convergent series where not only a_n but also na_n tends to 0

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Proposition. If the `http://planetmath.org/Series`terms a_n of the convergent series

$$a_1 + a_2 + \ldots$$

are positive and form a monotonically decreasing sequence, then

$$\lim_{n\to\infty} na_n \;=\; 0. \qquad (1)$$

Proof. Let ε be any positive number. By the Cauchy criterion for convergence and the positivity of the terms, there is a positive integer m such that

$$0 \;<\; a_{m+1} + \ldots + a_{m+p} \;<\; \frac{\varepsilon}{2} \qquad (p \;=\; 1,\,2,\,\ldots).$$

Since the sequence $a_1,\,a_2,\,\ldots$ is decreasing, this implies

$$0 \;<\; pa_{m+p} \;<\; \frac{\varepsilon}{2} \qquad (p \;=\; 1,\,2,\,\ldots). \qquad (2)$$

Choosing here especially $p := m$, we get

$$0 \;<\; ma_{m+m} \;<\; \frac{\varepsilon}{2},$$

whence again due to the decrease,

$$0 \;<\; ma_{m+p} \;<\; \frac{\varepsilon}{2} \qquad (p \;=\; m,\,m{+}1,\,\ldots). \qquad (3)$$

Adding the inequalities (2) and (3) with the common values $p = m,\,m{+}1,\,\ldots$ then yields

$$0 \;<\; (m{+}p)a_{m+p} \;<\; \varepsilon \qquad \text{for} \quad p \;\geqq\; m.$$

This may be written also in the form

$$0 \;<\; na_n \;<\; \varepsilon \qquad \text{for} \quad n \;\geqq\; 2m$$

which means that $\lim_{n\to\infty} na_n \;=\; 0.$

Remark. The assumption of monotonicity in the Proposition is essential. I.e., without it, one cannot gererally get the limit result (1). A counterexample would be the series $a_1{+}a_2{+}\ldots$ where $a_n := \frac{1}{n}$ for any perfect square n but 0 for other values of n. Then this series is convergent (cf. the over-harmonic series), but $na_n = 1$ for each perfect square n; so $na_n \not\to 0$ as $n \to \infty$.