# proof of Bolzano-Weierstrass Theorem

| | |
|---|---|
| Canonical name | ProofOfBolzanoWeierstrassTheorem |
| Date of creation | 2013-03-22 12:22:26 |
| Last modified on | 2013-03-22 12:22:26 |
| Owner | akrowne (2) |
| Last modified by | akrowne (2) |
| Numerical id | 5 |
| Author | akrowne (2) |
| Entry type | Proof |
| Classification | msc 40A05 |
| Classification | msc 26A06 |

To prove the Bolzano-Weierstrass theorem, we will first need two lemmas.

**Lemma 1.**

All bounded monotone sequences converge.

**proof.**

Let $(s_n)$ be a bounded, nondecreasing sequence. Let $S$ denote the set $\{s_n : n \in \mathbb{N}\}$. Then let $b = \sup S$ (the supremum of $S$.)

Choose some $\epsilon > 0$. Then there is a corresponding $N$ such that $s_N > b - \epsilon$. Since $(s_n)$ is nondecreasing, for all $n > N$, $s_n > b - \epsilon$. But $(s_n)$ is bounded, so we have $b - \epsilon < s_n \leq b$. But this implies $|s_n - b| < \epsilon$, so $\lim s_n = b$. $\square$

(The proof for nonincreasing sequences is analogous.)

**Lemma 2.**

Every sequence has a monotonic subsequence.

**proof.**

First a definition: call the $n$th term of a sequence *dominant* if it is greater than every term following it.

For the proof, note that a sequence $(s_n)$ may have finitely many or infinitely many dominant terms.

First we suppose that $(s_n)$ has infinitely many dominant terms. Form a subsequence $(s_{n_k})$ solely of dominant terms of $(s_n)$. Then $s_{n_{k+1}} < s_{n_k}$ $k$ by definition of "dominant", hence $(s_{n_k})$ is a decreasing (monotone) subsequence of $(s_n)$.

For the second case, assume that our sequence $(s_n)$ has only finitely many dominant terms. Select $n_1$ such that $n_1$ is beyond the last dominant term. But since $n_1$ is not dominant, there must be some $m > n_1$ such that $s_m > s_{n_1}$. Select this $m$ and call it $n_2$. However, $n_2$ is still not dominant, so there must be an $n_3 > n_2$ with $s_{n_3} > s_{n_2}$, and so on, inductively. The resulting sequence

$$s_1, s_2, s_3, \ldots$$

is monotonic (nondecreasing). $\square$

**proof of Bolzano-Weierstrass.**

The proof of the Bolzano-Weierstrass theorem is now simple: let $(s_n)$ be a bounded sequence. By Lemma 2 it has a monotonic subsequence. By Lemma 1, the subsequence converges. $\square$