proof of slower divergent series

Canonical name	ProofOfSlowerDivergentSeries
Date of creation	2013-03-22 15:08:42
Last modified on	2013-03-22 15:08:42
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	9
Author	rspuzio (6075)
Entry type	Proof
Classification	msc 40A05

Let us show that, if the series $\sum_{i=1}^{\infty} a_i$ of positive terms is divergent, then Abel's series

$$\sum_{i=1}^{\infty} \frac{a_i}{\sum_{j=1}^{i} a_j}$$

also diverges.

Since the series $\sum_{i=1}^{\infty} a_i$ diverges, we can find an increasing sequence $(n_i)_{i=0}^{\infty}$ of integers such that

$$\sum_{j=1}^{n_{i+1}} a_j > 2 \sum_{j=1}^{n_i} a_j$$

for all i. By convention, set $n_0 = 0$. Then we can group the sum like so:

$$\sum_{i=1}^{n_m} \frac{a_i}{\sum_{j=1}^{i} a_j} = \sum_{i=0}^{m-1} \sum_{k=n_i+1}^{n_{i-1}} \frac{a_k}{\sum_{j=1}^{k} a_j}$$

Because $\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{n_{i-1}} a_j$, we have

$$\sum_{k=n_i+1}^{n_{i-1}} \frac{a_k}{\sum_{j=1}^{k} a_j} \geq \sum_{k=n_i+1}^{n_{i-1}} \frac{a_k}{\sum_{j=1}^{n_{i-1}} a_j} = \frac{\sum_{k=n_i+1}^{n_{i-1}} a_k}{\sum_{j=1}^{n_{i-1}} a_j}$$

By the way we chose the sequence $(n_i)_{i=0}^{\infty}$, we have $2 \sum_{k=n_i+1}^{n_{i-1}} a_k > \sum_{j=1}^{n_{i-1}} a_j$ and, hence,

$$\sum_{k=n_i+1}^{n_{i-1}} \frac{a_k}{\sum_{j=1}^{k} a_j} \geq \frac{\sum_{k=n_i+1}^{n_{i-1}} a_k}{\sum_{j=1}^{n_{i-1}} a_j} > \frac{1}{2}.$$

Therefore,

$$\sum_{i=1}^{n_m} \frac{a_i}{\sum_{j=1}^{i} a_j} > \sum_{i=1}^{n_m} \frac{1}{2} = \frac{m}{2},$$

so the sum diverges in the limit $m \to \infty$.