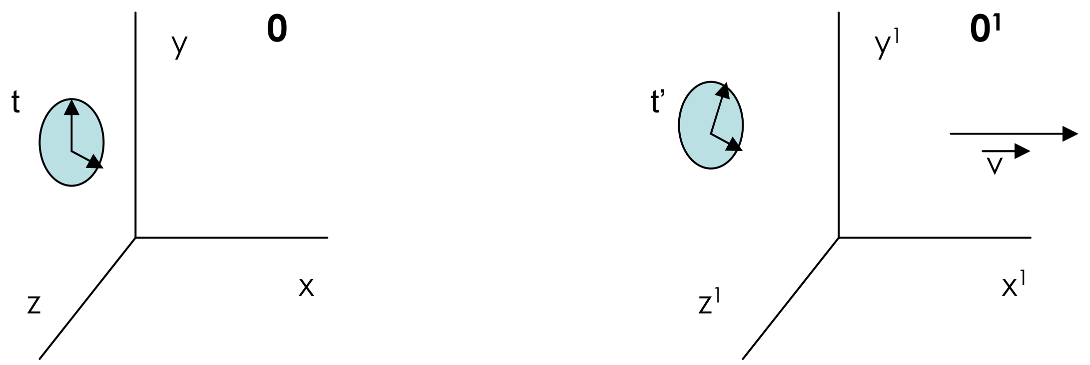
RELATIVISTIC KINEMATICS

Resumè of special relativity

Principle of Relativity

- the laws of physics are invariant in all inertial frames of reference (i.e., non-accelerating frames of reference);
- the speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.



$$\beta = \frac{\mathrm{v}}{c} \quad ; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x \equiv (ct; \vec{x}) \qquad\qquad x' \equiv (ct'; \vec{x}')$$

Lorentz transformation L(β) transform $x \equiv (ct; x)$ in $x' \equiv (ct'; x')$

Accordingly to $\quad x' = L(\beta)x$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{pmatrix}$$

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(x - \beta ct)$$

$$y' = y$$

$$z' = z$$

det(L) = $\gamma^2 - \beta^2\gamma^2$ **= 1**

The inverse **Trasformation:** $x = L^{-1}(\beta)x' = L(-\beta)x'$

Is defined by:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma ct' + \beta\gamma x' \\ \beta\gamma ct' + \gamma x' \\ y' \\ z' \end{pmatrix}$$

In the limit of v<<c we obtain the classical Galileo transformation

$x' = x - vt$

$t' = t$

$A = [a_0, a_1, a_2, a_3] = [a_0, \mathbf{a}]$

$B = [b_0, b_1, b_2, b_3] = [b_0, \mathbf{b}]$

$$A \cdot B = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$$

The product of two four-vectors is a relativistic invariant

$$A' \cdot B' = a'_0 b'_0 - a'_1 b'_1 - a'_2 b'_2 - a'_3 b'_3 =$$
$$= \gamma^2 (a_0 - \beta a_1)(b_0 - \beta b_1) - \gamma^2 (a_1 - \beta a_0)(b_1 - \beta b_0) -$$
$$a_2 b_2 - a_3 b_3 = \gamma^2 (1 - \beta^2)(a_0 b_0 - a_1 b_1) - a_2 b_2 - a_3 b_3 =$$
$$A \cdot B$$

Some important consequences for HEP

- Length contraction (Lorentz contraction)

if $d'=x'_2-x'_1$ is the distance measured in O' of a bar at rest in O' then

$x'_2-x'_1 = \gamma(x_2 - x_1) - \beta\gamma c(t_2 - t_1)$

But the Observer in O measures x_2 and x_1 simultaneously, at $t_2 = t_1$. Hence:

$$d' = \gamma(x_2 - x_1) = \gamma d$$

$$d = \frac{d'}{\gamma} \rightarrow d < d'$$

Some important consequences for HEP

- Time dilation

if $T' = t'_2 - t'_1$ is the time interval between two instants measured in O' in the same point $x'_2 = x'_1$, the Observer O measures

$cT = c(t_2 - t_1) = \gamma c(t'_2 - t'_1) + \beta\gamma(x'_2 - x'_1) = \gamma cT'$

$$\boxed{\begin{array}{l} T = \gamma T' \\ T > T' \end{array}}$$

The time interval measured in the rest frame is called «proper time»

- $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ is a relativistic invariant

If in O' the measurement is done in the same point

$ds^2 = cdt'^2$

Exercises

- Muons have mass of 106 MeV and lifetime of 2.2×10^{-6} s. If a muon is produced in cosmic rays with momentum 10 GeV/c and at a quote of 10 m, evaluate in the LAB and in the muon reference frame the probability that the muon reach the earth before decaying.

- In laboratories, beams of particles with known average lifetime τ are transported, before decaying, over distances many times greater than the Galilean limit of the decay distance: $c\tau$. Show that charged π produced at a primary target with a momentum of 200 GeV / c can be transported over distances, for example, of 300 m with a loss due to decays of less than 3%.

Four-vectors

- $$g^{\mu\nu} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \qquad x^{\mu} \equiv g^{\mu\nu} x_{\nu}$$

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2$$

- $$x^{\mu} \equiv \left(ct;\vec{r}\right) \quad ; \quad x_{\mu} \equiv \left(ct;-\vec{r}\right)$$

$$dx^{\mu} \cdot dx_{\mu} = ds^2$$

$$u^{\mu} = \frac{dx^{\mu}}{ds} \Rightarrow \begin{cases} u^{\mu} \equiv \left(\gamma;\gamma\vec{\beta}\right) \\ u^{\mu}u_{\mu} = 1 \end{cases}$$

$$p^{\mu} = m_0 c u^{\mu} \Rightarrow p^{\mu}p_{\mu} = \frac{E^2}{c^2} - p^2 = m_0^2 c^2$$

Energy-Momentum

The relativistic relation between the total energy E, the vector 3-momentum **p** and the rest mass m for a free particle is:

$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$

or, in natural units:

$$E^2 = \vec{p}^2 + m^2$$

The components of the 3-momentum and the energy can be written as components of an energy-momentum 4-vector P :

$P_0 = E \quad P_1 = p_x \quad P_2 = p_y \quad P_3 = p_z$

whose square modulus equals the squared rest mass:

$$P^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2 = E^2 - \vec{p}^2 = m^2$$

in units $\hbar = c = 1$.

Main relations between energy, momentum, velocity, etc.

$$i)\ \vec{p}=\gamma m_0\vec{v} \qquad ii)\ E=\gamma m_0 c^2 \qquad iii)\ \vec{v}=c^2\frac{\vec{p}}{E}$$

$$iv)\ \gamma=\frac{E}{m_0 c^2} \qquad v)\ \gamma\vec{v}=\frac{\vec{p}}{m_0} \qquad vi)\ \vec{\beta}=c\frac{\vec{p}}{E}$$

$$vii)\ \gamma\vec{\beta}=\frac{1}{c}\cdot\frac{\vec{p}}{m_0} \qquad viii)\ T=m_0 c^2(\gamma-1)$$

$$ix)\ E=T+m_0c^2=m_0c^2(\gamma-1)+m_0c^2$$

$$E=m_0c^2\gamma$$

Reference Frames

- LAB system: generally is the system in which measurement are performed. Is the R.F. of the Observer.

 In the LAB system, usually target is at rest

Lab: $p_1=(E_1;\mathbf{p}_1)$; $p_2(M;\mathbf{0})$;

$$P_{tot} = (E_1+M;\mathbf{p}_1)$$

- Center of Mass Frame C.M.: is defined by the condition

CdM = $$\sum_k \vec{p}_k = \vec{P}_{Tot} = \vec{0}$$

In the case we have just one particle, the CM is the system in which the particle is at rest and is the natural system to describe particle decay

$p_1^*=(E_1^*;\mathbf{p}^*)$; $p_2^*=(E_2^*;-\mathbf{p}^*)$; $P^*=(E^*;\mathbf{0})$

dove $E^* = E_1^* + E_2^*$

$$P^{*2}_{CdM} = E^{*2} = P_{LAB}^2 = m_1^2 + M^2 + 2E_1M$$

Invariant Mass

- Let's consider a system of N particles

$$p_k^\mu = \left(E_k; \vec{p}_k\right) \quad ; \quad \mathrm{P}^\mu = \sum_k p_k^\mu = \left(\sum_k E_k; \sum_k \vec{p}_k\right)$$

- $P^\mu P_\mu$ = Relativistic Invariant
- $P^\mu P_\mu$ = Invariant Mass of the N particles

$$\mathrm{P}^\mu \mathrm{P}_\mu = \left(E_1 + E_2 + \ldots. + E_N\right)^2 - \left(\vec{p}_1 + \vec{p}_2 + \ldots.. + \vec{p}_N\right)^2$$

- In the CMS, by definition:

$$\vec{P}_{Tot}^* = 0 \quad \Rightarrow \quad P^{\mu *} = \left(E_{Tot}^*; \vec{0}\right)$$

$$\text{Invariant Mass} = \mathrm{P}^\mu \mathrm{P}_\mu = \mathrm{E}^{*2}_{\mathrm{Tot}}$$

1947: Discovery of the π - meson (the "real" Yukawa particle)

Observation of the $\pi^+ \to \mu^+ \to e^+$ decay chain in nuclear emulsion exposed to cosmic rays at high altitudes

Nuclear emulsion: a detector sensitive to ionization with ~1 μm space resolution (AgBr microcrystals suspended in gelatin)

In all events the muon has a fixed kinetic energy (4.1 MeV, corresponding to a range of ~ 600 μm in nuclear emulsion) $\Rightarrow$ two-body decay

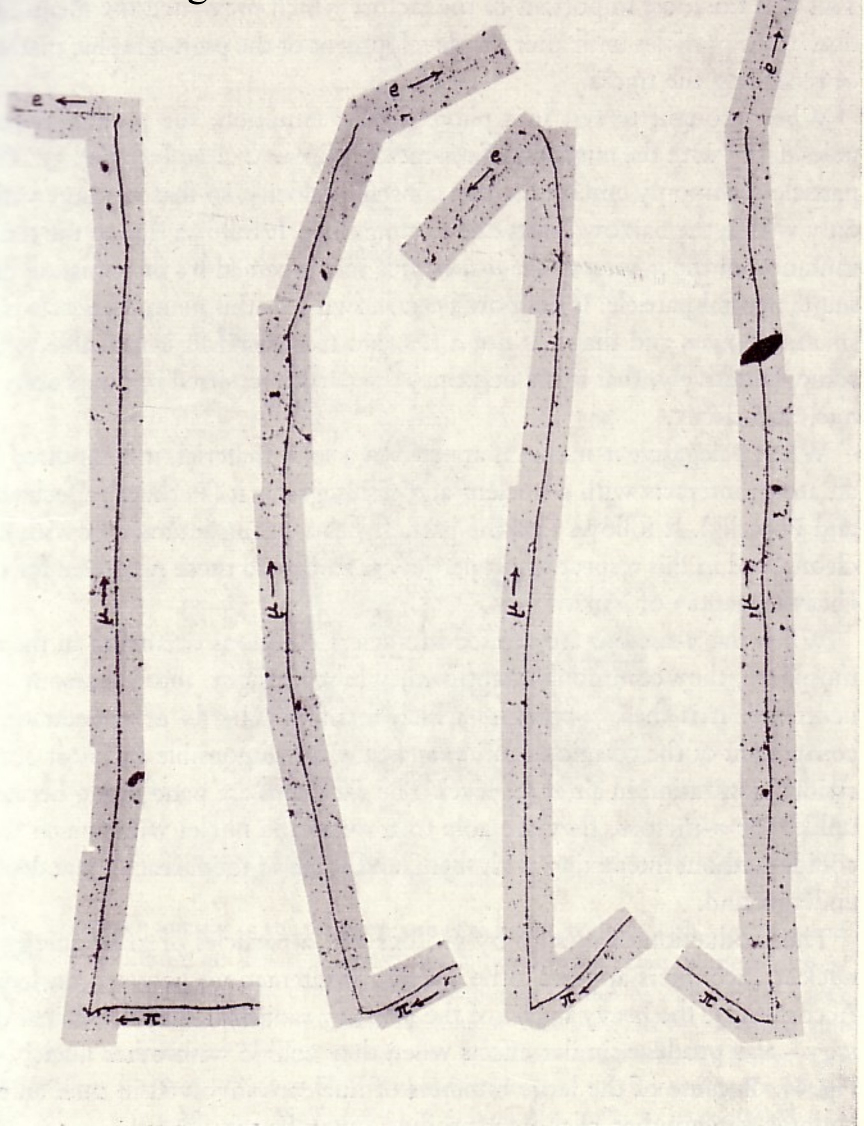
$m_\pi = 139.57$ MeV/c^2 ; spin = 0
Dominant decay mode: $\pi^+ \to \mu^+ + \nu$
(and $\pi^- \to \mu^- + \bar{\nu}$)
Mean life at rest: $\tau_\pi = 2.6 \times 10^{-8}$ s = 26 ns

π^- at rest undergoes nuclear capture, as expected for the Yukawa particle

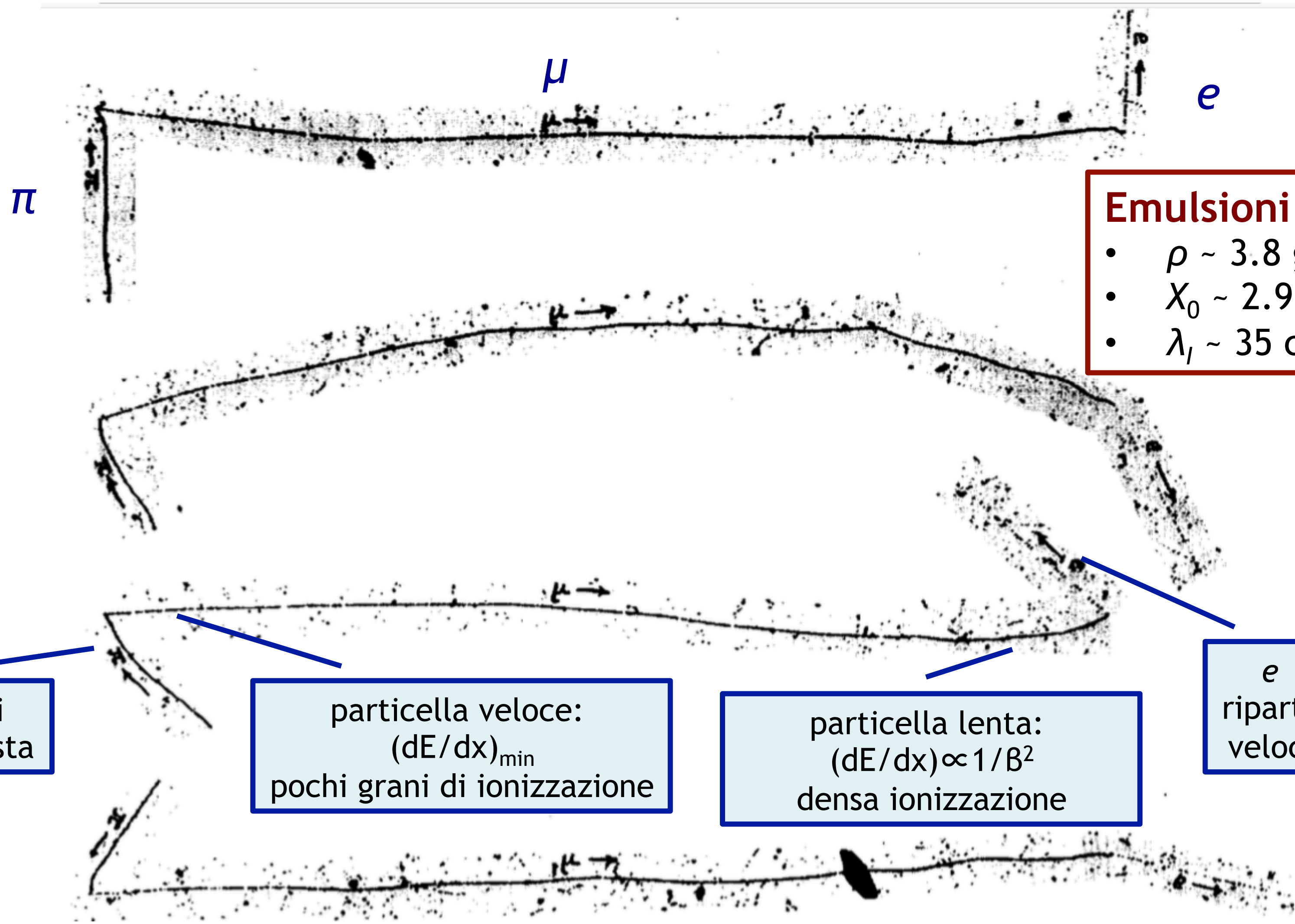
A neutral π – meson ($\pi°$) also exists:
m ($\pi°$) = 134. 98 MeV /c^2
Decay: $\pi° \to \gamma + \gamma$, mean life = 8.4×10^{-17} s

π – mesons are the most copiously produced particles in proton – proton and proton – nucleus collisions at high energies

Four events showing the decay of a π^+ coming to rest in nuclear emulsion



Osservazione del π (Lattes, Occhialini, Powell 1947)



Emulsioni nucleari
- ρ ~ 3.8 g/cm^2
- X_0 ~ 2.9 cm
- λ_I ~ 35 cm

π si arresta

particella veloce:
$(dE/dx)_{min}$
pochi grani di ionizzazione

particella lenta:
$(dE/dx) \propto 1/\beta^2$
densa ionizzazione

e riparte veloce

Invariant mass application

- We measure the muon momentum to be 29.3 MeV/c from nuclear emulsion
- If we assume the new particle decay $X \rightarrow \mu \nu_\mu$

$$p_\mu = p_\nu$$

- By applying the Invariant Mass definition we get:

$$(M_x c^2)^2 = E^{*2}_{Tot} = (E_\mu + E_{\nu\mu})^2$$

$$M_x c^2 = \sqrt{p_\mu^2 c^2 + m_\mu^2 c^4} + \mathrm{p}_\nu \mathrm{c} = \sqrt{29.3^2 + 105.6^2} + 29.3 =$$

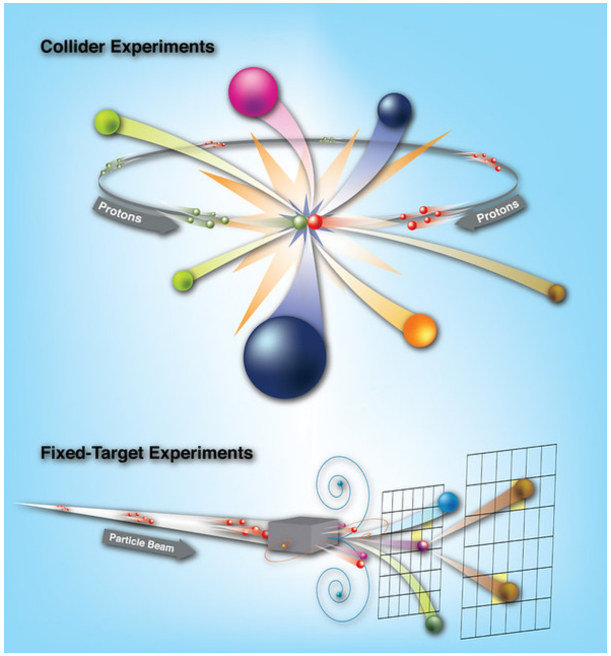
138.9 MeV/c^2

The experiments of Conversi, Pancini e Piccioni

L. Alvarez, Nobel Lecture, 1968:

As a personal opinion, I would suggest that modern particle physics started in the last days of World War II, when a group of young Italians, Conversi, Pancini, and Piccioni, who were hiding from the German occupying forces, initiated a remarkable experiment. In 1946, they showed that the 'mesotron' which had been discovered in 1937 by Neddermeyer and Anderson and by Street and Stevenson, was not the particle predicted by Yukawa as the mediator of nuclear forces, but was instead almost completely unreactive in a nuclear sense. Most nuclear physicists had spent the war years in military-related activities, secure in the belief that the Yukawa meson was available for study as soon as hostilities ceased. But they were wrong.

Fixed Target vs Collider experiment



- Fixed Target
  - P_2=0 and assume $m_{1,2}$<<E_1

$$P^\mu P_\mu = \Big|_{Nel\ Lab.} (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 =$$

$$= m_1^2 + p_1^2 + m_2^2 + 2m_2E_1 - p_1^2 \simeq 2m_2E_1$$

$$\mathbf{s} = 2m_2E_1$$

- At Collider, under the hypothesis of E_1>>$m_{1,2}$ $|\mathbf{p}_{1,2}| = |\mathbf{p}| \approx \mathbf{E}_{1,2}$

$$s = (p_1^\mu p_{1\mu} + p_2^\mu p_{2\mu} + 2p_1^\mu p_{2\mu})$$

$$= E_1^2 - |\vec{p}_1|^2 + E_2^2 - |\vec{p}_2|^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$$

$$= m_1^2 + m_2^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$$

$$= m_1^2 + m_2^2 + 2(E_1E_2 - |\vec{p}_1||\vec{p}_2|\cos\theta)$$

nella ipotesi che $E_1 >> m_1, m_2$ avremo $|p_{1,2}| = |p| \approx E_{1,2}$.

$$s = 2(E^2 - E^2\cos\theta)$$

$$s = 4E^2$$

Exercise

- A 100 GeV proton beam collide against a hydrogen target, evaluate √s and compare with the value obtained for colliding protons.
  - Fixed target

$$\sqrt{s} = \sqrt{2E_1 m_2} \approx \sqrt{2 \cdot 100 \cdot 1} \approx 14\ \mathrm{GeV}$$

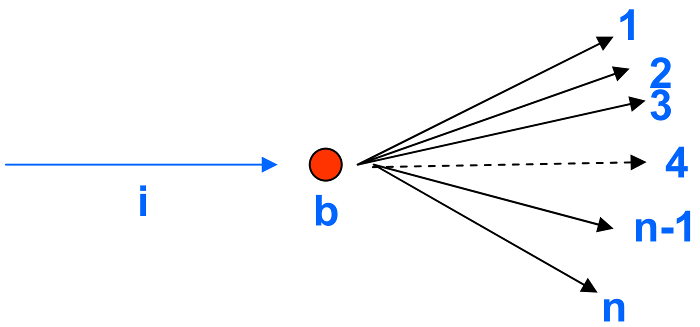
  - Collider

$$\sqrt{s} = 2 \cdot 100 = 200\ \mathrm{GeV}$$

In a collider the total cms energy increases linearly with the beam energy, whereas in fixed targed it increases as the square root of the beam energy.

Energy threshold

- In a nuclear/particle collision or reaction in general we can produce N particles



- In the initial state (lab sys)

$$E^{*2} = \left(E_i + m_b\right)^2 - \left(p\right)_i^2 = 2m_b E_i + m_i^2 + m_b^2 =$$

$$= 2\mathrm{m_b}\left(T_i + m_i\right) + m_i^2 + m_b^2 = 2\mathrm{m_b} T_i + \left(m_i + m_b\right)^2$$

- In the final state (CM system)

$$E^{*2} = \left[\sum_1^N {}_f \left(T_f^* + m_f\right)\right]^2$$

Energy threshold

- In order for the reaction to be energetically possible, it is necessary to have at least enough energy in the CM to produce the particles at rest

$$E^{*2} = \left[\sum_{1}^{N}{}_{f}\left(T_f^{*} + m_f\right)\right]^2 \geq \left[\sum{}_{f} m_f\right]^2$$

$$\boxed{T_i \geq T_{soglia} = \frac{\left(\sum_f m_f\right)^2 - \left(m_i + m_b\right)^2}{2m_b}}$$

- This relation defines the Threshold Energy of a reaction

Threshold energy: the discovery of antiprotons

$$T_s = \frac{\left[\left(4m_p\right)^2 - \left(2m_p\right)^2 \right]}{2m_p} = 6m_p = 5.6\ \text{GeV}$$

Threshold energy: the discovery of antiprotons

- Antiprotons have been discovered in the collision of a proton beam with an Hydrogen target. Proton in the hydron may considered in first aproximation as free

$$p + p \to p + p + p + \overline{p}$$

- The threshold energy for this process is:

$$T_s = \frac{\left[(4m_p)^2 - (2m_p)^2 \right]}{2m_p} = 6m_p = 5.6 \text{ GeV}$$

- If we consider the fermi momentum of the proton inside a Cu target, p_{FERMI}=0.24 GeV/c which is randomly distributed wrt the projectile direction

Threshold energy: the discovery of antiprotons

- The maximum value (minimum) of Ts will be for p_{Fermi} momentum parallel (antiparallel) to p_{proton}

$$E_F = \sqrt{p_F^2 + m_p^2} \quad ; \quad \mathrm{p}_{\mathrm{Lab}} = \left(E_p + E_F ; \vec{p}_p + \vec{p}_F\right)$$

$$p_{Lab}^2 = 2m_p^2 + 2E_pE_F - 2\vec{p}_p \bullet \vec{p}_F =$$

$$= 2\left(m_p^2 + E_pE_F \pm p_p p_F\right) \geq 16 m_p^2$$

$$m_p^2 + E_pE_F \pm p_p p_F \geq 8m_p^2 \;\rightarrow\; E_pE_F \pm p_p p_F \geq 7m_p^2$$

By approximating

$$\mathrm{E}_{\mathrm{F}} = m_p + \frac{p_F^2}{2m_p} \quad \text{and} \quad \mathrm{p}_{\mathrm{p}} \cong E_p$$

Threshold energy: the discovery of antiprotons

$$E_p \geq \frac{7m_p}{1 \pm \frac{p_F}{m_p} + \frac{p_F^2}{2m_p^2}} \quad ; \quad \mathrm{E_p} = T + m_p$$

$$T_{Min} = 4.2 \text{ GeV} \quad ; \quad \mathrm{T_{Max}} = 7.5 \text{ GeV}$$

$$E_{pMin} = 5.11 \text{ GeV} \qquad E_{pMax} = 8.46 \text{ GeV}$$

Exercises

- Consider the reaction

$$\overline{p} + p \rightarrow \Lambda + \overline{\Lambda} \text{ with } \left|\vec{\mathrm{p}}_{\overline{\mathrm{p}}}\right| = 0.65 \text{ GeV/c}$$

Tell if it is possible or not.

- Evaluate the energy threshold for the reactions

$$\gamma + N \rightarrow e^{+} + e^{-} + N$$

$$\pi^{-} + p \rightarrow \wedge^{0} + k^{0}$$

$$\nu_{\mu} + e^{-} \rightarrow \mu^{-} + \nu_{e}$$

$$e^{+} + e^{-} \rightarrow \mu^{+} + \mu^{-}$$ Consider also the case of a circular collider

CM vs LAB system

- Velocity of the CMF wrt LAB system

$$\mathbf{P}_{tot} = [(\textstyle\sum_k E_k)/c, \sum_k \overrightarrow{p_k}]$$

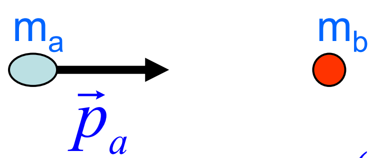
$$\mathbf{P}^*_{tot} = [(\textstyle\sum_k E^*_k)/c, \overrightarrow{0}] = [\sqrt{s}/c, \overrightarrow{0}]$$

$$\begin{pmatrix} \sqrt{s}/c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_{c.m.} & -\beta_{c.m.}\gamma_{c.m.} & 0 & 0 \\ -\beta_{c.m.}\gamma_{c.m.} & \gamma_{c.m.} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (\sum_k E_k)/c \\ |\sum_k \overrightarrow{p_k}| \\ 0 \\ 0 \end{pmatrix}$$

$$\beta_{c.m.} = \frac{|\sum_k \overrightarrow{p_k}|c}{\sum_k E_k} = \frac{|\overrightarrow{p}^{lab}_{tot}|c}{E^{lab}_{tot}} \qquad \gamma_{c.m.} = \frac{1}{\sqrt{1-\beta^2_{c.m.}}} = \frac{E^{lab}_{tot}}{\sqrt{s}}$$

Reference system

- In the lab:



In the CM

$$\vec{P}^{*}_{Tot} = 0$$

- We got

$$s = (E_a + E_b)^2 - p_a^2 = m_a^2 + p_a^2 + m_b^2 + 2E_a m_b - p_a^2 =$$
$$= m_a^2 + m_b^2 + 2E_a m_b$$

Evaluate β of CM

$$P^{\mu} \equiv \begin{pmatrix} E_a + m_b \\ \vec{p}_a \end{pmatrix} \quad ; \quad P^{*} \equiv \begin{pmatrix} \sqrt{s} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{E_a} + m_b \\ 0 \\ 0 \\ p_a \end{pmatrix} \equiv \begin{pmatrix} \gamma_{CdM} & 0 & 0 & \beta_{CdM}\gamma_{CdM} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_{CdM}\gamma_{CdM} & 0 & 0 & \gamma_{CdM} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E_a + m_b = \gamma_{CdM}\sqrt{s} \quad \mathrm{e} \quad \mathrm{p_a} = \beta_{CdM}\gamma_{CdM}\sqrt{s}$$

$$\beta_{CdM} = \frac{\mathrm{p_a}}{\gamma_{CdM}\sqrt{s}} \quad \mathrm{e} \quad \gamma_{CdM} = \frac{E_a + m_b}{\sqrt{s}}$$

$$\beta_{\mathrm{CdM}} = \frac{p_a}{\frac{(E_a + m_b)}{\sqrt{s}}\sqrt{s}} = \frac{p_a}{(E_a + m_b)}$$

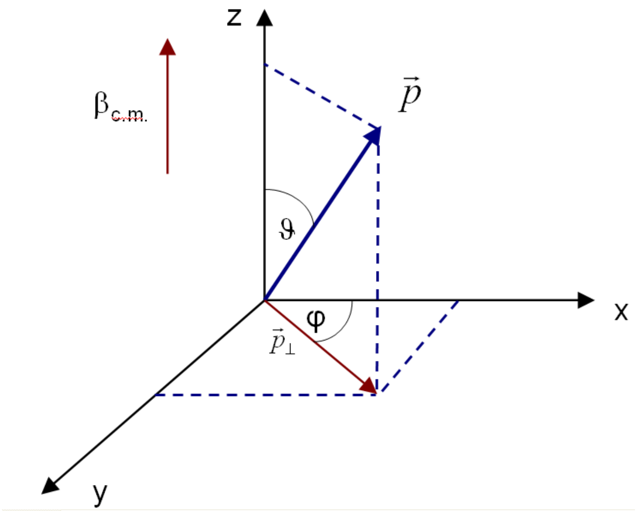
Compare CM and LAB system

Collision	Momentum Beam1 (GeV/c)	Momentum Beam2 (GeV/c)	E^*_{TOT}(GeV)	P*(GeV)	β	γ
e - e	2	0				
e - e	9	3				
e - e	2	2				
p-p	2	0				
e-p	2	0				
e-p	30	800				

E^*_{TOT} – total energy in CMF
P* - momentum in CMF
β_0 – velocity of the CM in the lab frame

Transverse momentum

- Consider the four-momentum of a particle in the transformation from the CMS to the LabS, assuming for simplicity that the CM moves parallel to the z axis



$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} E \\ |\overrightarrow{p}|sin\theta\ cos\phi \\ |\overrightarrow{p}|sin\theta\ sin\phi \\ |\overrightarrow{p}|cos\theta \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} E^* \\ |\overrightarrow{p}^*|sin\theta^* cos\phi^* \\ |\overrightarrow{p}^*|sin\theta^* sin\phi^* \\ |\overrightarrow{p}^*|cos\theta^* \end{pmatrix}$$

$$E = \gamma E^* + \beta\gamma p^* \cos\vartheta^* \qquad (1)$$

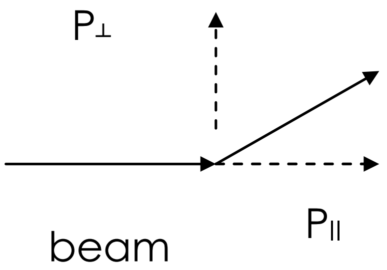
$$p \sin\vartheta \cos\varphi = p^* \sin\vartheta^* \cos\varphi^* \qquad (2)$$

$$p \sin\vartheta \sin\varphi = p^* \sin\vartheta^* \sin\varphi^* \qquad (3)$$

$$p \cos\vartheta = E^* \beta\gamma + \gamma p^* \cos\vartheta^* \qquad (4)$$

Transverse momentum

The Transverse Momentum = momentum component orthogonal to z axis
is a relativistic invariant



$$|\overrightarrow{p}|^2 sin^2\theta(cos^2\phi + sin^2\phi) = |\overrightarrow{p}^*|^2 sin^2\theta^*(cos^2\phi^* + sin^2\phi^*)$$

$0 \le \theta \le \pi$

$0 \le \theta^* \le \pi$:

$$p_\perp = |\overrightarrow{p}|sin\theta = |\overrightarrow{p}^*|sin\theta^* = p_\perp^*$$

cosφ = cosφ∗ and sinφ = sinφ∗
the azimuth angle φ around an axis is a relativistic invariant for Lorentz transformations along the axis itself.

Angle transformations

- $\cos\varphi = \cos\varphi*$ and $\sin\varphi = \sin\varphi*$

the azimuth angle φ around an axis is a relativistic invariant for Lorentz transformations along the axis itself.

$$tg\theta = \frac{sin\theta^*}{\gamma_{c.m.}\,(\beta_{c.m.}E^*/|\overrightarrow{p}^*| + cos\theta^*)}$$

$$\boxed{tg\,\vartheta = \frac{\sin\vartheta^*}{\gamma(\beta/\beta^* + \cos\vartheta^*)}}$$

The speed of the particle in the CM, $\beta^* = |-\mathbf{p}^*|/E^*$, is independent from $\beta_{c.m.}$

Angle transformations

- $\beta_{c.m.} > \beta^*$ denominator >0 for all $0 \leq \theta_* \leq \pi \rightarrow 0 \leq \theta < \pi/2$ the particle moves forward in the Lab system
- Since $\theta=0$, for $\theta^* =0$ and for $\theta^* =\pi$, → it should exist, in the LAB system a $\theta_{max} < \pi/2$. We can evaluate θ_{max} :

$$\frac{d(tg\theta)}{d\theta^*} = \frac{1 + cos\theta^* \left(\beta_{c.m.}/\beta^*\right)}{\gamma_{c.m.} \left(\beta_{c.m.}/\beta^* + cos\theta^*\right)^2} = 0$$

$$cos\theta^* = -\beta^*/\beta_{c.m.}$$

$$tg\theta_{max} = \frac{\beta^*}{\gamma_{c.m.}\sqrt{\beta_{c.m.}^2 - (\beta^*)^2}}$$

- The energy of the particle

$$E(\theta_{max}) = \gamma_{c.m.}(E^* + \beta_{c.m.}|\overrightarrow{p}^*|cos\theta^*) = \gamma_{c.m.}(E^* - |\overrightarrow{p}^*|\beta^*) =$$

$$= \gamma_{c.m.}\left(E^* - \frac{|\overrightarrow{p}^*|^2}{E^*}\right) = m^2\left(\frac{\gamma_{c.m.}}{E^*}\right) = m\frac{\gamma_{c.m.}}{\gamma^*}$$

Angle transformations

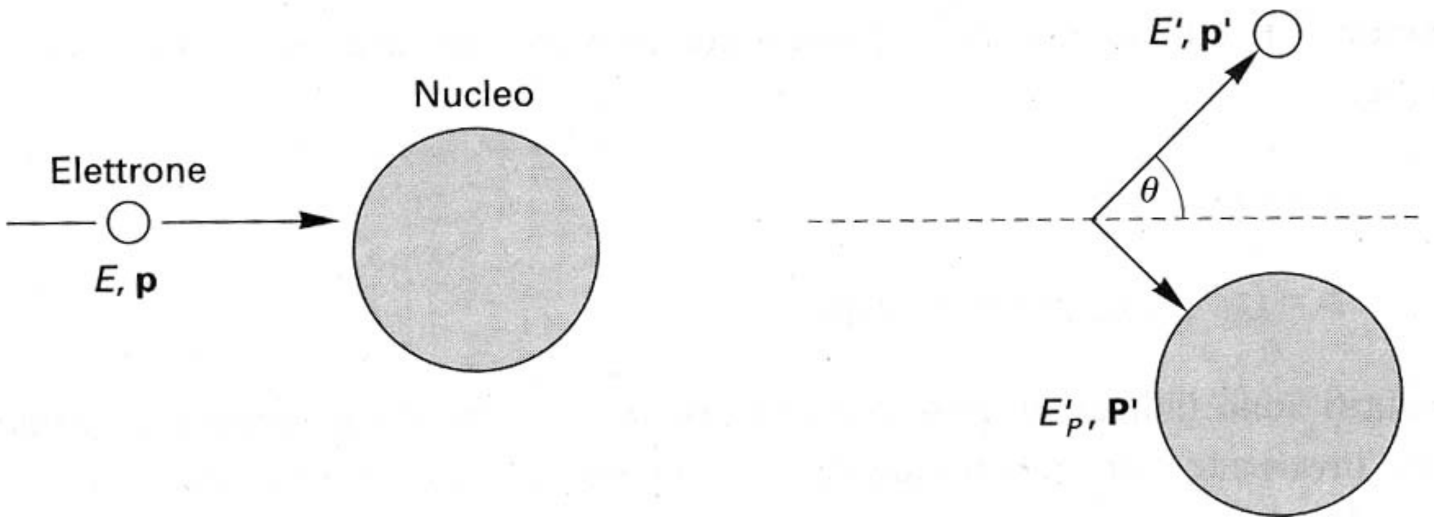
- $\beta_{c.m.} < \beta^*$ in this case the velocity of the particle in the CM can cancel the boost of the CM itself, allowing in the laboratory angles $\theta > \pi / 2$
  $\theta = \pi/2$ if $\cos\theta^* = -\beta_{c.m.}/\beta^*$

  It does not exist a θ_{max} since the derivative is always positive

- $\beta_{c.m.} = \beta^*$
  in this limit case, $\cos\theta^* = -1$ corresponds in the laboratory to the angle θ_{max} which is just $\pi/2$.

  It is interesting to note how in this configuration the particle, which in the CM travels in the opposite direction to the motion of the CM itself ($\theta^* = -\pi$), is instead at rest in the laboratory system; in fact its β^* exactly cancels the boost of the center of mass.

Elastic scattering



- From energy-momentum conservation
- p+P = p'+P' where: p=(E/c;**p**), p'=(E'/c;**p'**),
  P=(mc;**0**), P'=(E'/c;**P'**)

$p^2 + 2pP + P^2 = p'^2 + 2p'P' + P'^2$

- For an elastic collision
- $p^2 = p'^2 = m_e^2 c^2$; $P^2 = P'^2 = Mc^2$ per cui : pP = p'P'

Elastic scattering

- Experimentally we detect the deflected electron:

pP = p'(p+P-p') = p'p + p'P - $m_e^2c^2$

If we multiply each member for c^2

$EMc^2 = E'E - \mathbf{pp'}c^2 + E'Mc^2 - m_e^2c^4$

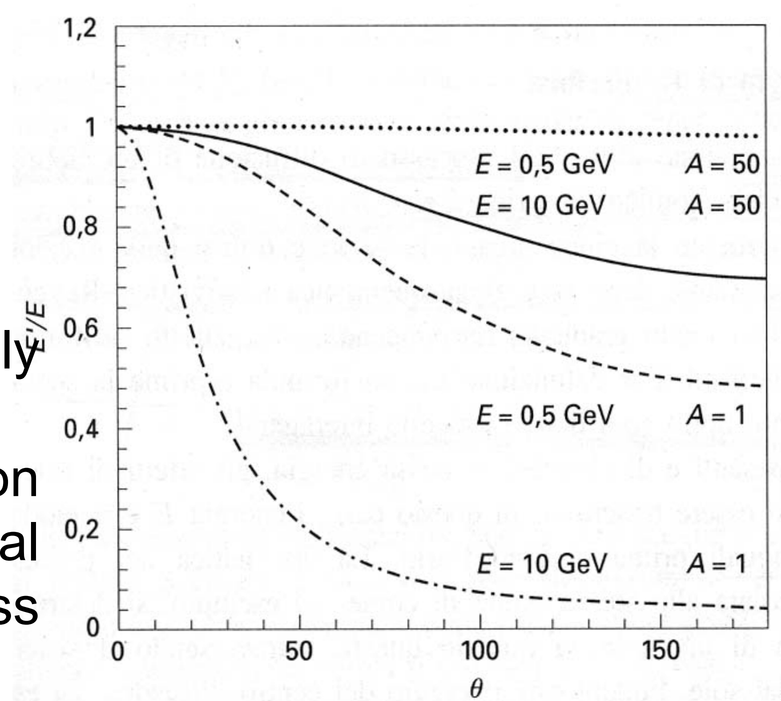
- For high energy we can neglet $m_e^2c^4$ and E≅pc

$EMc^2 = EE'(1-\cos\theta) + E'Mc^2$

$$E' = \frac{E}{1+\frac{E}{Mc^2}(1-\cos\vartheta)}$$

the energy of the diffused electron uniquely depends on the diffusion angle

the nucleus recoil energy, (E - E′), depends on E/M and in particular increases as the initial energy of the electron increases wrt the mass of the nucleus

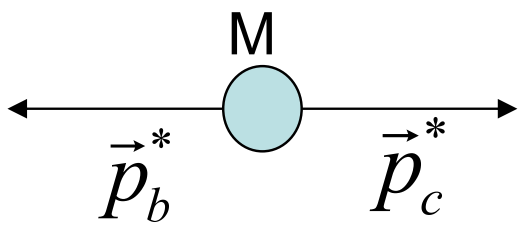


Decay

$$a \rightarrow b + c$$

with M_a, m_b and m_c masses of the three particles.
In the CMF a is at rest and the four-momentum conservation gives:

$$M_a = E_b^* + E_c^*$$



$$\vec{0} = \vec{p}_b^{\,*} + \vec{p}_c^{\,*}$$

$$P = (E^*_b + E^*_c; \mathbf{0}) \quad ; \; p^*_b = p^*_c = p^*$$

$$M_a = (p^{*2}_b + m_b^2)^{1/2} + (p^{*2}_c + m_c^2)^{1/2}$$

$$= (p^{*2} + m_b^2)^{1/2} + (p^{*2} + m_c^2)^{1/2}$$

Decay

$M_a^2 + p^{*2} + m_b^2 - 2M_a(p^{*2}+m_b^2)^{1/2} = p^{*2} + m_c^2$

$M_a^2 + m_b^2 - m_c^2 = 2M_a(p^{*2}+m_b^2)^{1/2}$

$\mathbf{M_a^4 + (m_b^2 - m_c^2)^2 + 2M_a^2(m_b^2 - m_c^2) = 4M_a^2(p^{*2} + m_b^2)}$

$$p^{*2} = \frac{M_a^4 - 2M_a^2\left(m_b^2 + m_c^2\right) + \left(m_b^2 - m_c^2\right)^2}{4M_a^2}$$

$$p^{*2} = \frac{\left[M_a^2 - \left(m_b + m_c\right)^2\right] \bullet \left[M_a^2 - \left(m_b - m_c\right)^2\right]}{4M_a^2}$$

$$E_b^* = \sqrt{p^{*2} + m_b^2} = \frac{M_a^2 + \left(m_b^2 - m_c^2\right)}{2M_a}$$

$$E_c^* = \sqrt{p^{*2} + m_c^2} = \frac{M_a^2 - \left(m_b^2 - m_c^2\right)}{2M_a}$$

Decay

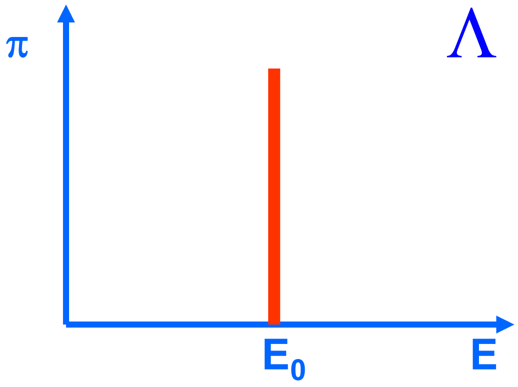
- In the case: $m_b = m_c = m$

$$E^*_b = E^*_c = E^*/2 \quad ; \quad p^{*2} = (E^*/2)^2 - m^2$$

$\Rightarrow$ In two body decays:

In the rest frame of the decaying particle, once the mass of the two daughters are known, the module of their momentum and hence their energy are fixed

The decay is mono-energetic



$$\Lambda \rightarrow p + \pi^-$$

π(E) = probability density function
The probability that π^- energy is E_0 is equal to 1 and it is equal to zero for any other value:
$\pi(E) = \delta(E - E_0)$

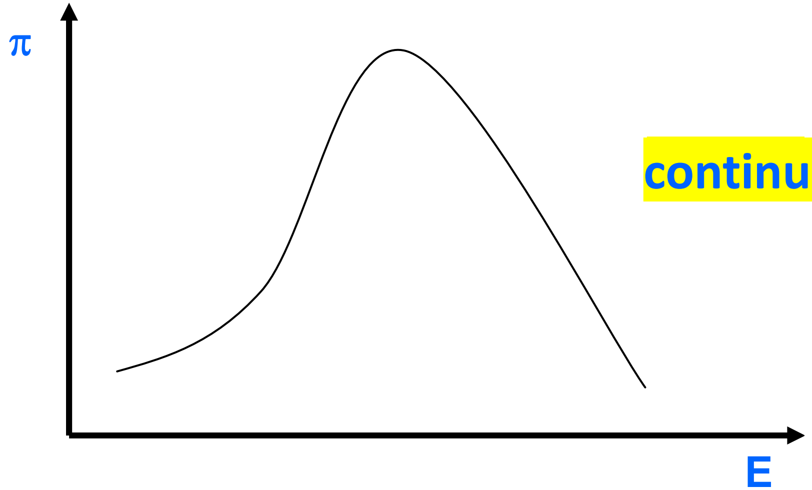
Decay

- Three body decay

$$n \rightarrow p + e^- + \overline{\nu}_e \qquad \mathrm{m}_{\overline{\nu}_e} = 0$$

- In the neutron reference frame

$$\vec{0} = \vec{\mathrm{p}}_1 + \vec{\mathrm{p}}_2 + \vec{\mathrm{p}}_3 \quad ; \quad \mathrm{m}_\mathrm{n} = \sqrt{p_1^2 + m_p^2} + \sqrt{p_2^2 + m_e^2} + p_3$$



continuous spectrum

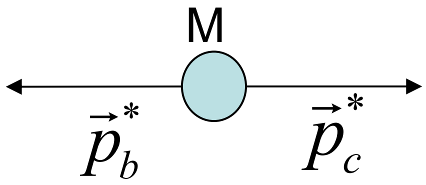
Exercise: limit angle in the lab system

- Let consider the decay

  a $\rightarrow$ b + c
- Being M_a, m_b and m_c the masses of the three particles. In the CM RF a is at rest and the conservation of four-momentum gives:

$$M_a = E_b^* + E_c^*$$

$$\vec{0} = \vec{p}_b^{\,*} + \vec{p}_c^{\,*}$$



Exercise: limit angle in the lab system

- If in the lab system a has momentum **p** the β value of the Lorentz transformation to the CdM will be

$$\beta = p/E \quad \text{e} \quad \beta\gamma = p/M_a$$

Exercise: limit angle in the lab system

$$\begin{pmatrix} E \\ P_x \\ P_y \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E^* \\ P_x^* \\ P_y^* \end{pmatrix}$$

$$\begin{cases} E_1 = \gamma E_1^* + P_{1x}^* \beta\gamma = \gamma E_1^* + \beta\gamma \cos\vartheta^* p^* \\ P_{1x} = P_1 \cos\vartheta_1 = \gamma P_{1x}^* + \beta\gamma E_1^* = \gamma p^* \cos\vartheta^* + \gamma\beta E_1^* \\ P_{1y} = P_1 \sin\vartheta_1 = P_{1y}^* = P^* \sin\vartheta^* \end{cases}$$

$$\begin{cases} E_2 = -\beta\gamma p^* \cos\vartheta^* + \gamma E_2^* \\ P_{2x} = P_2 \cos\vartheta_2 = -\gamma p^* \cos\vartheta^* + \beta\gamma E_2^* \\ P_{2y} = P_2 \sin\vartheta_2 = p^* \sin\vartheta^* \end{cases}$$

$$tg\vartheta_1 = \frac{p^* \sin\vartheta^*}{\gamma p^* \cos\vartheta^* + \gamma\beta E_1^*} = \frac{\sin\vartheta^*}{\gamma(\cos\vartheta^* + \beta/\beta^*)}$$

$$tg\vartheta_2 = \frac{\sin\vartheta^*}{\gamma(-\cos\vartheta^* + \beta/\beta_2^*)} \qquad \beta_1^* = \frac{P^*}{E_1^*}; \beta_2^* = \frac{p^*}{E_2^*}$$

If: $\beta^*_i < \beta$

Particle i will be emitted in the forward direction for every θ^* value

Exercise: limit angle in the lab system

The value of θ_{max}

$$\frac{dtg\vartheta_1}{d\vartheta^*} = \frac{1+\cos\vartheta^*\dfrac{\beta}{\beta_1^*}}{\gamma\left(\cos\vartheta^* + \dfrac{\beta}{\beta_1^*}\right)^2} = 0 \;\Rightarrow\; \cos\vartheta^* = -\frac{\beta_1^*}{\beta}$$

$$tg\vartheta_{\max} = \frac{\sqrt{1-\left(\dfrac{\beta_1^*}{\beta}\right)^2}}{\gamma\left(\dfrac{\beta}{\beta_1^*} - \dfrac{\beta_1^*}{\beta}\right)} = \frac{1}{\gamma\sqrt{\left(\dfrac{\beta}{\beta_1^*}\right)^2 - 1}}$$

$$\theta = \theta_1 + \theta_2$$

$$p^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\vartheta = M^2$$

$$\cos\vartheta = \frac{m_1^2 + m_2^2 + 2E_1E_2 - M^2}{2p_1p_2}$$

Exercise: limit angle in the lab system

In the limit $E_i >> m_i$ $\longrightarrow$ **$P_i \cong E_i$**

$$2E_1E_2(1-\cos\vartheta) = 4E_1E_2\sin^2\frac{\vartheta}{2} = M^2 - m_1^2 - m_2^2$$

$$\sin\frac{\vartheta}{2} = \frac{\sqrt{M^2 - m_1^2 - m_2^2}}{2\sqrt{E_1E_2}}$$

Θ is minimum for $E_1 = E_2$

Exercise: trasformation LAB⇒CdM

$$E^{*2} = (E_i + m_b)^2 - p_i^2 = E^2 - p^2$$

$$\beta_{CdM} = \frac{P}{E} = \frac{P_i}{\sqrt{p_i^2 + m_i^2} + m_b}$$

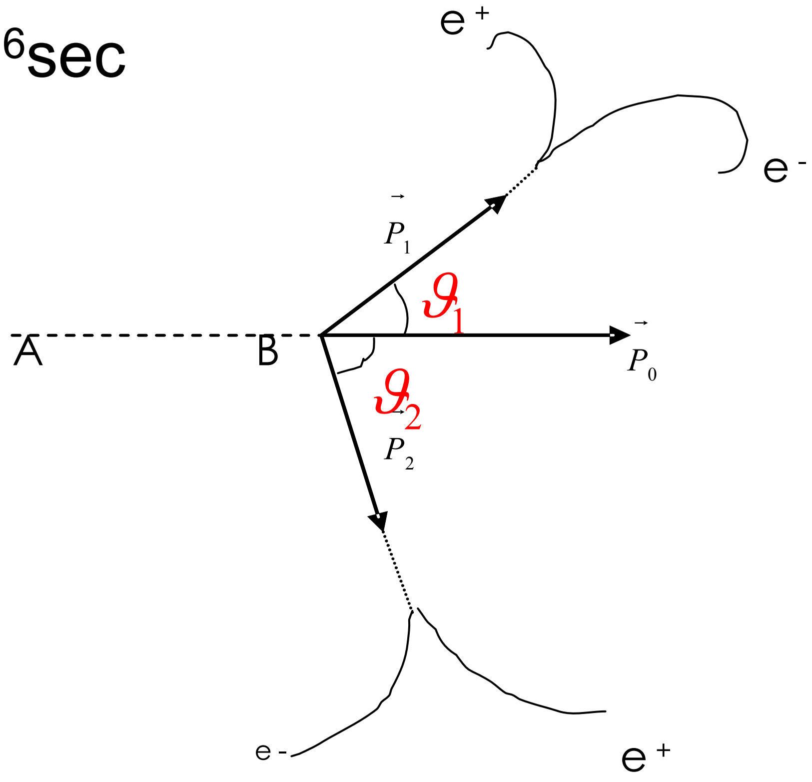
$$1 - \beta_{CDM}^2 = 1 - \frac{P^2}{E^2} = \frac{E^2 - P^2}{E^2} = \frac{E^{*2}}{E^2}$$

$$\gamma_{CdM} = \frac{1}{\sqrt{1 - \beta_{CdM}^2}} = \frac{E}{E^*}$$

$$\beta_{CdM}\gamma_{CdM} = \frac{P_i}{E^*}$$

π^0 Decay

- Decay: $\pi^0 \rightarrow \gamma + \gamma$
- mass: $m_{\pi 0}$ = 135 MeV
- lifetime: $\tau_{\pi 0}$= $0.828 \cdot 10^{-16}$sec



$$\vec{p}_0 = \vec{p}_1 + \vec{p}_2 \qquad ; \qquad \vartheta = \vartheta_1 + \vartheta_2$$

π^0 Decay

$$a)\quad \vec{p}_0 = \vec{p}_1 + \vec{p}_2 \quad \rightarrow \quad \mathrm{p}_0^2 = \mathrm{p}_1^2 + \mathrm{p}_2^2 + 2\mathrm{p}_1\mathrm{p}_2 \cos\vartheta$$

$$b)\quad \sqrt{m_{\pi^0}^2 + \mathrm{p}_0^2} = E_1 + E_2 = p_1 + p_2 \quad \rightarrow \quad m_{\pi^0}^2 + \mathrm{p}_0^2 = \mathrm{p}_1^2 + \mathrm{p}_2^2 + 2\mathrm{p}_1\mathrm{p}_2$$

- Summing up equations a) and b) we obtain:

$$2\mathrm{p}_1\mathrm{p}_2\left(1-\cos\vartheta\right) = m_{\pi^0}^2 \quad \rightarrow \quad 4\mathrm{p}_1\mathrm{p}_2 \sin^2\frac{\vartheta}{2}$$

$$\boxed{\sin^2\frac{\vartheta}{2} = \frac{m_{\pi^0}^2}{4E_1E_2}}$$

π^0 Decay

- Let's prove that the minimum value of θ occurs when:

$$E_1 = E_2 = \frac{E_0}{2} \text{ dove } E_0 = \sqrt{m_{\pi^0}^2 + p_0^2}$$

θ minimum corresponds to E_1E_2 = Max. Let's put
$y=E_1E_2=E_1(E_0-E_1)=E_1E_0-E_1^2=xE_0-x^2$

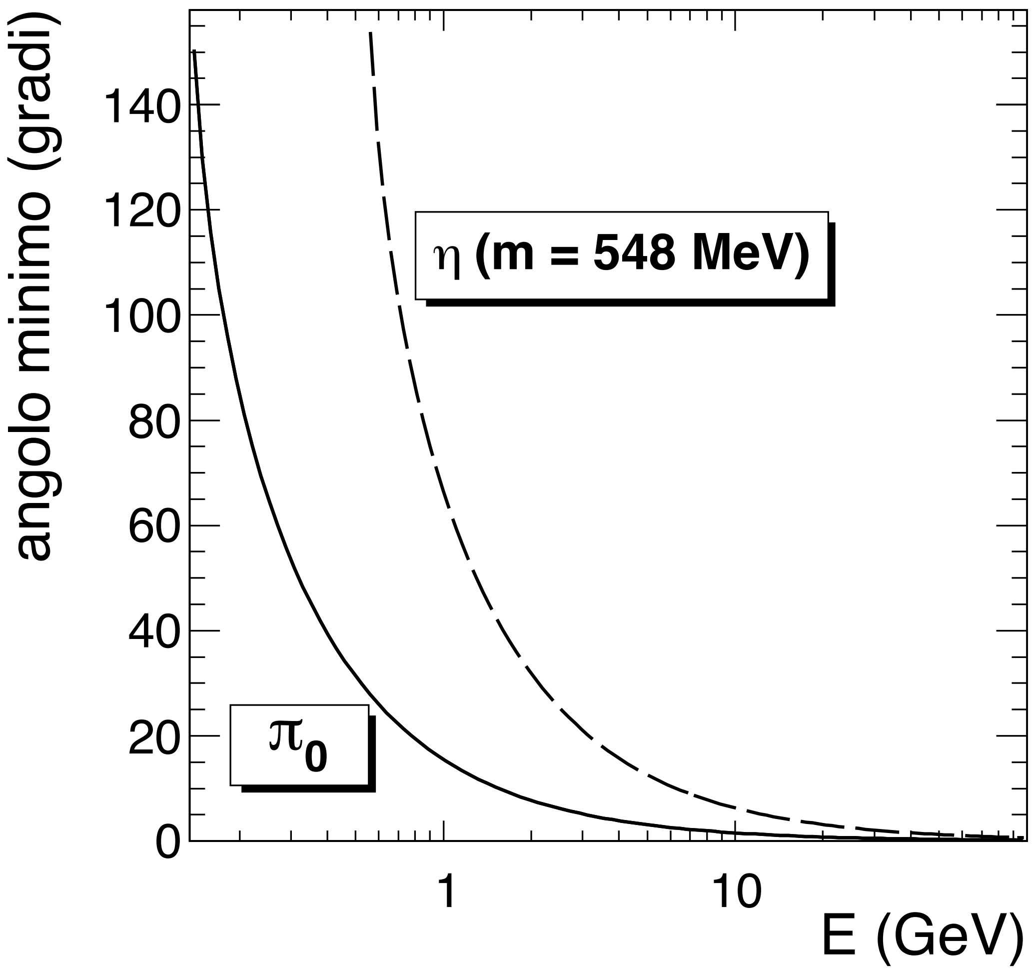
- Where we set $x=E_1$ which is the unknown variable

$$\frac{dy}{dx} = E_0 - 2x = 0 \quad \rightarrow \quad x = \frac{E_0}{2} \quad \rightarrow \quad E_1 = \frac{E_0}{2}$$

- when the angular aperture of the two photons is minimal:

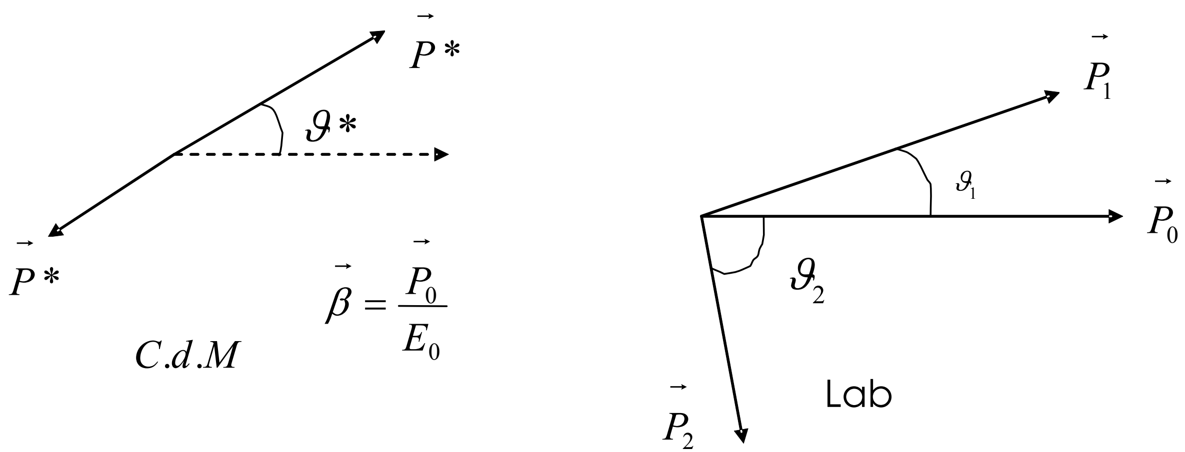
we have the equipartition of the energy of π^0 between the two photons.

π^0 Decay


angolo minimo (gradi)
140
120
100
80
60
40
20
0
η (m = 548 MeV)
π0
1
10
E (GeV)

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

- We know that in the reference system of π^0 at rest, photons have momentum equal to $M_{\pi 0}/2$.
- We derive the distribution of the momentum of the photons in the reference system of the laboratory.



- Since $M_\gamma = 0$, we will have that $p^* = p_1^* = p_2^* = E_1^* = E_2^* = M_{\pi 0}/2$
Moreover, if p_0 is the momentum of π^0 in the laboratory we will have:

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

$$\beta = \frac{p_0}{E_0} \quad ; \quad \gamma = \frac{\mathrm{E}_0}{\mathrm{E}^*} = \frac{\mathrm{E}_0}{M_{\pi^0}} \quad ; \quad \mathrm{p_y} = p_y^* = p^* \sin\vartheta^*$$

$$p_x = \gamma\left(p^* \cos\vartheta^* + \beta E^*\right)$$

$$E_1 = \gamma\left(E_1^* + \beta \cdot p_1^* \cdot \cos\vartheta^*\right) = \gamma \cdot \frac{m_{\pi^0}}{2}\left(1 + \beta \cdot \cos\vartheta^*\right) =$$

$$= \frac{E_0}{m_{\pi^0}} \cdot \frac{m_{\pi^0}}{2}\left(1 + \frac{p_0}{E_0} \cdot \cos\vartheta^*\right) = \frac{E_0}{2}\left(1 + \frac{p_0}{E_0} \cdot \cos\vartheta^*\right) =$$

$$= \frac{E_0}{2} \cdot \frac{E_0 + p_0 \cdot \cos\vartheta^*}{E_0} = \frac{E_0 + p_0 \cdot \cos\vartheta^*}{2}$$

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

- So in the laboratory:
- E_γ will vary between two minimum and maximum values given by: E_γ will be minimum for: $\cos\theta^* = -1 \Rightarrow \theta^* = \pi$

$$E_\gamma = \frac{E_0 - p_0}{2}$$

- E_γ will be maximum for: $\cos\theta^* = 1 \Rightarrow \theta^* = 0$

$$E_\gamma = \frac{E_0 + p_0}{2}$$

- We now derive the distribution function of E_γ between these minimum and maximum values.

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

- Recalling that the π^0 has spin zero, we will have that in the system at rest of π^0 the distribution in $\cos\theta^*$ must be flat. We will therefore have:

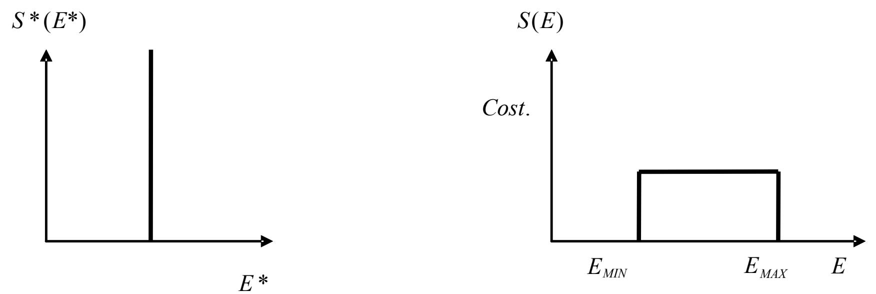
$$\int_{-1}^{+1} f(\cos\vartheta^*) d\cos\vartheta^* = 1 \Rightarrow f(\cos\vartheta^*) = \frac{1}{2}$$

$$dE_1 = \frac{1}{2} P_0 d\cos\vartheta^* = \beta\gamma \frac{m\pi^0}{2} d(\cos\vartheta^*)$$

$$dN = f(\cos\vartheta^*) d\cos\vartheta^* = \frac{1}{2} d\cos\vartheta^* = \frac{dE_1}{\beta\gamma M_{\pi^0}} = \frac{dE_1}{P_0}$$

$$\frac{dN}{dE_1} = \frac{1}{P_0}$$

Costant in the lab



CM

LAB

Angular distribution of photons in π^0 decay

- We now want to demonstrate that the decay configuration of π^0 at minimum angle is also the most probable configuration.
- In general we have obtained the following formula:

$$c)\quad \sin^2\frac{\vartheta}{2} = \frac{m_{\pi^0}^2}{4E_1E_2}$$

$$\rightarrow\quad 4E_1\cdot\left(E_0-E_1\right)=\frac{m_{\pi^0}^2}{\sin^2\dfrac{\vartheta}{2}}$$

- Differentiating left and right we have:

$$4(\mathrm{E}_0-2E_1)\cdot dE_1 = -2\frac{m_{\pi^0}^2}{\sin^3\dfrac{\vartheta}{2}}\cos\frac{\vartheta}{2}\cdot\frac{1}{2}\cdot d\vartheta$$

Angular distribution of photons in π^0 decay

d) $$\frac{dn}{d\vartheta} = \frac{dn}{dE_1}\frac{dE_1}{d\vartheta} = \frac{1}{p_0}\cdot\frac{m_{\pi^0}^2}{4(\mathrm{E}_0 - 2E_1)}\cdot\frac{\cos\frac{\vartheta}{2}}{\sin^3\frac{\vartheta}{2}}$$

- Let rewrite eq. c) as

$$\mathrm{E}_1\left(\mathrm{E}_0 - \mathrm{E}_1\right) = \frac{m_{\pi^0}^2}{4\cdot\sin^2\frac{\vartheta}{2}} \rightarrow \mathrm{E}_1^2 - \mathrm{E}_0\mathrm{E}_1 + A$$

$$\mathrm{A} = \frac{m_{\pi^0}^2}{4\cdot\sin^2\frac{\vartheta}{2}}$$

- The solution in E_1 will be given by:

$$\mathrm{E}_1 = \frac{E_0 \pm \sqrt{E_0^2 - 4\cdot A}}{2};$$

Angular distribution of photons in π^0 decay

$$\mathrm{E}_0 - 2E_1 = \mathrm{E}_0 - E_0 \mp \sqrt{E_0^2 - 4 \cdot A} =$$

$$= \mp \sqrt{E_0^2 - \frac{m_{\pi^0}^2}{\sin^2 \frac{\vartheta}{2}}} = \mp \frac{E_0}{\sin \frac{\vartheta}{2}} \sqrt{\sin^2 \frac{\vartheta}{2} - \frac{m_{\pi^0}^2}{E_0^2}}$$

- Substituting (E_0 −2E_1) into the equation d) we have:

$$\frac{\mathrm{dn}}{\mathrm{d}\vartheta} = \frac{m_{\pi^0}^2}{p_1 \cdot E_0} \cdot \frac{\cos \frac{\vartheta}{2}}{4 \sin^2 \frac{\vartheta}{2} \sqrt{\sin^2 \frac{\vartheta}{2} - \frac{m_{\pi^0}^2}{E_0}}}$$

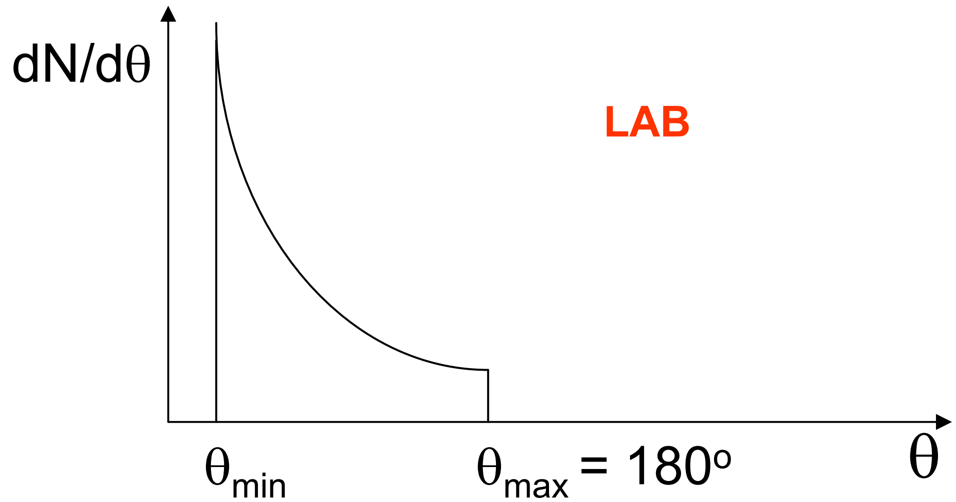
Angular distribution of photons in π^0 decay

The angular distribution will have a maximum

$$\sin^2\frac{\vartheta}{2} = \frac{m^2_{\pi^0}}{E_0}:$$

θ_{min} is also the most probable configuration

It follows that the most likely decay configuration is that with the smallest aperture angle between the two photons, with equipartition of initial energy and of the aperture angle



$E_\gamma = (E_0 + p_0 \cos\theta) / 2$

E_γ min for $\cos\theta^*=0 \Rightarrow \theta^*=\pi/2$

E_γ max for $\cos\theta^*=1 \Rightarrow \theta^*=0$

Exercise: pγ=pγ(θ∗) distribution

$$E_{\pi^0}^{LAB} = 4.05 Gev \qquad M_{\pi^0} = 135 MeV \qquad E^* = M_{\pi^0}$$

$$E_{\pi^0}^{LAB} = \gamma M_{\pi^0} \qquad \beta_{\gamma_1}^* = \frac{P_{\gamma_1}^*}{E_{\gamma_1}^*} = 1 \qquad E_{\gamma_1}^* = \frac{M_{\pi^0}}{2} = E_{\gamma_2}^* = 67.5 MeV$$

$$E_{\gamma_1} = \gamma \cdot E_{\gamma_1}^* \left(1 + \beta \cdot 1 \cdot \cos\vartheta^*\right)$$

$$E_{\gamma_1} = \frac{E_{\pi^0}^{Lab}}{M_{\pi^0}} \cdot E_{\gamma_1}^* \left(1 + \beta\cos\vartheta^*\right) = E_{\pi^0}^{Lab} \cdot \frac{\left(1 + \beta\cos\vartheta^*\right)}{2}$$

$$E_{\gamma_1} = E_{\pi^0}^{Lab} \cdot \frac{\left(1 + \beta\cos\vartheta^*\right)}{2}$$

$$\mathrm{tg}\,\vartheta_1 = \frac{\sin\vartheta_1^*}{\gamma\left(\beta + \cos\vartheta_1^*\right)}$$

Exercise: pγ=pγ(θ∗) distribution

A) $\theta^* = 90^\circ:$

$$E_{\gamma_1} = 4050\frac{(1+\beta\cdot 0)}{2} = 2025 MeV$$

$$E^*_{\gamma_1} = 67.5$$

$$\gamma = \frac{E_{\pi^0}}{m_{\pi^0}} = 30 GeV$$

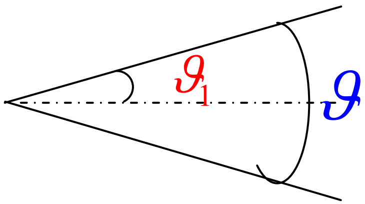
$$tg\vartheta_1(\theta^*_1 = 90^\circ) = \frac{1}{\beta\gamma}$$

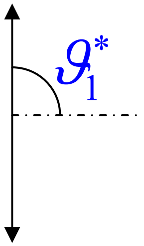
$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1-(\frac{1}{2\gamma^2})$$

$$(1-\beta) = \frac{1}{2\gamma^2}$$

$$\beta = 1-\frac{1}{1800} \approx 1$$

$$tg\vartheta_1 = \frac{1}{30} \Rightarrow \vartheta_1^{Lab} \approx 2^\circ \rightarrow \vartheta = 4^\circ$$





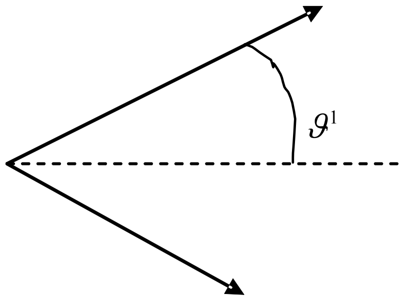
Exercise: pγ=pγ(θ∗) distribution

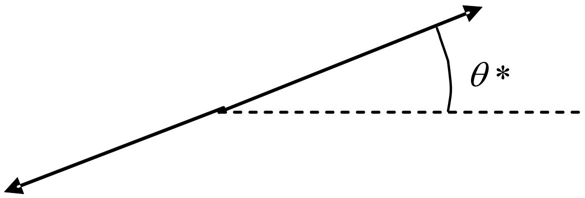
B)

$$\theta^* = 0; \theta^* = 180^\circ :$$

$$E_\gamma = \gamma E_\gamma^{*}(1 \pm \beta)\begin{cases} E_{\pi^0}^{Lab} / M_{\pi^0} \cdot \dfrac{M_{\pi^0}}{2}(1+1) = E_{\pi^0}^{Lab} = 4050 Mev \\ \gamma E_\gamma^*(1-\beta) \approx \gamma E_\gamma^* \cdot \dfrac{1}{2\gamma^2} = \dfrac{E_{\pi^0}^{Lab}}{4\gamma^2} \approx 1.1 Mev \end{cases}$$

C)





Exercise: pγ=pγ(θ∗) distribution

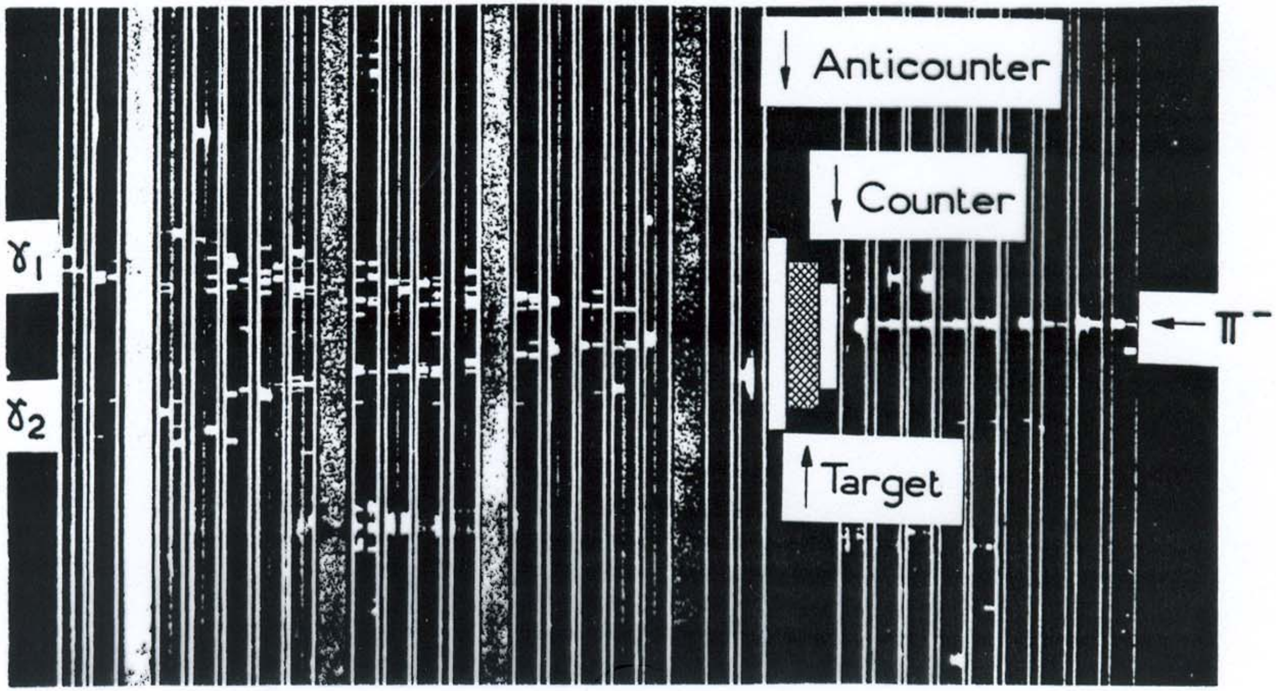


Fig. 1.6 In the target, a π^0 is produced by charge exchange $\pi^- + p \rightarrow \pi^0 + n$. The two gamma rays from the decay $\pi^0 \rightarrow \gamma_1 + \gamma_2$ produce "showers" γ_1 and γ_2 in a spark chamber. The energy of the π^0, E_{π^0} is about 3 GeV. The decay is slightly asymmetric ; (Photo CERN, SIS 16754)

Exercise

If p_{xi}, p_{yi}, p_{zi} (MeV/c) are the 3-momentum component of two charged pions measured in the lab system, evaluate the invariant mass of the mother particle and build a histogram with the obtained values.

px1	py1	pz1	px2	py2	pz2	px1	E1	px2	E2	M
-177.29	-304.79	354.31	165.10	303.89	-357.16					
-174.33	434.17	181.71	161.64	-434.58	-182.29					
-410.09	210.23	-141.90	393.96	-207.18	175.56					
379.79	-262.10	-94.39	-392.09	262.18	129.81					
152.14	-333.47	331.91	-163.16	329.89	-308.76					
160.12	-339.72	313.05	-176.50	342.35	-272.97					
149.02	289.33	373.46	-161.74	-289.49	-371.95					
-405.57	188.66	-215.35	396.61	-190.78	200.65					
-397.72	297.68	-77.99	385.72	-298.59	81.70					
352.08	-195.28	-277.04	-358.38	198.58	255.26					
-497.89	77.63	-41.46	485.30	-77.70	39.08					
-161.34	408.53	237.01	148.61	-408.47	-228.76					
-19.06	-485.24	-71.07	7.71	483.57	86.37					
-37.40	341.00	362.10	24.44	-339.96	-356.08					
-145.32	-385.43	257.33	134.65	388.17	-272.33					
-397.90	248.48	176.71	385.79	-249.21	-179.47					
-200.84	-345.58	299.61	188.81	346.11	-302.77					
384.25	-241.50	197.75	-396.71	240.57	-193.19					
339.64	-134.24	242.67	-353.52	134.83	-309.67					
-138.00	-395.18	248.14	125.94	393.68	-206.14					
-244.86	-309.62	314.18	231.41	309.11	-310.70					
-388.08	-267.87	-171.83	376.08	267.20	168.06					
277.04	-408.49	15.61	-290.59	408.85	-11.93					
-302.40	-286.31	258.94	292.24	285.12	-273.90					
-84.49	-382.48	308.71	70.70	383.49	-298.69					
457.14	155.49	79.29	-466.40	-155.31	-66.89					
151.35	373.26	294.61	-164.17	-373.08	-294.83					
-385.68	39.44	262.76	373.71	-42.03	-308.13					
-105.48	475.34	55.98	91.73	-475.53	-68.34					
-401.72	-260.86	111.61	387.35	261.93	-83.90					

Exercise: study of the decay $K^0 \rightarrow \pi^- \pi^+$

- Determine p*, the pion energies in the CM, their speed in the CM and in the LAB frame. Discuss about their maximum emission angles.

Exercise: study of the decay $\pi \rightarrow \mu\nu$

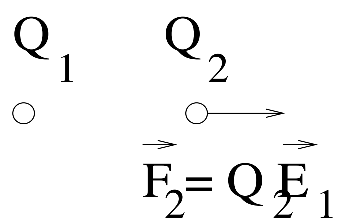
- Determine p*, the muon and neutrino energy in the CM, their speed in the CM and in the LAB frame. Determine the longitudinal momentum distribution of the two particles

Interactions

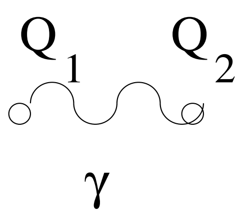
Classically interaction at a distance is described in terms of a potential or a *field*. In quantum theory it is viewed in terms of **exchange of quanta**. Quanta are bosons associated with the particular type of interaction.

Example: electrostatic interaction bewteen point charges.

Classica



Quantistica



we must have $\Delta E \cdot \Delta t \simeq \hbar$

Interactions

In nature there are four types of interaction.

- The strong interaction binds quarks in hadrons and protons and neutrons in nuclei. It is mediated by gluons.
- The electromagnetic interaction binds electrons and nuclei in atoms, and it is also responsible for the intermolecular forces in liquids and solids. It is mediated by the photon.
- The weak interaction is typified by radioactive decays, for example the slow β decay. The quanta of the weak field are the $W^{\pm}$ and Z^0 bosons.
- The gravitational interaction acts between all types of massive particles. It is by far the weakest of all the fundamental interactions.

Interactions

To indicate the relative magnitudes of the four types of interaction, the comparative strenghts of the force between two protons when just in contact are very roughly:

strong	electromagnetic	weak	gravity
1	10^{-2}	10^{-7}	10^{-39}

Ever since Einstein, physicists have speculated that the *4 interactions might be different manifestations of a unified force*. Up to now only electromagnetic and weak forces have been unified: these would have the same strength at very high energies, whereas at lower energies this symmetry is broken and the two forces have the same strength.

All four interactions play a fundamental role in our universe.

Electromagnetic Interaction

The coupling constant of the electromagnetic interaction is the fine structure constant α.

$$\alpha = \frac{\frac{1}{4\pi}\frac{e^2}{(\hbar / mc)}}{mc^2} = \frac{\text{electrostatic energy of two } e \text{ at a distance } (\hbar/mc)}{\text{rest mass of the electron}}$$

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

$\pi^0 \rightarrow \gamma\gamma$ $\tau = (8.4 \pm 0.6) \times 10^{-17}$ s

The quantum of the electromagnetic interaction is the photon.

The field theory of the electromagnetic interaction is Quantum ElectroDynamics (QED).

Week Interaction

$n \rightarrow p + e^- + \bar{\nu}_e$ β decay

$\bar{\nu}_e + p \rightarrow n + e^+$ $\bar{\nu}$ absorption

$\Sigma^- \rightarrow n + \pi^-$ $\tau \cong 10^{-10}$ s

(dds) (ddu)

$\Sigma^0 \rightarrow \Lambda + \gamma$ $\tau \cong 10^{-19}$ s

(e.m. viola isospin)

$$\frac{\alpha_W}{\alpha} \approx \sqrt{\frac{10^{-19}}{10^{-10}}} \approx 10^{-4} \div 10^{-5}$$

The quanta of the weak interaction are the $W^{\pm}$ and Z^0 bosons.

$M_W = (80.425 \pm 0.038)$ GeV/c^2

$M_Z = (91.1876 \pm 0.0021)$ GeV/c^2

Strong Interaction

$\Sigma^0(1385) \to \Lambda + \pi^0$ Γ=36 MeV $\tau \sim 10^{-23}$ s

$\Sigma^0(1192) \to \Lambda + \gamma$ $\tau \sim 10^{-19}$ s

$$\frac{\alpha_S}{\alpha} \approx \sqrt{\frac{10^{-19}}{10^{-23}}} \approx 100 \qquad \frac{e^2}{4\pi} = \frac{1}{137} \qquad \frac{g_s^2}{4\pi} \approx 1$$

The quanta of the strong interactions are the gluons. The strong charge is called color and it can assume 6 values R, G, B, $\bar{\text{R}}$, $\bar{\text{G}}$, $\bar{\text{B}}$.

Color symmetry is an exact symmetry, i.e. the force between quarks is color-independent.

The field theory of strong interactions is Quantum Chromodynamics (QCD).

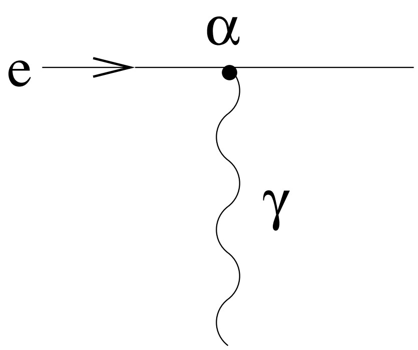
Asymptotic freedom $V_s \to \alpha_s/r$ $q^2 \to \infty$

Confinement $V_s \to kr$ $q^2 \to 0$ $[r \to \infty]$

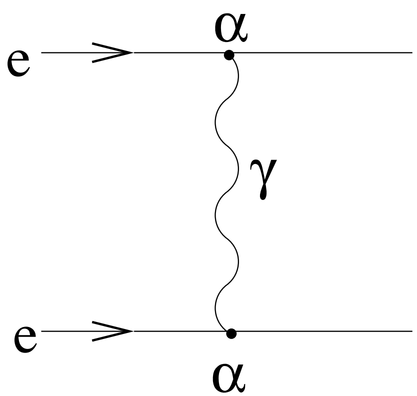
Feynman Diagram

- Feynman diagram are graphic way of representing the interactions between particles and fields.
- The solid lines represent the fermions.
- Wavy (curly or dashed) lines bosons.
- The **arrows** on the lines indicate the direction of time, with time running from left to right
- Fermionic and bosonic lines intersect at **vertices** where charge, energy and momentum are conserved.
- The intensity of the interaction is represented by a **coupling constant** associated with each vertex.
- Open lines represent real particles, closed lines represent virtual particles.

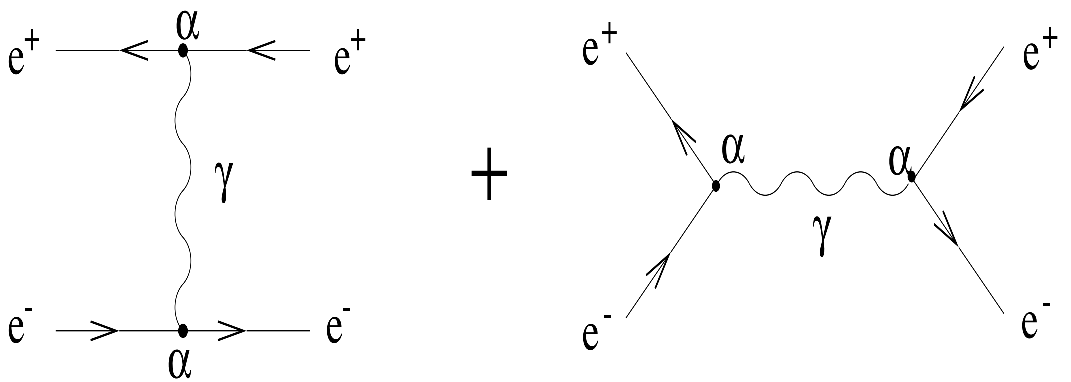
Feynman Diagrams



basic electron-photon vertex

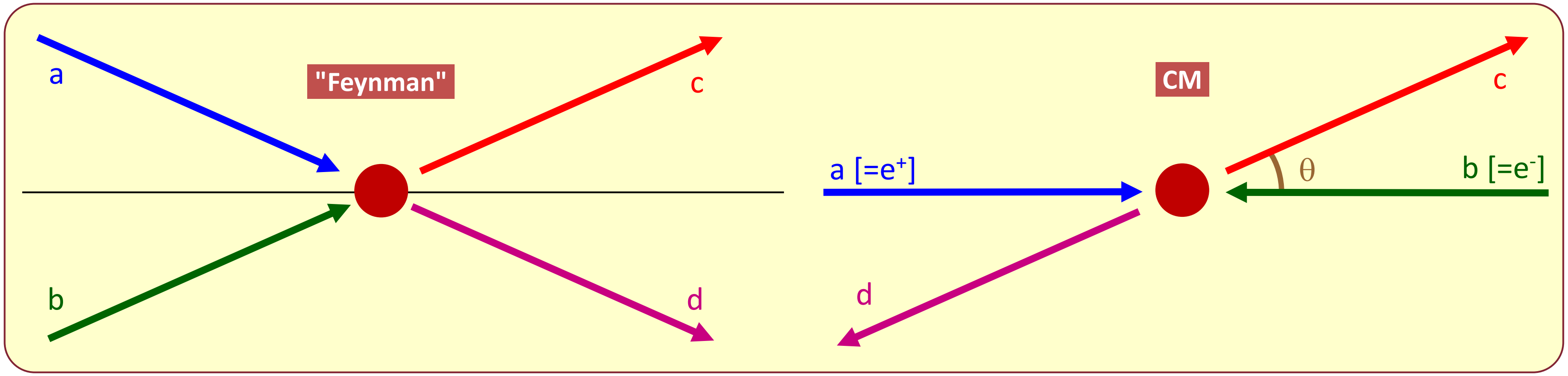


ee scattering via γ exchange



e^+e^- scattering via photon exchange with two diagrams contributing in first order

Mandelstam variables



The Mandelstam variables $\boxed{s, t, u}$:

CM system
- $p_a = [E, \quad p, \quad 0, \quad 0];$
- $p_b = [E, \quad -p, \quad 0, \quad 0];$
- $p_c = [E, p\cos\theta, p\sin\theta, \quad 0];$
- $p_d = [E, -p\cos\theta, -p\sin\theta, \; 0];$

s,t,u L-invariant
- $s \equiv (p_a + p_b)^2 = (p_c + p_d)^2 = 4E^2;$
- $t \equiv (p_a - p_c)^2 = (p_b - p_d)^2 = -\tfrac{1}{2}\, s\,(1 - \cos\theta) = -s\sin^2(\theta/2);$
- $u \equiv (p_a - p_d)^2 = (p_b - p_c)^2 = -\tfrac{1}{2}\, s\,(1 + \cos\theta) = -s\cos^2(\theta/2);$
- $s + t + u = 0$ ($\rightarrow$ 2 independent variables, e.g. $[E,\theta]$, $[s, t]$, $[\sqrt{s},\theta]$).

Lorentz-invariant variables for $2\rightarrow2$ processes.

Assume $E \gg m_i$, for the masses of all 4 bodies (otherwise, look for the formulas in [PDG]).

Q. : what about φ (the azimuth) ?
A. : if nothing in the dynamics is φ-dependent (e.g. the spin direction), then the cross-section must be φ-symmetric.

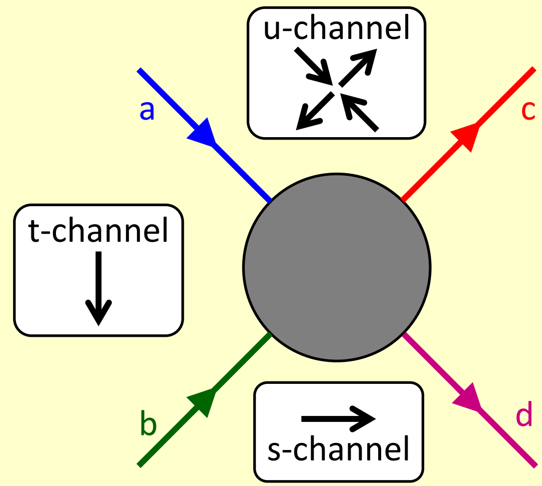
Mandelstam variables $m_i \neq 0$

General case $ab \to cd$, masses NOT negligible:

[p_i and p_j are 4-mom, $p_i p_j$ = dot product]

- $s \equiv (p_a + p_b)^2 = (p_c + p_d)^2 = m_a^2 + m_b^2 + 2p_a p_b;$
- $t \equiv (p_a - p_c)^2 = (p_b - p_d)^2 = p_a^2 + m_c^2 - 2p_a p_c;$
- $u \equiv (p_a - p_d)^2 = (p_b - p_c)^2 = p_a^2 + m_d^2 - 2\, p_a p_d;$
- $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 + 2p_a(p_a + p_b - p_c - p_d) = m_a^2 + m_b^2 + m_c^2 + m_d^2 = \sum_i m_i^2.$

In addition, the crossing symmetry correlates the processes which are symmetric wrt time (s-, t-, and u-channels [see box]). If the c.s. is conserved in the interaction, the same amplitude is valid for all the channels, in their appropriate physical domains (an example on next page).



s-channel	$ab \to cd$	$(\bar{p}p \to \bar{n}n)$
t-channel	$a\bar{c} \to \bar{b}d$	$(\bar{p}n \to \bar{p}n)$
u-channel	$a\bar{d} \to \bar{b}c$	$(\bar{p}\bar{n} \to \bar{p}\bar{n})$

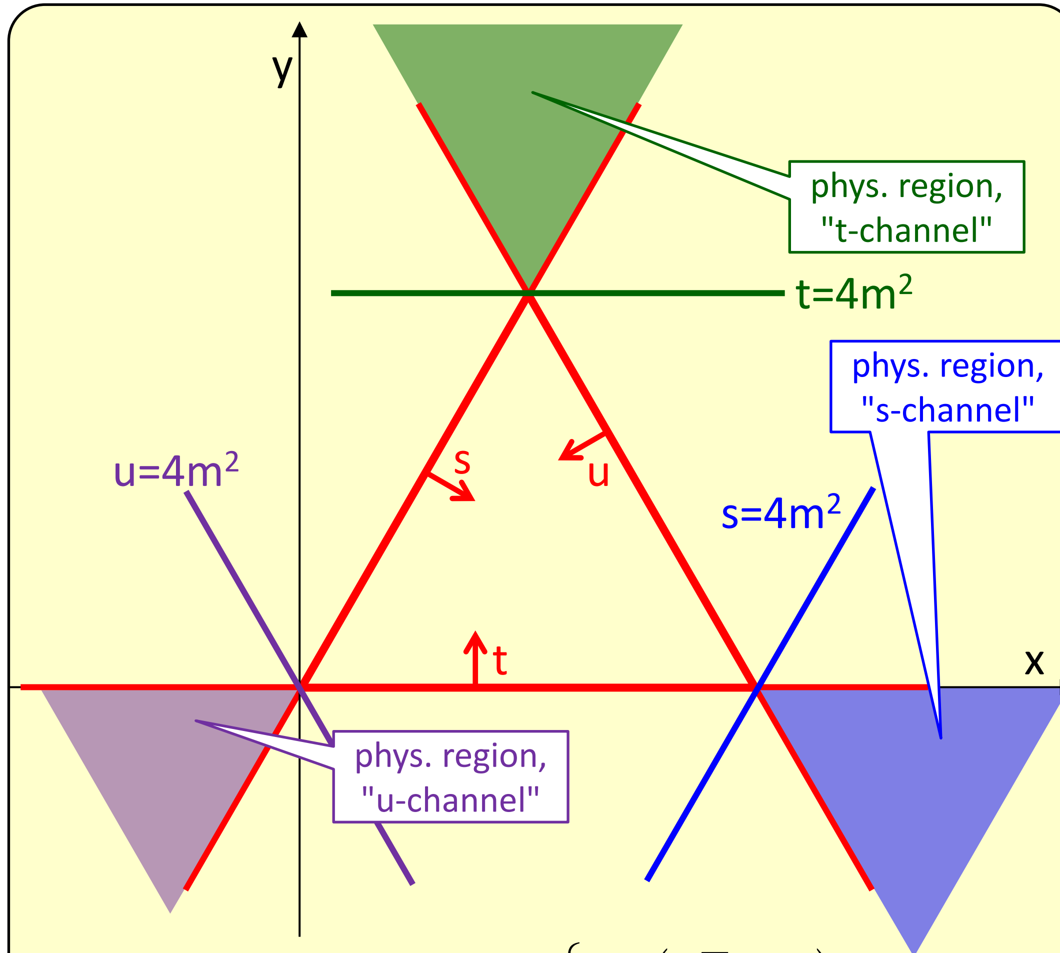
an old approach (1950-80), now almost forgotten, especially important for strong interactions at low energies (see the example $\bar{p}p \to \bar{n}n$), where the dynamics was not calculable (still is not).

Mandelstam variables

Example : $m_a = m_b = m_c = m_d = m$;

$$\left.\begin{array}{l} \bullet\ s = 4E^2 \geq 4m^2; \\ \bullet\ t = -4p^2\sin^2(\theta/2); \\ \bullet\ u = -4p^2\cos^2(\theta/2); \end{array}\right\} s + t + u = 4m^2;$$

- in a xy plane draw an equilateral triangle of height $4m^2$, and label s-t-u the three sides and the lines through them (drawn in red);
- remember Viviani's theorem and its extension ("*the sum of the signed distances between a point and the lines of a triangle is a constant*");
- find the physical regions (i.e. the allowed values of s-t-u) for the given process (i.e. the "s-channel") and for the t and u channels;
- among s-t-u, only two variables are independent → the "space of the parameters" is 2D.



$$\begin{cases} x = \dfrac{4m^2 + s - u}{\sqrt{3}} \\ y = t = 4m^2 - s - u \end{cases} \rightarrow \begin{cases} s = \left(\sqrt{3}x - y\right)/2 \\ t = y \\ u = 4m^2 - \left(\sqrt{3}x + y\right)/2. \end{cases}$$

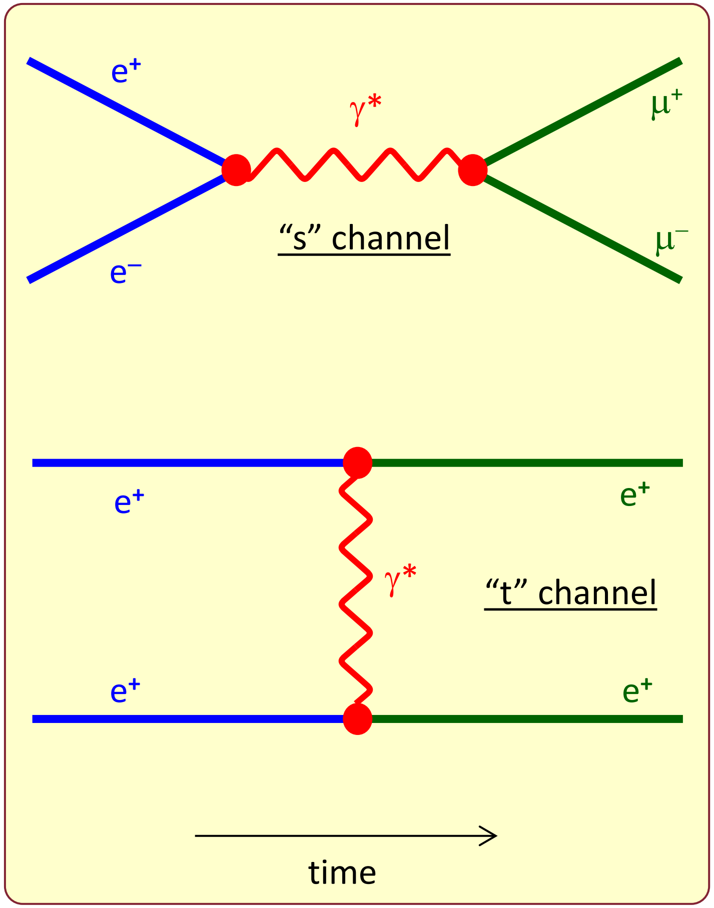
Mandelstam variables

- in a "s-channel" process (e.g. $e^+e^- \to \mu^+\mu^-$), the $|4\text{-momentum}|^2$ of the mediator γ^* is exactly s [i.e. $m(\gamma^*) = \sqrt{s}$, $\sqrt{s} > 0$];

- in a "t-channel" process (e.g. $e^+e^+ \to e^+e^+$), the $|4\text{-momentum}|^2$ of the mediator (γ^* also in this case) is t [t < 0 !!!] ;

- some processes (e.g. $e^+e^- \to e^+e^-$, called "Bhabha scattering") have more than one Feynman diagrams; some of them are of type s and some others of type t; in such a case we say it is a sum of "s-type diagrams" and "t-type diagrams" + the interference,

… although, *needless to say*, on an event-by-event basis, the observer does **NOT** know whether the event was *s* or *t*.



Mandelstam variables

- in absence of polarization, the cross sections of a process "X" does NOT depend on the azimuth φ :

$$\frac{d\sigma_{"X"}}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma_{"X"}}{d\cos\theta} = \frac{s}{4\pi}\frac{d\sigma_{"X"}}{dt}.$$

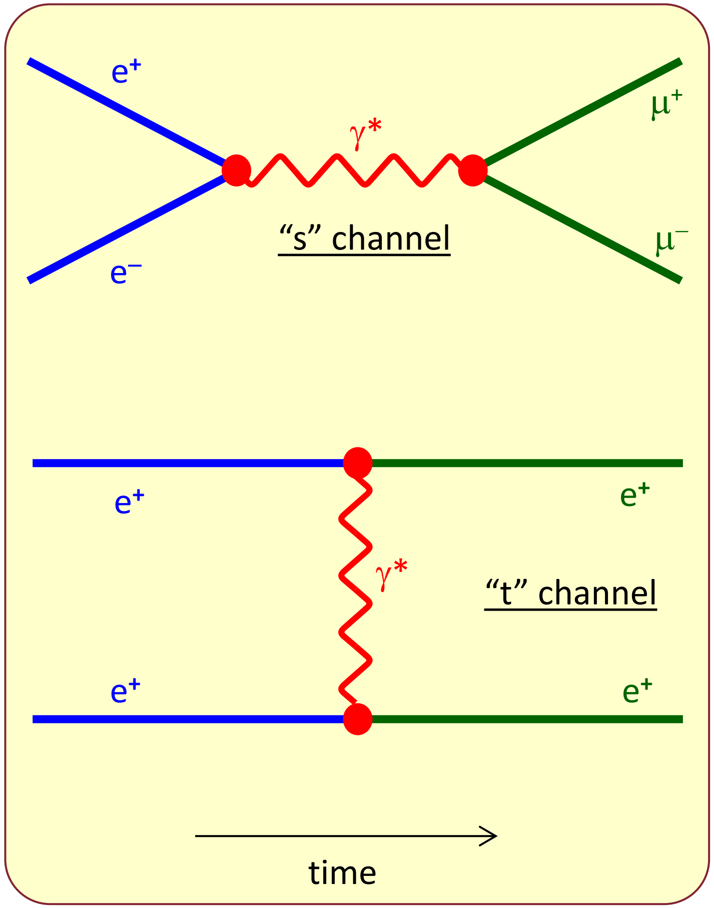
- for $m^2 << s$, if $\mathcal{M}_{"X"}$ is the matrix element of the process(*) :

$$\frac{d\sigma_{"X"}}{dt} = \frac{\left|\mathcal{M}_{"X"}\right|^2}{16\pi s^2}.$$

- in lowest order QED, if $m^2 << s$:

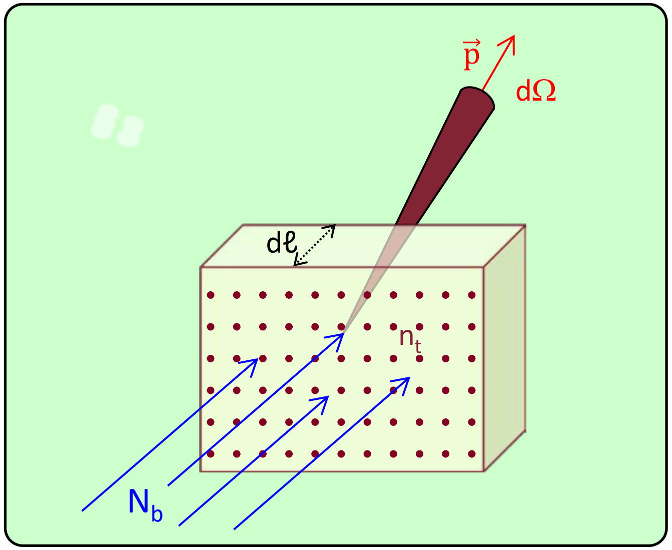
$$\frac{d\sigma_{"X"}}{d\cos\theta} = \frac{\left|\mathcal{M}_{"X"}\right|^2}{32\pi s} = \frac{\alpha^2}{s} f(\cos\theta).$$

- when $\theta \to 0$, $\cos\theta \to 1$:
  - s-channel : $f(\cos\theta) \to$ constant;
  - t-channel : $f(\cos\theta) \to \infty$.



(*) also by dimensional analysis :
[$c = \hbar = 1$], [σ] = [ℓ^2]; [t] = [s] = [ℓ^{-2}];
therefore, in absence of any other dimensional scale,
σ [and $d\sigma/d\Omega$] = [number] × 1/s.

The cross section σ



A beam of N_b particles is sent against a thin layer of thickness $d\ell$, containing dN_t scattering centers in a volume $\mathcal{V}$ ("target", density $n_t = dN_t/d\mathcal{V}$).

The number of scattered particles dN_b is:

$$dN_b \propto N_b\, n_t\, d\ell \quad \Rightarrow \quad dN_b = N_b\, n_t\, \sigma_T\, d\ell$$

the number of particles left after a finite length ℓ is

$$N_b(\ell) = N_b(0) \exp(-\, n_t\, \sigma_T\, \ell).$$

The parameter σ_T is the total **cross section** between the particles of the beam and those of the target; it can be interpreted as *the probability of an interaction when a single projectile enters in a region of unit volume containing a single target*.

If many exclusive processes may happen (simplest case : elastic or inelastic), σ_T is the sum of many σ_j, one for each process:

$\sigma_T = \Sigma_j\, \sigma_j$ [e.g. $\sigma_T = \sigma_{elastic} + \sigma_{inelastic}$];

in this case σ_j is proportional to the *probability of process j*.

Common differential dσ/d... 's:

$$\frac{d\sigma}{d\Omega} = \frac{d^2\sigma}{d\cos\theta\, d\varphi} \xrightarrow[\text{dependence}]{\text{no } \varphi} \frac{1}{2\pi} \frac{d\sigma}{d\cos\theta};$$

$$\frac{d\sigma}{d\vec{p}} = \frac{d^3\sigma}{dp_x dp_y dp_z} = \frac{d^3\sigma}{p_T dp_T dp_\ell d\varphi} \to \frac{1}{\pi} \frac{d^2\sigma}{dp_T^2 dp_\ell};$$

+ others.

The cross section σ: $\sigma_{inclusive}$

In a process ($a\,b \to c\,X$), assume:

- we are only interested in "c" and not in the rest of the final state ["X"];
- "c" can be a single particle (e.g. $W^\pm$, Z, Higgs) or a system (e.g. $\pi^+\pi^-$).

Define:

$\sigma_{inclusive}(ab \to cX) = \sum_k \sigma_{exclusive}(ab \to cX_k)$,
where the sum runs on all the **exclusive** processes which in the final state contain "c" + anything else [define also $d\sigma_{inclusive}/d\Omega$ wrt angles of "c", etc.].

The word *inclusive* may be explicit or implicit from the context. E.g., "*the cross-section for Higgs production at LHC*" is obviously $\sigma_{inclusive}(pp \to HX)$.

From the definition, if $\sigma_{inclusive} << \sigma_{total}$:

$\mathcal{P}_c$ = probability of "c" in the final state =
$= \sigma_{inclusive}(ab \to cX) / \sigma_{total}(ab)$.

Instead, if "c" is common:

$\langle n_c \rangle$ = <number of "c" in the final state> =
$= \sigma_{inclusive}(ab \to cX) / \sigma_{total}(ab)$.

e.g.

σ_{Higgs}(LHC, 8 TeV) $= \sigma_{incl}(pp \to HX, \sqrt{s}=8\text{ TeV}) =$
≈ 22.3 pb;

$\sigma_{total}(pp, \sqrt{s} = 8\text{ TeV}) = 101.7 \pm 2.9$ mb;

$\to\ \mathcal{P}_{Higgs}$(LHC) $\approx 2 \times 10^{-10}$;

[§ LHC]

$\sigma_{incl}(pp \to \pi^0 X, p_{LAB}=24\text{ GeV}) = 53.5 \pm 3.1$ mb;

$\sigma_{total}(pp, p_{LAB}=24\text{ GeV}) = 38.9$ mb;

$\to \langle n_{\pi^\circ}(pp, p_{LAB}=24\text{ GeV})\rangle \approx 1.37$

[V.Blobel et al. - Nucl. Phys., B69 (1974) 454].

Mutatis mutandis, define

- "inclusive width" $\Gamma(A \to BX)$;
- "inclusive BR" $BR(A \to BX)$.

The cross section σ: Fermi 2nd golden rule

- N_b, N_t : particles in beam(b) / target(t);
- $\mathcal{V}$: volume element;
- n_b, n_t : density of particles [= $dN_{b,t}/d\mathcal{V}$];
- v_b : velocity of incident particles;
- ϕ : flux of incident particles [= $n_b\, v_b$];
- p', E' : 4-mom. of scattered particles;
- $\rho(E')$: density of final states;
- $\mathcal{M}_{fi}$: matrix element between $i \to f$ state;
- dN/dt : number of events / time [= $\phi N_t \sigma$];
- W : rate of process [= (dN/dt) / $(N_b N_t)$].

Fermi second golden rule

$$\boxed{W = \frac{2\pi}{\hbar}\left|\mathcal{M}_{fi}\right|^2 \rho(E');}$$

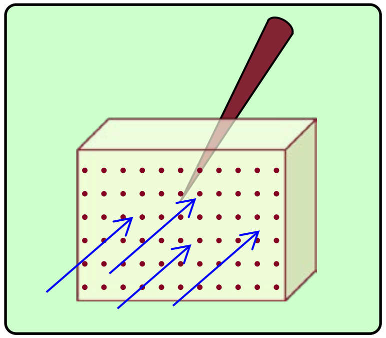
$$\rho(E') = \frac{dn(E')}{dE'} = \frac{\mathcal{V} 4\pi p'^2}{v'(2\pi\hbar)^3};$$

$$W = \frac{dN}{dt}\frac{1}{N_b N_t} = \frac{\phi N_t \sigma}{N_b N_t} = \frac{v_b \sigma}{\mathcal{V}}.$$

$$dn(p') = \frac{\mathcal{V} 4\pi p'^2}{(2\pi\hbar)^3} dp' = \frac{\mathcal{V} 4\pi p'^2}{(2\pi\hbar)^3}\frac{dE'}{v'}$$

$$\boxed{\sigma = \frac{W\mathcal{V}}{v_b} = \frac{2\pi}{\hbar}\left|\mathcal{M}_{fi}\right|^2 \rho(E')\frac{\mathcal{V}}{v_b}}$$

- the rule is THE essential connection (experiment $\leftrightarrow$ theory);
- experiments measure event numbers $\to$ cross-sections;
- theories predict matrix elements $\to$ cross-sections;
- when we check a prediction, we are actually applying the rule;
- properly normalized, the rule is valid also for differential cases (i.e. $d\sigma/dk$, $d\mathcal{M}/dk$, dW/dk), where k is any kinematical variable, e.g. $\cos\theta$].



Excited states : decay pdf

Consider N (N large) unstable particles :

- independent decays;
- decay probability time-independent (e.g. no internal structure, like a timer);

Then :

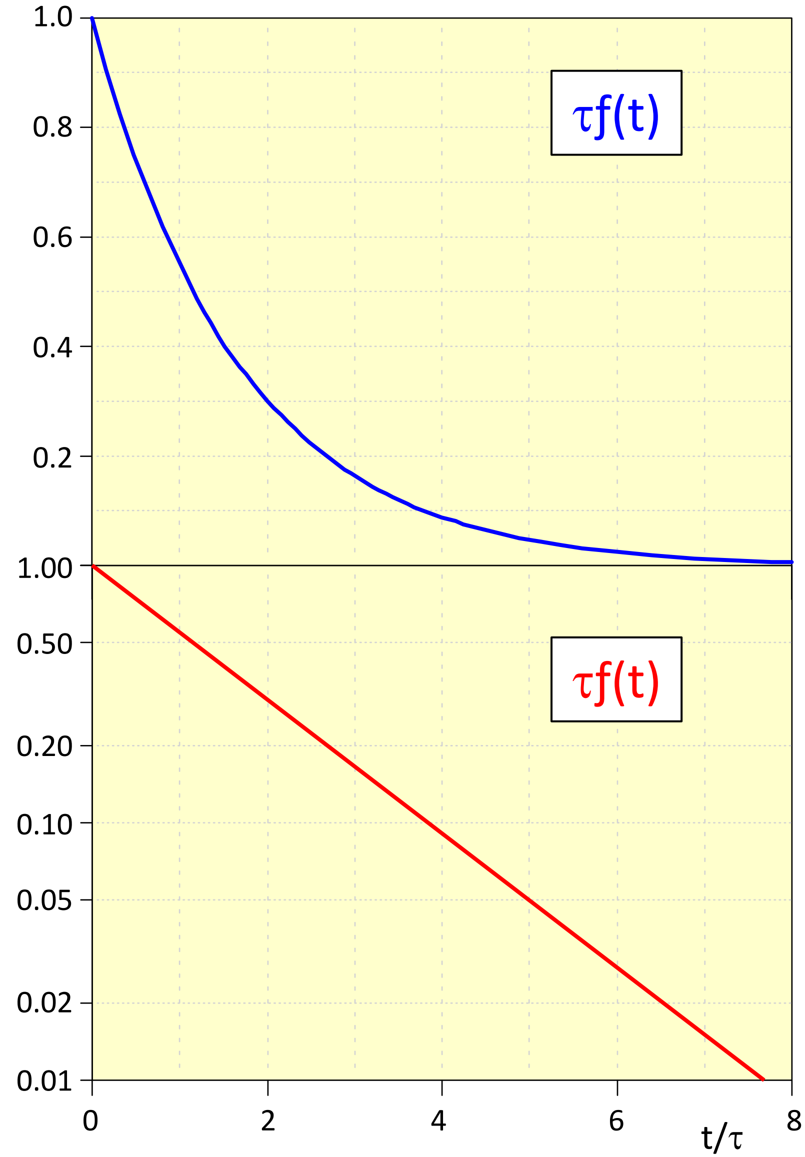
$$dN = -N\Gamma dt; \qquad \Gamma \equiv \frac{1}{\tau} = \text{const.} \qquad \Rightarrow$$

$$N(t) = N_0\, e^{-\Gamma t} = N_0 e^{-t/\tau}.$$

The pdf of the decay for a single particle is

$$\int_0^\infty f(t)dt = 1 \quad \Rightarrow \quad \boxed{f(t) = \frac{1}{\tau} e^{-t/\tau}.}$$

- average decay time : $\left(\sum t_j\right)/n$ = <t> = = τ;
- likelihood estimate of τ, after n decays observed : τ^* = <t>.



Excited states : Breit-Wigner

If τ is small, the energy at rest (= mass) of a state is not unique (= δ_{Dirac}), but may vary as $\tilde{f}(E)$ around the nominal value E_0 = m :

Define $\psi(t<0) = 0$; $\psi(t=0) = \psi_0$;
width Γ [unstable] ;

$$\psi(t) = \psi_0 e^{(-im-\Gamma/2)t};$$

$$|\psi(t)|^2 = |\psi_0|^2 e^{-\Gamma t} = |\psi_0|^2 e^{-t/\tau};$$

$$\tilde{f}(E) = |\tilde{\psi}(E)|^2 = \frac{|\psi_0|^2}{2\pi} \frac{1}{(E-E_0)^2 + \Gamma^2/4}.$$

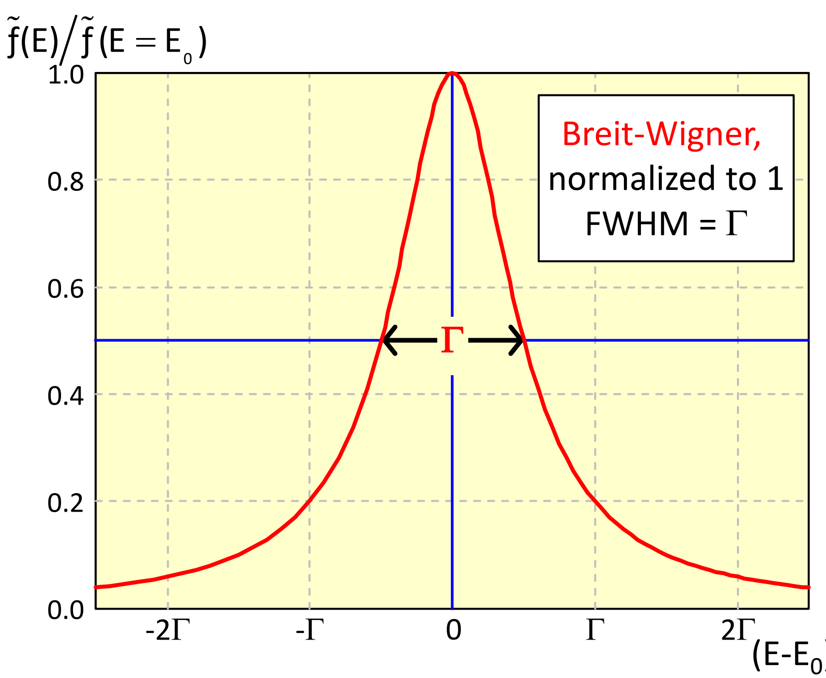
$$\tilde{\psi}(E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iEt} \psi(t) dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{iEt} \psi_0 e^{-i(E_0 - i\Gamma/2)t} dt =$$

$$= \frac{\psi_0}{\sqrt{2\pi}} \frac{-1}{i(E-E_0) - \Gamma/2} = \frac{\psi_0}{\sqrt{2\pi}} \frac{i(E-E_0) + \Gamma/2}{(E-E_0)^2 + \Gamma^2/4}.$$

The curve $(1 + x^2)^{-1}$ is called "Lorentzian" or "Cauchy" in math and "**Breit-Wigner**" in physics; it describes a RESONANCE and appears in many other phenomena:

- forced mechanical oscillations;
- electric circuits;
- accelerators;
- …



Excited states : BW properties

Cauchy (or Lorentz, or BW) distribution :

$$f(x) = BW(x\,|\,x_0,\gamma) = \frac{1}{\pi\gamma}\frac{\gamma^2}{(x-x_0)^2+\gamma^2};$$

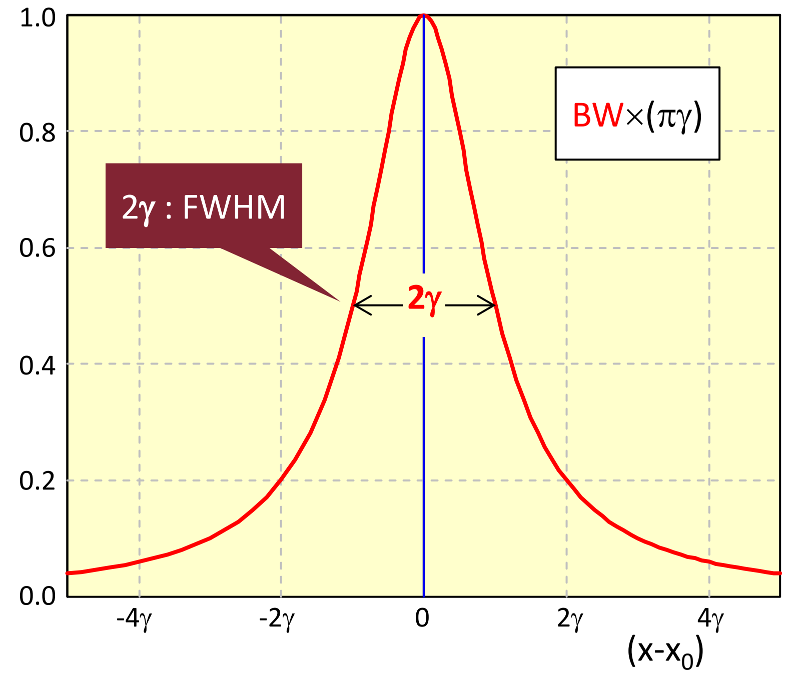
- median = mode = x_0;
- mean = math undefined [but use x_0];
- variance = really undefined [divergent]

This anomaly is due to

$$\langle x \rangle = \int_{-\infty}^{+\infty} x f(x) dx = \infty;$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x) dx = \infty$$

The anomaly does NOT conflict with physics : the BW is an approximation valid only if $\gamma \ll x_0$ and in the proximity of x_0, e.g. in case of an excited state (mass m, width Γ), for ($\Gamma \ll m$) and ($|\sqrt{s} - m|$ < few Γ's).



The "relativistic BW" is usually defined as

$$BW_{rel}(x\,|\,x_0,\gamma) = \frac{x_0^2\gamma^2}{(x^2-x_0^2)^2+x_0^2\gamma^2} \quad \left[\begin{array}{c}\text{properly}\\ \text{normalized}\end{array}\right].$$

The formula comes from the requirement to be Lorentz invariant [see *Berends et al., CERN 89-08, vol 1*].

Excited states : BW properties

From first principles of QM

$$\sigma_{ab\to R\to X}(E_{CM}=\sqrt{s}) = \frac{\pi}{|\vec{p}_{a,b}|^2}\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\frac{\Gamma_{ab}\Gamma_X}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4} \approx$$

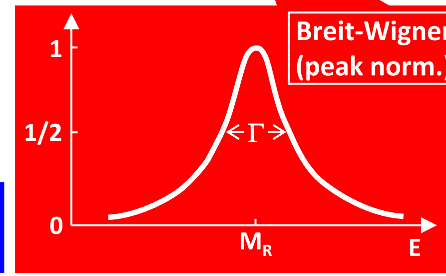
$$\approx \left[\frac{16\pi}{s}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_X}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

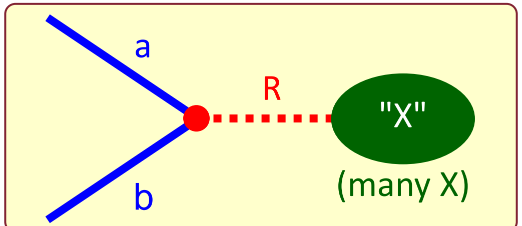
scale factor (1/s)

statistical factor (particle spins)

= BR(R→ab)

= BR(R→X)





$(E, \vec{p})$: CM 4-mom.

Γ_R : constant width

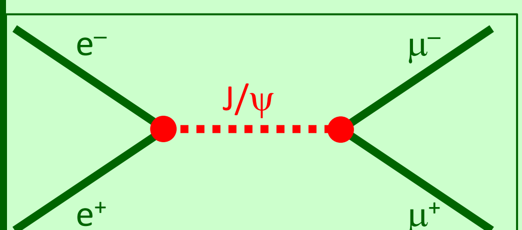
$\Gamma_{ab,\,X}$: couplings

M_R : E_0, mass

e.g.
$e^+e^- \to J/\psi \to \mu^+\mu^-$

$\sigma_{peak} \propto 1/s$ ($\approx M_R^{-2}$), independent from coupling strength.

$$\sigma(e^+e^-\to J/\psi\to\mu^+\mu^-) = \left[\frac{16\pi}{s}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_{tot}}\right]\left[\frac{\Gamma_{\mu\mu}}{\Gamma_{tot}}\right]\left[\frac{(\Gamma_{tot}/2)^2}{\left(\sqrt{s}-M\right)^2+(\Gamma_{tot}/2)^2}\right] =$$



$$= \frac{12\pi}{s}BR_{J/\psi\to e^+e^-}BR_{J/\psi\to\mu^+\mu^-}\left[\frac{(\Gamma_{tot}/2)^2}{\left(\sqrt{s}-M\right)^2+(\Gamma_{tot}/2)^2}\right].$$

Resonance : different functions

Many more parameterizations used in literature (semi-empirical or *theory inspired*), e.g.:

$$\sigma_0 = \left[\frac{16\pi}{(2p)^2}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

original, non-relativistic

$$\sigma_1 = \left[\frac{16\pi}{s}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

m_a, m_b << p

$$\sigma_2 = \left[\frac{16\pi}{M_R^2}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

if $M_R \gg \Gamma_R$, neglect s-dependence

$$\sigma_3 = \left[\frac{16\pi}{M_Z^2}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_Z}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_Z}\right]\left[\frac{M_Z^2\Gamma_Z^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}\right]$$

relativistic BW for $e^+e^- \to Z \to f\bar{f}$

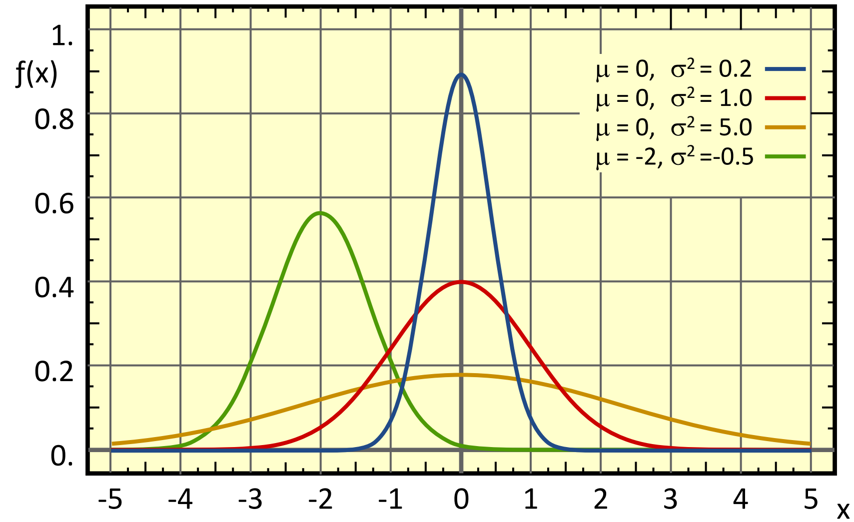
$$\sigma_4 = \left[\frac{16\pi}{M_Z^2}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_Z}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_Z}\right]\left[\frac{s\Gamma_Z^2}{\left(s-M_Z^2\right)^2+s^2\Gamma_Z^2/M_Z^2}\right]$$

"s-dependent Γ_Z" (used at LEP for the Z lineshape)

Gauss Distribution

$$f(x) = G(x\,|\,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- mean = median = mode = μ;
- variance = σ^2;
- symmetric : $G(\mu+x) = G(\mu-x)$
- central limit theorem* : the limit of processes arising from multiple random fluctuations is a single G(x);
- similarly, in the large number limit, both the binomial and the Poisson distributions converge to a Gaussian;
- therefore $\boxed{G(x\,|\;\mu=x_{meas},\;\sigma=error_{meas})}$ is often used as the resolution function of a given experimental observation [*but as a good (?) first approx. only*].



* Consider n independent random variables $x = \{x_1, x_2, \ldots, x_n\}$, each with mean μ_i and variance $\sigma^2{}_i$; the variable

$$t = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{x_i - \mu_i}{\sigma_i}$$

can be shown to have a distribution that, in the large-n limit, converges to $G(t\,|\,\mu=0,\sigma=1)$.

Gauss Distribution: hypothesis test

Given a measurement x with an expected value μ and an error σ, the value

$$F(x) = \int_x^{+\infty} G(t\,|\,\mu,\sigma)dt$$

is often used as a "hypothesis test" of the expectation.

E.g. (see the ◣ plot): if the observation is at 2σ from the expectation, one speaks of a "2σ fluctuation" (not dramatic, it happens once every 44 trials – or 22 trials if both sides are considered).

The value of "5σ" * has assumed a special value in modern HEP [*see later*].

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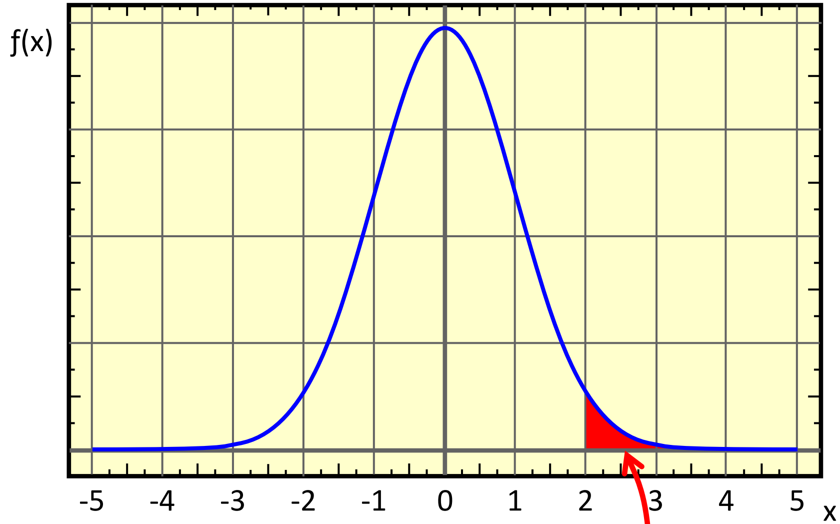
* if the expectation is not gaussian, one speaks of "5σ" when there is a fluctuation ≤ 2.87 E-7 in the tail of the probability, even in the <u>non-gauss case</u>.



x	G(x|0,1)	F(x)	$=1/n_{trial}$
0	3.989 E-01	5.000 E-01	2
1	2.420 E-01	1.587 E-01	6.3
2	5.399 E-02	**2.275 E-02**	44.0
3	4.432 E-03	1.350 E-03	741
4	1.338 E-04	3.167 E-05	31,500
5	**1.487 E-06**	**2.867 E-07**	**3.5 E+06**
6	6.076 E-09	9.866 E-10	1.0 E+09
7	9.135 E-12	1.280 E-12	7.8 E+11

Gauss Distribution: the «Voigtian»

Assume :

- a physical effect (e.g. a resonance) of intrinsic width described by a BW;
- a detector with a gaussian resolution;

→ the measured shape is a convolution "Voigtian" (after Woldemar Voigt).

- the V. is expressed by an integral and has no analytic form if $\underline{\gamma > 0 \text{ AND } \sigma > 0}$.
- however modern computers have all the stuff necessary for the numerical computations;
- mean = mathematically undefined [use x_0];
- variance = really undefined [divergent].

→ for real physicists : check carefully if resolution is gaussian, dynamics is BW, and γ and σ are uncorrelated .

$$f(x) = V(x\,|\,x_0,\gamma,\sigma) =$$

$$= \int_{-\infty}^{+\infty} dt\, G(t\,|\,0,\sigma)\, BW(x-t\,|\,x_0,\gamma) =$$

$$= \int_{-\infty}^{+\infty} dt \left[\frac{e^{\left(-\frac{t^2}{2\sigma^2}\right)}}{\sigma\sqrt{2\pi}} \right] \left[\frac{1}{\pi\gamma} \frac{\gamma^2}{(x-t-x_0)^2+\gamma^2} \right].$$