arithmetic-geometric series

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It is well known that a finite geometric series is given by

$$G_n(q) = \sum_{k=1}^{n} q^k = \frac{q}{1-q}(1-q^n), \qquad q \neq 1, \tag{1}$$

where in general $q = re^{i\theta}$ is complex. When we are dealing with such sums it is common to consider the expression

$$H_n(q) := \sum_{k=1}^{n} kq^k, \qquad q \neq 1, \tag{2}$$

which we shall call an *arithmetic-geometric series.* Let us derive a formula for $H_n(q)$.

$$H_n(q) = \sum_{k=1}^{n} kq^k, \qquad qH_n(q) = \sum_{k=1}^{n} kq^{k+1}.$$

Subtracting,

$$(1-q)H_n(q) = \sum_{k=1}^{n} kq^k - \sum_{k=1}^{n} kq^{k+1} = \sum_{k=1}^{n} kq^k - \sum_{k=2}^{n+1} (k-1)q^k = \sum_{k=1}^{n} kq^k - \sum_{k=2}^{n} (k-1)q^k - nq^{n+1}.$$

We will proceed to eliminate the right-hand side sums.

$$(1-q)H_n(q) = q + \sum_{k=2}^{n} q^k - nq^{n+1} = \sum_{k=1}^{n} q^k - nq^{n+1}.$$

By using (1) and solving for $H_n(q)$, we obtain

$$H_n(q) = \sum_{k=1}^{n} kq^k = \frac{q}{(1-q)^2}(1-q^n) - \frac{nq^{n+1}}{1-q} \cdot \tag{3}$$

The formula (3) holds in any commutative ring with 1, as long as $(1-q)$ is invertible. If q is a complex number and $|q| < 1$, (3) is the partial sum of the convergent series

$$H(q) = \lim_{n\to\infty} H_n(q) = \lim_{n\to\infty} \sum_{k=1}^{n} kq^k = \lim_{n\to\infty} \left[\frac{q}{(1-q)^2}(1-q^n) - \frac{nq^{n+1}}{1-q}\right],$$

that is,

$$H(q) = \sum_{k=1}^{\infty} kq^k = \frac{q}{(1-q)^2}, \qquad |q| < 1. \tag{4}$$

This last result giving the sum of a converging arithmetic-geometric series may be, naturally, obtained also from the sum formula of the converging geometric series, i.e.

$$1+q+q^2+q^3+\ldots = \frac{1}{1-q},$$

when one differentiates both sides with respect to q and then multiplies them by q:

$$1+2q+3q^2+\ldots = \frac{1}{(1-q)^2},$$

$$q+2q^2+3q^3+\ldots = \frac{q}{(1-q)^2}$$

(A power series can be differentiated termwise on the open interval of convergence.)