proof of integral test

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Owner	paolini (1187)
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Author	paolini (1187)
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Consider the function (see the definition of floor)

$$g(x) = a_{\lfloor x \rfloor}.$$

Clearly for $x \in [n, n+1)$, being f non increasing we have

$$g(x+1) = a_{n+1} = f(n+1) \leq f(x) \leq f(n) = a_n = g(x)$$

hence

$$\int_M^{+\infty} g(x+1)\,dx = \int_{M+1}^{+\infty} g(x)\,dx \leq \int_M^{+\infty} f(x) \leq \int_M^{+\infty} g(x)\,dx.$$

Since the integral of f and g on $[M, M+1]$ is finite we notice that f is integrable on $[M, +\infty)$ if and only if g is integrable on $[M, +\infty)$.

On the other hand g is locally constant so

$$\int_n^{n+1} g(x)\,dx = \int_n^{n+1} a_n\,dx = a_n$$

and hence for all $N \in \mathbb{Z}$

$$\int_N^{+\infty} g(x) = \sum_{n=N}^{\infty} a_n$$

that is g is integrable on $[N, +\infty)$ if and only if $\sum_{n=N}^{\infty} a_n$ is convergent.

But, again, $\int_M^N g(x)\,dx$ is finite hence g is integrable on $[M, +\infty)$ if and only if g is integrable on $[N, +\infty)$ and also $\sum_{n=0}^{N} a_n$ is finite so $\sum_{n=0}^{\infty} a_n$ is convergent if and only if $\sum_{n=N}^{\infty} a_n$ is convergent.