Kronecker's lemma

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Kronecker's lemma gives a condition for convergence of partial sums of real numbers, and for example can be used in the proof of Kolmogorov's strong law of large numbers.

Lemma (Kronecker). *Let* $x_1, x_2, \dots$ *and* $0 < b_1 < b_2 < \cdots$ *be sequences of real numbers such that* b_n *increases to infinity as* $n \to \infty$. *Suppose that the sum* $\sum_{n=1}^{\infty} x_n/b_n$ *converges to a finite limit. Then,* $b_n^{-1} \sum_{k=1}^{n} x_k \to 0$ *as* $n \to \infty$.

Proof. Set $u_n = \sum_{k=1}^{n} x_k/b_k$, so that the limit $u_\infty = \lim_{n\to\infty} u_n$ exists. Also set $a_n = \sum_{k=1}^{n-1} (b_{k+1} - b_k) u_k$ so that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = u_n \to u_\infty$$

as $n \to \infty$. Then, the Stolz-Cesaro theorem says that a_n/b_n also converges to u_∞, so

$$b_n^{-1} \sum_{k=1}^{n} x_k = b_n^{-1} \sum_{k=1}^{n} b_k (u_k - u_{k-1}) = u_n - b_n^{-1} a_n \to 0.$$

□