sequentially compact

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A topological space X is *sequentially compact* if every sequence in X has a convergent subsequence.

Every sequentially compact space is countably compact. Conversely, every first countable countably compact space is sequentially compact. The ordinal space $W(2\omega_1)$ is sequentially compact but not first countable, since ω_1 has not countable local basis.

Next, compactness and sequential compactness are not compatible. In other words, neither one implies the other. Here's an example of a compact space that is not sequentially compact. Let $X = I^I$, where I is the closed unit interval (with the usual topology), and X is equipped with the product topology. Then X is compact (since I is, together with Tychonoff theorem). However, X is not sequentially compact. To see this, let $f_n : I \to I$ be the function such that for any $r \in I$, $f(r)$ is the n-th digit of r in its binary expansion. But the sequence $f_1, \ldots, f_n, \ldots$ has no convergent subsequences: if $f_{n_1}, \ldots, f_{n_k}, \ldots$ is a subsequence, let $r \in I$ such that its binary expansion has its k-th digit 0 iff k is odd, and 1 otherwise. Then $f_{n_1}(r), \ldots, f_{n_k}(r), \ldots$ is the sequence $0, 1, 0, 1, \ldots$, and is clearly not convergent. The ordinal space $\Omega_0 := W(\omega_1)$ is an example of a sequentially compact space that is not compact, since the cover $\{W(\alpha) \mid \alpha \in \Omega_0\}$ has no finite subcover.

When X is a metric space, the following are equivalent:

- X is sequentially compact.
- X is limit point compact.
- X is compact.
- X is totally bounded and complete.