proof that a metric space is compact if and only if it is complete and totally bounded

Canonical name	ProofThatAMetricSpaceIsCompactIfAndOnlyIfItIsCompleteAndTotallyBound
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Theorem: A metric space is compact if and only if it is complete and totally bounded.

Proof. Let X be a metric space with metric d. If X is compact, then it is sequentially compact and thus complete. Since X is compact, the covering of X by all ϵ-balls must have a finite subcover, so that X is totally bounded.

Now assume that X is complete and totally bounded. For metric spaces, compact and sequentially compact are equivalent; we prove that X is sequentially compact. Choose a sequence $p_n \in X$; we will find a Cauchy subsequence (and hence a convergent subsequence, since X is complete).

Cover X by finitely many balls of radius 1 (since X is totally bounded). At least one of those balls must contain an infinite number of the p_i. Call that ball B_1, and let S_1 be the set of integers i for which $p_i \in B_1$.

Proceeding inductively, it is clear that we can define, for each positive integer $k > 1$, a ball B_k of radius $1/k$ containing an infinite number of the p_i for which $i \in S_{k-1}$; define S_k to be the set of such i.

Each of the S_k is infinite, so we can choose a sequence $n_k \in S_k$ with $n_k < n_{k+1}$ for all k. Since the S_k are nested, we have that whenever $i, j \geq k$, then $n_i, n_j \in S_k$. Thus for all $i, j \geq k$, p_{n_i} and p_{n_j} are both contained in a ball of radius $1/k$. Hence the sequence p_{n_k} is Cauchy.

References

[1] J. Munkres, *Topology* , Prentice Hall, 1975.