example of improper integral

Canonical name	ExampleOfImproperIntegral
Date of creation	2014-11-07 11:47:42
Last modified on	2014-11-07 11:47:42
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	6
Author	pahio (2872)
Entry type	Example
Classification	msc 40A10
Related topic	SubstitutionNotation

The integrand of

$$I = \int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\,dx \tag{1}$$

is undefined both at the lower and the upper limit. However, the value of the improper integral exists and may be found via the more general integral

$$I(y) = \int_0^1 \frac{\arctan xy}{x\sqrt{1-x^2}}\,dx. \tag{2}$$

Denote the integrand of (2) by $f(x,\,y)$. For any fixed real value y,

$$f(x,\,y) \in O(1) \;\text{ as }\; x \to 0, \quad f(x,\,y) \in O(\frac{1}{\sqrt{1-x^2}}) \;\text{ as }\; x \to 1,$$

where the http://planetmath.org/formaldefinitionoflandaunotationLandau big ordo notation has been used. Accordingly, the integral (2) converges for every y.

The inequality

$$\left|\frac{\partial f(x,\,y)}{\partial y}\right| = \frac{1}{(1+x^2y^2)\sqrt{1-x^2}} \leqq \frac{1}{\sqrt{1-x^2}}$$

and the convergence of the integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$$

imply that the integral

$$\int_0^1 \frac{\partial f(x,\,y)}{\partial y}\,dx \tag{3}$$

http://planetmath.org/node/6277converges uniformly on the whole y-axis and equals $I'(y)$. For expressing this derivative in a http://planetmath.org/ExpressibleInClose form, one may utilise the http://planetmath.org/ChangeOfVariableInDefiniteIntegralchang of variable

$$x := \cos\varphi, \quad \tan\varphi := t$$

which yield

$$\begin{aligned} I'(y) &= \int_0^1 \frac{dx}{(1+x^2y^2)\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{1+y^2\cos^2\varphi} \\ &= \int_0^\infty \frac{dt}{1+y^2+t^2} = \Big/_{t=0}^{\infty} \frac{1}{\sqrt{1+y^2}} \arctan \frac{t}{\sqrt{1+y^2}} \\ &= \frac{\pi}{2\sqrt{1+y^2}}. \end{aligned}$$

Hence,

$$I(y) = \frac{\pi}{2} \int_0^y \frac{dy}{\sqrt{1+y^2}} = \Big/_{0}^{y} \ln(y + \sqrt{1+y^2})$$

and the integral (1) equals $I = I(1) = \frac{\pi}{2}\ln(1+\sqrt{2})$, i.e.

$$\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\ln(1+\sqrt{2}). \tag{4}$$