derivative of limit function diverges from limit of derivatives

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For a function sequence, one cannot always change the of http://planetmath.org/node/6209t limit and http://planetmath.org/Differentiatedifferentiating, i.e. it may well be

$$\lim_{n\to\infty}\frac{d}{dx}f_n(x) \;\neq\; \frac{d}{dx}\lim_{n\to\infty}f_n(x),$$

in the case that a sequence of continuous (and differentiable) functions converges uniformly; cf. Theorem 2 of the http://planetmath.org/LimitFunctionOfSequenceparent entry.

Example. The function sequence

$$f_n(x) \;:=\; \sum_{j=1}^{n}\frac{x^3}{(1+x^2)^j} \;=\; x-\frac{x}{(1+x^2)^n} \qquad (n \;=\; 1,\,2,\,3,\,\ldots) \qquad (1)$$

provides an instance; we consider it on the interval $[-1,\,1]$. It's a question of partial sum the converging geometric series

$$\frac{x^3}{1+x^2}+\frac{x^3}{(1+x^2)^2}+\frac{x^3}{(1+x^2)^2}+\ldots$$

(although one cannot use Weierstrass' criterion of uniform convergence). Since the limit function is

$$f(x) \;:=\; \lim_{n\to\infty}\left(x-\frac{x}{(1+x^2)^n}\right) \;=\; x \quad \forall x\in[-1,\,1],$$

we have

$$\sup_{[-1,\,1]}|f_n(x)-f(x)| \;=\; \sup_{[-1,\,1]}\frac{|x|}{(1+x^2)^n} \;\longrightarrow 0 \quad \text{as} \;\; n\to\infty,$$

which means by Theorem 1 of the http://planetmath.org/LimitFunctionparent entry that the sequence (1) converges uniformly on the interval to the identity function. Further, the members of the sequence are continuous and differentiable. Furthermore,

$$f_n'(x) \;=\; 1-\frac{1+\,(1-2n)x^2}{(1+x^2)^{n+1}},$$

whence

$$\lim_{n\to\infty}f_n'(x) \;=\; 1 \quad (x \;\neq\; 0).$$

But in the point $x = 0$ we have

$$\lim_{n\to\infty} f'_n(0) \;=\; \lim_{n\to\infty} 0 \;=\; 0,$$

which says that the limit of derivative sequence of (1) is discontinuous in the origin. Because

$$f'(x) \;\equiv\; 1,$$

we may write

$$\lim_{n\to\infty} \frac{d}{dx} f_n \;\neq\; \frac{d}{dx} \lim_{n\to\infty} f_n.$$