Lab Assignment 1

I. Basic Understanding for CVX

CVX is a one of the most used solvers for convex optimization, and the aim of this lab is to understand how to use CVX. In addition, the lab assignment is to use CVX to confirm the theory of convex optimization learned during the lectures. Please refer to http://cvxr.com/cvx/ for more details about this toolbox.

Task 1: Study the example, equality_constr_norm_min.m.

In this Matlab code, the problem of norm minimization with equality constraints is solved

$$\begin{aligned} \min . \quad & ||\mathbf{A}\mathbf{x} - \mathbf{b}||_p \\ \text{s.t.} \quad & \mathbf{C}\mathbf{x} = \mathbf{d} \end{aligned} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{R}^{q\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{d} \in \mathbb{R}^q$ and $\mathbf{p} \in \mathbb{R}_{++}$.

The code file consists of three parts. The first part is to generate the variables $\mathbf{A}$, $\mathbf{b}$, $\mathbf{C}$, $\mathbf{d}$ and p. The second part is to use the cvx solver to solve problem (1), where $\mathbf{x}$ is the main output, denoted in the code by x. It is worth pointing out that the optimal dual solution $\boldsymbol{\nu}$ is also provided as one of the outcomes, denoted by y in the code. The third part is to display the outcome. For this task, you need to learn how to use the cvx solver, understand the outputs of the solver, and how to manipulate the parameters of the optimization, including $\mathbf{A}$, $\mathbf{b}$, $\mathbf{C}$, $\mathbf{d}$ and p, particularly the impact of the choice of p on the problem. The aim of Task 1 is to get familiar with CVX and there is no need to be included into your report.

II. Least Square Problem

Recall that the least square problem can be formulated as follows:

$$\min . \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 \tag{2}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$.

Task 2.1: Follow the example in the previous section and write a code to solve problem (2) by using CVX. In the report, please include the code and the results (please use $n = 10$).

n = 100;

```
A = randn(2*n,n);
b = randn(2*n,1);
cvx_begin
    variable x(n)
    minimize( norm( A*x-b ) ) cvx_end
solution_ana=inv(A'*A)*A'*b
```

Task 2.2: Use the KKT conditions to solve the least square problem and find p^* and $\mathbf{x}^*$. In the report, provide the steps to find the closed-form expression for the optimal solution.

Task 2.3: In your report, compare the solution found via CVX and the analytical solution and state your findings.

III. Quadratic Program Problem

Recall that the quadratic program problem can be formulated as follows:

$$\text{min.} \quad \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x} + r \tag{3}$$

$$\text{s.t.} \quad \mathbf{G}\mathbf{x} \preceq \mathbf{h}, \tag{4}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{5}$$

where $\mathbf{P} \in \mathbb{S}_+^n$, $\mathbf{q} \in \mathbb{R}^n$, $\mathbf{G} \in \mathbb{R}^{m\times n}$, $\mathbf{h} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{p\times n}$, $\mathbf{b} \in \mathbb{R}^p$.

Task 3.1: Follow the example in the previous section and write a code to solve problem (3) by using CVX. In the report, please include the code and the results (please use $n = 10$).

Task 3.2: Write the KKT conditions for the quadratic program problem.

Task 3.3: In your report, provide the code which is to verify the KKT conditions by using the outcomes of CVX, and state your findings.

IV. Water Filling Problem

In communication systems, the power allocation problem can be formulated as a type of water filling problems. Particularly, denote P_i by the power allocated to the i-th subchannel, and h_i ($h_i > 0$) by the channel gain. The power allocation problem can be formulated as follows:

$$\max. \quad \sum_{i=1}^{n} \log(1 + h_i P_i) \tag{6}$$

$$\text{s.t.} \quad P_i \geq 0, \tag{7}$$

$$\sum_{i=1}^{n} P_i = P \tag{8}$$

where n denotes the overall number of the subchannels, and P denotes the power budget.

Task 4.1: Reformulate it as the water-filling problem and write a code to solve problem (6) by using CVX. In the report, please include the code and the results (please use $n = 10$).

Task 4.2: Use the KKT conditions to solve the problem and find p^* and $\mathbf{x}^*$. In the report, provide the steps to find the closed-form expression for the optimal solution.

Task 4.3: In your report, provide the code which is to verify the KKT conditions by using the outcomes of CVX, and state your findings.

Task 4.5: In your report, plot an interpretation figure following the one shown in the lecture notes (by using your generated h_i).