proof of Abel lemma (by expansion)

Canonical name	ProofOfAbelLemmabyExpansion
Date of creation	2013-03-22 17:28:14
Last modified on	2013-03-22 17:28:14
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Last modified by	perucho (2192)
Numerical id	7
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Entry type	Proof
Classification	msc 40A05

1 Abel lemma

$$\sum_{i=0}^{n} a_i b_i = \sum_{i=0}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n, \tag{1}$$

where, $A_i = \sum_{k=0}^{i} a_k$. Sequences $\{a_i\}$, $\{b_i\}$, $i = 0, \ldots, n$, are real or complex one.

2 Proof

We consider the expansion of the sum

$$\sum_{i=0}^{n} A_i (b_i - b_{i+1})$$

on two different forms, namely:

1. On the short way.

$$\sum_{i=0}^{n} A_i (b_i - b_{i+1}) = \sum_{i=0}^{n-1} A_i (b_i - b_{i+1}) + A_n b_n - A_n b_{n+1}. \tag{2}$$

2. On the long way.

$$\sum_{i=0}^{n} A_i (b_i - b_{i+1}) = \sum_{i=0}^{n} A_i b_i - \sum_{i=0}^{n} A_i b_{i+1} = \sum_{i=0}^{n} A_i b_i - \sum_{i=1}^{n+1} A_{i-1} b_i =$$

$$A_0 b_0 + \sum_{i=1}^{n} (A_{i-1} + a_i) b_i - \sum_{i=1}^{n} A_{i-1} b_i - A_n b_{n+1} = \sum_{i=0}^{n} a_i b_i - A_n b_{n+1}, \tag{3}$$

where a simplification has been performed. Notice that $A_0 = a_0$. By equating (2), (3), the last two terms cancel, [1] and then, (1) follows. □

[1] Without loss of generality, b_{n+1} may be assumed finite. Indeed we don't need b_{n+1}, but the proof is a couple lines larger. It is left as an exercise.