proof that Euler's constant exists

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Theorem 1 *The limit*

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$$

exists.

Proof. Let

$$C_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

and

$$D_n = C_n - \frac{1}{n}$$

Then

$$C_{n+1} - C_n = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right)$$

and

$$D_{n+1} - D_n = \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)$$

Now, by considering the Taylor series for $\ln(1+x)$, we see that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

and so

$$C_{n+1} - C_n < 0 < D_{n+1} - D_n$$

Thus, the C_n decrease monotonically, while the D_n increase monotonically, since the differences are negative (positive for D_n). Further, $D_n < C_n$ and thus $D_1 = 0$ is a lower bound for C_n. Thus the C_n are monotonically decreasing and bounded below, so they must converge.

References

[1] E. Artin, *The Gamma Function*, Holt, Rinehart, Winston 1964.