proof of identity theorem of power series

Canonical name	ProofOfIdentityTheoremOfPowerSeries1
Date of creation	2013-03-22 16:48:46
Last modified on	2013-03-22 16:48:46
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	11
Author	rspuzio (6075)
Entry type	Proof
Classification	msc 40A30
Classification	msc 30B10

We can prove the identity theorem for power series using divided differences. From amongst the points at which the two series are equal, pick a sequence $\{w_k\}_{k=0}^{\infty}$ which satisfies the following three conditions:

1. $\lim_{k \to \infty} w_k = z_0$
2. $w_m = w_n$ if and only if $m = n$.
3. $w_k \neq z_0$ for all k.

Let f be the function determined by one power series and let g be the function determined by the other power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

Because formation of divided differences involves finite sums and dividing by differences of w_k's (which all differ from zero by condition 2 above, so it is legitimate to divide by them), we may carry out the formation of finite differences on a term-by-term basis. Using the result about divided differences of powers, we have

$$\Delta^m f[w_k, \dots, w_{k+m}] = \sum_{n=m}^{\infty} a_n D_{mnk}$$
$$\Delta^m f[w_k, \dots, w_{k+m}] = \sum_{n=m}^{\infty} b_n D_{mnk}$$

where

$$D_{mnk} = \sum_{j_0 + \dots j_m = n-m} (w_k - z_0)^{j_0} \cdots (w_{k+m} - z_0)^{j_m}.$$

Note that $\lim_{k \to infty} D_{mnk} = 0$ when $m > n$, but $D_{mmk} = 1$. Since power series converge uniformly, we may intechange limit and summation to conclude

$$\lim_{k \to \infty} \Delta^m f[w_k, \dots, w_{k+m}] = \sum_{n=m}^{\infty} a_n \lim_{k \to \infty} D_{mnk} = a_m$$
$$\lim_{k \to \infty} \Delta^m g[w_k, \dots, w_{k+m}] = \sum_{n=m}^{\infty} b_n \lim_{k \to \infty} D_{mnk} = b_m.$$

Since, by design, $f(w_k) = g(w_k)$, we have

$$\Delta^m f[w_k, \ldots, w_{k+m}] = \Delta^m g[w_k, \ldots, w_{k+m}],$$

hence $a_m = b_m$ for all m.