relationship among different kinds of compactness

Canonical name	RelationshipAmongDifferentKindsOfCompactness
Date of creation	2013-03-22 18:00:57
Last modified on	2013-03-22 18:00:57
Owner	rm50 (10146)
Last modified by	rm50 (10146)
Numerical id	6
Author	rm50 (10146)
Entry type	Theorem
Classification	msc 40A05
Classification	msc 54D30

The goal of this article is to prove

Theorem 1. *If X is second countable and T_1, or if X is a metric space, then the following are equivalent:*

1. *X is compact;*
2. *X is limit point compact;*
3. *X is sequentially compact.*

We prove this using several subsidiary theorems, which prove the various implications in stronger settings.

Theorem 2. *A compact topological space T is limit point compact. (Here we make no assumptions about the topology on T).*

Proof. Choose an subset $A \subset T$, and suppose A has no limit points. Then A contains its (vacuous set of) limit points and is therefore closed. But closed subsets of compact spaces are compact, so A is compact. Since A has no limit points, we may choose a neighborhood U_a of each $a \in A$ such that U_a intersects A only in a. But this cover clearly has a finite subcover only if A is finite. So any set without limit points is finite, and thus any infinite set has a limit point. This concludes the proof.

Theorem 3. *If T is first countable, T_1, and limit point compact, then T is sequentially compact.*

Proof. Let x_i be any sequence of points in T, and assume that x_i takes infinitely many values (otherwise it obviously has a convergent subsequence). Choose a limit point x for the sequence; we may assume wlog that x_i is equal to x for only finitely many i (otherwise again the result holds trivially). So by ignoring a finite number of leading terms of the sequence, we may assume that $x_i \neq x$ for every i. Since T is first countable, choose a countable basis B_i at x; by replacing B_n with $B_1 \cap \ldots \cap B_n$, we may assume that $B_{i+1} \subset B_i$ for all i.

Now, choose n_1 such that $p_{n_1} \in B_1$. Inductively, assume we have chosen $n_1, \ldots, n_k$ with $p_{n_k} \in B_k$. Since T is T_1, we may choose a neighborhood U of q that is disjoint from $p_{n_1}, \ldots, p_{n_k}$; choose $p_{n_{k+1}}$ to be any point in $U \cap B_{k+1}$. Then inductively the p_{n_i} form a subsequence with $p_{n_i} \in B_i$, and clearly the p_{n_i} converge to q. This concludes the proof.

Note that every metric space and every second countable T_1 space is also first countable and T_1.

Proposition 1. *Any sequentially compact metric space M is second countable.*

Proof. It clearly suffices to show that M has a countable dense subset.

Claim first that for $\epsilon > 0$, the set of ϵ-balls in M has a finite subcover. Suppose this is false for some particular ϵ. Let $p_1 \in M$ be any point, and construct inductively points p_k with $p_k \notin B_\epsilon(p_1) \cup \ldots B_\epsilon(p_{k-1})$. Since M is sequentially compact, we may replace the p_i by a convergent subsequence, which we also call p_i, with $p_i \to p \in M$. But convergent sequences are Cauchy, so for n large enough, we have $d(p_n, p_m) < \epsilon$, which contradicts the construction of the p_i. This proves the claim.

Then for each positive integer n, let $p_{n,n_1}, \ldots, p_{n,n_k} \in M$ be a finite set of points such that the $\frac{1}{n}$-balls around those points cover M. This set of points is countable, and is obviously dense in M. This concludes the proof.

Theorem 4. *If T is second countable or is a metric space, and sequentially compact, then T is compact.*

Proof. Assume first that T is second countable. Choose any open cover of T; it has a countable subcover U_i. We use an argument very similar to that used in the above proposition. Suppose no finite subset of the U_i covers T, and choose $p_k \in T \backslash (U_1 \cup \ldots \cup U_k)$. Since T is sequentially compact, the p_k have a convergent subsequence p_{n_k} converging to $p \in T$. But $p \in U_n$ for some n; since the p_{n_k} converge to p, all $p_{n_k} \in U_n$ for k large enough. But this is a contradiction to the construction of the p_k, so that a finite subset of the U_i cover T and T is compact.

Since any sequentially compact metric space is second countable by the above proposition, we are done.

The main theorem follows trivially from the above. Note that we have in fact proven the following set of implications:

- Compact $\Rightarrow$ limit point compact for general topological spaces;
- Limit point compact $\Rightarrow$ sequentially compact for first countable T_1 spaces;
- Sequentially compact $\Rightarrow$ compact for second countable or metrizable spaces.