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A formula for the number of partitions of *n* in terms of the partial Bell polynomials

Sumit Kumar Jha[1]

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Abstract
We derive a formula for $p(n)$ (the number of partitions of n) in terms of the partial Bell polynomials using Faà di Bruno's formula and Euler's pentagonal number theorem.

Keywords Integer partitions · Partial Bell polynomials · Pentagonal numbers · Faà di Bruno's formula · Ramanujan's tau function

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1 Main result

Recall the classical partition function, denoted by $p(n)$, gives the number of ways of writing the integer n as a sum of positive integers, where the order of summands is not considered significant. For example, $p(4) = 5$, since there are 5 ways to represent 4 as sum of positive integers, namely, $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$.

We also recall another classical statistic, the (n, k)*th partial Bell polynomial* in the variables $x_1, x_2, \ldots, x_{n-k+1}$, denoted by $B_{n,k} \equiv \mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ ([2, p. 134], [1, Ch. 12]), defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

✉ Sumit Kumar Jha
kumarjha.sumit@research.iiit.ac.in

1 International Institute of Information Technology, Hyderabad 500032, India

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Cvijović [3] gives the following formula for calculating these polynomials:

$$B_{n,k+1} = \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k} \overbrace{\binom{n}{\alpha_1}\binom{\alpha_1}{\alpha_2}\cdots\binom{\alpha_{k-1}}{\alpha_k}}^{k} \times \cdots$$
$$\times\, x_{n-\alpha_1} x_{\alpha_1-\alpha_2} \cdots x_{\alpha_{k-1}-\alpha_k} x_{\alpha_k} \qquad (n \geq k+1, k = 1, 2, \ldots) \tag{1}$$

We prove the following here.

Theorem 1 *We have*

$$p(n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, k! \, B_{n,k}(\lambda_1, \lambda_2, \cdots, \lambda_{n-k+1}) \tag{2}$$

where

$$\lambda_m = \begin{cases} (-1)^{\frac{1+\sqrt{1+24m}}{6}} \, m! & \text{if } \frac{1+\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ (-1)^{\frac{1-\sqrt{1+24m}}{6}} \, m! & \text{if } \frac{1-\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Proof We begin by the following generating function [4, Eq. 22.13]

$$\sum_{n \geq 0} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}. \tag{4}$$

We recall the Euler's pentagonal number theorem [4, Eq. 7.8]

$$\begin{aligned} E(q) := \prod_{j=1}^{\infty} (1-q^j) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} \\ &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots. \end{aligned} \tag{5}$$

Let $f(q) = 1/q$. Using Faà di Bruno's formula ([2, p. 137], [1, Ch. 12]) we have

$$\frac{d^n}{dq^n} f(E(q)) = \sum_{k=0}^{n} f^{(k)}(E(q)) \cdot B_{n,k}\left(E'(q), E''(q), \ldots, E^{(n-k+1)}(q)\right). \tag{6}$$

Since $f^{(k)}(q) = \frac{(-1)^k k!}{q^{k+1}}$ and $E(0) = 1$, letting $q \to 0$ in the above equation gives

$$p(n)\, n! = \sum_{k=0}^{n} (-1)^k \, k! \, B_{n,k}\left(E'(0), E''(0), \ldots, E^{(n-k+1)}(0)\right).$$

Then Euler's pentagonal number theorem (5) gives us

$$E^{(m)}(0) = \lambda_m,$$

where λ_m is as defined in (3). □

Combining equations (1) and (2) we can conclude that

$$p(n) = -\theta_n + \sum_{k=1}^{n-1}(-1)^{k-1} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k} \theta_{n-\alpha_1}\theta_{\alpha_1-\alpha_2}\cdots\theta_{\alpha_{k-1}-\alpha_k}\theta_{\alpha_k},$$

where

$$\theta_m = \begin{cases} (-1)^{\frac{1+\sqrt{1+24m}}{6}} & \text{if } \frac{1+\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ (-1)^{\frac{1-\sqrt{1+24m}}{6}} & \text{if } \frac{1-\sqrt{1+24m}}{6} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1 *Let* $E(q)^r := \prod_{j=1}^{\infty}(1-q^j)^r = \sum_{n=0}^{\infty} p_r(n)q^n$ *with* $p_r(0) = 1$ *(see [5]). Then*

$$p(n) = \sum_{r=0}^{n}(-1)^r \binom{n+1}{r+1} p_r(n),$$

where by the virtue of the Faà di Bruno's formula (6) with $f(q) = q^l$ *we have*

$$p_l(n) = \frac{1}{n!}\sum_{k=0}^{l}\binom{l}{k} k!\, B_{n,k}(\lambda_1, \ldots, \lambda_{n-k+1}).$$

Proof We start with the generating function for the partial Bell polynomials [2, Eq. (3a') on p. 133] as follows:

$$\begin{aligned}
\sum_{n=k}^{\infty} B_{n,k}(\lambda_1, \ldots, \lambda_{n-k+1})\frac{q^n}{n!} &= \frac{1}{k!}\left(\sum_{j=1}^{\infty}\lambda_j \frac{q^j}{j!}\right)^k \\
&= \frac{1}{k!}(E(q) - 1)^k \\
&= \frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}E(q)^r \\
&= \frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\sum_{n=0}^{\infty} p_r(n)q^n
\end{aligned}$$

to conclude that

$$B_{n,k}(\lambda_1, \ldots, \lambda_{n-k+1}) = \frac{n!}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} p_r(n).$$

The above equation, together with the formula (2), gives us

$$\begin{aligned} p(n) &= \sum_{k=0}^{n} \sum_{r=0}^{k} (-1)^r \binom{k}{r} p_r(n) \\ &= \sum_{r=0}^{n} (-1)^r p_r(n) \sum_{k=r}^{n} \binom{k}{r} \\ &= \sum_{r=0}^{n} (-1)^r \binom{n+1}{r+1} p_r(n). \end{aligned}$$

□

Remark 1 Note that similar ideas are used in [6] with relation to partition zeta functions.

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