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AN IDENTITY FOR THE SUM OF INVERSES OF ODD DIVISORS OF n IN TERMS OF THE NUMBER OF REPRESENTATIONS OF n AS A SUM OF SQUARES

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Let $c_r(n)$ be the number of representations of a positive integer n as a sum of r squares, where representations with different orders and different signs are counted as distinct. We prove a combinatorial identity relating this quantity to the sum of inverses of odd divisors of n:

$$\sum_{\text{odd } d \mid n} \frac{1}{d} = \frac{1}{2} \sum_{r=1}^{n} \frac{(-1)^{n+r}}{r} \binom{n}{r} c_r(n).$$

1. Main result

Let n be a positive integer. We derive here the following identity for the sum of inverses of odd divisors of n, in terms of the the number $c_r(n)$ of representations of n as a sum of r squares (r ranging over the positive integers, and representations with different orders or different signs being counted as distinct).

$$\sum_{\text{odd } d \mid n} \frac{1}{d} = \frac{1}{2} \sum_{r=1}^{n} \frac{(-1)^{n+r}}{r} \binom{n}{r} c_r(n). \tag{1}$$

Recall that the $c_r(n)$ occur in the expansion of the r-th power of the function

$$\theta(q) := \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j} = \sum_{i=-\infty}^{\infty} (-1)^i q^{i^2} \quad (|q| < 1)$$

(see [2, 7.324], for example). More precisely, we have

$$\theta(q)^r = \sum_{n=0}^{\infty} c_r(n)(-1)^n q^n.$$

Let $B_{n,k} \equiv \mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ be the partial Bell polynomials [1, p. 134], defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

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Lemma 1. *For all positive integers n, we have*

$$2\sum_{\text{odd } d\,|\,n} = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k\,(k-1)!\,B_{n,k}(\theta'(0),\theta''(0),\dots,\theta^{(n-k+1)}(0)), \tag{2}$$

Proof. It is easy to see that

$$\log\theta(q)=\sum_{j=1}^{\infty}\log(1-q^j)-\sum_{j=1}^{\infty}\log(1+q^j)=-\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{q^{lj}}{l}+\sum_{j'=1}^{\infty}\sum_{l'=1}^{\infty}\frac{q^{l'j'}(-1)^{l'}}{l'}$$
$$=-\sum_{n=1}^{\infty}q^n\Bigg(\sum_{d|n}\frac{1-(-1)^d}{d}\Bigg).$$

Let $f(q)=\log q$. Using Faà di Bruno's formula (see [1, p. 137] or [3]), we have

$$\frac{d^n}{dq^n}f(\theta(q))=\sum_{k=1}^{n}f^{(k)}(\theta(q))\cdot B_{n,k}(\theta'(q),\theta''(q),\dots,\theta^{(n-k+1)}(q)). \tag{3}$$

Since $f^{(k)}(q)=\dfrac{(-1)^{k-1}\,(k-1)!}{q^k}$ and $\theta(0)=1$, letting $q\to 0$ in (3) gives the lemma. □

Lemma 2. *We have, for positive integers n,k,*

$$B_{n,k}(\theta'(0),\theta''(0),\dots,\theta^{(n-k+1)}(0))=(-1)^n\,\frac{n!}{k!}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}c_r(n). \tag{4}$$

Proof. We start with the generating function for the partial Bell polynomials [1, Equation (3a′) on p. 133], and proceed as follows to conclude (4):

$$\sum_{n=k}^{\infty}B_{n,k}(\theta'(0),\theta''(0),\dots,\theta^{(n-k+1)}(0))\frac{q^n}{n!}=\frac{1}{k!}\Bigg(\sum_{j=1}^{\infty}\theta^{(j)}(0)\frac{q^j}{j!}\Bigg)^k=\frac{1}{k!}(\theta(q)-1)^k$$
$$=\frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\theta(q)^r$$
$$=\frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{n=0}^{\infty}(-1)^n c_r(n)q^n \qquad \square$$

Proof of Equation (1). We combine (2) and (4) to obtain

$$2\sum_{\text{odd } d\,|\,n}\frac{1}{d}=(-1)^n\sum_{k=1}^{n}\frac{1}{k}\sum_{r=1}^{k}(-1)^r\binom{k}{r}c_r(n)=(-1)^n\sum_{r=1}^{n}(-1)^r c_r(n)\sum_{k=r}^{n}\frac{1}{k}\binom{k}{r}$$
$$=(-1)^n\sum_{r=1}^{n}\frac{(-1)^r}{r}\binom{n}{r}c_r(n),$$

having used that $\sum_{k=r}^{n}\frac{1}{k}\binom{k}{r}=\frac{1}{r}\binom{n}{r}$, a consequence of Pascal's formula $\binom{k}{r-1}=\binom{k+1}{r}-\binom{k}{r}$. □

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