one-sided continuity by series

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Theorem. If the function series

$$\sum_{n=1}^{\infty} f_n(x) \tag{1}$$

is uniformly convergent on the interval $[a,\, b]$, on which the $f_n(x)$ are continuous from the right or from the left, then the sum function $S(x)$ of the series has the same property.

Proof. Suppose that the terms $f_n(x)$ are continuous from the right. Let ε be any positive number and

$$S(x) \ := \ S_n(x) + R_{n+1}(x),$$

where $S_n(x)$ is the n^{th} partial sum of (1) $(n = 1,\, 2,\, \ldots)$. The uniform convergence implies the existence of a number n_ε such that on the whole interval we have

$$|R_{n+1}(x)| < \frac{\varepsilon}{3} \quad \text{when} \ \ n > n_\varepsilon.$$

Let now $n > n_\varepsilon$ and $x_0,\, x_0{+}h \in [a,\, b]$ with $h > 0$. Since every $f_n(x)$ is continuous from the right in x_0, the same is true for the finite sum $S_n(x)$, and therefore there exists a number δ_ε such that

$$|S_n(x_0{+}h) - S_n(x_0)| < \frac{\varepsilon}{3} \quad \text{when} \ \ 0 < h < \delta_\varepsilon.$$

Thus we obtain that

$$\begin{aligned}
|S(x_0{+}h) - S(x_0)| \ &= \ |[S_n(x_0{+}h) - S_n(x_0)] + R_{n+1}(x_0{+}h) - R_{n+1}(x_0| \\
&\leqq \ |S_n(x_0{+}h) - S_n(x_0)| + |R_{n+1}(x_0{+}h)| + |R_{n+1}(x_0)| \\
&< \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ = \ \varepsilon
\end{aligned}$$

as soon as

$$0 < h < \delta_\varepsilon.$$

This means that S is continuous from the right in an arbitrary point x_0 of $[a,\, b]$.

Analogously, one can prove the assertion concerning the continuity from the left.