Cesàro summability

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Cesàro summability is a generalized convergence criterion for infinite series. We say that a series $\sum_{n=0}^{\infty} a_n$ is Cesàro summable if the Cesàro means of the partial sums converge to some limit L. To be more precise, letting

$$s_N = \sum_{n=0}^{N} a_n$$

denote the N^{th} partial sum, we say that $\sum_{n=0}^{\infty} a_n$ Cesàro converges to a limit L, if

$$\frac{1}{N+1}(s_0 + \ldots + s_N) \to L \quad \text{as} \quad N \to \infty.$$

Cesàro summability is a generalization of the usual definition of the limit of an infinite series.

Proposition 1 *Suppose that*

$$\sum_{n=0}^{\infty} a_n = L,$$

in the usual sense that $s_N \to L$ as $N \to \infty$. Then, the series in question Cesàro converges to the same limit.

The converse, however is false. The standard example of a divergent series, that is nonetheless Cesàro summable is

$$\sum_{n=0}^{\infty} (-1)^n.$$

The sequence of partial sums $1, 0, 1, 0, \ldots$ does not converge. The Cesàro means, namely

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \ldots$$

do converge, with $1/2$ as the limit. Hence the series in question is Cesàro summable.

There is also a relation between Cesàro summability and Abel summability[1].

[1]This and similar results are often called Abelian theorems.

Theorem 2 (Frobenius) *A series that is Cesàro summable is also Abel summable. To be more precise, suppose that*

$$\frac{1}{N+1}(s_0 + \ldots + s_N) \to L \quad \text{as} \quad N \to \infty.$$

Then,

$$f(r) = \sum_{n=0}^{\infty} a_n r^n \to L \quad \text{as} \quad r \to 1^-$$

as well.