limit of nondecreasing sequence

Canonical name	LimitOfNondecreasingSequence
Date of creation	2013-03-22 17:40:31
Last modified on	2013-03-22 17:40:31
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	13
Author	pahio (2872)
Entry type	Theorem
Classification	msc 40-00
Synonym	nondecreasing sequence with upper bound
Synonym	limit of increasing sequence
Related topic	MonotonicallyIncreasing
Related topic	MonotoneIncreasing
Related topic	Supremum
Related topic	Infimum
Related topic	ConvergenceOfTheSequence11nn

Theorem. A monotonically nondecreasing sequence of real numbers with upper bound a number M converges to a limit which does not exceed M.

Proof. Let $a_1 \leqq a_2 \leqq \ldots \leqq a_n \leqq \ldots \leqq M$. Therefore the set $\{a_1, a_2, \ldots\}$ has a finite supremum $s \leqq M$. We show that

$$\lim_{n\to\infty} a_n = s. \tag{1}$$

Let ε an arbitrary positive number. According to the definition of supremum we have $a_n \leqq s$ for all n and on the other hand, there exists a member $a_{n(\varepsilon)}$ of the sequence that is $> s-\varepsilon$. Then we have $s-\varepsilon < a_{n(\varepsilon)} \leqq s$, and since the sequence is nondecreasing,

$$0 \leqq s-a_n \leqq s-a_{n(\varepsilon)} < \varepsilon \quad \text{for all } n \geqq n(\varepsilon).$$

Thus the equation (1) and the whole theorem has been proven.

For the nonincreasing sequences there is the corresponding

Theorem. A monotonically nonincreasing sequence of real numbers with lower bound a number L converges to a limit which is not less than L.

Note. A good application of the latter theorem is in the proof that Euler's constant exists.