proof of Silverman-Toeplitz theorem

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First, we shall show that the series $\sum_{n=0}^{\infty} a_{mn} z_n$ converges. Since the sequence $\{z_n\}$ converges, it must be bounded in absolute value — there must exist a constant $K > 0$ such that $|z_n| \leq K$ for all n. Hence, $|a_{mn} z_n| \leq K |a_{mn}|$. Summing this gives

$$\sum_{n=0}^{\infty} |a_{mn} z_n| \leq K \sum_{n=0}^{\infty} |a_{mn}| \leq KB.$$

Hence, the series $\sum_{n=0}^{\infty} a_{mn} z_n$ is absolutely convergent which, in turn, implies that it converges.

Let z denote the limit of the sequence $\{z_n\}$ as $n \to \infty$. Then $|z| \leq K$. We need to show that, for every $\epsilon > 0$, there exists an integer M such that

$$\left| \sum_{n=0}^{\infty} a_{mn} z_n - z \right| < \epsilon$$

whenever $m > M$.

Since the sequence $\{z_n\}$ converges, there must exist an integer n_1 such that $|z_n - z| < \frac{\epsilon}{4B}$ whenever $n > n_1$.

By condition 3, there must exist constants $m_o, m_1, \ldots, m_{n_1}$ such that

$$|a_{mn}| < \frac{\epsilon}{4(n_1 + 1)(K + 1)} \quad \text{for} \quad 0 \leq n \leq n_1 \quad \text{and} \quad m > m_n. \tag{1}$$

Choose $m' = \max\{m_0, m_1, \ldots, m_{n_1}\}$. Then

$$\left| \sum_{n=0}^{n_1} a_{mn} z_n \right| \leq \sum_{n=0}^{n_1} |a_{mn} z_n| < \sum_{n=0}^{n_1} \frac{|z_n| \epsilon}{4(n_1 + 1)(K + 1)} < \frac{\epsilon}{4} \tag{2}$$

when $m > m'$.

By condition 2, there exists a constant m'' such that

$$\left| \sum_{n=0}^{\infty} a_{mn} - 1 \right| < \frac{\epsilon}{6(|z| + 1)}$$

whenever $m > m''$. By (1),

$$\left| \sum_{n=0}^{n_1} a_{mn} \right| < \frac{\epsilon}{4(K + 1)} \leq \frac{\epsilon}{4(|z| + 1)}$$

when $m > m'$. Hence, if $m > m'$ and $m > m''$, we have

$$\left| \sum_{n=n_1+1}^{\infty} a_{mn} - 1 \right| < \frac{\epsilon}{2(|z|+1)}. \tag{3}$$

and

$$\left| \sum_{n=n_1+1}^{\infty} a_{mn}(z_n - z) \right| \leq \sum_{n=n_1+1}^{\infty} |a_{mn}|\ |z_n - z| < \frac{\epsilon}{4B} \sum_{n=0}^{\infty} |a_{mn}| \leq \frac{\epsilon}{4}. \tag{4}$$

By the triangle inequality and (2), (3), (4) it follows that

$$\left| \sum_{n=0}^{\infty} a_{mn} z_n - z \right| \leq \left| \sum_{n=0}^{n_1} a_{mn} z_n \right| + \left| \sum_{n=n_1+1}^{\infty} a_{mn}(z_n - z) \right| + |z| \left| \sum_{n=n_1+1}^{\infty} a_{mn} - 1 \right|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.$$