example of telescoping sum

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Some trigonometric sums, as $\sum_{k=1}^{n}\cos k\alpha$ and $\sum_{k=1}^{n}\sin k\alpha$, may be telescoped if the terms are first edited by a suitable http://planetmath.org/GoniometricFormulaego formula ("product formula"). E.g. we may write:

$$\sum_{k=1}^{n}\cos k\alpha \;=\; \frac{1}{\sin\frac{\alpha}{2}}\sum_{k=1}^{n}\cos k\alpha\sin\frac{\alpha}{2}$$

The product formula $\cos x\sin y \;=\; \frac{1}{2}[\sin(x+y)-\sin(x-y)]$ alters this to

$$\sum_{k=1}^{n}\cos k\alpha \;=\; \frac{1}{2\sin\frac{\alpha}{2}}\sum_{k=1}^{n}\left(\sin\frac{(2k+1)\alpha}{2}-\sin\frac{(2k-1)\alpha}{2}\right),$$

or

$$\sum_{k=1}^{n}\cos k\alpha \;=\; \frac{1}{2\sin\frac{\alpha}{2}}\left(\sin\frac{3\alpha}{2}-\sin\frac{\alpha}{2}+\sin\frac{5\alpha}{2}-\sin\frac{3\alpha}{2}+-\ldots+\sin\frac{(2n+1)\alpha}{2}-\sin\frac{(2n-1)\alpha}{2}\right).$$

After cancelling the opposite numbers we obtain the formula

$$\sum_{k=1}^{n}\cos k\alpha \;=\; \frac{\sin\frac{(2n+1)\alpha}{2}-\sin\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}. \tag{1}$$

The corresponding formula

$$\sum_{k=1}^{n}\sin k\alpha \;=\; \frac{-\cos\frac{(2n+1)\alpha}{2}+\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}. \tag{2}$$

is derived analogously.

Note. The formulae (1) and (2) are gotten also by adding the left side of the former and i times the left side of the latter and then applying de Moivre identity.

References

[1] Л. Д. Кудрявцев: *Математический анализ. II том.* Издательство "Высшая школа". Москва (1970).