power series

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Defines	constant term

A *power series* is a series of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k,$$

with $a_k, x_0 \in \mathbb{R}$ or $\in \mathbb{C}$. The a_k are called the coefficients and x_0 the center of the power series. a_0 is called the *constant term*.

Where it converges the power series defines a function, which can thus be represented by a power series. This is what power series are usually used for. Every power series is convergent at least at $x = x_0$ where it converges to a_0. In addition it is absolutely and uniformly convergent in the region $\{x \mid |x - x_0| < r\}$, with

$$r = \liminf_{k \to \infty} \frac{1}{\sqrt[k]{|a_k|}}$$

It is divergent for every x with $|x - x_0| > r$. For $|x - x_0| = r$ no general predictions can be made. If $r = \infty$, the power series converges absolutely and uniformly for every real or complex x. The real number r is called the **radius of convergence** of the power series.

Examples of power series are:

- Taylor series, for example:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

- The geometric series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,$$

  with $|x| < 1$.

Power series have some important :

- If a power series converges for a $z_0 \in \mathbb{C}$ then it also converges for all $z \in \mathbb{C}$ with $|z - x_0| < |z_0 - x_0|$.
- Also, if a power series diverges for some $z_0 \in \mathbb{C}$ then it diverges for all $z \in \mathbb{C}$ with $|z - x_0| > |z_0 - x_0|$.

- For $|x - x_0| < r$ Power series can be added by adding coefficients and multiplied in the obvious way:

$$\sum_{k=0}^{\infty} a_k(x-x_o)^k \cdot \sum_{l=0}^{\infty} b_j(x-x_0)^j = a_0b_0+(a_0b_1+a_1b_0)(x-x_0)+(a_0b_2+a_1b_1+a_2b_0)(x-x_0)^2 \ldots.$$

- (Uniqueness) If two power series are equal and their are the same, then their coefficients must be equal.

- Power series can be termwise differentiated and integrated. These operations keep the radius of convergence.