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Halley's formula

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The following formula is due to the English scientist and mathematician Edmond Halley (1656 à 1742):

$$\ln x = \lim_{n\to\infty} (\sqrt[n]{x} - 1)n \tag{1}$$

Proof. We change the nth root to power of e and use the power series expansion of exponential function:

$$\begin{aligned}
(\sqrt[n]{x} - 1)n &= (e^{\frac{\ln x}{n}} - 1)n \\
&= \left(\sum_{m=0}^{\infty} \frac{(\ln x/n)^m)}{m!} - 1\right)n \\
&= \sum_{m=1}^{\infty} \frac{(\ln x/n)^m n}{m!} \\
&= \ln x + \frac{1}{n} \sum_{m=2}^{\infty} \frac{(\ln x)^m}{m! n^{m-2}}
\end{aligned}$$

The last converging series has a finite sum, and as $n \to \infty$, the asserted formula follows.

Note. The formula (1) was known also by Leonhard Euler, who used it for defining the natural logarithm. Using (1), one can easily prove the well-known laws of logarithm, e.g.

$$\begin{aligned}
\ln xy &= \lim_{n\to\infty} (\sqrt[n]{x}\,\sqrt[n]{y} - 1)n \\
&= \lim_{n\to\infty} (\sqrt[n]{x}\,\sqrt[n]{y} - \sqrt[n]{y} + \sqrt[n]{y} - 1)n \\
&= \lim_{n\to\infty} y^{\frac{1}{n}} (\sqrt[n]{x} - 1)n + \lim_{n\to\infty} (\sqrt[n]{y} - 1)n \\
&= y^0 \ln x + \ln y \\
&= \ln x + \ln y.
\end{aligned}$$

References

[1] Paul Loya: *Amazing and Aesthetic Aspect of Analysis: On the incredible infinite*. A course in undergraduate analysis, fall 2006. Available `http://www.math.binghamton.edu/dennis/478.f07/EleAna.pdf`here.