Discrete-Time Markov Chains (DTMCs)

Definition

A stochastic process $\{X_n, n \geq 0\}$ is said to be a Discrete-Time Markov Chain (DTMC) or simply a Markov Chain (MC) if

$$P\{X_{n+1} = j | X_n = i,\, X_{n-1} = i_{n-1},\, \ldots,\, X_0 = i_0\} = P\{X_{n+1} = j | X_n = i\}$$

for all states $i_0,\, i_1,\, \ldots,\, i_{n-1},\, i$ and j and all $n \geq 0$ (provided the conditional probabilities are defined).

► For a MC, both parameter and state spaces are discrete. The property given above is referred to as **Markov property**.
► For a MC, the conditional distribution of X_{n+1} (future) given the past $X_0,\, X_1,\, \ldots,\, X_{n-1}$ and present X_n depends only on the present X_n and not on the past $X_0,\, X_1,\, \ldots,\, X_{n-1}$.
► An MC describes a simple and highly useful form of dependency among the random variables (dependency extending only to the last known state of the process).

- The quantity $P\{X_{n+1} = j | X_n = i\} = p_{ij}(n)$ is referred to as the (one-step) transition probability and gives the conditional probability of making a transition from state i at time n to state j at time $n+1$.
- If $p_{ij}(n) = p_{ij}$ for all $n \geq 1$, then the MC is said to have stationary transition probabilities or the MC is a time-homogeneous MC (and we consider only time-homogeneous MCs).
- The matrix $P = ((p_{ij}))_{i,j \in S}$ is called (one-step) transition probability matrix (TPM). We have that $p_{ij} \geq 0$ for all i and j and $\sum_{j \in S}^{\infty} p_{ij} = 1$.

  ♦ P is a stochastic matrix.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & \cdots & \\ p_{21} & p_{22} & \cdots & \\ \vdots & & & \end{pmatrix} \end{array}$$

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Examples

- An MC whose state space is given by set of integers is said to be simple random walk (SRW) if for some $0 < p < 1$, $p_{i,i+1} = p = 1 - p_{i,i-1}$. It is simple symmetric random walk (SSRW) if $p = 1/2$.
- In a sequence of coin tosses, the number X_n of heads in the first n tosses.
- Consider a communication system that transmit the digits 0 and 1. Each digit, transmitted must pass several stages, at each stage of which there is a probability p that the digit entered will remain unchanged when it leaves.
- (A Gambling Model) Consider a gambler who at each play of the game either wins Re. 1 with probability p or losses Re. 1 with probability $1 - p$. Suppose that the gambler quits plays either when he goes broke or he attains a fortune of Rs. N.

- Fact: A random variable is probabilistically specified by its distribution. Likewise, a stochastic process is specified by its finite dimensional distributions (FDDs).
- An MC is specified by its initial distribution and its transition probabilities. Let $P\{X_0 = i\} = \mu_i$ for $i \in S$. Then

$$P\{X_0 = i_0,\, X_1 = i_1,\, \ldots,\, X_n = i_n\} = \left(\prod_{k=0}^{n-1} p_{i_k i_{k+1}}\right) \mu_{i_0}.$$

$$\text{LHS} = \underbrace{P\{X_n = i_n \mid X_0 = i_0, \ldots X_{n-1} = i_{n-1}\}} \cdot P\{X_0 = i_0, \ldots X_{n-1} = i_{n-1}\}$$

$$= p_{i_{n-1} i_n} \; P\{X_0 = i_0, \ldots X_{n-1} = i_{n-1}\}$$

$$= \vdots$$

$$= p_{i_{n-1} i_n}\, p_{i_{n-2} i_{n-1}} \cdots p_{i_0 i_1}\, P\{X_0 = i_0\}.$$

Chapman-Kolmogorov Equations

Consider an MC having state space S and one-step transition probabilities p_{ij} for $i,\, j \in S$. Let us define

$$p_{ij}^{(n)} = P\left\{X_n = j | X_0 = i\right\} = P\left\{X_{n+k} = j | X_k = i\right\}.$$

These are known as the n-step transition probabilities.

The Chapman-Kolmogorov (CK) equations are given by

$$p_{ij}^{(m+n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n)}$$

for all $m, n \geq 0$ and all $i,\, j \in S$.

If we denote n-step transition probability matrix by $P^{(n)}$, then

$$P^{(n+m)} = P^{(n)} P^{(m)} \implies P^{(n)} = P^n.$$

Note: $P^{(0)} = I$.

Example

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather condition. Suppose that if it is raining today, then it will rain tomorrow with probability 0.75. If it is not raining today, then it will rain tomorrow with probability 0.40. Calculate the probability that it will rain four days from today given that it is raining today.

$$P = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{bmatrix} 0.75 & 0.25 \\ 0.40 & 0.60 \end{bmatrix} \end{array}, \quad P^{(4)} = P^4 = \begin{bmatrix} \mathbf{0.6212} & 0.3788 \\ 0.6061 & 0.3938 \end{bmatrix}$$

states : $1 \to$ rain

$2 \to$ no rain

$p_{11}^{(4)} = ?$

Example

Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be put in any of these urns. What is the probability that there will be exactly 3 occupied urns after 9 balls have been distributed?

If X_n is the number of nonempty urns after n balls have been distributed, then $\{X_n\}$ is a MC with states $\{0, 1, 2, \ldots, 8\}$ and with $p_{ii} = i/8 = 1 - p_{i,i+1}, i = 0, 1, 2, \ldots, 8$ and the desired probability is $p_{03}^{(9)}$ which can be computed as 0.00756 using P^9.

$$P = \left(\cdots \right) \qquad P^9 = \left(\cdots \right)$$

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But, for our problem, observe that the first transition is deterministic (from 0 to 1) and hence the required probability is equal to $p_{13}^{(8)}$, we can simplify the problem by letting $Y_n = \max\{X_n + 1, 4\}, n \geq 0$ with state space $\{1, 2, 3, 4\}$ and TPM P as given below.

$$P = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad P^{(4)} = P^4 = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0 & 0.0039 & 0.0952 & 0.9009 \\ 0 & 0 & 0.0198 & 0.9802 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, $p_{13}^{(8)} = \sum_{j=1}^{4} p_{1j}^{(4)} p_{j3}^{(4)} = \mathbf{0.00756}$.

State Probabilities

Consider an MC having state space S and one-step transition probabilities p_{ij} for $i,\, j \in S$. Let us define

$$\pi_j^{(n)} = P\left\{X_n = j\right\}$$

as the probability of finding the system in state j at time n (also known as state probabilities). These are known as the n-step transition probabilities.

It can be shown that

$$\pi_j^{(m)} = \sum_{i \in S} \pi_i^{(m-1)} p_{ij}$$

for all $m \geq 1$ and all $i,\, j \in S$. In matrix notation,

$$\boldsymbol{\pi}^{(m)} = \boldsymbol{\pi}^{(m-1)}\mathbf{P}.$$

This means that

$$\boldsymbol{\pi}^{(m)} = \boldsymbol{\pi}^{(m-1)}\mathbf{P} = \boldsymbol{\pi}^{(m-2)}\mathbf{P}^2 = \cdots = \boldsymbol{\pi}^{(0)}\mathbf{P}^m,$$

where $\boldsymbol{\pi}^{(0)}$ is the initial state distribution.

Properties of MCs

- Accessibility: State j is said to be accessible from state i if there exists $n \geq 0$ such that $p_{ij}^{(n)} > 0$, where $p_{ij}^{(0)} = \delta_{ij}$.

  ► If j is not accessible from i, then $P\,(\text{Ever be in } j|\text{starting from } i) = 0$.

- Communication: Two states i and j are said to communicate if i and j are accessible from each other, *i.e.*, there exist $m \geq 0$ and $n \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.

  Notation: $i \to j$: j is accessible from i.
  $i \leftrightarrow j$: i and j communicate.

  ► Communication is an equivalence relation (i.e., it satisfies (Reflexivity) $i \leftrightarrow i$, (Symmetry) $i \leftrightarrow j \iff j \leftrightarrow i$, and (Transitivity) $i \leftrightarrow k$ and $k \leftrightarrow j \implies i \leftrightarrow j$).
  ► This relation partitions the state space into equivalence classes (known as communicating classes).

- Irreducible/Reducible: An MC is said to be **irreducible** if all states communicate with each other, *i.e.*, there is a single communicating class. A chain is **reducible** otherwise.

- Closed: A subset A of the state space S is said to be **closed** if no one-step transition is possible from any state in A to any state in A^c.
- Absorbing State: If a closed set A contains only a single state, then the state is called as **absorbing** state.
  ▶ A state i is absorbing if and only if $p_{ii} = 1$.
- If S is closed and does not contain any proper subset which is closed, then we have an *irreducible* MC. If S contains proper subsets that are closed, then the chain is *reducible*.
  ■ If a closed subset of a reducible MC contains no closed subsets of itself, then it is referred to as an *irreducible sub-MC* (and these may be studied independently of other states).

Examples

$$P_1 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Hitting Time: For any $A \subseteq S$, the hitting time T_A is defined by

$$T_A = \inf \{n \geq 1 : X_n \in A\},$$

with the convention that $\inf \phi = \infty$.

  - T_A is the first time after 0, when the chain enters A.
  - T_A is also called **first passage/return time** to A.
  - $T_{\{i\}}$ will be denoted by T_i, $i \in S$.

- Classification of States:
  - A state i is called **recurrent** (or persistent) if $P\{T_i < \infty | X_0 = i\} = 1$.
    - State i is recurrent if and only if $f_{ii} = P\{X_n = i \text{ for some } n \geq 1 | X_0 = i\} = 1$.
  - A state i is called **transient** if $P(T_i < \infty | X_0 = i) < 1$.
  - A recurrent state i is called **null recurrent** if $E(T_i | X_0 = i) = \infty$ and **positive recurrent** (or non-null recurrent) if $E(T_i | X_0 = i) < \infty$.

- Let $f_{ii}^{(n)}$ be the probability that a chain starting in state i returns for the first time to i in n transitions. Then $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$. For a recurrent state i, since $f_{ii} = 1$, $\{f_{ii}^{(n)}\}$ defines the first-return time or recurrence time distribution and the mean recurrence time is $M_{ii} = E(T_i | X_0 = i) = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$.
  - Recurrent state i is positive recurrent if $M_{ii} < \infty$ and null recurrent if $M_i = \infty$.

- Periodicity: The period of a state i is defined by the greatest common divisor of all integers $n \geq 1$ for which $p_{ii}^{(n)} > 0$, *i.e.*,

$$d(i) = \begin{cases} \text{gcd}\left\{n \geq 1 : p_{ii}^{(n)} > 0\right\} & \text{if } \left\{n \geq 1 : p_{ii}^{(n)} > 0\right\} \neq \phi \\ 0 & \text{if } \left\{n \geq 1 : p_{ii}^{(n)} > 0\right\} = \phi. \end{cases}$$

If $d(i) = 1$, then the state i is said to be aperiodic.
If $d(i) = \gamma > 1$, then the state i is said to be periodic with period γ.

Example

Consider an MC with $S = \{0, \pm 1, \pm 2, \ldots\}$ and with $p_{i,\,i+1} = a$, $p_{i,\,i-1} = b$, $p_{ii} = c$, where $a + b + c = 1$, $a > 0$, $b > 0$, $c \geq 0$.
Determine the period of states (different cases).

$c > 0 \; : \; d(i) = 1$

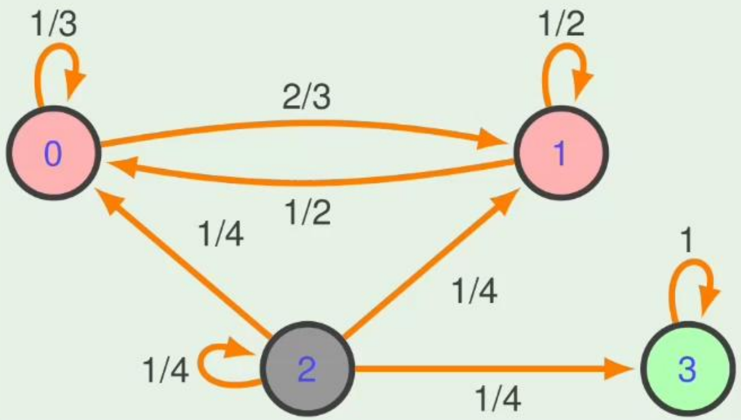
$c = 0 \; : \; d(i) = 2$

State Diagram

- A state transition diagram (or simply state diagram) for an MC is a directed graph where the nodes represent the states and the edges represent possible one-step transitions. More precisely, the state diagram contains an edge from node i to node j if and only if $p_{ij} > 0$.
  ♦ Very useful tool and we will use extensively.

Example

Consider an MC with state space $S = \{0, 1, 2, 3\}$ and with TPM $P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.



MC has three classes $\{0, 1\}$, $\{2\}$ and $\{3\}$ and hence reducible.

Some Results

1. (Number of Visits) For any state i, let N_i be the number of visits to state i. Then,
   a i recurrent implies $P\{N_i = \infty | X_0 = i\} = 1$.
   b i transient implies $P\{N_i = n | X_0 = i\} = f_{ii}^n (1 - f_{ii})$ for $n = 0, 1, 2, \ldots$, where $f_{ii} = P\{T_i < \infty | X_0 = i\}$ is the probability of returning to i starting from i. Thus $N_i | X_0 = i \sim Geo(1 - f_{ii})$.

   Corollary: A state i is recurrent iff $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ and transient iff $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$.

   - $P\{X_n = i \text{ for infinitely many } n | X_0 = i\} = 1$ or 0 iff recurrent or transient.
2. If the state space S is finite, then at least one state must be recurrent.
3. Positive recurrence, null recurrence and transience are all class properties. Also, all the states in a class have the same period.
4. All states of a finite irreducible MC are positive recurrent.
5. Let i be recurrent and $i \to j$. Then $f_{ji} = P\{T_i < \infty | X_0 = j\} = 1$ and j is recurrent. [Note: Not true if i is transient.]
6. Suppose that $\{X_n\}$ is irreducible and recurrent. Then for all $i \in S$, $P_\mu \{T_i < \infty\} = 1$ for any initial distribution μ.