summation by parts

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The following corollaries apply Abel's lemma to allow estimation of certain bounded sums:

Corollary 1 *(Summation by parts)*
Let $\{a_i\}, \{b_i\}$ be sequences of complex numbers. Suppose the partial sums of the a_i are bounded in magnitude by h, that $\sum_0^\infty |b_i - b_{i+1}|$ converges, and that $\lim_{i\to\infty} b_i = 0$*. Then $\sum_0^\infty a_i b_i$ converges, and*

$$\left|\sum_0^\infty a_i b_i\right| \leq h \sum_0^\infty |b_i - b_{i+1}|$$

Proof. By Abel's lemma,

$$\sum_{i=0}^{N} a_i b_i = \sum_{i=0}^{N-1} A_i(b_i - b_{i+1}) + A_N b_N$$

so that

$$\begin{aligned}\left|\sum_{i=0}^{N} a_i b_i\right| &= \left|\sum_{i=0}^{N-1} A_i(b_i - b_{i+1}) + A_N b_N\right| \leq \sum_{i=0}^{N-1} |A_i(b_i - b_{i+1})| + |A_N b_N| \\ &\leq h \sum_{i=0}^{N-1} |b_i - b_{i+1}| + h\,|b_N|\end{aligned}$$

The condition that the $b_i \to 0$ is easily seen to imply that the sequence $\left|\sum_{i=0}^{N} a_i b_i\right|$ is Cauchy hence convergent, so that

$$\left|\sum_{i=0}^\infty a_i b_i\right| \leq h \sum_{i=0}^\infty |b_i - b_{i+1}|$$

since $b_N \to 0$.

Corollary 2 *(Summation by parts for real sequences)*
Let $\{a_i\}$ be a sequence of complex numbers. Suppose the partial sums are bounded in magnitude by h. Let $\{b_i\}$ be a sequence of decreasing positive real numbers such that $\lim_{i\to\infty} b_i = 0$*. Then $\sum_1^\infty a_i b_i$ converges, and $|\sum_1^\infty a_i b_i| \leq hb_1$.*

Proof. This follows immediately from the above, since $|b_i - b_{i+1}| = b_i - b_{i+1}$.