proof of convergence of a sequence with finite upcrossings

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We show that a sequence $x_1, x_2, \ldots$ of real numbers converges to a limit in the extended real numbers if and only if the number of upcrossings $U[a,b]$ is finite for all $a < b$.

Denoting the infimum limit and supremum limit by

$$l = \liminf_{n\to\infty} x_n, \; u = \limsup_{n\to\infty} x_n,$$

then $l \le u$ and the sequence converges to a limit if and only if $l = u$.

We first show that if the sequence converges then $U[a,b]$ is finite for $a < b$. If $l > a$ then there is an N such that $x_n > a$ for all $n \ge N$. So, all upcrossings of $[a,b]$ must start before time N, and we may conclude that $U[a,b] \le N$ is finite. On the other hand, if $l \le a$ then $u = l < b$ and we can infer that $x_n < b$ for all $n \ge N$ and some N. Again, this gives $U[a,b] \le N$.

Conversely, suppose that the sequence does not converge, so that $u > l$. Then choose $a < b$ in the interval (l,u). For any integer n, there is then an $m > n$ such that $x_m > b$ and an $m > n$ with $x_m < a$. This allows us to define infinite sequences s_k, t_k by $t_0 = 0$ and

$$\begin{aligned} s_k &= \inf\left\{m \ge t_{k-1} \colon X_m < a\right\}, \\ t_k &= \inf\left\{m \ge s_k \colon X_m > b\right\}, \end{aligned}$$

for $k \ge 1$. Clearly, $s_1 < t_1 < s_2 < \cdots$ and $x_{s_k} < a < b < x_{t_k}$ for all $k \ge 1$, so $U[a,b] = \infty$.