adding and removing parentheses in series

Canonical name	AddingAndRemovingParenthesesInSeries
Date of creation	2013-03-22 18:54:09
Last modified on	2013-03-22 18:54:09
Owner	pahio (2872)
Last modified by	pahio (2872)
Numerical id	13
Author	pahio (2872)
Entry type	Topic
Classification	msc 40A05
Related topic	EmptySum

We consider series with real or complex terms.

- If one groups the terms of a convergent series by adding parentheses but not changing the order of the terms, the series remains convergent and its sum the same. (See theorem 3 of the `http://planetmath.org/node/6517`parent entry.)
- A divergent series can become convergent if one adds an infinite amount of parentheses; e.g.
  $1-1+1-1+1-1+-\ldots$ diverges but $(1-1)+(1-1)+(1-1)+\ldots$ converges.
- A convergent series can become divergent if one removes an infinite amount of parentheses; cf. the preceding example.
- If a series parentheses, they can be removed if the obtained series converges; in this case also the original series converges and both series have the same sum.
- If the series

  $$(a_1+\ldots+a_r)+(a_{r+1}+\ldots+a_{2r})+(a_{2r+1}+\ldots+a_{3r})+\ldots \tag{1}$$

  converges and

  $$\lim_{n\to\infty} a_n \;=\; 0, \tag{2}$$

  then also the series

  $$a_1+a_2+a_3\ldots \tag{3}$$

  converges and has the same sum as (1).

  Proof. Let S be the sum of the (1). Then for each positive integer n, there exists an integer k such that $kr < n \leqq (k+1)r$. The partial sum of (3) may be written

  $$a_1+\ldots+a_n \;=\; \underbrace{(a_1+\ldots+a_{kr})}_{s}+\underbrace{(a_{kr+1}+\ldots+a_n)}_{s'}.$$

When $n \to \infty$, we have

$$s \to S$$

by the convergence of (1) to S, and

$$s' \to 0$$

by the condition (2). Therefore the whole partial sum will tend to S, Q.E.D.

Note. The parenthesis expressions in (1) need not be "equally long" — it suffices that their lengths are under an finite bound.