real part series and imaginary part series

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Theorem 1. Given the series

$$z_1 + z_2 + z_3 + \ldots \tag{1}$$

with the real parts of its terms $\Re z_n = a_n$ and the imaginary parts of its terms $\Im z_n = b_n$ ($n = 1, 2, 3, \ldots$). If the series (1) converges and its sum is $A + iB$, where A and B are real, then also the series

$$a_1 + a_2 + a_3 + \ldots \quad \text{and} \quad b_1 + b_2 + b_3 + \ldots$$

converge and their sums are A and B, respectively. The converse is valid as well.

Proof. Let ε be an arbitrary positive number. Denote the partial sum of (1) by

$$S_n = z_1 + \ldots + z_n = (a_1 + ib_1) + \ldots + (a_n + ib_n) = (a_1 + \ldots + a_n) + i(b_1 + \ldots + b_n) := A_n + iB_n$$

($n = 1, 2, 3, \ldots$). When (1) converges to the sum $A + iB$, then there is a number n_ε such that for any integer $n > n_\varepsilon$ we have

$$|(A_n - A) + i(B_n - B)| = |(A_n + iB_n) - (A + iB)| < \varepsilon.$$

But a complex number is always absolutely at least equal to the real part (see the inequalities in modulus of complex number), and therefore $|A_n - A| \leqq |(A_n - A) + i(B_n - B)| < \varepsilon$, similarly $|B_n - B| \leqq |(A_n - A) + i(B_n - B)| < \varepsilon$ as soon as $n > n_\varepsilon$. Hence, $A_n \to A$ and $B_n \to B$ as $n \to \infty$. This means the convergences

$$a_1 + a_2 + a_3 + \ldots = A \quad \text{and} \quad b_1 + b_2 + b_3 + \ldots = B,$$

Q.E.D. The converse part is straightforward.

Theorem 2. Notations same as in the preceding theorem. The series

$$|z_1| + |z_2| + |z_3| + \ldots$$

converges if and only if the series

$$a_1 + a_2 + a_3 + \ldots \quad \text{and} \quad b_1 + b_2 + b_3 + \ldots$$

`http://planetmath.org/AbsoluteConvergence`converge absolutely.

Proof. Use the inequalities

$$0 \leqq |a_n| \leqq |z_n|, \quad 0 \leqq |b_n| \leqq |z_n|$$

and

$$0 \leqq |z_n| \leqq |a_n| + |b_n|$$

for using the comparison test.

Theorem 3. If the series $\sum_{n=1}^{\infty} |z_n|$ converges, then also the series $\sum_{n=1}^{\infty} z_n$ converges and we have

$$\left| \sum_{n=1}^{\infty} z_n \right| \leqq \sum_{n=1}^{\infty} |z_n|.$$

Proof. By theorem 2, the convergence of $\sum |z_n|$ implies the convergence of $\sum a_n$ and $\sum b_n$, which, by theorem 1, in turn imply the convergence of $\sum z_n$. Since for every n the triangle inequality guarantees the inequality

$$\left| \sum_{j=1}^{n} z_j \right| \leqq \sum_{j=1}^{n} |z_j|,$$

then we must have the asserted limit inequality, too.