Laplace integrals

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The improper integrals

$$\int_{-\infty}^{\infty}\frac{a\cos x}{x^2+a^2}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}\,dx,$$

where a is a positive , are called *Laplace integrals*. Both of them have the same value πe^{-a}.

The evaluation of the Laplace integrals can be performed by first determining the integrals

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x-ia}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{e^{ix}}{x+ia}\,dx$$

where one integrates along the real axis. Therefore one has to determine the integrals

$$\oint\frac{e^{iz}}{z-ia}\,dz \quad \text{and} \quad \oint\frac{e^{iz}}{z+ia}\,dz$$

around the perimeter of the half-disk with the arc in the upper half-plane, centered in the origin and with the diameter $(-R,\,+R)$. The residue theorem yields the values

$$\oint\frac{e^{iz}}{z-ia}\,dz \;=\; 2i\pi e^{-a} \quad \text{and} \quad \oint\frac{e^{iz}}{z+ia}\,dz \;=\; 0.$$

As in the entry example of using residue theorem, the parts of these contour integrals along the half-circle tend to zero when $R\to\infty$. Consequently,

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x-ia}\,dx \;=\; 2i\pi e^{-a} \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{e^{ix}}{x+ia}\,dx \;=\; 0.$$

These equations imply by adding and subtracting and then taking the `http://planetmath.org/Rea` and the imaginary parts, the

$$\int_{-\infty}^{\infty}\frac{a\cos x}{x^2+a^2}\,dx \;=\; \int_{-\infty}^{\infty}\frac{x\sin x}{x^2+a^2}\,dx \;=\; \pi e^{-a}.$$

References

[1] R. Nevanlinna & V. Paatero: *Funktioteoria*. Kustannusosakeyhtiö Otava. Helsinki (1963).