slower convergent series

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Theorem. If

$$a_1 + a_2 + a_3 + \cdots \tag{1}$$

is a converging series with positive , then one can always form another converging series

$$g_1 + g_2 + g_3 + \cdots$$

such that

$$\lim_{n\to\infty} \frac{g_n}{a_n} = \infty \tag{2}$$

Proof. Let S be the sum of (1), $S_n = a_1 + a_2 + \cdots + a_n$ the n^{th} partial sum of (1) and $R_{n+1} = S - S_n = a_{n+1} + a_{n+2} + \cdots$ the corresponding remainder term. Then we have

$$a_n = R_n - R_{n+1} = (\sqrt{R_n} + \sqrt{R_{n+1}})(\sqrt{R_n} - \sqrt{R_{n+1}}).$$

We set

$$g_n := \frac{a_n}{\sqrt{R_n} + \sqrt{R_{n+1}}} = \sqrt{R_n} - \sqrt{R_{n+1}} \quad \forall n = 1,\, 2,\, 3,\, \ldots$$

Then the series $g_1 + g_2 + g_3 + \cdots$ fulfils the requirements in the theorem. Its g_n are positive. Further, it converges because its n^{th} partial sum is equal to $\sqrt{R_1} - \sqrt{R_{n+1}}$ which tends to the limit $\sqrt{R_1} = \sqrt{S}$ as $n \to \infty$ since $R_{n+1} \to 0$; this implies also (2).