Rapid Communication

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An elementary proof of Euler's product expansion for the sine

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Abstract: We give a proof of the Euler's infinite product for the sine using elementary trigonometric identities, and Tannery's theorem for infinite products.

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MSC 2010: 33B10, 26A09, 40A20

Euler's infinite product for the sine is, for $x \in \mathbb{R}$, the identity

$$\frac{\sin x}{x} = \prod_{j=1}^{\infty}\Big(1 - \frac{x^2}{j^2\pi^2}\Big).$$

In the article [1], the author gives an elementary proof of the Euler's product expansion for the sine using a trigonometric identity and Tannery's theorem for infinite products. However, the author also makes use of the Wallis's product formula. Our proof here makes use of only trigonometric identities, and not Wallis's product formula, which is often obtained from the Euler's product expansion for the sine.

We begin with the following identity.

Theorem 1. *For all complex numbers z and all integers n, we have*

$$\sin nz = 2^{n-1}\sin z \sin\Big(z + \frac{\pi}{n}\Big)\sin\Big(z + \frac{2\pi}{n}\Big)\cdots\sin\Big(z + \frac{(n-1)\pi}{n}\Big). \tag{1}$$

Proof. We can write

$$\begin{aligned}\sin nz &= \frac{e^{niz} - e^{-niz}}{2i}\\ &= \frac{e^{-niz}(e^{2niz} - 1)}{2i}\\ &= \frac{e^{-niz}(e^{2iz} - 1)(e^{2iz} - e^{\frac{-2i\pi}{n}})\cdots(e^{2iz} - e^{\frac{-2(n-1)i\pi}{n}})}{2i}\\ &= k\cdot\sin z\sin\Big(z + \frac{\pi}{n}\Big)\sin\Big(z + \frac{2\pi}{n}\Big)\cdots\sin\Big(z + \frac{(n-1)\pi}{n}\Big),\end{aligned}$$

where

$$k = (2i)^{n-1}e^{\frac{-i\pi}{n}(1+2\cdots+(n-1))} = (2i)^{n-1}e^{\frac{-i\pi(n-1)}{2}} = 2^{n-1}. \qquad \square$$

Corollary 1. *For all integers n, we have*

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}. \tag{2}$$

Proof. To obtain the result, we divide both sides of (1) by $\sin z$ and let $z \to 0$. $\square$

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Now, letting $n = 2m+1$ in (1), we get

$$\begin{aligned}\sin nz &= 2^{n-1}\prod_{j=0}^{2m}\sin\Big(z+\frac{j\pi}{n}\Big) = 2^{n-1}(-1)^m\prod_{j=0}^{m}\sin\Big(z+\frac{j\pi}{n}\Big)\prod_{j=m+1}^{2m}\sin\Big(z+\frac{j\pi}{n}-\pi\Big)\\ &= 2^{n-1}(-1)^m\prod_{j=-m}^{m}\sin\Big(z+\frac{j\pi}{n}\Big)\\ &= (-1)^m\,2^{n-1}\sin z\prod_{j=1}^{m}\sin\Big(z+\frac{j\pi}{n}\Big)\sin\Big(z-\frac{j\pi}{n}\Big)\\ &= (-1)^m\,2^{n-1}\sin z\prod_{j=1}^{m}\Big(\sin^2 z-\sin^2\frac{j\pi}{n}\Big).\end{aligned}$$

Now, from equation (2) we can conclude $\prod_{j=1}^{m}\sin^2\frac{j\pi}{n} = \frac{n}{2^{n-1}}$. Using this and replacing z by $\frac{z}{n}$, we obtain

$$\sin z = n\sin\frac{z}{n}\prod_{j=1}^{m}\Big(1-\frac{\sin^2\frac{z}{n}}{\sin^2\frac{j\pi}{n}}\Big).$$

Next, we state and use the Tannery's theorem for infinite products

Theorem 2 (Tannery's theorem for infinite products [2]). *For all natural numbers $n\in\mathbb{N}$, let $\prod_{j=1}^{\infty}(1+a_j(n))$ be a convergent infinite product. If for all j, $\lim_{n\to\infty}a_j(n)=a_j$ and $|a_j(n)|\le M_j$ for all n where the series $\sum M_j$ converges, then*

$$\lim_{n\to\infty}\prod_{j=1}^{\infty}(1+a_j(n)) = \prod_{j=1}^{\infty}(1+a_j).$$

Now, let

$$a_j(n) = \frac{-\sin^2\frac{z}{n}}{\sin^2\frac{j\pi}{n}}.$$

Since

$$\frac{\sin x}{x}\ge 1-\frac{x^2}{6}\ge\frac{1}{2}\quad\text{for } 0\le x\le\frac{\pi}{2},$$

we have

$$n\sin\frac{j\pi}{n}\ge\frac{j\pi}{2}\quad\text{for } 1\le j\le\frac{n}{2},$$

that is,

$$\frac{1}{n^2\sin^2\frac{j\pi}{n}}\le\frac{4}{j^2\pi^2}\quad\text{for } 1\le j\le\frac{n}{2}.$$

But since

$$\sin^2\frac{z}{n}\le\frac{z^2}{n^2}\le\frac{l^2}{n^2},$$

where l is a constant greater than z for given real z, it follows that

$$|a_j(n)| = \frac{\sin^2\frac{z}{n}}{\sin^2\frac{j\pi}{n}}\le\frac{4l^2}{j^2\pi^2}.$$

Using the Tannery's theorem for infinite products, we can write

$$\sin z = z\lim_{n\to\infty}\prod_{j=1}^{\infty}\Big(1-\frac{\sin^2\frac{z}{n}}{\sin^2\frac{j\pi}{n}}\Big) = z\prod_{j=1}^{\infty}\lim_{n\to\infty}\Big(1-\frac{\sin^2\frac{z}{n}}{\sin^2\frac{j\pi}{n}}\Big) = z\prod_{j=1}^{\infty}\Big(1-\frac{z^2}{j^2\pi^2}\Big).$$

References

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