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prime theorem of a convergent sequence, a

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Theorem. *Suppose* (a_n) *is a positive real sequence that converges to* L. *Then the sequence of arithmetic means* $(b_n) = (n^{-1}\sum_{k=1}^{n} a_k)$ *and the sequence of geometric means* $(c_n) = (\sqrt[n]{a_1 \cdots a_n})$ *also converge to* L.

Proof. We first show that (b_n) converges to L. Let $\varepsilon > 0$. Select a positive integer N_0 such that $n \geq N_0$ implies $|a_n - L| < \varepsilon/2$. Since (a_n) converges to a finite value, there is a finite M such that $|a_n - L| < M$ for all n. Thus we can select a positive integer $N \geq N_0$ for which $(N_0 - 1)M/N < \varepsilon/2$.

By the triangle inequality,

$$\begin{aligned} |b_n - L| &\leq \frac{1}{n}\sum_{k=1}^{n} |a_k - L| \\ &< \frac{(N_0 - 1)M}{n} + \frac{(n - N_0 + 1)\varepsilon}{2n} \\ &< \varepsilon/2 + \varepsilon/2. \end{aligned}$$

Hence (b_n) converges to L.

To show that (c_n) converges to L, we first define the sequence (d_n) by $d_n = c_n^n = a_1 \cdots a_n$. Since d_n is a positive real sequence, we have that

$$\liminf \frac{d_{n+1}}{d_n} \leq \liminf \sqrt[n]{d_n} \leq \limsup \sqrt[n]{d_n} \leq \limsup \frac{d_{n+1}}{d_n},$$

a proof of which can be found in [**?**]. But $d_{n+1}/d_n = a_{n+1}$, which by assumption converges to L. Hence $\sqrt[n]{d_n} = c_n$ must also converge to L. □

References

[1] Rudin, W., *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, 1976.