absolute convergence of integral and boundedness of derivative

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Theorem. Assume that we have an `http://planetmath.org/node/11865`absolutely converging integral

$$\int_a^\infty f(x)\,dx$$

where the real function f and its derivative f' are continuous and f' additionally bounded on the interval $[a,\,\infty)$. Then

$$\lim_{x\to\infty} f(x) \;=\; 0. \tag{1}$$

Proof. If $c > a$, we obtain

$$\int_a^c f(x)f'(x)\,dx \;=\; \frac{1}{2}\Big/_{\!a}^{\,c}(f(x))^2 \;=\; \frac{(f(c))^2-(f(a))^2}{2},$$

from which

$$(f(c))^2 \;=\; (f(a))^2 + 2\int_a^c f(x)f'(x)\,dx. \tag{2}$$

Using the boundedness of f' and the absolute convergence, we can estimate upwards the integral

$$\int_a^c |f(x)f'(x)|\,dx \;=\; \int_a^c |f(x)||f'(x)|\,dx \;\leqq\; M\int_a^c |f(x)|\,dx \;\leqq\; M\int_a^\infty |f(x)|\,dx \quad \forall c\in[a,\,\infty)$$

whence $\int_a^\infty |f(x)f'(x)|\,dx$ is finite and thus $\int_a^\infty f(x)f'(x)\,dx$ converges absolutely. Hence (2) implies

$$\lim_{c\to\infty}(f(c))^2 \;=\; (f(a))^2 + 2\int_a^\infty f(x)f'(x)\,dx,$$

i.e. $\lim\limits_{x\to\infty}(f(x))^2$ exists as finite, therefore also

$$\lim_{x\to\infty}|f(x)| \;:=\; A.$$

Antithesis: $A > 0$. It implies that there is an x_0 $(\geqq a)$ such that

$$|f(x)| \;\geqq\; \frac{A}{2} \quad \forall x \geqq x_0.$$

If now $b > x_0$, then we had

$$\int_{x_0}^{b} |f(x)|\,dx \;\geqq\; \frac{A}{2}(b-x_0) \quad\longrightarrow \infty \quad \text{as } b \to \infty.$$

This means that $\int_{x_0}^{\infty} |f(x)|\,dx$ and consequently also $\int_a^{\infty} |f(x)|\,dx$ would be divergent. Since it is not true, we infer that $A = 0$, i.e. that the assertion (1) is true.