ON AN EXPRESSION FOR BERNOULLI NUMBERS IN TERMS OF STIRLING NUMBERS OF THE SECOND KIND

SUMIT KUMAR JHA AND J. LÓPEZ-BONILLA

Abstract. We give a combinatorial proof of an interesting expression for the Bernoulli numbers in terms of the Stirling numbers of the second kind.

1. Introduction

Definition 1. The *Bernoulli numbers* B_n can be defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 2. The *Stirling number of the second kind*, denoted by $\left\{ {n \atop k} \right\}$, is the number of ways of partitioning a set of n elements into m nonempty sets.

Jha [1] obtained the following expression for the Bernoulli numbers:

$$B_{m+n} = \sum_{k=0}^{n} \sum_{r=0}^{m} \frac{(-1)^{k+r}(k!\, r!)^2}{(k+r+1)!} \left\{ {m \atop r} \right\} \left\{ {n \atop k} \right\} \qquad (m, n \geq 0) \tag{1}$$

using an integral expression for the Riemann zeta function in terms of the polylogarithm function. The proof requires analytic continuation of both the Riemann zeta function and the polylogarithm function [2].

We give of a combinatorial proof of the above expression in the following section.

2. Proof of Main result

Proof of expression (1). When $m = n = 0$ the expression is trivial since we know that $B_0 = 1$.

When $m = 0$ with arbitrary n the expression (1) takes form of the following well known formula [3, 2, 4]

$$B_n = \sum_{k=0}^{n} \frac{(-1)^k\, k!}{k+1} \left\{ {n \atop k} \right\} \qquad (n \geq 0).$$

2010 *Mathematics Subject Classification.* 11B68, 11B73.

Key words and phrases. Bernoulli numbers; Fukuhara-Kawazumi-Kuno's relation; Stirling numbers.

Thus it is sufficient to prove the expression for $m, n \geq 1$. We first recall the Fukuharda-Kawazumi-Kuno identity [5, Theorem 1]

$$B_N = (-1)^M \sum_{j=1}^{Q+1} \frac{(-1)^{j+1}}{j} \binom{Q+1}{j} \sum_{q=1}^{j-1} q^M (j-q)^{N-M} \tag{2}$$

which is valid for all integers $N \geq 2$ and $0 \leq M \leq N \leq Q$.

Letting $M = 0$, $N = k + r$, $Q = m + n$ with $0 \leq k + r \leq Q = m + n$ in Eq. (2) gives us

$$B_{k+r} = \sum_{j=1}^{m+n+1} \frac{(-1)^{j+1}}{j} \binom{m+n+1}{j} \sum_{q=1}^{j-1} (j-q)^{k+r} \tag{3}$$

which can be multiplied by Stirling numbers of the first kind to obtain

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{r=0}^{m} B_{k+r} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ r \end{bmatrix} &= \sum_{q=1}^{(m+n+1)} \sum_{l=0}^{(m+n+1-q)} \frac{(-1)^{l+q+1}}{l+q} \binom{m+n+1}{l+q} \sum_{k=0}^{n}\sum_{r=0}^{m} l^{k+r} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ r \end{bmatrix} \\
&= m!\, n! \sum_{q=1}^{(m+n+1)} \sum_{l=0}^{(m+n+1-q)} \frac{(-1)^{l+q+1}}{l+q} \binom{m+n+1}{l+q} \binom{l}{m} \binom{l}{n} \\
&= m!\, n! \sum_{q=1}^{(m+n+1)} \sum_{j=q}^{(m+n+1)} \frac{(-1)^{j+1}}{j} \binom{m+n+1}{j} \binom{j-q}{m} \binom{j-q}{n} \\
&= m!\, n! \sum_{j=\max(m+1,n+1)}^{m+n+1} \frac{(-1)^{j+1}}{j} \binom{m+n+1}{j} \sum_{t=\max(m,n)}^{j-1} \binom{t}{m} \binom{t}{n} \\
&= \frac{(-1)^{m+n} (m!\, n!)^2}{(m+n+1)!}.
\end{aligned} \tag{4}$$

The double sum in the second last step was evaluated in Maple with the following code:

```
B:=proc(m,n) local j,t:add((-1)^j/j*binomial(n+m+1,j)
*add(binomial(t,n)*binomial(t,m),t=n..j-1),j=2..n+m+1):end:

ForB:=proc(m,n): (-1)^(m+n+1)*m!*n!/(m+n+1)!:end:
```

Using the Stirling inversion formula [6, 7]

$$\sum_{k=0}^{n} f(k) \begin{bmatrix} n \\ k \end{bmatrix} = g(n) \qquad \sum_{r=0}^{m} g(r) \begin{Bmatrix} m \\ r \end{Bmatrix} = f(m) \tag{5}$$

with Eq. (4) we get

$$\sum_{r=0}^{m} B_{m+r} \begin{bmatrix} m \\ r \end{bmatrix} = (-1)^m (m!)^2 \sum_{r=0}^{m} \frac{(-1)^r (r!)^2}{(m+r+1)!} \begin{Bmatrix} m \\ r \end{Bmatrix}. \tag{6}$$

Finally, the use of Eq. (5) in Eq. (6) gives the expression Eq. (1). $\square$

Acknowledgement

The authors would like to thank Prof. Doron Zeilberger (Rutgers University) for his kind help in evaluating an essential double sum using Maple.

References

[1] S. K. Jha, An identity involving the Bernoulli numbers and the Stirling numbers of the second kind, *Notes on Number Theory and Discrete Mathematics*, **26** (3), 160-162.
[2] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier, London (2012)
[3] J. Quaintance and H. W. Gould, *Combinatorial identities for Stirling numbers*, World Scientific, Singapore (2016).
[4] F. Qi and B. N. Guo, Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers, *Analysis* (Berlin) **34**, No. 3 (2014) 311–317.
[5] S. Fukuhara, N. Kawazumi, and Y. Kuno, Self-intersections of curves on a surface and Bernoulli numbers, *Osaka J. Math.*, **55**, No. 4 (2018), 761–768.
[6] H. W. Gould, Explicit formulas for the Bernoulli and Euler numbers, *J. London Math. Soc.* **2**, No. 2 (1972) 44–51.
[7] M. Z. Spivey, *The art of proving binomial identities*, CRC Press, Boca Raton, Fl, USA (2019).

International Institute of Information Technology, Hyderabad-500032, India
Email address: kumarjha.sumit@research.iiit.ac.in

ESIME-Zacatenco, Instituto Politécnico Nacional,, Edif. 4, 1er. Piso, Col. Lindavista 07738, CDMX, Mexico
Email address: jlopezb@ipn.mx