example using Stolz-Cesaro theorem

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Example: We try to determine the value of

$$\lim_{n\to\infty} \frac{1^k + 2^k + ... + n^k}{n^{k+1}},\ k \in \mathbb{N}.$$

We consider the sequences $\alpha_{n\geq 1} = 1^k + 2^k + ... + n^k$ and $\beta_{n\geq 1} = n^k$ and using the Stolz-Cesaro theorem we have that

$$\lim_{n\to\infty} \frac{1^k + 2^k + ... + n^k}{n^{k+1}} = \tag{1}$$

$$\lim_{n\to\infty} \frac{(1^k + 2^k + ... + (n+1)^k) - (1^k + 2^k + ... + n^k)}{(n+1)^{k+1} - n^{k+1}} = \tag{2}$$

$$\lim_{n\to\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}}. \tag{3}$$

Now we try to get the expression in the indeterminate form $\frac{0}{0}$ as n approaches ∞, dividing numerator and denominator of (3) by $(n+1)^k$.

$$\lim_{n\to\infty} \frac{1}{(n+1) - n^{k+1}(n+1)^{-k}} = \tag{4}$$

$$\lim_{n\to\infty} \frac{1}{n(1 + n^{-1} - (1 + n^{-1})^{-k})} = \tag{5}$$

$$\lim_{n\to\infty} \frac{n^{-1}}{1 + n^{-1} - (1 + n^{-1})^{-k}}. \tag{6}$$

By applying L'Hôpital's rule once we get

$$\lim_{n\to\infty} \frac{n^{-1}}{1 + n^{-1} - (1 + n^{-1})^{-k}} = \tag{7}$$

$$\lim_{n\to\infty} \frac{-n^{-2}}{-n^{-2} - k(1 + n^{-1})^{-k-1}n^{-2}} = \tag{8}$$

$$\lim_{n\to\infty} \frac{1}{1 + k(1 + n^{-1})^{-k-1}} = \tag{9}$$

$$\frac{1}{1+k}. \tag{10}$$