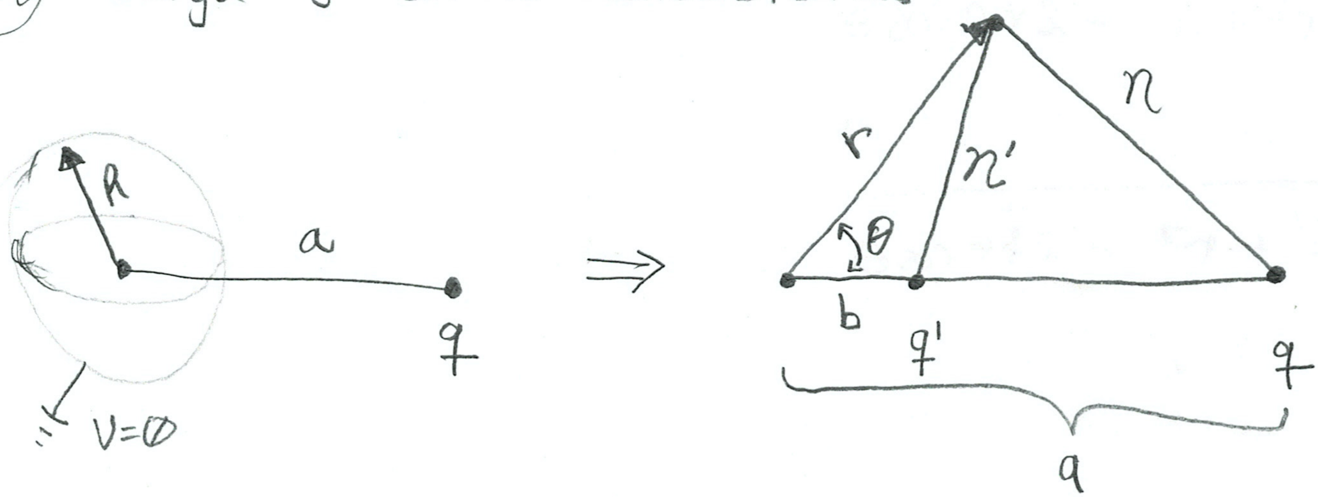
2º Carga y esfera conductora:



el potencial generado por las dos cargas es

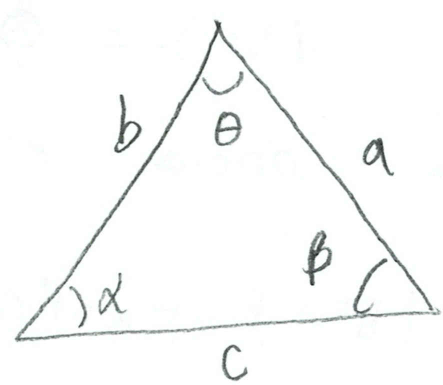
$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{n} + \frac{q'}{n'}\right)$$

debe cumple las condiciones de frontera:

1º $V(R) = \emptyset$

2º $\lim\limits_{|r|\to\infty} V(r) = \emptyset$

Nota: recordar la ley de cosenos:



- $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$
- $b^2 = a^2 + c^2 - 2ac\cos(\beta)$
- $c^2 = a^2 + b^2 - 2ab\cos\theta$

$$r^{*} = \sqrt{r^2 + a^2 - 2ra\cos\theta}$$

$$r' = \sqrt{b^2 + r^2 - 2br\cos\theta}$$

entonces

$$V(R) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{R^2 + a^2 - 2Ra\cos\theta}} + \frac{q'}{\sqrt{b^2 + R^2 - 2bR\cos\theta}}\right) = 0$$

$$\therefore \quad q = -q'$$

y $\sqrt{R^2 + a^2 - 2Ra\cos\theta} = \sqrt{R^2 + b^2 - 2bR\cos\theta}$

$$\rightarrow \quad a^2 - 2Ra\cos\theta = b^2 - 2bR\cos\theta$$

$\cancel{(a^2 - b^2)}$ $a = b$ pero esto nos provocaría

que $V(r) = 0$.

→ otra opción es

$$q^2(b^2 + R^2 - 2bR\cos\theta) = q'^2(R^2 + a^2 - 2Ra\cos\theta)$$

Factorizando:

$$R^2 q^2\left(1-\left(\frac{q'}{q}\right)^2\right) + b^2 q^2\left(1-\left(\frac{q'a}{qb}\right)^2\right)$$

$$-2Rb\cos\theta\, q^2\left(1-\left(\frac{q'}{q}\right)^2\frac{a}{b}\right) = 0$$

debe ser válido para todo θ:

$$\therefore \quad 1-\left(\frac{q'}{q}\right)^2\frac{a}{b} = 0$$

$\hookrightarrow \quad q'^2 = \frac{b}{a}q^2$, esto hace cero el útimo término

sostituyendo entonces en la ecuación anterior

$$R^2\left(1-\left(\frac{q'}{q}\right)^2\right) + b^2\left(1-\left(\frac{q'a}{qb}\right)^2\right) = R^2\left(1-\frac{b}{a}\right) + b^2\left(1-\frac{a}{b}\right) = 0$$

por lo tanto

$$R^2 - \frac{R^2 b}{a} + b^2 - ab = 0$$

$$\hookrightarrow \quad b^2 - b\left(\frac{R^2}{a}+a\right) + R^2 = 0$$

$$\lambda = + \left(\frac{R^2}{a} + a\right) \pm \frac{\sqrt{\left(\frac{R^2}{a} + a\right)^2 - 4R^2}}{2}$$

$$= \left(\frac{R^2}{a} + a\right) \pm$$

$$a\left[R^2 - \frac{R^2 b}{a} + b^2 - ab\right] = \emptyset$$

$$\hookrightarrow \quad aR^2 - R^2 b + ab^2 - a^2 b = \emptyset$$

$$ab^2 - (R^2 + a^2)b + aR^2 = \emptyset$$

$$b = + \frac{(R^2 + a^2)}{2a} \pm \frac{\sqrt{(R^2 + a^2)^2 - 4a^2R^2}}{2a} = \frac{R^2 + a^2}{2a} \pm \frac{\sqrt{(R^2 - a^2)^2}}{2a}$$

$$= \frac{R^2 + a^2}{2a} \pm \frac{R^2 - a^2}{2a} = \frac{R^2}{a} \; / \; a \quad //$$

entonces la carga es

$$q'^2 = \frac{b}{a} q^2 = \frac{R^2}{a^2} q^2$$ por lo que

$$q' = \frac{R}{a} q \quad y \quad q' = -\frac{R}{a} q$$

por lo que

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \left(\frac{R}{a}\right)\frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}}\right]$$

con $b = \frac{R^2}{a}$ reescribimos

$$-\left(\frac{R}{a}\right)\frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} = \frac{-q}{\left(\frac{a}{R}\right)\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}} = \frac{-q}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-ra\cos\theta}}$$

finalmente

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{\sqrt{R^2+\left(\frac{ra}{R}\right)^2-2ra\cos\theta}}\right\}$$

- Calculando la ~~densidad de~~ carga superficial inducida:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

$$\sigma = \frac{q}{4\pi R}\left(R^2-a^2\right)\left(R^2+a^2-2Ra\cos\theta\right)^{-3/2}$$

- La carga total inducida

$$q_{ind} = \int \sigma \, da = \frac{q}{4\pi R}\left(R^2-a^2\right)\int \left(R^2+a^2-2Ra\cos\theta\right)^{-3/2} R^2 \operatorname{sen}\theta \, d\theta d\phi$$

$$= \frac{q}{2a}\left(a^2-R^2\right)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}} - \frac{1}{\sqrt{R^2+a^2-2Ra}}\right]$$

$$= \frac{q}{2a}\left(a^2-R^2\right)\left[\frac{1}{(a+R)} - \frac{1}{a-R}\right]$$

con $a > R$

$$= \frac{q}{2a}\left(a^2-R^2\right)\left[\frac{a-R-a-R}{\left(a^2-R^2\right)}\right] = -\frac{R}{a}q \equiv q'$$

- Trabago de la configuración:

La fuerza que siente q debido a q' es

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q\,q'}{(a-b)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{R}{a}q^2\frac{1}{\left(a-\frac{R^2}{a}\right)^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2\,Ra}{(a^2-R^2)^2}$$

$$W = \frac{q^2R}{4\pi\varepsilon_0}\int_{\infty}^{a}\frac{a'}{(a'^2-R^2)^2}\,da' = \frac{q^2R}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{1}{(a'^2-R^2)}\right]_{\infty}^{a}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{2(a^2-R^2)}$$

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