proof of alternating series test

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The series has partial sum

$$S_{2n+2} = a_1 - a_2 + a_3 - +... - a_{2n} + a_{2n+1} - a_{2n+2},$$

where the a_j's are all nonnegative and nonincreasing. From above, we have the following:

$$S_{2n+1} = S_{2n} + a_{2n+1};$$

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2});$$

$$\begin{aligned} S_{2n+3} &= S_{2n+1} - (a_{2n+2} - a_{2n+3}) \\ &= S_{2n+2} + a_{2n+3} \end{aligned}$$

Since $a_{2n+1} \geq a_{2n+2} \geq a_{2n+3}$, we have $S_{2n+1} \geq S_{2n+3} \geq S_{2n+2} \geq S_{2n}$. Moreover,

$$S_{2n+2} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1}) - a_{2n+2}.$$

Because the a_j's are nonincreasing, we have $S_n \geq 0$ for any n. Also, $S_{2n+2} \leq S_{2n+1} \leq a_1$. Thus, $a_1 \geq S_{2n+1} \geq S_{2n+3} \geq S_{2n+2} \geq S_{2n} \geq 0$. Hence, the even partial sums S_{2n} and the odd partial sums S_{2n+1} are bounded. Also, the even partial sums S_{2n}'s are monotonically nondecreasing, while the odd partial sums S_{2n+1}'s are monotonically nonincreasing. Thus, the even and odd series both converge.

We note that $S_{2n+1} - S_{2n} = a_{2n+1}$. Therefore, the sums converge to the *same* limit if and only if $a_n \to 0$ as $n \to \infty$. The theorem is then established.