proof of composition limit law for uniform convergence

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Theorem 1. *Let X, Y, Z be metric spaces, with X compact and Y locally compact. If $f_n \colon X \to Y$ is a sequence of functions converging uniformly to a continuous function $f \colon X \to Y$, and $h \colon Y \to Z$ is continuous, then $h \circ f_n$ converge to $h \circ f$ uniformly.*

Proof. Let K denote the compact set $f(X) \subseteq Y$. By local compactness of Y,for each point $y \in K$, there is an open neighbourhood U_y of y such that $\overline{U_y}$ is compact. The neighbourhoods U_y cover K, so there is a finite subcover $U_{y_1}, \ldots, U_{y_n}$ covering K. Let $U = \bigcup_i U_{y_i} \supseteq K$. Evidently $\overline{U} = \bigcup_i \overline{U_{y_i}}$ is compact.

Next, let V be the δ_0-neighbourhood of K contained in U, for some $\delta_0 > 0$. $\overline{V}$ is compact, since it is contained in $\overline{U}$.

Now let $\epsilon > 0$ be given. h is uniformly continuous on $\overline{V}$, so there exists a $\delta > 0$ such that when $y, y' \in \overline{V}$ and $d(y, y') < \delta$, we have $d(g(y), g(y')) < \epsilon$.

From the uniform convergence of f_n, choose N so that when $n \geq N$, $d(f_n(x), f(x)) < \min(\delta, \delta_0)$ for all $x \in X$. Since $f(x) \in K$, it follows that $f_n(x)$ is inside the δ_0-neighbourhood of K, i.e. both $y = f_n(x)$ and $y' = f(x)$ are both in V. Thus $d(g(f_n(x)), g(f(x))) < \epsilon$ when $n \geq N$, uniformly for all $x \in X$. $\square$