unconditional convergence

Canonical name	UnconditionalConvergence
Date of creation	2013-03-22 15:29:57
Last modified on	2013-03-22 15:29:57
Owner	kompik (10588)
Last modified by	kompik (10588)
Numerical id	11
Author	kompik (10588)
Entry type	Definition
Classification	msc 40A05
Synonym	unconditionally convergent
Related topic	AbsoluteConvergence
Related topic	ConditionallyConvergentSeriesOfRealNumbersCanBeRearrangedToConvergeT

A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is *unconditionally convergent* if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges.

Alternatively, for every chain of finite subsets $S_1 \subseteq S_2 \subseteq \cdots$ of $\mathbb{N}$, the partial sums

$$\sum_{k \in S_1} x_k, \ \sum_{k \in S_2} x_k, \ , \ldots$$

converges. The trick to see this equivalence is to realize two facts: 1. every subsequence of a convergent sequence is convergent, and 2. every chain $\{S_i\}$ can be enlarged to a maximal chain $\{T_i\}$, such that $|T_i| = i$. Then the series indexed by $\{S_i\}$ is a subseries indexed by $\{T_i\}$, which is a subseries of a permutation of the original convergent series.

Yet a third `http://planetmath.org/Equivalent3`equivalent definition is given as follows: A series is unconditionally convergent if for every sequence $(\varepsilon_n)_{n=1}^{\infty}$, with $\varepsilon_n \in \{\pm 1\}$, the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges.

Every absolutely convergent series is unconditionally convergent, the converse implication does not hold in general.

When $X = \mathbb{R}^n$ then by a famous theorem of Riemann $(\sum x_n)$ is unconditionally convergent if and only if it is absolutely convergent.

References

[1] K. Knopp: *Theory and application of infinite series.*

[2] K. Knopp: *Infinite sequences and series.*

[3] P. Wojtaszczyk: *Banach spaces for analysts.*

[4] Ch. Heil: `http://www.math.gatech.edu/ heil/papers/bases.pdf`A basis theory primer.