Riemann's theorem on rearrangements

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If the map $n \mapsto n'$ is a bijection on $\mathbb{N}$, we say that the sequence $(a_{n'})$ is a rearrangement of (a_n).

The following theorem, which is due to Riemann, shows that the convergence of a conditionally convergent series depends so much on the order of its terms; in particular, a conditionally convergent series can be made to converge to any real number by changing the order of its terms.

Theorem (Riemann series theorem). Let (a_n) be a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n| = \infty$, i.e, $\sum a_n$ is conditionally convergent. Let $-\infty \leq \alpha < \beta \leq \infty$ be arbitrary. Then there exists a rearrangement $(a_{n'})$ such that

$$\liminf_N \sum_{n'=1}^{N} a_{n'} = \alpha \quad \text{and} \quad \limsup_N \sum_{n'=1}^{N} a_{n'} = \beta.$$

Proof. Let $a_n^+ = \max\{0, a_n\}$ and $a_n^- = \min\{0, -a_n\}$. Then we have $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$. Since $\sum a_n < \infty$, both $\sum a_n^+$ and $\sum a_n^-$ diverge or converge simultaneously. But since $\sum |a_n| = \infty$, we see that at least one of $\sum a_n^+$ and $\sum a_n^-$ must diverge. It follows that $\sum a_n^+ = \infty$ and $\sum a_n^- = \infty$.

Also by the nth term test, $\lim_n a_n^+ = \lim_n a_n^- = 0$.

Now we pass to subsequence of (a_n^+) by removing all terms with $a_n^+ = 0$ and $a_n \neq 0$. For a_n^-, we remove all terms with $a_n^- = 0$. Let us denote the subsequences still as (a_n^+) and (a_n^-). Since only zeros have been removed, $\sum a_n^+$ and $\sum a_n^-$ are still divergent.

Now we will define integers m_j and k_j for $j \in \mathbb{N}$, and consider the series

$$\begin{aligned} & a_1^+ + a_2^+ + \cdots + a_{m_1}^+ \\ & \quad - a_1^- - a_2^- - \cdots - a_{k_2}^- \\ & \quad + a_{m_1+1}^+ + a_2^+ + \cdots + a_{m_2}^+ \\ & \quad - a_{k_1+1}^- - a_2^- - \cdots - a_{k_2}^- + \\ & \quad \vdots \end{aligned}$$

This series is clearly a rearrangement of $\sum a_n$, by our choice of the subsequences a_n^+ and a_n^-.

We pick up two sequences α_j and β_j such that $\alpha_j \to \alpha$, $\beta_j \to \beta$, $\alpha_n \leq \beta_n$ and $\beta_1 > 0$. We choose m_1 such that $\sum_{n=1}^{m_1} \geq \beta_1$ but $\sum_{n=1}^{m_1-1} < \beta_1$. We choose

k_1 such that $\sum_{n=1}^{m_1} - \sum_{n=1}^{k_1} \leq \alpha_1$ but $\sum_{n=1}^{m_1} - \sum_{n=1}^{k_1-1} > \alpha_1$. We continue this way, inductively.

Since $\lim_n a_n^+ = \lim_n a_n^- = 0$, the subsequences of the sequence of partial sums that end with $a_{m_j}^+$ and $a_{k_j}^-$ converge to β and α, and it can be seen that no subsequence can be found with a limit larger than β or lower than α. $\square$

References

[1] Rudin, W., *Principles of Mathematical Analysis*, McGraw Hill, 1976.