Dirichlet's convergence test

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Theorem. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\{\sum_{i=0}^{n} a_i\}$ is bounded and $\{b_n\}$ decreases with 0 as limit. Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

Proof. Let $A_n := \sum_{i=0}^{n} a_n$ and let M be an upper bound for $\{|A_n|\}$. By Abel's lemma,

$$\begin{aligned}
\sum_{i=m}^{n} a_i b_i &= \sum_{i=0}^{n} a_i b_i - \sum_{i=0}^{m-1} a_i b_i \\
&= \sum_{i=0}^{n-1} A_i(b_i - b_{i+1}) - \sum_{i=0}^{m-2} A_i(b_i - b_{i+1}) + A_n b_n - A_{m-1} b_{m-1} \\
&= \sum_{i=m-1}^{n-1} A_i(b_i - b_{i+1}) + A_n b_n - A_{m-1} b_{m-1} \\
|\sum_{i=m}^{n} a_i b_i| &\leq \sum_{i=m-1}^{n-1} |A_i(b_i - b_{i+1})| + |A_n b_n| + |A_{m-1} b_{m-1}| \\
&\leq M \sum_{i=m-1}^{n-1} (b_i - b_{i+1}) + |A_n b_n| + |A_{m-1} b_{m-1}|
\end{aligned}$$

Since $\{b_n\}$ converges to 0, there is an $N(\epsilon)$ such that both $\sum_{i=m-1}^{n-1}(b_i - b_{i+1}) < \frac{\epsilon}{3M}$ and $b_i < \frac{\epsilon}{3M}$ for $m, n > N(\epsilon)$. Then, for $m, n > N(\epsilon)$, $|\sum_{i=m}^{n} a_i b_i| < \epsilon$ and $\sum a_n b_n$ converges.