# limit of geometric sequence

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As mentionned in the geometric sequence entry,

$$\lim_{n\to\infty} ar^n = 0 \tag{1}$$

for $|r| < 1$. We will prove this for real or complex values of $r$.

We first remark, that for the values $s > 1$ we have $\lim_{n\to\infty} s^n = \infty$ (cf. limit of real number sequence). In fact, if $M$ is an arbitrary positive number, the binomial theorem (or Bernoulli's inequality) implies that

$$s^n = (1+s-1)^n > 1^n + \binom{n}{1}(s-1) = 1 + n(s-1) > n(s-1) > M$$

as soon as $n > \dfrac{M}{s-1}$.

Let now $|r| < 1$ and $\varepsilon$ be an arbitrarily small positive number. Then $|r| = \dfrac{1}{s}$ with $s > 1$. By the above remark,

$$|r^n| = |r|^n = \frac{1}{s^n} < \frac{1}{n(s-1)} < \varepsilon$$

when $n > \dfrac{1}{(s-1)\varepsilon}$. Hence,

$$\lim_{n\to\infty} r^n = 0,$$

which easily implies (1) for any real number $a$.