# every bounded sequence has limit along an ultrafilter

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**Theorem 1.** *Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$ and $(x_n)$ be a real bounded sequence. Then $\mathcal{F}\text{-}\lim x_n$ exists.*

*Proof.* Let $(x_n)$ be a bounded sequence. Choose $a_0$ and $b_0$ such that $a_0 \leq x_n \leq b_0$. Put $c_0 := \frac{a_0+b_0}{2}$. Then precisely one of the sets $\{n \in \mathbb{N}; x_n \in \langle a_0, c_0\rangle\}$, $\{n \in \mathbb{N}; x_n \in \langle c_0, b_0\rangle\}$ belongs to the filter $\mathcal{F}$. (Their union is $\mathbb{N}$ and the filter $\mathcal{F}$ is an ultrafilter.) We choose $\langle a_1, b_1\rangle$ as that subinterval from $\langle a_0, c_0\rangle$ and $\langle c_0, b_0\rangle$ for which $C := \{n \in \mathbb{N}; x_n \in \langle a_1, b_1\rangle\}$ belongs to $\mathcal{F}$.

Now we again bisect the interval $\langle a_1, b_1\rangle$ by putting $c_1 = \frac{a_1+b_1}{2}$. Denote $A := \{n \in \mathbb{N}; x_n \in \langle a_1, c_1\rangle\}$, $B := \{n \in \mathbb{N}; x_n \in \langle c_1, b_1\rangle\}$. It holds $B \cup A \cup (\mathbb{N} \setminus C) = \mathbb{N}$. By the alternative characterization of ultrafilters we get that one of these sets is in $\mathcal{F}$. The set $\mathbb{N} \setminus C$ doesn't belong to $\mathcal{F}$, therefore it must be one of the sets $A$ and $B$. We choose the corresponding interval for $\langle a_2, b_2\rangle$.

By induction we obtain the monotonous sequences $(a_n)$, $(b_n)$ with the same limit $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n := L$ such that for any $n \in \mathbb{N}$ it holds $\{n \in \mathbb{N}; x_n \in \langle a_1, b_1\rangle\} \in \mathcal{F}$.

We claim that $\mathcal{F}\text{-}\lim x_n = L$. Indeed, for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\langle a_n, b_n\rangle \subseteq (L-\varepsilon, L+\varepsilon)$, thus $\{n \in \mathbb{N}; x_n \in \langle a_n, b_n\rangle\} \subseteq A(\varepsilon)$. The set $\{n \in \mathbb{N}; x_n \in \langle a_1, b_1\rangle\}$ belongs to $\mathcal{F}$, hence $A(\varepsilon) \in \mathcal{F}$ as well. $\square$

Note that, if we modify the definition of $\mathcal{F}$-limit in a such way that we admit the values $\pm\infty$, then every sequence has $\mathcal{F}$-limit along an ultrafilter $\mathcal{F}$. (The limit is $+\infty$ if for each neighborhood $V$ of infinity, the set $\{n \in \mathbb{N}; x_n \in V\}$ belongs to $\mathcal{F}$. Similarly for $-\infty$.)

# References


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