A CONGRUENCE FOR THE NUMBER OF ALTERNATING PERMUTATIONS

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Abstract. We present a new proof of a result of Knuth and Buckholtz concerning the period of the number of alternating congruences modulo an odd prime. The proof is based on properties of special functions, specifically the polylogarithm, Dirichlet eta and beta functions, and Stirling numbers of the second kind.

1. Introduction

Definition 1.1. Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$. A permutation $a_1a_2\cdots a_n \in S_n$ is called *alternating* if $a_1 > a_2 < a_3 > a_4 < \cdots$, that is, we have $a_i < a_{i+1}$ for i even, and $a_i > a_{i+1}$ for i odd. Let A_n denote the number of alternating permutations in S_n.

The sequence $\{A_n\}$ has been studied by many authors and is listed as sequence A000111 [9] in the On-Line Encyclopedia of Integer Sequences. The first few values of A_n are $1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots$. The odd indexed terms of the sequence $\{A_n\}$ are the *tangent numbers* [10], and the even indexed terms are the *secant numbers* [11]. Knuth and Buckholtz [6, Theorem 1, 2, and 3] have studied the periodicity of these two sequences and obtained the same results which are in this paper, with the exception of the half-period result. Kummer [8, p. 372] obtained the periodicity and half-period results for the secant numbers. He did not discuss the congruence for tangent numbers, but his methods would work for the tangent numbers also.

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2. Main result

Our main result is the following theorem.

Theorem 2.1. *For every odd prime p and positive integers n and l,*

$$A_{l(p-1)+n} \equiv (-1)^{l\left(\frac{p-1}{2}\right)} A_n \pmod{p}. \tag{2.1}$$

To prove this theorem we first revisit some known related results.

3. Some Known Results

The exponential generating function for the number of alternating permutations is given in the following theorem.

Theorem 3.1 ([12])**.** *We have*

$$\sec z + \tan z = \sum_{n=0}^{\infty} \frac{A_n\, z^n}{n!}.$$

Using the expansions for $\tan z$ and $\sec z$ in terms of the Bernoulli numbers B_r and the Euler numbers E_r, respectively [5, equation (2.3) and equation (2.7)], we can conclude that

$$A_r = \begin{cases} (-1)^{\frac{r-1}{2}} \frac{2^{r+1}\left(2^{r+1}-1\right)B_{r+1}}{r+1}, & \text{when } r \text{ is odd;} \\ (-1)^{\frac{r}{2}} E_r, & \text{when } r \text{ is even.} \end{cases} \tag{3.1}$$

Definition 3.2. The *polylogarithm function* $\mathrm{Li}_s(z)$ is defined by the following power series in z

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \tag{3.2}$$

where $|z| < 1$, or $|z| = 1$ with $\Re(s) > 1$, where $\Re(s)$ denotes the real part of s.

The polylogarithm function is a special case of the *Lerch transcendent* $\Phi(\lambda, s, a)$ defined by

$$\Phi(\lambda, s, a) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+a)^s} \tag{3.3}$$

for any real $a > 0$, and complex λ and s with either $\lambda < 1$ for all $s \in \mathbb{C}$ or $|\lambda| = 1$ with $\Re(s) > 1$. Thus, we have

$$\mathrm{Li}_s(\lambda) = \lambda\, \Phi(\lambda, s, 1).$$

The series (3.3) extends to a function $\Phi(\lambda, s, a)$ which is analytic in λ and s for $\lambda \in \mathbb{C} \setminus [1, \infty)$ and for all complex s [4, Lemma 2.2].

For the negative integer values of s, the Lerch transcendent function is given in terms of the *Apostol–Bernoulli polynomials* $\beta_n(a, \lambda)$ [1, equation (6.5)] as

$$\Phi(\lambda, -r, a) = \frac{-\beta_{r+1}(a, \lambda)}{r+1}, \tag{3.4}$$

where the Apostol–Bernoulli polynomials can be defined by

$$\frac{z\, e^{az}}{\lambda\, e^z - 1} = \sum_{n=0}^{\infty} \beta_n(a, \lambda)\, \frac{z^n}{n!} \qquad (|z + \log \lambda| < 2\pi).$$

Thus, we have

$$\mathrm{Li}_{-r}(\lambda) = (-\lambda)\, \frac{\beta_{r+1}(a, \lambda)}{r+1}. \tag{3.5}$$

It is also known that [2, formula (3.7)]

$$\mathrm{Li}_{-r}(z) = \left(z \frac{\partial}{\partial z}\right)^r \frac{z}{1-z} = \sum_{k=1}^{r} k!\, S(r, k) \left(\frac{1}{1-z}\right)^{k+1} z^k, \tag{3.6}$$

where $S(r, k)$ are Stirling numbers of the second kind which can be defined by the recurrence $S(r+1, k+1) = (k+1)S(r, k+1) + S(r, k)$ with $S(n, 1) = 1$ and $S(1, r) = 0$ for $r > 1$.

We also require the following fact from [3, p. 937].

Lemma 3.3. *We have, for an odd prime p,*

$$S(n + l(p-1), k) \equiv S(n, k) \pmod{p}$$

with integers $1 \le k \le p-1$, $n \ge 1$, and $l \ge 1$.

4. Two Formulas for A_r

Lemma 4.1. *Let $i = \sqrt{-1}$. We have*

$$A_r = 2\, i^{r+1}\, \mathrm{Li}_{-r}(-i), \tag{4.1}$$

where $\mathrm{Li}_{-r}(\cdot)$ is a negative polylogarithm function.

Proof. Following equation (3.5), we have

$$\mathrm{Li}_{-r}(-i) = i\, \frac{\beta_{r+1}(1, -i)}{r+1}.$$

Let $[z^n]f(z)$ denote the coefficient of $\frac{z^n}{n!}$ in the Taylor series expansion of $f(z)$ within the circle of convergence. Then

$$\begin{aligned} \beta_r(1, -i) = [z^r] \left(\frac{z\, e^z}{e^z\,(-i) - 1}\right) &= [z^r](i) \left(\frac{z}{2}(\tanh z + 1) + \frac{iz}{2} \operatorname{sech} z\right) \\ &= i \left(2^{r-1}(2^r - 1)B_r + \frac{i\, r}{2} E_{r-1}\right), \end{aligned}$$

where in the last step we used the expansions of $\tanh z$ and $\operatorname{sech} z$ in terms of the Bernoulli numbers B_r and the Euler numbers E_r, respectively [5, equation (2.3) and equation (2.7)]. The above result allows us to conclude that

$$\mathrm{Li}_{-r}(-i) = -\left(2^r \frac{2^{r+1}-1}{r+1} B_{r+1} + i\frac{E_r}{2}\right).$$

As $B_{r+1} = 0$ when r is even and $E_r = 0$ when r is odd, an application of equation (3.1) assures the validity of equation (4.1). □

Corollary 4.2. *Let* $i = \sqrt{-1}$. *We have*

$$A_r = i^r \sum_{k=1}^{r} \frac{(-1)^k\, k!\, S(r,k)}{2^k} (i+1)^{k+1}. \tag{4.2}$$

Proof. We substitute $z = -i$ in equation (3.6), multiply by $2\, i^{r+1}$, and simplify to conclude equation (4.2). □

5. Proof of Theorem 2.1

Proof of Theorem 2.1. Letting $r = l(p-1)+n$, where p is an odd prime in equation (4.2), we get

$$\begin{aligned} A_{l(p-1)+n} = {} & i^{l(p-1)+n} \sum_{k=1}^{p-1} \frac{(-1)^k\, k!\, S(l(p-1)+n,k)}{2^{\frac{k-1}{2}}} \left(\frac{i+1}{\sqrt{2}}\right)^{k+1} \\ & + i^{l(p-1)+n} \sum_{k=p}^{l(p-1)+n} \frac{(-1)^k\, k!\, S(l(p-1)+n,k)}{2^{\frac{k-1}{2}}} \left(\frac{i+1}{\sqrt{2}}\right)^{k+1}. \end{aligned} \tag{5.1}$$

The first term on the right side of equation (5.1) is clearly congruent to $(-1)^{l\left(\frac{p-1}{2}\right)} A_n \pmod{p}$ in light of Lemmas 4.2 and 3.3.

Let $p \le k \le l(p-1)+n$. It is clear that $k!$ is a multiple of p and that the power of 2 in the prime factorization of $k!$ is at least $[k/2]$ ($[x]$ denotes the greatest integer $\le x$). Since p is an odd prime, the power of 2 in the prime factorization of the integer $k!/p$ is still at least $[k/2]$. Thus when k is odd, $k!/2^{(k-1)/2}$ is an integer which is a multiple of p, and the term $((i+1)/\sqrt{2})^{k+1} = e^{\frac{i\,\pi(k+1)}{4}}$ is a complex number with both real and imaginary parts being integers.

Suppose k is even. Then $k!/2^{k/2}$ is an integer which is a multiple of p, and the term

$$\sqrt{2}\left(\frac{i+1}{\sqrt{2}}\right)^{k+1} = \sqrt{2}\, e^{\frac{i\,\pi(k+1)}{4}}$$

is a complex number with both real and imaginary parts being integers. This allows us to conclude that the second term on the right side of the equation (5.1) is a multiple of p, and hence gives our result, equation (2.1). $\square$

6. Related Results

Remark 6.1 (Zhang's congruence)**.** Zhang [13] proved that $A_{p-1} \equiv -1 + (-1)^{\frac{p-1}{2}} \pmod{p}$ for all odd primes p. This congruence can also be proved by our method. We have

$$A_{p-1} = (-1)^{\frac{p-1}{2}} \sum_{k=1}^{p-1} \frac{(-1)^k\, k!\, S(p-1,k)}{2^k} (i+1)^{k+1}.$$

Using the explicit form for the Stirling numbers of the second kind [3], we have

$$(-1)^k\, k!\, S(p-1,k) = \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^{p-1} \equiv \sum_{j=1}^{k} (-1)^j \binom{k}{j} \equiv -1 \pmod{p}$$

for all $1 \le k \le p-1$. Thus, operating in the field $\mathbb{Z}/p\mathbb{Z}$, we see that

$$\begin{aligned} 2^{p-1} A_{p-1} &= (-1)^{\frac{p+1}{2}} \sum_{k=1}^{p-1} (i+1)^{k+1}\, 2^{p-1-k} \\ &= (-1)^{\frac{p+1}{2}} 2^{p-1} ((i-1) - 2i\, 2^{-\frac{p}{2}}\, e^{\frac{i\pi p}{4}}). \end{aligned}$$

Discarding the imaginary part in the above, we have

$$A_{p-1} \equiv (-1)^{\frac{p+1}{2}} (2^{\frac{p-1}{2}} \sqrt{2}\, \sin(p\pi/4) - 1) \pmod{p}.$$

Noting that $2^{\frac{p-1}{2}} \equiv (-1)^{\frac{p^2-1}{8}} \pmod{p}$ [7, p. 44] and $\sqrt{2}\, \sin(p\pi/4) = (-1)^{\frac{-(p-1)(p-3)}{8}}$ completes our alternative proof of Zhang's congruence.

Remark 6.2 (Generalization)**.** Theorem 2.1 can be generalized using Euler's generalization for Fermat's little theorem as follows. Let N be any natural number. Suppose n and l are positive integers with $n \ge N$. Then we have

$$A_{l\cdot\phi(p^N)+n} \equiv (-1)^{l\cdot\frac{\phi(p^N)}{2}} A_n \pmod{p^N}.$$

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