a sufficient condition for convergence of integral

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Suppose that the real function f is positive and continuous on the interval $[a,\,\infty)$. A sufficient condition for the http://planetmath.org/ConvergentIntegralconvergence of the improper integral

$$\int_a^\infty f(x)\,dx \tag{1}$$

is that

$$\lim_{x\to\infty}\frac{f(x\!+\!1)}{f(x)} \;=\; q \;<\; 1. \tag{2}$$

Proof. Assume that the condition (2) is in . For an http://planetmath.org/ReductioAdAbsur proof, make the antithesis that the http://planetmath.org/RiemannIntegralintegral (1) http://planetmath.org/DivergentIntegraldiverges.

Because of the positiveness, we have $\int_a^\infty f(x)\,dx \;=\; \infty$. We can use http://planetmath.org/LHpitalsRulel'Hôpital's rule:

$$\lim_{c\to\infty}\frac{\int_a^c f(x\!+\!1)\,dx}{\int_a^c f(x)\,dx} \;=\; \lim_{c\to\infty}\frac{f(c\!+\!1)}{f(c)}.$$

Using the http://planetmath.org/node/11373substitution $x\!+\!1 \,=\, t$ we get

$$\int_a^c f(x\!+\!1)\,dx \;=\; \int_{a-1}^{c-1} f(t)\,dt \;=\; \int_{a-1}^{a} f(t)\,dt + \int_a^c f(t)\,dt - \int_{c-1}^{c} f(t)\,dt,$$

and dividing this equation by $\int_a^c f(t)\,dt$ and taking http://planetmath.org/ImproperLimitslimits yield (f is bounded!)

$$1 \;>\; q \;=\; \lim_{c\to\infty}\frac{\int_a^c f(x\!+\!1)\,dx}{\int_a^c f(x)\,dx} \;=\; 0+1-0 \;=\; 1.$$

This contradictory result shows that the antithesis is wrong; thus (1) must be http://planetmath.org/ConvergentIntegralconvergent.

Note. The condition (2) is not necessary for the convergence of (1). This is seen e.g. in the case of the converging of (2) equals 1.