# absolute convergence of double series

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| Related topic | DoubleSeries |
| Related topic | DiagonalSumming |
| Defines | row series |
| Defines | column series |
| Defines | diagonal series |

Let us consider the double series

$$\sum_{i,j=1}^{\infty} u_{ij} \tag{1}$$

of real or complex numbers $u_{ij}$. Denote the *row series* $u_{k1}+u_{k2}+\dots$ by $R_k$, the *column series* $u_{1k}+u_{2k}+\dots$ by $C_k$ and the *diagonal series* $u_{11}+u_{12}+u_{21}+u_{13}+u_{22}+u_{31}+\dots$ by DS. Then one has the

**Theorem.** All row series, all column series and the diagonal series converge absolutely and

$$\sum_{k=1}^{\infty} R_k \;=\; \sum_{k=1}^{\infty} C_k \;=\; \mathrm{DS},$$

if one of the following conditions is true:

- The diagonal series converges absolutely.
- There exists a positive number $M$ such that every finite sum of the numbers $|u_{ij}|$ is $\leqq M$.
- The row series $R_k$ converge absolutely and the series $W_1+W_2+\dots$ with

  $$\sum_{j=1}^{\infty} |u_{kj}| = W_k$$

  is convergent. An analogical condition may be formulated for the column series $C_k$.

**Example.** Does the double series

$$\sum_{m=2}^{\infty}\sum_{n=3}^{\infty} n^{-m}$$

converge? If yes, determine its sum.

The column series $\displaystyle\sum_{m=2}^{\infty}\left(\frac{1}{n}\right)^m$ have positive terms and are absolutely converging geometric series having the sum

$$\frac{(1/n)^2}{1-1/n} \;=\; \frac{1}{n(n-1)} \;=\; \frac{1}{n-1}-\frac{1}{n} \;=\; W_n.$$

The series $W_3+W_4+\ldots$ is convergent, since its partial sum is a telescoping sum

$$\sum_{n=3}^{N} W_n = \sum_{n=3}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right) = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{N-1}-\frac{1}{N}\right)$$

equalling simply $\frac{1}{2}-\frac{1}{N}$ and having the limit $\frac{1}{2}$ as $N \to \infty$. Consequently, the given double series converges and its sum is $\frac{1}{2}$.