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Euler product formula

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Theorem (Euler). If $s > 1$, the infinite product

$$\prod_p \frac{1}{1-\frac{1}{p^s}} \tag{1}$$

where p runs the positive rational primes, converges to the sum of the over-harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s). \tag{2}$$

Proof. Denote the sequence of prime numbers by $p_1 < p_2 < p_3 < \ldots$ For any $s > 0$, we can form convergent geometric series

$$\frac{1}{1-\frac{1}{p_1^s}} = 1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \ldots = \sum_{\nu_1=0}^{\infty} \frac{1}{p_1^{\nu_1 s}},$$

$$\frac{1}{1-\frac{1}{p_2^s}} = 1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \ldots = \sum_{\nu_2=0}^{\infty} \frac{1}{p_2^{\nu_2 s}}.$$

Since these series are absolutely convergent, their product (see multiplication of series) may be written as

$$\frac{1}{1-\frac{1}{p_1^s}} \cdot \frac{1}{1-\frac{1}{p_2^s}} = \sum_{\nu_1,\nu_2=0}^{\infty} \frac{1}{p_1^{\nu_1 s}} \cdot \frac{1}{p_2^{\nu_2 s}} = \sum_{\nu_1,\nu_2=0}^{\infty} \frac{1}{(p_1^{\nu_1} p_2^{\nu_2})^s}$$

where ν_1 and ν_2 independently on each other run all nonnegative integers. This equation can be generalised by induction to

$$\prod_{\nu=1}^{k} \frac{1}{1-\frac{1}{p_\nu^s}} = \sum_{\nu_1,\nu_2,\ldots,\nu_k=0}^{\infty} \frac{1}{(p_1^{\nu_1} p_2^{\nu_2} \cdots p_k^{\nu_k})^s} \tag{3}$$

for $s > 0$ and for arbitrarily great k; the exponents $\nu_1, \nu_2, \ldots, \nu_k$ run independently all nonnegative integers.

Because the prime factorization of positive integers is `http://planetmath.org/FundamentalTh`
we can rewrite (3) as

$$\prod_{\nu=1}^{k} \frac{1}{1-\frac{1}{p_\nu^s}} = \sum_{(n)} \frac{1}{n^s}, \tag{4}$$

where n runs all positive integers not containing greater prime factors than p_k. Then the inequality

$$\sum_{n=1}^{p_k} \frac{1}{n^s} < \prod_{\nu=1}^{k} \frac{1}{1-\frac{1}{p_\nu^s}}, \tag{5}$$

holds for every k, since all the terms $1, \frac{1}{p_1^s}, \ldots, \frac{1}{p_k^s}$ are in the series of the right hand side of (4). On the other hand, this series contains only a part of the terms of (2). Thus, for $s > 1$, the product (3) is less than the sum $\zeta(s)$ of the series (2), and consequently

$$\sum_{n=1}^{p_k} \frac{1}{n^s} < \prod_{\nu=1}^{k} \frac{1}{1-\frac{1}{p_\nu^s}} < \zeta(s). \tag{6}$$

Letting $k \to \infty$, we have $p_k \to \infty$, and the sum on the left hand side of (6) tends to the limit $\zeta(s)$, therefore also tends the product (3). Hence we get the limit equation

$$\prod_{p} \frac{1}{1-\frac{1}{p^s}} = \zeta(s) \qquad (s > 1). \tag{7}$$

References

[1] E. Lindelöf: *Differentiali- ja integralilasku ja sen sovellutukset III.2.* Mercatorin Kirjapaino Osakeyhtiö, Helsinki (1940).