Abel summability

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http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Abel.htmlAbel summability is a generalized convergence criterion for power series. It extends the usual definition of the sum of a series, and gives a way of summing up certain divergent series. Let us start with a series $\sum_{n=0}^{\infty} a_n$, convergent or not, and use that series to define a power series

$$f(r) = \sum_{n=0}^{\infty} a_n r^n.$$

Note that for $|r| < 1$ the summability of $f(r)$ is easier to achieve than the summability of the original series. Starting with this observation we say that the series $\sum a_n$ is *Abel summable* if the defining series for $f(r)$ is convergent for all $|r| < 1$, and if $f(r)$ converges to some limit L as $r \to 1^-$. If this is so, we shall say that $\sum a_n$ Abel converges to L.

Of course it is important to ask whether an ordinary convergent series is also Abel summable, and whether it converges to the same limit? This is true, and the result is known as Abel's limit theorem, or simply as Abel's theorem.

Theorem 1 (Abel) *Let $\sum_{n=0}^{\infty} a_n$ be a series; let*

$$s_N = a_0 + \cdots + a_N, \quad N \in \mathbb{N},$$

denote the corresponding partial sums; and let $f(r)$ be the corresponding power series defined as above. If $\sum a_n$ is convergent, in the usual sense that the s_N converge to some limit L as $N \to \infty$, then the series is also Abel summable and $f(r) \to L$ as $r \to 1^-$.

The standard example of a divergent series that is nonetheless Abel summable is the alternating series

$$\sum_{n=0}^{\infty} (-1)^n.$$

The corresponding power series is

$$\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n.$$

Since

$$\frac{1}{1+r} \to \frac{1}{2} \quad \text{as} \quad r \to 1^-,$$

this otherwise divergent series Abel converges to $\frac{1}{2}$.

Abel's theorem is the prototype for a number of other theorems about convergence, which are collectively known in analysis as Abelian theorems. An important class of associated results are the so-called Tauberian theorems. These describe various convergence criteria, and sometimes provide partial converses for the various Abelian theorems.

The general converse to Abel's theorem is false, as the example above illustrates[1]. However, in the 1890's `http://www-groups.dcs.st-and.ac.uk/ history/Mathematic` proved the following partial converse.

Theorem 2 (Tauber) *Suppose that $\sum a_n$ is an Abel summable series and that $na_n \to 0$ as $n \to \infty$. Then, $\sum_n a_n$ is convergent in the ordinary sense as well.*

The proof of the above theorem is not hard, but the same cannot be said of the more general Tauberian theorems. The more famous of these are due to Hardy, Hardy-Littlewood, Weiner, and Ikehara. In all cases, the conclusion is that a certain series or a certain integral is convergent. However, the proofs are lengthy and require sophisticated techniques. Ikehara's theorem is especially noteworthy because it is used to prove the prime number theorem.

[1] We want the converse to be false; the whole idea is to describe a method of summing certain divergent series!