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Bohr's theorem

Canonical name	BohrsTheorem
Date of creation	2015-04-13 12:52:55
Last modified on	2015-04-13 12:52:55
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	12
Author	pahio (2872)
Entry type	Theorem
Classification	msc 40A30
Classification	msc 30B10

(Bohr 1914). If the power series $\sum_{n=0}^{\infty} a_n z^n$ satisfies

$$\left| \sum_{n=0}^{\infty} a_n z^n \right| \; < \; 1 \tag{1}$$

in the unit disk $|z| < 1$, then (1) and the inequality

$$\sum_{n=0}^{\infty} |a_n z^n| \; < \; 1 \tag{2}$$

is true in the disk $|z| < \frac{1}{3}$. Here, the radius $\frac{1}{3}$ is the best possible.

Proof. One needs *Carathéodory's inequality* which says that if the real part of a holomorphic function

$$g(z) \; := \; \sum_{n=0}^{\infty} b_n z^n$$

is positive in the unit disk, then

$$|b_n| \; \leqq \; 2\,\mathrm{Re}\, b_0 \quad \text{for } n = 1,\, 2,\, \ldots$$

Choosing now $g(z) := 1 - e^{i\varphi} f(z)$ where φ is any real number and $f(z)$ the sum function of the series in the theorem, we get

$$|a_n| \; \leqq \; 2\,\mathrm{Re}\,(1 - e^{i\varphi} a_0) \; = \; 2(1 - a_0 \cos\varphi),$$

and especially

$$|a_n| \; \leqq \; 2(1 - |a_0|), \quad \text{for } n = 1,\, 2,\, \ldots$$

If $f(z) \not\equiv a_0$, in the disk $|z| < \frac{1}{3}$ we thus have

$$\sum_{n=0}^{\infty} |a_n z^n| \; < \; |a_0| + 2(1 - |a_0|) \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \; = \; 1.$$

Take then in particular the function defined by

$$f(z) \; := \; \frac{z - c}{1 - cz}$$

with $0 < c < 1$. Its series expansion

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n \;=\; -c + (1-c^2)z + (1-c^2)cz^2 + (1-c^2)c^2z^3 + \ldots$$

shows that

$$\sum_{n=0}^{\infty} |a_n z^n| \;=\; f(|z|) + 2c,$$

which last form can be seen to become greater than 1 for $|z| > \dfrac{1}{1+2c}$. Because c may come from below arbitrarily to 1, one sees that the value $\frac{1}{3}$ in the theorem cannot be increased.

References

[1] Harald Bohr: "A theorem concerning power series". – *Proc. London Math. Soc.* **13** (1914).

[2] Harold P. Boas: "Majorant series". – *J. Korean Math. Soc.* **37** (2000).