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Abel's lemma

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Theorem 1 Let $\{a_i\}_{i=0}^N$ and $\{b_i\}_{i=0}^N$ be sequences of real (or complex) numbers with $N \geq 0$. For $n = 0, \ldots, N$, let A_n be the partial sum $A_n = \sum_{i=0}^n a_i$. Then

$$\sum_{i=0}^{N} a_i b_i = \sum_{i=0}^{N-1} A_i (b_i - b_{i+1}) + A_N b_N.$$

In the trivial case, when $N = 0$, then sum on the right hand side should be interpreted as identically zero. In other words, if the upper limit is below the lower limit, there is no summation.

An inductive proof can be found `http://planetmath.org/ProofOfAbelsLemmaByInduction`her
The result can be found in [?] (Exercise 3.3.5).

If the sequences are indexed from M to N, we have the following variant:

Corollary Let $\{a_i\}_{i=M}^N$ and $\{b_i\}_{i=M}^N$ be sequences of real (or complex) numbers with $0 \leq M \leq N$. For $n = M, \ldots, N$, let A_n be the partial sum $A_n = \sum_{i=M}^n a_i$. Then

$$\sum_{i=M}^{N} a_i b_i = \sum_{i=M}^{N-1} A_i (b_i - b_{i+1}) + A_N b_N.$$

Proof. By defining $a_0 = \ldots = a_{M-1} = b_0 = \ldots = b_{M-1} = 0$, we can apply Theorem 1 to the sequences $\{a_i\}_{i=0}^N$ and $\{b_i\}_{i=0}^N$. $\square$

References

[1] R.B. Guenther, L.W. Lee, *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications, 1988.