y)^2 \leq f(x, y + 1)$.*
5. *$f(x, y)^{f(x,y)} \leq f(x, y + 2)$*
6. *$f(x, y) + f(x, z) \leq f(x, 1 + \max\{y, z\})$.*
7. *$f(x, y) \cdot f(x, z) \leq f(x, 1 + \max\{y, z\})$.*
8. *$f(x, y)^{f(x,z)} \leq f(x, 2 + \max\{y, z\})$.*
9. *$f(f(x, y), z) \leq f(x, y + 2z)$.*
10. *$y < f(x, y)$.*

Proof. Most of the proofs are done by induction.

1. The case when $x = 0$ is obvious. Assume now that $x \neq 0$. Induct on y. The case $y = 0$ is clear. Suppose $x \leq f(x, y)$. Then $x \leq x^x \leq x^{f(x,y)} = f(x, y + 1)$.

2. To see $f(x,y) < f(x,y+1)$ for $x > 1$, induct on y. First, $f(x,0) = x < x^x = f(x,1)$. Next, assume $f(x,y) < f(x,y+1)$. Then $f(x,y+1) = x^{f(x,y)} < x^{f(x,y+1)} = f(x,y+1)$.

   To see $f(x,y) < f(x+1,y)$ for $x > 1$, again induct on y. First, $f(x,0) = x < x+1 = f(x+1,y)$. Next, assume $f(x,y) < f(x+1,y)$. Then $f(x,y+1) = x^{f(x,y)} < (x+1)^{f(x,y)} < (x+1)^{f(x+1,y)} = f(x+1,y+1)$.

3. Induct on y: if $y = 0$, then $2f(x,0) = 2x \leq x^2 \leq x^x = f(x,1)$. Next, assume $2f(x,y) \leq f(x,y+1)$. Then $2f(x,y+1) = x^{f(x,y)} \leq x^{2f(x,y)} \leq x^{f(x,y+1)} = f(x,y+2)$.

4. If $y = 0$, then $f(x,0)^2 = x^2 \leq x^x = x^{f(x,0)} = f(x,1)$. Otherwise, $y = z+1$. Then $f(x,y)^2 = f(x,z+1)^2 = x^{2f(x,z)} \leq x^{f(x,z+1)} = x^{f(x,y)} = f(x,y+1)$. The inequality $2f(x,z) \leq f(x,z+1)$ is derived previously.

5. If $y = 0$, then $f(x,0)^{f(x,0)} = x^x = f(x,1) \leq f(x,2)$. Otherwise, $y = z+1$. Then $f(x,y)^{f(x,y)} = f(x,z+1)^{f(x,z+1)} = x^{f(x,z)f(x,z+1)} \leq x^{f(x,z+1)^2} = x^{f(x,z+2)} = f(x,z+3) = f(x,y+2)$.

From the last three statements, the next three proofs can be easily settled, fisrt, let $t = \max\{y,z\}$. Then

6. $f(x,y) + f(x,z) \leq 2f(x,t) \leq f(x,t+1)$.

7. $f(x,y)f(x,z) = f(x,t)^2 \leq f(x,t+1)$.

8. $f(x,y)^{f(x,z)} \leq f(x,t)^{f(x,t)} \leq f(x,t+2)$.

9. Induct on z. If $z = 0$, then $f(f(x,y),0) = f(x,y)$. Next, assume $f(f(x,y),z) \leq f(x,y+2z)$. Then $f(f(x,y),z+1) = f(x,y)^{f(f(x,y),z)} = f(x,y)^{f(x,y+2z)} \leq f(x,y+1)^{f(x,y+2z)} \leq x^{f(x,y)f(x,y+2z)} \leq x^{f(x,y+2z+1)} = f(x,y+2z+2)$.

10. Induct on y. The case when $y = 0$ is obvious. Next, if $y < f(x,y)$, then $y+1 < f(x,y)+1 < f(x,y)+f(x,0) \leq f(x,y+1)$.

$\square$

Concerning the recursiveness of f, here is another basic property of f:

Proposition 2. *f is primitive recursive.*

Proof. Since $f(m, 0) = m = p_1^1(m)$ and $f(m, n+1) = \exp(m, f(m, n)) = g(m, n, f(m, n))$, where $g(x, y, z) = \exp(p_1^3(x, y, z), p_3^3(x, y, z))$, are defined by primitive recursion via functions p_1^1 and g, and since the projection functions p_i^j, the exponential function exp, and consequently g, are primitive recursive (g obtained by composition), we see that f is primitive recursive. $\square$

Remark. Another recursive property of f is that f is not elementary recursive. The proof uses the properties listed above. It is a bit lengthy, and is done in the link below.