uniform convergence on union interval

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Owner	pahio (2872)
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Author	pahio (2872)
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Theorem. If $a < b < c$ and the sequence $f_1,\, f_2,\, f_3,\, \ldots$ of real functions converges uniformly both on the interval $[a,\, b]$ and on the interval $[b,\, c]$, then the function sequence converges uniformly also on the http://planetmath.org/Unionunion interval $[a,\, c]$.

Proof. We have the limit functions $f_{ab} := \lim\limits_{n\to\infty} f_n$ on $[a,\, b]$ and $f_{bc} := \lim\limits_{n\to\infty} f_n$. It follows that

$$f_{ab}(b) = \lim_{n\to\infty} f_n(b) = f_{bc}(b).$$

Define the new function

$$f(x) := \begin{cases} f_{ab}(x) & \forall x \in [a,\, b], \\ f_{bc}(x) & \forall x \in [b,\, c]. \end{cases}$$

Choose an arbitrary positive number ε. The supposed uniform convergences on the intervals $[a,\, b]$ and $[b,\, c]$ imply the existence of the numbers $n_1(\varepsilon)$ and $n_2(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \;\; \forall x \in [a,\, b], \quad \text{when } n > n_1(\varepsilon)$$

and

$$|f_n(x) - f(x)| < \varepsilon \;\; \forall x \in [b,\, c], \quad \text{when } n > n_2(\varepsilon).$$

If one takes $n > \max\{n_1(\varepsilon),\, n_2(\varepsilon)\}$, then one has simultaneously on both intervals $[a,\, b]$ and $[b,\, c]$, i.e. on the whole greater interval $[a,\, c]$, the condition

$$|f_n(x) - f(x)| < \varepsilon.$$