The Discovery of the Antiproton

Dirac’s Equation and Antimatter

- In 1928, Dirac formulated a theory describing the behavior of relativistic electrons in electric and magnetic fields.
  - Dirac’s equation has negative energy solutions, implying the existence of antimatter.
- The positron was discovered in 1932 from cosmic ray experiments.
  - This method would not work for discovering antiprotons.
  - No accelerator existing at that time was energetic enough to produce antiprotons.

Requirements to Make an Antiproton

- Creating an antiproton would also require the simultaneous production of a proton or neutron.
  - Since the mass of the proton is 938 MeV, the minimum energy required to get an antiproton is two times that, or about 2 GeV (In those days, physicists typically said BeV instead of GeV.)
  - Using the fixed target technology of the time, this would require striking the target with a 6 GeV proton.
- A new accelerator that had an energy of several GeV (or BeV) was required, hence the name Bevatron.

Gli "eventi strani" nei raggi cosmici

E. Hayward, "Ionization of High Energy Cosmic-Ray Electrons", Physical Review 72 (1947)

E. W. Cowan, "A V-Decay Event with a Heavy Negative Secondary, and Identification of the Secondary V-Decay Event in a Cascade", Physical Review 94 (1954)

M. Schein, D.M. Haskin, and R.G. Glasser, "Narrow Shower of Pure Photons at 100000 Feet", Physical Review 95 (1954)

H.S. Bridge, H. Courant, H. DeStaebler, Jr., and B. Rossi, "Possible Example of the Annihilation of a Heavy Particle", Physical Review 95 (1954)

E. Amaldi, C. Castagnoli, G. Cortini, C. Franzinetti and A. Manfredini, "Unusual Event Produced by Cosmic Rays", Il Nuovo Cimento Vol. I N. 3 (1955)

PHYSICAL REVIEW VOLUME 94, NUMBER 1 APRIL 1, 1954

A V-Decay Event with a Heavy Negative Secondary, and Identification of the Secondary V-Decay Event in a Cascade*

E. W. COWAN
California Institute of Technology, Pasadena, California
(Received December 31, 1953)

Two cosmic-ray decay events have been photographed in a cloud chamber under conditions that yield mass values from combined magnetic-field momentum measurements and ionization measurements from droplet counting. A method has been developed for assigning meaningful probable errors to the ionization measurements. The first event is interpreted as the decay of a neutral V particle into a positive π meson and a negative particle of mass $1850\pm250m_e$. On the assumption of a two-body decay, the Q value for the decay is 11.7 ± 4 Mev. The second event is a cascade decay that can be summarized by the following reaction:

$$Y^- \rightarrow 67\pm12 \text{ Mev} + \pi^- + \Lambda^0$$
$$\searrow 40\pm13 \text{ Mev} + \pi^- + p.$$

The proton of the Λ^0 decay is identified by a measured mass of $2050\pm350m_e$. On the assumption of a two-body decay, the mass of the primary V particle is $2600\pm34m_e$.

"The mass of this particle is near or equal to that of a proton and is not consistent with the mass of any negative particle that has been identified… there is no clear evidence that the particle is actually an antiparticle to the proton. No annihilation phenomenon is observed…" (Cowan 1953)

E. W. Cowan, "A V-Decay Event with a Heavy Negative Secondary, and Identification of the Secondary V-Decay Event in a Cascade", Physical Review 94 (1954)

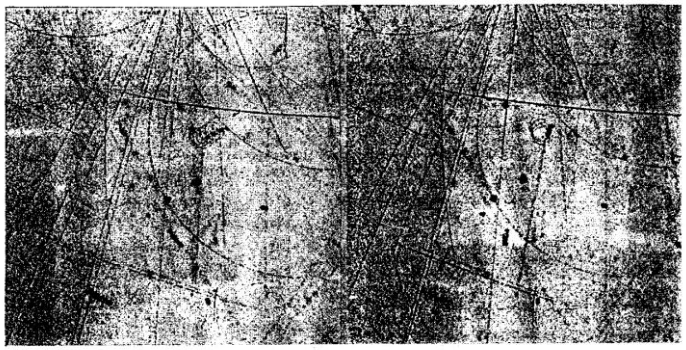


FIG. 1. A V-decay event with a heavy negative secondary.

"... The mass of this particle is near or equal to that of a proton and is not consistent with the mass of any negative particle that has been identified... there is no clear evidence that the particle is actually an antiparticle to the proton. No annihilation phenomenon is observed... "

E. Hayward, "Ionization of High Energy Cosmic-Ray Electrons", Physical Review 72 (1947)

(a) (b) (c)

FIG. 5. Cloud-chamber photographs of unusual events, as described in the text.

"... is a photograph of the track of a particle that ionized above five times as much as an average mesotron and also seems to have produced a huge shower in the lead below.... Other possible explanations are that... it is a negative proton giving up all of its energy in interacting with the lead plate"

M. Schein, D.M. Haskin, and R.G. Glasser, "Narrow Shower of Pure Photons at 100000 Feet", Physical Review 95 (1954)

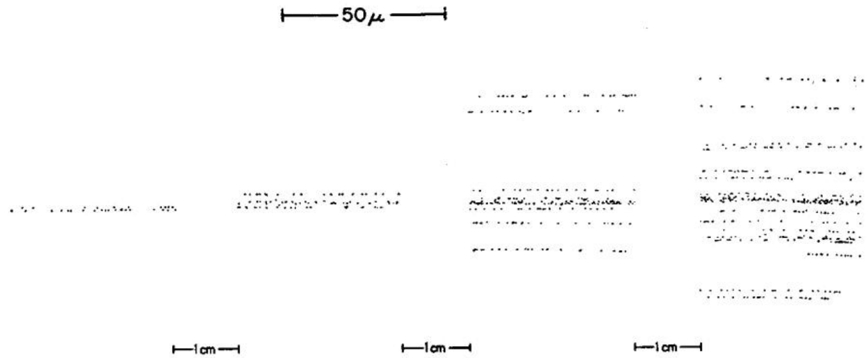


FIG. 1. Narrow shower of pure photons. Sections at arbitrary intervals to show development of shower. Note pair starting in last section.

" Such a phenomenon would appear to be incompatible with the production of these photons by any conventional electromagnetic process... One possibility... is that it may be produced by an annihilation process in flight at very high energy"

"Other possible explanations are that… it is a negative proton" (Hayward 1947)

"One possibility… is that it may be produced by an annihilation process" (Schein et al. 1954)

Possible Example of the Annihilation of a Heavy Particle*

H. S. BRIDGE, H. COURANT, H. DESTAEBLER, JR.,† AND B. ROSSI

Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received June 21, 1954)

THE picture in Fig. 1 and the sketch in Fig. 2 show an unusual cosmic-ray event photographed with the M.I.T. multiplate cloud chamber at Echo Lake, Colorado. The chamber contained eleven brass plates, each 0.50 inch thick (11.1 g cm^{-2}) and was triggered by a penetrating-shower detector placed above it. Two additional views, taken at different angles, are available.

Three electron showers, *b*, *c*, *d*, appear to be associated with the stopping of a charged particle, *a*, in one of the plates. Within the experimental errors, the axes of the three showers and the direction of the last visible segment of track (*a*) intersect at one point in the plate.

From the number of small showers with no apparent origin occurring in our cloud chamber, we found an upper limit of 10^{-3} for the probability that either (*c*) or (*d*) may be a case of chance association. It is practically impossible to explain shower (*b*) in a similar way for a survey of about 10 000 pictures has not revealed a single shower of the size of (*b*), with no apparent origin and going upward.

H.S. Bridge, H. Courant, H. DeStaebler, Jr., and B. Rossi, "Possible Example of the Annihilation of a Heavy Particle", Physical Review 95 (1954)

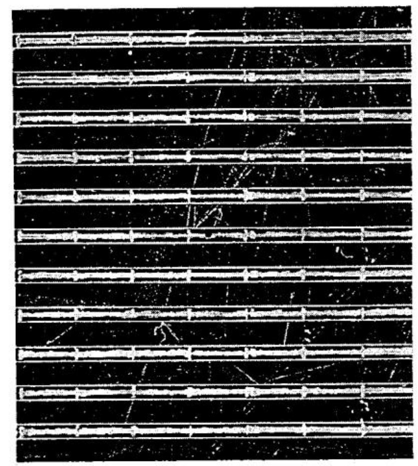


FIG. 1. Cloud-chamber photograph of the cosmic-ray event.

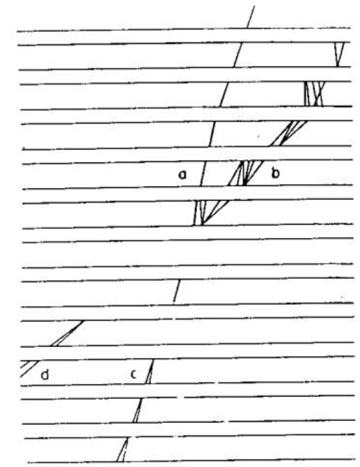


FIG. 2. Sketch of the cosmic-ray event.

"In view of the difficulties of interpreting the event as a decay or an absorption process, one should consider the possibility that the event represents the annihilation process of two heavy fermions. For example, the incident particle might be an antiproton (or an antihyperon) that undergoes annihilation with an ordinary proton. A large fraction of the energy liberated in such a process may well be changed into π^0 mesons and thus ultimately appear in the form of γ rays"

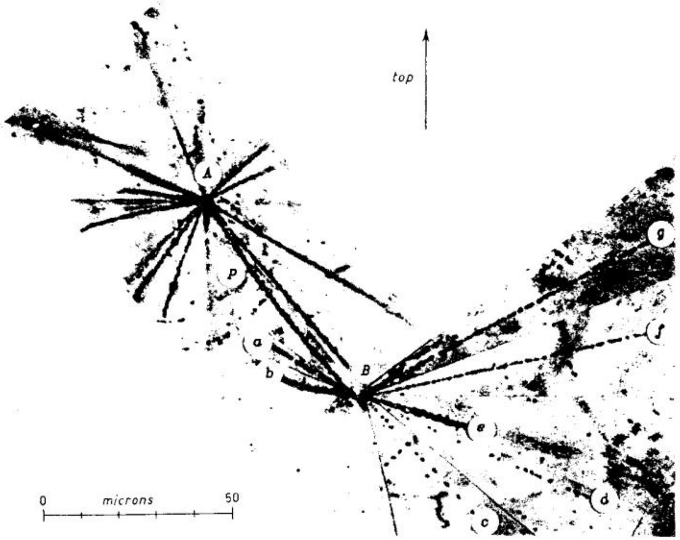
B. Rossi, Rochester Conference, 1956:

"... we used the photometric method to re-analyze the M.I.T. antiproton event, and found a value of 823 ± 155 Mev for the rest energy of this primary particle.... there is thus little doubt that the M.I.T. event was indeed the annihilation of an antiproton"

"… there is thus little doubt that the M.I.T. event was indeed the annihilation of an antiproton" (Rossi 1956)

"Unusual Event Produced by a Heavy Particle at Rest"

"Unusual Event Produced by Cosmic Rays"



"... the interpretation of this track in terms of a high energy fragment... is very improbable. Such a conclusion is definitely confirmed by the fact that the deflection of a fast fragment through an angle of 90° should be associated with a rather long recoil track, even in the case of a target nucleus as heavy as silver. No recoil is observed in the present case.... the track is due to a low energy particle.

... the event could also be due to an accidental coincidence in space. Therefore we have evaluated the probability for such a coincidence... the value is sufficiently small to entitle us to look for an interpretation of the observed event in terms of a physical process... We are left to consider the star B as produced by the track p. Then the corresponding particle either has rest energy of the order of 1.5 $\div$ 2 GeV, or, being an antiproton, it has been annihilated by a nucleon, releasing 2 $m_p c^2$ = 1876 MeV.

One can conclude that the probability of an accidental coincidence can not be disregarded although it is rather small. If one excludes this possibility the more likely interpretation seems to be that of an annihilation process of a heavy particle... the many questions raised by the discussion of this event will obviously find their final answer only if other similar events will be observed."

IL NUOVO CIMENTO Vol. I, N. 3 1° Marzo 1955

Unusual Event Produced by Cosmic Rays.

E. Amaldi, C. Castagnoli, G. Cortini, C. Franzinetti and A. Manfredini

Istituto di Fisica dell'Università - Roma

Istituto Nazionale di Fisica Nucleare - Sezione di Roma

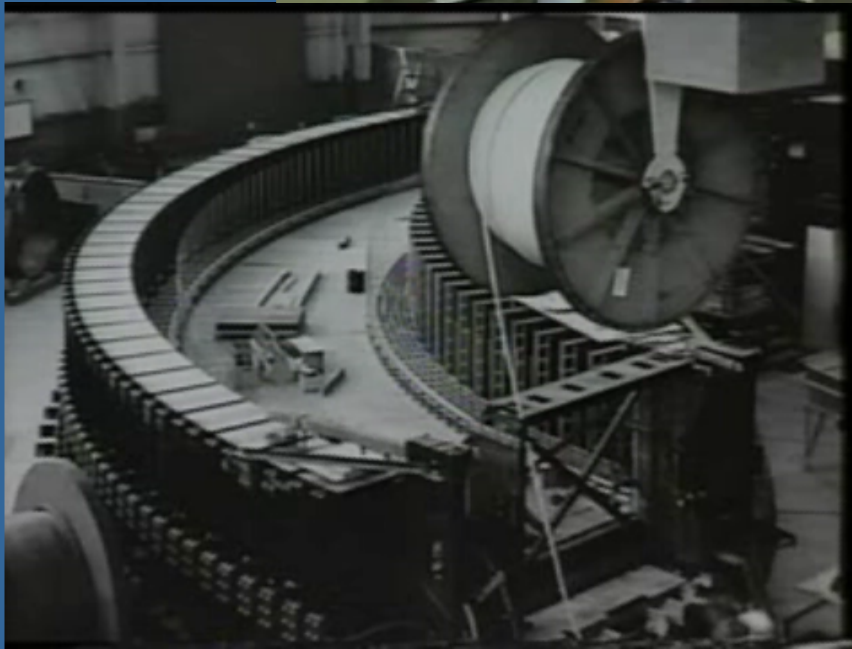
(ricevuto il 18 Febbraio 1955)

Summary. – The authors describe an event consisting of two stars respectively of about 5 and 1-2 GeV energy. The probable value of the number of accidental space coincidences that one expects to observe in the scanned volume, is about $4\cdot10^{-4}$. This value, although it does not allow us to exclude an accidental process, justifies the consideration of interpretations in terms of some physical process. Special attention is devoted to the production, capture and annihilation of a negative proton.

“Faustina”, l’evento “strano” rintracciato all’inizio del 1955 dal gruppo di Roma nelle lastre esposte alla radiazione cosmica durante la spedizione di Sardegna del 1953

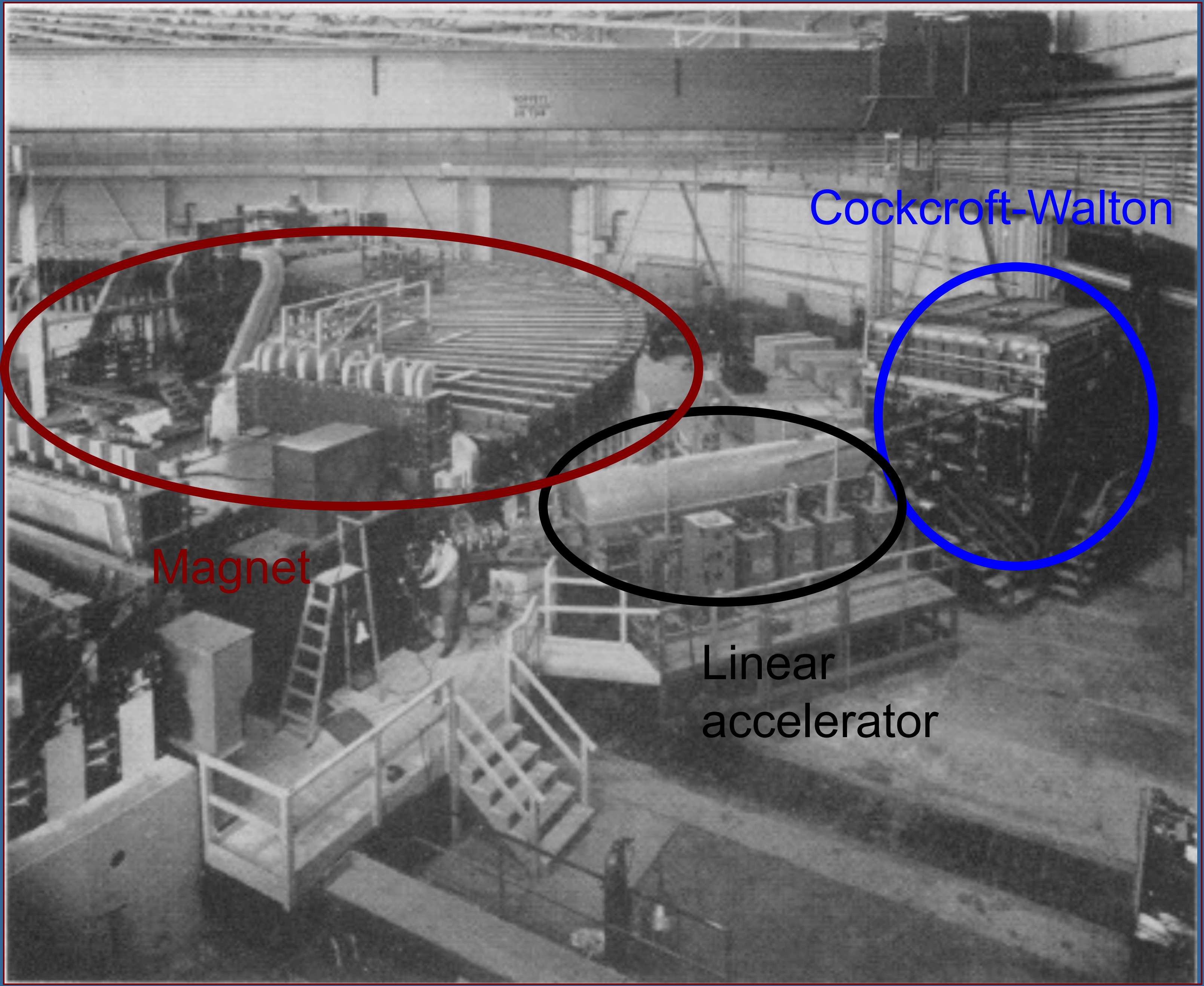
The Beginning

- Design started in 1947 under the direction of Ernest Lawrence. The primary designer was engineer William Brobeck.
- Construction began in 1949 at The University of California Radiation Laboratory at Berkeley. (The lab was later named the Lawrence Berkeley National Laboratory).
- The first beam at the full energy of 6.2 BeV (GeV) was delivered on April 1, 1954.



The Bevatron

- The protons are held in a circular path by a magnetic field.
- An accelerating electrode is used to give repeated increments of energy to the protons.
  - Electrodes are exited with radio frequency in synchronism with the particles.
- Unlike earlier accelerators, the radius of the particle is approximately constant.
  - The magnetic field varies during the accelerating cycle.
  - The frequency of the accelerating voltage increases with particle speed.

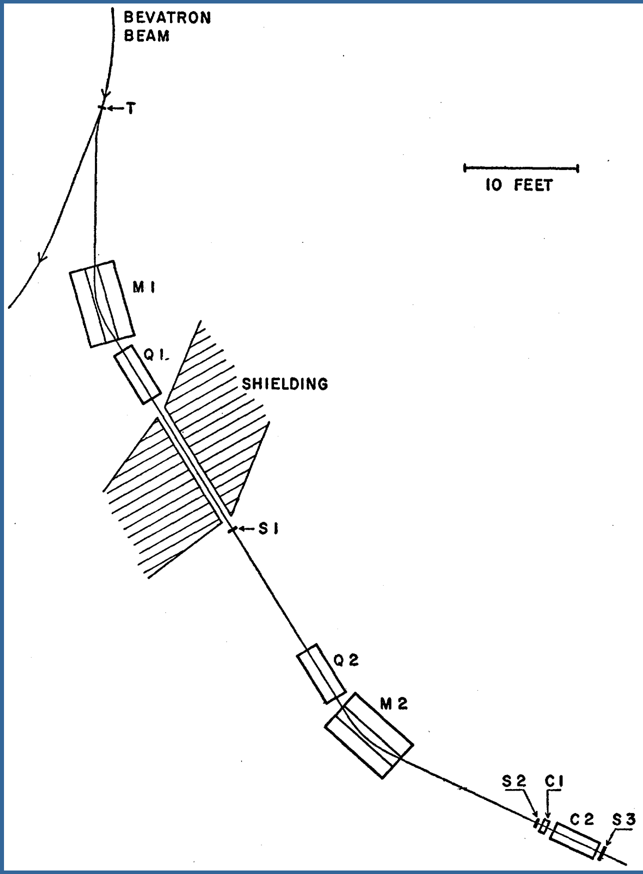

Cockcroft-Walton
Magnet
Linear
accelerator

Acceleration in the Bevatron

- At the time of injection into the Bevatron, the magnetic field is 300 gauss.
- Radio frequency power is applied to the accelerating electrode.
  - Each time the protons pass through the accelerating electrode, they gain 1500 eV.
  - The magnetic field and frequency of the accelerating power are continuously increased.
- After 2 seconds, the magnetic field has increased to 15,500 gauss, and the protons have an energy of 6.2 BeV (GeV).

Antiprotons or Pions?

- The antiprotons had to be found in large background of π^-.
- The negative particles were deflected and focused by magnet M1 and quadrupole focusing magnet Q1.
- The particles passed through scintillation counter S1.
- The particles were again focused and deflected by Q2 and M2 on their way to S2.
- By measuring the time of flight between S1 and S2, antiprotons could be distinguished from π^-.
- *S1* and *S2* are *12 m* apart in the beam. The time of flight for π (with $\beta > 0.96$) for this distance was *40 ns*, while for antiprotons ($\beta \sim 0.76$) it was *51 ns*.

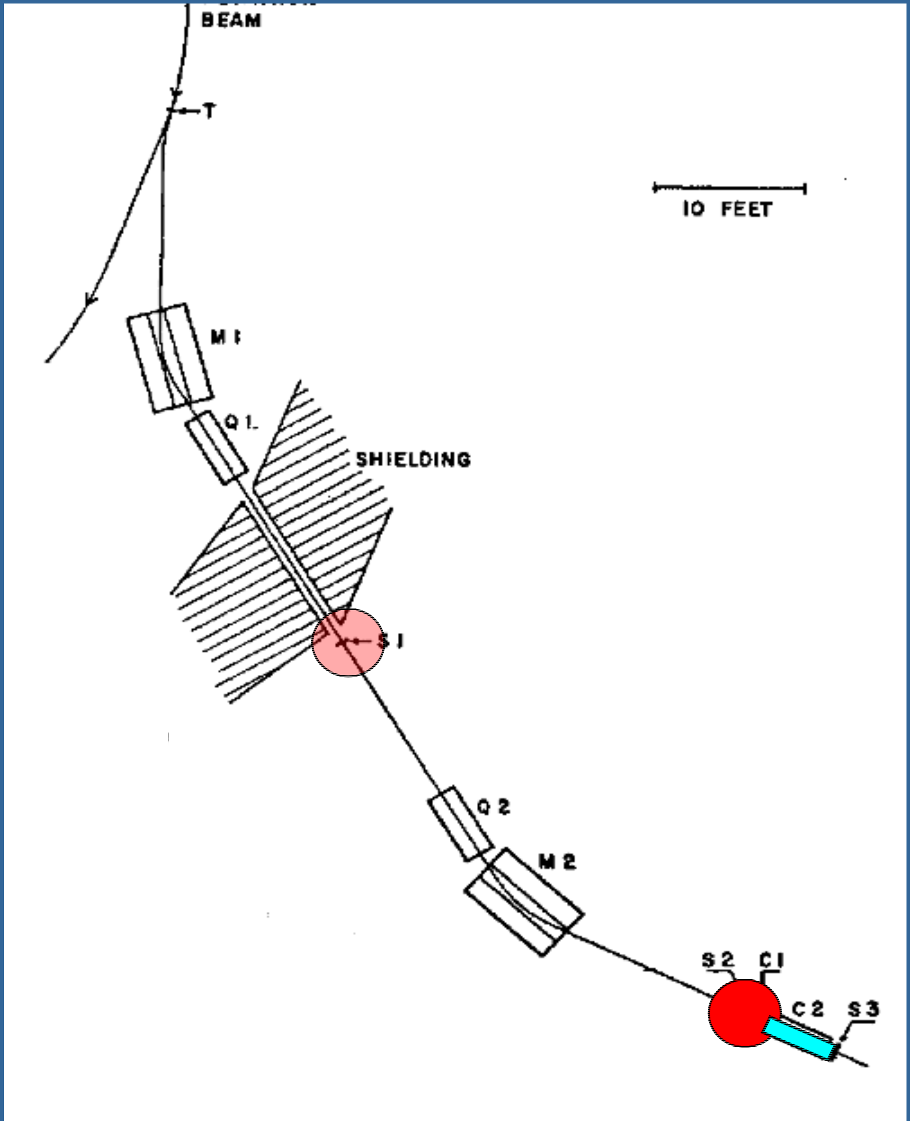


Antiprotons or pions?

Since the antiprotons must be selected from a heavy background of pions it has been necessary to measure the velocity by more than one method.... C2 is a Cerenkov counter that counts particles only within a narrow velocity interval 0.75 < β < 0.78 ... the requirement that the particle be counted in this counter constituted one of the determinations of the velocity of the particle....

the apparatus has some shortcomings ... Accidental coincidences between S1 and S2 cause some mesons to count .... [and] C2 could be actuated by .... [one of these] if the meson suffered a nuclear scattering in the radiator of the counter [C2] . ... Both of these deficiencies have been eliminated by the insertion of the guard counter C1 , which records all particles of β > 0.79. A pulse from C1 indicates a particle (meson) moving too fast to be an antiproton of the selected momentum and indicates that this event should be rejected

Chamberlain et al., 1955 [10]



Observation of Antiprotons*

OWEN CHAMBERLAIN, EMILIO SEGRÈ, CLYDE WIEGAND, AND THOMAS YPSILANTIS

Radiation Laboratory, Department of Physics, University of California, Berkeley, California

(Received October 24, 1955)

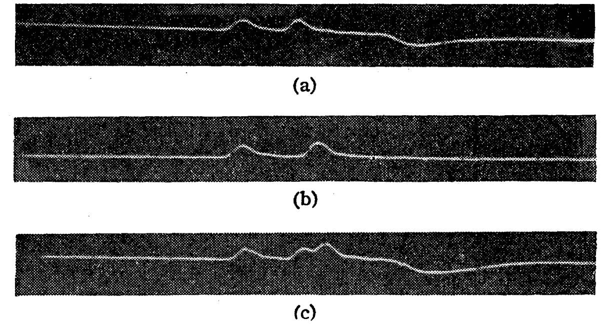


FIG. 2. Oscilloscope traces showing from left to right pulses from $S1$, $S2$, and $C1$. (a) meson, (b) antiproton, (c) accidental event.

- Antiprotons were discovered in 1955.
  - 1959 Nobel Prize in Physics for Chamberlain and Segre.
- Antiprotons have a time of flight over the 40 ft interval of 51 ns.
  - 40 ns for π^-.

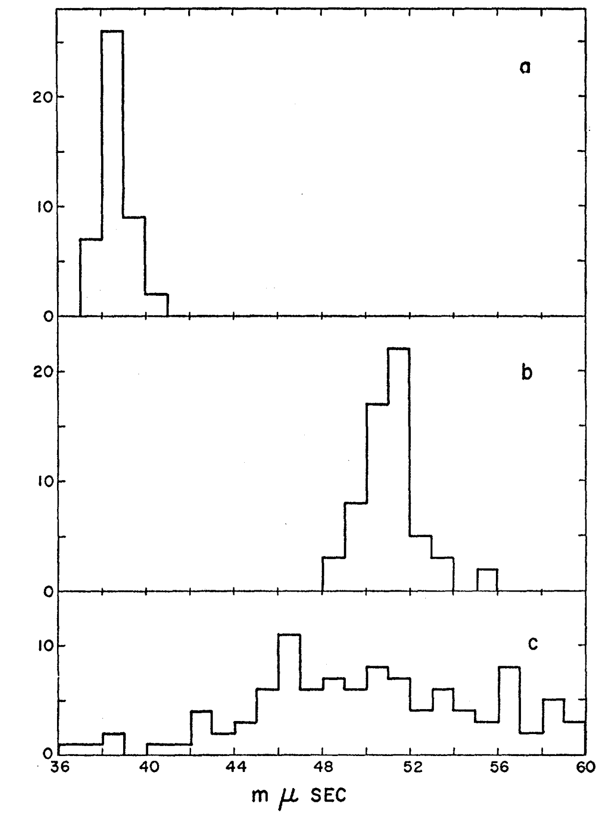


FIG. 3. (a) Histogram of meson flight times used for calibration. (b) Histogram of antiproton flight times. (c) Apparent flight times of a representative group of accidental coincidences. Times of flight are in units of 10^{-9} sec. The ordinates show the number of events in each 10^{-10}-sec intervals.

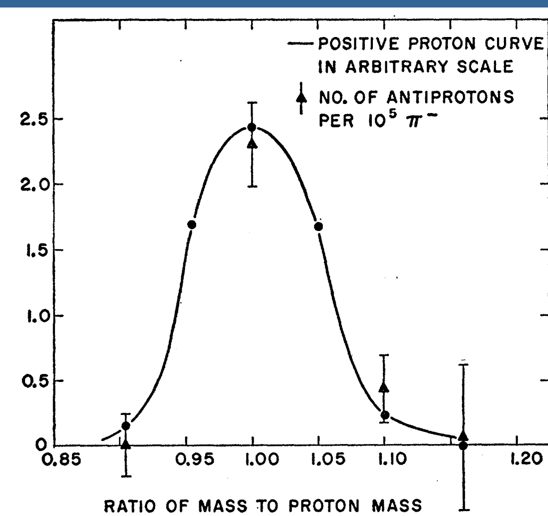
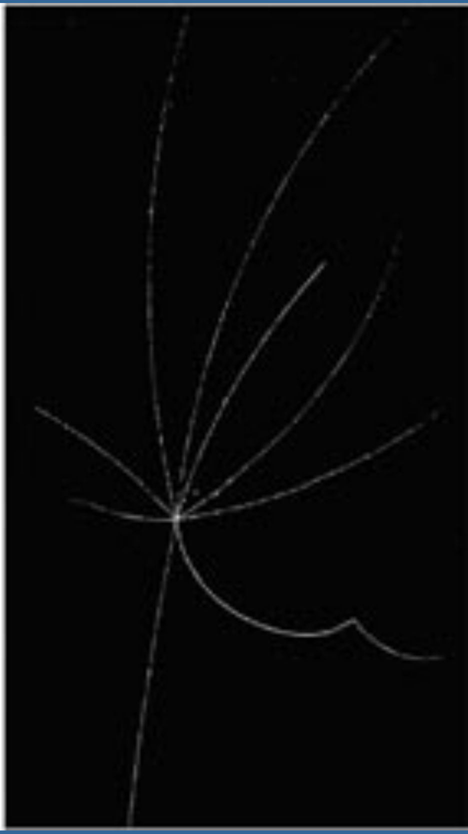


FIG. 4. The solid curve represents the mass resolution of the apparatus as obtained with protons. Also shown are the experimental points obtained with antiprotons.

Visual Confirmation

- The left picture is the annihilation star from an antiproton, viewed in photographic-emulsion stack experiments.
  - Led by Gerson Goldhaber of Segre’s group.
- The right picture is a bubble chamber image.
  - The antiproton enters from the bottom.
    - Upon striking a proton, four positive and four negative pions are created.



Pions and Muons

Conversi, Pancini, Piccioni (CPP) experiment

Marcello Conversi

Ettore Pancini

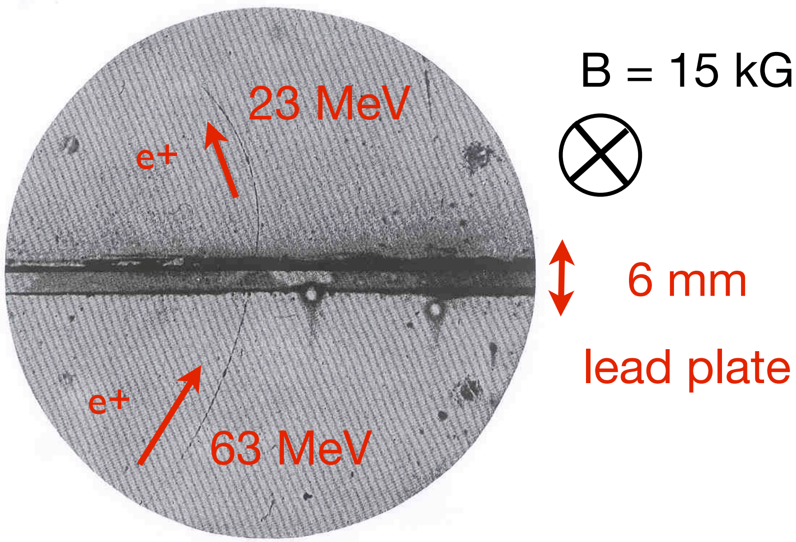
Oreste Piccioni

Historical introduction

1912 Discovery of cosmic rays (V. Hess)

1932 Discovery of positron (C. Anderson)

predicted by Dirac in 1928



Measurement of the curvature of the track

Range not compatible with proton hypothesis (~5 mm in Air)

From ΔE infer upper limit on particle mass (< 20 mass electron)

Historical Introduction II

1935 Yukawa theory for strong interaction

predicts mediator particle with mass ~ 100 MeV. Yukawa meson is supposed to decay into electron and neutrino with decay time of ~1 μs

1937 Discovery of the “mesotron” (C. Anderson, S. Neddermeyer)

mass ~110 MeV → associated with Yukawa particle

1940 Decay and absorption properties of Yukawa mesons (S. Tomonaga, G. Araki)

Yukawa meson - strong interaction with matter

nuclear capture and decay rates are different for positive and negative mesons at rest. Different interaction with positive nuclei.

Decay of the mesotron

1940 Observation of mesotron decays into positrons (William, Roberts)

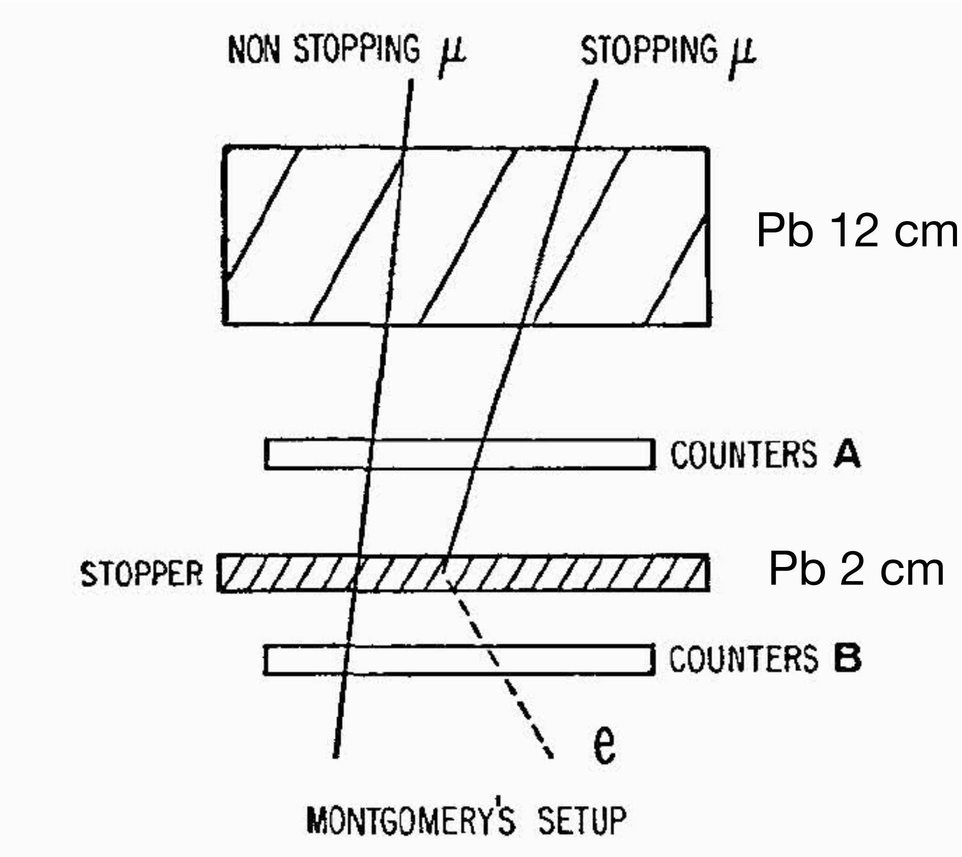
1941 Measurement of decay time (Rasetti) $\tau = (1.5 \pm 0.3)\mu s$

1941 Piccioni and Conversi decided to work together and improve the precision on the decay time measurement...

Piccioni ed io, quando sul finire del 1941 decidemmo di lavorare insieme, avevamo in mente la determinazione diretta della vita media del mesotrone. Piccioni, con alcuni anni di esperienza più di me, aveva una profonda conoscenza ed un grande entusiasmo per l'elettronica, e la maggior parte dello sviluppo che ne seguì fu dovuto alla sua grande competenza ed ingegnosità in questo campo." (M.C.)

Pioneer experiments for decay time measurement

1939 C. Montgomery, W. Ramsey, D. Cowie, D. Montgomery



Pb 12 cm

Pb 2 cm

Register delayed coincidence between A and B, after a time interval t_1 and before t_2

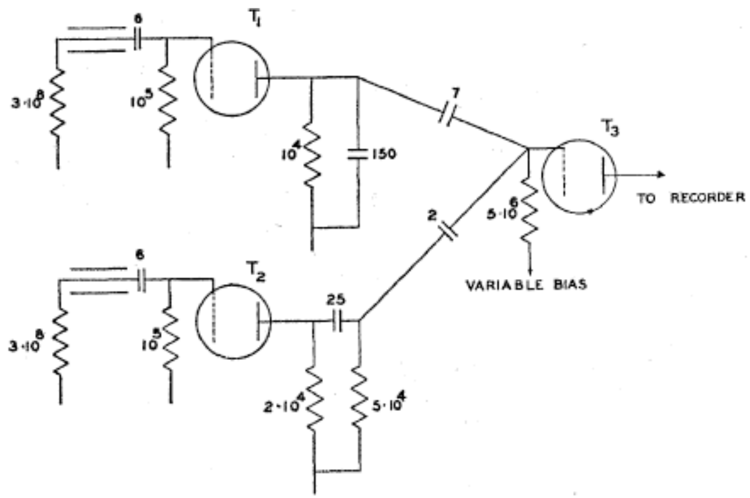


FIG. 2. Circuit for selecting the delayed coincidences.

Extract decay time from intensities of delayed coincidences with and without stopper.
Too spurious coincidences in counter B.
Experiment was not successful

Mesotron decay time measurement

1940 F. Rasetti

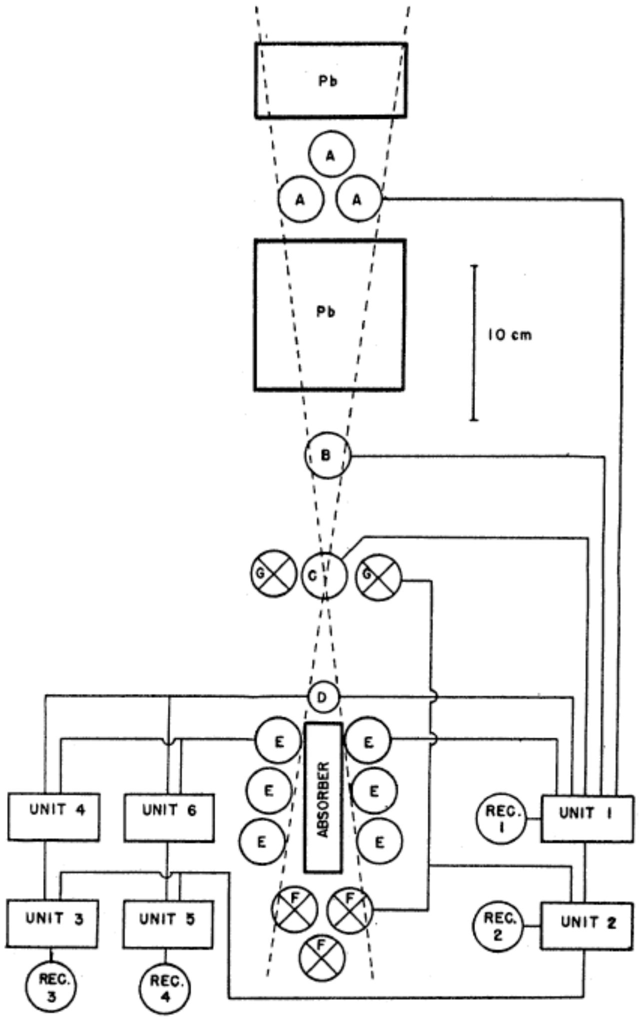


FIG. 1. Arrangement of counters, illustrating connections to amplifier units.

Experimental Procedure

Define a beam of mesotrons with the coincidence (ABCD)
Anticounter G (anticoincidence) discriminates against e.m. showers
Anticounter F selects mesotrons stopped in the absorber
Counter E detects particles emitted in the absorber

Signal events in Rasetti experiment

No delayed coincidences but “prompt” coincidences with different resolving times.

Signal + Noise events

n_2: defined by 5 the coincidence and 1 anticoincidence (ABCDE-F). Resolving time t_2=12μs

Noise events

n_3, n_4: defined by (ABCDE-F) (in coincidence with (DE), {(ABCDE-F) (DE)}. Resolving time of (DE), t_3= 1.95μs, t_4= (0.95, 0.76) μs.

It is not possible to prove the existence of an exponential decay.

The mean life τ of the disintegration process is calculated from the differences of the numbers of counts, according to the formula:

$$\exp\left(-\frac{t_3-t_4}{\tau}\right)=\frac{n_2-n_3}{n_2-n_4}$$

coincidence resolving times

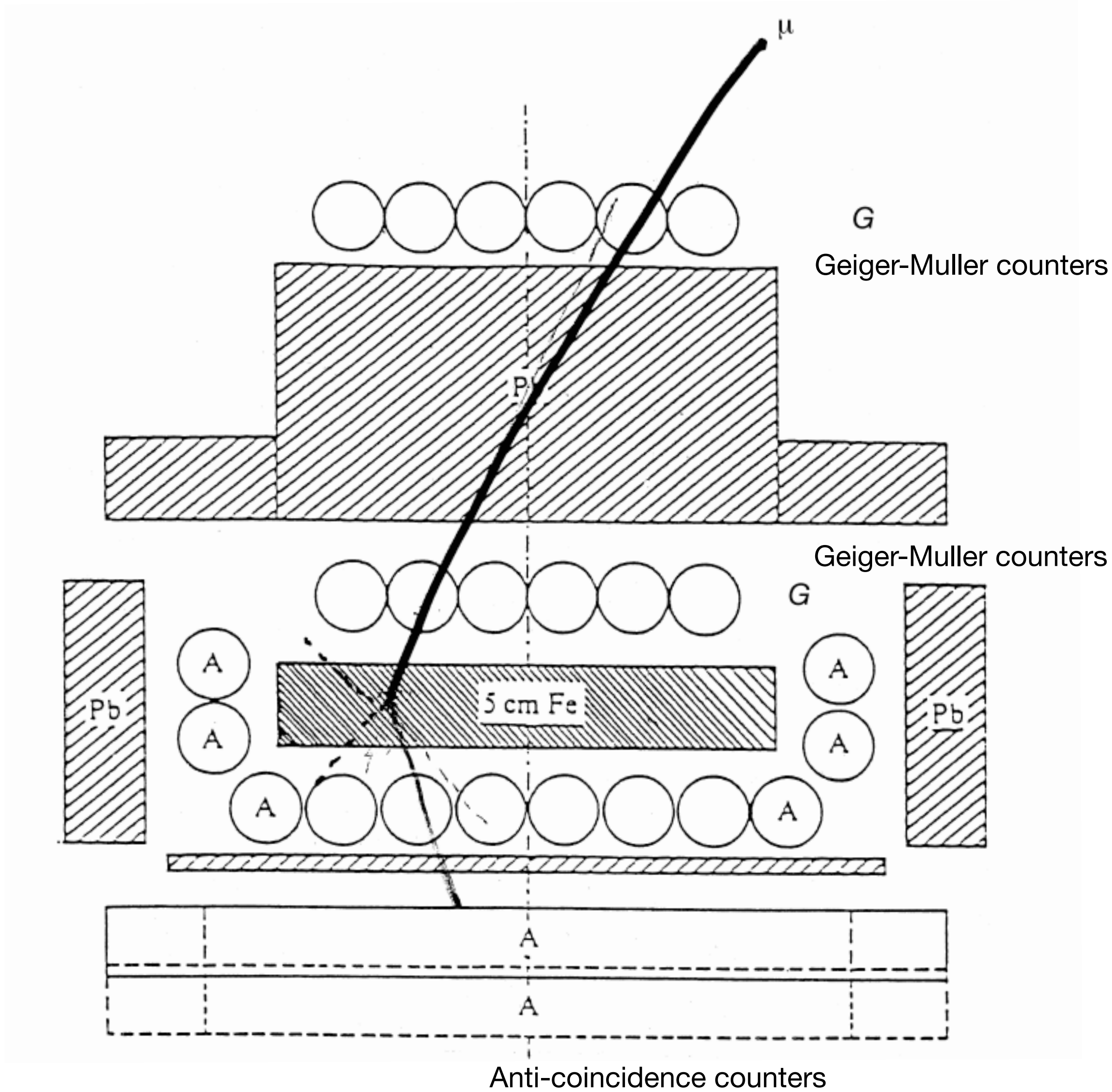
registered events

$$\tau = (1.5 \pm 0.3)\mu s$$

Piccioni e Conversi appreciated Rasetti measurement and in 1941 started working together to design an improved experiment.

Mesotron decay time, improved measurement

1942 M. Conversi, O. Piccioni



"*Il compito di sviluppare una tecnica adeguata era già arduo di per sé e venne reso ancora più gravoso, ovviamente, dalle condizioni particolari imposte dalla guerra*." (M.C.)

Signal events

a mesotron stops in the absorber (Fe) and then decay.

Use of delayed coincidences between top and bottom counters and anti-coincidences (A).

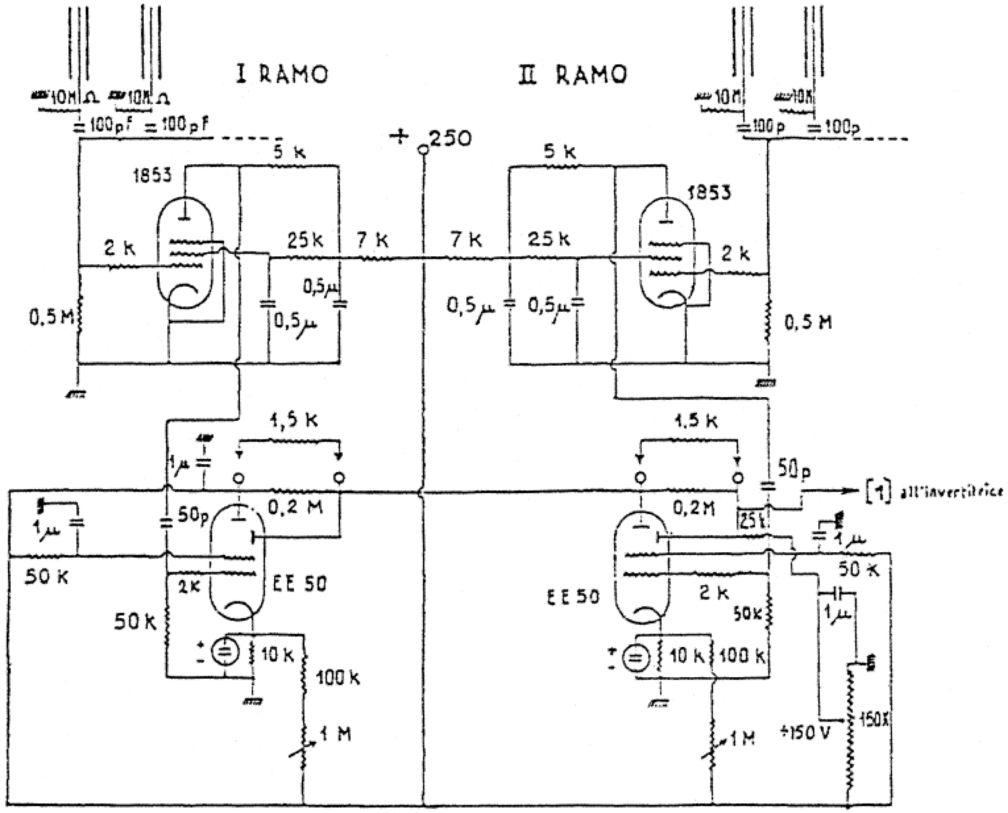
Electron are stopped by the Pb absorber.

Anti-coincidence signal reject the event.

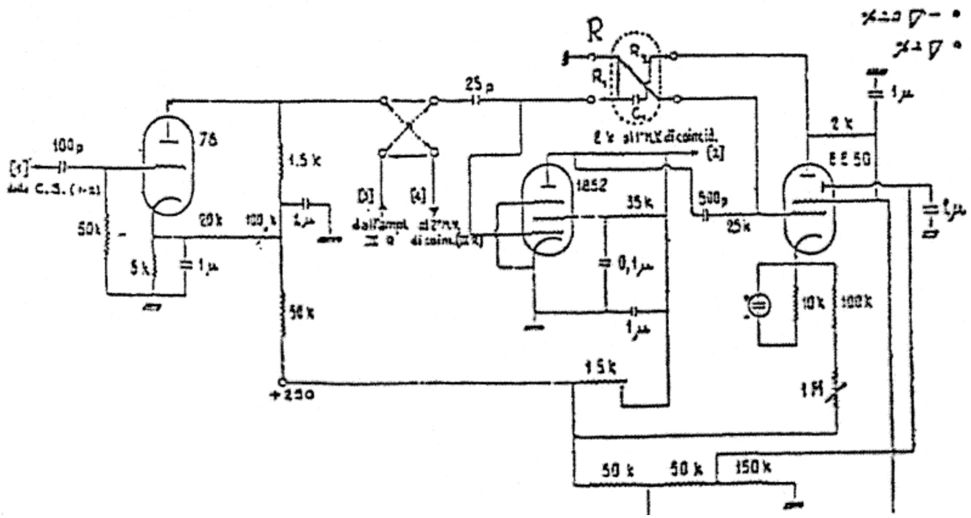
Design of precise and fast electronics by Piccioni and Conversi

M. Conversi, 1943. *Sul ritardo degli impulsi nei contatori di Geiger-Müller.* Atti della SIPS, Roma.
M. Conversi, O. Piccioni, 1943. *Sulle registrazioni di coincidenze a piccoli tempi di separazione.* Nuovo Cimento, 1; 279 ss.

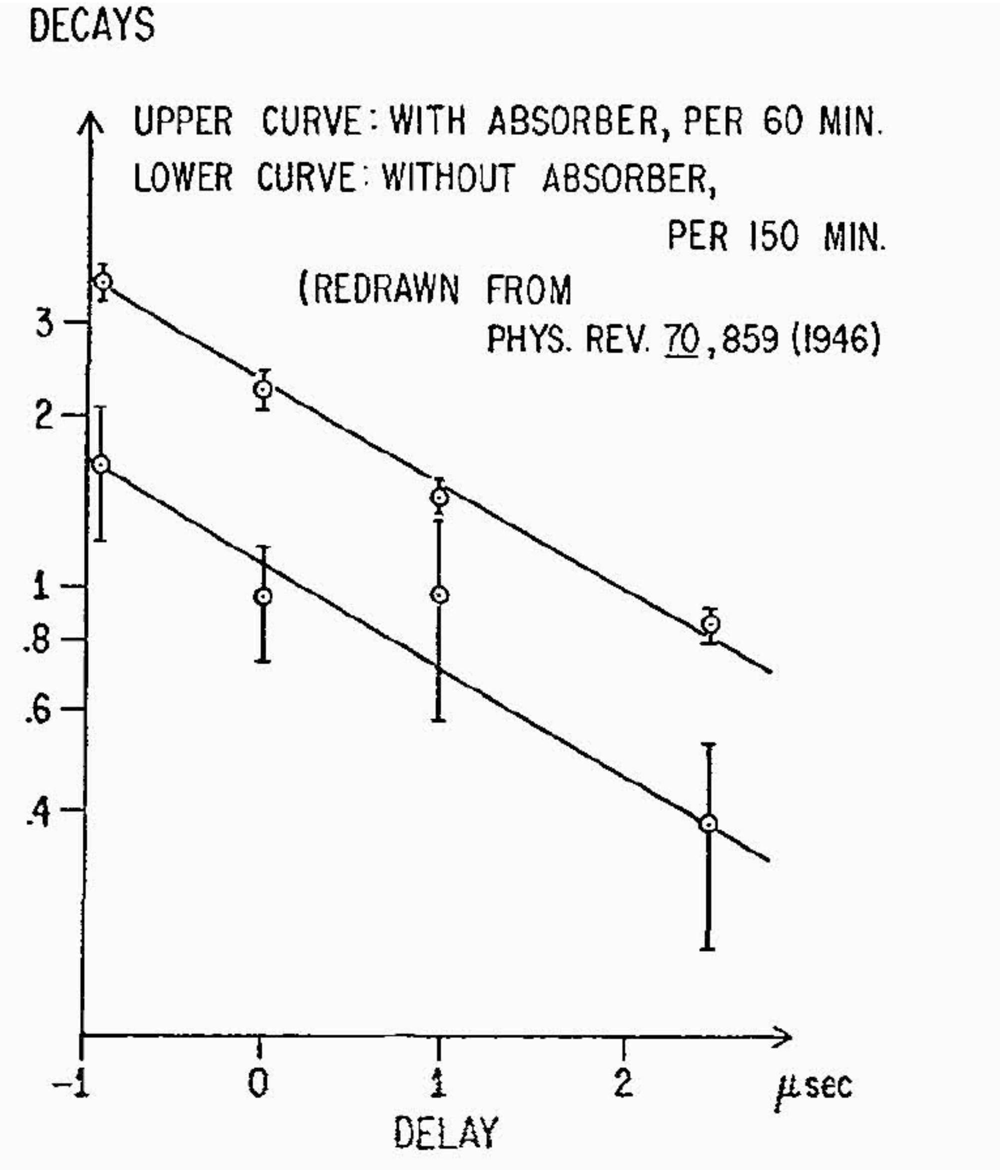
Input amplifiers designed by Piccioni and Conversi



Original scheme of the electronic circuit for the measurement of the coincidence delays



Results



First experimental evidence of exponential decay of the mesotron. Rossi and Nerenson obtained similar results independently.

Extract the decay time value from the slope of the distribution of the counts vs coincidence delay

$$N(t) = N_0 e^{-t/\tau}$$

$$\tau = (2.33 \pm 0.15)\mu s$$

Test of Tomonaga - Araki theory

S. TOMONAGA - G. ARAKI, Effect of the Nuclear Coulomb Field on the Capture of Slow Mesons. Phys.Rev., 58, 1940, 90-91.

In consequence of the Coulomb attraction the capture probability will increase for negative mesons, while for positives it will be greatly reduced by the potential barrier. The competition between nuclear capture and spontaneous disintegration must in this way be different for mesons of different signs.

Since the probability for negative mesons being captured is seen always to be larger than the probability of disintegration, which is of the order of 10^6 sec.$^{-1}$, the negative mesons will be much more likely captured by nuclei than disintegrate spontaneously,

Strong interaction

Yukawa meson interacts with nucleus mainly via strong interaction. According to calculations, the nuclear capture depends mildly from the Z value of the material

Nuclear capture process

slow positive mesotrons are repulsed by positive nuclei, while slow negative mesons can be captured

Decay process

slow positive mesotrons can only decay

The CPP experiment and the previous tests

"*La moderna fisica delle particelle ebbe inizio durante gli ultimi giorni della seconda guerra mondiale, quando un gruppo di giovani italiani, Conversi, Pancini e Piccioni, iniziarono un notevole esperimento*" (L. Alvares, Nobel Lecture 1968).

1944 First test of T-A theory

use same apparatus as for the previous decay time measurement. Thinner absorber (0.6 cm Fe instead of 5 cm) to improve electron detection efficiency. Measure ratio of mesotrons that decay inside Fe, $h = 0.49 \pm 0.07$ in agreement with predicted value $h=0.55$ due to 20% excess of positive mesotrons at sea level from cosmic rays.

1945 The CPP experiment

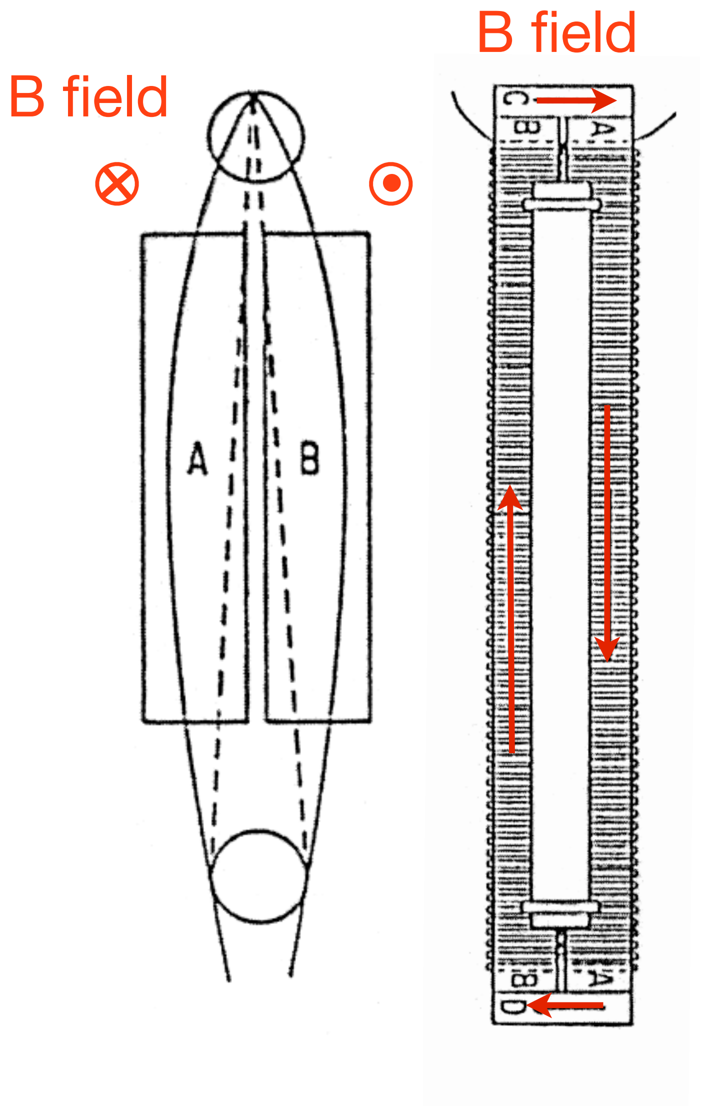
improved apparatus with magnetic lens for separation of negative and positive mesotrons. Use Fe (high Z material) as absorber. Confirmation of T-A theory.

1946 The CPP experiment

Use carbon (low Z) as absorber, for *experimental completeness sake.* Also for detecting photons emitted from nuclear capture of negative mesotrons. Results are in disagreement with the T-A theory.

Magnetic lens

1930 B. Rossi

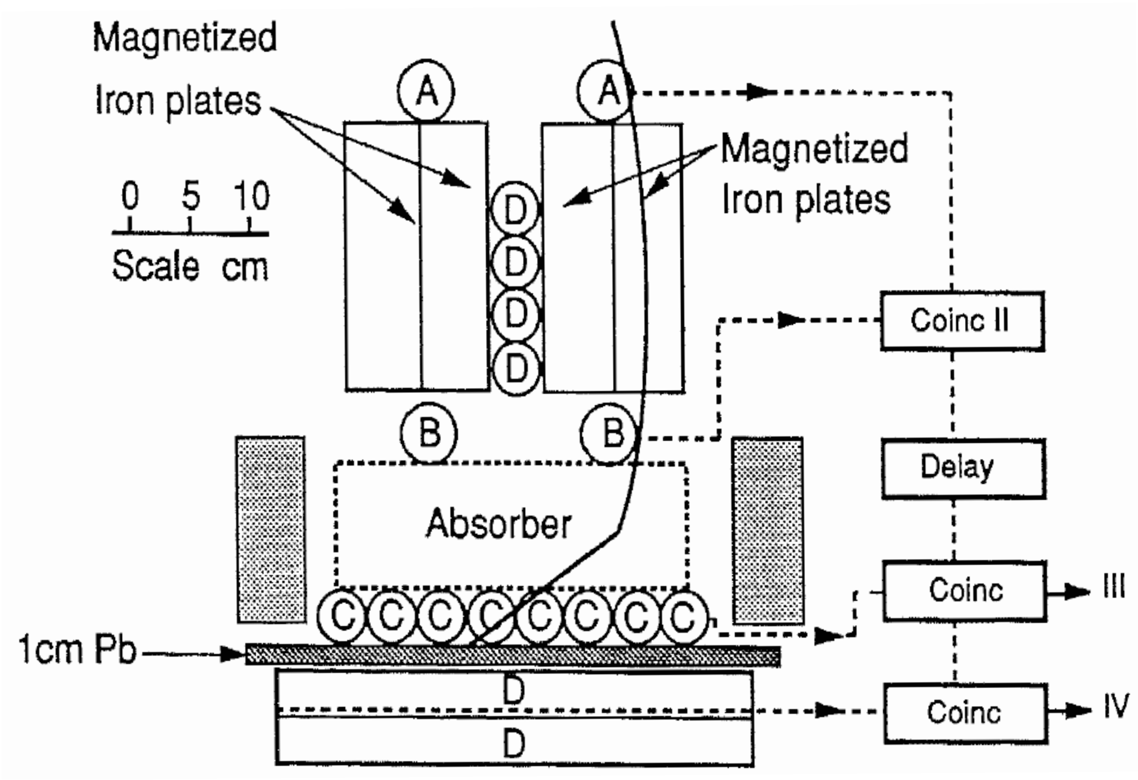


Closed magnetic circuit made of iron plates with coils wound around to produce a magnetic field

According to the direction of the B field the magnets focussed in one plane either negative or positive particles

1931 Rossi, Puccianti measured the relative abundance of positive (55%) and negative (45%) particles in cosmic rays.

Experimental setup



B field 15 kGauss to select negative and positive particles

Magnetic lens ~ 20 cm of Fe

Absorber Fe (5 cm), C (4 cm) graphite cylinders

III 3-fold delayed coincidence,
IV 4-fold delayed coincidence.
Signal = III - IV (decay electrons)

Delay: 1 - 4.5 μs

Pb (1cm) absorber for electrons from decays

1. Which is the typical energy for muons that are focussed and then stopped in the absorber?

2. Contamination from opposite sign particles selected by the magnetic lens is negligible. Use approximate measurements from the drawings and show how it depends on the geometry of the lens.

3. Which kind of events can fire the delayed 4-fold coincidence IV?

Main result of the CPP experiment

carica dei muoni fermati	assorbitore	frequenza di disintegrazione (eventi / 10 ore)
positiva	5 cm Fe	6.7 ± 0.65
negativa	5 cm Fe	0.3 ± 0.25
negativa	nessuno	-0.1
positiva	4 cm C	3.6 ± 0.45
negativa	4 cm C + 5 cm Fe	2.7 ± 0.35
negativa	6.2 cm Fe	0

The decay rate in Fe for negative particles is compatible with zero. Negative particles are captured by the nuclei before they decay

Decay rate of negative particles in C is different from zero and similar to positive particle rate

Results are in disagreement with T-A calculations for a strongly interacting Yukawa particle

Fermi and the CPP experiment

Fermi (with Teller and Weisskopf) concluded that there were about 12 orders of magnitude difference between the predicted capture time for a negative Yukawa particle and the results of the experiment for negative particles

The mesotron is not the strongly interacting Yukawa particle (today pion)

The mesotron was renominated the μ "meson"

«La prima volta (continua Piccioni) che io sentii questa affermazione fu da Enrico Fermi, nel suo studio in Chicago. Marcello era arrivato da poco a Chicago, ed era anch'egli presente, come anche Wick. Wick chiese a Fermi come egli si rappresentasse matematicamente la cattura del muone, e Fermi rispose con estrema naturalezza "con un potenziale immaginario". Quando uscimmo dallo studio di Fermi, Marcello mi disse di essersi goduto la mia faccia terrorizzata davanti a quelle parole. Forse esagerò; ma è vero che la mia innocenza in meccanica quantistica era al di là di ogni elogio (*it*

Problem 5: When the muon was originally discovered back in the 30s, people thought that this was the Yukawa particle. The Yukawa particle was considered then to be the mediator of the strong interaction. This incorrect interpretation of the nature of the muon was due to the fact that the muon mass (106 MeV) was not very different from the expected mass of the particle predicted by Yukawa. Later on it turned out that the Yukawa particle was the pion which was discovered in the 40s at Bristol.

This problem[2] relates to the calculations done by Tomonaga and Araki who predicted that if the muon was the mediator of the strong interaction then negative muons passing through matter would be more likely to be captured by the nuclei rather than decay.

(a) Show that a negative muon captured in an S-state by a nucleus of charge Ze and mass A will spend a fraction $f \simeq 0.25\,A(Z/137)^3$ of its time inside the nuclear matter and that in time t it will travel a total distance $fct \quad (Z/137)$ in the nuclear matter. The hydrogen atom ground state wave function can be used in these calculations with modifications to account for the fact that the muon mass is of the order of 200 times larger than the electron mass:

$$\Psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{\alpha_0}\right)^{3/2} e^{-\frac{Zr}{\alpha_0}} \text{ where } \alpha_0 = \frac{\hbar^2}{M_R e^2} \text{ and } M_R = \frac{m_p m_\mu}{m_p + m_\mu}$$ is the reduced mass of the proton muon system. The proton and muon masses are $m_p = 938\,MeV$ and $m_\mu = 106\,MeV$.

(b) The law of radioactive decay of free muons is $dN/dt = -\Gamma_d N$, where $\Gamma = 1/\tau$ is the decay constant (width) and the lifetime is $\tau = 2.16\,\mu\,sec$. For a negative muon captured in an atom Z the decay constant is $\Gamma_{TOT} = \Gamma_d + \Gamma_c$, where Γ_c is the width for nuclear capture i.e. the probability per unit time of nuclear capture. For aluminium $(Z=13, A=27)$ the mean lifetime of negative muons is $\tau = 0.88\,\mu\,sec$. Calculate Γ_c and using he expression for f in (a), compute the interaction mean free path Λ for a muon in nuclear matter.

(c) From the magnitude of Λ estimate the magnitude of the coupling constant of the interaction that caused the nuclear capture $\mu^- + p \rightarrow n + \nu$ given that the strong interaction coupling constant is α_S and corresponds to a mean free path of $1\,fm$.

Conversi, Pancini and Piccioni[3] did experiments in Rome in the 40s to test Tomonaga's and Araki's hypothesis and found that positive muons traversing different materials always decay rather than being captured (not surprising). They also found that negative muons undergo nuclear capture in iron. However, in carbon negative muons decay and do not get captured by the nucleus in direct contradiction of the Tomonaga-Araki hypothesis. Hence, the muons have nothing to do with the strong interaction.

Answers:

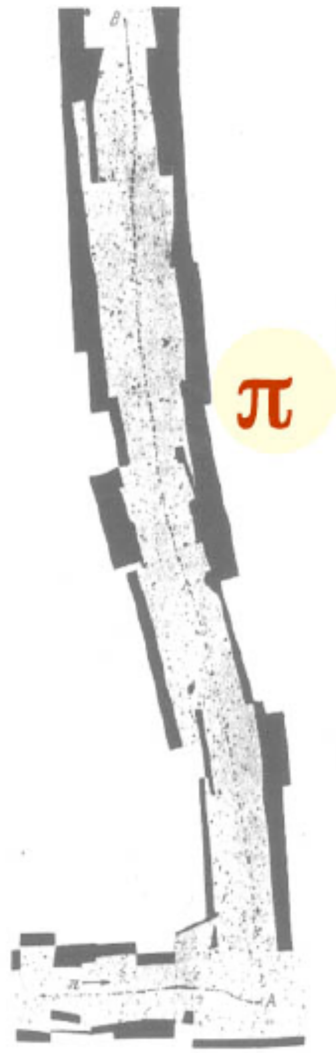
(b) 26.5 cm

(c) 10^{-7}

Finally the pion discovery

1947 Hypothesis of existence of the pion and the muon (Marshak, Bethe)

the Yukawa particle (the pion) is produced high in the atmosphere and decay into the muon which arrives on the earth



1947 Nuclear capture theory for muons (Wheeler)

$$\mu^- + p \rightarrow n + \nu_\mu$$

nuclear capture probability for muons is proportional to Z^4 (atomic number). Carbon (Z=6) has smaller capture probability than Fe (Z=26) in agreement with CPP experiment results.

1947 Discovery of the pion (Lattes, Muirhead, Powell, Occhialini,)

evidence of pion into muon decays in photographic plates

Bibliography

Marcello Conversi
Carlo Rubbia
RENDICONTI LINCEI **2**, 3 (1991)

La vita di Oreste Piccioni e la sua attività scientifica in Italia
Giorgio Salvini
RENDICONTI LINCEI **15**, 289 (2004)

Effect of the nuclear coulomb field on the capture of slow mesons
S. Tomonaga and G. Araki
Phys.Rev., 58, 90-91 (1940),

Sulla disintegrazione dei mesoni lenti
M. Conversi, O. Piccioni
Il Nuovo Cimento **2**, 71 (1944)

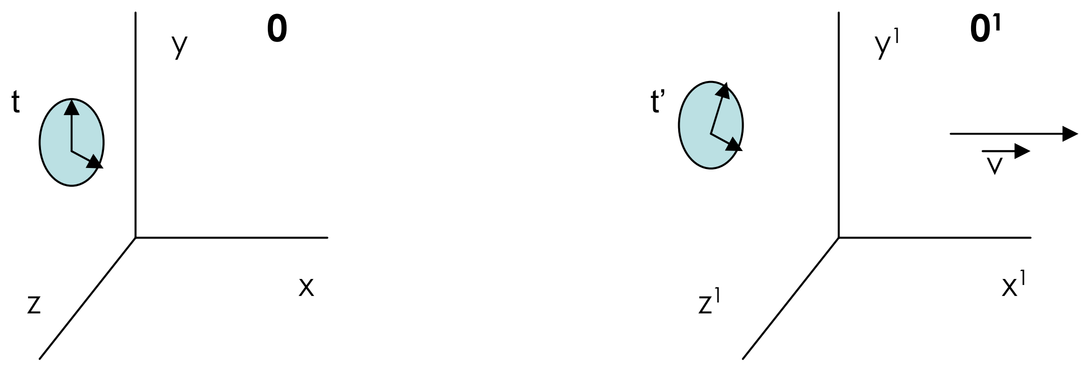
Sull'assorbimento e sulla disintegrazione dei mesoni alla fine del loro percorso
M. Conversi, E. Pancini, O. Piccioni
Il Nuovo Cimento **3**, 372 (1946)

RELATIVISTIC KINEMATICS

Resumè of special relativity

Principle of Relativity

- the laws of physics are invariant in all inertial frames of reference (i.e., non-accelerating frames of reference);
- the speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.



$$\beta = \frac{v}{c} \quad ; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x \equiv (ct; \vec{x}) \qquad\qquad x' \equiv (ct'; \vec{x}\,')$$

Lorentz transformation **L(β) transform** $x \equiv (ct; x)$ in $x' \equiv (ct'; x')$

Accordingly to $\quad x' = L(\beta)x$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{pmatrix}$$

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(x - \beta ct)$$

$$y' = y$$

$$z' = z$$

det(L) = $\gamma^2 - \beta^2\gamma^2$ = 1

The inverse **Trasformation:** $x = L^{-1}(\beta)x' = L(-\beta)x'$

Is defined by:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma ct' + \beta\gamma x' \\ \beta\gamma ct' + \gamma x' \\ y' \\ z' \end{pmatrix}$$

In the limit of v<<c we obtain the classical Galileo transformation

x' = x –vt
t' = t

$A = [a_0, a_1, a_2, a_3] = [a_0, \mathbf{a}]$

$B = [b_0, b_1, b_2, b_3] = [b_0, \mathbf{b}]$

$$A \cdot B = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$$

The product of two four-vectors is a relativistic invariant

$$A' \cdot B' = a'_0 b'_0 - a'_1 b'_1 - a'_2 b'_2 - a'_3 b'_3 =$$
$$= \gamma^2 (a_0 - \beta a_1)(b_0 - \beta b_1) - \gamma^2 (a_1 - \beta a_0)(b_1 - \beta b_0) -$$
$$a_2 b_2 - a_3 b_3 = \gamma^2 (1 - \beta^2)(a_0 b_0 - a_1 b_1) - a_2 b_2 - a_3 b_3 =$$
$$A \cdot B$$

Some important consequences for HEP

- Length contraction (Lorentz contraction)

if $d' = x'_2 - x'_1$ is the distance measured in O' of a bar at rest in O' then

$x'_2 - x'_1 = \gamma(x_2 - x_1) - \beta\gamma c(t_2 - t_1)$

But the Observer in O measures x_2 and x_1 simultaneously, at $t_2 = t_1$. Hence:

$$d' = \gamma(x_2 - x_1) = \gamma d$$

$$d = \frac{d'}{\gamma} \rightarrow d < d'$$

Some important consequences for HEP

- Time dilation

if $T'=t'_2-t'_1$ is the time interval between two instants measured in O' in the same point $x'_2=x'_1$, the Observer O measures

$cT= c(t_2-t_1)= \gamma c(t'_2 - t'_1) + \beta\gamma(x'_2 - x'_1) = \gamma cT'$

$$T = \gamma T'$$
$$T > T'$$

The time interval measured in the rest frame is called «proper time»

- $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ is a relativistic invariant

If in O' the measurement is done in the same point

$ds^2 = cdt'^2$

Exercises

- Muons have mass of 106 MeV and lifetime of 2.2×10^{-6} s. If a muon is produced in cosmic rays with momentum 10 GeV/c and at a quote of 10 m, evaluate in the LAB and in the muon reference frame the probability that the muon reach the earth before decaying.

- In laboratories, beams of particles with known average lifetime τ are transported, before decaying, over distances many times greater than the Galilean limit of the decay distance: $c\tau$. Show that charged π produced at a primary target with a momentum of 200 GeV / c can be transported over distances, for example, of 300 m with a loss due to decays of less than 3%.

Four-vectors

- $$g^{\mu\nu} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \qquad x^{\mu} \equiv g^{\mu\nu} x_{\nu}$$

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2$$

- $$x^{\mu} \equiv \left(ct; \vec{r}\right) \quad ; \quad x_{\mu} \equiv \left(ct; -\vec{r}\right)$$

$$dx^{\mu} \cdot dx_{\mu} = ds^2$$

$$u^{\mu} = \frac{dx^{\mu}}{ds} \Longrightarrow \begin{cases} u^{\mu} \equiv \left(\gamma; \gamma\vec{\beta}\right) \\ u^{\mu} u_{\mu} = 1 \end{cases}$$

$$p^{\mu} = m_0 c u^{\mu} \Longrightarrow p^{\mu} p_{\mu} = \frac{E^2}{c^2} - p^2 = m_0^2 c^2$$

Energy-Momentum

The relativistic relation between the total energy E, the vector 3-momentum **p** and the rest mass m for a free particle is:

$$E^2 = \vec{p}^{\,2} c^2 + m^2 c^4$$

or, in natural units:

$$E^2 = \vec{p}^{\,2} + m^2$$

The components of the 3-momentum and the energy can be written as components of an energy-momentum 4-vector P :

$P_0 = E \quad P_1 = p_x \quad P_2 = p_y \quad P_3 = p_z$

whose square modulus equals the squared rest mass:

$$P^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2 = E^2 - \vec{p}^{\,2} = m^2$$

in units $\hbar = c = 1$.

Main relations between energy, momentum, velocity, etc.

$i)\ \vec{p} = \gamma m_0 \vec{v}$ $\quad$ $ii)\ E = \gamma m_0 c^2$ $\quad$ $iii)\ \vec{v} = c^2 \frac{\vec{p}}{E}$

$iv)\ \gamma = \frac{E}{m_0 c^2}$ $\quad$ $v)\ \gamma \vec{v} = \frac{\vec{p}}{m_0}$ $\quad$ $vi)\ \vec{\beta} = c \frac{\vec{p}}{E}$

$vii)\ \gamma \vec{\beta} = \frac{1}{c} \cdot \frac{\vec{p}}{m_0}$ $\quad$ $viii)\ T = m_0 c^2 (\gamma - 1)$

$$ix)\ E = T + m_0 c^2 = m_0 c^2 (\gamma - 1) + m_0 c^2$$

$$E = m_0 c^2 \gamma$$

Reference Frames

- LAB system: generally is the system in which measurement are performed. Is the R.F. of the Observer.

  In the LAB system, usually target is at rest

Lab: $p_1=(E_1;\mathbf{p}_1)$; $p_2(M;\mathbf{0})$;

$P_{tot} = (E_1+M;\mathbf{p}_1)$

- Center of Mass Frame C.M.: is defined by the condition

CdM = $\sum_k \vec{p}_k = \vec{P}_{Tot} = \vec{0}$

In the case we have just one particle, the CM is the system in which the particle is at rest and is the natural system to describe particle decay

$p_1^*=(E_1^*;\mathbf{p}^*)$; $p_2^*=(E_2^*;-\mathbf{p}^*)$; $P^*=(E^*;\mathbf{0})$

dove $E^* = E_1^* + E_2^*$

$P^{*2}_{CdM} = E^{*2} = P_{LAB}^2 = m_1^2 + M^2 + 2E_1M$

Invariant Mass

- Let's consider a system of N particles

$$p_k^\mu = \left(E_k ; \vec{p}_k\right) \quad ; \quad \mathrm{P}^\mu = \sum_k p_k^\mu = \left(\sum_k E_k ; \sum_k \vec{p}_k\right)$$

- $\mathrm{P}^\mu \mathrm{P}_\mu$ = Relativistic Invariant
- $\mathrm{P}^\mu \mathrm{P}_\mu$ = Invariant Mass of the N particles

$$\mathrm{P}^\mu \mathrm{P}_\mu = \left(E_1 + E_2 + \ldots. + E_N\right)^2 - \left(\vec{p}_1 + \vec{p}_2 + \ldots.. + \vec{p}_N\right)^2$$

- In the CMS, by definition:

$$\vec{P}^*_{Tot} = 0 \quad \Rightarrow \quad P^{\mu *} = \left(E^*_{Tot} ; \vec{0}\right)$$

Invariant Mass = $\mathrm{P}^\mu \mathrm{P}_\mu = \mathrm{E}^{*2}_{\mathrm{Tot}}$

1947: Discovery of the π - meson (the "real" Yukawa particle)

Observation of the $\pi^+ \rightarrow \mu^+ \rightarrow e^+$ decay chain in nuclear emulsion exposed to cosmic rays at high altitudes

Nuclear emulsion: a detector sensitive to ionization with ~1 μm space resolution (AgBr microcrystals suspended in gelatin)

In all events the muon has a fixed kinetic energy (4.1 MeV, corresponding to a range of ~ 600 μm in nuclear emulsion) $\Rightarrow$ two-body decay

$m_\pi = 139.57$ MeV/c^2 ; spin = 0

Dominant decay mode: $\pi^+ \rightarrow \mu^+ + \nu$

(and $\pi^- \rightarrow \mu^- + \bar{\nu}$)

Mean life at rest: $\tau_\pi = 2.6 \times 10^{-8}$ s = 26 ns

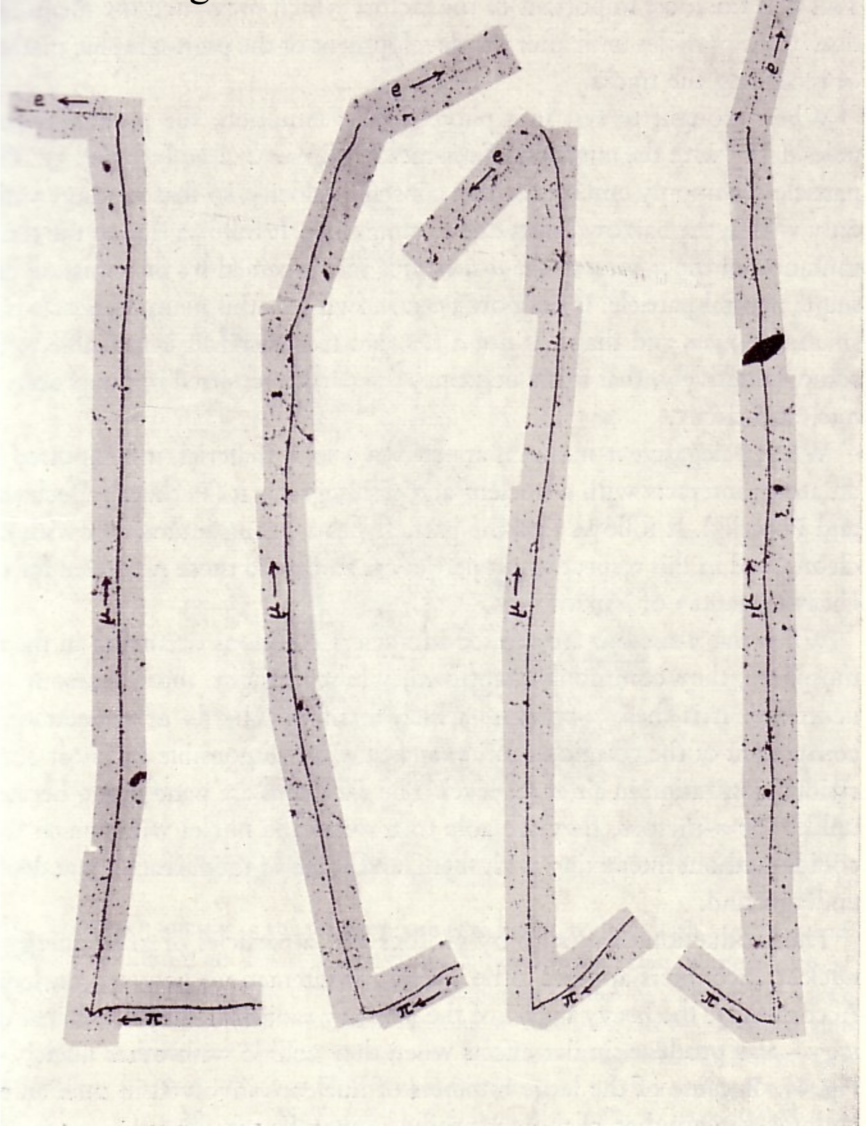
π^- at rest undergoes nuclear capture, as expected for the Yukawa particle

A neutral π – meson ($\pi°$) also exists:

m ($\pi°$) = 134. 98 MeV /c^2

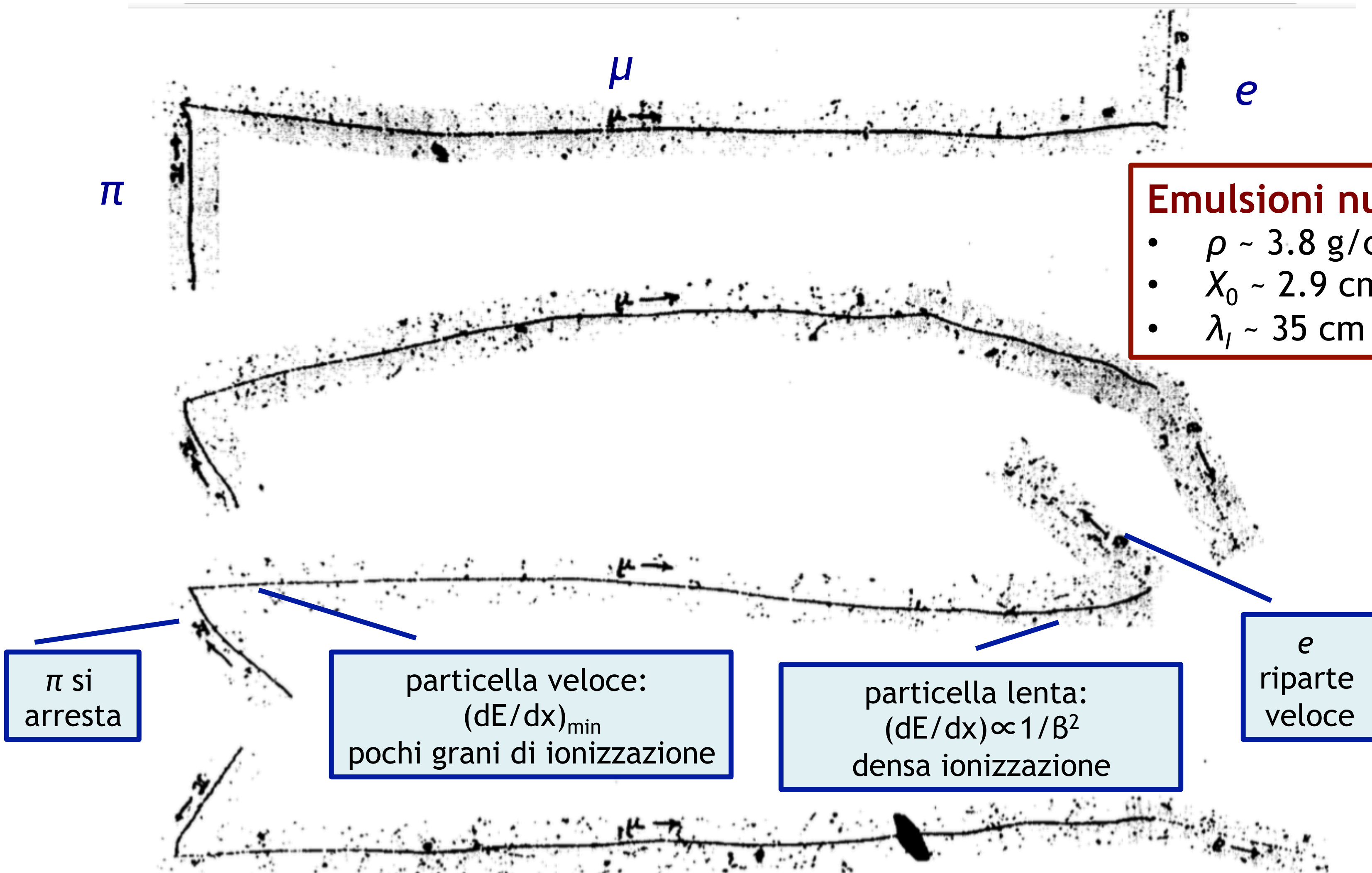
Decay: $\pi° \rightarrow \gamma + \gamma$, mean life = 8.4×10^{-17} s

π – mesons are the most copiously produced particles in proton – proton and proton – nucleus collisions at high energies



Four events showing the decay of a π^+ coming to rest in nuclear emulsion

Osservazione del π (Lattes, Occhialini, Powell 1947)



Emulsioni nucleari

- ρ ~ 3.8 g/cm^2
- X_0 ~ 2.9 cm
- λ_I ~ 35 cm

Invariant mass application

- We measure the muon momentum to be 29.3 MeV/c from nuclear emulsion
- If we assume the new particle decay $X \rightarrow \mu \nu_\mu$

$$p_\mu = p_\nu$$

- By applying the Invariant Mass definition we get:

$$(M_x c^2)^2 = E^{*2}_{Tot} = (E_\mu + E_{\nu\mu})^2$$

$$M_x c^2 = \sqrt{p_\mu^2 c^2 + m_\mu^2 c^4} + \mathrm{p}_\nu \mathrm{c} = \sqrt{29.3^2 + 105.6^2} + 29.3 =$$

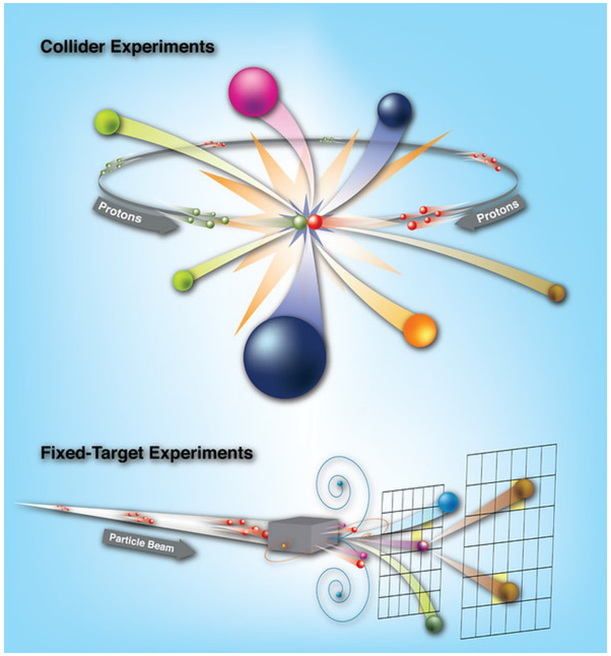
138.9 MeV/c^2

The experiments of Conversi, Pancini e Piccioni

L. Alvarez, Nobel Lecture, 1968:

As a personal opinion, I would suggest that modern particle physics started in the last days of World War II, when a group of young Italians, Conversi, Pancini, and Piccioni, who were hiding from the German occupying forces, initiated a remarkable experiment. In 1946, they showed that the ‘mesotron’ which had been discovered in 1937 by Neddermeyer and Anderson and by Street and Stevenson, was not the particle predicted by Yukawa as the mediator of nuclear forces, but was instead almost completely unreactive in a nuclear sense. Most nuclear physicists had spent the war years in military-related activities, secure in the belief that the Yukawa meson was available for study as soon as hostilities ceased. But they were wrong.

Fixed Target vs Collider experiment



- Fixed Target
  - $P_2=0$ and assume $m_{1,2} << E_1$

$$P^\mu P_\mu = \Big|_{Nel\ Lab.} (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 =$$

$$= \mathrm{m}_1^2 + p_1^2 + \mathrm{m}_2^2 + 2m_2E_1 - p_1^2 \simeq 2m_2E_1$$

$$\mathbf{S} = 2m_2E_1$$

- At Collider, under the hypothesis of $E_1 >> m_{1,2}$ $\mathbf{|p_{1,2}| = |p| \approx E_{1,2}}$

$$s = (p_1^\mu p_{1\mu} + p_2^\mu p_{2\mu} + 2p_1^\mu p_{2\mu})$$

$$= E_1^2 - |\vec{p}_1|^2 + E_2^2 - |\vec{p}_2|^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$$

$$= m_1^2 + m_2^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$$

$$= m_1^2 + m_2^2 + 2(E_1E_2 - |\vec{p}_1||\vec{p}_2|\cos\theta)$$

nella ipotesi che $E_1 >> m_1, m_2$ avremo $|p_{1,2}| = |p| \approx E_{1,2}$.

$$\boldsymbol{s = 2(E^2 - E^2\cos\theta)}$$

$$\boldsymbol{s = 4E^2}$$

Exercise

- A 100 GeV proton beam collide against a hydrogen target, evaluate √s and compare with the value obtained for colliding protons.
  - Fixed target

$$\sqrt{s} = \sqrt{2E_1 m_2} \approx \sqrt{2 \cdot 100 \cdot 1} \approx 14\ \text{GeV}$$

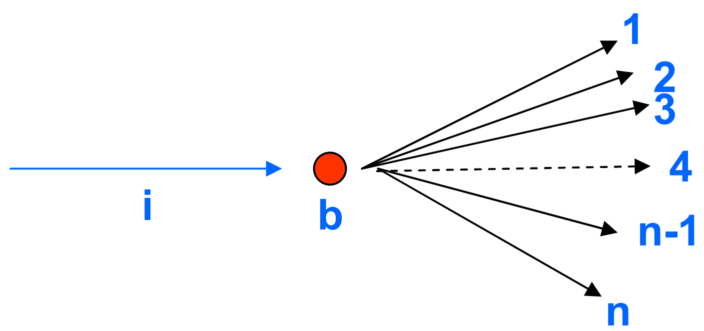
  - Collider

$$\sqrt{s} = 2 \cdot 100 = 200\ \text{GeV}$$

In a collider the total cms energy increases linearly with the beam energy, whereas in fixed targed it increases as the square root of the beam energy.

Energy threshold

- In a nuclear/particle collision or reaction in general we can produce N particles



- In the initial state (lab sys)

$$E^{*2} = \left(E_i + m_b\right)^2 - \left(p\right)_i^2 = 2m_b E_i + m_i^2 + m_b^2 =$$

$$= 2\mathrm{m_b}\left(T_i + m_i\right) + m_i^2 + m_b^2 = 2\mathrm{m_b} T_i + \left(m_i + m_b\right)^2$$

- In the final state (CM system)

$$E^{*2} = \left[\sum_1^N {}_f \left(T_f^* + m_f\right)\right]^2$$

Energy threshold

- In order for the reaction to be energetically possible, it is necessary to have at least enough energy in the CM to produce the particles at rest

$$E^{*2} = \left[\sum_{1}^{N} {}_{f} \left(T_f^* + m_f \right) \right]^2 \geq \left[\sum {}_{f} m_f \right]^2$$

$$\boxed{T_i \geq T_{soglia} = \frac{\left(\sum_f m_f \right)^2 - \left(m_i + m_b \right)^2}{2 m_b}}$$

- This relation defines the Threshold Energy of a reaction

Threshold energy: the discovery of antiprotons

$$T_s = \frac{\left[\left(4m_p\right)^2 - \left(2m_p\right)^2\right]}{2m_p} = 6m_p = 5.6 \text{ GeV}$$

Threshold energy: the discovery of antiprotons

- Antiprotons have been discovered in the collision of a proton beam with an Hydrogen target. Proton in the hydron may considered in first aproximation as free

$$p + p \to p + p + p + \overline{p}$$

- The threshold energy for this process is:

$$T_s = \frac{\left[(4m_p)^2 - (2m_p)^2\right]}{2m_p} = 6m_p = 5.6 \text{ GeV}$$

- If we consider the fermi momentum of the proton inside a Cu target, p_{FERMI}=0.24 GeV/c which is randomly distributed wrt the projectile direction

Threshold energy: the discovery of antiprotons

- The maximum value (minimum) of Ts will be for p_{Fermi} momentum parallel (antiparallel) to p_{proton}

$$E_F = \sqrt{p_F^2 + m_p^2} \quad ; \quad \mathrm{p}_{\mathrm{Lab}} = \left(E_p + E_F ; \vec{p}_p + \vec{p}_F\right)$$

$$p_{Lab}^2 = 2m_p^2 + 2E_pE_F - 2\vec{p}_p \bullet \vec{p}_F =$$

$$= 2\left(m_p^2 + E_pE_F \pm p_p p_F\right) \geq 16m_p^2$$

$$m_p^2 + E_pE_F \pm p_p p_F \geq 8m_p^2 \rightarrow E_pE_F \pm p_p p_F \geq 7m_p^2$$

By approximating

$$\mathrm{E}_{\mathrm{F}} = m_p + \frac{p_F^2}{2m_p} \quad \text{and} \quad \mathrm{p}_{\mathrm{p}} \cong E_p$$

Threshold energy: the discovery of antiprotons

$$E_p \geq \frac{7m_p}{1 \pm \frac{p_F}{m_p} + \frac{p_F^2}{2m_p^2}} \quad ; \quad \mathrm{E_p} = T + m_p$$

$$T_{Min} = 4.2\ \mathrm{GeV} \quad ; \quad \mathrm{T_{Max}} = 7.5\ \mathrm{GeV}$$

$$E_{pMin} = 5.11\ \mathrm{GeV} \qquad E_{pMax} = 8.46\ \mathrm{GeV}$$

Exercises

- Consider the reaction

$$\overline{p} + p \rightarrow \Lambda + \overline{\Lambda} \text{ with } \left|\vec{\mathrm{p}}_{\overline{\mathrm{p}}}\right| = 0.65 \text{ GeV/c}$$

Tell if it is possible or not.

- Evaluate the energy threshold for the reactions

$$\gamma + N \rightarrow e^{+} + e^{-} + N$$

$$\pi^{-} + p \rightarrow \wedge^{0} + k^{0}$$

$$\nu_{\mu} + e^{-} \rightarrow \mu^{-} + \nu_{e}$$

$$e^{+} + e^{-} \rightarrow \mu^{+} + \mu^{-}$$ Consider also the case of a circular collider

CM vs LAB system

- Velocity of the CMF wrt LAB system

$$\mathbf{P}_{tot} = [(\sum_k E_k)/c, \sum_k \overrightarrow{p_k}]$$

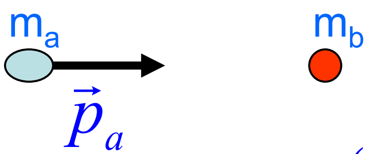
$$\mathbf{P}^*_{tot} = [(\sum_k E^*_k)/c, \overrightarrow{0}] = [\sqrt{s}/c, \overrightarrow{0}]$$

$$\begin{pmatrix} \sqrt{s}/c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_{c.m.} & -\beta_{c.m.}\gamma_{c.m.} & 0 & 0 \\ -\beta_{c.m.}\gamma_{c.m.} & \gamma_{c.m.} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (\sum_k E_k)/c \\ |\sum_k \overrightarrow{p_k}| \\ 0 \\ 0 \end{pmatrix}$$

$$\beta_{c.m.} = \frac{|\sum_k \overrightarrow{p_k}|c}{\sum_k E_k} = \frac{|\overrightarrow{p}^{lab}_{tot}|c}{E^{lab}_{tot}} \qquad \gamma_{c.m.} = \frac{1}{\sqrt{1-\beta^2_{c.m.}}} = \frac{E^{lab}_{tot}}{\sqrt{s}}$$

Reference system

- In the lab:



In the CM

$$\vec{P}^*_{Tot} = 0$$

- We got

$$s = (E_a + E_b)^2 - p_a^2 = m_a^2 + p_a^2 + m_b^2 + 2E_a m_b - p_a^2 =$$
$$= m_a^2 + m_b^2 + 2E_a m_b$$

Evaluate β of CM

$$P^\mu \equiv \begin{pmatrix} E_a + m_b \\ \vec{p}_a \end{pmatrix} \quad ; \quad P^* \equiv \begin{pmatrix} \sqrt{s} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{E_a} + m_b \\ 0 \\ 0 \\ p_a \end{pmatrix} \equiv \begin{pmatrix} \gamma_{CdM} & 0 & 0 & \beta_{CdM}\gamma_{CdM} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_{CdM}\gamma_{CdM} & 0 & 0 & \gamma_{CdM} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E_a + m_b = \gamma_{CdM}\sqrt{s} \quad \text{e} \quad \mathrm{p_a} = \beta_{CdM}\gamma_{CdM}\sqrt{s}$$

$$\beta_{CdM} = \frac{\mathrm{p_a}}{\gamma_{CdM}\sqrt{s}} \quad \text{e} \quad \gamma_{CdM} = \frac{E_a + m_b}{\sqrt{s}}$$

$$\beta_{\mathrm{CdM}} = \frac{p_a}{\frac{(E_a + m_b)}{\sqrt{s}}\sqrt{s}} = \frac{p_a}{(E_a + m_b)}$$

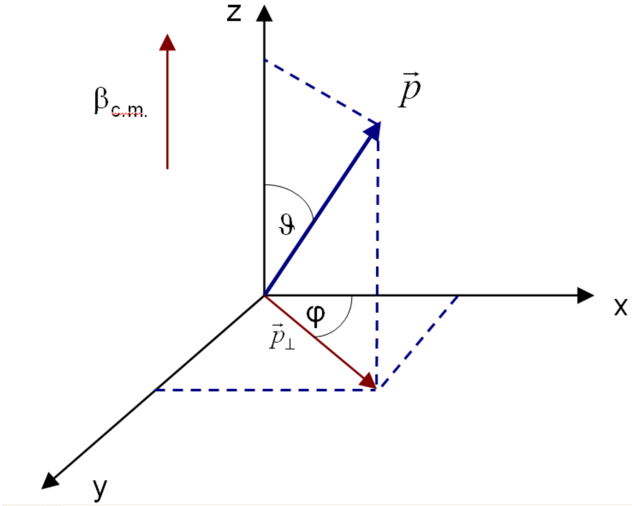
Compare CM and LAB system

Collision	Momentum Beam1 (GeV/c)	Momentum Beam2 (GeV/c)	E^*_{TOT}(GeV)	P*(GeV)	β	γ
e - e	2	0				
e - e	9	3				
e - e	2	2				
p-p	2	0				
e-p	2	0				
e-p	30	800				

E^*_{TOT} – total energy in CMF
P* - momentum in CMF
β_0 – velocity of the CM in the lab frame

Transverse momentum

- Consider the four-momentum of a particle in the transformation from the CMS to the LabS, assuming for simplicity that the CM moves parallel to the z axis



$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} E \\ |\overrightarrow{p}|sin\theta\ cos\phi \\ |\overrightarrow{p}|sin\theta\ sin\phi \\ |\overrightarrow{p}|cos\theta \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} E^* \\ |\overrightarrow{p}^*|sin\theta^* cos\phi^* \\ |\overrightarrow{p}^*|sin\theta^* sin\phi^* \\ |\overrightarrow{p}^*|cos\theta^* \end{pmatrix}$$

$$E = \gamma E^* + \beta\gamma p^* \cos\vartheta^* \qquad (1)$$

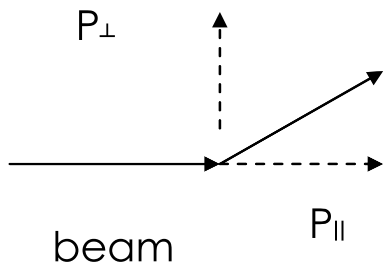
$$p \sin\vartheta \cos\varphi = p^* \sin\vartheta^* \cos\varphi^* \qquad (2)$$

$$p \sin\vartheta \sin\varphi = p^* \sin\vartheta^* \sin\varphi^* \qquad (3)$$

$$p \cos\vartheta = E^* \beta\gamma + \gamma p^* \cos\vartheta^* \qquad (4)$$

Transverse momentum

The Transverse Momentum = momentum component orthogonal to z axis
is a relativistic invariant



$$|\overrightarrow{p}|^2 sin^2\theta(cos^2\phi + sin^2\phi) = |\overrightarrow{p}^*|^2 sin^2\theta^*(cos^2\phi^* + sin^2\phi^*)$$

$0 \leq \theta \leq \pi$

$0 \leq \theta^* \leq \pi$: $\quad p_\perp = |\overrightarrow{p}|sin\theta = |\overrightarrow{p}^*|sin\theta^* = p^*_\perp$

cosφ = cosφ∗ and sinφ = sinφ∗
the azimuth angle φ around an axis is a relativistic invariant for Lorentz transformations along the axis itself.

Angle transformations

- $\cos\varphi = \cos\varphi*$ and $\sin\varphi = \sin\varphi*$

the azimuth angle φ around an axis is a relativistic invariant for Lorentz transformations along the axis itself.

$$tg\theta = \frac{sin\theta^*}{\gamma_{c.m.}(\beta_{c.m.}E^*/|\overrightarrow{p}^*| + cos\theta^*)}$$

$$\boxed{tg\,\vartheta = \frac{\sin\vartheta^*}{\gamma(\beta/\beta^* + \cos\vartheta^*)}}$$

The speed of the particle in the CM, $\beta^* = |-\mathbf{p}^*|/E^*$, is independent from $\beta_{c.m.}$

Angle transformations

- $\beta_{c.m.} > \beta^*$ denominator >0 for all $0 \le \theta_* \le \pi \rightarrow 0 \le \theta < \pi/2$ the particle moves forward in the Lab system
- Since θ=0, for θ^* =0 and for θ^* =π, → it should exist, in the LAB system a $\theta_{max} < \pi/2$. We can evaluate θ_{max} :

$$\frac{d(tg\theta)}{d\theta^*} = \frac{1 + cos\theta^* \left(\beta_{c.m.}/\beta^*\right)}{\gamma_{c.m.}\left(\beta_{c.m.}/\beta^* + cos\theta^*\right)^2} = 0$$

$$cos\theta^* = -\beta^*/\beta_{c.m.}$$

$$tg\theta_{max} = \frac{\beta^*}{\gamma_{c.m.}\sqrt{\beta^2_{c.m.} - (\beta^*)^2}}$$

- The energy of the particle

$$E(\theta_{max}) = \gamma_{c.m.}(E^* + \beta_{c.m.}|\overrightarrow{p}^*|cos\theta^*) = \gamma_{c.m.}(E^* - |\overrightarrow{p}^*|\beta^*) =$$

$$= \gamma_{c.m.}\left(E^* - \frac{|\overrightarrow{p}^*|^2}{E^*}\right) = m^2\left(\frac{\gamma_{c.m.}}{E^*}\right) = m\frac{\gamma_{c.m.}}{\gamma^*}$$

Angle transformations

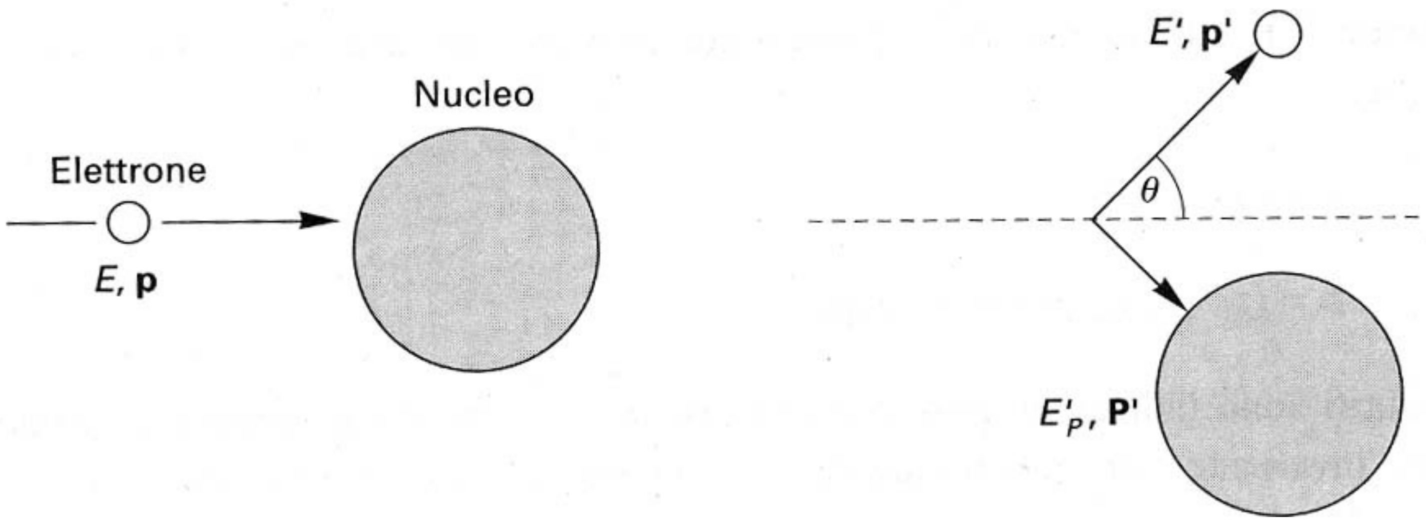
- $\beta_{c.m.} < \beta^*$ in this case the velocity of the particle in the CM can cancel the boost of the CM itself, allowing in the laboratory angles $\theta > \pi / 2$
  $\theta = \pi/2$ if $\cos\theta^* = -\beta_{c.m.}/\beta^*$

  It does not exist a θ_{max} since the derivative is always positive

- $\beta_{c.m.} = \beta^*$
  in this limit case, $\cos\theta^* = -1$ corresponds in the laboratory to the angle θ_{max} which is just $\pi/2$.

  It is interesting to note how in this configuration the particle, which in the CM travels in the opposite direction to the motion of the CM itself ($\theta^* = -\pi$), is instead at rest in the laboratory system; in fact its β^* exactly cancels the boost of the center of mass.

Elastic scattering



- From energy-momentum conservation
- $p+P = p'+P'$ where: $p=(E/c;\mathbf{p})$, $p'=(E'/c;\mathbf{p'})$,
  $P=(mc;\mathbf{0})$, $P'=(E'/c;\mathbf{P'})$

$p^2 + 2pP + P^2 = p'^2 + 2p'P' + P'^2$

- For an elastic collision
- $p^2=p'^2=m_e^2c^2$; $P^2=P'^2=Mc^2$ per cui : $pP = p'P'$

Elastic scattering

- Experimentally we detect the deflected electron:

$pP = p'(p+P-p') = p'p + p'P - m_e^2c^2$

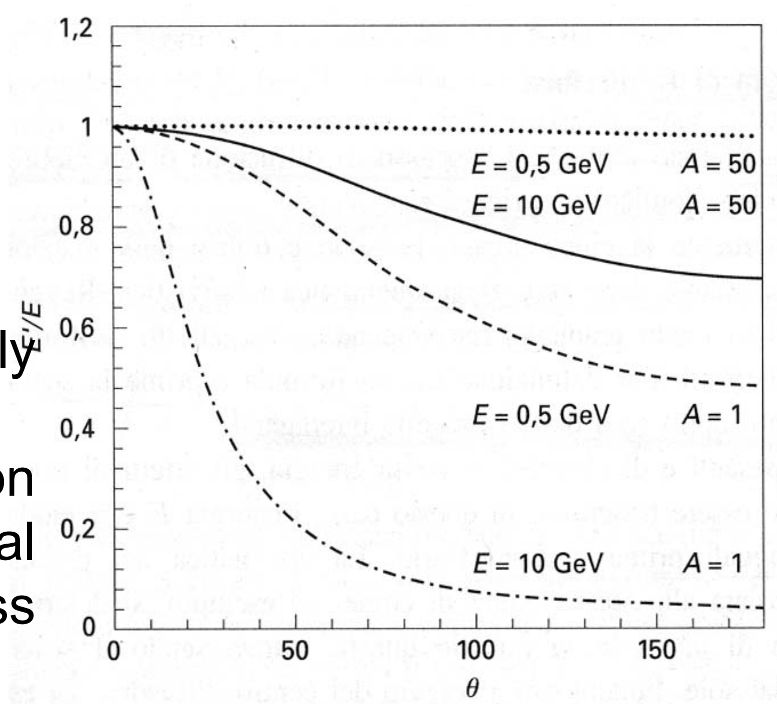
If we multiply each member for c^2

$EMc^2 = E'E - \mathbf{pp'}c^2 + E'Mc^2 - m_e^2c^4$

- For high energy we can neglet $m_e^2c^4$ and $E \cong pc$

$EMc^2 = EE'(1-\cos\theta) + E'Mc^2$

$$E' = \frac{E}{1 + \frac{E}{Mc^2}(1-\cos\vartheta)}$$



the energy of the diffused electron uniquely depends on the diffusion angle

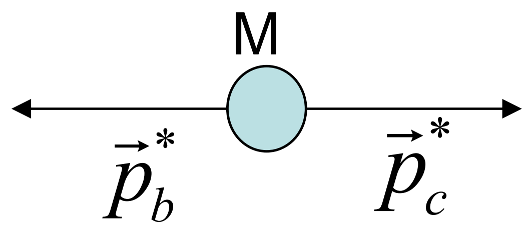
the nucleus recoil energy, (E - E′), depends on E/M and in particular increases as the initial energy of the electron increases wrt the mass of the nucleus

Decay

$$a \rightarrow b + c$$

with M_a, m_b and m_c masses of the three particles.
In the CMF a is at rest and the four-momentum conservation gives:

$$M_a = E_b^* + E_c^*$$



$$\vec{0} = \vec{p}_b^{\,*} + \vec{p}_c^{\,*}$$

$$P = (E^*_b + E^*_c; \mathbf{0}) \quad ; \; p^*_b = p^*_c = p^*$$

$$M_a = (p^{*2}_b + m_b^2)^{1/2} + (p^{*2}_c + m_c^2)^{1/2}$$

$$= (p^{*2} + m_b^2)^{1/2} + (p^{*2} + m_c^2)^{1/2}$$

Decay

$M_a^2 + p^{*2} + m_b^2 - 2M_a(p^{*2}+m_b^2)^{1/2} = p^{*2} + m_c^2$

$M_a^2 + m_b^2 - m_c^2 = 2M_a(p^{*2}+m_b^2)^{1/2}$

$\mathbf{M_a^4 + (m_b^2 - m_c^2)^2 + 2M_a^2(m_b^2 - m_c^2) = 4M_a^2(p^{*2} + m_b^2)}$

$$p^{*2} = \frac{M_a^4 - 2M_a^2\left(m_b^2 + m_c^2\right) + \left(m_b^2 - m_c^2\right)^2}{4M_a^2}$$

$$p^{*2} = \frac{\left[M_a^2 - \left(m_b + m_c\right)^2\right] \bullet \left[M_a^2 - \left(m_b - m_c\right)^2\right]}{4M_a^2}$$

$$E_b^* = \sqrt{p^{*2} + m_b^2} = \frac{M_a^2 + \left(m_b^2 - m_c^2\right)}{2M_a}$$

$$E_c^* = \sqrt{p^{*2} + m_c^2} = \frac{M_a^2 - \left(m_b^2 - m_c^2\right)}{2M_a}$$

Decay

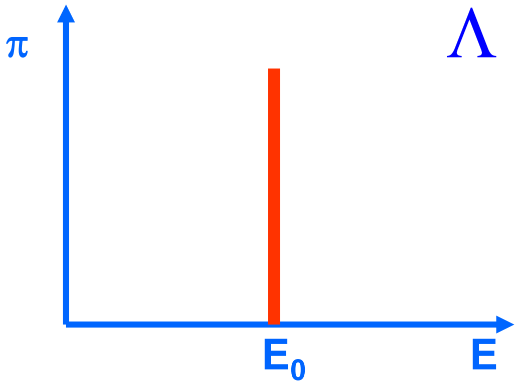
- In the case: $m_b = m_c = m$

$$E^*_b = E^*_c = E^*/2 \quad ; \quad p^{*2} = (E^*/2)^2 - m^2$$

$\Rightarrow$ In two body decays:

In the rest frame of the decaying particle, once the mass of the two daughters are known, the module of their momentum and hence their energy are fixed

The decay is mono-energetic



$$\Lambda \rightarrow p + \pi^-$$

$\pi(E)$ = probability density function
The probability that π^- energy is E_0 is equal to 1 and it is equal to zero for any other value:
$\pi(E) = \delta(E - E_0)$

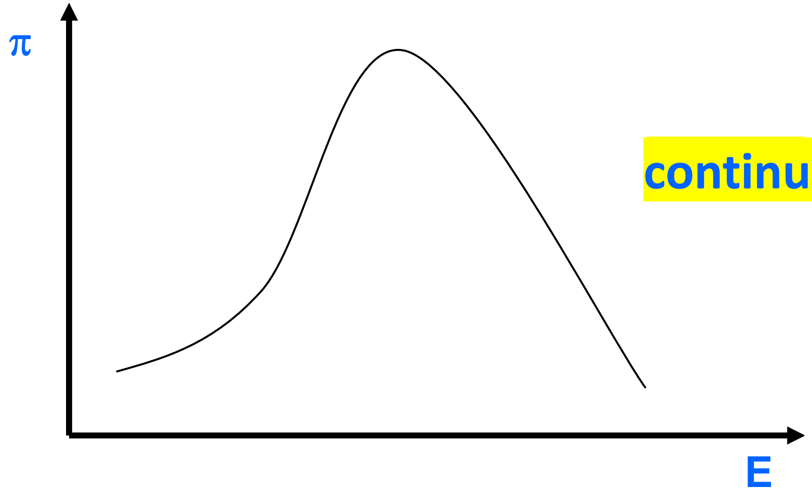
Decay

- Three body decay

$$n \rightarrow p + e^{-} + \overline{\nu}_e \qquad\qquad m_{\overline{\nu}_e} = 0$$

- In the neutron reference frame

$$\vec{0} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \quad ; \quad m_n = \sqrt{p_1^2 + m_p^2} + \sqrt{p_2^2 + m_e^2} + p_3$$



continuous spectrum

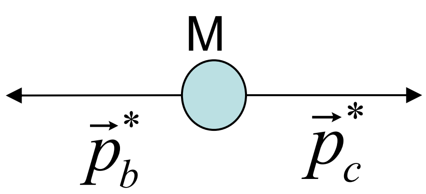
Exercise: limit angle in the lab system

- Let consider the decay

  a $\rightarrow$ b + c
- Being M_a, m_b and m_c the masses of the three particles. In the CM RF a is at rest and the conservation of four-momentum gives:

$$M_a = E_b^* + E_c^*$$

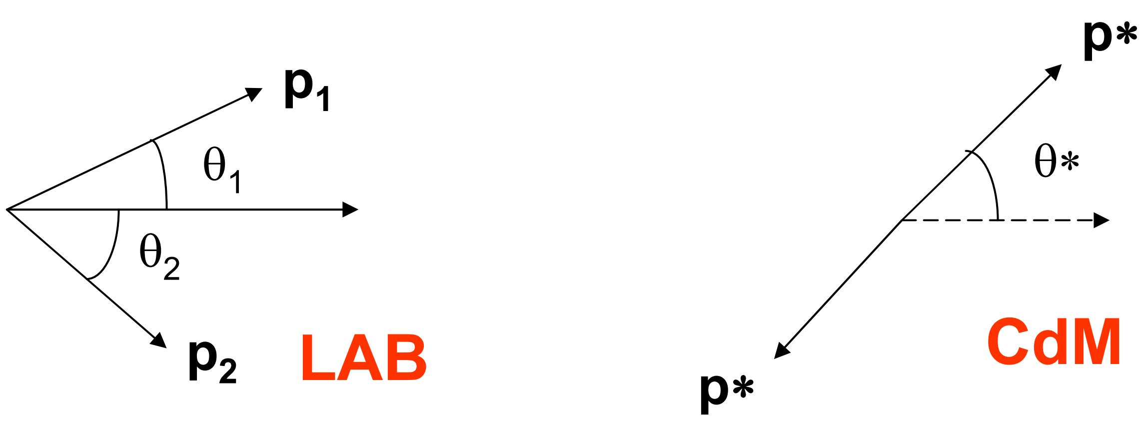
$$\vec{0} = \vec{p}_b^{\,*} + \vec{p}_c^{\,*}$$



Exercise: limit angle in the lab system

- If in the lab system a has momentum **p** the β value of the Lorentz transformation to the CdM will be

$$\beta = p/E \quad \text{e} \quad \beta\gamma = p/M_a$$



Exercise: limit angle in the lab system

$$\begin{pmatrix} E \\ P_x \\ P_y \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E^* \\ P_x^* \\ P_y^* \end{pmatrix}$$

$$\begin{cases} E_1 = \gamma E_1^* + P_{1x}^* \beta\gamma = \gamma E_1^* + \beta\gamma \cos\vartheta^* p^* \\ P_{1x} = P_1 \cos\vartheta_1 = \gamma P_{1x}^* + \beta\gamma E_1^* = \gamma p^* \cos\vartheta^* + \gamma\beta E_1^* \\ P_{1y} = P_1 \sin\vartheta_1 = P_{1y}^* = P^* \sin\vartheta^* \end{cases}$$

$$\begin{cases} E_2 = -\beta\gamma p^* \cos\vartheta^* + \gamma E_2^* \\ P_{2x} = P_2 \cos\vartheta_2 = -\gamma p^* \cos\vartheta^* + \beta\gamma E_2^* \\ P_{2y} = P_2 \sin\vartheta_2 = p^* \sin\vartheta^* \end{cases}$$

$$tg\vartheta_1 = \frac{p^* \sin\vartheta^*}{\gamma p^* \cos\vartheta^* + \gamma\beta E_1^*} = \frac{\sin\vartheta^*}{\gamma(\cos\vartheta^* + \beta/\beta^*)}$$

$$tg\vartheta_2 = \frac{\sin\vartheta^*}{\gamma(-\cos\vartheta^* + \beta/\beta_2^*)} \qquad \beta_1^* = \frac{P^*}{E_1^*}; \beta_2^* = \frac{p^*}{E_2^*}$$

If: $\beta^*_i < \beta$

Particle i will be emitted in the forward direction for every θ^* value

Exercise: limit angle in the lab system

The value of θ_{max}

$$\frac{dtg\vartheta_1}{d\vartheta^*} = \frac{1+\cos\vartheta^*\dfrac{\beta}{\beta_1^*}}{\gamma\left(\cos\vartheta^* + \dfrac{\beta}{\beta_1^*}\right)^2} = 0 \;\Rightarrow\; \cos\vartheta^* = -\frac{\beta_1^*}{\beta}$$

$$tg\vartheta_{max} = \frac{\sqrt{1-\left(\dfrac{\beta_1^*}{\beta}\right)^2}}{\gamma\left(\dfrac{\beta}{\beta_1^*} - \dfrac{\beta_1^*}{\beta}\right)} = \frac{1}{\gamma\sqrt{\left(\dfrac{\beta}{\beta_1^*}\right)^2 - 1}}$$

$$\theta = \theta_1 + \theta_2$$

$$p^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\vartheta = M^2$$

$$\cos\vartheta = \frac{m_1^2 + m_2^2 + 2E_1E_2 - M^2}{2p_1p_2}$$

Exercise: limit angle in the lab system

In the limit $E_i >> m_i$ ⟶ **$P_i \cong E_i$**

$$2E_1E_2(1-\cos\vartheta) = 4E_1E_2\sin^2\frac{\vartheta}{2} = M^2 - m_1^2 - m_2^2$$

$$\sin\frac{\vartheta}{2} = \frac{\sqrt{M^2 - m_1^2 - m_2^2}}{2\sqrt{E_1E_2}}$$

Θ is minimum for $E_1 = E_2$

Exercise: trasformation LAB⇒CdM

$$E^{*2} = (E_i + m_b)^2 - p_i^2 = E^2 - p^2$$

$$\beta_{CdM} = \frac{P}{E} = \frac{P_i}{\sqrt{p_i^2 + m_i^2} + m_b}$$

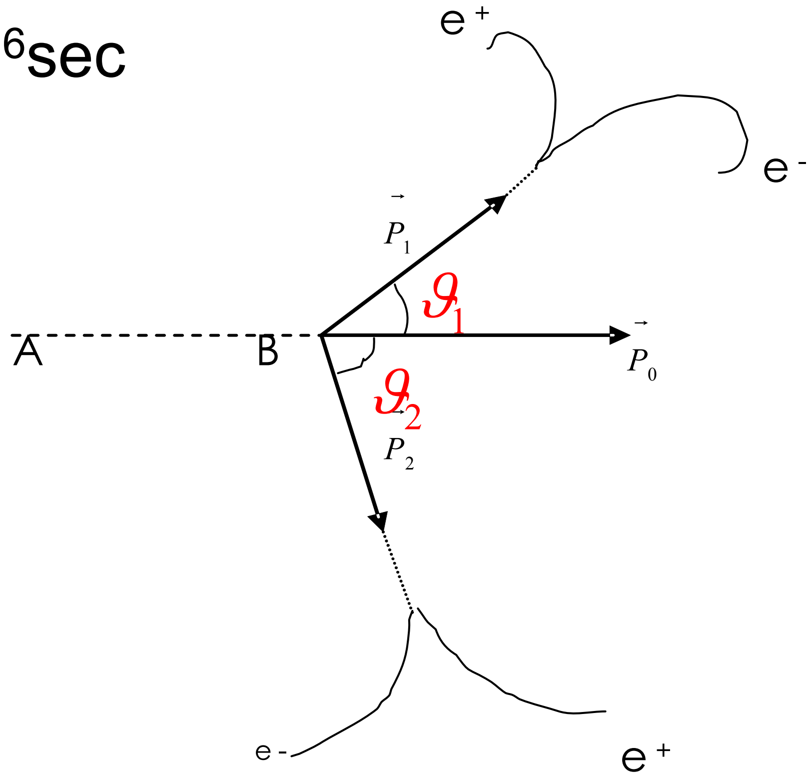
$$1 - \beta_{CDM}^2 = 1 - \frac{P^2}{E^2} = \frac{E^2 - P^2}{E^2} = \frac{E^{*2}}{E^2}$$

$$\gamma_{CdM} = \frac{1}{\sqrt{1 - \beta_{CdM}^2}} = \frac{E}{E^*}$$

$$\beta_{CdM}\gamma_{CdM} = \frac{P_i}{E^*}$$

π^0 Decay

- Decay: $\pi^0 \rightarrow \gamma + \gamma$
- mass: $m_{\pi 0}$ = 135 MeV
- lifetime: $\tau_{\pi 0}$= $0.828 \cdot 10^{-16}$sec



$$\vec{p}_0 = \vec{p}_1 + \vec{p}_2 \quad ; \quad \vartheta = \vartheta_1 + \vartheta_2$$

π^0 Decay

$$a)\quad \vec{p}_0 = \vec{p}_1 + \vec{p}_2 \quad \rightarrow \quad \mathrm{p}_0^2 = \mathrm{p}_1^2 + \mathrm{p}_2^2 + 2\mathrm{p}_1\mathrm{p}_2 \cos\vartheta$$

$$b)\quad \sqrt{m_{\pi^0}^2 + \mathrm{p}_0^2} = E_1 + E_2 = p_1 + p_2 \quad \rightarrow \quad m_{\pi^0}^2 + \mathrm{p}_0^2 = \mathrm{p}_1^2 + \mathrm{p}_2^2 + 2\mathrm{p}_1\mathrm{p}_2$$

- Summing up equations a) and b) we obtain:

$$2\mathrm{p}_1\mathrm{p}_2\left(1-\cos\vartheta\right) = m_{\pi^0}^2 \quad \rightarrow \quad 4\mathrm{p}_1\mathrm{p}_2 \sin^2\frac{\vartheta}{2}$$

$$\boxed{\sin^2\frac{\vartheta}{2} = \frac{m_{\pi^0}^2}{4E_1E_2}}$$

π^0 Decay

- Let's prove that the minimum value of θ occurs when:

$$E_1 = E_2 = \frac{E_0}{2} \text{ dove } E_0 = \sqrt{m_{\pi^0}^2 + p_0^2}$$

θ minimum corresponds to E_1E_2 = Max. Let's put
$y=E_1E_2 = E_1(E_0 - E_1) = E_1E_0 - E_1^2 = xE_0 - x^2$

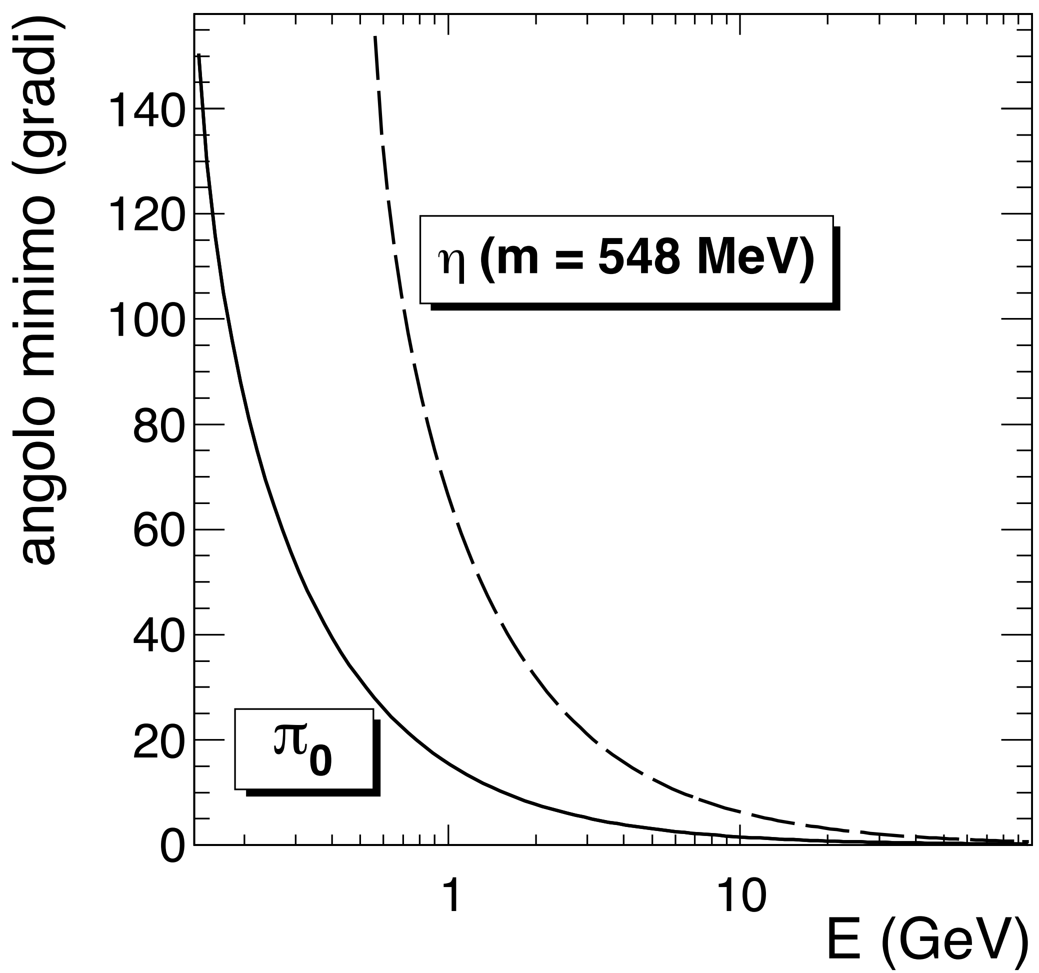
- Where we set $x=E_1$ which is the unknown variable

$$\frac{\mathrm{dy}}{\mathrm{dx}} = E_0 - 2x = 0 \quad \Rightarrow \quad x = \frac{E_0}{2} \quad \Rightarrow \quad E_1 = \frac{E_0}{2}$$

- when the angular aperture of the two photons is minimal:

we have the equipartition of the energy of π^0 between the two photons.

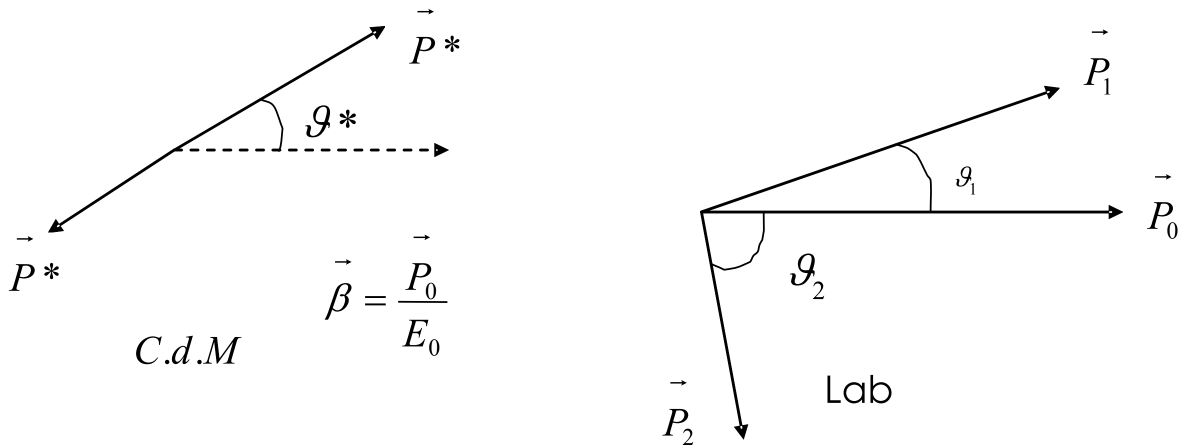
π^0 Decay



angolo minimo (gradi)
140
120
100
80
60
40
20
0
1
10
E (GeV)
η (m = 548 MeV)
π0

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

- We know that in the reference system of π^0 at rest, photons have momentum equal to $M_{\pi 0}/2$.
- We derive the distribution of the momentum of the photons in the reference system of the laboratory.



- Since $M_\gamma = 0$, we will have that $p^*=p_1^*=p_2^* = E_1^* = E_2^* = M_{\pi 0}/2$ Moreover, if p_0 is the momentum of π^0 in the laboratory we will have:

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

$$\beta = \frac{p_0}{E_0} \quad ; \quad \gamma = \frac{\mathrm{E}_0}{\mathrm{E}^*} = \frac{\mathrm{E}_0}{M_{\pi^0}} \quad ; \quad \mathrm{p_y} = p_y^* = p^* \sin\vartheta^*$$

$$p_x = \gamma\left(p^* \cos\vartheta^* + \beta E^*\right)$$

$$E_1 = \gamma\left(E_1^* + \beta \cdot p_1^* \cdot \cos\vartheta^*\right) = \gamma \cdot \frac{m_{\pi^0}}{2}\left(1 + \beta \cdot \cos\vartheta^*\right) =$$

$$= \frac{E_0}{m_{\pi^0}} \cdot \frac{m_{\pi^0}}{2}\left(1 + \frac{p_0}{E_0} \cdot \cos\vartheta^*\right) = \frac{E_0}{2}\left(1 + \frac{p_0}{E_0} \cdot \cos\vartheta^*\right) =$$

$$= \frac{E_0}{2} \cdot \frac{E_0 + p_0 \cdot \cos\vartheta^*}{E_0} = \frac{E_0 + p_0 \cdot \cos\vartheta^*}{2}$$

Momentum of photons from $\pi^0 \rightarrow \gamma + \gamma$ in the Laboratory

- So in the laboratory:
- E_γ will vary between two minimum and maximum values given by: E_γ will be minimum for: $\cos\theta^* = -1 \Rightarrow \theta^* = \pi$

$$E_\gamma = \frac{E_0 - p_0}{2}$$

- E_γ will be maximum for: $\cos\theta^* = 1 \Rightarrow \theta^* = 0$

$$E_\gamma = \frac{E_0 + p_0}{2}$$

- We now derive the distribution function of E_γ between these minimum and maximum values.

Momentum of photons from π^0 → γ + γ in the Laboratory

- Recalling that the π^0 has spin zero, we will have that in the system at rest of π^0 the distribution in cosθ* must be flat. We will therefore have:

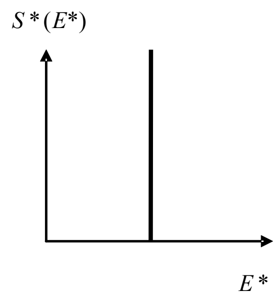
$$\int_{-1}^{+1} f(\cos\vartheta^*)\,d\cos\vartheta^* = 1 \Rightarrow f(\cos\vartheta^*) = \tfrac{1}{2}$$

$$dE_1 = \frac{1}{2}P_0\,d\cos\vartheta^* = \beta\gamma\frac{m\pi^0}{2}d(\cos\vartheta^*)$$

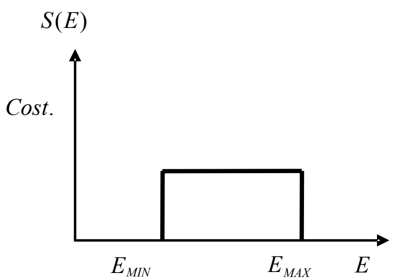
$$dN = f(\cos\vartheta^*)\,d\cos\vartheta^* = \frac{1}{2}d\cos\vartheta^* = \frac{dE_1}{\beta\gamma M_{\pi^0}} = \frac{dE_1}{P_0}$$

$$\frac{dN}{dE_1} = \frac{1}{P_0}$$

Costant in the lab



CM



LAB

Angular distribution of photons in π^0 decay

- We now want to demonstrate that the decay configuration of π^0 at minimum angle is also the most probable configuration.
- In general we have obtained the following formula:

$$c)\quad \sin^2\frac{\vartheta}{2} = \frac{m^2_{\pi^0}}{4E_1E_2}$$

$$\rightarrow\quad 4E_1\cdot\left(E_0 - E_1\right) = \frac{m^2_{\pi^0}}{\sin^2\frac{\vartheta}{2}}$$

- Differentiating left and right we have:

$$4(\mathrm{E}_0 - 2E_1)\cdot dE_1 = -2\frac{m^2_{\pi^0}}{\sin^3\frac{\vartheta}{2}}\cos\frac{\vartheta}{2}\cdot\frac{1}{2}\cdot d\vartheta$$

Angular distribution of photons in π^0 decay

d) $$\frac{dn}{d\vartheta}=\frac{dn}{dE_1}\frac{dE_1}{d\vartheta}=\frac{1}{p_0}\cdot\frac{m_{\pi^0}^2}{4(\mathrm{E}_0-2E_1)}\cdot\frac{\cos\frac{\vartheta}{2}}{\sin^3\frac{\vartheta}{2}}$$

- Let rewrite eq. c) as

$$\mathrm{E}_1\left(\mathrm{E}_0-\mathrm{E}_1\right)=\frac{m_{\pi^0}^2}{4\cdot\sin^2\frac{\vartheta}{2}}\quad\rightarrow\quad \mathrm{E}_1^2-\mathrm{E}_0\mathrm{E}_1+A \qquad\qquad \mathrm{A}=\frac{m_{\pi^0}^2}{4\cdot\sin^2\frac{\vartheta}{2}}$$

- The solution in E_1 will be given by:

$$\mathrm{E}_1=\frac{E_0\pm\sqrt{E_0^2-4\cdot A}}{2};$$

Angular distribution of photons in π^0 decay

$$\mathrm{E}_0 - 2E_1 = \mathrm{E}_0 - E_0 \mp \sqrt{E_0^2 - 4 \cdot A} =$$

$$= \mp \sqrt{E_0^2 - \frac{m_{\pi^0}^2}{\sin^2 \frac{\vartheta}{2}}} = \mp \frac{E_0}{\sin \frac{\vartheta}{2}} \sqrt{\sin^2 \frac{\vartheta}{2} - \frac{m_{\pi^0}^2}{E_0^2}}$$

- Substituting (E_0 −2E_1) into the equation d) we have:

$$\frac{\mathrm{dn}}{\mathrm{d}\vartheta} = \frac{m_{\pi^0}^2}{p_1 \cdot E_0} \cdot \frac{\cos \frac{\vartheta}{2}}{4 \sin^2 \frac{\vartheta}{2} \sqrt{\sin^2 \frac{\vartheta}{2} - \frac{m_{\pi^0}^2}{E_0}}}$$

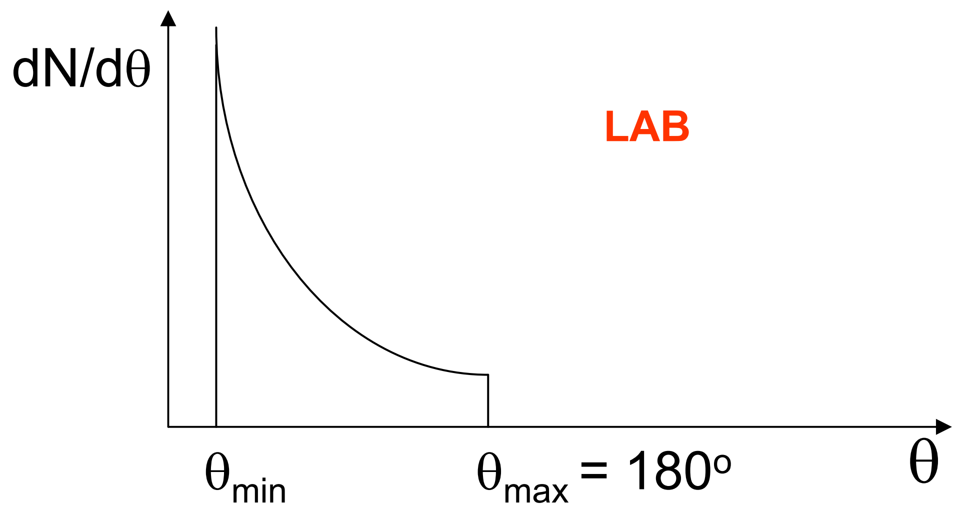
Angular distribution of photons in π^0 decay

The angular distribution will have a maximum

$$\sin^2\frac{\vartheta}{2} = \frac{m_{\pi^0}^2}{E_0}:$$

θ_{min} is also the most probable configuration

It follows that the most likely decay configuration is that with the smallest aperture angle between the two photons, with equipartition of initial energy and of the aperture angle



$E_\gamma = (E_0 + p_0 \cos\theta) / 2$

E_γ min for $\cos\theta^*=0 \Rightarrow \theta^*=\pi/2$

E_γ max for $\cos\theta^*=1 \Rightarrow \theta^*=0$

Exercise: pγ=pγ(θ∗) distribution

$$E_{\pi^0}^{LAB} = 4.05 Gev \qquad M_{\pi^0} = 135\,MeV \qquad E^* = M_{\pi^0}$$

$$E_{\pi^0}^{LAB} = \gamma M_{\pi^0} \qquad \beta_{\gamma_1}^* = \frac{P_{\gamma_1}^*}{E_{\gamma_1}^*} = 1 \qquad E_{\gamma_1}^* = \frac{M_{\pi^0}}{2} = E_{\gamma_2}^* = 67.5\,MeV$$

$$E_{\gamma_1} = \gamma \cdot E_{\gamma_1}^* \left(1 + \beta \cdot 1 \cdot \cos\vartheta^*\right)$$

$$E_{\gamma_1} = \frac{E_{\pi^0}^{Lab}}{M_{\pi^0}} \cdot E_{\gamma_1}^* \left(1 + \beta \cos\vartheta^*\right) = E_{\pi^0}^{Lab} \cdot \frac{\left(1 + \beta \cos\vartheta^*\right)}{2}$$

$$E_{\gamma_1} = E_{\pi^0}^{Lab} \cdot \frac{\left(1 + \beta \cos\vartheta^*\right)}{2}$$

$$\mathrm{tg}\,\vartheta_1 = \frac{\sin\vartheta_1^*}{\gamma\left(\beta + \cos\vartheta_1^*\right)}$$

Exercise: pγ=pγ(θ∗) distribution

A) $\theta * = 90^\circ :$

$$E_{\gamma_1} = 4050\frac{(1+\beta\cdot 0)}{2} = 2025 MeV$$

$$E^*_{\gamma_1} = 67.5$$

$$\gamma = \frac{E_{\pi^0}}{m_{\pi^0}} = 30 GeV$$

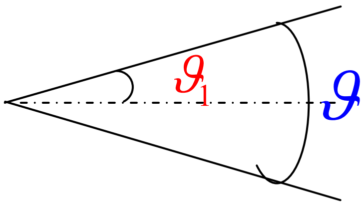
$$tg\vartheta_1(\theta^*_1 = 90^\circ) = \frac{1}{\beta\gamma}$$

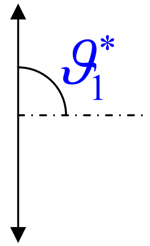
$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1-(\frac{1}{2\gamma^2})$$

$$(1-\beta) = \frac{1}{2\gamma^2}$$

$$\beta = 1-\frac{1}{1800} \approx 1$$

$$tg\vartheta_1 = \frac{1}{30} \Rightarrow \vartheta_1^{Lab} \approx 2^\circ \to \vartheta = 4^\circ$$



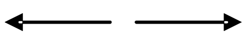


Exercise: pγ=pγ(θ∗) distribution

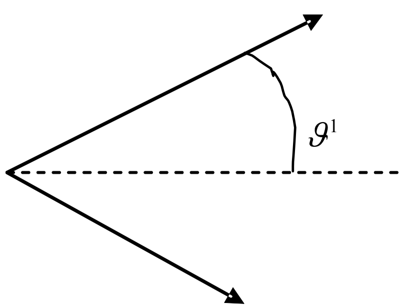
B)

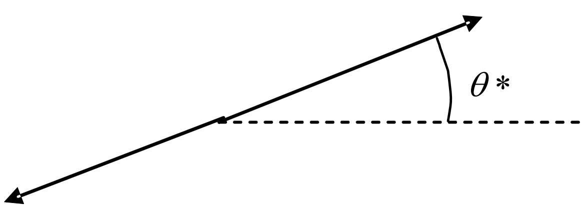
$\theta * = 0; \theta * = 180^\circ:$

$$E_\gamma = \gamma E^*_\gamma (1 \pm \beta) \begin{cases} E^{Lab}_{\pi^0} \big/ M_{\pi^0} \cdot \dfrac{M_{\pi^0}}{2}(1+1) = E^{Lab}_{\pi^0} = 4050 Mev \\ \gamma E^*_\gamma (1-\beta) \approx \gamma E^*_\gamma \cdot \dfrac{1}{2\gamma^2} = \dfrac{E^{Lab}_{\pi^0}}{4\gamma^2} \approx 1.1 Mev \end{cases}$$



C)





Exercise: pγ=pγ(θ∗) distribution

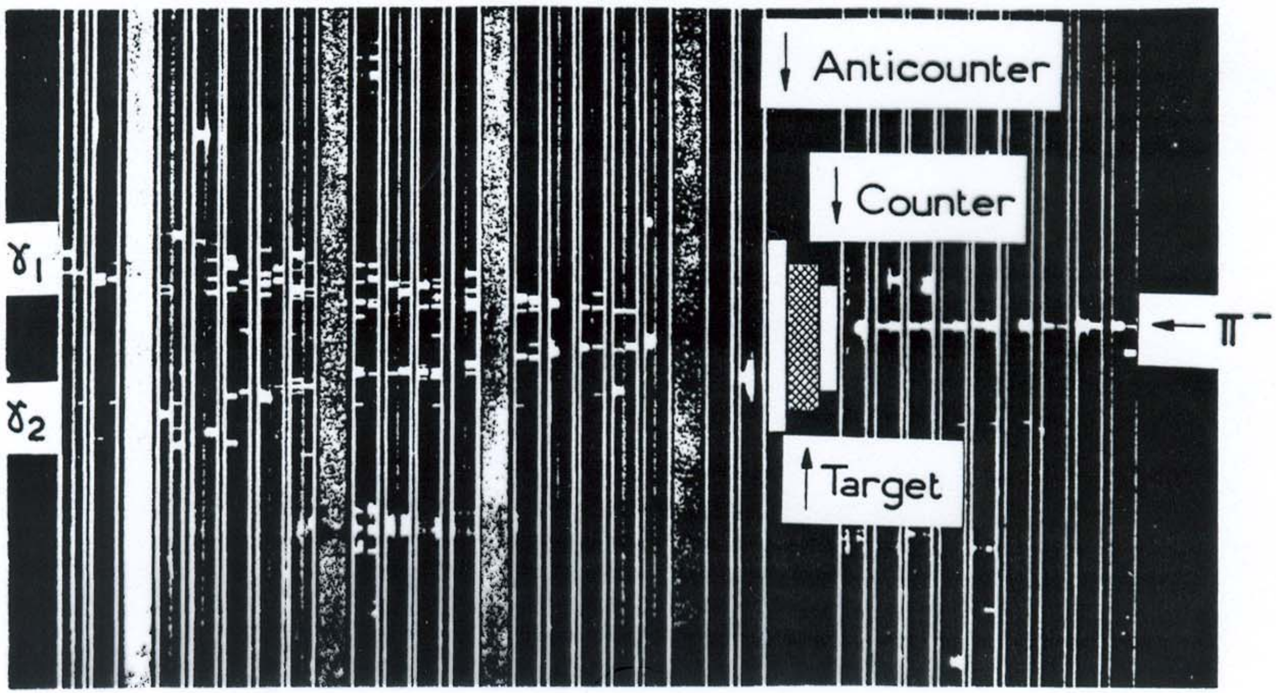
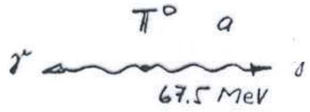


Fig. 1.6 In the target, a π^0 is produced by charge exchange $\pi^- + p \rightarrow \pi^0 + n$. The two gamma rays from the decay $\pi^0 \rightarrow \gamma_1 + \gamma_2$ produce "showers" γ_1 and γ_2 in a spark chamber. The energy of the π^0, E_{π^0} is about 3 GeV. The decay is slightly asymmetric ; (Photo CERN, SIS 16754)



Exercise

If p_{xi},p_{yi},p_{zi} (MeV/c) are the 3-momentum component of two charged pions measured in the lab system, evaluate the invariant mass of the mother particle and build a histogram with the obtained values.

px1	py1	pz1	px2	py2	pz2	px1	E1	px2	E2	M
-177.29	-304.79	354.31	165.10	303.89	-357.16					
-174.33	434.17	181.71	161.64	-434.58	-182.29					
-410.09	210.23	-141.90	393.96	-207.18	175.56					
379.79	-262.10	-94.39	-392.09	262.18	129.81					
152.14	-333.47	331.91	-163.16	329.89	-308.76					
160.12	-339.72	313.05	-176.50	342.35	-272.97					
149.02	289.33	373.46	-161.74	-289.49	-371.95					
-405.57	188.66	-215.35	396.61	-190.78	200.65					
-397.72	297.68	-77.99	385.72	-298.59	81.70					
352.08	-195.28	-277.04	-358.38	198.58	255.26					
-497.89	77.63	-41.46	485.30	-77.70	39.08					
-161.34	408.53	237.01	148.61	-408.47	-228.76					
-19.06	-485.24	-71.07	7.71	483.57	86.37					
-37.40	341.00	362.10	24.44	-339.96	-356.08					
-145.32	-385.43	257.33	134.65	388.17	-272.33					
-397.90	248.48	176.71	385.79	-249.21	-179.47					
-200.84	-345.58	299.61	188.81	346.11	-302.77					
384.25	-241.50	197.75	-396.71	240.57	-193.19					
339.64	-134.24	242.67	-353.52	134.83	-309.67					
-138.00	-395.18	248.14	125.94	393.68	-206.14					
-244.86	-309.62	314.18	231.41	309.11	-310.70					
-388.08	-267.87	-171.83	376.08	267.20	168.06					
277.04	-408.49	15.61	-290.59	408.85	-11.93					
-302.40	-286.31	258.94	292.24	285.12	-273.90					
-84.49	-382.48	308.71	70.70	383.49	-298.69					
457.14	155.49	79.29	-466.40	-155.31	-66.89					
151.35	373.26	294.61	-164.17	-373.08	-294.83					
-385.68	39.44	262.76	373.71	-42.03	-308.13					
-105.48	475.34	55.98	91.73	-475.53	-68.34					
-401.72	-260.86	111.61	387.35	261.93	-83.90					

Exercise: study of the decay $K^0 \rightarrow \pi^- \pi^+$

- Determine p*, the pion energies in the CM, their speed in the CM and in the LAB frame. Discuss about their maximum emission angles.

Exercise: study of the decay $\pi \rightarrow \mu\nu$

- Determine p*, the muon and neutrino energy in the CM, their speed in the CM and in the LAB frame. Determine the longitudinal momentum distribution of the two particles

Interactions

Classically interaction at a distance is described in terms of a potential or a *field*. In quantum theory it is viewed in terms of **exchange of quanta**. Quanta are bosons associated with the particular type of interaction.

<u>Example</u>: electrostatic interaction bewteen point charges.

we must have $\Delta E \cdot \Delta t \simeq \hbar$

Interactions

In nature there are four types of interaction.

- The strong interaction binds quarks in hadrons and protons and neutrons in nuclei. It is mediated by gluons.
- The electromagnetic interaction binds electrons and nuclei in atoms, and it is also responsible for the intermolecular forces in liquids and solids. It is mediated by the photon.
- The weak interaction is typified by radioactive decays, for example the slow β decay. The quanta of the weak field are the $W^{\pm}$ and Z^0 bosons.
- The gravitational interaction acts between all types of massive particles. It is by far the weakest of all the fundamental interactions.

Interactions

To indicate the relative magnitudes of the four types of interaction, the comparative strenghts of the force between two protons when just in contact are very roughly:

strong	electromagnetic	weak	gravity
1	10^{-2}	10^{-7}	10^{-39}

Ever since Einstein, physicists have speculated that the *4 interactions might be different manifestations of a unified force*. Up to now only electromagnetic and weak forces have been unified: these would have the same strength at very high energies, whereas at lower energies this symmetry is broken and the two forces have the same strength.

All four interactions play a fundamental role in our universe.

Electromagnetic Interaction

The coupling constant of the electromagnetic interaction is the fine structure constant α.

$$\alpha = \frac{\frac{1}{4\pi}\frac{e^2}{(\hbar/mc)}}{mc^2} = \frac{\text{electrostatic energy of two } e \text{ at a distance } (\hbar/mc)}{\text{rest mass of the electron}}$$

$$\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

$\pi^0 \rightarrow \gamma\gamma$ $\tau = (8.4 \pm 0.6) \times 10^{-17}$ s

The quantum of the electromagnetic interaction is the photon.

The field theory of the electromagnetic interaction is Quantum ElectroDynamics (QED).

Week Interaction

$n \to p + e^- + \bar{\nu}_e$ β decay

$\bar{\nu}_e + p \to n + e^+$ $\bar{\nu}$ absorption

$\Sigma^- \to n + \pi^-$ $\tau \cong 10^{-10}$ s

(dds) (ddu)

$\Sigma^0 \to \Lambda + \gamma$ $\tau \cong 10^{-19}$ s

(e.m. viola isospin)

$$\frac{\alpha_W}{\alpha} \approx \sqrt{\frac{10^{-19}}{10^{-10}}} \approx 10^{-4} \div 10^{-5}$$

The quanta of the weak interaction are the $W^\pm$ and Z^0 bosons.

$M_W = (80.425 \pm 0.038)$ GeV/c^2

$M_Z = (91.1876 \pm 0.0021)$ GeV/c^2

Strong Interaction

$\Sigma^0(1385) \to \Lambda + \pi^0$ Γ=36 MeV $\tau \sim 10^{-23}$ s

$\Sigma^0(1192) \to \Lambda + \gamma$ $\tau \sim 10^{-19}$ s

$$\frac{\alpha_S}{\alpha} \approx \sqrt{\frac{10^{-19}}{10^{-23}}} \approx 100 \qquad \frac{e^2}{4\pi} = \frac{1}{137} \qquad \frac{g_s^2}{4\pi} \approx 1$$

The quanta of the strong interactions are the gluons. The strong charge is called color and it can assume 6 values R, G, B, $\overline{R}$, $\overline{G}$, $\overline{B}$.

Color symmetry is an exact symmetry, i.e. the force between quarks is color-independent.

The field theory of strong interactions is Quantum Chromodynamics (QCD).

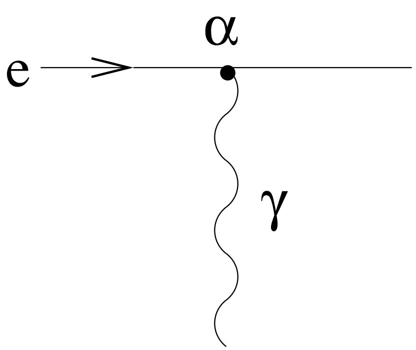
Asymptotic freedom $V_s \to \alpha_s/r$ $q^2 \to \infty$

Confinement $V_s \to kr$ $q^2 \to 0$ $[r \to \infty]$

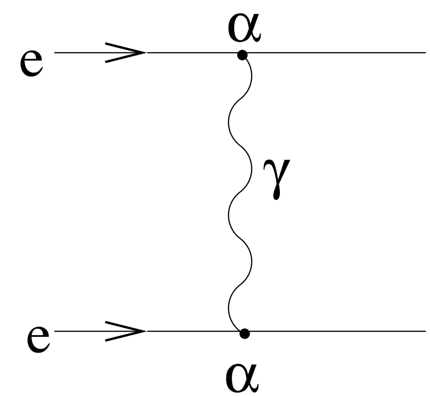
Feynman Diagram

- Feynman diagram are graphic way of representing the interactions between particles and fields.
- The solid lines represent the fermions.
- Wavy (curly or dashed) lines bosons.
- The **arrows** on the lines indicate the direction of time, with time running from left to right
- Fermionic and bosonic lines intersect at **vertices** where charge, energy and momentum are conserved.
- The intensity of the interaction is represented by a **coupling constant** associated with each vertex.
- Open lines represent real particles, closed lines represent virtual particles.

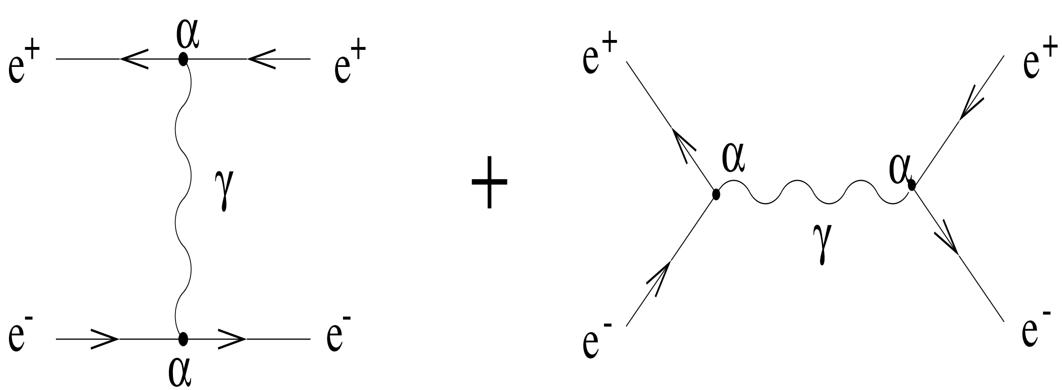
Feynman Diagrams



basic electron-photon vertex

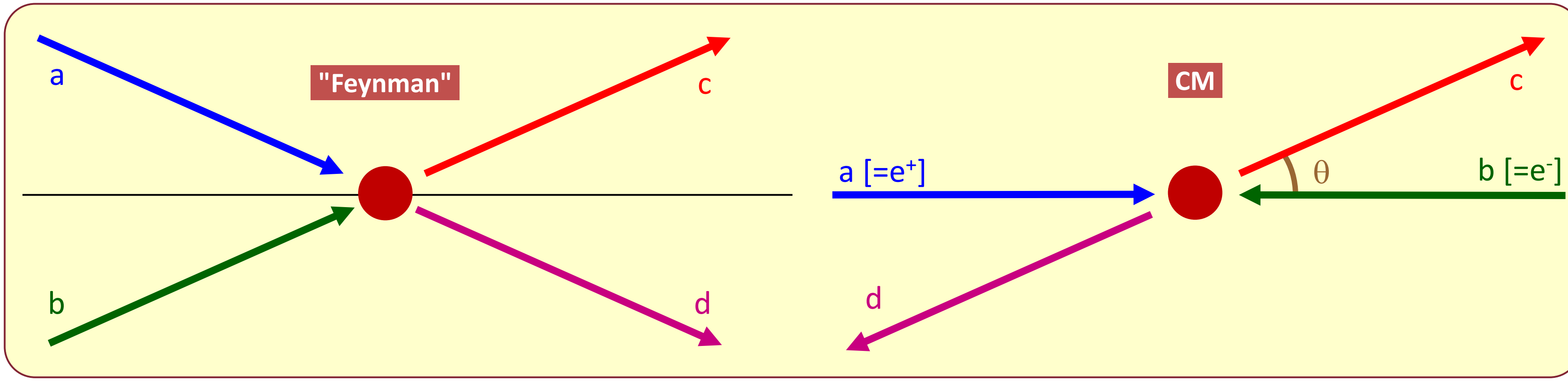


ee scattering via γ exchange



e^+e^- scattering via photon exchange with two diagrams contributing in first order

Mandelstam variables



The Mandelstam variables $\boxed{s, t, u}$:

CM system

- $p_a = [E, \quad p, \quad 0, \quad 0];$
- $p_b = [E, \quad -p, \quad 0, \quad 0];$
- $p_c = [E, p\cos\theta, p\sin\theta, \quad 0];$
- $p_d = [E, -p\cos\theta, -p\sin\theta, 0];$

s,t,u L-invariant

- $s \equiv (p_a + p_b)^2 = (p_c + p_d)^2 = 4E^2;$
- $t \equiv (p_a - p_c)^2 = (p_b - p_d)^2 = -\tfrac{1}{2}\, s\,(1 - \cos\theta) = -s\sin^2(\theta/2);$
- $u \equiv (p_a - p_d)^2 = (p_b - p_c)^2 = -\tfrac{1}{2}\, s\,(1 + \cos\theta) = -s\cos^2(\theta/2);$
- $s + t + u = 0$ ($\rightarrow$ 2 independent variables, e.g. $[E,\theta]$, $[s, t]$, $[\sqrt{s},\theta]$).

Lorentz-invariant variables for 2→2 processes.

Assume $E \gg m_i$, for the masses of all 4 bodies (otherwise, look for the formulas in [PDG]).

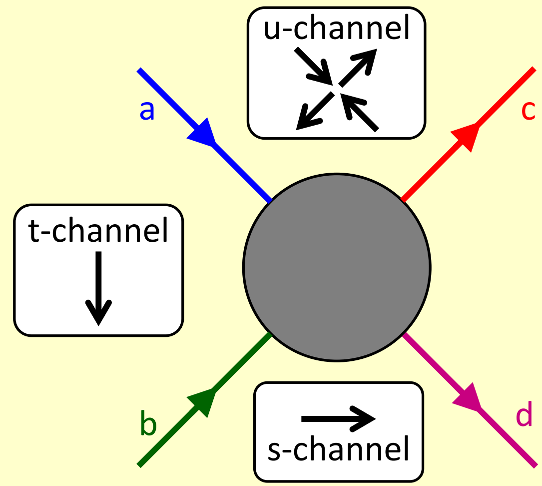
Q. : what about φ (the azimuth) ?
A. : if nothing in the dynamics is φ-dependent (e.g. the spin direction), then the cross-section must be φ-symmetric.

Mandelstam variables $m_i \neq 0$

General case ab → cd, masses NOT negligible:

[p_i and p_j are 4-mom, $p_i p_j$ = dot product]

- $s \equiv (p_a + p_b)^2 = (p_c + p_d)^2 = m_a^2 + m_b^2 + 2 p_a p_b$;
- $t \equiv (p_a - p_c)^2 = (p_b - p_d)^2 = p_a^2 + m_c^2 - 2 p_a p_c$;
- $u \equiv (p_a - p_d)^2 = (p_b - p_c)^2 = p_a^2 + m_d^2 - 2\, p_a p_d$;
- $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 + {}$
  $+ 2 p_a (p_a + p_b - p_c - p_d) =$
  $= m_a^2 + m_b^2 + m_c^2 + m_d^2 = \Sigma_i\, m_i^2$.



s-channel	ab → cd	($\bar{p}p \to \bar{n}n$)
t-channel	a$\bar{c}$ → $\bar{b}$d	($\bar{p}n \to \bar{p}n$)
u-channel	a$\bar{d}$ → $\bar{b}$c	($\bar{p}\bar{n} \to \bar{p}\bar{n}$)

In addition, the <u>crossing symmetry</u> correlates the processes which are symmetric wrt time (s-, t-, and u-channels [see box]). If the c.s. is conserved in the interaction, the same amplitude is valid for all the channels, in their appropriate physical domains (an example on next page).

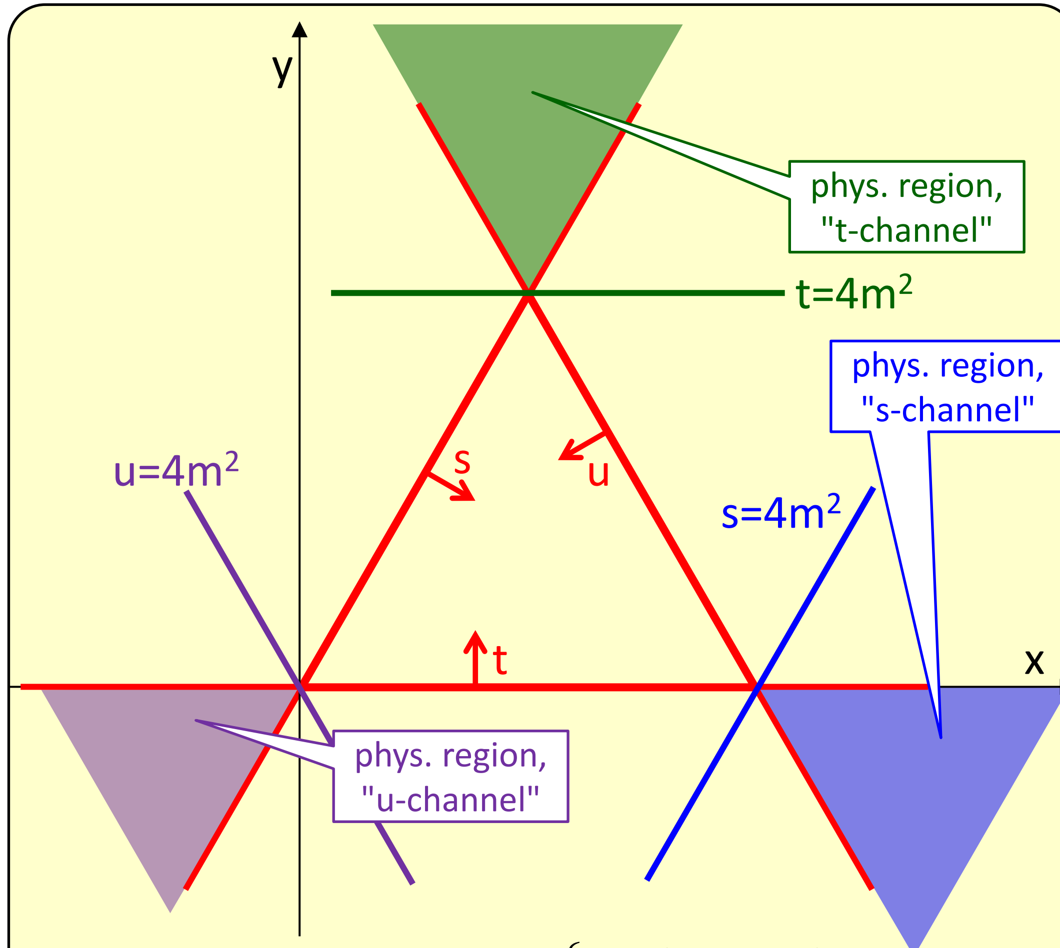
an old approach (1950-80), now almost forgotten, especially important for strong interactions at low energies (see the example $\bar{p}p \to \bar{n}n$), where the dynamics was not calculable (still is not).

Mandelstam variables

Example : $m_a = m_b = m_c = m_d = m$;

$$\left.\begin{array}{l} \bullet\ s = 4E^2 \geq 4m^2; \\ \bullet\ t = -4p^2\sin^2(\theta/2); \\ \bullet\ u = -4p^2\cos^2(\theta/2); \end{array}\right\} s + t + u = 4m^2;$$

- in a xy plane draw an equilateral triangle of height $4m^2$, and label s-t-u the three sides and the lines through them (drawn in red);
- remember Viviani's theorem and its extension ("*the sum of the signed distances between a point and the lines of a triangle is a constant*");
- find the physical regions (i.e. the allowed values of s-t-u) for the given process (i.e. the "s-channel") and for the t and u channels;
- among s-t-u, only two variables are independent $\rightarrow$ the "space of the parameters" is 2D.



$$\begin{cases} x = \dfrac{4m^2 + s - u}{\sqrt{3}} \\ y = t = 4m^2 - s - u \end{cases} \rightarrow \begin{cases} s = \left(\sqrt{3}x - y\right)/2 \\ t = y \\ u = 4m^2 - \left(\sqrt{3}x + y\right)/2. \end{cases}$$

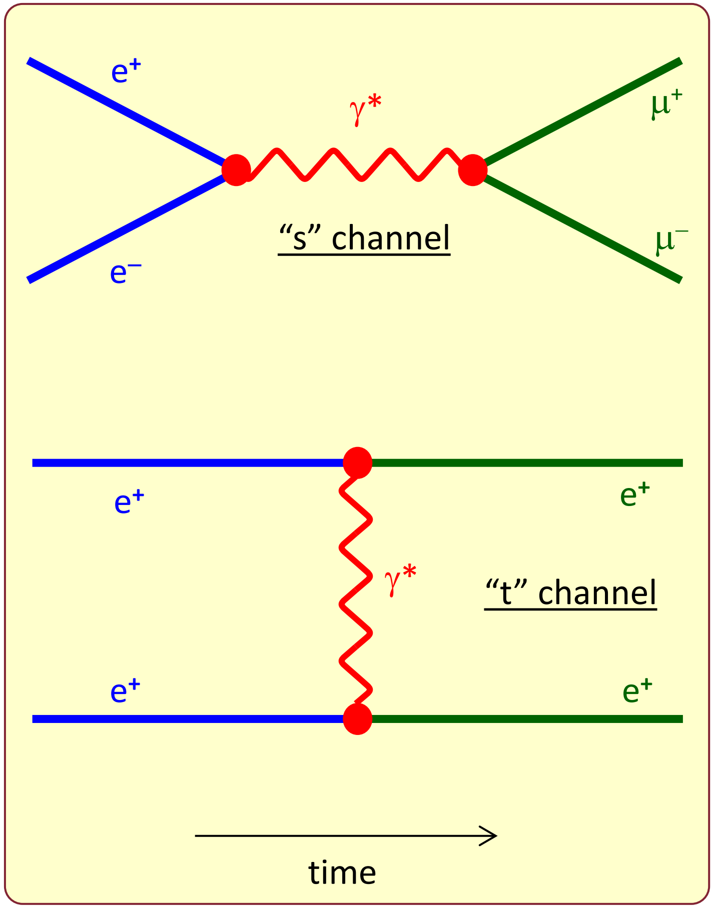
Mandelstam variables

- in a "s-channel" process (e.g. $e^+e^- \to \mu^+\mu^-$), the $|\text{4-momentum}|^2$ of the mediator γ^* is exactly s [i.e. $m(\gamma^*) = \sqrt{s}$, $\sqrt{s} > 0$];

- in a "t-channel" process (e.g. $e^+e^+ \to e^+e^+$), the $|\text{4-momentum}|^2$ of the mediator (γ^* also in this case) is t [$t < 0$!!!] ;

- some processes (e.g. $e^+e^- \to e^+e^-$, called "Bhabha scattering") have more than one Feynman diagrams; some of them are of type s and some others of type t; in such a case we say it is a sum of "s-type diagrams" and "t-type diagrams" + the interference,

... although, *needless to say*, on an event-by-event basis, the observer does **NOT** know whether the event was *s* or *t*.



Mandelstam variables

- in absence of polarization, the cross sections of a process "X" does NOT depend on the azimuth φ :

$$\frac{d\sigma_{"X"}}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma_{"X"}}{d\cos\theta} = \frac{s}{4\pi}\frac{d\sigma_{"X"}}{dt}.$$

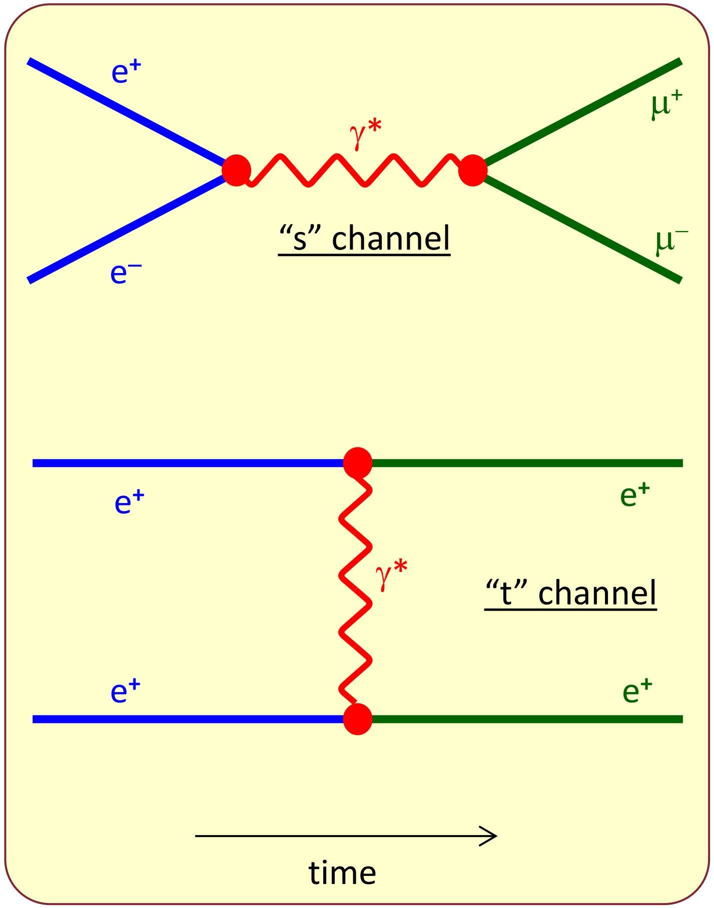
- for $m^2 \ll s$, if $\mathcal{M}_{"X"}$ is the matrix element of the process(*) :

$$\frac{d\sigma_{"X"}}{dt} = \frac{\left|\mathcal{M}_{"X"}\right|^2}{16\pi s^2}.$$

- in lowest order QED, if $m^2 \ll s$:

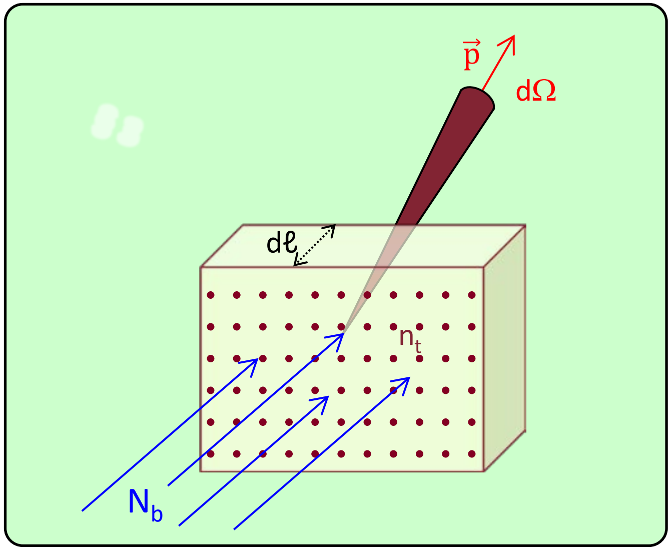
$$\frac{d\sigma_{"X"}}{d\cos\theta} = \frac{\left|\mathcal{M}_{"X"}\right|^2}{32\pi s} = \frac{\alpha^2}{s} f(\cos\theta).$$

- when $\theta \to 0$, $\cos\theta \to 1$:
  - s-channel : $f(\cos\theta) \to$ constant;
  - t-channel : $f(\cos\theta) \to \infty$.



(*) also by dimensional analysis :
$[c = \hbar = 1]$, $[\sigma] = [\ell^2]$; $[t] = [s] = [\ell^{-2}]$;
therefore, in absence of any other dimensional scale,
σ [and $d\sigma/d\Omega$] = [number] × 1/s.

The cross section σ



A beam of N_b particles is sent against a thin layer of thickness $d\ell$, containing dN_t scattering centers in a volume $\mathcal{V}$ ("target", density $n_t = dN_t/d\mathcal{V}$).

The number of scattered particles dN_b is:

$dN_b \propto N_b\, n_t\, d\ell \;\; \Rightarrow \;\; dN_b = N_b\, n_t\, \sigma_T\, d\ell$

the number of particles left after a finite length ℓ is

$N_b(\ell) = N_b(0) \exp(-\, n_t\, \sigma_T\, \ell)$.

The parameter σ_T is the total **cross section** between the particles of the beam and those of the target; it can be interpreted as *the probability of an interaction when a single projectile enters in a region of unit volume containing a single target*.

If many exclusive processes may happen (simplest case : elastic or inelastic), σ_T is the sum of many σ_j, one for each process:

$\sigma_T = \Sigma_j\, \sigma_j$ [e.g. $\sigma_T = \sigma_{elastic} + \sigma_{inelastic}$];

in this case σ_j is proportional to the *probability of process j*.

Common differential $d\sigma/d...$'s:

$$\frac{d\sigma}{d\Omega} = \frac{d^2\sigma}{d\cos\theta\, d\varphi} \xrightarrow{\text{no } \varphi \text{ dependence}} \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta};$$

$$\frac{d\sigma}{d\vec{p}} = \frac{d^3\sigma}{dp_x dp_y dp_z} = \frac{d^3\sigma}{p_T dp_T dp_\ell d\varphi} \rightarrow \frac{1}{\pi}\frac{d^2\sigma}{dp_T^2 dp_\ell};$$

+ others.

The cross section σ: $\sigma_{inclusive}$

In a process ($a\, b \to c\, X$), assume:

- we are only interested in "c" and not in the rest of the final state ["X"];
- "c" can be a single particle (e.g. $W^{\pm}$, Z, Higgs) or a system (e.g. $\pi^+\pi^-$).

Define:

$\sigma_{inclusive}(ab \to cX) = \sum_k \sigma_{exclusive}(ab \to cX_k)$,
where the sum runs on all the **exclusive** processes which in the final state contain "c" + anything else [define also $d\sigma_{inclusive}/d\Omega$ wrt angles of "c", etc.].

The word *inclusive* may be explicit or implicit from the context. E.g., "*the cross-section for Higgs production at LHC*" is obviously $\sigma_{inclusive}(pp \to HX)$.

From the definition, if $\sigma_{inclusive} << \sigma_{total}$:

$\mathcal{P}_c$ = probability of "c" in the final state =
$= \sigma_{inclusive}(ab \to cX) \,/\, \sigma_{total}(ab)$.

Instead, if "c" is common:

$\langle n_c \rangle$ = <number of "c" in the final state> =
$= \sigma_{inclusive}(ab \to cX) \,/\, \sigma_{total}(ab)$.

e.g.

σ_{Higgs}(LHC, 8 TeV) $= \sigma_{incl}(pp \to HX, \sqrt{s}=8 \text{ TeV}) = \approx 22.3$ pb;

$\sigma_{total}(pp, \sqrt{s} = 8 \text{ TeV}) = 101.7 \pm 2.9$ mb;

$\to \mathcal{P}_{Higgs}$(LHC) $\approx 2 \times 10^{-10}$;

[§ LHC]

$\sigma_{incl}(pp \to \pi^0 X, p_{LAB}=24 \text{ GeV}) = 53.5 \pm 3.1$ mb;

$\sigma_{total}(pp, p_{LAB}=24 \text{ GeV}) = 38.9$ mb;

$\to \langle n_{\pi^\circ}(pp, p_{LAB}=24 \text{ GeV})\rangle \approx 1.37$

[V.Blobel et al. - Nucl. Phys., B69 (1974) 454].

Mutatis mutandis, define

- "inclusive width" $\Gamma(A \to BX)$;
- "inclusive BR" $BR(A \to BX)$.

The cross section σ: Fermi 2nd golden rule

- N_b, N_t : particles in beam(b) / target(t);
- $\mathcal{V}$: volume element;
- n_b, n_t : density of particles [= $dN_{b,t}/d\mathcal{V}$];
- v_b : velocity of incident particles;
- ϕ : flux of incident particles [= $n_b\, v_b$];
- p', E' : 4-mom. of scattered particles;
- $\rho(E')$: density of final states;
- $\mathcal{M}_{fi}$: matrix element between i→f state;
- dN/dt : number of events / time [= $\phi N_t \sigma$];
- W : rate of process [= $(dN/dt) / (N_b N_t)$].

Fermi second golden rule

$$\boxed{W = \frac{2\pi}{\hbar}\left|\mathcal{M}_{fi}\right|^2 \rho(E');}$$

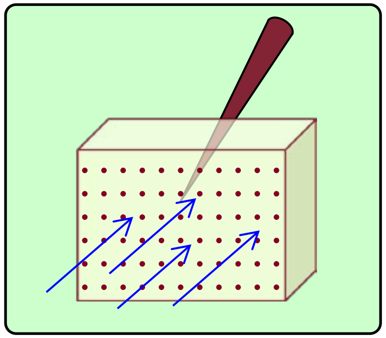
$$\rho(E') = \frac{dn(E')}{dE'} = \frac{\mathcal{V} 4\pi p'^2}{v'(2\pi\hbar)^3};$$

$$W = \frac{dN}{dt}\frac{1}{N_b N_t} = \frac{\phi N_t \sigma}{N_b N_t} = \frac{v_b \sigma}{\mathcal{V}}.$$

$$dn(p') = \frac{\mathcal{V} 4\pi p'^2}{(2\pi\hbar)^3} dp' = \frac{\mathcal{V} 4\pi p'^2}{(2\pi\hbar)^3}\frac{dE'}{v'}$$

$$\boxed{\sigma = \frac{W\mathcal{V}}{v_b} = \frac{2\pi}{\hbar}\left|\mathcal{M}_{fi}\right|^2 \rho(E')\frac{\mathcal{V}}{v_b}}$$

- the rule is THE essential connection (experiment ↔ theory);
- experiments measure event numbers → cross-sections;
- theories predict matrix elements → cross-sections;
- when we check a prediction, we are actually applying the rule;
- properly normalized, the rule is valid also for differential cases (i.e. $d\sigma/dk$, $d\mathcal{M}/dk$, dW/dk), where k is any kinematical variable, e.g. $\cos\theta$].



Excited states : decay pdf

Consider N (N large) unstable particles :

- independent decays;
- decay probability time-independent (e.g. no internal structure, like a timer);

Then :

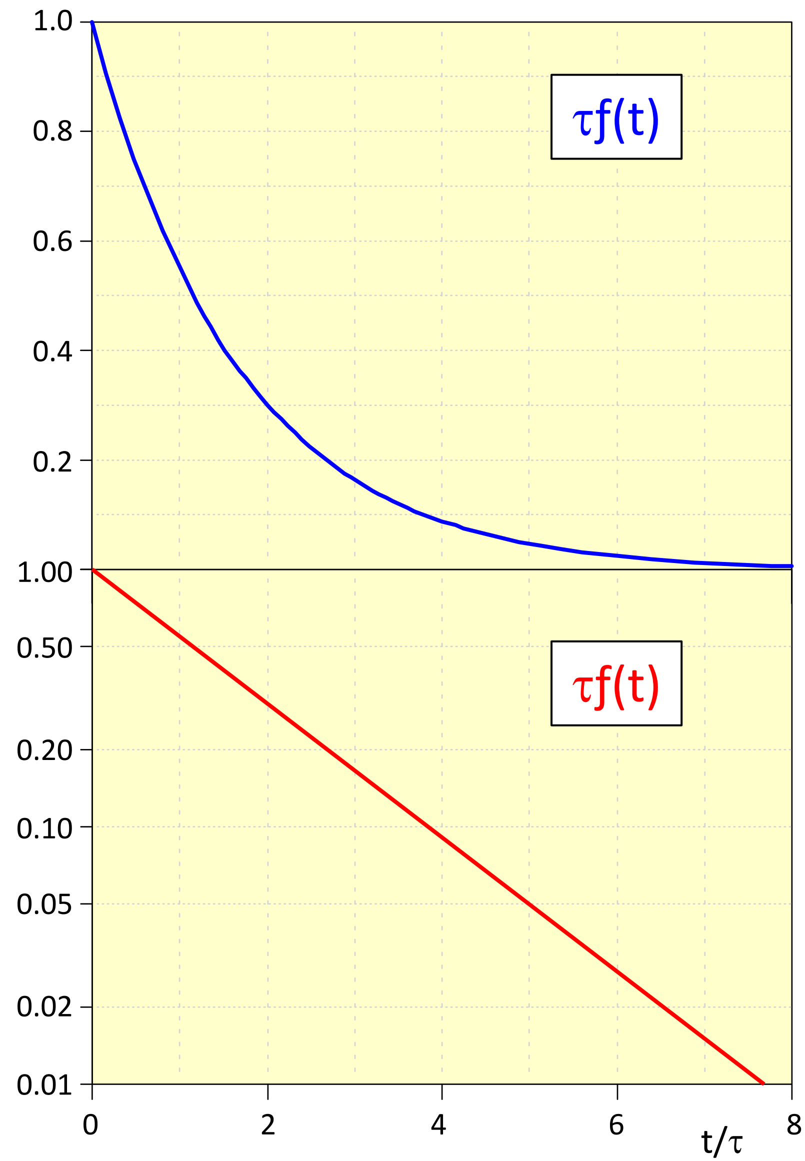
$$dN = -N\Gamma dt; \qquad \Gamma \equiv \frac{1}{\tau} = \text{const.} \qquad \Rightarrow$$

$$N(t) = N_0\, e^{-\Gamma t} = N_0 e^{-t/\tau}.$$

The pdf of the decay for a single particle is

$$\int_0^\infty f(t)dt = 1 \quad \Rightarrow \quad \boxed{f(t) = \frac{1}{\tau} e^{-t/\tau}.}$$

- average decay time : $\left(\sum t_j\right)/n = \langle t \rangle =$
  $= \tau;$
- likelihood estimate of τ, after n decays observed : τ^* $= \langle t \rangle$.



Excited states : Breit-Wigner

If τ is small, the energy at rest (= mass) of a state is not unique (= δ_{Dirac}), but may vary as $\tilde{f}(E)$ around the nominal value $E_0 = m$:

Define $\psi(t<0) = 0$; $\psi(t=0) = \psi_0$;
width Γ [unstable] ;

$$\psi(t) = \psi_0 e^{(-im-\Gamma/2)t};$$

$$|\psi(t)|^2 = |\psi_0|^2 e^{-\Gamma t} = |\psi_0|^2 e^{-t/\tau};$$

$$\tilde{f}(E) = |\tilde{\psi}(E)|^2 = \frac{|\psi_0|^2}{2\pi} \frac{1}{(E-E_0)^2 + \Gamma^2/4}.$$

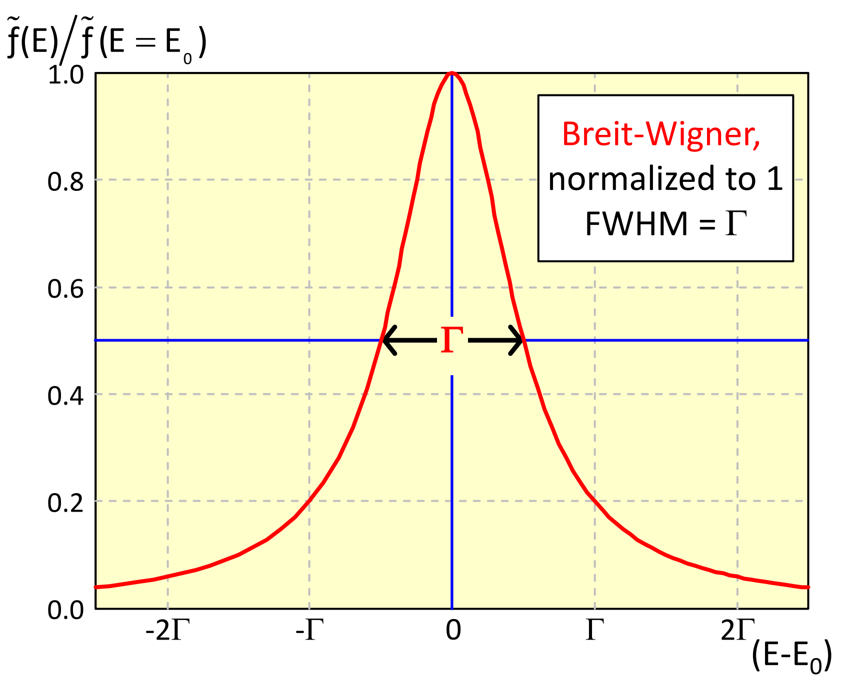
$$\tilde{\psi}(E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iEt} \psi(t) dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{iEt} \psi_0 e^{-i(E_0 - i\Gamma/2)t} dt =$$

$$= \frac{\psi_0}{\sqrt{2\pi}} \frac{-1}{i(E-E_0) - \Gamma/2} = \frac{\psi_0}{\sqrt{2\pi}} \frac{i(E-E_0) + \Gamma/2}{(E-E_0)^2 + \Gamma^2/4}.$$

The curve $(1 + x^2)^{-1}$ is called "Lorentzian" or "Cauchy" in math and "**Breit-Wigner**" in physics; it describes a RESONANCE and appears in many other phenomena:

- forced mechanical oscillations;
- electric circuits;
- accelerators;
- …



Excited states : BW properties

Cauchy (or Lorentz, or BW) distribution :

$$\boxed{f(x) = BW(x \mid x_0, \gamma) = \frac{1}{\pi\gamma} \frac{\gamma^2}{(x - x_0)^2 + \gamma^2};}$$

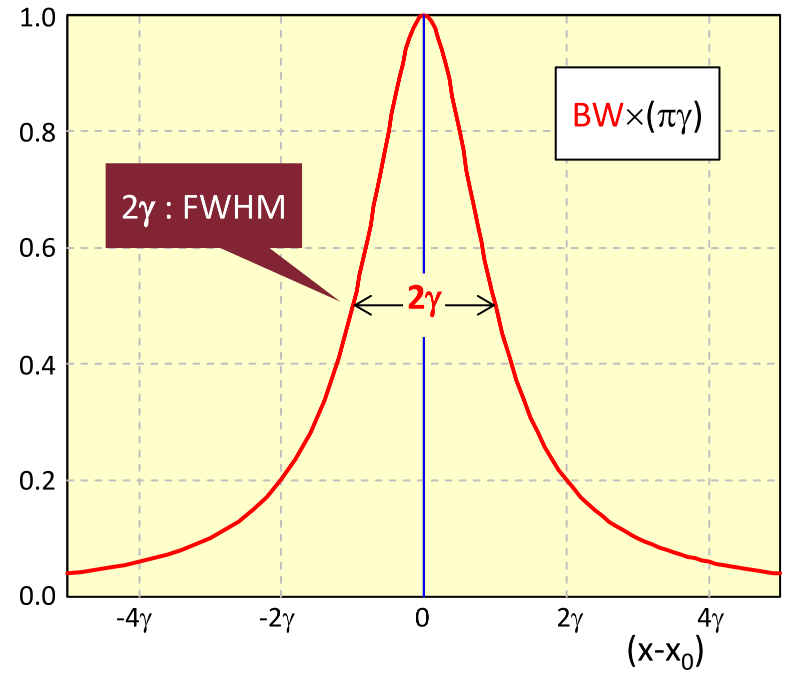
- median = mode = x_0;
- mean = math undefined [but use x_0];
- variance = really undefined [divergent]

This anomaly is due to

$$\langle x \rangle = \int_{-\infty}^{+\infty} x f(x) dx = \infty;$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 f(x) dx = \infty$$

The anomaly does NOT conflict with physics : the BW is an approximation valid only if $\gamma << x_0$ and in the proximity of x_0, e.g. in case of an excited state (mass m, width Γ), for (Γ<<m) and ($|\sqrt{s}$−m$|$ < few Γ's).



The "relativistic BW" is usually defined as

$$\boxed{BW_{rel}(x \mid x_0, \gamma) = \frac{x_0^2 \gamma^2}{(x^2 - x_0^2)^2 + x_0^2 \gamma^2}} \left[\begin{array}{c}\text{properly} \\ \text{normalized}\end{array}\right].$$

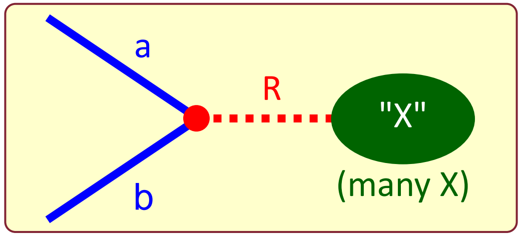
The formula comes from the requirement to be Lorentz invariant [see *Berends et al., CERN 89-08, vol 1*].

Excited states : BW properties

From first principles of QM

$$\sigma_{ab\to R\to X}(E_{CM}=\sqrt{s}) = \frac{\pi}{|\vec{p}_{a,b}|^2}\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\frac{\Gamma_{ab}\Gamma_X}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4} \approx$$

$$\approx \left[\frac{16\pi}{s}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_X}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$



$(E, \vec{p})$: CM 4-mom.

Γ_R : constant width

$\Gamma_{ab,\,X}$: couplings

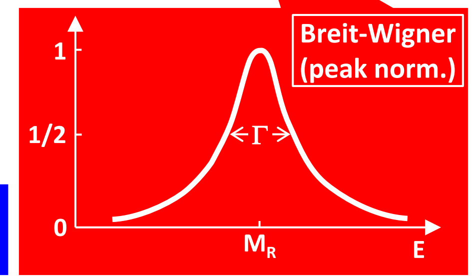
M_R : E_0, mass

scale factor (1/s)

statistical factor (particle spins)

= BR(R→ab)

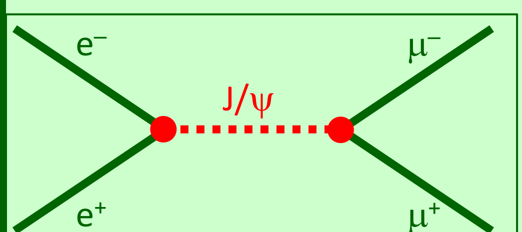
= BR(R→X)



e.g.
$e^+e^- \to J/\psi \to \mu^+\mu^-$

$\sigma_{peak} \propto 1/s$ ($\approx M_R^{-2}$), independent from coupling strength.

$$\sigma(e^+e^-\to J/\psi\to\mu^+\mu^-) = \left[\frac{16\pi}{s}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_{tot}}\right]\left[\frac{\Gamma_{\mu\mu}}{\Gamma_{tot}}\right]\left[\frac{(\Gamma_{tot}/2)^2}{\left(\sqrt{s}-M\right)^2+(\Gamma_{tot}/2)^2}\right] =$$



$$= \frac{12\pi}{s}BR_{J/\psi\to e^+e^-}BR_{J/\psi\to\mu^+\mu^-}\left[\frac{(\Gamma_{tot}/2)^2}{\left(\sqrt{s}-M\right)^2+(\Gamma_{tot}/2)^2}\right].$$

Resonance : different functions

Many more parameterizations used in literature (semi-empirical or *theory inspired*), e.g.:

$$\sigma_0 = \left[\frac{16\pi}{(2p)^2}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

original, non-relativistic

$$\sigma_1 = \left[\frac{16\pi}{s}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

m_a, m_b << p

$$\sigma_2 = \left[\frac{16\pi}{M_R^2}\right]\left[\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\right]\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{final}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

if $M_R >> \Gamma_R$, neglect s-dependence

$$\sigma_3 = \left[\frac{16\pi}{M_Z^2}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_Z}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_Z}\right]\left[\frac{M_Z^2\Gamma_Z^2}{\left(s-M_Z^2\right)^2+M_Z^2\Gamma_Z^2}\right]$$

relativistic BW for $e^+e^- \to Z \to f\bar{f}$

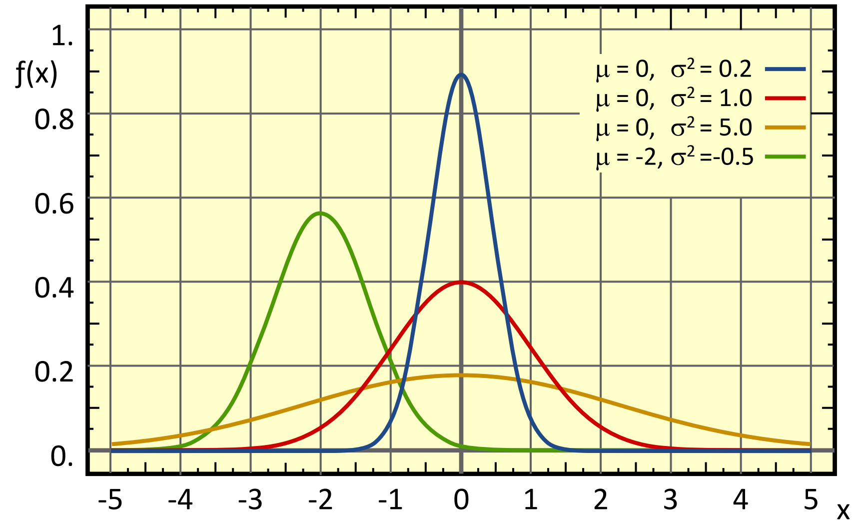
$$\sigma_4 = \left[\frac{16\pi}{M_Z^2}\right]\left[\frac{3}{4}\right]\left[\frac{\Gamma_{ee}}{\Gamma_Z}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_Z}\right]\left[\frac{s\Gamma_Z^2}{\left(s-M_Z^2\right)^2+s^2\Gamma_Z^2/M_Z^2}\right]$$

"s-dependent Γ_Z" (used at LEP for the Z lineshape)

Gauss Distribution

$$f(x) = G(x\,|\,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- mean = median = mode = μ;
- variance = σ^2;
- symmetric : $G(\mu+x) = G(\mu-x)$
- central limit theorem* : the limit of processes arising from multiple random fluctuations is a single $G(x)$;
- similarly, in the large number limit, both the binomial and the Poisson distributions converge to a Gaussian;
- therefore $\boxed{G(x\,|\,\mu=x_{meas}, \sigma=error_{meas})}$ is often used as the resolution function of a given experimental observation [*but as a good (?) first approx. only*].



* Consider n independent random variables $x = \{x_1, x_2, \ldots, x_n\}$, each with mean μ_i and variance σ^2_i; the variable

$$t = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{x_i - \mu_i}{\sigma_i}$$

can be shown to have a distribution that, in the large-n limit, converges to $G(t\,|\,\mu=0,\sigma=1)$.

Gauss Distribution: hypothesis test

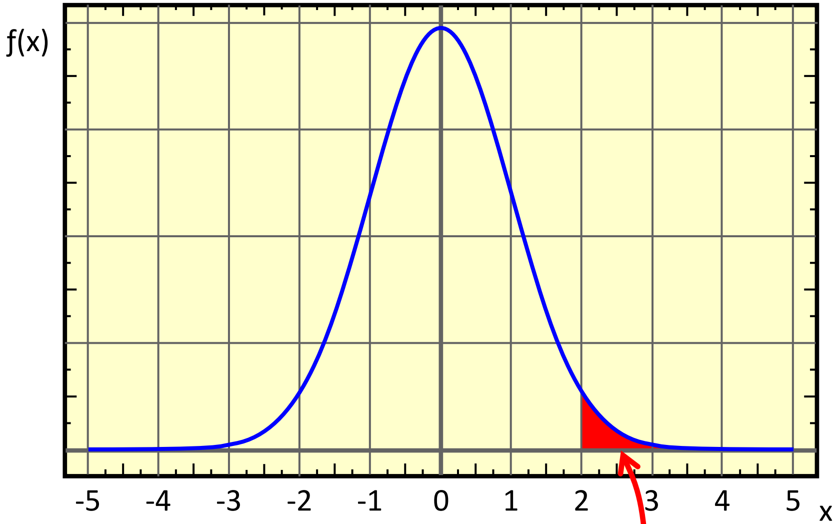
Given a measurement x with an expected value μ and an error σ, the value

$$F(x) = \int_x^{+\infty} G(t\,|\,\mu,\sigma)\,dt$$

is often used as a "hypothesis test" of the expectation.

E.g. (see the ◣ plot): if the observation is at 2σ from the expectation, one speaks of a "2σ fluctuation" (not dramatic, it happens once every 44 trials – or 22 trials if both sides are considered).

The value of "5σ" * has assumed a special value in modern HEP [*see later*].



x	G(x|0,1)	F(x)	=$1/n_{trial}$
0	3.989 E-01	5.000 E-01	2
1	2.420 E-01	1.587 E-01	6.3
2	5.399 E-02	**2.275 E-02**	44.0
3	4.432 E-03	1.350 E-03	741
4	1.338 E-04	3.167 E-05	31,500
5	**1.487 E-06**	**2.867 E-07**	**3.5 E+06**
6	6.076 E-09	9.866 E-10	1.0 E+09
7	9.135 E-12	1.280 E-12	7.8 E+11

---

* if the expectation is not gaussian, one speaks of "5σ" when there is a fluctuation $\leq$ 2.87 E-7 in the tail of the probability, even in the non-gauss case.

Gauss Distribution: the «Voigtian»

Assume :

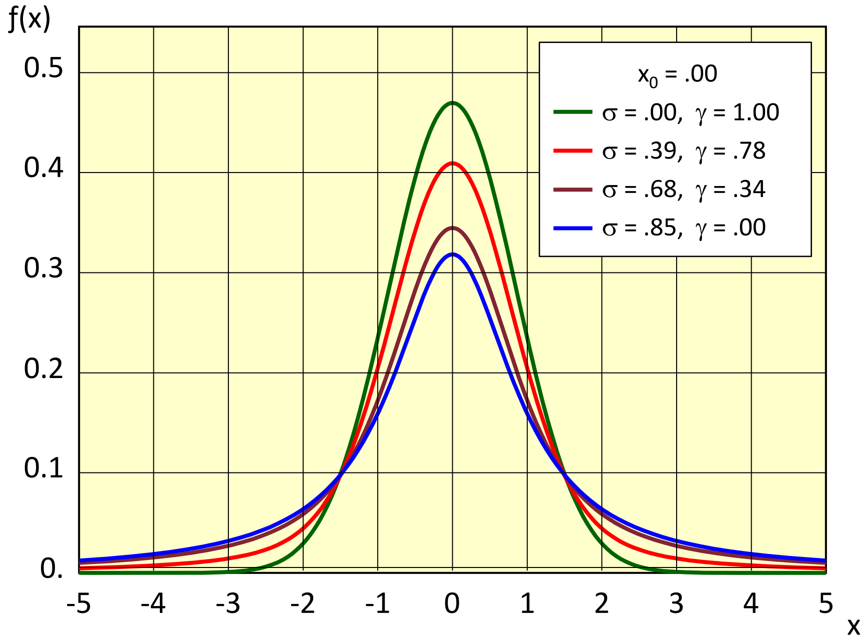
- a physical effect (e.g. a resonance) of intrinsic width described by a BW;
- a detector with a gaussian resolution;

→ the measured shape is a convolution "Voigtian" (after Woldemar Voigt).

- the V. is expressed by an integral and has no analytic form if $\gamma > 0$ AND $\sigma > 0$.
- however modern computers have all the stuff necessary for the numerical computations;
- mean = mathematically undefined [use x_0];
- variance = really undefined [divergent].

→ for real physicists : check carefully if resolution is gaussian, dynamics is BW, and γ and σ are uncorrelated .

$$
\begin{aligned}
f(x) &= V(x\,|\,x_0,\gamma,\sigma) = \\
&= \int_{-\infty}^{+\infty} dt\, G(t\,|\,0,\sigma)\, BW(x-t\,|\,x_0,\gamma) = \\
&= \int_{-\infty}^{+\infty} dt \left[\frac{e^{\left(-\frac{t^2}{2\sigma^2}\right)}}{\sigma\sqrt{2\pi}}\right]\left[\frac{1}{\pi\gamma}\frac{\gamma^2}{(x-t-x_0)^2+\gamma^2}\right].
\end{aligned}
$$



THE STATIC QUARK MODEL

Outline

- Symmetry and Quantum numbers
- Hadrons : elementary or composite ?
- The eightfold way
- The discovery of the Ω^-
- The static quark model
- The mesons
- Meson quantum numbers
- Meson mixing
- The baryons
- SU(3)
- Color

Simmetry and Conservation Law

In Heisenberg representation the time dependence of the operator *Q(t)* is given by:

$$i\hbar\frac{dQ}{dt} = i\hbar\frac{\partial Q}{\partial t} + \left[Q, H\right]$$

An operator with no explicit time dependance is a constant of the motion if it commutes with the hamiltonian operator. In general, conserved quantum numbers are associated to operators commuting with the hamiltonian.

Simmetry and Conservation Law

Example: space translations

$$\psi(r+\delta r) = \psi(r) + \delta r \frac{\partial \psi}{\partial r} = \underbrace{\left(1 + \delta r \frac{\partial}{\partial r}\right)}\psi$$

For a finite translation:

$$D = \left(1 + \frac{ip}{\hbar}\delta r\right)$$

$$D = \lim_{n\to\infty}\left(1 + \frac{ip\Delta r}{n\hbar}\right)^n = e^{\frac{ip\Delta r}{\hbar}}$$

P is the generator of the operator D of space translations.
IF *H* is invariant under transations **[D,H] =0** hence:

[p,H] = 0

The following three statements are equivalent:
- Momentum is conserved for an isolated system.
- The hamiltonian is invariant under space translations.
- The momentum operator commutes with the hamiltonian.

Parity (P)

The operation of spatial inversion of coordinates is produced by the **parity** operator P:

$$P\psi(\vec{r}) = \psi(-\vec{r})$$

Repetition of this operation implies **P^2=1** so that P is a unitary operator.

$$\psi(\vec{r}) \xrightarrow{P} \psi(-\vec{r}) \xrightarrow{P} \psi(\vec{r})$$

Therefore if there are parity eigenvalues they must be: **P = ±1**

Examples:

P = +1 $\quad \psi(x) = \cos x \xrightarrow{P} \cos(-x) = \cos x = \psi(x)$

P = -1 $\quad \psi(x) = \sin x \xrightarrow{P} \sin(-x) = -\sin x = -\psi(x)$

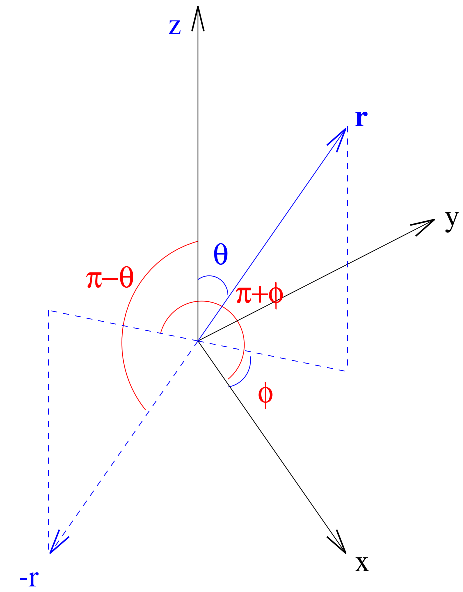
$$\psi(x) = \sin x + \cos x \xrightarrow{P} = -\sin x + \cos x \neq \pm\psi(x)$$

Parity (P): an example

Example: **the hydrogen atom**

$$\psi(r,\theta,\varphi)=\chi(r)\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_m^l(\cos\vartheta)e^{im\varphi}$$

$$\vec{r}\rightarrow-\vec{r}\Leftrightarrow\begin{cases}\theta\rightarrow\pi-\theta\\ \varphi\rightarrow\pi+\varphi\end{cases}$$



$$e^{im\varphi}\rightarrow e^{im(\varphi+\pi)}=(-1)^m e^{im\varphi}$$

$$P_l^m(\cos\theta)\rightarrow(-1)^{l+m}P_l^m(\cos\theta)$$

$$Y_l^m\rightarrow(-1)^l Y_l^m$$

Parity (P): an example

Hence the spherical harmonics have parity ***$P=(-1)^l$*.**

For example, in electric dipole transitions, which obey the selection rule $\Delta l = \pm 1$, the atomic parity changes. Therefore the parity of the emitted radiation must be negative, in in order to conserve the total parity of the system atom+photon.

$$P(\gamma) = -1$$

P is a multiplicative quantum number. It is conserved in strong and elettromagnetic interactions, but it is not conserved in weak interactions.

Parity conservation law requires the assignment of an intrinsic parity to each particle.
Protons and neutrons are conventionally assigned positive parity

$$P_p=P_n=+1$$

Charged pion parity

The pion (π) is a spin 0 meson. Consider the reaction

$$\pi^- + d \rightarrow n + n$$

(where the deuteron *d* is a *pn* bound state).
In the initial state *l=0*; since $s_\pi=0$, $s_d=1$ the total angular momentum must be J=1 (**J=L+S**). Therefore *also in the final state* we must have J=1. The symmetry of the final state wave function (under interchange of the 2 neutrons) is given by:

$$K = \underbrace{(-1)^{S+1}}_{spin}\underbrace{(-1)^{L}}_{orbitale} = (-1)^{L+S+1}$$

Since we have two identical fermions it must be K=-1, which implies L+S is even.
With the condition J=1 we have the following possibilities:

L=0 S=1 no L=1 S=0 no L=2 S=1 no

L=1 S=1 OK

Therefore the parity of the final state is $P=(-1)^L=-1$. Since the parity of the deuteron is $P_d=+1$ we obtain for the π intrinsic parity $P_\pi = -1$.
The π is therefore a pseudoscalar meson.

Parity of the neutral pion (π^0)

$\pi^0 \to \gamma\,\gamma$ B.R. = (99.798 ± 0.032) %

Let **k** and **-k** be the momentum vectors of the two γ; $\mathbf{e}_1$ and $\mathbf{e}_2$ their polarization vectors. The simplest linear combinations one can form which satisfy requirements of exchange symmetry for identical bosons are:

$$\psi_1(2\gamma) = A(\vec{e}_1 \cdot \vec{e}_2) \propto \cos\phi$$

$$\psi_2(2\gamma) = B(\vec{e}_1 \times \vec{e}_2) \cdot \vec{k} \propto \sin\phi$$

ψ_1 is a scalar and therefore even under space inversion, ψ_2 is a pseudoscalar and therefore it has odd parity.

$$P_{\pi^0} = +1 \quad |\psi|^2 \propto \cos^2\phi \qquad\qquad P_{\pi^0} = -1 \quad |\psi|^2 \propto \sin^2\phi$$

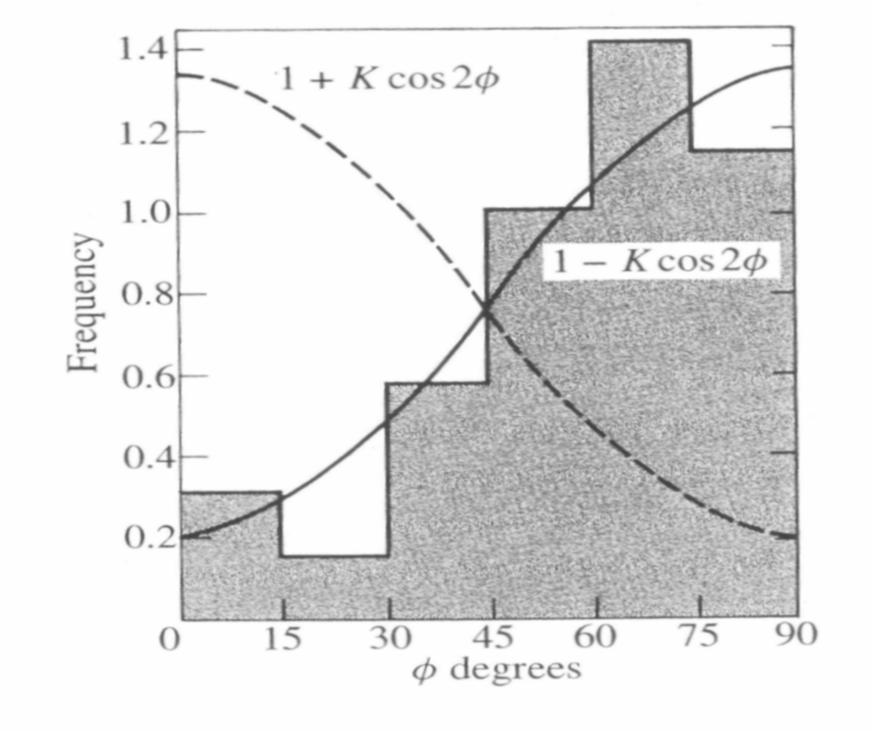
where ϕ is the angle between the polarization planes of the two γ. The experiment was done using the decay:

$$\pi^0 \to e^+ + e^- + e^+ + e^-$$

(double Dalitz; B.R. = $(3.14 \pm 0.30)\times 10^{-5}$) in which each Dalitz pair lies predominantly in the polarization plane of the “internally converting” photon. The result is $P_{\pi 0}=-1$.

Parity of the neutral pion (π^0)

$$\pi^0 \to e^+ + e^- + e^+ + e^-$$



Parity

The assignment of an intrinsic parity is meaningful when particles interact with one another (as in the case of electric charge).

The nucleon intrinsic parity is a matter of convention.

The relative parity of particle and antiparticle is not a matter of convention.

Fermions and antifermions are created in pairs, for instance:

$$p + p \to p + p + p + \overline{p}$$

whereas this is not the case for bosons.

Fermions: particle and antiparticle have **opposite parity.**

Bosons: particle and antiparticle have **equal parity.**

$$\left.\begin{array}{l}\vec{r} \to -\vec{r} \\ \vec{p} \to -\vec{p}\end{array}\right\} \quad polar \quad vectors$$

$$\vec{\sigma} \to \vec{\sigma} \qquad axial \quad vector \quad (\vec{r} \times \vec{p})$$

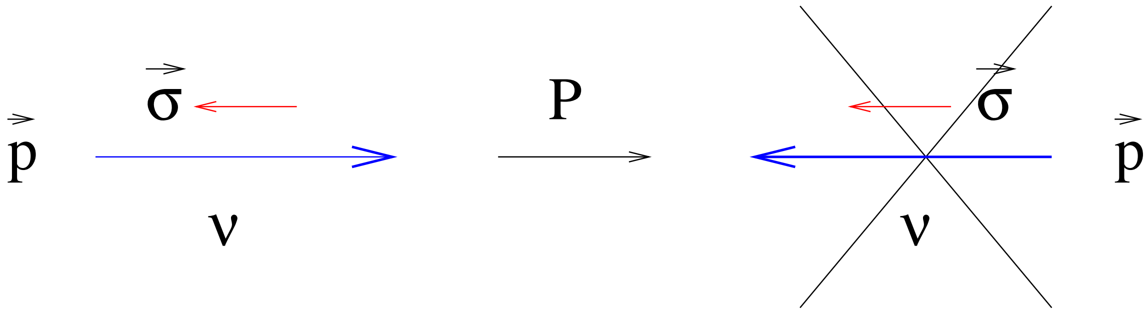
$$\vec{E} \to -\vec{E}$$

$$\vec{B} \to \vec{B}$$

Parity conservation

Parity is conserved in strong and electromagnetic interactions, whereas it is violated in weak interactions. (V-A theory, maximal parity violation)

Example:



In experimental studies of strong and electromagnetic interactions tiny degrees of parity violation are in fact observed, due to contributions from the weak interactions: $H = H_s + H_{em} + H_w$. Atomic transitions:

$$^{16}O^* \longrightarrow {}^{12}C + \alpha$$

$$J^P=2^- \qquad\qquad J^P=2^+$$

with total witdth Γ_α=(1.0±0.3)×10^{-10} eV, to be compared with $^{16}O^* \rightarrow {}^{16}O + \gamma$ of width 3×10^{-3} eV.

Particle and antiparticles

The relativistic relation between the total energy *E*, momentum *p* and rest mass *m* of a particle is:

$$E^2 = p^2c^2 + m^2c^4$$

The total energy can assume negative as well as positive values:

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

In quantum mechanics we represent the amplitude of an infinite stream of particles, e.g. electrons, travelling along the positive *x*-axis with 3-momentum *p* by the plane wavefunction:

$$\psi = Ae^{-i(Et-px)/\hbar}$$

Formally this expression can also represent particles of **energy *-E* and momentum *-p* travelling in the negative *x*-direction and backwards in time**.

Particle and antiparticles

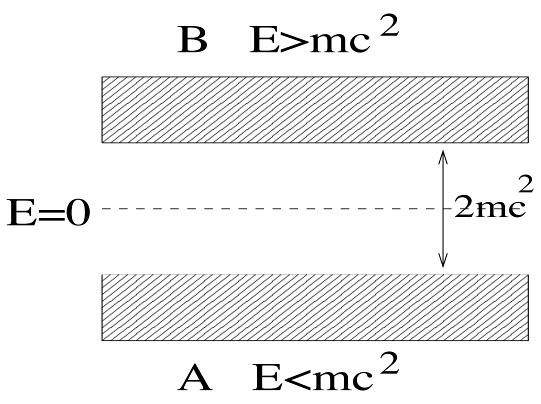
E > 0 $\longrightarrow$ t

E < 0 $\longleftarrow$ t

Such a stream of negative electrons flowing backwards in time is equivalent to positive charges flowing forward, and thus having E>0.

The negative energy particle states are connected with the existence of positive energy antiparticles of exactly equal but opposite electrical charge and magnetic moment, and otherwise identical.

the positron, the antiparticle of the electron, was discovered experimentally in 1932 in cloud chamber experiments with cosmic rays.



Charge Conjugation (c)

Charge conjugation reverses the charge and magnetic moment of a particle.

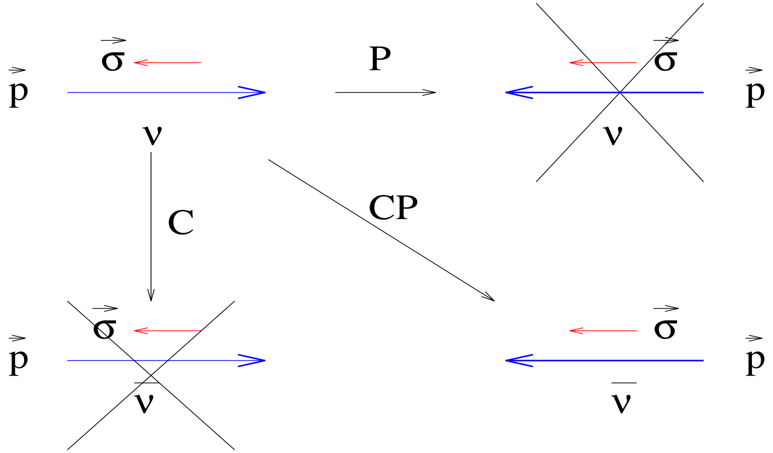
In classical physics Maxwell's equations are invariant under:

$$q \to -q \quad \vec{j} \to -\vec{j} \quad \vec{E} \to -\vec{E} \quad \vec{H} \to -\vec{H}$$

In relativistic quantum mechanics: particle $\leftrightarrow$ antiparticle

$$p \xrightarrow{\quad C \quad} \bar{p}$$

	p	$\bar{p}$
Q	+e	-e
B	+1	-1
μ	+2.79(eℏ/2mc)	-2.79(eℏ/2mc)
σ	½ℏ	½ℏ



Eigenstates of the C Operator

Only neutral bosons which are their own antiparticle can be eigenstates of C.

$C|\pi^+\rangle \rightarrow |\pi^-\rangle \neq \pm|\pi^+\rangle$ π^+ e π^- are not C eigenstates. For the π°:

$$C|\pi^0\rangle = \eta|\pi^0\rangle$$

$$\eta^2 = 1 \quad \Rightarrow \quad C|\pi^0\rangle = \pm|\pi^0\rangle$$

$$\pi^0 \rightarrow \gamma\gamma \quad \Rightarrow \quad C_{\pi^0} = +1$$

Electromagnetic interactions conserve C, therefore the decay

$$\pi^0 \rightarrow 3\gamma$$

should be forbidden. Experimentally we find:

$$\frac{BR(\pi^0 \rightarrow 3\gamma)}{BR(\pi^0 \rightarrow 2\gamma)} < 3.1 \times 10^{-8}$$

Conservation of C

Charge conjugation C is conserved in strong and electromagnetic interactions, but not in weak interaction.

- Spectra of particle and antiparticle, for example:

$$p + \bar{p} \to \pi^+ + \pi^- + \ldots.$$

$$p + \bar{p} \to K^+ + K^- + \ldots.$$

- η meson decay ($J^P = 0^-$, M = 550 MeV/c^2)

$\eta \to \gamma\gamma$	B.R. = (39.21 $\pm$ 0.34) %
$\eta \to \pi^+\pi^-\pi^0$	B.R. = (23.1 $\pm$ 0.5) %
$\eta \to \pi^+\pi^-\gamma$	B.R. = (4.77 $\pm$ 0.13) %
$\eta \to \pi^0 e^+ e^-$	B.R. < 4 $\times$ 10^{-5}

$$B.R. \equiv \frac{\Gamma_{ch}}{\Gamma_{tot}}$$

Since $\eta \to \gamma\gamma$ we must have $C_\eta = +1$. Hence the decay $\eta \to \pi^0 e^+ e^-$ is forbidden by C conservation.

Positronium decay

Positronium is an e^+e^- bound state which possesses energy levels similar to the hydrogen atom (with about half the spacing). Wave function:

$\psi(e^+e^-) = \phi(\text{space})\times\alpha(\text{spin})\times\chi(\text{charge})$

ϕ(space) Particle interchange is equivalent to space inversion introducing a factor $(-1)^L$ where L is orbital angular momentum

α(spin)

$$\begin{cases} \alpha(1,1) = \psi_1(\tfrac{1}{2},\tfrac{1}{2})\psi_2(\tfrac{1}{2},\tfrac{1}{2}) \\ \alpha(1,0) = \frac{1}{\sqrt{2}}\left[\psi_1(\tfrac{1}{2},\tfrac{1}{2})\psi_2(\tfrac{1}{2},-\tfrac{1}{2}) + \psi_1(\tfrac{1}{2},-\tfrac{1}{2})\psi_2(\tfrac{1}{2},\tfrac{1}{2})\right] \\ \alpha(1,-1) = \psi_1(\tfrac{1}{2},-\tfrac{1}{2})\psi_2(\tfrac{1}{2},-\tfrac{1}{2}) \end{cases}$$

Triplet S=1
Symmetric

$$\alpha(0,0) = \frac{1}{\sqrt{2}}\left[\psi_1(\tfrac{1}{2},\tfrac{1}{2})\psi_2(\tfrac{1}{2},-\tfrac{1}{2}) - \psi_1(\tfrac{1}{2},-\tfrac{1}{2})\psi_2(\tfrac{1}{2},\tfrac{1}{2})\right]$$

Singlet S=0
Antisymmetric

<u>The symmetry of α is therefore</u> $(-1)^{S+1}$

Let the charge wave function acquire a factor C.

The total symmetry of the wave function for the interchange of e^+ and e^- is

$$K = (-1)^L(-1)^{S+1}C$$

Positronium decay

Two decays are observed for positronium annihilation from L=0:

$(e^+e^-) \rightarrow 2\gamma$ $(e^+e^-) \rightarrow 3\gamma$

The two-photon decay must have J=0, so the three-photon decays has to be assigned J=1.

	S=J	L	C	K
2γ	0	0	+1	-1
3γ	1	0	-1	-1

(C = $(-1)^n$ for a system consisting of n photons).

In QED the widths of these states can be calculated very accurately:

	Γ	τ (theory)	τ (experiment)
2γ	$\frac{1}{2} mc^2 \alpha^5$	$1.252 \times 10^{-10} s$	$(1.252 \pm 0.017) \times 10^{-10} s$
3γ	$\frac{2}{9\pi}(\pi^2 - 9)\alpha^6 mc^2$	$1.374 \times 10^{-7} s$	$(1.377 \pm 0.004) \times 10^{-7} s$

Charge conservation and gauge invariance

Electric charge is known to be very accurately conserved in all processes.

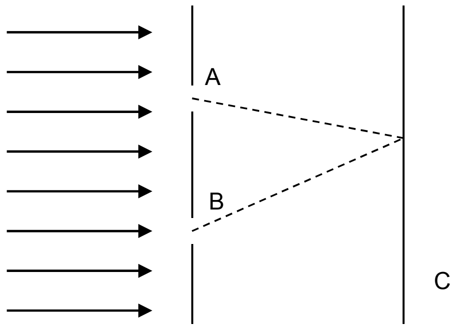
$$\frac{n \to p \nu_e \bar{\nu}_e}{n \to p e^- \bar{\nu}_e} < 9 \times 10^{-24}$$

The conservation of electric charge is related to the gauge invariance of the electromagnetic interaction.

Wigner(1949): Suppose we create a charge Q at a point where the potential is ϕ. Let us now move the charge to a point where the potential is ϕ'. $\Delta W = Q(\phi-\phi')$. Suppose we destroy the charge in this point. If W was the work done to create the charge, this work will be recovered when the charge is destroyed. Therefore we gain a net energy $W-W+\phi-\phi'$ because W does not depend on ϕ.

The conservation of energy implies that we cannot create or destroy charge if the scale of electrostatic potential is arbitrary.

Charge conservation and gauge invariance



Suppose we send a beam of electrons on a screen in which there are two slits *A* and *B* and that we observe the interference on a second screen *C* located at a distance d from the first.

$$\psi = e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\psi = e^{i(\vec{p}\cdot\vec{x}-Et)} = e^{ipx} \qquad p \equiv (E,\vec{p}) \quad x \equiv (t,\vec{x}) \quad \hbar = c = 1$$

Let us redefine ψ by adding a phase $-e\alpha$.

$$\psi = e^{(ipx-e\alpha)}$$

The interference pattern on *C* depends only on phase differences and it is independent on the global phase ***e***α. If however $e\alpha = e\alpha(x)$:

$$\frac{\partial}{\partial x} i(\ px - e\alpha\ ) = i(\ p - e\frac{\partial \alpha}{\partial x}\ )$$

And the result *would seem to depend on the local phase transformation*.

Charge conservation and gauge invariance

Electrons however are charged and they interact via an electromagnetic potential, which we write as a 4-vector *A*:

$$A \equiv (\phi, \vec{A})$$

The effect of the potential is to change the phase of an electron:

$$p \rightarrow p + eA$$

So the derivative now becomes:

$$\frac{\partial}{\partial x} i\big(px + eAx - e\alpha(x)\big) = i\left(p + eA - e\frac{\partial \alpha}{\partial x}\right)$$

The potential scale is also arbitrary and we can change it by adding to *A* the gradient of any scalar function (Gauge transformation):

$$A \rightarrow A + \frac{\partial \alpha}{\partial x}$$

With this transformation the derivative becomes *ip,* independent of $\alpha(x)$.

The effect of the original local phase transformation is cancelled exactly by the gauge transformation.

$$A \rightarrow A + \frac{\partial \alpha(x)}{\partial x}$$

Time Reversal (T)

$$p + {}^{27}Al \rightleftarrows \alpha + {}^{24}Mg$$

$$t \rightarrow -t$$

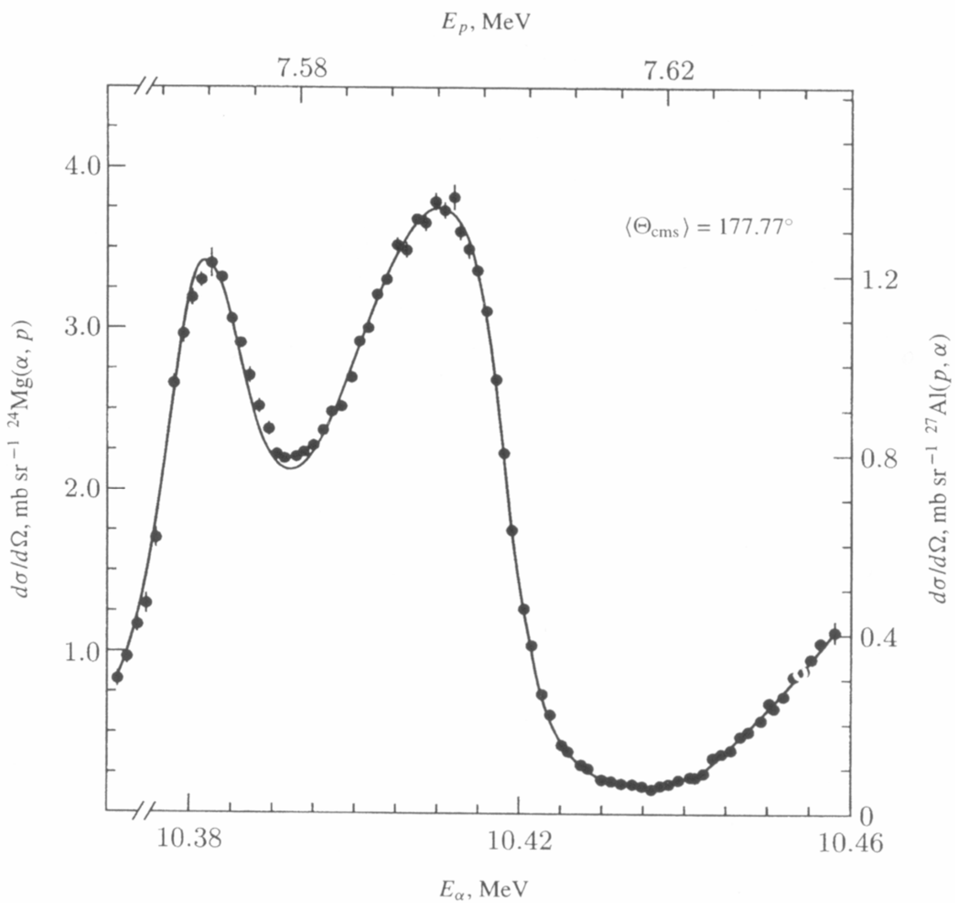
$$\vec{r} \longrightarrow +\vec{r}$$

$$\vec{p} \longrightarrow -\vec{p}$$

$$\vec{\sigma} \longrightarrow -\vec{\sigma}$$

$$\vec{E} \longrightarrow +\vec{E}$$

$$\vec{B} \longrightarrow -\vec{B}$$



T violation < 5×10^{-4}

CPT

CPT theorem:

All interactions are invariant under the succession of the three operation C, P and T taken in any order.

m(particle) = *m*(antiparticle)

$$\frac{m_{K^0} - m_{\overline{K^0}}}{m_{K^0} + m_{\overline{K^0}}} < 10^{-19}$$

mass

τ (particle) = τ (antiparticle)

$$\frac{\tau_{\mu^+} - \tau_{\mu^-}}{\tau_{\mu^+} + \tau_{\mu^-}} < 10^{-4}$$

lifetime

μ (particle)=-μ (antiparticle)

$$\frac{\mu_{e^+} - \mu_{e^-}}{\mu_{e^+} + \mu_{e^-}} < 10^{-12}$$

magnetic moment

CP

In 1964 it was discovered that the long lived K^0_L, which normally decays into three pions (CP = -1), could occasionally decay into two pions (CP=+1). This result represents the discovery of CP violation.

CP violation is at the origin of the asymmetry between matter and antimatter in our universe.

CP violation is equivalent to T violation (via the CPT theorem).

The following observables are sensitive to T violation:

- Transverse polarization $\sigma \cdot (\mathbf{p_1} \times \mathbf{p_2})$ in weak decays such as $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$. Upper limits from these studies $< 10^{-3}$.
- Electric dipole moment $\sigma \cdot \mathbf{E}$. Upper limit for the neutron:

$$\text{EDM}(n) < 1.0 \times 10^{-25}\ e \cdot \text{cm}$$

Spin of the charged pion $\pi^{\pm}$

The spin of the charged pion was determined by applying detailed balance to the reversible reaction:

$$p + p \rightleftarrows \pi^{+} + d$$

$$\sigma_{pp\to\pi^{+}d} = \left|M_{if}\right|^{2} \frac{(2s_{\pi}+1)(2s_{d}+1)}{v_{i}v_{f}} p_{\pi}^{2}$$

$$\sigma_{\pi^{+}d\to pp} = \frac{1}{2}\left|M_{fi}\right|^{2} \frac{(2s_{p}+1)^{2}}{v_{f}v_{i}} p_{p}^{2}$$

(the factor ½ comes from the integration over half solid angle, due to the fact that there are two identical bosons in the final state).

$$\frac{\sigma_{pp\to\pi^{+}d}}{\sigma_{\pi^{+}d\to pp}} = 2\frac{(2s_{\pi}+1)(2s_{d}+1)}{(2s_{p}+1)^{2}} \frac{p_{\pi}^{2}}{p_{p}^{2}}$$

Measuring the cross sections for the direct and reverse reactions one obtains:

$$\boxed{s_{\pi} = 0}$$

Spin of the neutral pion π^0

For the neutral pion the decay $\pi^0 \to \gamma\gamma$ proves that the spin must be integral and that it cannot be one.

For a photon (zero mass, spin 1) $s_z = \pm 1$. Taking the common line of flight of the photons in the π^0 rest frame as the quantization axis, if S is the total spin of the two photons we can have: $S_z = 0$ oppure $S_z = 2$.

If the π^0 spin is 1, then $S_z = 0$. In this case the two-photon amplitude must behave under spatial rotations like the polynomial $P^0{}_1(\cos\theta)$, which is odd under the interchange of the two photons.

But the wave function must be symmetric under the interchange of the two identical bosons, hence the π^0 spin cannot be 1.

In conclusion $s_\pi = 0$ or $s_\pi \geq 2$.

Isospin

$$m_p = 938.27\,MeV \quad m_n = 939.57\,MeV$$

$$m_p \approx m_n$$

Heisenberg (1932):

Proton and neutron considered as different charge substates of one particle, the ***Nucleon***.

A nucleon is ascribed a quantum number, **isospin**, conserved in the strong interaction, not conserved in electromagnetic interactions.

Nucleon is assigned isospin $I = \frac{1}{2}$

$$I_3 = +\tfrac{1}{2} \quad p$$

$$I_3 = -\tfrac{1}{2} \quad n$$

$$\frac{Q}{e} = \frac{1}{2} + I_3$$

Isospin

The nucleon has an internal degree of freedom with two allowed states (the proton and the neutron) which are not distinguished by the nuclear force.

Let us write the nucleon states as $|I, I_3\rangle$

$$|p\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \quad |n\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

For a two-nucleon system we have therefore:

Triplet
(symmetric)

$$\begin{cases} \chi(1,1) = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \\ \chi(1,0) = \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\right) \\ \chi(1,-1) = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \end{cases}$$

Singlet
(antisymmetric)

$$\left\{\chi(0,0) = \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\right)\right.$$

Gell-Mann Nishijima formula

The third component of the isospin distinguish the electrical charge within an isospin multiplet

charge

strangeness

$$Q = I_3 + \frac{1}{2}(B+S)$$

Baryonic number

N.B. $B+S = Y$ (hypercharge)

$$\Rightarrow \quad Q = I_3 + \frac{1}{2}Y$$

The electromagnetic interaction breaks the isospin symmetry; as a consequence the masses within a multiplet are different (m_p different from m_n)

Isospin: deuteron

Example: deuteron (S-wave *pn* bound state)

$$\psi = \phi(spazio) \times \alpha(spin) \times \chi(isospin)$$

$$(-1)^l = +1 \qquad (-1)^{S+1} = +1 \qquad (-1)^{I+1}$$

$$\underbrace{(l = 0) \qquad (S = 1)}$$

$$\boxed{(-1)^{I+1} = -1 \quad \Rightarrow \quad I = 0}$$

ψ is the wave function for *two identical fermions* (two nucleons), hence it must be *globally antisymmetric*. This implies that the deuteron must have **zero isospin**:

$$I_d = 0$$

Isospin: deuteron

Let's see a dynamical consequence of the isospin conservation

- Let's suppose to have two nucleons. From the rule of the addition of angular momentum we know that the total isospin can be 1 or 0.

Symmetric triplet; I = 1

a) $|1,1\rangle = pp$

b) $|1,0\rangle = \frac{1}{\sqrt{2}}(pn+np)$

c) $|1,-1\rangle = nn$

Antisymmetric isosinglet; I = 0

$$|0,0\rangle = \frac{1}{\sqrt{2}}(pn - np)$$

- It exists a bound state proton-neutron (deuteron), but do not exist bound states proton-proton or neutron-neutron, hence the deuteron must be an isospin singlet, otherwise they should exist also the other two states that differ by a rotation in the isospin space.

Isospin

As an example let us consider the two reactions

$$p + p \rightarrow \pi^+ + d$$
$$p + n \rightarrow \pi^0 + d \qquad (I_\pi = 1)$$

Since I_d=0 in each case the final state has isospin 1.

Let us now consider the initial states:

$$pp = |1{,}1\rangle$$
$$np = \tfrac{1}{\sqrt{2}}\left(|1{,}0\rangle - |0{,}0\rangle\right)$$

The cross section

$$\sigma \propto |ampiezza|^2 \approx \sum_I \left|\langle I', I'_3 | A | I, I_3 \rangle\right|^2$$

Isospin conservation implies

$$I = I' = 1 \quad I_3 = I'_3$$

The reaction $np \rightarrow \pi^0 d$ proceeds with probability $\left(\tfrac{1}{\sqrt{2}}\right)^2$ with respect to $pp \rightarrow \pi^+ d$ hence:

$$\frac{\sigma\left(pp \rightarrow \pi^+ d\right)}{\sigma\left(np \rightarrow \pi^0 d\right)} = 2$$

Isospin: nucleon-nucleon scattering

Let's consider the following processes:

a) $p + p \rightarrow d + \pi^+$

b) $p + n \rightarrow d + \pi^0$

c) $n + n \rightarrow d + \pi^-$

the π has isospin 1 because it exists in three different states

- Since the deuteron has I=0, for the right hand processes we have:

$$d + \pi^+ = |1,1\rangle \; ; \; d + \pi^0 = |1,0\rangle \; ; \; d + \pi^- = |1,-1\rangle$$

while for the ones on the left we have:

$$p + p = |1,1\rangle \; ; \; p + n = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right) \; ; \; n + n = |1,-1\rangle$$

- Since the total isospin I must be conserved, only the state with I=1 will contribute. The scattering amplitudes have to be in the ratio:

$$1 : \frac{1}{\sqrt{2}} : 1$$

and the σ $\quad 2 : 1 : 2$

- The processes a) and b) have been measured, and once we take into account the e.m. interaction, they are in the predicted ratio.

Isospin in the πN system

The π meson exists in three charge states of roughly the same mass:

$$m_{\pi^\pm} = 139.57\, MeV$$

$$m_{\pi^0} = 134.98\, MeV$$

Consequently it is assigned $I_\pi=1$, with the charge given by $Q/e=I_3$.

$$\left|\pi^+\right\rangle = \left|1,1\right\rangle \quad \left|\pi^0\right\rangle = \left|1,0\right\rangle \quad \left|\pi^-\right\rangle = \left|1,-1\right\rangle$$

For the π $B=0$:

$$\boxed{\frac{Q}{e} = I_3 + \frac{B}{2}}$$

Isospin in the πN system

For the πN system the total isospin can be either I=1/2 or I=3/2

$$\left.\begin{array}{l} \pi^+ p \rightarrow \pi^+ p \\ \pi^- n \rightarrow \pi^- n \end{array}\right\}$$

pure
I=3/2

$$\left.\begin{array}{l} \pi^- p \rightarrow \pi^- p \\ \pi^- p \rightarrow \pi^0 n \\ \pi^+ n \rightarrow \pi^+ n \\ \pi^+ n \rightarrow \pi^0 p \end{array}\right\}$$

combination of
I=1/2 and I=3/2

		$I=\frac{3}{2}$				$I=\frac{1}{2}$	
$1\times\frac{1}{2}$	I_3	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\pi^+ p$		1					
$\pi^+ n$			$\sqrt{\frac{1}{3}}$			$\sqrt{\frac{2}{3}}$	
$\pi^0 p$			$\sqrt{\frac{2}{3}}$			$-\sqrt{\frac{1}{3}}$	
$\pi^0 n$				$\sqrt{\frac{2}{3}}$			$\sqrt{\frac{1}{3}}$
$\pi^- p$				$\sqrt{\frac{1}{3}}$			$-\sqrt{\frac{2}{3}}$
$\pi^- n$					1		

The coefficients in the linear combinations, i.e. the relative weights of the 1/2 and 3/2 amplitudes, are given by *Clebsch-Gordan coefficients*

$$\left|\pi^+ n\right\rangle = \left|1,1\right\rangle \times \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$
$$= \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|\pi^+ n\right\rangle + \sqrt{\tfrac{2}{3}}\left|\pi^0 p\right\rangle$$
$$= \sqrt{\tfrac{1}{3}}\left|1,1\right\rangle \times \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|1,0\right\rangle \times \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$|\pi^+,p\rangle = |\tfrac{3}{2},+\tfrac{3}{2}\rangle \quad ; \quad |\pi^-,p\rangle = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\pi^-,n\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle \quad ; \quad |\pi^0,n\rangle = \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Isospin in the πN system

$$(1) \quad \pi^+ p \to \pi^+ p$$
$$(2) \quad \pi^- p \to \pi^- p$$

Elastic scattering

$$(3) \quad \pi^- p \to \pi^0 n$$

Charge exchange

$$\sigma \propto \left|\langle f|H|i\rangle\right|^2 = \left|M_{if}\right|^2$$

H = { H_1 if it acts between states of *I=1/2*; H_3 if it acts between states of *I=3/2* }

let $M_1 = \langle I = \tfrac{1}{2}|H_1|I = \tfrac{1}{2}\rangle$

$M_3 = \langle I = \tfrac{3}{2}|H_3|I = \tfrac{3}{2}\rangle$

(1) $\sigma_1 = K|M_3|^2$

$|i\rangle = |f\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$

(2) $\sigma_2 = K|\langle f|(H_1 + H_3)|i\rangle|^2$

$\sigma_2 = K|\tfrac{1}{3}M_3 + \tfrac{2}{3}M_1|^2$

(3) $|i\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$

$|f\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$

$\sigma_3 = K\left|\sqrt{\tfrac{2}{9}}M_3 - \sqrt{\tfrac{2}{9}}M_1\right|^2$

$$\boxed{\sigma_1 : \sigma_2 : \sigma_3 = |M_3|^2 : \tfrac{1}{9}|M_3 + 2M_1|^2 : \tfrac{2}{9}|M_3 - M_1|^2}$$

$M_3 >> M_1 \quad \sigma_1 : \sigma_2 : \sigma_3 = 9 : 1 : 2$

$M_1 >> M_3 \quad \sigma_1 : \sigma_2 : \sigma_3 = 0 : 2 : 1$

Isospin in the πN system

a) $\pi^+ + p \rightarrow \pi^+ + p$

b) $\pi^- + p \rightarrow \pi^0 + n$

c) $\pi^- + p \rightarrow \pi^- + p$

d) $\pi^- + n \rightarrow \pi^- + n$

$$1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$$

We express the various states in the basis of the total isospin using the Clebsch Gordan coefficients

$$|\pi^+,p\rangle = |\tfrac{3}{2},+\tfrac{3}{2}\rangle \quad ; \quad |\pi^-,p\rangle = \sqrt{\frac{1}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$|\pi^-,n\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle \quad ; \quad |\pi^0,n\rangle = \sqrt{\frac{2}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

Isospin in the πN system

- The 4 processes in the new basis

a) $|\frac{3}{2},+\frac{3}{2}\rangle \to |\frac{3}{2},+\frac{3}{2}\rangle$

b) $\sqrt{\frac{1}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle \to \sqrt{\frac{2}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle$

c) $\sqrt{\frac{1}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle \to \sqrt{\frac{1}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle$

d) $|\frac{3}{2},-\frac{3}{2}\rangle \to |\frac{3}{2},-\frac{3}{2}\rangle$

- To estimate the probability amplitude:

$$\langle\frac{3}{2},I_3|S|\frac{3}{2},I_3\rangle = A_{3/2} \;;\; \langle\frac{1}{2},I_3|S|\frac{1}{2},I_3\rangle = A_{1/2}$$

N.B. $A_{1/2} \neq A_{3/2}$

a) $A_{tot}=A_{3/2}$

b) $A_{tot}=\left(\frac{\sqrt{2}}{3}A_{3/2} - \frac{\sqrt{2}}{3}A_{1/2}\right)$

c) $A_{tot}=\left(\frac{1}{3}A_{3/2} + \frac{2}{3}A_{1/2}\right)$

d) $A_{tot}=A_{3/2}$

Scattering nucleon-pion

- The cross sections of the 4 processes will be proportional, by means of a K factor equal for all (takes into account the phase space, factors 2, etc ...), to:

$$\text{a)}\quad \sigma\left(\pi^{+} + \mathrm{p} \rightarrow \pi^{+} + \mathrm{p}\right) = K\left|A_{3/2}\right|^2$$

$$\text{b)}\quad \sigma\left(\pi^{-} + \mathrm{p} \rightarrow \pi^{0} + \mathrm{n}\right) = K\left|\frac{\sqrt{2}}{3}\mathrm{A}_{3/2} - \frac{\sqrt{2}}{3}\mathrm{A}_{1/2}\right|^2$$

$$\text{c)}\quad \sigma\left(\pi^{-} + \mathrm{p} \rightarrow \pi^{-} + \mathrm{p}\right) = K\left|\frac{1}{3}\mathrm{A}_{3/2} + \frac{2}{3}\mathrm{A}_{1/2}\right|^2$$

$$\text{d)}\quad \sigma\left(\pi^{-} + \mathrm{n} \rightarrow \pi^{-} + \mathrm{n}\right) = K\left|A_{3/2}\right|^2$$

- Processes a) and d) must have the same cross section at the same energy. This has been verified experimentally.
- For the other processes it is necessary to know $A_{1/2}$ and the relative phase between the amplitudes.

Formation resonance

$a + b \rightarrow R \rightarrow a + b$ **Elastic Cross-Section**

$a + b \rightarrow R \rightarrow X$ **Total Cross-Section**

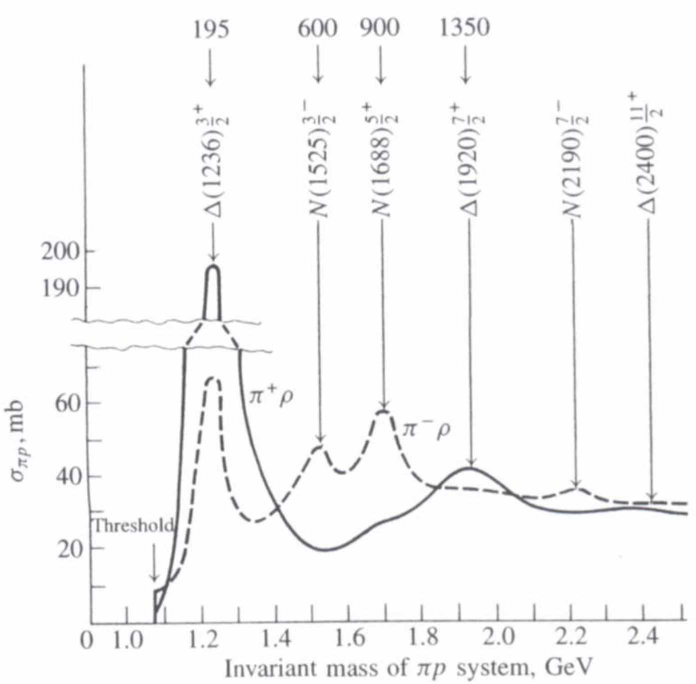
- The scattering process happens through the "formation" of an intermediate resonant state R;
- The resonance can decay in:
  - same particles of the initial state (elastic scattering)
  - other particles (anelastic scattering)
- The resonance is described by the Breit-Wigner formula:

$$\sigma(E) = \frac{4\pi\hbar^2}{p_{cm}^2} \frac{2J+1}{(2S_a+1)\cdot(2S_b+1)} \left[\frac{\Gamma_{in} \cdot \Gamma_{fin}}{(E-M_R)^2 + \Gamma^2/4} \right]$$

- P_{cm}: beam momentum in the center of mass reference frame
- E: center of mass energy ($\sqrt{s}$)
- M_R: resonance mass

- S_a, S_b :initial state spins
- J : resonance spin
- Γ, Γ_{in}, Γ_{fin} : resonance total and partial widths

Δ resonance



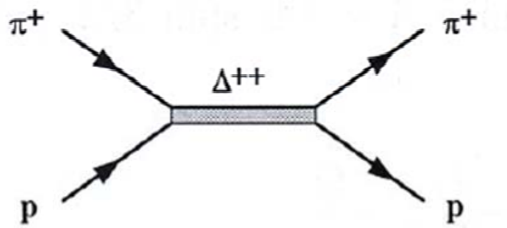
$$\Delta(1236) \quad \Gamma = 120\,MeV$$

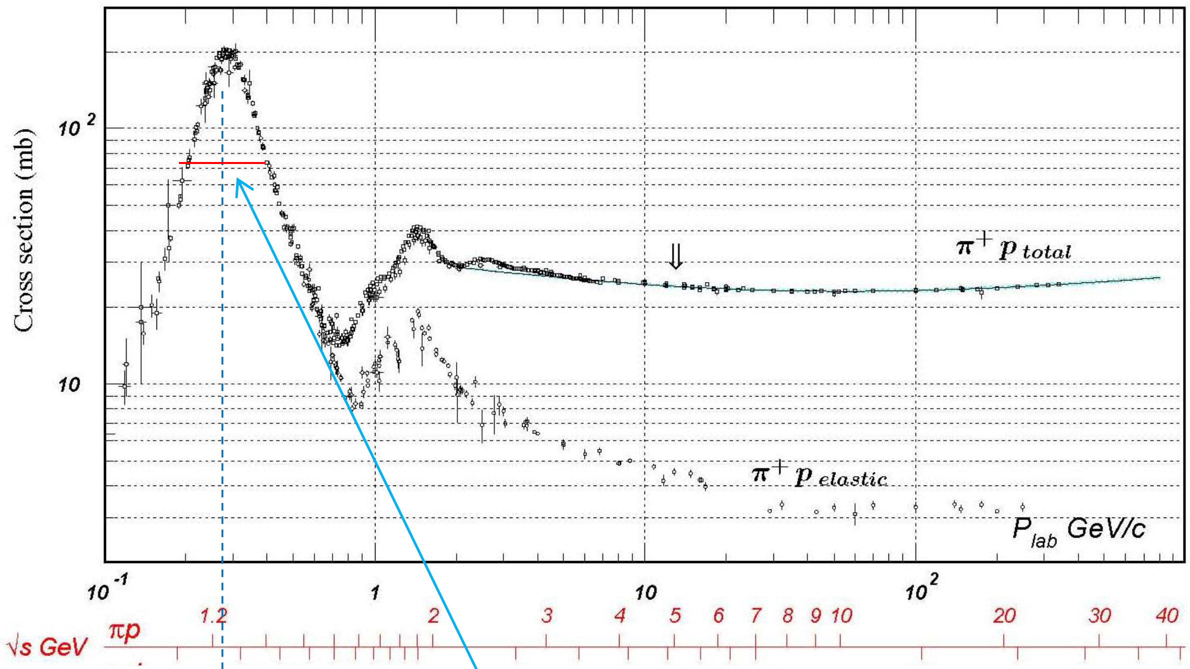
$$J^P = \frac{3}{2}^+ \quad I = \frac{3}{2} \quad (3,3)$$

$$\sigma(E) = \frac{2J+1}{(2s_1+1)(2s_2+1)} \frac{\pi}{k^2} \frac{\Gamma_{ab}\Gamma_{cd}}{(E-M_R)^2 + \frac{\Gamma^2}{4}}$$

a+b → R → c+d

Δ resonance





From the angular distribution of the decay products it turns out that the spin of the Δ is 3/2

- for √s <1.4 GeV the elastic and the total σ coincide.

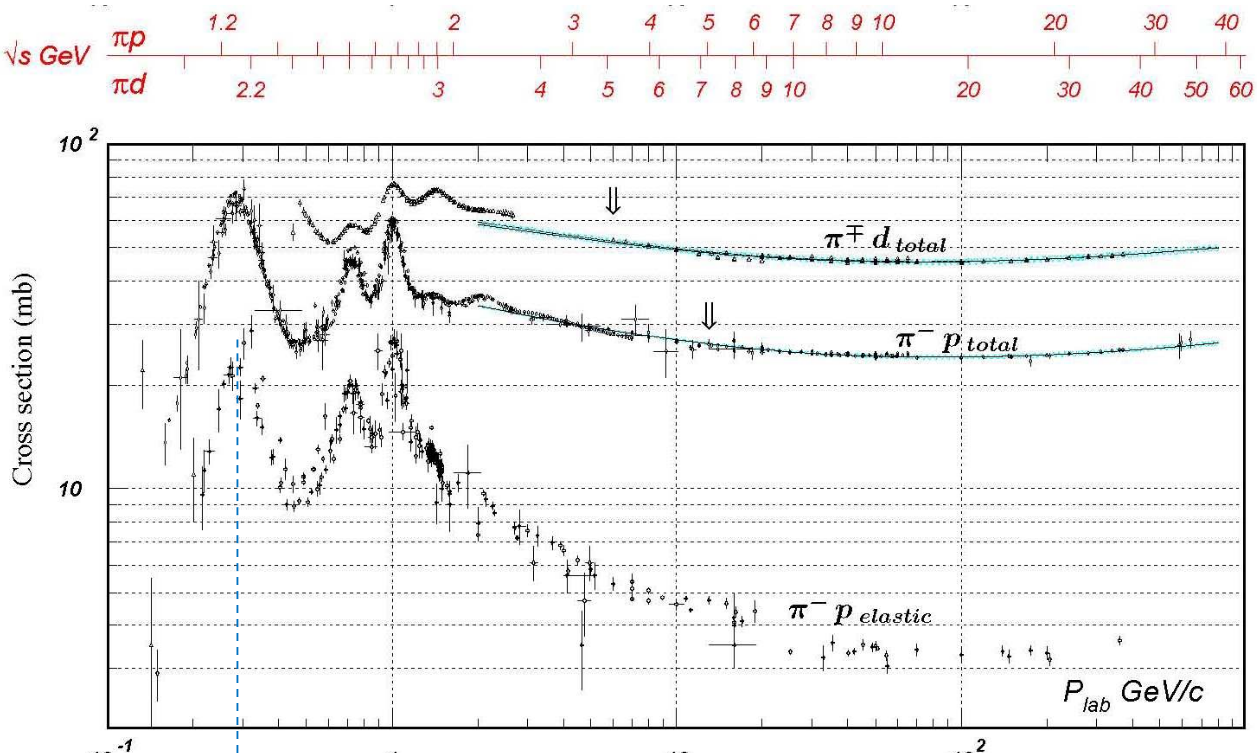
$M = 1232$ MeV

$\Gamma_{tot} = 118$ MeV

$\sigma_{picco} = 195$ mb

$$\tau = \frac{\hbar}{\Gamma_{tot}} = \frac{6.58 \cdot 10^{-16}\ eV \cdot s}{118 \cdot 10^{6}\ eV} = 5.6 \cdot 10^{-24}\ \mathrm{s}$$

Δ resonance



M = 1232 MeV

Same peak position

... Same Γ_{tot}

$\pi^+ p \to \Delta^{++}$

$\pi^+ n \to \Delta^{+}$

$\pi^- p \to \Delta^{0}$

$\pi^- n \to \Delta^{-}$

$\sigma_{picco}\ (\pi^- p \to \pi^- p) = 22$ mb

$\sigma_{picco}\ (\pi^- p \to \pi^0 n) = 45$ mb

Question: why are the elastics σ of $\pi^- p$ and $\pi^+ p$ different? The answer is in the isospin of.

Δ resonance

The resonance Δ has isospin 3/2 (it exists in 4 states of different charge), so the processes in which the Δ appears as resonance of formation can only proceed through the channel with I = 3/2, therefore:

$$\text{a)}\quad \sigma\left(\pi^+ + p \rightarrow \pi^+ + p\right) = K\left|A_{3/2}\right|^2$$

$$\text{b)}\quad \sigma\left(\pi^- + p \rightarrow \pi^0 + n\right) = K\left|\frac{\sqrt{2}}{3}A_{3/2}\right|^2 = \frac{2}{9}\left|A_{3/2}\right|^2$$

$$\text{c)}\quad \sigma\left(\pi^- + p \rightarrow \pi^- + p\right) = K\left|\frac{1}{3}A_{3/2}\right|^2 = \frac{1}{9}\left|A_{3/2}\right|^2$$

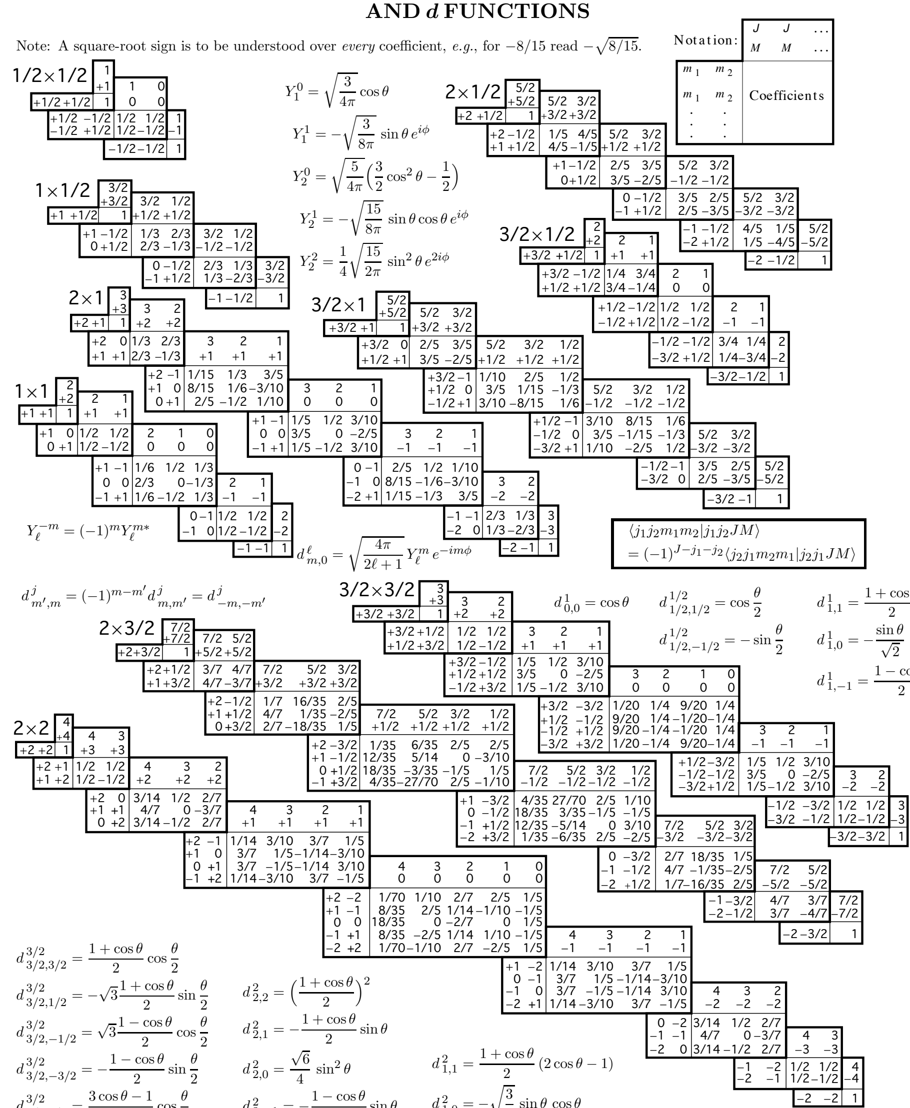
$$\text{d)}\quad \sigma\left(\pi^- + n \rightarrow \pi^- + n\right) = K\left|A_{3/2}\right|^2$$

$$\frac{\sigma\left(\pi^+ + p \rightarrow \pi^+ + p\right)}{\sigma\left(\pi^- + p \rightarrow \pi^- + p\right)} = 9 \quad ; \quad \frac{\sigma\left(\pi^- + p \rightarrow \pi^0 + n\right)}{\sigma\left(\pi^- + p \rightarrow \pi^- + p\right)} = 2$$

Clebsch-Gordan Coefficient

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

Exercises

- We have a system composed of one Σ^- and one proton. Write the wave function of the system in terms of the total isospin states of the system and calculate the probability of finding the system in a total isotope spin state 1/2

  - The Σ^- has I=1 e I_3=-1 (see Gell-mann-Nishijima formula) , while the proton has I=1/2 e I_3 = +1/2 . Combining together the two states we can have has total isospin ½ or 3/2 with the third component I_3 equal to -1/2.

$$|\Sigma^- p\rangle = \sqrt{\frac{1}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle$$

  - The probability to find the system in a state of total isospin ½ is 2/3.

Exercises

- The baryon Λ decays into proton - π^- or into neutron - π^0 In the decay the s quark of the Λ changes into a u quark of the nucleon, so its strong isospin varies by ½. Assuming that in the Λ decay this selection rule is respected and neglecting other corrections, what is the relationship that would be expected between the B.R. in p- π^- compared to that in n- π^0?

Exercises

- The barion Λ decays in proton– π^- or neutron-π^0. In the decay a s-quark of the Λ transforms into a u-quark of the nucleon, therefore its strong isospin changes by ½. Assuming that in the Λ decays this selection rule is retained, and ignoring other small correction, deduce the ratio of the B.R. of the p- π^- decay with respect to the n- π^0 decay.

- The nucleone has isospin ½ while the pion has isospin 1, therefore a nucleon plus a pion can give total isospin equal to ½ or 3/2. The Λ has isospin zero, so in the total wave function of the system nucleon-pion we need to take into account only the component with isospin ½ , due to the selection rule ΔI = ½.

$$p+\pi^- = \left|\frac{1}{2};\frac{1}{2}\right\rangle + \left|1;-1\right\rangle = -\sqrt{\frac{2}{3}}\left|\frac{1}{2};-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{3}{2};-\frac{1}{2}\right\rangle \qquad n+\pi^0 = \left|\frac{1}{2};-\frac{1}{2}\right\rangle + \left|1;0\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2};-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{3}{2};-\frac{1}{2}\right\rangle$$

- The transition probability is equal to the square of the w.f.:

$$\frac{B.R.\left(\Lambda \to p+\pi^-\right)}{B.R.\left(\Lambda \to n+\pi^0\right)} = \frac{\left|\left\langle p+\pi^- \mid \frac{1}{2};-\frac{1}{2}\right\rangle\right|^2}{\left|\left\langle n+\pi^0 \mid \frac{1}{2};-\frac{1}{2}\right\rangle\right|^2} = \frac{2/3}{1/3} = 2$$

- The experimental values are:

$$B.R.\left(\Lambda \to p+\pi^-\right) = 63.9\% \quad ; \; B.R.\left(\Lambda \to n+\pi^0\right) = 35.8\%$$

- $$\frac{B.R.\left(\Lambda \to p+\pi^-\right)}{B.R.\left(\Lambda \to n+\pi^0\right)} = \frac{63.9}{35.8} = 1.78$$

(most likely there is a higher order contribution with ΔI=3/2)

Exercises

- The K^0_S can decay into two charged pions or into two neutral pions. Find the relationship between the B.R. of the decay in neutral pions compared to that in charged pions. Remember that for symmetry reasons the final state must have zero total isospin

- Deduce through which isospin channels the following two reactions may occur:

$$\text{a) } K^- + p \rightarrow \Sigma^0 + \pi^0 \quad ; \quad \text{b) } K^- + p \rightarrow \Sigma^+ + \pi^-$$

- In the event that the dominant channel is the one with isospin 0 for both reactions, find the ratio between the cross sections σ_a / σ_b

Exercises

Il K^0_S può decadere in due pioni carichi oppure in due pioni neutri. Trovare il rapporto tra il B.R. del decadimento in pioni neutri rispetto a quello in pioni carichi. Si ricorda che per ragioni di simmetria lo stato finale deve avere isospin totale zero

Nei decadimento deboli con $\Delta S=1$ si ha $\Delta I=1/2$, quindi dato che il K ha $I=1/2$, lo stato finale dei due pioni deve avere $I=0$ oppure $I=1$. La funzione d'onda dei due pioni deve essere simmetrica rispetto allo scambio delle due particelle, quindi dato che essi hanno spin zero e si trovano in uno stato di momento angolare $l=0$, anche la parte di isospin deve essere simmetrica, quindi $I=0$.

Utilizzando i coefficienti di Clebsh-Gordan si ha:

$$|0;0\rangle = +\sqrt{\frac{1}{3}}|1,+1;1-1\rangle - \sqrt{\frac{1}{3}}|1,0;1,0\rangle + \sqrt{\frac{1}{3}}|1,-1;1+1\rangle = \quad = +\sqrt{\frac{1}{3}}\pi^+\pi^- - \sqrt{\frac{1}{3}}\pi^0\pi^0 + \sqrt{\frac{1}{3}}\pi^-\pi^+$$

Di conseguenza abbiamo:

$$\frac{B.R.\left(K_S^0 \to \pi^0 + \pi^0\right)}{B.R.\left(K_S^0 \to \pi^+ + \pi^-\right)} = \frac{\left|\langle \pi^0\pi^0 \mid 0;0\rangle\right|^2}{\left|\langle \pi^+\pi^- \mid 0;0\rangle\right|^2} = \frac{1}{2}$$

I valori sperimentali sono:

$$B.R.\left(K_S^0 \to \pi^0 + \pi^0\right) = 30.7\% \;\; ; \; B.R.\left(K_S^0 \to \pi^+ + \pi^-\right) = 69.2\%$$

$$\frac{B.R.\left(K_S^0 \to \pi^0 + \pi^0\right)}{B.R.\left(K_S^0 \to \pi^+ + \pi^-\right)} = \frac{30.7}{69.2} = 0.44$$

Probabilmente vi è un contributo di ordine superiore con $\Delta I=3/2$

Exercises

Dedurre attraverso quali canali di isospin possono avvenire le seguenti due reazioni:

$$a)\ K^- + p \to \Sigma^0 + \pi^0 \quad ; \quad b)\ K^- + p \to \Sigma^+ + \pi^-$$

Nel caso in cui il canale dominante sia quello con isospin 0 per entrambe le reazioni, trovare il rapporto tra le sezioni d'urto σ_a/σ_b

Ricordiamo l'isospin totale e la terza componente delle particelle coinvolte nella reazione e scriviamo lo stato iniziale ed i due stati finali in termini degli autostati di isospin utilizzando i coefficienti di Clebsh-Gordan.

$$K^- = \left|I = \tfrac{1}{2}; I_3 = -\tfrac{1}{2}\right\rangle \; ; \; p = \left|I = \tfrac{1}{2}; I_3 = \tfrac{1}{2}\right\rangle \Rightarrow K^- + p = +\sqrt{\frac{1}{2}}\left|1;0\right\rangle - \sqrt{\frac{1}{2}}\left|0;0\right\rangle$$

$$\Sigma^0 = \left|I = 1; I_3 = 0\right\rangle \; ; \; \pi^0 = \left|I = 1; I_3 = 0\right\rangle \Rightarrow \Sigma^0 + \pi^0 = +\sqrt{\frac{2}{3}}\left|2;0\right\rangle - \sqrt{\frac{1}{3}}\left|0;0\right\rangle$$

$$\Sigma^+ = \left|I = 1; I_3 = 1\right\rangle \; ; \; \pi^- = \left|I = 1; I_3 = -1\right\rangle \Rightarrow \Sigma^+ + \pi^- = +\sqrt{\frac{1}{6}}\left|2;0\right\rangle + \sqrt{\frac{1}{2}}\left|1;0\right\rangle + \sqrt{\frac{1}{3}}\left|0;0\right\rangle$$

Di conseguenza la reazione a) può avvenire soltanto attraverso il canale di isospin totale 0, mentre la reazione b) può avvenire attraverso il canale con isospin 0 ed anche con isospin 1.

Nel caso in cui il canale dominante sia quello con isospin 0 per entrambe le reazioni, allora il rapporto tra le sezioni d'urto è pari al rapporto dei quadrati dei coefficienti di C.G. dell'autostato di isospin 0 nei due stati finali:

$$\frac{\sigma_a}{\sigma_b} = \left|\frac{\left\langle \Sigma^0 + \pi^0 \middle| 0;0 \right\rangle}{\left\langle \Sigma^+ + \pi^- \middle| 0;0 \right\rangle}\right|^2 = \left|\frac{-\sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}}}\right|^2 = 1$$

Exercises

Verify the conserved quantities in the reaction $\pi^{-}+p \rightarrow \pi^{0}+n$. Is the process allowed?

Does the reaction $d+d \rightarrow {}^{4}\mathrm{He}+\pi^{0}$ conserve isospin?

Compute the isospin balance for $\Sigma^{0} \rightarrow \Lambda+\gamma$.

Strangeness S

Strange particles are copiously produced in strong interactions

They have a long lifetime, typical of a weak decay.

S quantum number: strangeness *conserved in strong and electromagnetic interactions*, *not conserved in weak interactions*.

Example: $\pi^- + p \rightarrow \Lambda + K^0$

$$\Lambda \rightarrow p + \pi^- \qquad \tau = 2.6 \times 10^{-10}\, s$$

$$\begin{array}{lcccc} & \Lambda & \rightarrow & p & + \pi^- \\ I & 0 & & \tfrac{1}{2} & 1 \\ I_3 & 0 & & \tfrac{1}{2} & -1 \end{array}$$

I = 0, because the Λ has no charged counterparts

$$\begin{array}{lcccc} & \pi^- & + p & \rightarrow \Lambda & + K^0 \\ I & 1 & \tfrac{1}{2} & 0 & \tfrac{1}{2} \\ I_3 & -1 & \tfrac{1}{2} & 0 & -\tfrac{1}{2} \end{array}$$

Strangeness S

$$\left.\begin{aligned} K^0, K^+ \quad \frac{Q}{e} &= I_3 + \frac{1}{2} \\ \bar{K}^0, K^- \quad \frac{Q}{e} &= I_3 - \frac{1}{2} \end{aligned}\right\} \qquad \frac{Q}{e} = I_3 + \frac{B+S}{2}$$

(Gell-Mann Nishijima)

Y = B+S hypercharge

Using the Gell-Mann Nishijima formula strangeness is assigned together with isospin.

Example.:

$$\begin{array}{lll} n, p & S = 0 & I = \frac{1}{2} \\ \Lambda & S = -1 & I = 0 \\ K^0, K^+ & S = 1 & I = \frac{1}{2} \\ K^-, \bar{K}^0 & S = -1 & I = \frac{1}{2} \end{array}$$

Example of strangeness conservation:

$$\begin{array}{lcccc} & K^- & +\, p \rightarrow & \Lambda & +\, \pi^0 \\ S & -1 & 0 & -1 & 0 \\ I_3 & -\frac{1}{2} & +\frac{1}{2} & 0 & 0 \end{array}$$

$$\begin{array}{lcccc} & \pi^\pm & +\, p \rightarrow & \Sigma^\pm & +\, K^+ \\ S & 0 & 0 & -1 & +1 \\ I & 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ I_3 & \pm 1 & \frac{1}{2} & \pm 1 & \frac{1}{2} \end{array}$$

$$\begin{array}{lccc} & \Sigma^0 \rightarrow & \Lambda & +\, \gamma \quad e.m. \\ S & -1 & -1 & 0 \end{array}$$

$$\begin{array}{lccc} & \Sigma^+ \rightarrow & n & +\, \pi^+ \quad weak \\ S & -1 & 0 & 0 \end{array}$$

$$\begin{array}{lccc} & \Xi^- \rightarrow & \Lambda & +\, \pi^- \quad weak \\ S & -2 & -1 & 0 \end{array}$$

G-parity G

$$G = Ce^{i\pi I_2}$$

Rotation of π around the 2 axis in isospin space followed by charge conjugation.

$$I_3 \xrightarrow{e^{i\pi I_2}} -I_3 \xrightarrow{C} I_3$$

Consider an isospin state $\chi(I,I_3=0)$: under isospin rotations this state behaves like $Y^0_l(\theta,\varphi)$ (under rotations in ordinary space)

The rotation around the 2 axis implies:

$$\vartheta \to \pi - \vartheta \quad \varphi \to \pi - \varphi$$

$$Y_l^0 \to (-1)^l Y_l^0$$

therefore

$$\chi(I,0) \to (-1)^I \chi(I,0)$$

G-parity G

Example: for a nucleon-antinucleon state the effect of C is to give a factor $(-1)^{l+s}$ (just is in the case of positronium). Therefore:

$$G\left|\psi(N\overline{N})\right\rangle = (-1)^{l+s+I}\left|\psi(N\overline{N})\right\rangle$$

This formula has **general validity**, not limited to the $I_3=0$ case.

For the π

$$G\left|\pi^+\right\rangle = \pm\left|\pi^+\right\rangle$$

$$G\left|\pi^-\right\rangle = \pm\left|\pi^-\right\rangle$$

$$G\left|\pi^0\right\rangle = \pm\left|\pi^0\right\rangle$$

For the π^0 C=+1 ($\pi^0 \to \gamma\gamma$), the rotation gives $(-1)^I=-1$ (I=1) so that G= -1.

$$G_{\pi^0} = -1$$

It is the practice to assign the phases so that *all members of an isospin triplet have the same G-parity as the neutral member*.

$$G\left|\pi^\pm\right\rangle = -\left|\pi^\mp\right\rangle \qquad \text{with} \quad C\left|\pi^\pm\right\rangle = -\left|\pi^\mp\right\rangle$$

G-parity G

Since the C operation reverses the sign of the baryon number B, **the eigenstates of G-parity must have baryon number zero B=0**.

G is a multiplicative quantum number, so for a system of n π

$$G=(-1)^n$$

$$\rho \to \pi\pi \qquad G_\rho = +1$$

$$\omega \to \pi\pi\pi \qquad G_\omega = -1 \qquad B.R. = 89\%$$

$$\omega \to \pi\pi \qquad G_f = +1 \qquad B.R. = 2.2\%$$

$\eta \to \gamma\gamma$ C=+1 which, with I=0, yields G=+1.

$$\eta \not\to \pi\pi \qquad viola\ P$$

$$\eta \to \pi\pi\pi \qquad viola\ G \Rightarrow e.m.$$

Conservation Laws

	Strong	E.M.	Weak
Energy/Momentum	✓	✓	✓
Electric Charge	✓	✓	✓
Baryon Number	✓	✓	✓
Lepton Number	✓	✓	✓
Isospin (I)	✓	✘	✘
Strangeness (S)	✓	✓	✘
Charm (C)	✓	✓	✘
Parity (P)	✓	✓	✘
Charge Conjugation (C)	✓	✓	✘
CP (or T)	✓	✓	✘
CPT	✓	✓	✓

Exercises

For each of the following reactions (a) establish whether it is allowed
(b) if it is not, give the reasons (there may be more than one), (c) establish the types of interaction that allow it: (1) $\pi^{-}p\rightarrow\pi^{0}+n$; (2) $\pi^{+}\rightarrow\mu^{+}+\nu_{\mu}$;(3) $\pi^{+}\rightarrow\mu^{+}+\nu^{-}{}_{\mu}$; (4) $\pi^{0}\rightarrow2\gamma$; (5) $\pi^{0}\rightarrow3\gamma$; (6) $e^{+}+e^{-}\rightarrow\gamma$; (7) $p+{}^{-}p\rightarrow\Lambda+\Lambda$; (8) $p+p\rightarrow\Sigma^{+}+\pi^{+}$; (9) $n\rightarrow p+e^{-}$; (10) $n\rightarrow p+\pi^{-}$.

(1) OK, S; (2) OK, W; (3) Violates $\mathcal{L}_{\mu}$; (4) OK, EM; (5) Violates $\mathcal{C}$; (6) Cannot conserve both energy and momentum; (7) Violates $\mathcal{B}$ and S (8) Violates $\mathcal{B}$ and S; (9) Violates J and Le; (10) Violates energy conservation.

THE STATIC QUARK MODEL

Outline

- Symmetry and Quantum numbers
- Hadrons : elementary or composite ?
- The eightfold way
- The discovery of the Ω^-
- The static quark model
- The mesons
- Meson quantum numbers
- Meson mixing
- The baryons
- SU(3)
- Color

Particle classification

- In the 1950s, new particles and resonances were discovered which were themselves regarded as new particles.
- An attempt was made to classify all these particles in such a way as to reveal their true nature (a similar work was done by Rydberg who found the formula to describe atomic spectra, or by Mendeleiev)
- A first symmetry found was associated with isotopic spin; particles with the same isospin are exactly the same particle for strong interactions, but e.m. interactions break the symmetry and cause a mass difference of a few% between the particles of the same multiplet.

Hadrons : "elementary" or composite?

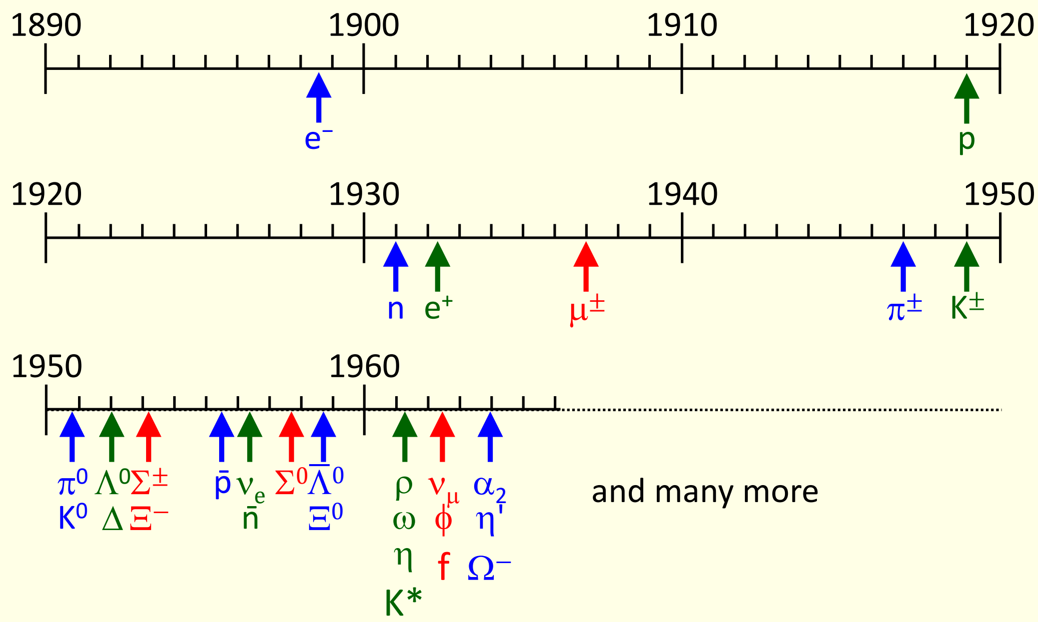
Over time the very notion of "*elementary* (???) particle" entered a deep crisis.

The existence of (too) many hadrons was seen as a contradiction with the elementary nature of the fundamental component of matter.

It was natural to interpret the hadrons as consecutive resonances of elementary components.

The main problem was then to measure the properties of the components and possibly to observe them.

[... and the leptons ? ...]

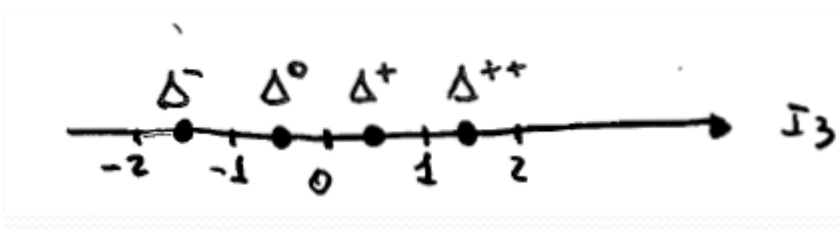


too many hadronic states: resonances ?

the figure shows the particle discoveries from 1898 to the '60s; their abundance and regularity, as a function of quantum numbers like charge and strangeness, were suggesting a possible sequence, similar to the Mendeleev table.

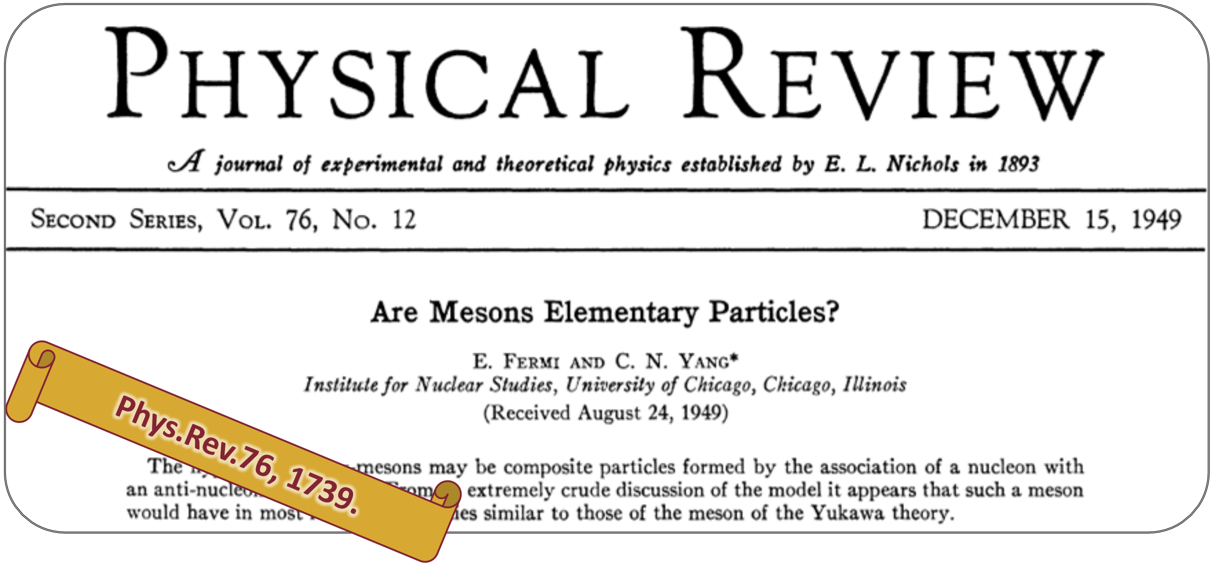
Particle Classification

- To extend the symmetry, an attempt was made to group several isospin multiplets into a larger group that had the same spin and parity but with different strangeness (or hypercharge).
- There are other possible a priori choices, such as same oddity but different spin and parity, but these don't work.
- The components of the isospin multiplets are represented as spaced points of a unit on the horizontal axis I_3. For example for the Δ (1232) we have:



$$\boxed{Q = I_3 + \frac{1}{2}(B+S)}$$

Hadrons : "elementary" or composite?


PHYSICAL REVIEW
A journal of experimental and theoretical physics established by E. L. Nichols in 1893
SECOND SERIES, VOL. 76, NO. 12 DECEMBER 15, 1949

Are Mesons Elementary Particles?

E. FERMI AND C. N. YANG*
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois
(Received August 24, 1949)

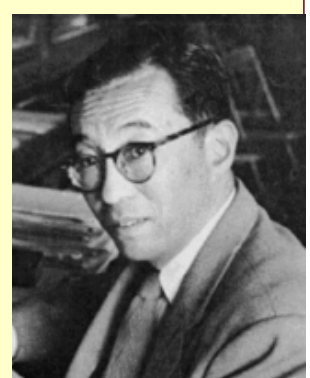
The ... mesons may be composite particles formed by the association of a nucleon with an anti-nucleo... From ... extremely crude discussion of the model it appears that such a meson would have in most ...ies similar to those of the meson of the Yukawa theory.

1949 : E.Fermi and C.N. Yang proposed that ALL the resonances were bound state p-n.

Enrico Fermi

Chen-Ning Yang (杨振宁 - 楊振寧, *Yáng Zhènníng*)

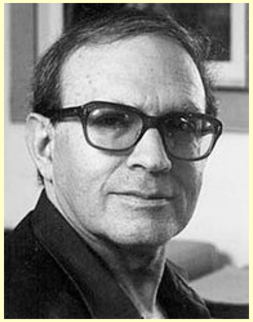
Shoiki Sakata (坂田 昌一, *Sakata Shōichi*)



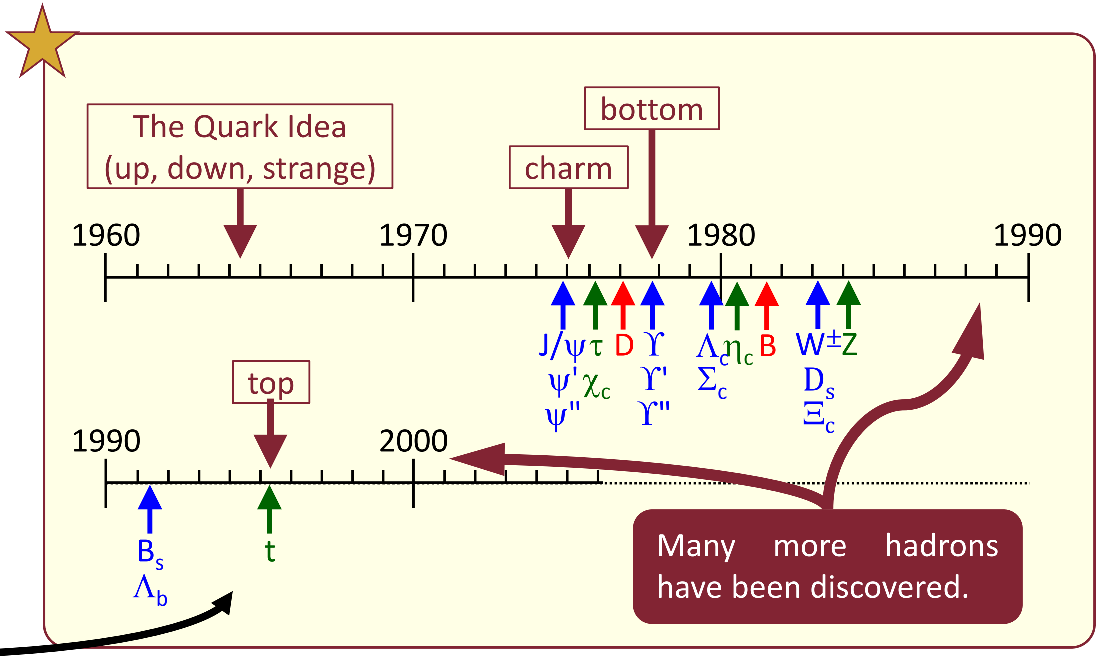
1956 : Sakata extended the Fermi-Yang model including the Λ, to account for strangeness : all hadronic states were then composed by (p, n, Λ) and their antiparticles.

Hadrons : "elementary" or composite?

1961 : M. Gell-Mann and Y. Ne'eman (independently) proposed a new classification, the **Eightfold Way**, based on the symmetry group SU(3). The classification did **NOT** explicitly mention an **internal structure**. The name was invented by Gell-Mann and comes from the "eight commandments" of the Buddhism.



Yuval Ne'eman
(יובל נאמן)



Warning : "*t*" is a quark, not a hadron (in modern language).

Simmetries and groups (I)

Illustrative example: the rotation group

- Two successive rotations R_1 followed by R_2 are equivalent to a single rotation $R=R_2R_1$. The group is closed under multiplication.
- There is an identity element (no rotation).
- Every rotation R has an inverse R^{-1} (rotate back again).
- The product is not necessarily commutative $R_1R_2 \neq R_2R_1$, but the associative law always holds: $R_3(R_2R_1) = (R_3R_2)R_1$.
- It is a continuous group: each rotation can be labeled by a set of continuously varying parameters $(\alpha_1, \alpha_2, \alpha_3)$ which can be regarded as the components of a vector $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ directed along the axis of rotation with magnitude given by the angle of rotation.
- The rotation group is a Lie group: every rotation can be expressed as the product of a succession of infinitesimal rotations (arbitrarily close to the identity). The group is then completely defined by the “neighborhood of the identity”.

Simmetries and groups (II)

- Rotations are a subset of the Lorentz transformations and they form a symmetry group of a physical system: Physics is invariant under rotations.
  For example suppose that under a rotation R the state of a system transforms as
  $$|\psi\rangle \xrightarrow{R} |\psi'\rangle = U|\psi\rangle$$
  Probabilities must be unchanged by R:
  $$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2 = |\langle\phi|U^+U|\psi\rangle|^2$$
  $U^+U = 1$, U must be a unitary operator.
  The operators $U(R)$ form a group, with exactly the same structure as the original group (R_1,R_2,): they are said to form a unitary representation of the rotation group.
- The Hamiltonian is unchanged by a symmetry operation R of the system and the matrix elements are preserved

$$\langle\phi'|H|\psi'\rangle = \langle\phi|U^+HU|\psi\rangle = \langle\phi|H|\psi\rangle \qquad U^+HU = H \qquad [U,H] = 0$$

Simmetries and groups (III)

- The transformation U has no explicit time dependance and the equation of motion

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

is unchanged by the the symmetry operation. As a consequence the expectation value of U is a constant of the motion

$$i\frac{d}{dt}\left\langle\psi\right|U\left|\psi\right\rangle = \left\langle\psi\right|UH - HU\left|\psi\right\rangle = 0$$

Simmetries and groups (IV)

- All group properties follow from considering infinitesimal rotations in the neighborhood of the identity. Example rotation through ε around the 3-axis:

$$U = 1 - i\varepsilon J_3$$

J_3 is called the generator of rotations around the 3-axis.

$$1 = U^+U = (1 + i\varepsilon J_3^+)(1 - i\varepsilon J_3)$$
$$= 1 + i\varepsilon(J_3^+ - J_3) + O(\varepsilon^2)$$

$J_3^+ = J_3$ is therefore hermitian, and hence is an observable.

- Consider the effect of a rotation R on the wave function. Invariance under rotations requires:

$$\psi'(\vec{r}) = \psi(R^{-1}\vec{r}) = U\psi(\vec{r})$$

For an infinitesimal rotation ε around the 3-axis:

$$U\psi(x,y,z) = \psi(R^{-1}\vec{r}) = \psi(x + \varepsilon y, y - \varepsilon x, z)$$
$$= \psi(x,y,z) + \varepsilon\left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right)$$
$$= \psi\left[1 - i\varepsilon(xp_y - yp_x)\right]$$

$$\boxed{U\psi = \left(1 - i\varepsilon J_3\right)\psi}$$

J_3 component along the 3-axis of the angular momentum. Invariance under rotations corresponds to the conservation of angular momentum.

Simmetries and groups (V)

For a rotation through a finite angle θ:

$$U(\theta) = [U(\varepsilon)]^n = \left(1 - i\frac{\theta}{n}J_3\right)^n \xrightarrow[n\to\infty]{} e^{-i\theta J_3}$$

The commutator algebra of the generators is:

$$\left[J_m, J_n\right] = i\varepsilon_{mnl}J_l$$

The J's are said to form a Lie Algebra

ε_{ijk} = structure constants of the group

Nonlinear functions of the generators which commute with all the generators are called invariants or Casimir operators. For the rotation group the only Casimir operator is:

$$J^2 = J_1^2 + J_2^2 + J_3^2 \qquad\qquad [J^2, J_l] = 0 \quad l = 1,2,3$$

It follows that we can construct simultaneous eigenstates of J^2 and one of the generators, e.g. J_3:

$$J^2|j,m\rangle = j(j+1)|j,m\rangle$$

$$J_3|j,m\rangle = m|j,m\rangle$$

The group SU(2)

In the lowest-dimension nontrivial representation of the rotation group (*j*=½) the generators may be written:

$$J_k = \frac{1}{2}\sigma_k \quad k = 1,2,3 \quad \sigma_k = matrici\ di\ Pauli$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The basis for this representation is given by the eigenvectors of σ_3: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ describing a spin ½ particle of spin projections up and down.

The transformation matrices:

$$U(\theta_i) = e^{-i\theta_i \frac{\sigma_i}{2}}$$

are unitary. The set of all unitary 2×2 matrices is **U(2)** (**Unitary Group**). However U(2) is larger than the group $U(\theta_i)$, since the σ_i all have zero trace. For any hermitian traceless matrix σ it can be shown that:

$$\det e^{i\sigma} = e^{i\mathrm{Tr}(\sigma)} = 1$$

This property is preserved in matrix multiplication. The set of traceless unitary 2×2 matrices form a subgroup of U(2) called **SU(2)** (**Special Unitary**). The two-dimensional representation is the fundamental representation.

The group SU(2)

For a composite system $|j_A,j_B,m_A,m_B\rangle$ the operator $\mathbf{J=J_A+J_B}$ satisfies the Lie algebra and the eigenvalues $J(J+1)$ and M of J^2 and J_3 are conserved quantum numbers. The *product* of the two irreducible representations $(2j_A+1)$ and $(2j_B+1)$ may be decomposed into the sum of irreducible representations of dimension $(2J+1)$ with basis $|j_A,j_B,J,M\rangle$, where

$$|j_A-j_B| \le J \le |j_A+j_B| \qquad M=m_A+m_B$$

$$\left|j_A j_B J M\right\rangle = \sum_{m_A,m_B} C(m_A,m_B,J,M)\left|j_A j_B m_A m_B\right\rangle$$

Clebsch-Gordan coefficients

Example: from two two-dimensional representations (j=½) we obtain one 3-dimensional (J=1, triplet) and one 1-dimensional 1 (J=0, singolet) representation.

$$2\otimes 2=3\oplus 1$$

$$2\otimes 2\otimes 2=(3\otimes 2)\oplus(1\otimes 2)$$

$$=4 \quad \oplus \quad 2 \quad \oplus \quad 2$$

quadruplet
spin 3/2

doublet
spin 1/2

SU(2) Isospin

$$\left[I_m, I_n\right] = i\varepsilon_{mnl} I_l$$

Generators in the fundamental representation:

$$I_k = \frac{1}{2}\tau_k \quad k = 1,2,3$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

basis

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$I = \tfrac{1}{2}, I_3 = +\tfrac{1}{2} \qquad I = \tfrac{1}{2}, I_3 = -\tfrac{1}{2}$$

Isospin for antiparticles

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = e^{-i\pi\frac{\tau_2}{2}} \begin{pmatrix} p \\ n \end{pmatrix} = -i\tau_2 \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix}$$

$$Cp = \overline{p} \qquad Cn = \overline{n}$$

$$\begin{pmatrix} \overline{p}' \\ \overline{n}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{p} \\ \overline{n} \end{pmatrix}$$

In order for the antidoublet to transform in the same way as the doublet we must:
- Reorder the doublet
- Introduce a minus sign

$$\begin{pmatrix} -\overline{n}' \\ \overline{p}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{p} \\ \overline{n} \end{pmatrix}$$

For N $\overline{\text{N}}$

$$\begin{cases} \left|I=1, I_3=1\right\rangle = -p\overline{n} \\ \left|I=1, I_3=0\right\rangle = \sqrt{\tfrac{1}{2}}(p\overline{p} - n\overline{n}) \\ \left|I=1, I_3=-1\right\rangle = n\overline{p} \end{cases}$$

$$\left|I=0, I_3=0\right\rangle = \sqrt{\tfrac{1}{2}}(p\overline{p} + n\overline{n})$$

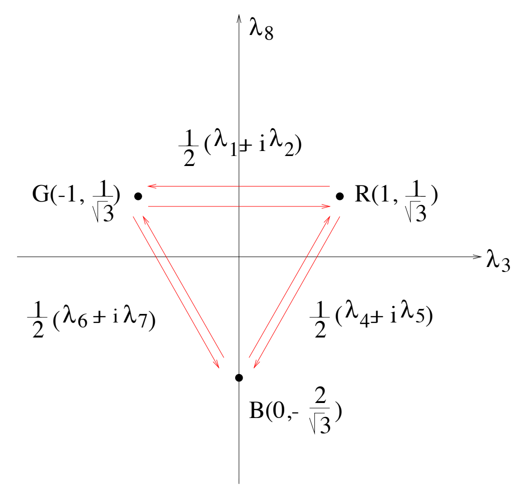
The group SU(3)

It is the group of unitary 3×3 matrices with det*U=1*. The generators may be taken to be any 3^2-1=8 linearly independent traceless hermitian 3×3 matrices. There are therefore 8 generators, of which 2 are diagonal. This is also the *maximum number of mutually commuting generators* : **Rank of the group**.
It can be shown that the rank of the group is equal to the number of Casimir operators.
The fundamental representation of SU(3) is a triplet (e.g. the three color charges of a quark). *The generators are 3×3 matrices*: λ_i, *i=1,..,8* (Gell-Mann matrices).

$$\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

simultaneous eigenvectors of $\lambda_3$and λ_8.



Isospin and strangeness: flavour SU(3)

The introduction of a second additive quantum number S in addition to I_3 suggests to enlarge isospin symmetry to a larger group, a group of rank 2. In 1961 SU(3) was proposed. The assignment of particles to SU(3) multiplets is not straightforward due to the high mass differences between the various particles (strange and non strange).

For example, the baryonic octet group particles with mass differences up to *400 MeV*, over an average octet mass of *1100 MeV*.

SU(3) flavor symmetry is much more approximate than SU(2) of isospin. We will see that this is due to the fact that the strange quark *s* is much heavier than *u* and *d*. SU(3) symmetry forms the basis of the quark model and it turns out to be very useful to classify hadrons and to understand some of their properties.

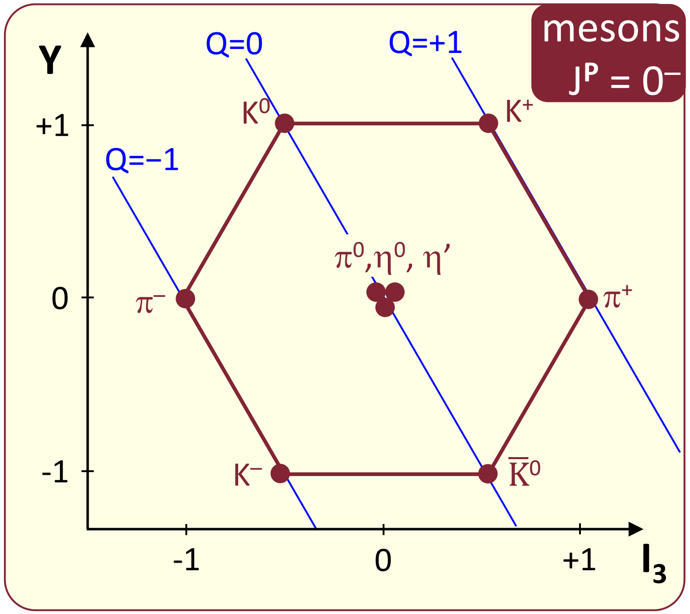
Color SU(3) on the other hand is an exact symmetry of fundamental origin.

The Eightfold Way: 1961-64

All hadrons (known in the '60s) are classified in the plane **(I_3 – Y)**, (Y = strong hypercharge):

$I_3 = I_z$ = third component of isospin;
$Y = \mathscr{B} + S$ **[baryon number + strangeness]**.

The strangeness ***S***, which contributes to ***Y***, had the effect to enlarge the isospin symmetry group **SU(2)** to the larger **SU(3): Special Unitarity group**, with dimension=3.



The Gell-Mann – Nishijima formula (1956) was :

$$Q = I_3 + ½(\mathscr{B}+S)$$

including heavy flavors [$\mathscr{B}$:baryon, B:bottom] :

$$Q = I_3 + ½(\mathscr{B}+S+C+B+T)$$

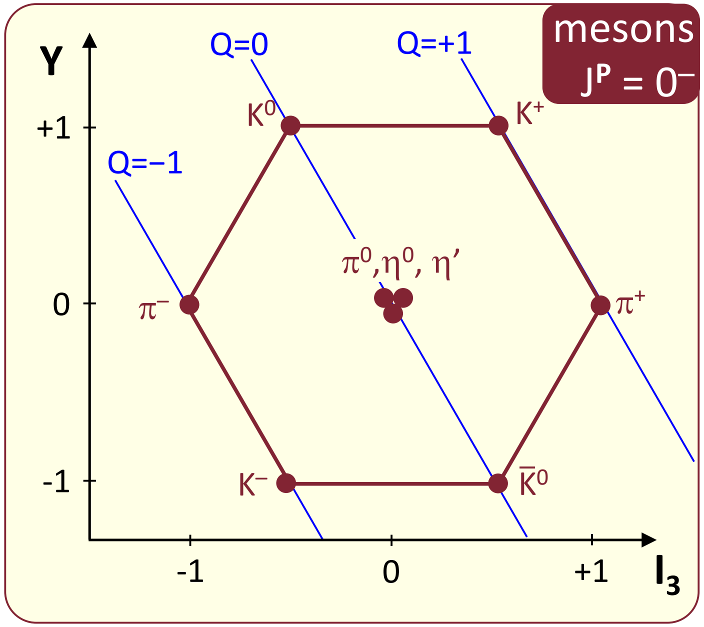
This symmetry is now called "**flavor SU(3) [SU(3)$_F$]**", to distinguish it from the "**color SU(3) [SU(3)$_C$]**", which is the exact symmetry of the strong interactions in QCD.

The Eightfold Way: SU(3)

The particles form the multiplets of **SU(3)$_F$**. Each multiplet contains particles that have the same spin and intrinsic parity. The basic multiplicity for mesons is nine ($3\times\overline{3}$), which splits in two SU(3) multiplets: (octet + singlet). For baryons there are octects + decuplets.

The gestation of SU(3) was long and difficult. It both explained the multiplets of known particles/resonances, and (more exciting) predicted new states, before they were actually discovered (*really a triumph*).

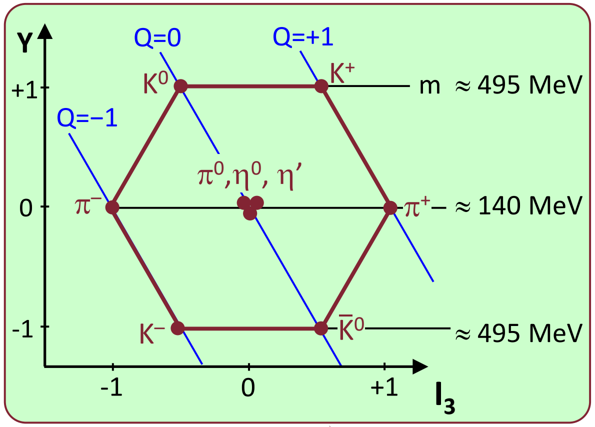
However, the mass difference p – n (or $\pi^\pm - \pi^0$) is < few MeV, while the π – K (or p – Λ) is much larger. Therefore, while the isospin symmetry **SU(2)** is almost exact, the symmetry **SU(3)$_F$**, grouping together strange and non-strange particles, is substantially violated.

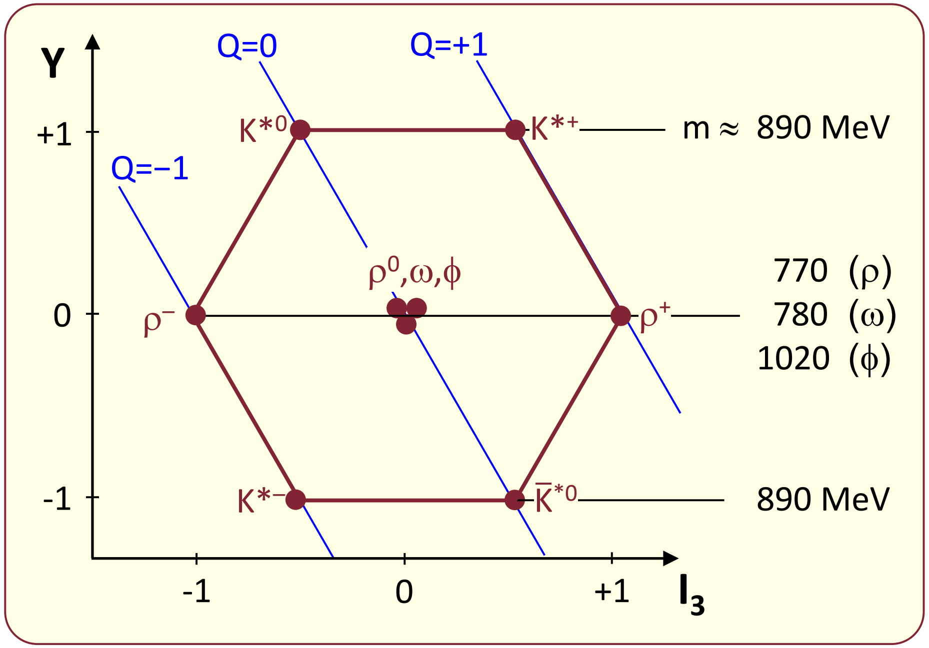


In principle, in a similar way, the discovery of heavier flavors could be interpreted with higher groups (e.g. **SU(4)$_F$** to incorporate the charm quark, and so on). However, these higher symmetries are broken even more, as demonstrated by the mass values. Therefore, **SU(6)$_F$** for all known mesons $J^P = 0^-$ is (almost) never used.

The Eightfold Way: mesons $J^P=1^-$

Another example of a multiplet: the octet of vector mesons :

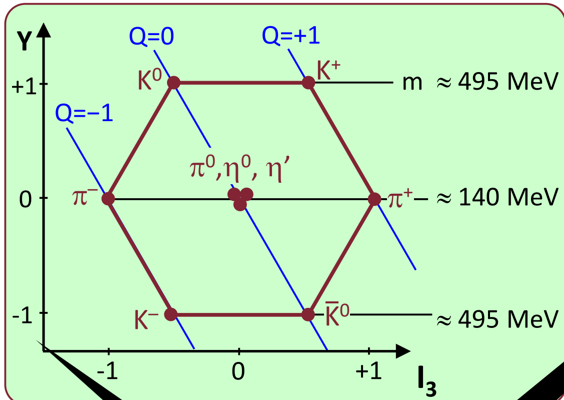


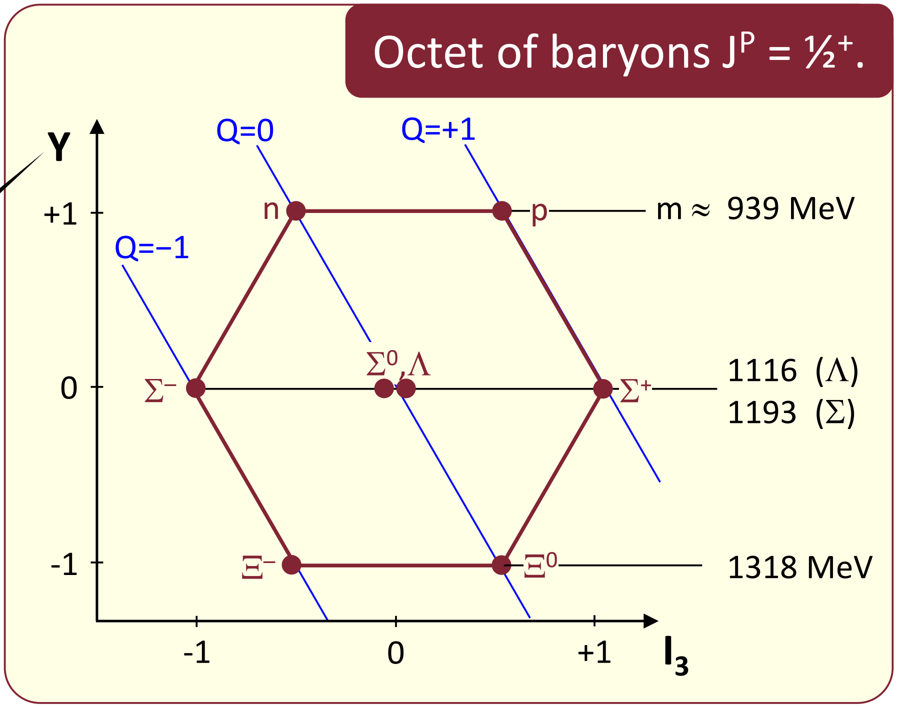


[mesons $J^P = 0^-$]

meson resonances $J^P = 1^-$ (all discovered by 1961).

The Eightfold Way: baryions $J^P=1/2^+$





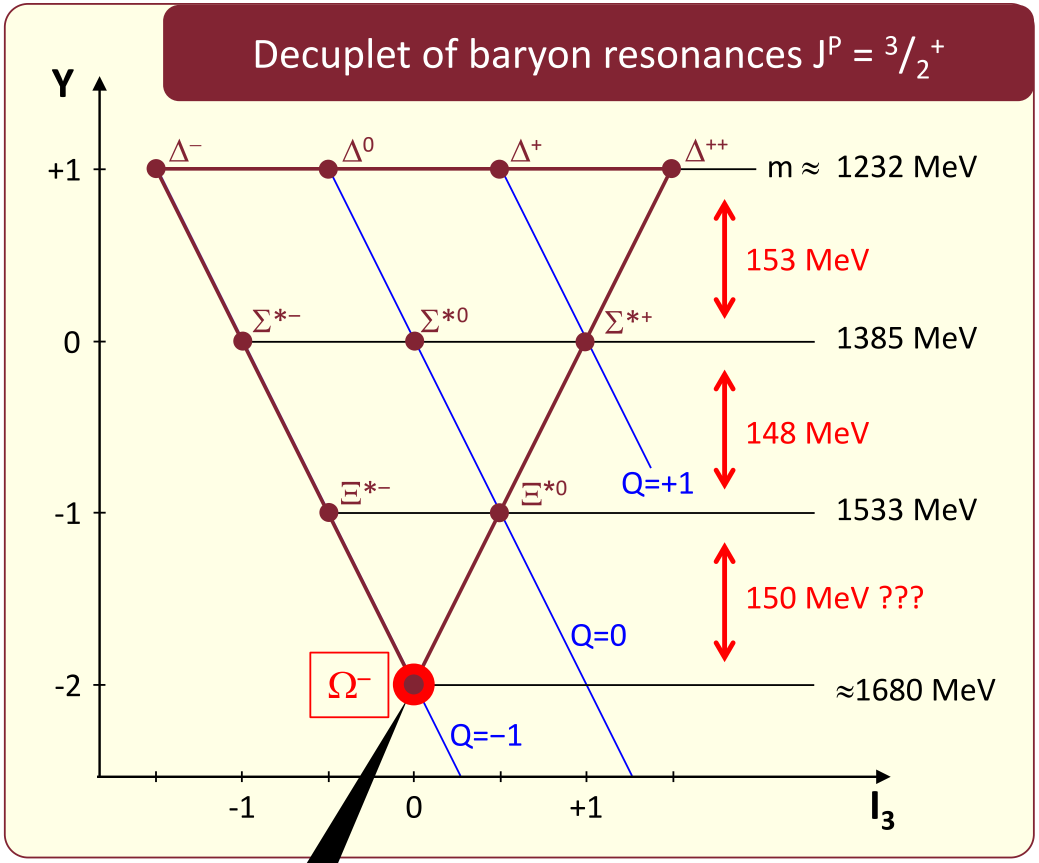
mesons: $Y = S$

baryons: $Y = S + \mathscr{B}$

notice the masses: for mesons, because of $\mathbb{CPT}$ ($K \leftrightarrow \bar{K}$) the masses of an octet are symmetric wrt (S=0, I_3=0), while for baryons the mass increases as $-S$

[*because the s-quark (S= −1) is heavier than u/d, but they did not know it*]

The Eightfold Way: baryons $J^P=3/2^+$



when the Eightfold Way was first proposed, this particle (now called Ω^-) was not known → see next slide.

The next multiplet of baryons is a decuplet $J^P = {}^3/_2{}^+$.

When the E.W. was proposed, they knew only 9 members of the multiplet, but can predict the last member:

- it is a decuplet, because of E.W.;
- the state $Y = -2$, $I_3 = 0$ ($\rightarrow Q = -1$, $S = -3$, $\mathscr{B}=1$) must exists;
- call it Ω^-;
- look the mass differences vs Y:
- mass linear in Y → $m_{\Omega^-} \approx 1680$ MeV (*NOT an E.W. requirement, but a reasonable assumption*);
- the conservation laws set the dynamics of production and decay of the Ω^-.

The discovery of the Ω^-

The particle Ω^-, predicted (★) in 1962, was discovered in 1964 by N.Samios et al., using the 80-inch hydrogen bubble chamber at Brookhaven (next slide).

The Ω^- can only decay weakly to an S = –2 final state [(1)] :

$\Omega^- \rightarrow \Xi^0\,\pi^-$; $\rightarrow \Xi^-\,\pi^0$; $\rightarrow \Lambda^0\,K^-$;

[a posteriori confirmed by the measurement $\tau_{\Omega^-} \cong 0.82 \times 10^{-10}$ s]

———————————

(1) Since the electromagnetic and strong interactions conserve the strangeness, the lightest (non-weak) S- and $\mathscr{B}$- conserving decay is :

$\Omega^- \rightarrow \Xi^0\,K^-$ [S : –3 → –2 –1, $\mathscr{B}$: +1 → +1 +0]

which is impossible, because

$m(\Omega) \approx 1700$ MeV < $m(\Xi) + m(K) \approx 1800$ MeV.

Therefore the Ω^- must decay via strangeness-violating weak interactions : <u>the Ω^- lifetime reflects its weak (NOT strong NOR e.m.) decay</u>.

(★) ***<u>From a 1962 report:</u>***

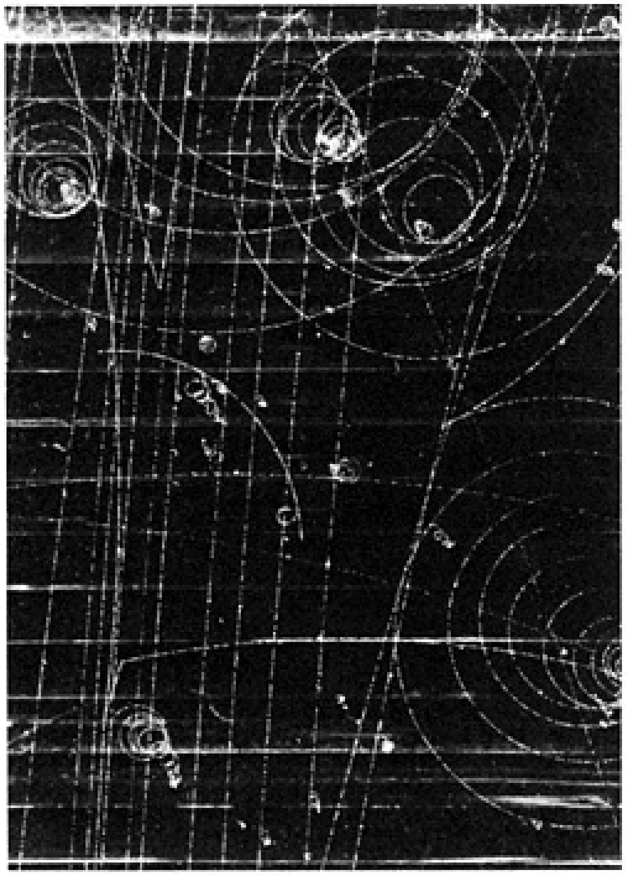
Discovery of Ξ^* resonance with mass ~1530 MeV is announced […].

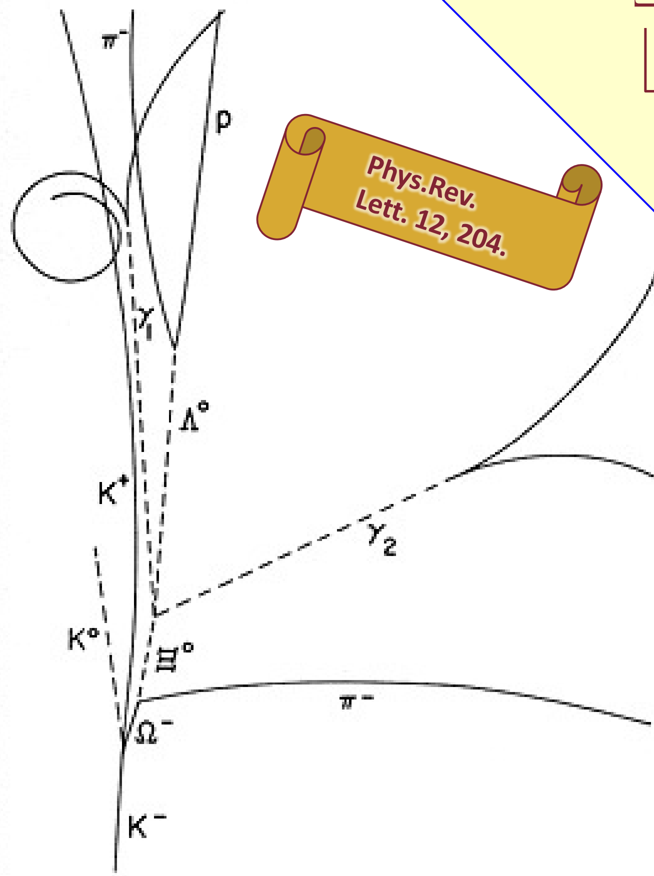
[As a consequence,] **Gell-Mann and Ne'eman** […] predicted a new particle and all its properties:

- Name = Ω^- (*Omega* because this particle is the last in the decuplet);
- Mass ≈ 1680 MeV (the masses of Δ, Σ^* and Ξ^* are about equidistant ~150 MeV);
- Charge = –1;
- Spin = $^3/_2$;
- Strangeness = –3, Y = –2;
- Isospin = 0 (no charge-partners);
- Lifetime ~10^{-10} s, because of its weak decay, since strong decay is forbidden**[(1)]**;
- Decay modes: $\Omega^- \rightarrow \Xi^0\,\pi^-$ or $\Omega^- \rightarrow \Xi^-\,\pi^0$.

The discovery of the Ω^-: the event

the Ω^- observation required both genius and luck (e.g. compute the probability of the two γ conversions in H_2):





Phys.Rev. Lett. 12, 204.

$K^- + p \to \Omega^- + K^+ + K^0$

↳ $\Xi^0 + \pi^-$ ($\Delta S = 1$ weak decay)

↳ $\pi^0 + \Lambda$ ($\Delta S = 1$ w.d.)

↳ $\pi^- + p$ ($\Delta S = 1$ w.d.)

↳ $\gamma + \gamma$ (e.m. decay)

↳ e^+e^-

↳ e^+e^-

Nick Samios

Brookhaven National Laboratory 80-inch hydrogen bubble chamber - 1964

The static quark model

In 1964 M. Gell-Mann and G. Zweig proposed independently that all the hadrons are composed of three constituents, that Gell-Mann called[1] quarks.

This model, enriched by both extensions (other quarks) and dynamics (electroweak interactions and QCD) is still the basis of our understanding of the elementary particles, the **Standard Model**[2].

In this chapter we consider only the static properties of the three original quarks. Sometimes, in the literature, it is referred as the *naïve quark model*.

1969 : Gell-Mann is awarded Nobel Prize *"for his contributions and discoveries concerning the classification of elementary particles and their interactions"*.

---

[1] The name so whimsical was taken from the (now) famous quote "*Three quarks for Muster Mark !*", from James Joyce's novel "*Finnegans Wake*" (book 2, chapt. 4).

[2] At that time it was not clear whether the quark hypothesis was a mathematical convenience or reality. Today, as shown in the following, our understanding is clearer, but complicated: the quarks are real (to the extent that all QM particles are), but they cannot be seen as isolated single objects.

The static quark model

- They must be fermions in order to construct both fermions and bosons.
- In analogy to the idea of Fermi and Yang, mesons are $q_1 \bar{q}_2$ pairs; baryons (antibaryons) are $q_1q_2q_3$ ($\bar{q}_1 \bar{q}_3 \bar{q}_3$) states.
- In order to form nonstrange particles with charges 0,$\pm$1 at least two quarks are needed. These must form an isospin doublet (in order to have both *I=0* and *I=1*).
- In order to form strange particles a third quark is needed, to which by convention is assigned *S=-1*. *The minimal number of constituents is thus 3*.
- To properly account for baryon numbers quarks are assigned *B=1/3*.
- The simplest spin-parity assignment is $J^P=\frac{1}{2}^+$.
- From the Gell-Mann and Nishijima formula $Q = I_3 + \frac{B+S}{2}$ it follows

$$Q(u) = +\frac{2}{3} \qquad Q(d) = Q(s) = -\frac{1}{3}$$

Quarks have a **fractional electric charge.**

The static quark model

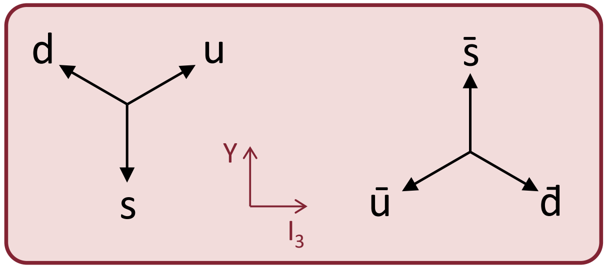
The hypothesis:

- three quarks **u**, **d**, and **s** (up, down, strange);
- quarks (q): standard Dirac fermions with spin ½ and fractional charge ($\pm\frac{1}{3}$e $\pm\frac{2}{3}$e);
- antiquarks ($\bar{q}$): according to Dirac theory, the q-antiparticles;
- baryons: combinations qqq (e.g. uds, uud);
- antibaryons: three antiquarks (e.g $\bar{u}\bar{u}\bar{d}$);
- mesons: pairs $q\bar{q}$ (e.g $u\bar{u}$, $u\bar{d}$, $s\bar{u}$);
- "*antimesons*": a $\bar{q}q$ pair: the mesons are their own antiparticles, i.e. "anti-mesons" = mesons.

		u	d	s	c	b	t
$\mathscr{B}$	baryon	⅓	⅓	⅓			
J	spin	½	½	½			
I	isospin	½	½	0			
I_3	3rd i-spin	½	−½	0			
S	strang.	0	0	−1			
Y	$\mathscr{B}$+S	⅓	⅓	−⅔			
Q	$I_3 + \frac{1}{2}Y$	⅔	−⅓	−⅓			

c, **b**, **t** not yet discovered in the '60 !!! see § 3

The quarks form a triplet, which is a basic representation of the group SU(3). Quarks may be represented in a vector shape in the plane $I_3 - Y$; their combinations (= hadrons) are the sums of such vectors.



The static quark model

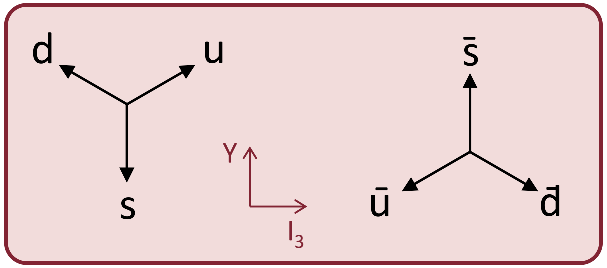
The hypothesis:

- three quarks **u**, **d**, and **s** (up, down, strange);
- quarks (q): standard Dirac fermions with spin ½ and fractional charge (±⅓e ±⅔e);
- antiquarks ($\bar{q}$): according to Dirac theory, the q-antiparticles;
- baryons: combinations qqq (e.g. uds, uud);
- antibaryons: three antiquarks (e.g $\bar{u}\bar{u}\bar{d}$);
- mesons: pairs $q\bar{q}$ (e.g $u\bar{u}$, $u\bar{d}$, $s\bar{u}$);
- "*antimesons*": a $\bar{q}q$ pair: the mesons are their own antiparticles, i.e. "anti-mesons" = mesons.

		u	d	s	c	b	t
$\mathscr{B}$	baryon	⅓	⅓	⅓			
J	spin	½	½	½			
I	isospin	½	½	0			
$\mathbf{I_3}$	3rd i-spin	½	−½	0			
S	strang.	0	0	−1			
Y	$\mathscr{B}$+S	⅓	⅓	−⅔			
Q	I_3 + ½Y	⅔	−⅓	−⅓			

c, **b**, **t** not yet discovered in the '60 !!! see § 3

The quarks form a triplet, which is a basic representation of the group SU(3). Quarks may be represented in a vector shape in the plane I_3 – Y; their combinations (= hadrons) are the sums of such vectors.

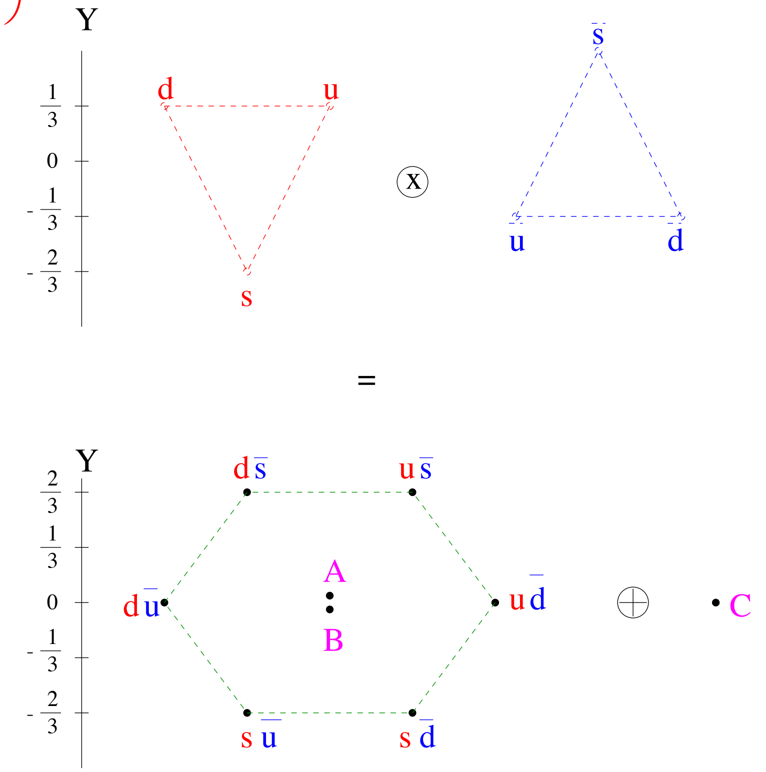


The mesons

Let us start with two quarks (*u* and *d*) $\begin{pmatrix} u \\ d \end{pmatrix}$ $\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$

$$\begin{cases} |1,1\rangle = -u\bar{d} \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right) \\ |1,-1\rangle = \bar{u}d \end{cases}$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)$$



Adding a third quark *s* there are 9 possible combinations: 1 octet and 1 singlet (under transformations in SU(3) the 8 states transform among themselves, but they never mix with the singlet).

$$3 \otimes \bar{3} = 8 \oplus 1$$

The mesons

The singlet state, *C*, is symmetric in flavor:

$$C = \sqrt{\tfrac{1}{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$$

A is the neutral member of the isospin triplet:

$$\left(d\bar{u},\, A, -u\bar{d}\right) \quad \Longrightarrow \quad A = \sqrt{\tfrac{1}{2}}\left(u\bar{u} - d\bar{d}\right)$$

Quarks have *spin ½*, therefore the total spin of the $q_1\,\bar{q}_2$ pair can be *S=0* or *S=1*. The spin *J* of the mesons results as the combination of S and of the relative angular momentum L. The parity P of the meson is thus:

$$P = \underset{\uparrow}{-}(-1)^L = (-1)^{L+1}$$

Product of the intrinsic parities
of fermion and antifermion

The value of *C* is obtained like in positronium :

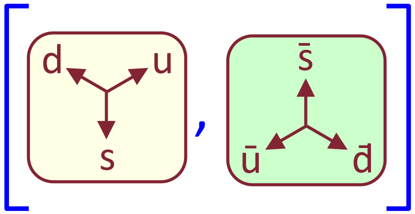
$$C = \underset{\uparrow}{-}(-1)^{S+1}(-1)^L = (-1)^{L+S}$$

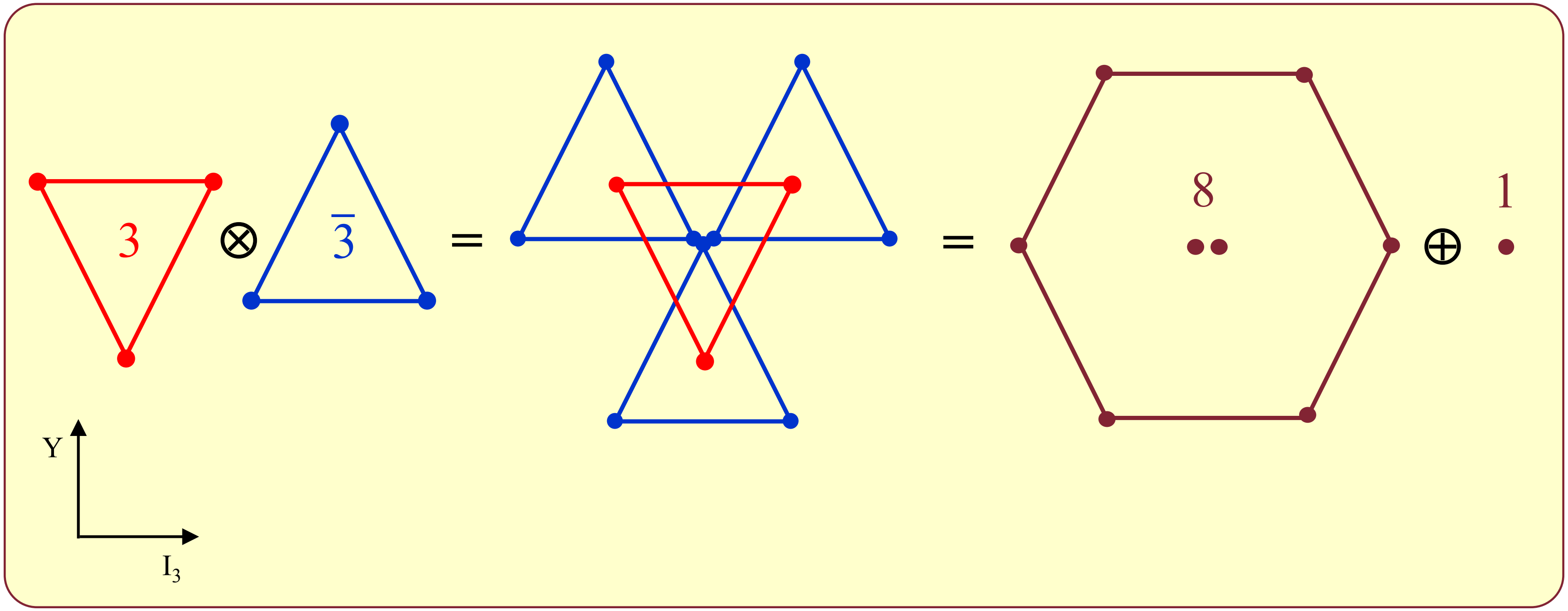
Fermion interchange

The mesons

"Build" the mesons $q\bar{q}$ with these rules :

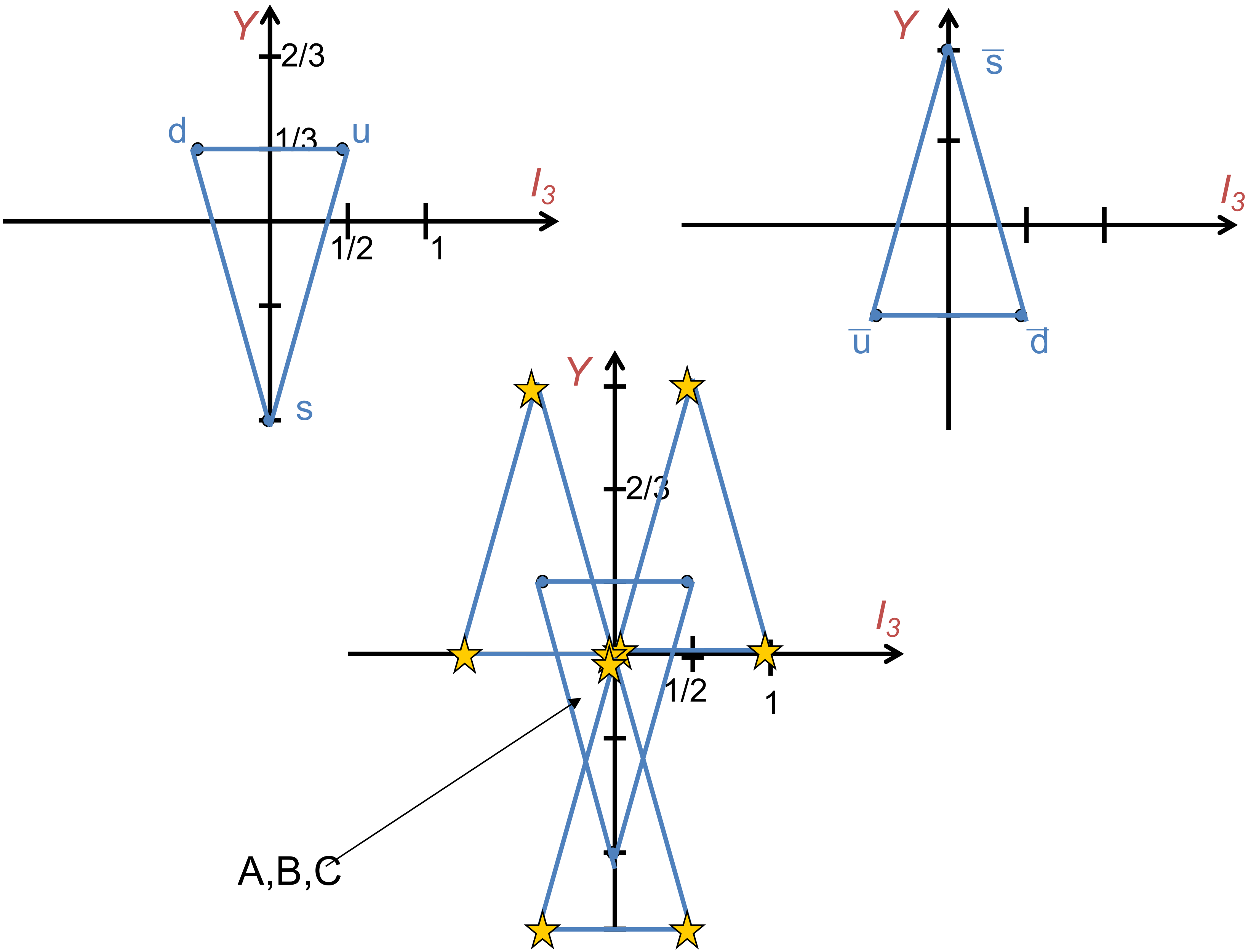
- in the space I_3 - Y, sum "vectors" (i.e. quarks and antiquarks) to produce $q\bar{q}$ pairs, i.e. mesons;
- all the combinations are allowed:



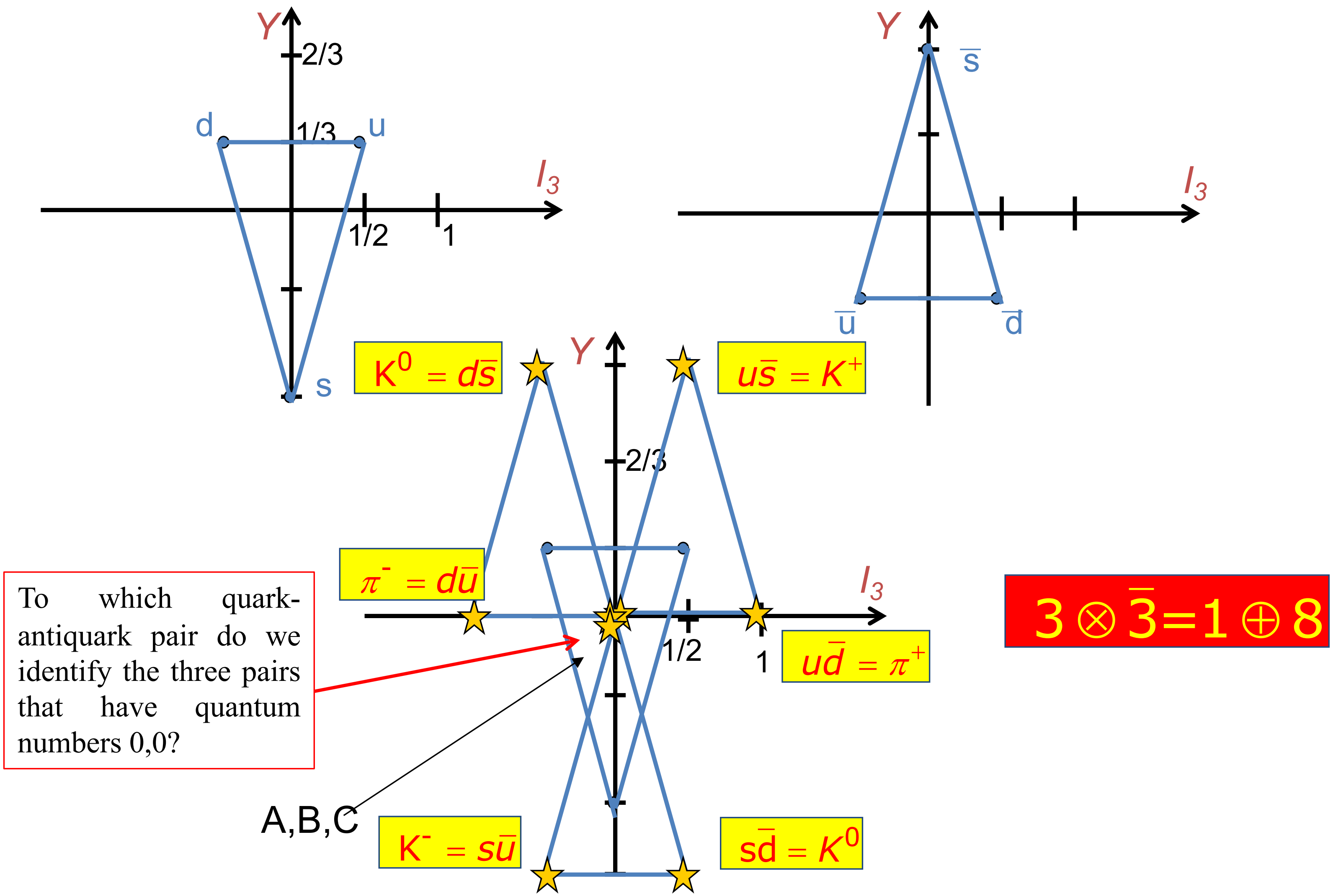


- the pseudoscalar mesons ($J^P=0^-$) are $q\bar{q}$ states in s-wave with opposite spins (⇑⇓).

Graphical construction of the meson 0⁻

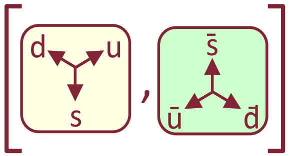


Graphical construction of the meson 0⁻

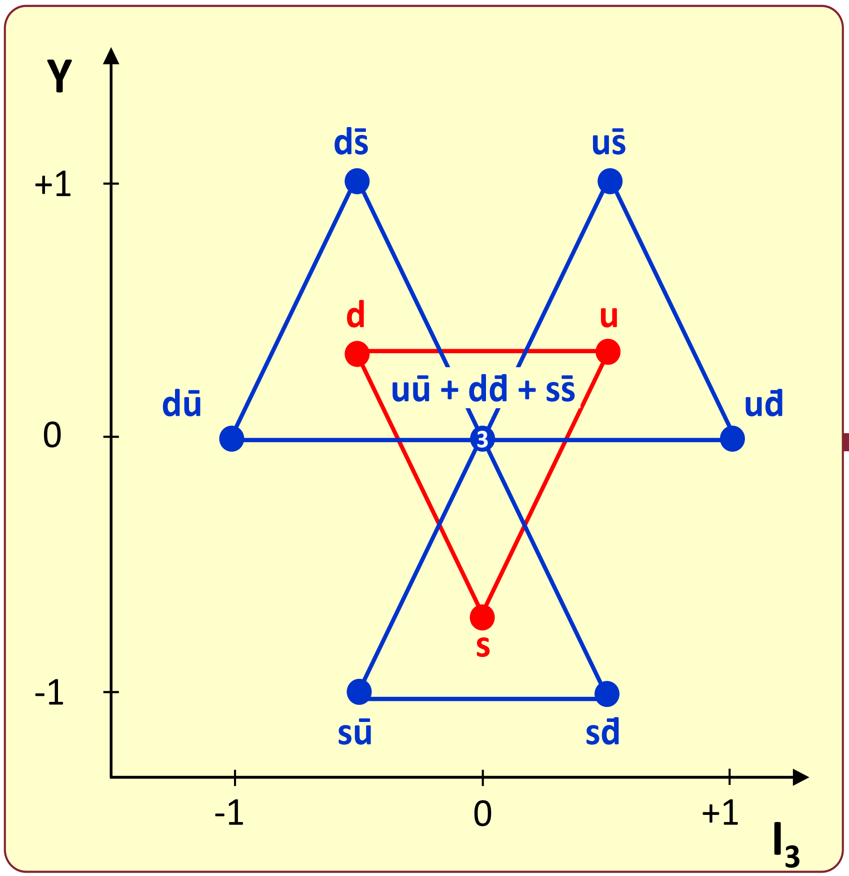


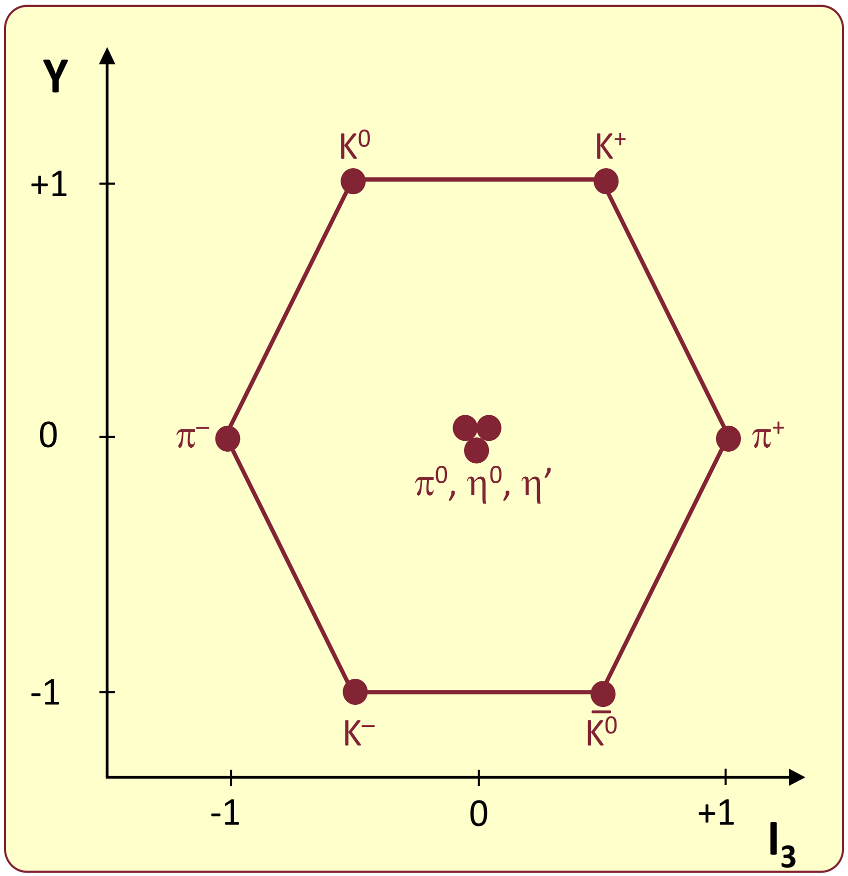
The mesons: $J^{PC}=0^{-+}$

More specifically, with s-wave ($J^{PC}=0^{-+}$), we get the "pseudoscalar" nonet :



Notice that π^0, η, η' are combinations (mixing) of the three possible $q\bar{q}$ states (for the mixing parameters see later) :





Mesons 0^-

- The three states A, B, C with $I_3=0$ and $Y=0$ are orthogonal linear combinations of the states $u\bar{u} + d\bar{d} + s\bar{s}$

- Let's identify a state with $\{n, |\, I, I_3\rangle\}$ where n is the dimension of the representation.

- The SU(3) singlet must contain, because of the symmetry, all three states with the same weight (a rotation in the SU(3) space must not change the state):

$$\boxed{\eta_1 = \{1, |\, 0, 0\rangle\} = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)}$$

- One of the other two states with $I_3=0$ must belong to the triplet with isospin=1, therefore it can be obtained with the ladder operators, in this case with the operators of lowering or raising the charge.

Charge conjugation of the nucleons

- Let's remind the behaviour of some nucleons through the charge conjugation operation. They appear a few "minus" signs in according to the Condon-Shortley convention.

$$\begin{array}{ccccc} I_3 & & & & \\ +\frac{1}{2} & |p\rangle & |\bar{n}\rangle & |u\rangle & |\bar{d}\rangle \\ -\frac{1}{2} & |n\rangle & -|\bar{p}\rangle & |d\rangle & -|\bar{u}\rangle \end{array}$$

$$\begin{pmatrix} u \\ d \end{pmatrix} \text{ e } \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix} \Rightarrow \begin{cases} I^- \,|\, \bar{d}\rangle = |-\bar{u}\rangle \\ I^+ \,|\, \bar{u}\rangle = |-\bar{d}\rangle \end{cases}$$

$I^{\pm}$: operator of isospin shift (raising and lowering of the charge)

- N.B. the quark s is an isospin singlet, therefore when we add it to an isospin doublet, it will not change the doublet properties:

$$\begin{pmatrix} u\bar{s} = K^+ \\ d\bar{s} = K^0 \end{pmatrix} \text{ e } \begin{pmatrix} s\bar{d} = \bar{K}^0 \\ -s\bar{u} = -K^- \end{pmatrix}$$

- Combining d with ū (or viceversa) we can have I=0 or I=1

Wave function of the π^0

- Let's apply the isospin shift operator that has the following property:

$$\mathrm{I}^{\pm} \,|\, \Psi(I, I_3)\rangle = \sqrt{I(I+1) - I_3(I_3 \pm 1)} \,|\, \Psi(I, I_3 \pm 1)\rangle$$

- If we apply it to a quark we get: $\begin{cases} \mathrm{I}^+ \,|\, d \,\rangle = |\, u\rangle \;\; ; \;\; \mathrm{I}^+ \,|\, \bar{u} \,\rangle = |\, -\bar{d}\rangle \;\; ; \\ \mathrm{I}^+ \,|\, u \,\rangle = \mathrm{I}^+ \,|\, \bar{d}\rangle = 0 \end{cases}$

- Moreover: $\begin{cases} \mathrm{I}^- \,|\, \Psi(1,1)\rangle = I^+ \,|\, \Psi(1,-1)\rangle = \sqrt{2} \,|\, \Psi(1,0)\rangle \\ \mathrm{I}^+ \,|\, \Psi(1,0)\rangle = \sqrt{2} \,|\, \Psi(1,1)\rangle \;\; ; \;\; \mathrm{I}^- \,|\, \Psi(1,0)\rangle = \sqrt{2} \,|\, \Psi(1,-1)\rangle \\ \mathrm{I}^+ \,|\, \Psi(1,1)\rangle = I^- \,|\, \Psi(1,-1)\rangle = 0 \end{cases}$

- By convention the wave function of the π^- is: $-d\bar{u}$

$$\mathrm{I}^+ \,|\, \pi^- \,\rangle = \mathrm{I}^+ \,|\, -d\bar{u} \,\rangle = |\, -\left[\left(\mathrm{I}^+ d\right)\bar{u} + d\left(\mathrm{I}^+ \bar{u}\right)\right]\rangle = |\, -u\bar{u} + d\bar{d}\rangle = \sqrt{2}\left(\frac{1}{\sqrt{2}} \,|\, -u\bar{u} + d\bar{d}\rangle\right) = \sqrt{2} \;|\, \pi^0\rangle$$

- The π^0 is identified with the state: $\boxed{\pi^0 = \frac{1}{\sqrt{2}}\left(d\bar{d} - u\bar{u}\right)}$

- Indeed: $\mathrm{I}^+ \,|\, \pi^0 \,\rangle = \mathrm{I}^+ \dfrac{|\, d\bar{d} - u\bar{u} \,\rangle}{\sqrt{\;}} = \dfrac{|\, u\bar{d} + 0 - 0 + u\bar{d}\rangle}{\sqrt{2}} = \sqrt{2} \;\; |\, u\bar{d}\rangle = \sqrt{2} \;\; |\, \pi^+\rangle$

Wave functions and physics states

- In order to find the singlet of the octet $\eta_8 = \{8, |\,0,0\rangle\}$ we need to find the quark composition that it is orthogonal to $\eta_1 = \{1, |\,0,0\rangle\}$ and to the π^0:

$$\eta_1 = \{1, |\,0,0\rangle\} = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$$

$$\pi^0 = \frac{1}{\sqrt{2}}\left(d\bar{d} - u\bar{u}\right)$$

$$\eta_8 = \{8, |\,0,0\rangle\} = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$$

N.B. $I^{\pm}\,|\,\eta_8\rangle = 0$

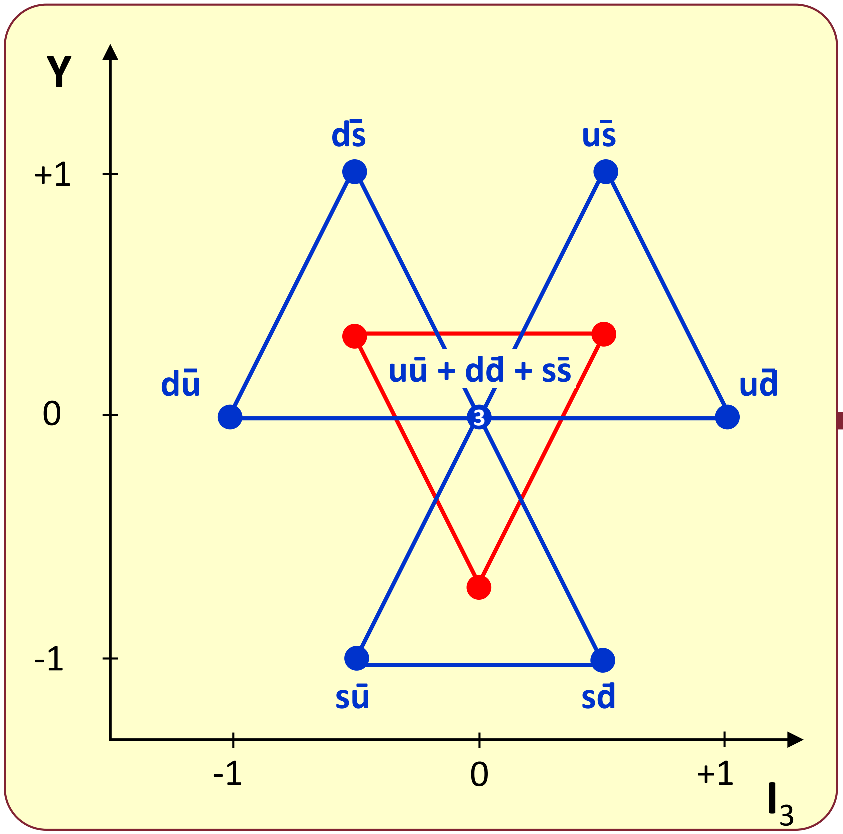
- The physical states η and η' are a linear combination of η_1 e η_8, but since the mixing angle is small (~11°), we can do the identification:

$$\eta_8 \equiv \eta \quad ; \quad m_\eta = 548 \text{ MeV}$$

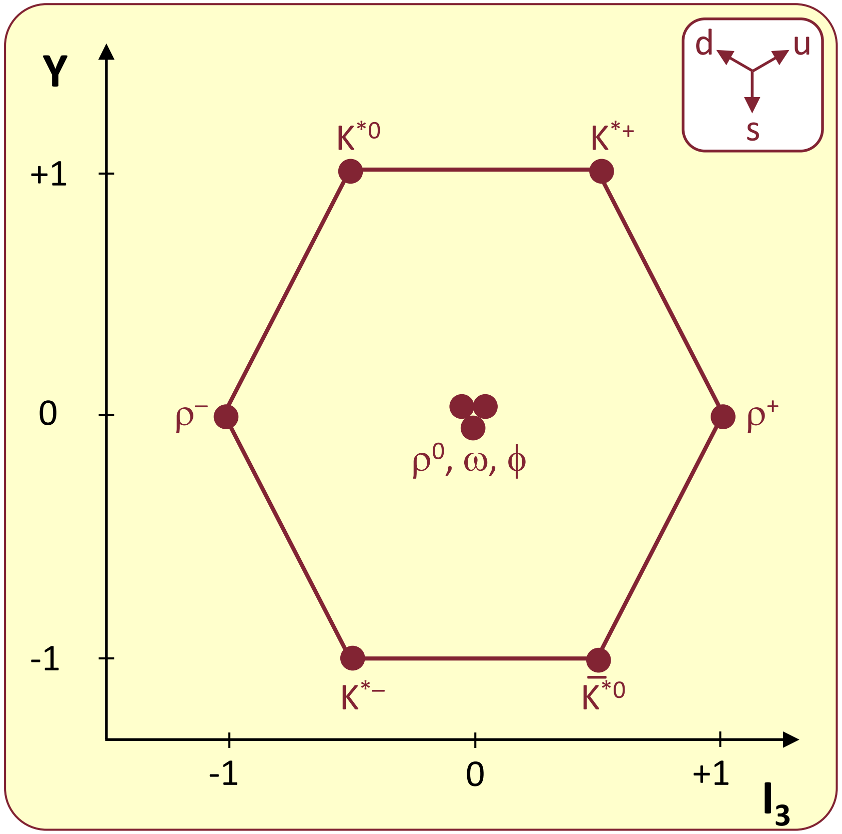
$$\eta_1 \equiv \eta' \quad ; \quad m_{\eta'} = 958 \text{ MeV}$$

The mesons: $J^{PC}=1^{--}$

If $J^{PC} = 1^{--}$ (i.e. spin ⇈), the "vector" nonet :



Notice that ρ^0, ω, φ are combinations (mixing) of the three possible $q\bar{q}$ states :



Meson quantum numbers: J^{PC}

- Parity : the quarks and the antiquarks have opposite P :
  $P_{q\bar{q}} = P_1 P_2 (-1)^L = -1\,(-1)^L = (-1)^{L+1}$.

- Charge conjugation : for mesons, which are also $\mathbb{C}$ eigenstates, $\mathbb{C} = \mathbb{P}\mathbb{S}$, parity followed by spin swap (see before).

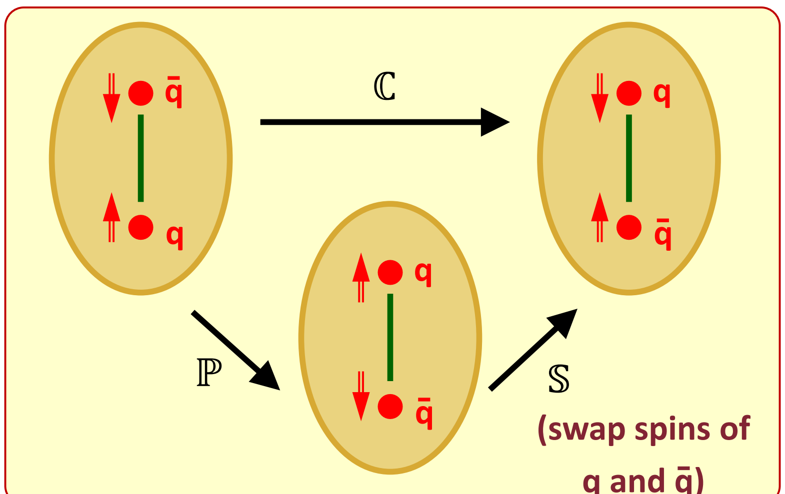
$P = (-1)^{L+1}$;

$S = (-1)^{S+1}$ (Pauli principle, [BJ, 263]);

$C = P \times S = (-1)^{L+S}$;

$G = (-1)^{L+S+I}$ (see before).

$$\underbrace{\frac{1}{\sqrt{2}}(\Uparrow\Downarrow - \Downarrow\Uparrow);}_{\substack{S=0\\ \text{antisymmetric}}} \qquad \underbrace{\Downarrow\Downarrow; \quad \frac{1}{\sqrt{2}}(\Uparrow\Downarrow + \Downarrow\Uparrow); \quad \Uparrow\Uparrow}_{\substack{S=1\\ \text{symmetric}}}$$



L	S	J=L⊕S	P	C	I	G
0	0	0	–	+	0	+
					1	–
	1	1	–	–	0	+
					1	–
1	0	1	+	–	0	–
					1	+
	1	0,1,2	+	+	0	–
					1	+

$J^{PC} = 0^{-+}, 1^{--}, 1^{+-}, 0^{++}, 1^{++}, 2^{++}, \ldots$

Meson quantum numbers: multiplets

- For the lowest state nonets, these are the quantum numbers :

L	S	J^{PC}	$^{2s+1}L_J$	I=1 state
0	0	0^{-+}	$^1S_0$	$\pi(140)$
	1	1^{--}	$^3S_1$	$\rho(770)$
1	0	1^{+-}	$^1P_1$	$b_1(1235)$
	1	0^{++}	$^3P_0$	$a_0(1450)$
		1^{++}	$^3P_1$	$a_1(1260)$
		2^{++}	$^3P_2$	$a_2(1320)$

- all these multiplets have main qn n = 1;
- as of today ~20 meson multiplets have been (partially) discovered [PDG].
- important activity from the ’50 to the ’70; still some addict;

- method (mainly bubble chambers) :
  - measure (zillions of) events; e.g. :

    $\bar{p}p \to \pi^+ \pi^+ \pi^- \pi^- \pi^0;$
  - look for "peaks" in final state combined mass, e.g. m($\pi^+ \pi^- \pi^0$);
  - the peaks are associated with high mass resonances, decaying via strong interactions (width $\to \Gamma \to$ strength);
  - the scattering properties (e.g. the angular distribution) and decay modes identify the other quantum numbers;
- result : an overall consistent picture;
- Great success !!!

“If I could remember the names of all these particles, I'd be a botanist.”
Enrico Fermi

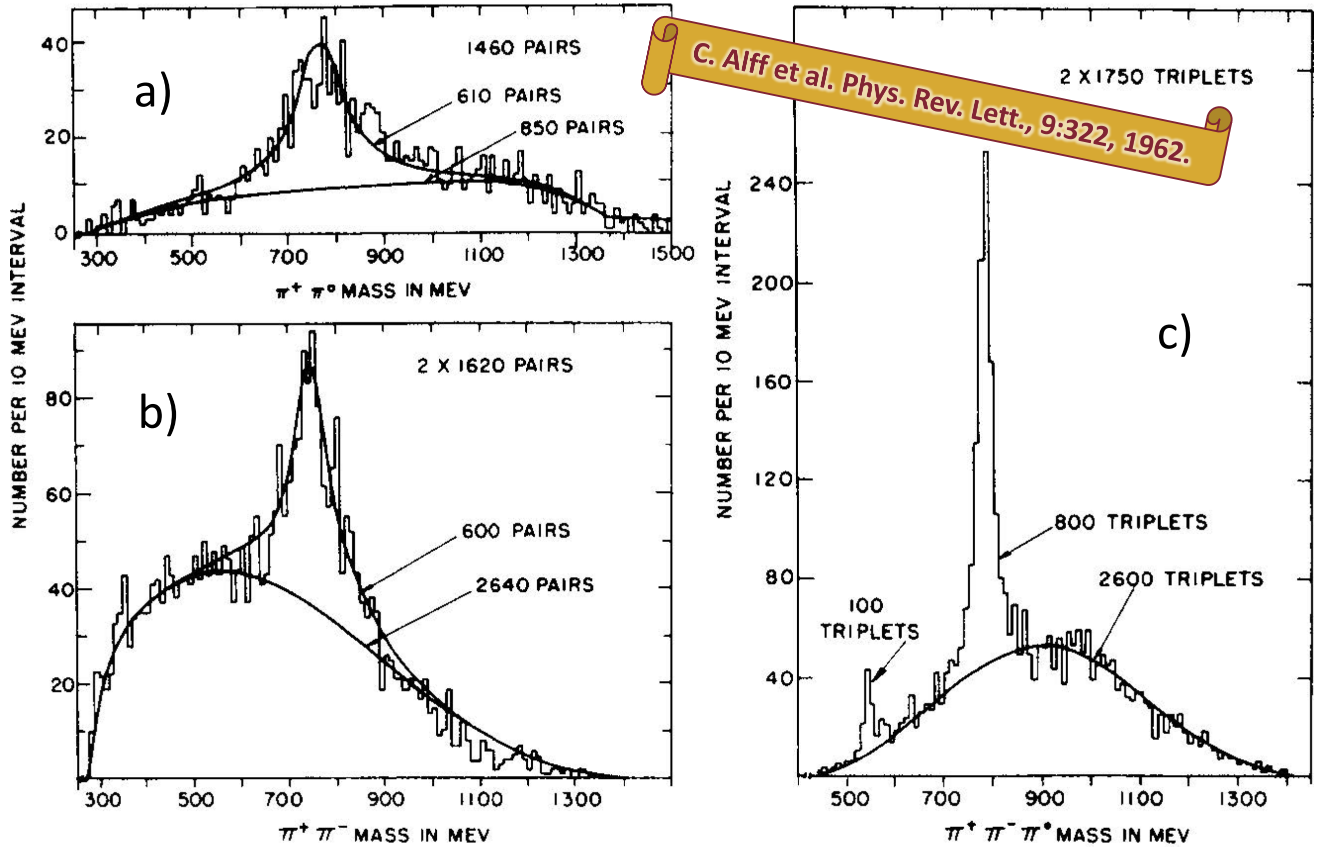
Meson quantum numbers: example

Three examples in $\pi^+ p \to X$

a) $m(\pi^+\pi^0)$ for $X = \pi^+\pi^0 p$

b) $m(\pi^+\pi^-)$ for $X = \pi^+\,\pi^+\,\pi^- p$

c) $m(\pi^+\,\pi^-\pi^0)$ for $X = \pi^+\,\pi^+\,\pi^-\pi^0 p$



Q: which resonances ?

a) $\rho^+(770) \to \pi^+\pi^0$; b) $\rho^0(770) \to \pi^+\pi^-$; c) $\eta(548) \to \pi^+\pi^-\pi^0$
$\omega(782) \to \pi^+\pi^-\pi^0$.

why not the ρ^0 ?

Meson quantum numbers : $\rho^0 \nrightarrow \pi^0\pi^0$

Problem: $\rho^0 \to \pi^0\pi^0$ is allowed ? **NO**, because of :

a) $\mathbb{C}$-parity

$C(\rho^0) = -1$; $C(\pi^0) = +1$

therefore, since the initial state is a C-eigenstate,

$-1 = (+1) \times (+1) \ \to$ **NO**

NB. A general rule : "a vector cannot decay into two equal (pseudo-)scalars".

But (a) and (b) do not hold for weak decays. Instead (c) is due to statistics + angular momentum conservation, and is valid for all interactions.

[(c) also forbids **Z** $\to$ **HH**]

b) Clebsch-Gordan coeff. in isospin space

$|\rho^0\rangle = |I=1, I_3=0\rangle$;

$|\pi^0\rangle = |1, 0\rangle$;

therefore the decay is

$\langle\pi^0\pi^0|\rho^0\rangle = \langle j_1 j_2 m_1 m_2 | J\ M\rangle =$
$= \langle 1\ 1\ 0\ 0\ |\ 1\ 0\ \rangle = 0$;

$\to$ **NO**.

[PDG, § 44 :

$1 \otimes 1$		...	1
		...	0
...	...	...	...
0	0	...	**0**

c) Spin-statistics

[Povh, problem 15-1]

- $S(\rho^0) = 1$, $S(\pi^0) = 0$ $\to L(\pi^0\pi^0) = 1$;
- ρ^0 is a boson $\to$ wave function symmetric;
- the π^0's are two equal bosons $\to$ space wave function symmetric;
- L=1 makes the wave function anti-symmetric

$\to$ **NO**.

Meson mixing

Light mesons	$q\bar{q}$	J^{PC} (1)	I	I_3	S	Q (1)	mass (MeV)	$q\bar{q}$ of I_3=0 (2)
π^+, π^0, π^-	$u\bar{d}$, $q\bar{q}^{(2)}$, $d\bar{u}$	0^{-+}	1	1, 0, -1	0	1, 0, -1	140	$\sim(u\bar{u}-d\bar{d})/\sqrt{2}$
η	$q\bar{q}^{(2)}$	0^{-+}	0	0	0	0	550	$\sim(u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}$
η'	$q\bar{q}^{(2)}$	0^{-+}	0	0	0	0	960	$\sim(u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{6}$
K^+, K^0 (3)	$u\bar{s}$, $d\bar{s}$	0^-	½	½,-½	+1	1, 0	495	
$\bar{K}^0, K^-$ (3)	$s\bar{d}$, $s\bar{u}$	0^-	½	½, -½	-1	0, -1	495	
ρ^+, ρ^0, ρ^-	$u\bar{d}$, $q\bar{q}^{(2)}$, $d\bar{u}$	1^{--}	1	1, 0, -1	0	1, 0, -1	770	$\sim(u\bar{u}-d\bar{d})/\sqrt{2}$
ω	$q\bar{q}^{(2)}$	1^{--}	0	0	0	0	780	$\sim(u\bar{u}+d\bar{d})/\sqrt{2}$
ϕ	$q\bar{q}^{(2)}$	1^{--}	0	0	0	0	1020	$\sim s\bar{s}$
K^{*+}, K^{*0} (3)	$u\bar{s}$, $d\bar{s}$	1^-	½	½,-½	+1	1, 0	890	
$\bar{K}^{*0}, K^{*-}$ (3)	$s\bar{d}$, $s\bar{u}$	1^-	½	½, -½	-1	0, -1	890	

Notes :

(1) (L=0, $\mathscr{B}$=0) $\rightarrow$ $P = (-)^{L+1} = -$; $C = (-)^{L+S} = (-)^S$; $Q = I_3 + ½Y = I_3 + ½S$;

(2) The mesons π^0, η, η', ρ^0, ω, ϕ are mixing of $u\bar{u} \oplus d\bar{d} \oplus s\bar{s}$ (see next);

(3) States with strangeness $\neq 0$ are NOT eigenstates of C; since they have I=½, no I_3=0 exists.

Meson mixing: $J^P=1^-$

- **The vector mesons 1^- have the same quark composition of the mesons 0^-, they are in the S-state (L=0) but the two quarks (actually quark-antiquark) have the spin that are parallel (S=1).**

- There are three mesons with $I_3=0$ and Y=0; one of them belongs to the isospin triplet ρ : ρ^+, ρ^-, ρ^0.
- ρ^0 has the same wave function of the π^0 (besides a factor "-1"):

$$\rho^0 = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$$

- The SU(3) singlet ϕ_1 and the isospin singlet of the octet ϕ_8 are mixed to give the mass eigenstates ϕ and ω:

$$\boxed{\begin{aligned}\omega &= \phi_1 \cos\vartheta + \phi_8 \sin\vartheta \\ \phi &= \phi_1 \sin\vartheta - \phi_8 \cos\vartheta\end{aligned}}$$

- N.B. in this case the mixing angle is $\theta \sim 35^{\circ}$

Meson mixing: $J^P=0^-, 1^-$

Mesons are bound states $q\bar{q}$. Consider only uds quarks (+ $\bar{u}\bar{d}\bar{s}$) in the nonets ($J^P = 0^-\ 1^-$, the *pseudo-scalar* and *vector* nonets) :

- the states ($\pi^+=u\bar{d}$, $\pi^-=d\bar{u}$, $K^+=u\bar{s}$, $K^0=d\bar{s}$, $K^-=s\bar{u}$, $\bar{K}^0=s\bar{d}$) have no quark ambiguity;
- but ($u\bar{u}$ $d\bar{d}$ $s\bar{s}$) have the same quantum numbers and the three states ($\psi_{8,0}$ $\psi_{8,1}$ ψ_1) mix together ($\rightarrow$ 2 angles per nonet);
- the physical particles (π^0, η, η' for 0^-, ρ^0, ω, ϕ for 1^-) are linear combinations $q\bar{q}$;
- ($\psi_{8,1}$) decuples (π^0 ρ^0) ($\rightarrow$ 1 angle only);
- θ_{ps} and θ_v are computed from the mass matrices* [PDG, §15.2];
- notice: the vector mixing $\theta_v \approx 36° \approx \tan^{-1}(1/\sqrt{2})$, i.e. the ϕ meson is almost $s\bar{s}$ only [i.e. $\phi \rightarrow K\bar{K}$, see KLOE exp.];

(... continue)

$$\left.\begin{aligned} \psi_{8,1}[\text{oct}, I=1] &= (u\bar{u}-d\bar{d})/\sqrt{2} \\ \psi_{8,0}[\text{oct}, I=0) &= (u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6} \\ \psi_1[\text{sing}] &= (u\bar{u}+d\bar{d}+s\bar{s})/\sqrt{3} \end{aligned}\right\} \begin{array}{c} \psi_{\text{multi},I} \\ \text{ideal case} \end{array}$$

$$\left.\begin{aligned} \pi^0(140) &\approx \psi^{ps}_{8,1} = (u\bar{u}-d\bar{d})/\sqrt{2} \\ \eta(550) &= \psi^{ps}_{8,0}\cos\theta_{ps} - \psi^{ps}_1 \sin\theta_{ps} \\ \eta'(960) &= \psi^{ps}_{8,0}\sin\theta_{ps} + \psi^{ps}_1 \cos\theta_{ps} \end{aligned}\right\} \begin{array}{c} J^P = 0^-, \\ \theta_{\text{pseudo-scalar}} \approx -25°; \end{array}$$

$$\left.\begin{aligned} \rho^0(770) &\approx \psi^{v}_{8,1} = (u\bar{u}-d\bar{d})/\sqrt{2} \\ \phi(1020) &= \psi^{v}_{8,0}\cos\theta_{v} - \psi^{v}_1 \sin\theta_{v} \approx s\bar{s} \\ \omega(780) &= \psi^{v}_{8,0}\sin\theta_{v} + \psi^{v}_1 \cos\theta_{v} \approx \\ &\approx (u\bar{u}+d\bar{d})/\sqrt{2} \end{aligned}\right\} \begin{array}{c} J^P = 1^-, \\ \theta_{\text{vector}} \approx 36°. \end{array}$$

* in principle, both the mass spectra and the mixing angles can be computed from QCD lagrangian $\mathscr{L}_{QCD}$ ... waiting for substantial improvements in computation methods.

Meson 1^- computation of the mixing angle

- Let'assume that the Hamiltonian matrix element between two states is equal to the "mass" squared:

$$M_\omega^2 = \langle \omega | H | \omega \rangle = M_1^2 \cos^2\vartheta + M_8^2 \sin^2\vartheta + 2M_{18}^2 \sin\vartheta \cos\vartheta$$
$$M_\phi^2 = \langle \phi | H | \phi \rangle = M_1^2 \sin^2\vartheta + M_8^2 \cos^2\vartheta - 2M_{18}^2 \sin\vartheta \cos\vartheta$$

- Since ω and ϕ are mass eigenstates, they are orthogonal:

$$M_{\omega\phi}^2 = \langle \phi | H | \omega \rangle = 0 = \left(M_1^2 - M_8^2\right)\sin\vartheta\cos\vartheta + M_{18}^2\left(\sin^2\vartheta - \cos^2\vartheta\right)$$

- If we get rid of of M_{18} and M_1 from these three equations, we get:

$$\tan^2\vartheta = \frac{M_\phi^2 - M_8^2}{M_8^2 - M_\omega^2}$$

- From the mass formula of Gell-Mann – Okubo we have:

$$M_8^2 = \frac{1}{3}\left(4M_{K^*}^2 - M_\rho^2\right)$$

$M_\rho = 776$ MeV
$M_{K^*} = 892$ MeV
$M_\omega = 783$ MeV
$M_\phi = 1020$ MeV

- If we put in the formula the measured values of the masses, we get:

$$\boxed{\vartheta \approx 40^o}$$

N.B. $\sin\,\vartheta = \frac{1}{\sqrt{3}}$ if $\vartheta \approx 35^o$

Meson mixing: $J^P=1^-$

- If we use $\sin\vartheta = \frac{1}{\sqrt{3}}$ we have:

$$\omega = \frac{1}{\sqrt{3}}\left(\phi_8 + \sqrt{2}\phi_1\right)$$
$$\phi = \frac{1}{\sqrt{3}}\left(\phi_1 - \sqrt{2}\phi_8\right)$$

- since:

$$\phi_1 = \frac{1}{\sqrt{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$$
$$\phi_8 = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$$

- We have

$$\phi = s\bar{s} \quad ; \ \omega = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right)$$

$M_\rho = 776$ MeV
$M_{K^*} = 892$ MeV
$M_\omega = 783$ MeV
$M_\phi = 1020$ MeV

- In this case of "ideal mixing", that is almost true in practice, the φ is composed entirely from quark s and the ω from quarks u and d
- This implies that the mass of the ω should be similar to the one of the ρ^0 and the mass of φ should be higher, as it is observed experimentally.

Meson mixing summary

- We have the mixing in the states with I=0:
  - η,η' are linear combinations of η_1, η_8 that can mix between themselves because they have the same quantum numbers: $(I=I_3=S=0)$
  - The same thing happens to the physical states ω and ϕ that are the results of the mixing of ϕ_1 and ϕ_8

We need to introduce new parameters: the mixing angles between the states

$$|\omega\rangle = \sqrt{\frac{1}{2}}\left(u\bar{u} + d\bar{d}\right)$$

$$|\phi\rangle = s\bar{s}$$

pure state $s\bar{s}$

$$|\eta\rangle = \sqrt{\frac{1}{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right)$$

$$|\eta'\rangle = \sqrt{\frac{1}{3}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right)$$

Almost exact, there is just a small mixing

Φ Meson Decay and the OZI (Okubo, Zweig, Iizuka) Rule

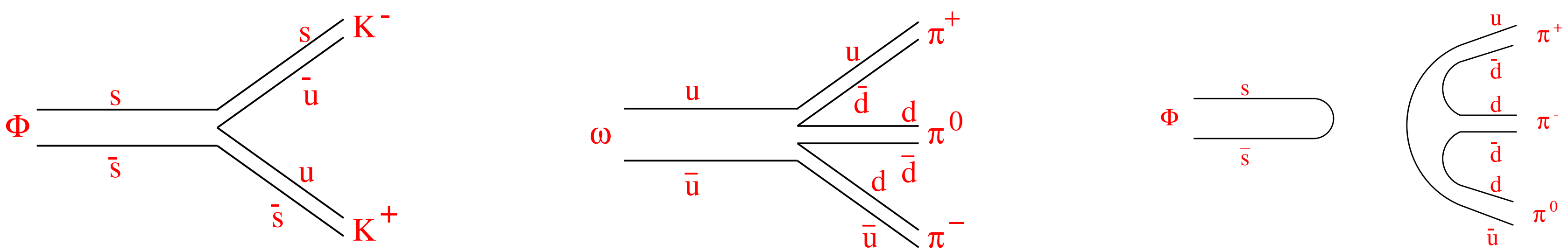
$$\phi \to K^+K^- \quad 49.1\%$$
$$\to K_L^0 K_S^0 \quad 34.4\%$$
$$\to \pi^+\pi^-\pi^0 \quad 15.3\%$$

$$\omega \to \pi^+\pi^-\pi^0 \quad 88.8\%$$
$$\to \pi^0\gamma \quad 8.5\%$$
$$\to \pi^+\pi^- \quad 2.2\%$$

For the $\phi(1020)$ phase space would favor the 3π decay with respect to 2K:

$$Q_{3\pi} \equiv M_\phi - 2M_{\pi^\pm} - M_{\pi^0} \approx 600\ MeV$$

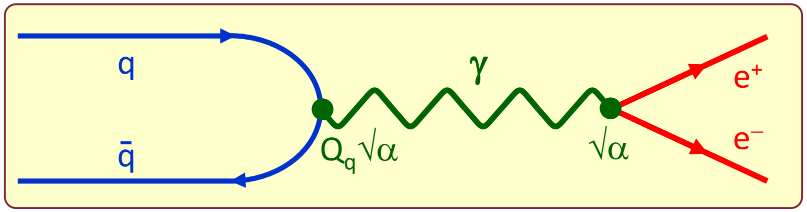
$$Q_{2K^0} \equiv M_\phi - 2M_{K^0} \approx 24\ MeV \quad Q_{2K^\pm} \approx 32\ MeV$$



The diagram for the 3π decay is suppressed because it contains disconnected quark lines (Zweig Rule). The kinematical suppression of the 2K decays is reflected in the small total width of the ϕ: Γ_ϕ = (4.26 ± 0.05) MeV to be compared, for example, with Γ_ρ = (150.3 ± 1.6) MeV.

Meson mixing: $J^P=1^-$

The decay amplitudes in the e.m. channels may be computed, up to a common factor, and compared to the experiment;



Few problems :

- the values are small*, e.g. $BR(\rho^0 \to e^+e^-) \approx 4.7 \times 10^{-5}$;
- the phase-space factor is important, especially for ϕ, which is very close to the $s\bar{s}$ threshold ($m_\phi - 2\,m_K$ = few MeV).

However, the overall picture is clear: the theory explains the data ***very well***.

———————————

* warning: the dominant $\rho^0\omega\phi$ decay modes are strong; however, the <u>e.m.</u> decays $\rho^0\omega\phi \to e^+e^-$, with a much smaller BR, are detectable $\to \Gamma_{e.m.}$ measurable $\to$ quark charges compared.

$$\left.\begin{array}{l} \rho^0(770) = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}); \\ \omega(780) = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}); \\ \phi(1020) = s\bar{s}; \end{array}\right\} \to \begin{array}{l} \mathcal{M}_{fi}(\rho^0\omega\phi \to e^+e^-) \\ \quad \propto \alpha \sum_j Q_q^j; \end{array}$$

$$\to \left\{\begin{array}{l} \Gamma(\rho^0 \to e^+e^-) \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3} - \frac{-1}{3}\right)\right]^2 = \frac{1}{2}; \\ \Gamma(\omega \to e^+e^-) \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3} + \frac{-1}{3}\right)\right]^2 = \frac{1}{18}; \\ \Gamma(\phi \to e^+e^-) \propto \left[\frac{1}{3}\right]^2 = \frac{1}{9}; \end{array}\right.$$

$$\to \Gamma_\rho : \Gamma_\omega : \Gamma_\phi = \begin{cases} 9 : 1 : 2 & \text{(theo)} \\ 8.8 \pm 2.6 : 1 : 1.7 \pm 0.4 & \text{(exp).} \end{cases}$$

Leptonic decay of the vector mesons

$$\rho^0 = \tfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

$$\omega = \tfrac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

$$\phi = s\bar{s}$$

Consider the decays

$$V \to l^+l^- \quad V = \rho, \omega, \phi \quad l = e, \mu$$

The partial width is given by:

$$\Gamma(V \to l^+l^-) = \frac{16\pi\alpha^2 Q^2}{M_V^2}\left|\psi(0)\right|^2$$

$$Q^2 = \left|\sum a_i Q_i\right|^2; \quad M_V = V\ mass; \quad m_l^2 << M_V^2$$

ρ,ω,ϕ have similar masses

$$\Longrightarrow \quad \frac{\left|\psi(0)\right|^2}{M_V^2} \approx \text{costante}$$

$$\Gamma(V \to l^+l^-) \propto Q^2$$

$$\rho^0 : \left[\sqrt{\tfrac{1}{2}}\left(\tfrac{2}{3} - \left(-\tfrac{1}{3}\right)\right)\right]^2 = \tfrac{1}{2}$$

$$\omega : \left[\sqrt{\tfrac{1}{2}}\left(\tfrac{2}{3} + \left(-\tfrac{1}{3}\right)\right)\right]^2 = \tfrac{1}{18} \qquad \phi : \left(\tfrac{1}{3}\right)^2 = \tfrac{1}{9}$$

$$\Longrightarrow \quad \Gamma(\rho^0) : \Gamma(\omega) : \Gamma(\phi) = 9 : 1 : 2$$

Van Royen - Weisskopf

Experimentally

$$\frac{\Gamma(\rho^0 \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} = 11.3 \qquad \frac{\Gamma(\phi \to e^+e^-)}{\Gamma(\omega \to e^+e^-)} = 2.3$$

3 Quark states: the baryons

The construction looks complicated, but in fact is quite simple :

- add the three quarks one after the other;
- count the resultant multiplicity.

In group's theory language :

$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8' \oplus 1$

i.e. a decuplet, two octects and a singlet.

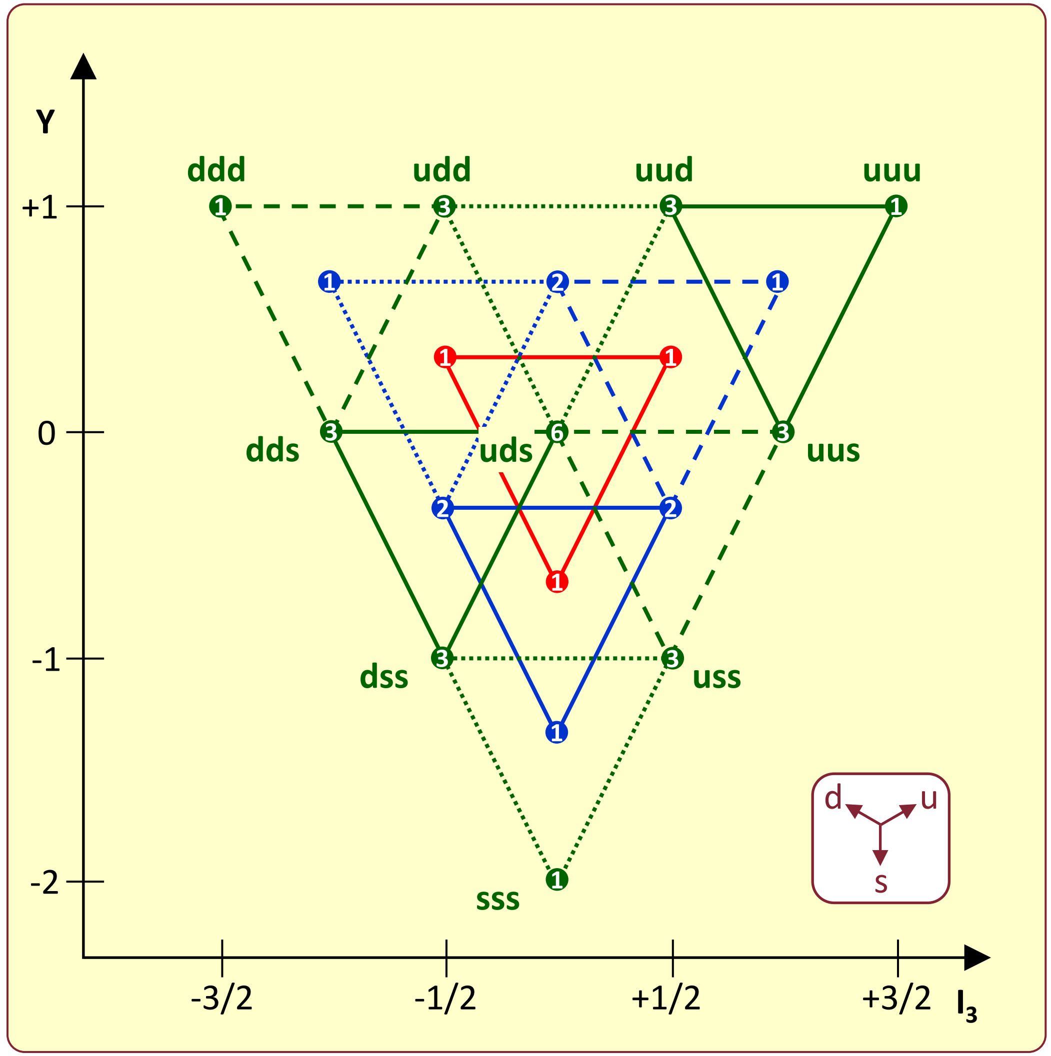
[proof. :

$3 \otimes 3 = \ 6 \oplus \bar{3};$

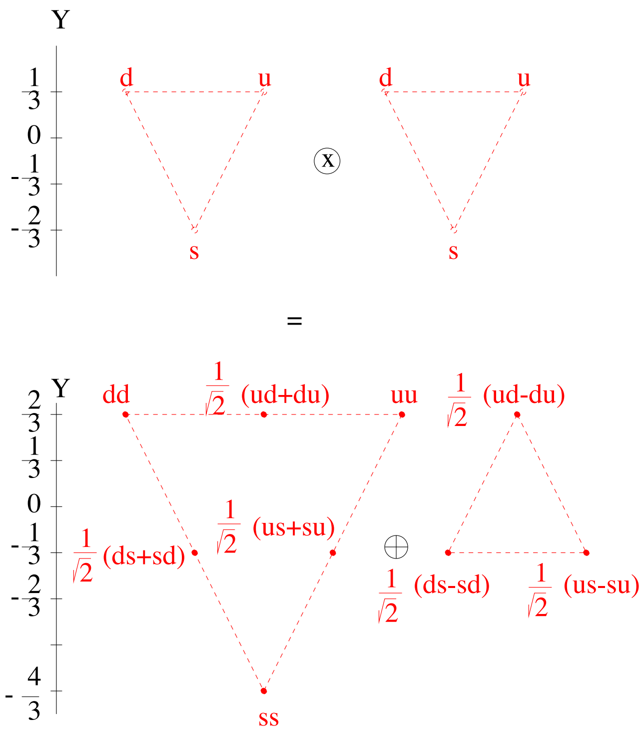
$6 \otimes 3 = 10 \oplus 8;$

$\bar{3} \otimes 3 = \ 8 \oplus 1.$ q.e.d.]

Both for 10, 8, 8' and 1 the three quarks have L = 0.

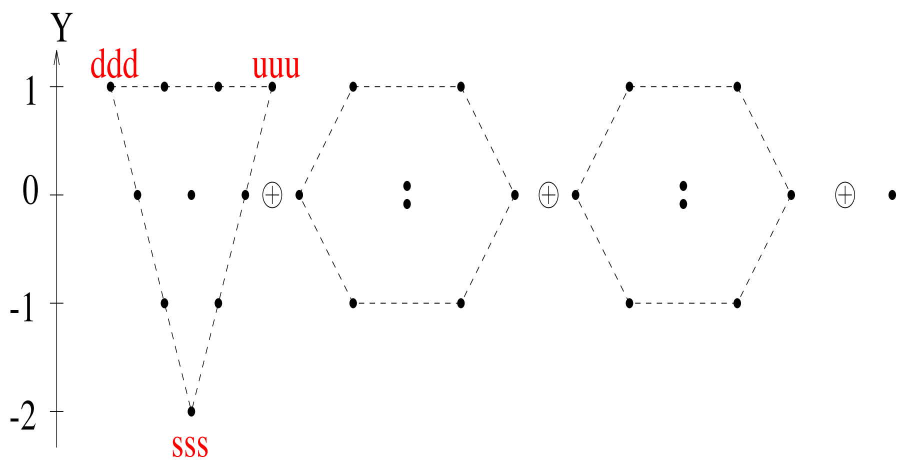


3 Quark states: the baryons



$$3 \otimes 3 = 6 \oplus \bar{3}$$

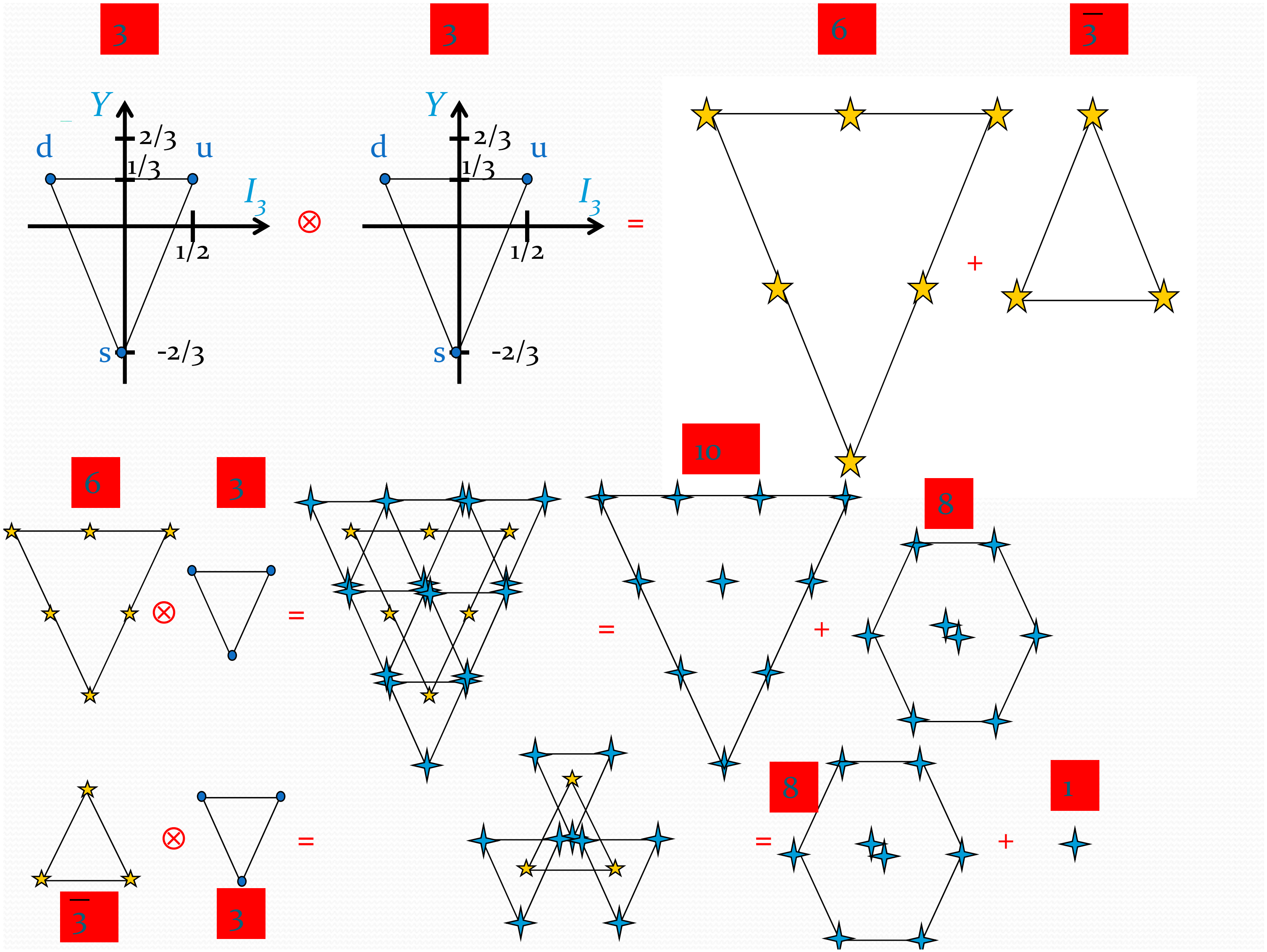
Let us now add the third quark



$$3 \otimes 3 \otimes 3 = (6 \otimes 3) \oplus (\bar{3} \otimes 3)$$

$$= 10 \oplus 8 \oplus 8 \oplus 1$$

3 Quark states: the baryons



The baryon quantum numbers

Baryons	qqq	J^P	I	I_3	S	$Q^{(1)}$	mass (MeV)
p, n	uud, udd	½⁺	½	½, -½	0	1, 0	940
Λ	uds	½⁺	0	0	-1	0	1115
$\Sigma^+, \Sigma^0, \Sigma^-$	uus, uds, dds	½⁺	1	1, 0, -1	-1	1, 0, -1	1190
Ξ^0, Ξ^-	uss, dss	½⁺	½	½,-½	-2	1, 0	1320
$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$	uuu, uud, udd, ddd	$^3/_2{}^+$	$^3/_2$	$^3/_2$, $^1/_2$, $-^1/_2$, $-^3/_2$	0	2, 1, 0, -1	1230
$\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}$	uus, uds, dds	$^3/_2{}^+$	1	1, 0, -1	-1	1, 0, -1	1385
Ξ^{*0}, Ξ^{*-}	uss, dss	$^3/_2{}^+$	½	½,-½	-2	1, 0	1530
Ω^-	sss	$^3/_2{}^+$	0	0	-3	-1	1670

10
$\Uparrow\Uparrow\Uparrow$

8
$\Uparrow\Uparrow\Downarrow$

Notes :

(1) $Q = I_3 + \frac{1}{2}Y = I_3 + \frac{1}{2}(\mathscr{B} + S)$; $\mathscr{B} = 1$.

The baryons: the octet $J^P = 1/2^+$

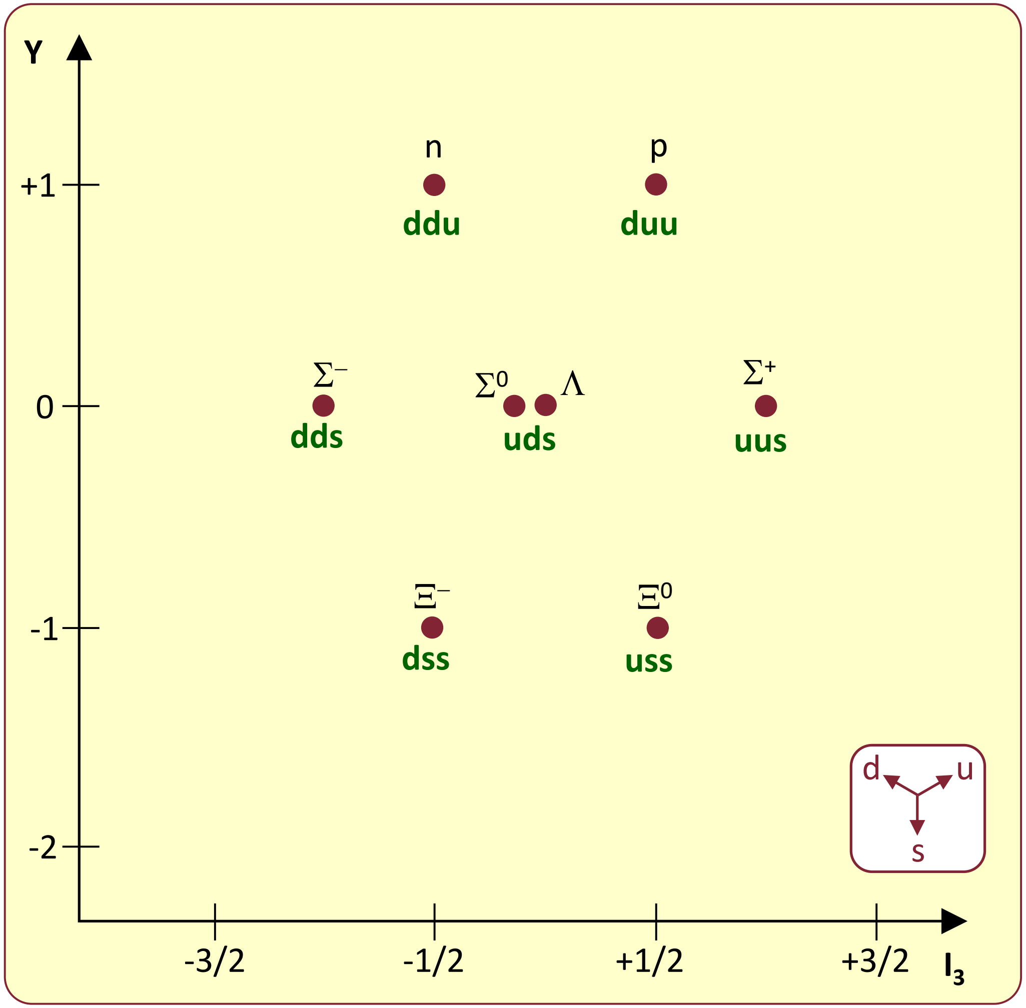
The lowest mass multiplet is an octet, which contains the familiar p and n, a triplet of S=-1 (the Σ's) a singlet S=-1 (the Λ) and a doublet of S=-2 (the Ξ's, sometimes called "cascade baryons").

The three quarks have ℓ = 0 and spin ($\Uparrow\Uparrow\Downarrow$), i.e. a total spin of ½.

The masses are :

- ~ 940 MeV for p and n;
- ~1115 MeV for the Λ;
- ~1190 MeV for the Σ's;
- ~1320 MeV for the Ξ's;

(difference of < few MeV in the isospin multiplet, due to e-m interactions.)



The baryons: the decuplet $J^P = 3/2^+$

The decuplet is rather simple (*but there is a spin/statistics problem, see later*). The spins are aligned (⇑ ⇑ ⇑), to produce an overall $J={}^3/_2$.

The masses, at percent level, are :

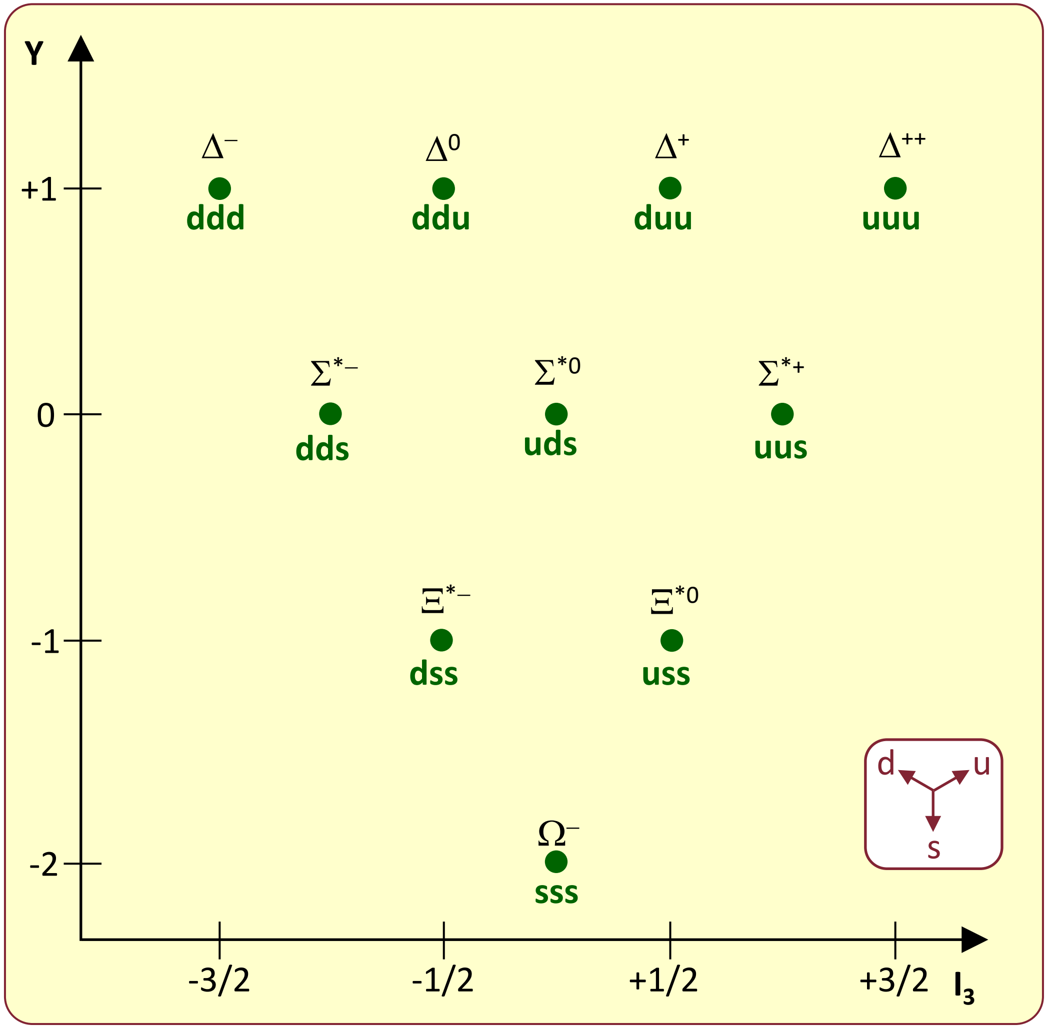
~ 1230 MeV for the Δ's;

~ 1385 MeV for the Σ^*'s,

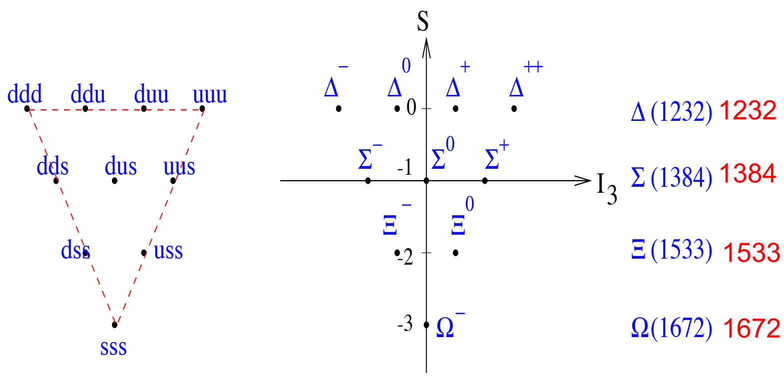
~ 1530 MeV for the Ξ^*'s

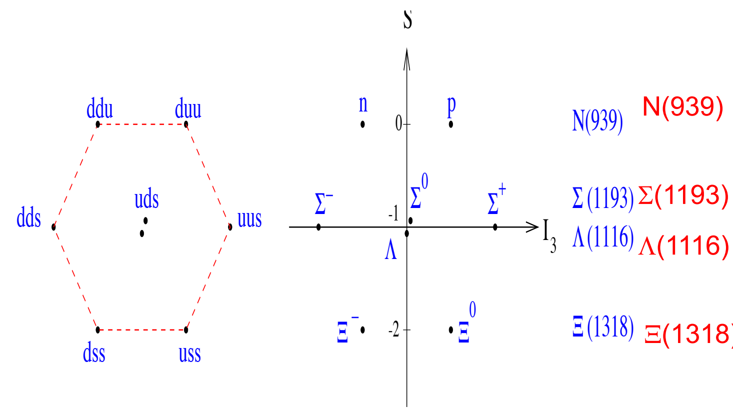
~ 1670 MeV for the Ω^-.

Notice that the mass split among multiplets is very similar, ~150 MeV (important for the Ω^- discovery, lot of speculations, no real explanation).



The baryons: the decuplet and the octet





If mass differences were solely due to the fact that the *s* quark is heavier than the *u* and *d* quarks we should have:

$$J^P = \tfrac{3}{2}^+ \quad \Sigma(1384) - \Delta(1232) = \Xi(1533) - \Sigma(1384) = \Omega(1672) - \Xi(1533)$$

$$152\ MeV \qquad\qquad 149\ MeV \qquad\qquad 139\ MeV$$

$$J^P = \tfrac{1}{2}^+ \quad \Sigma(1193) = \Lambda(1116)$$

$$\Lambda(1116) - N(939) = \Xi(1318) - \Lambda(1116)$$

$$152\ MeV \qquad\qquad 149\ MeV$$

The order of magnitude is correct, but discrepancies are still significant.
A quantitative understanding of hadron masses must take into account the effects of the hyperfine splitting in quark interactions.

The hadron masses

If flavor SU(3) symmetry were exact all members of a given multiplet would have exactly the same mass. Yet it is not so.

$$m_\omega \approx m_\rho(u\bar{u}) = 0.78\ GeV$$

$$m_\phi(s\bar{s}) = 1.02\ GeV \qquad\qquad J^P = 1^-$$

$$m_{K^*}(s\bar{u}) = 0.89\ GeV$$

If we consider hadron masses as the sum of the masses of the constituent quarks we obtain:

$$m_u \approx m_d \approx 0.39\ GeV$$

$$m_s \approx 0.51\ GeV$$

Effective masses of quarks bound in hadrons. Constituent masses.

There are further problems:

$$m_\Delta(J = \tfrac{3}{2}) > m_N(J = \tfrac{1}{2})$$

$$m_\rho(J = 1) > m_\pi(J = 0)$$

Δ and N contain the same quarks, as do π and ρ.

The hadron masses

Since hadron masses cannot be explained solely in terms of the masses of the constituent quarks it is necessary to consider the effects of quark interaction.

In the hydrogen atom the spin-spin interaction leads to the hyperfine structure of levels.

For two pointlike fermions of magnetic moments $\vec{\mu}_i$ and $\vec{\mu}_j$ the interaction energy is $\propto \frac{\vec{\mu}_i \cdot \vec{\mu}_j}{r_{ij}^3}$

Dirac theory gives: $\vec{\mu} = \frac{e}{2m}\vec{\sigma}$

The hyperfine separation is given by:

$$\Delta E_{hf} = -\frac{2}{3}\vec{\mu}_1 \cdot \vec{\mu}_2 |\psi(0)|^2$$

$$= \frac{2\pi\alpha}{3}\frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{m_1 m_2}|\psi(0)|^2$$

It is a contact interaction: it contains the square of the wave function at zero separation and therefore it only applies to *L=0* states.

The hadron masses

For quarks the magnetic interaction associated to charge and spin is of the order of the MeV. But quarks interact through their color charges with a potential of the form:

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + kr$$

At small distances the term in *1/r* dominates and α_s is small enought to make the strong hyperfine splitting important:

$$\Delta E(Q\overline{Q}) = \frac{8\pi\alpha_s}{9m_1m_2}\vec{\sigma}_1\cdot\vec{\sigma}_2|\psi(0)|^2 \qquad \Delta E(QQ) = \frac{4\pi\alpha_s}{9m_1m_2}\vec{\sigma}_1\cdot\vec{\sigma}_2|\psi(0)|^2$$

In this scheme hadron masses are given by:

$$m(q_1\overline{q}_2) = m_1 + m_2 + \frac{a}{m_1m_2}\vec{\sigma}_1\cdot\vec{\sigma}_2$$

$$m(q_1q_2q_3) = m_1 + m_2 + m_3 + \frac{a'}{2}\sum_{i>j}\frac{\vec{\sigma}_i\cdot\vec{\sigma}_j}{m_im_j}$$

The hadron masses

For two quarks (or for quark-antiquark):

$$\vec{\sigma}_i = 2\vec{s}_i$$

$$\vec{s} = \vec{s}_1 + \vec{s}_2 \quad \Rightarrow \quad \vec{s}_1 \cdot \vec{s}_2 = \tfrac{1}{2}(\vec{S}^2 - \vec{s}_1^{\,2} - \vec{s}_2^{\,2})$$

Hence the eigenvalues of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ are:

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 4\vec{s}_1 \cdot \vec{s}_2 = 2[S(S+1) - s_1(s_1+1) - s_2(s_2+1)]$$

$$= \begin{cases} +1 & S = 1 \\ -3 & S = 0 \end{cases}$$

Similarly for 3-quark systems:

$$\sum \vec{\sigma}_i \cdot \vec{\sigma}_j = 4\sum \vec{s}_i \cdot \vec{s}_j = 2[S(S+1) - 3s(s+1)]$$

$$= \begin{cases} +3 & S = \frac{3}{2} \\ -3 & S = \frac{1}{2} \end{cases}$$

$$\Rightarrow$$

$$m_\pi = m_u + m_d - \frac{3a}{m_u m_d} \qquad (S = 0)$$

$$m_{K^*} = m_u + m_s + \frac{a}{m_u m_s} \qquad (S = 1)$$

$$(\Delta E)_\Delta = +\frac{3}{m_u^2}\frac{a'}{2}$$

$$(\Delta E)_N = -\frac{3}{m_u^2}\frac{a'}{2}$$

The hadron masses

Using the experimentally measured mass values it is possible to fit the parameters m_u, m_d, m_s, a and a'. The results are:

$$m_u = m_d = 363\ MeV \qquad \frac{a'}{m_u^2} = 100\ MeV$$

$$m_s = 538\ MeV \qquad \frac{a}{m_u^2} = 160\ MeV$$

In this way the agreement with experimental data is of the order of 1 % or better.

Electromagnetic mass differences

A further contribution to hadron mass comes from the electromagnetic interaction. Let us take as an example the baryons in the octet and let us assume that the charge distributions are similar. We expect similar *electromagnetic contributions* Δm:

$$p(uud) \quad \Delta m_p = \Delta m_{\Sigma^+} \quad \Sigma^+(uus)$$

$$\Sigma^-(dds) \quad \Delta m_{\Sigma^-} = \Delta m_{\Xi^-} \quad \Xi^-(dss)$$

$$\Xi^0(uss) \quad \Delta m_{\Xi^0} = \Delta m_n \quad n(udd)$$

Let us add the bare hadron masses and sum these equations:

$$m_p + m_{\Sigma^-} + m_{\Xi^0} = m_{\Sigma^+} + m_{\Xi^-} + m_n \qquad (m_p - m_n) = (m_{\Sigma^+} - m_{\Sigma^-}) + (m_{\Xi^-} - m_{\Xi^0})$$

$$-1.3\ MeV \qquad \underbrace{-8\ MeV \qquad +6.4\ MeV}_{-1.6\ MeV}$$

Coleman-Glashow. Mass differences are associated with isospin symmetry breaking.

Electromagnetic mass differences

Electromagnetic mass differences are due to three effects:

- Difference in mass of the *u* and *d* quarks; since $m_n > m_p$ we expect $m_d > m_u$.
- Coulomb energy difference associated with the electrical energy between pairs of quarks, of the order of:

$$\frac{e^2}{R_0} \approx 2\ MeV$$

- Magnetic energy difference associated with the magnetic moment (hyperfine) interaction between quark pairs:

$$\left(\frac{e\hbar}{mc}\right)^2 \frac{1}{R_0^3} \approx 1 - 2\ MeV$$

Fitting the exact forms of these terms to the data it is found that:

$$m_d - m_u = 2\ MeV$$

The approximate isospin invariance can be associated with the near equality of the *u* and *d* quark masses.

Baryon Magnetic Moments

Baryon magnetic moments can be calculated as the vector sums of the moments of the constituent quarks.

For a Dirac pointlike particle of mass *m* and charge *e*:

$$\vec{\mu} = \frac{e}{2m}\vec{\sigma}$$

As and example let us calculate the magnetic moment of the proton (uud). The two u quarks are in a triplet state. Combining with a further $\left(\frac{1}{2},-\frac{1}{2}\right)$ we get:

$$\psi(\tfrac{1}{2},\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}\underbrace{\chi(1,1)\phi(\tfrac{1}{2},-\tfrac{1}{2})}_{\mu_u+\mu_u-\mu_d} - \sqrt{\tfrac{1}{3}}\underbrace{\chi(1,0)\phi(\tfrac{1}{2},\tfrac{1}{2})}_{\mu_u-\mu_u+\mu_d}$$

$$\mu_p = \tfrac{2}{3}(2\mu_u - \mu_d) + \tfrac{1}{3}\mu_d \qquad \mu_p = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d$$

$$\boxed{\mu_p = 2.79\frac{e}{2m_p}}$$

Baryon Magnetic Moments

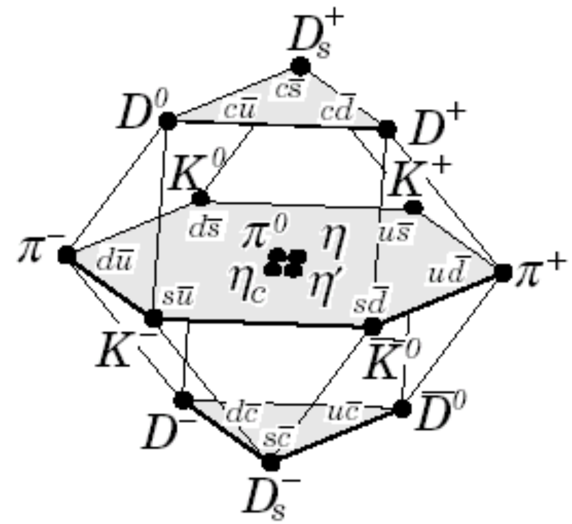
Comparison between predicted and measured magnetic moments for some baryons:

		th	$\exp$
p	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	2.79	2.793
n	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	-1.86	-1.913
Λ	μ_s	-0.58	-0.614
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.68	2.33
Σ^-	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	-1.05	-1.00
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.40	-1.25
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.47	-1.85

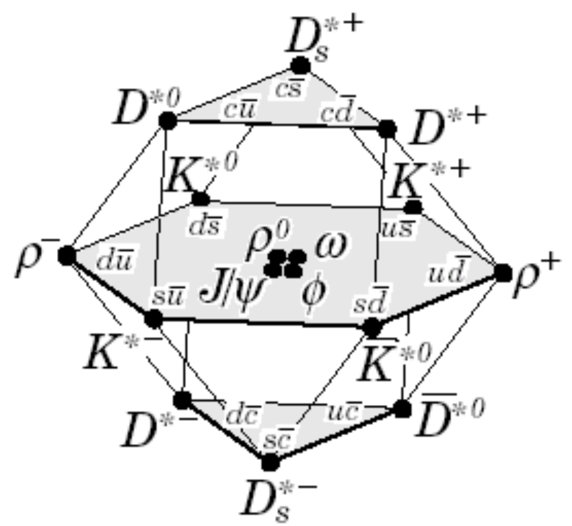
Mesons: SU(4)

SU(4): u, d, s, c $(L = 0)$

$J = 0$



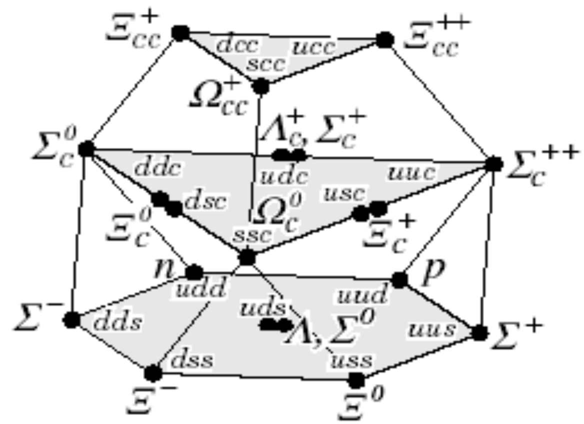
$J = 1$



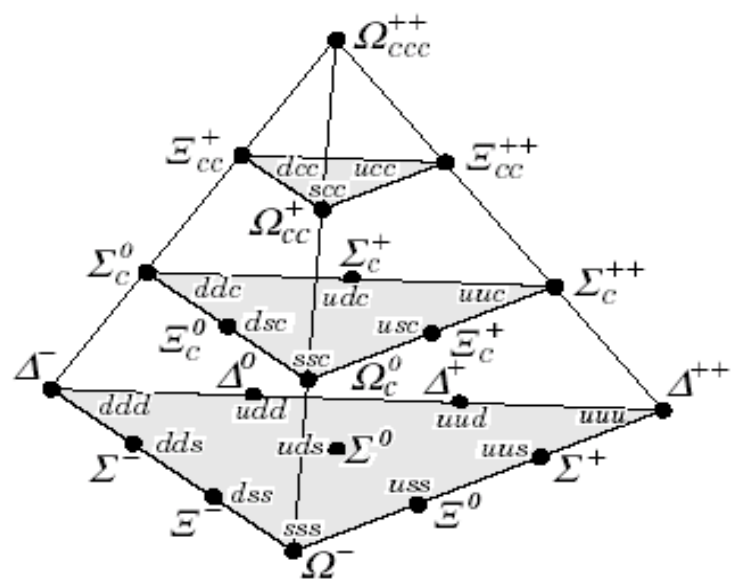
Baryons: SU(4)

SU(4): u, d, s, c $(L=0)$

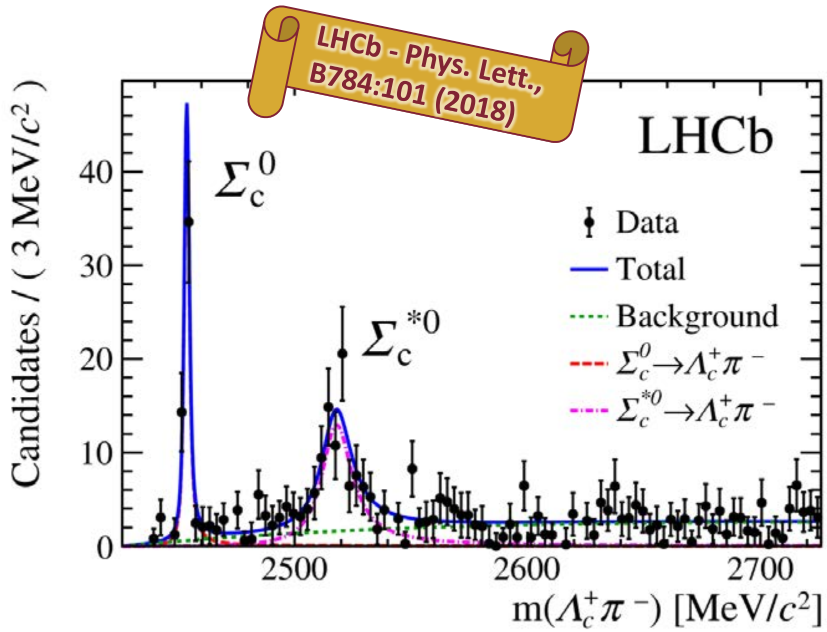
$J=\frac{1}{2}$



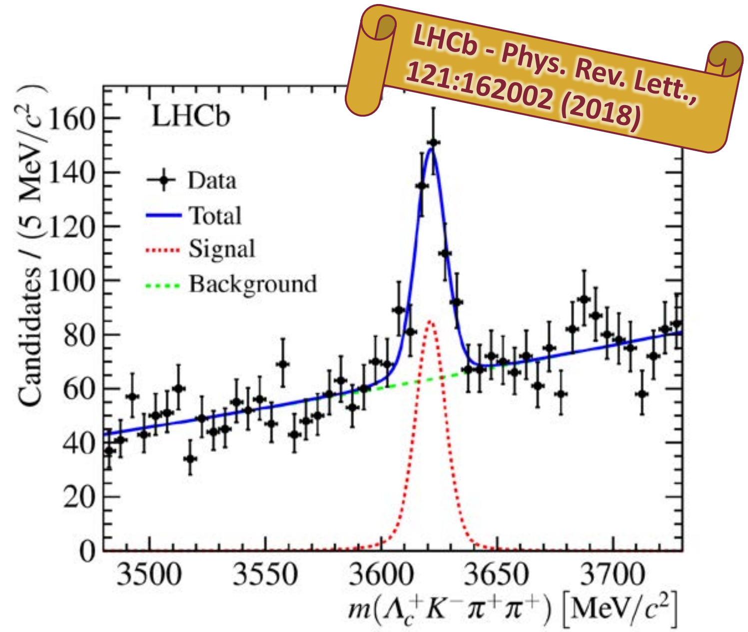
$J=\frac{3}{2}$



The baryons: newcomers!



A - $\boxed{\Sigma_c^0, \Sigma_c^{0*} \rightarrow \Lambda_c^+\pi^-}$



B - $\boxed{\Xi_{cc}^{++} \rightarrow \Lambda_c^+K^-\pi^+\pi^+}$

Recently, the LHCb Collaboration at LHC has realized a nice search for baryons made with heavy quarks.

quark content:

Ξ_{cc}^{++} : ucc;
$\Sigma_c^0, \Sigma_c^{0*}$: ddc;
Λ_c^+ : udc;
K^- : $\bar{u}$s;
$\pi^\pm$: u$\bar{d}$, $\bar{u}$d.

Check conservation of:
✓✓ a. baryon n.
✓✓ b. charge
✓✗ c. charm
– ✗ d. strangeness
Write the Feynman diagrams of the decays

SU(3)

- For the SU(2) symmetry, the generators are the Pauli matrices. The third one is associated to the conserved quantum number I_3.
- For SU(3), the Gell-Mann matrices T_j (j=1-8) are defined (next page).
- The two diagonal ones are associated to the operators of the third component of isospin (T_3) and hypercharge (T_8).
- The eigenvectors $|u\rangle$ $|d\rangle$ $|s\rangle$ are associated with the quarks (u, d, s).

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \psi_1^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \psi_1^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

$$\sigma_2 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad \psi_2^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}; \quad \psi_2^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix};$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \psi_3^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \psi_3^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = I;$$

$$\left[\sigma_i, \sigma_j\right] = 2i\sum_k \varepsilon_{ijk}\sigma_{k.}$$

Pauli matrices and eigenvectors

in the following, some of the properties of SU(3) in group theory: no rigorous math, only results useful for our discussions

SU(3): Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_2 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\lambda_7 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Gell-Mann matrices λ_i

$$T_i = \frac{1}{2}\lambda_i$$

$$\sum_{j=1}^{8} \lambda_j^2 = \frac{16}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ diagonal.}$$

$$U = 1 + \frac{i}{2}\sum_{j=1}^{8} \varepsilon_j \lambda_j \text{ unitary matrix, } \det U = 1.$$

SU(3): eigenvectors

Definition of I_3, Y, quark eigenvectors

and related relations :

$$\hat{T}_3 = \frac{1}{2}\lambda_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad Y = \frac{1}{\sqrt{3}}\lambda_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

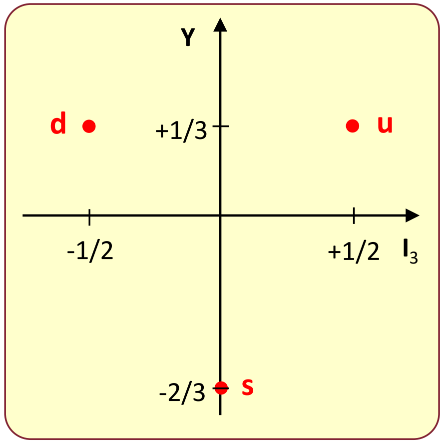
$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad |s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

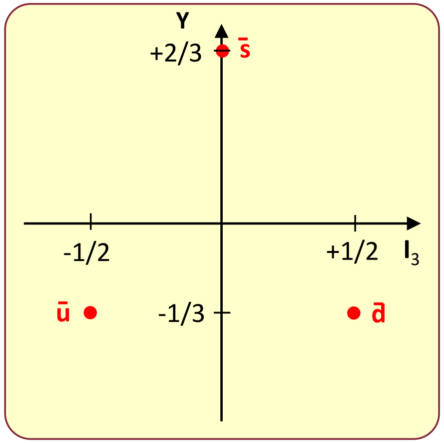
$$\hat{T}_3|u\rangle = +\frac{1}{2}|u\rangle; \qquad \hat{T}_3|d\rangle = -\frac{1}{2}|d\rangle; \qquad \hat{T}_3|s\rangle = 0;$$

$$\hat{Y}|u\rangle = +\frac{1}{3}|u\rangle; \qquad \hat{Y}|d\rangle = +\frac{1}{3}|d\rangle; \qquad \hat{Y}|s\rangle = -\frac{2}{3}|s\rangle;$$

$$\hat{T}_3|\bar{u}\rangle = -\frac{1}{2}|\bar{u}\rangle; \qquad \hat{T}_3|\bar{d}\rangle = +\frac{1}{2}|\bar{d}\rangle; \qquad \hat{T}_3|\bar{s}\rangle = 0;$$

$$\hat{Y}|\bar{u}\rangle = -\frac{1}{3}|\bar{u}\rangle; \qquad \hat{Y}|\bar{d}\rangle = -\frac{1}{3}|\bar{d}\rangle; \qquad \hat{Y}|\bar{s}\rangle = +\frac{2}{3}|\bar{s}\rangle.$$



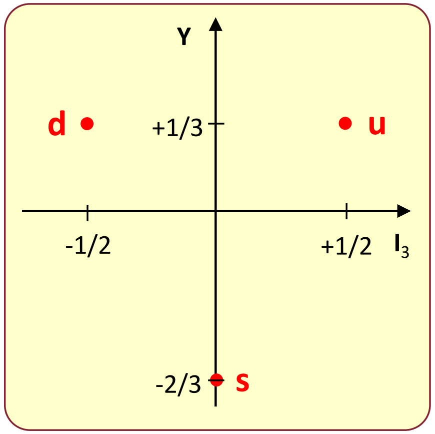


SU(3): operators

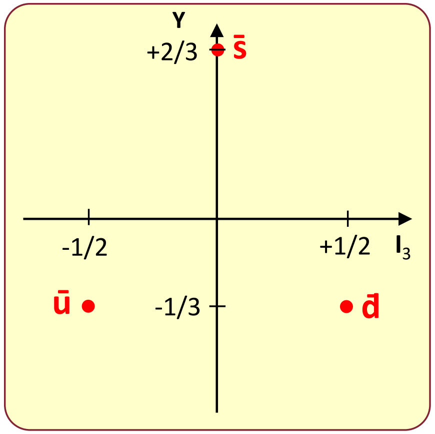
The ladder operators $T_{\pm}$, $U_{\pm}$, $V_{\pm}$:

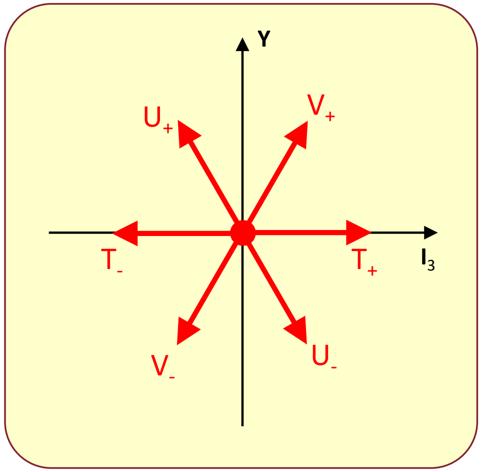
As an example, take V_+ :

$$T_{\pm} = T_1 \pm iT_2; \quad U_{\pm} = T_6 \pm iT_7; \quad V_{\pm} = T_4 \pm iT_5;$$



$$V_+ = T_4 + iT_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \begin{cases} V_+|\bar{u}\rangle = -|\bar{s}\rangle; \\ V_+|\bar{d}\rangle = 0; \\ V_+|\bar{s}\rangle = 0; \end{cases}$$



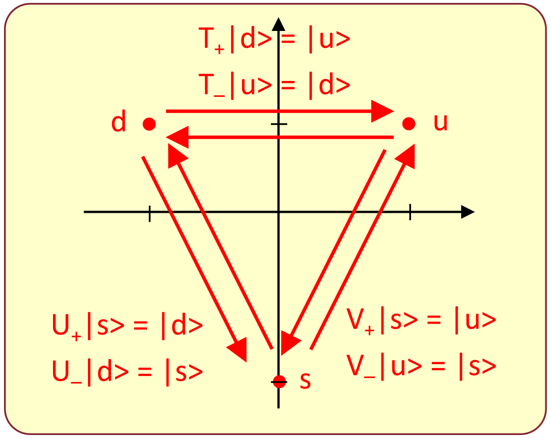


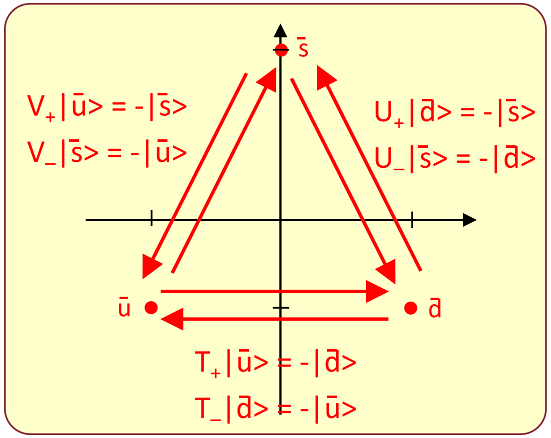
$$V_+|u\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0;$$

$$V_+|d\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0;$$

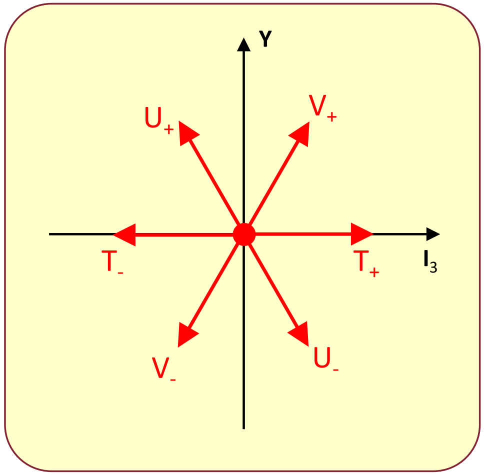
$$V_+|s\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |u\rangle.$$

SU(3): ladder operators





The ladder operators $T_\pm$, $U_\pm$, $V_\pm$.



§ QCD

Exercises

In the K^0 + proton, reaction in which the K^0 have kinetic energy of 800 MeV, explain if strange baryons can be produced. If this is not the case, explain why.

The D_s meson has charm=1, strangeness=1 and spin zero. Write its quark composition, its isotopic splin and its charge. The D^*_s meson has the same quark composition of the D_s but spin 1. The D^*_s decay in $D_s + \pi$ with a B.R. of 6%. Explain which is the ineraction responsible of this decay.

The octect of $1/2^+$ SU(3) baryons is composed by p, n,Σ^+, Σ^0, Σ^-, Ξ^-, Ξ^0, Λ. The lifetime of the Λ is $2.6 \cdot 10^{-10}$s, the one of the Σ is $7.4 \cdot 10^{-20}$s and the one of the Ξ_0 es $2.9 \cdot 10^{-10}$ s. Why he lifetime of the Σ^0 is smaller than the one of the Λ and Ξ^0.

Exercises

ρ^0 (770) and f_2^0(1270) mesons decay by strong interaction in a $\pi^+\pi^-$ pair, and have spin spin 1 and spin 2 respectively

One of the two cannot decay into two neutral pions. Identify who is the meson that cannot decay into $\pi^0\pi^0$ and explain why.

Color: a new quantum number

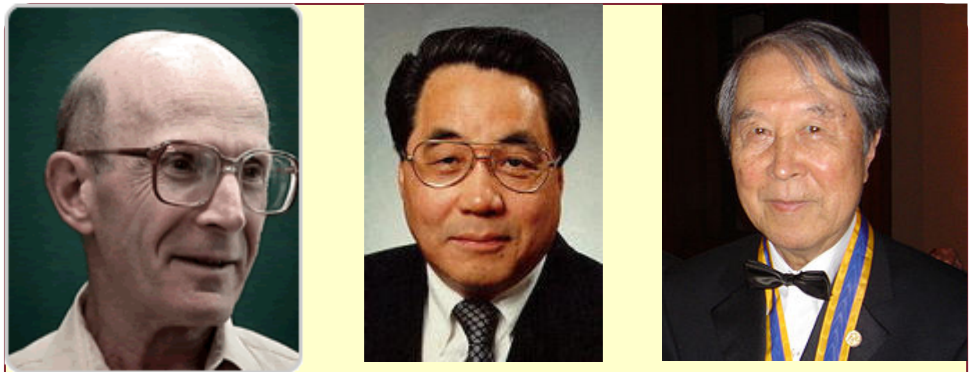
Consider the Δ^{++} resonance:

- $J^P=3/2^+$ (measured);

$\rightarrow$ quark/spin content [*no choice*]:
$|\Delta^{++}\rangle = |\, u\Uparrow\, u\Uparrow\, u\Uparrow \rangle$

- wave function :

$\boxed{\psi(\Delta^{++}) = \psi_{space} \times \psi_{flavor} \times \psi_{spin}}$ NO !!!



Oskar W. Greenberg

Moo-Young Han (한무영)

Yoichiro Nambu (南部 陽一郎, *Nambu Yōichirō*)

Why "NO" ?
Consider the symmetry of $\psi(\Delta^{++})$:

- it is lightest uuu state $\rightarrow$ L = 0

$\rightarrow \psi_{space}$ symmetric;

$\rightarrow \psi_{flavor}$ and ψ_{spin} symmetric;

$\rightarrow \psi(\Delta^{++})$ = sym.$\times$sym.$\times$sym. = sym.

... but the Δ^{++} is a fermion ...NO

Anomaly : the Δ^{++} is a spin $^3/_2$ fermion and its function MUST be antisymmetric for the exchange of two quarks (Pauli principle). However, this function is the product of three symmetric functions, and therefore is symmetric $\rightarrow$ **???**.

The solution was suggested in 1964 by Greenberg, later also by Han and Nambu. They introduced a **new quantum number** for strongly interacting particles, composed by quarks : the **COLOR**.

Color: why and how

The idea:

1. quarks exist in three colors (say Red, Green and Blue, like the TV screen(*);
2. they sum like in a TV-screen : e.g. when RGB are all present, the screen is white;
3. the "anticolor" is such that, color + anticolor gives white (e.g. $\bar{R}$ = G + B);
4. anti-quarks bring ANTI-colors (see previous point);
5. Mesons and Baryons, which are made of quarks, are white and have no color: they are a "**color singlet**".

Therefore, we have to include the color in the complete wave function; e.g. for Δ^{++} :

$$\psi(\Delta^{++}) = \psi_{space} \times \psi_{flavor} \times \psi_{spin} \times \psi_{color}$$

$$\psi_{color} = (1/\sqrt{6})\,(u_r^1 u_g^2 u_b^3 + u_g^1 u_b^2 u_r^3 + u_b^1 u_r^2 u_g^3 - u_g^1 u_r^2 u_b^3 - u_r^1 u_b^2 u_g^3 - u_b^1 u_g^2 u_r^3)$$

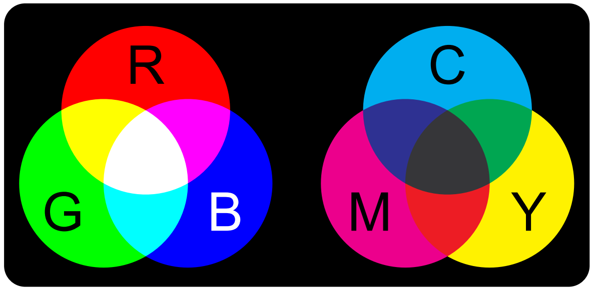
(where u_r, u_g, u_b are the color functions for u quarks of red, green, blue type).

Then ψ_{color} is antisymmetric for the exchange of two quarks and so is the global wave function.

The introduction of the color has many other experimental evidences and theoretical implications, *which we will discuss in the following*.

____

(*) *however, these colors are in no way similar to the real colors; therefore the names "red-green-blue" are totally irrelevant.*



Summary: Simmetries and Multiplets

1. Since the strong interactions conserve isotopic spin ("$\mathbb{I}$"), hadrons gather in I-multiplets. Within each multiplet, the states are identified by the value of I_3.

2. If no effect breaks the symmetry, the members of each multiplet would be mass-degenerate. The electromagnetic interactions, which do not respect the I-symmetry, split the mass degeneration (at few %) in I-multiplets.

3. Since the strong interactions conserve I, $\mathbb{I}$-operators must commute with the strong interactions Hamiltonian ("$\mathbb{H}_s$") and with all the operators which in turn commute with $\mathbb{H}_s$.

4. Among these operators, consider the angular momentum $\mathbb{J}$ and the parity $\mathbb{P}$. As a result, all the members of an isospin multiplet must have the same spin and the same parity.

5. $\mathbb{H}_s$ is also invariant with respect to unitary representations of SU(2). The quantum numbers which identify the components of the multiplets are as many as the number of generators, which can be diagonalized simultaneously, because are mutually commuting. This number is the *rank* of the Group. In the case of SU(2) the rank is 1 and the operator is $\mathbb{I}_3$.

6. Since $[\mathbb{I}_j\ ,\ \mathbb{I}_k] = i\varepsilon_{jkm}\mathbb{I}_m$, each of the generators commutes with $\mathbb{I}^2$:

   $\mathbb{I}^2 = \mathbb{I}_1{}^2 + \mathbb{I}_2{}^2 + \mathbb{I}_3{}^2$.

   Therefore $\mathbb{I}^2$, obviously hermitian, can be diagonalized at the same time as $\mathbb{I}_3$.

(continue …)

Summary: Simmetries and Multiplets

7. The eigenvalues of $\mathbb{I}$ and $\mathbb{I}_3$, can "tag" the eigenvectors and the particles.
8. This fact gives the possibility to regroup the states into multiplets with a given value of *I*.
9. We can generalize this mechanism from the isospin case to any operator : if we can prove that $\mathbb{H}$ is invariant for a given kind of transformations, then:
   a. look for an appropriate symmetry group;
   b. identify its irreducible representations and derive the possible multiplets,
   c. verify that they describe physical states which actually exist.
10. This approach suggested the idea that Baryons and Mesons are grouped in two octets, composed of multiplets of isotopic spin.
11. In reality, since the differences in mass between the members of the same multiplet are ~20%, the symmetry is "broken" (i.e. approximated).
12. Since the octets are characterized by two quantum numbers (I_3 and Y), the symmetry group has rank = 2, i.e. two of the generators commute between them.
13. We are interested in the "irreducible representations" of the group, such that we get any member of a multiplet from everyone else, using the transformations.

(… continue …)

Summary: Simmetries and Multiplets

14. The non-trivial representation (non-trivial = other than the Singlet) of lower dimension is called "Fundamental representation".

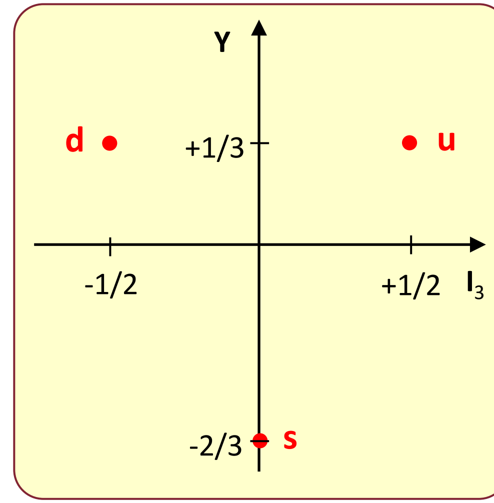
15. In SU(3) there are eight symmetry generators. Two of them are diagonal and associated to I_3 and Y.

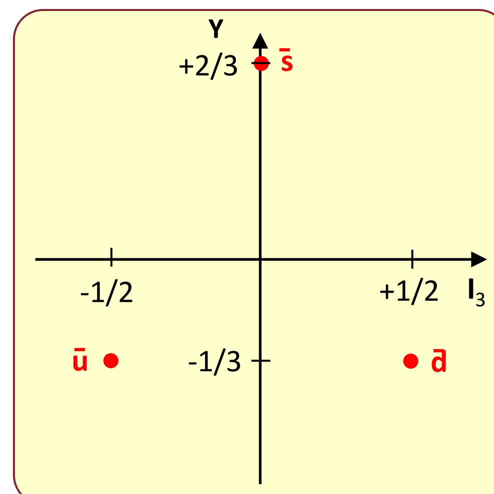
16. The fundamental representations are triplets ($\rightarrow$ quarks), from which higher multiplets ($\rightarrow$ hadrons) are derived :

    mesons: $3 \otimes \bar{3} = 1 \oplus 8$;

    baryons: $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$.

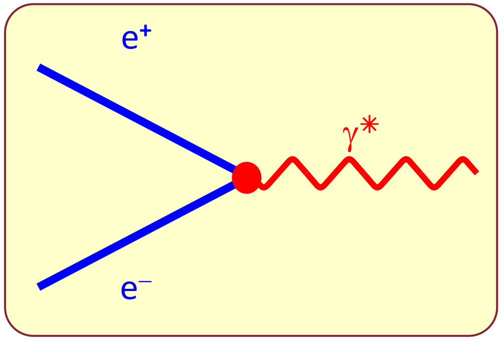
17. This purely mathematical scheme has two relevant applications:
    a. "flavour SU(3)", $SU(3)_F$ with Y_F and I_{3F} for the quarks uds – this symmetry is approximate (i.e. "broken");
    b. "color SU(3)", $SU(3)_C$ with Y_C and I_{3C} for the colors rgb; this symmetry is exact.





Collision e^+e^-: initial state

- At low energy$^{(*)}$, the main processes happen with annihilation into a virtual γ^*.
- The initial state is :
  - charge = 0;
  - lepton (+ baryon + other additive) number = 0;
  - spin = 1 ("γ^*");



- CM kinematics :
  - e^+ [E, p, 0, 0];
  - e^- [E, −p, 0, 0];
  - γ^* [2E,0, 0, 0];
  - $m(\gamma^*) = \sqrt{s} = 2E$ [virtual photon, short lived].

Remember…Mandelstam variables

- in absence of polarization, the cross sections of a process "X" does NOT depend on the azimuth φ :

$$\frac{d\sigma_{"X"}}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma_{"X"}}{d\cos\theta} = \frac{s}{4\pi}\frac{d\sigma_{"X"}}{dt}.$$

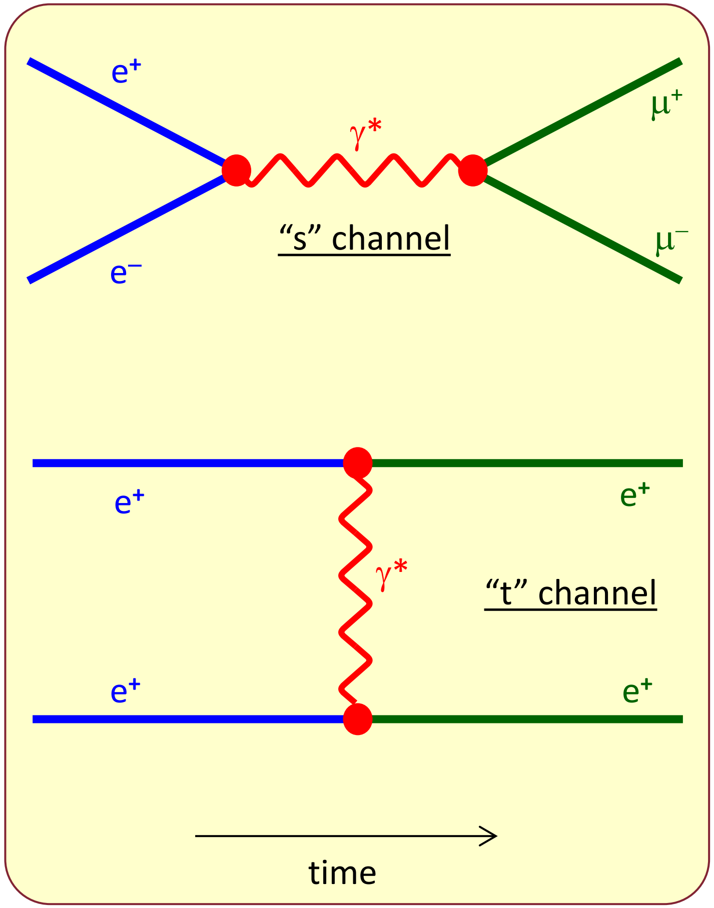
- for $m^2 << s$, if $\mathcal{M}_{"X"}$ is the matrix element of the process$^{(*)}$:

$$\frac{d\sigma_{"X"}}{dt} = \frac{\left|\mathcal{M}_{"X"}\right|^2}{16\pi s^2}.$$

- in lowest order QED, if $m^2 << s$:

$$\frac{d\sigma_{"X"}}{d\cos\theta} = \frac{\left|\mathcal{M}_{"X"}\right|^2}{32\pi s} = \frac{\alpha^2}{s} f(\cos\theta).$$

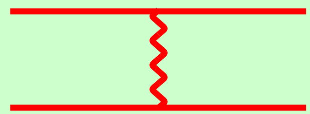
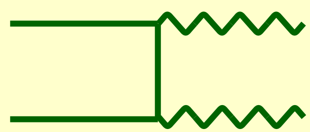
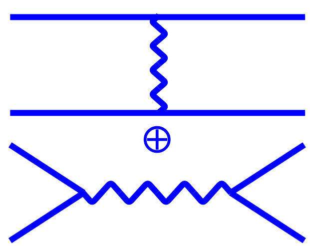
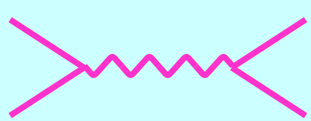
- when $\theta \to 0$, $\cos\theta \to 1$:
  - s-channel : $f(\cos\theta) \to$ constant;
  - t-channel : $f(\cos\theta) \to \infty$.



(*) also by dimensional analysis :
$[c = \hbar = 1]$, $[\sigma] = [\ell^2]$; $[t] = [s] = [\ell^{-2}]$;
therefore, in absence of any other dimensional scale,
σ [and $d\sigma/d\Omega$] = [number] × $1/s$.

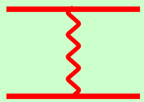
Collision e^+e^-: QED cross-section

Consider some QED processes in lowest order [$\sqrt{s} << m_Z$, only γ^* exchange] :

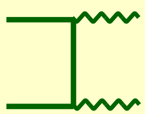
➢ $e^\pm e^\pm \to e^\pm e^\pm$		$\frac{d\sigma(e^\pm e^\pm \to e^\pm e^\pm)}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \times \left(\frac{3+\cos^2\theta}{1-\cos^2\theta}\right)^2;$
➢ $e^+e^- \to \gamma\gamma$		$\frac{d\sigma(e^+e^- \to \gamma\gamma)}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \times \frac{1+\cos^2\theta}{1-\cos^2\theta};$
➢ $e^+e^- \to e^+e^-$		$\frac{d\sigma(e^+e^- \to e^+e^-)}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \times \left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2;$
➢ $e^+e^- \to \mu^+\mu^-$		$\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \times \left(1+\cos^2\theta\right);$

Collision e^+e^-: QED $d\sigma/d\cos\theta$

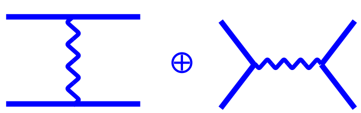
$$\frac{d\sigma(e^\pm e^\pm \to e^\pm e^\pm)}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \times \left(\frac{3+\cos^2\theta}{1-\cos^2\theta}\right)^2;$$



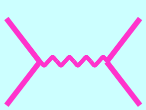
$$\frac{d\sigma(e^+e^- \to \gamma\gamma)}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \times \frac{1+\cos^2\theta}{1-\cos^2\theta};$$

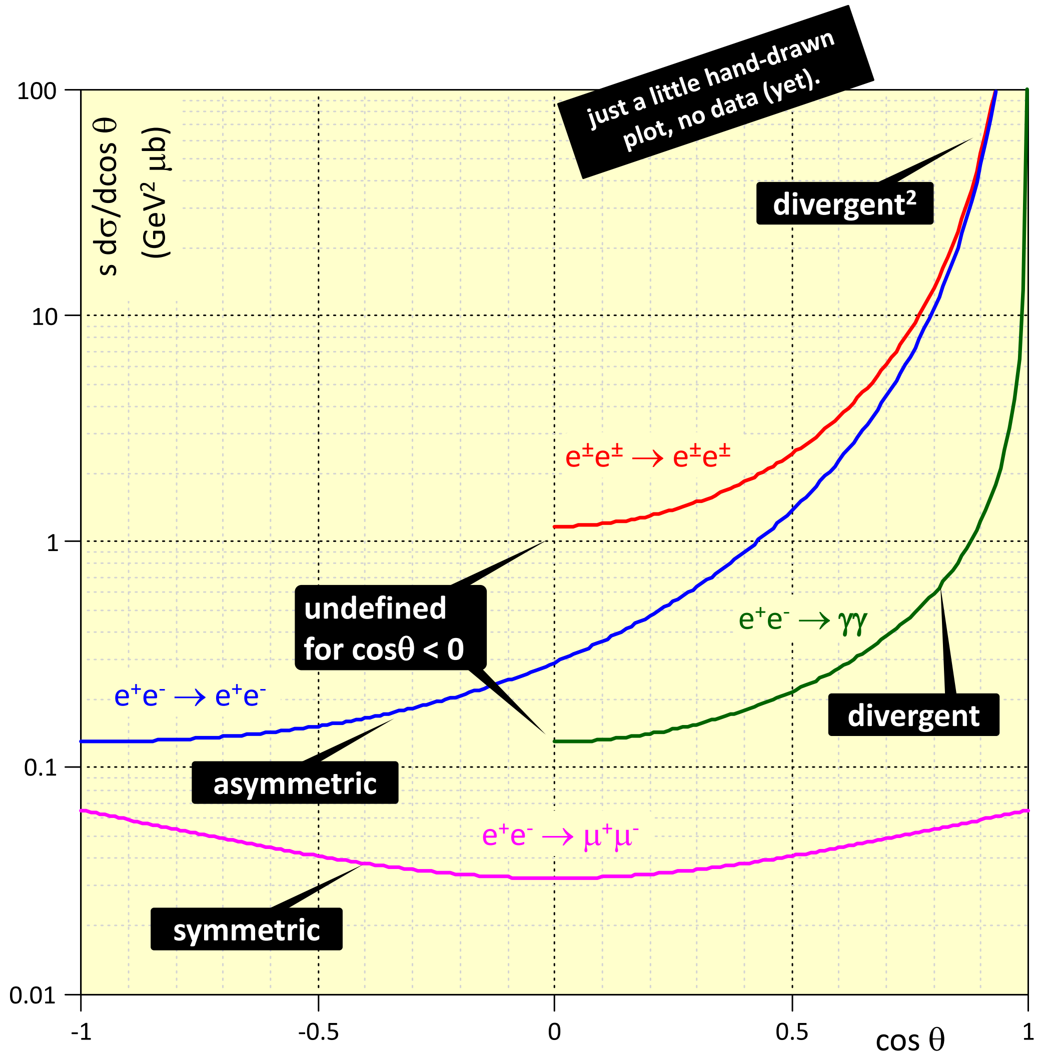


$$\frac{d\sigma(e^+e^- \to e^+e^-)}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \times \left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2;$$



$$\frac{d\sigma(e^+e^- \to \mu^+\mu^-)}{d\cos\theta} = \frac{\pi\alpha^2}{2s} \times \left(1+\cos^2\theta\right);$$





Collision e^+e^-: $e^+e^- \rightarrow \mu^+ \mu^-$/qqbar

- kinematics, computed in CM sys, $\sqrt{s} >> m_e, m_\mu$:

e^+ $(E, \quad p, \quad 0, \; 0)$;

e^- $(E, \quad -p, \quad 0, \; 0)$;

μ^+ $(E, \; p\cos\theta, \; p\sin\theta, \; 0)$;

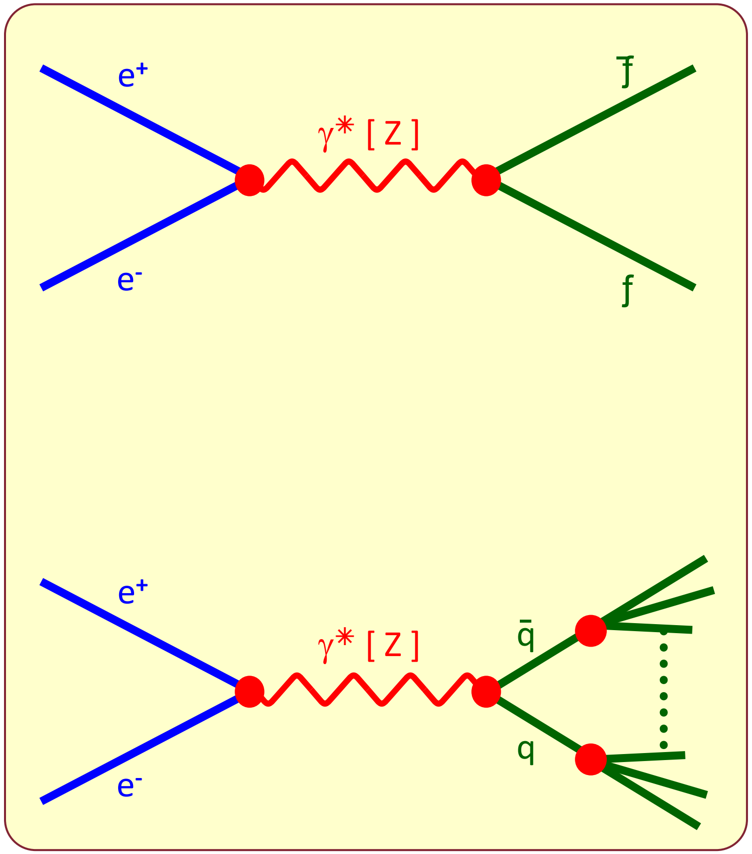
μ^- $(E, -p\cos\theta, -p\sin\theta, \; 0)$;

$p \approx E = \sqrt{s}/2$;

$\vec{p}(e^+) \cdot \vec{p}(\mu^+) \approx E^2 \cos\theta \approx s\cos\theta / 4$;

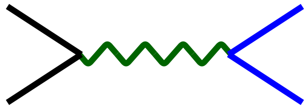
$p(e^+)\, p(\mu^+) \approx E^2 (1 - \cos\theta) = s \sin^2(\theta/2) = -t$;

- the case $e_+e_- \rightarrow q\bar{q}$ is similar at parton level; however free (anti-)quarks do NOT exist → quarks hadronize, producing collimated jets of hadrons



Collisions e^+e^- : $\sigma(e^+e^- \to \mu^+\mu^-, q\bar{q})$

- $e^+e^- \to \mu^+\mu^-$



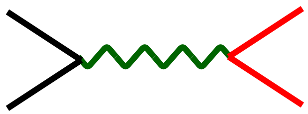
$$\sigma_{\mu\mu} = \int_{-1}^{1} d\cos\theta \left[\frac{d\sigma_{\mu\mu}}{d\cos\theta}\right] = \frac{\pi\alpha^2}{2s}\int_{-1}^{1} d\cos\theta\left(1+\cos^2\theta\right) =$$

$$= \frac{4\pi\alpha^2}{3s} = \frac{86.8\text{ nb}}{s[\text{GeV}^2]} = \frac{21.7\text{ nb}}{E^2_{beam}[\text{GeV}^2]}.$$

$[1+\cos^2\theta] = P_1^{Legendre}(\cos\theta)$

[spin 1 $\to$ 2 spin ½]

- $e^+e^- \to q\bar{q}$

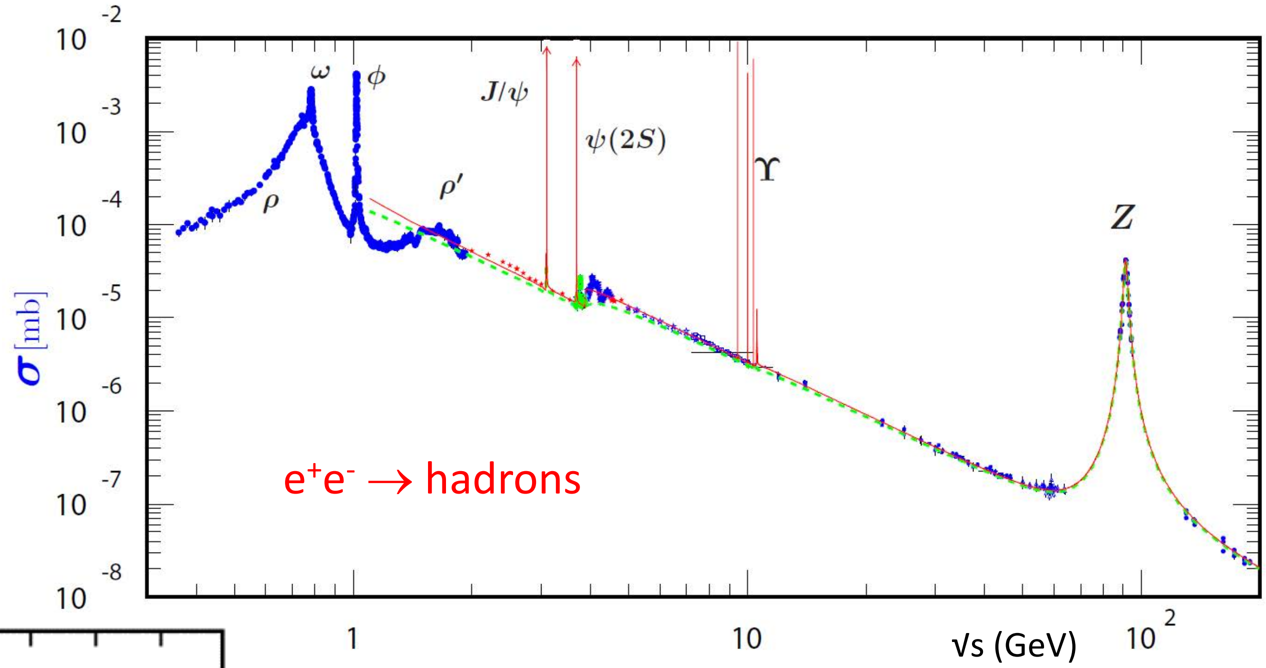


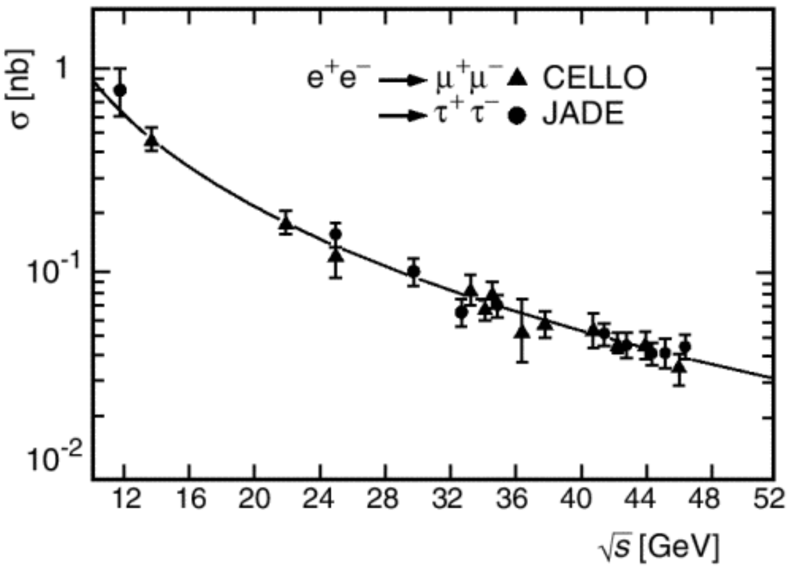
$$\frac{d\sigma_{q\bar{q}}}{d\cos\theta} = \frac{d\sigma_{\mu\mu}}{d\cos\theta} \times c_f e_f^2 = \frac{\pi\alpha^2}{2s} c_f e_f^2 \left(1+\cos^2\theta\right); \quad c_f = \begin{cases} 3 & \text{quarks} \\ 1 & \text{leptons} \end{cases} \quad [\text{color}]$$

$$\sigma_{q\bar{q}} = \sigma_{\mu\mu} c_f e_f^2 = \frac{4\pi\alpha^2}{3s} c_f e_f^2; \quad e_f = \begin{cases} 1 & \text{leptons} \\ 2/3 & u\ c\ t \\ -1/3 & d\ s\ b \end{cases} \quad [\text{charge}].$$

Collisions e^+e^- : $\sigma_{large\sqrt{s}}(e^+e^- \to \mu^+\mu^-, q\bar{q})$

$$\sigma_{\mu\mu} = \frac{4\pi\alpha^2}{3s} = \frac{86.8\ \text{nb}}{s[\text{GeV}^2]} = \frac{21.7\ \text{nb}}{E^2[\text{GeV}^2]}.$$



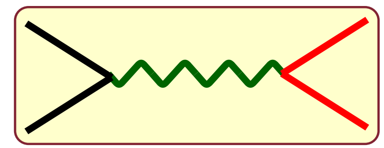


- the continuum, for $0.5 \le \sqrt{s} \le 50$ GeV, agrees well with the predicted $1/s$ [the line in log-log scale];
- + resonances $q\bar{q}$ [the bumps];
- for $\sqrt{s} > 50$ GeV it is dominated by the Z formation in the s-channel.

Collisions e^+e^- : $R = \sigma(q\bar{q})/\sigma(\mu^+\mu^-)$

- define the quantity, both simple conceptually and easy to measure:

$$\boxed{R = \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\sum_{\text{quarks}} e_i^2 = R\left(\sqrt{s}\right);}$$



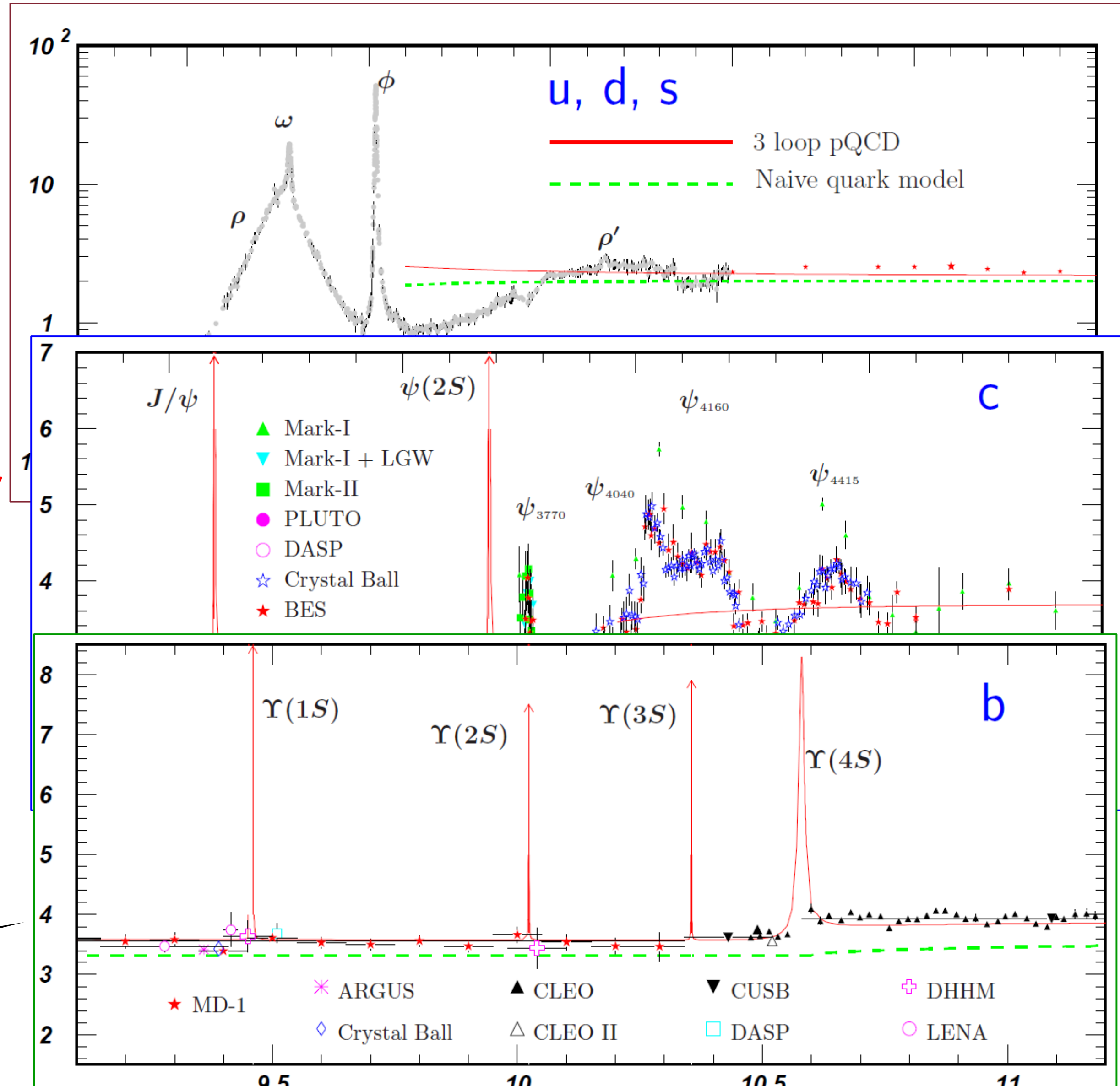
- sum over all the quarks, produced at energy $\sqrt{s}$ (i.e. $2m_q < \sqrt{s}$) :
  - $0 < \sqrt{s} < 2\,m_c$: $R = R_{uds} = 3 \times [\,(^2/_3)^2 + (^{-1}/_3)^2 + (^{-1}/_3)^2\,] = 2;$
  - $2\,m_c < \sqrt{s} < 2\,m_b$: $R = R_{udsc} = R_{uds} + 3 \times (^2/_3)^2 = 3 + {}^1/_3;$
  - $2\,m_b < \sqrt{s} < 2\,m_t$: $R = R_{udscb} = R_{udsc} + 3 \times (^{-1}/_3)^2 = 3 + {}^2/_3;$
  - ~~$2\,m_t < \sqrt{s} < \infty$: $R = R_{udscbt} = R_{udscb} + 3 \times (^2/_3)^2 = 5$~~;

Collisions e⁺e⁻ : R vs √s (small √s)

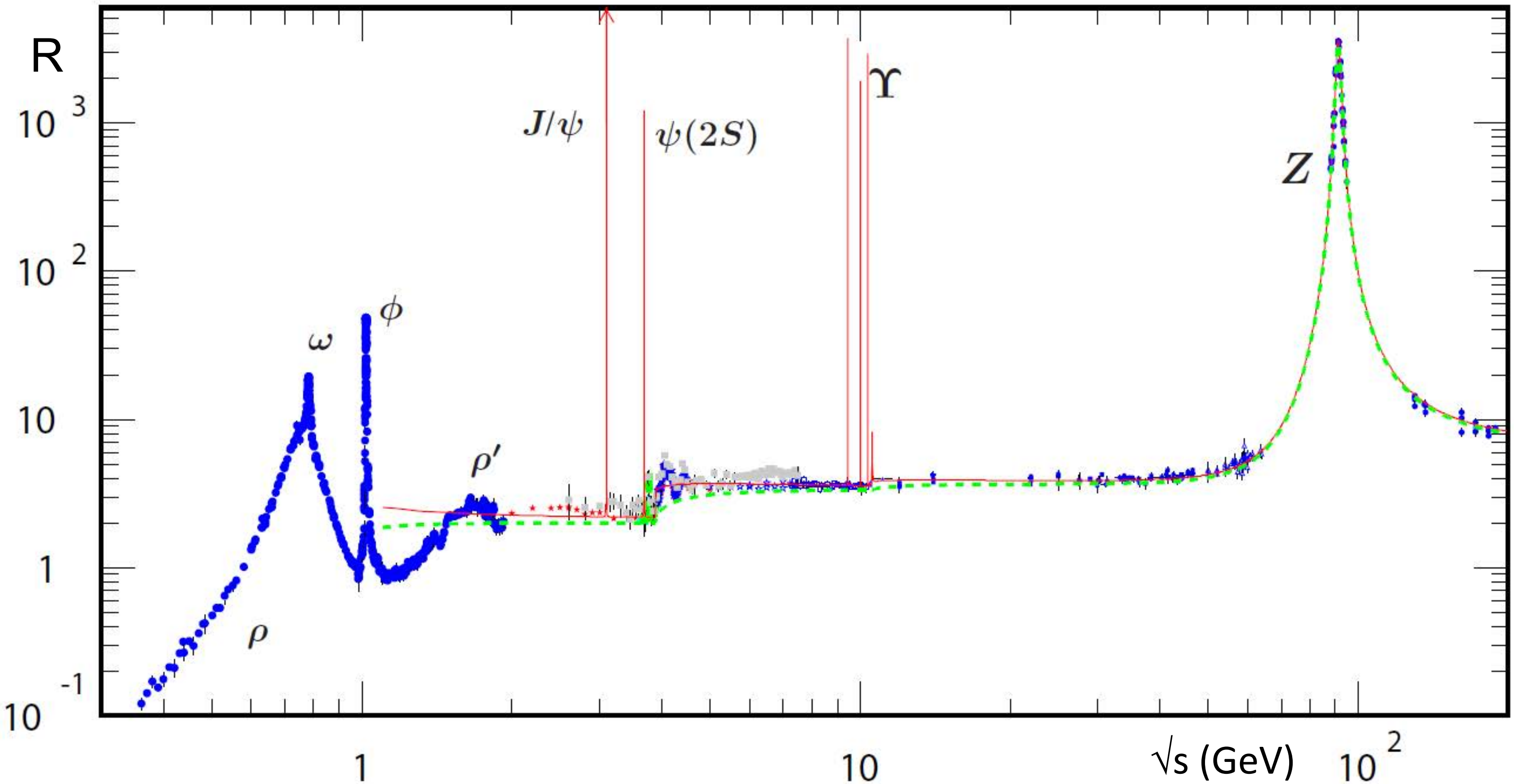
Plot R vs √s (=2E):

- resonances $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ at 1-2 GeV (only those with $J^P=1^-$) (→"vector dominance");
- step at $2m_c$ (J/ψ);
- step at $2m_b$ (Υ);
- slow increase at √s > 50 GeV (Z, next slide);
- [lot of effort required, as demonstrated by the number of detectors and accelerators];
- strong evidence for the color (factor 3 necessary).

plots from [PDG, 588]



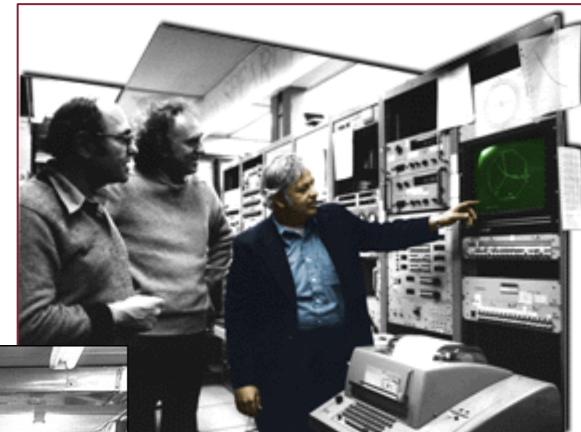
Collisions e^+e^- : R vs √s (large √s)

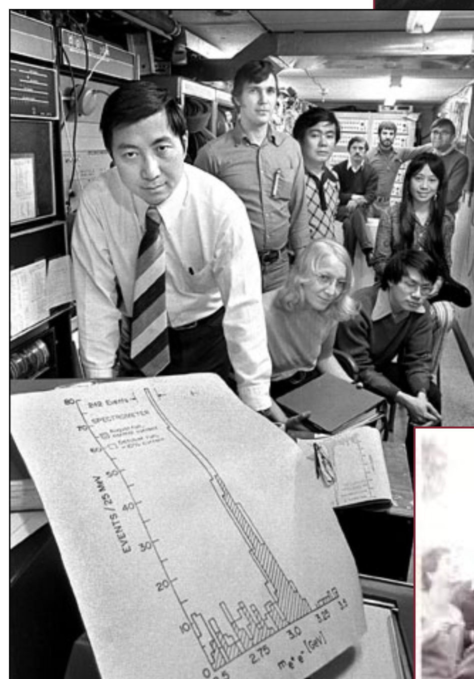


- The full range 200 MeV < √s < 200 GeV (3 orders of magnitude !!!).
- For √s > 50 GeV new phenomenon: electroweak interactions and the Z pole.

The November revolution

- The u,d,s quarks have not been predicted; in fact the mesons and baryons have been discovered, and later interpreted in terms of their quark content [§ 1];
- ***Some theoreticians had foreseen another quark, based on** (no $K^0 \to \mu^+\mu^-$)**, but people did not believe it.***
- In November 1974, the groups of Burton Richter (SLAC) and Samuel Ting (Brookhaven) discovered simultaneously a new state with a mass of $\approx$ 3.1 GeV and a tiny width, much smaller than their respective mass resolution.
- Ting & coll. had the name "J", while Richter & coll. called it "ψ". Today's name is "J/ψ".
- We split the discussion : start with the hadronic experiment.
- The width was measured, after some time, to be 0.087 MeV, a surprisingly small value for a resonance of 3 GeV mass.





the two experiments are quite different: we will review first the "J" and then the "ψ".

The November revolution: J

- The group of Ting at the AGS proton accelerator measured the inclusive production of e^+e^- pairs in interactions of 30 GeV protons on a plate of beryllium :

  $$p\,Be \to e^+e^-X.$$

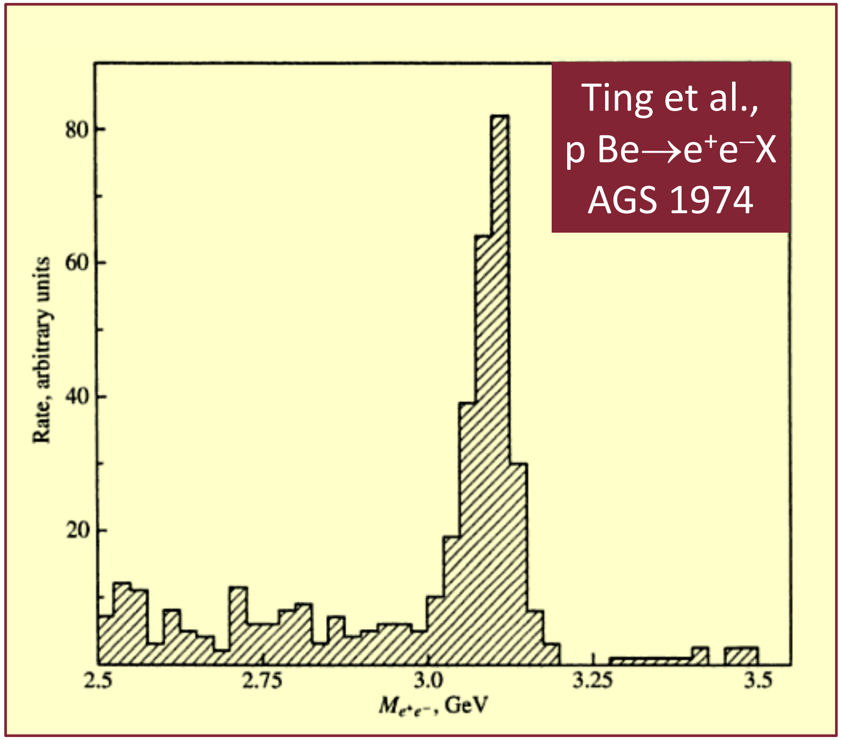
- The detector was designed to search for high mass resonances with $J^P = 1^-$ ($= \gamma$), decaying into (e^+e^-) pairs.
- They were very clever in minimizing the multiple scattering $\to$ the resolution for the invariant mass was good:

  $$\Delta m(e^+e^-) \approx 20 \text{ MeV}.$$

- This resolution allowed for a much higher sensitivity wrt another previous exp. (Leon Lederman), which studied $\mu^+\mu^-$ pairs in the same range. Lederman had a "shoulder" in $d\sigma/dm(\mu^+\mu^-)$, but no conclusive evidence [next slide].

- Ting called the new particle "J", because of the e.m. current.

Measured quantum numbers of the J:
- mass ~3.1 GeV;
- width << 20 MeV (upper limit, not meas.);
- charge = 0;
- $J^P = 1^-$;
- no isospin, Γ, other decay modes ...

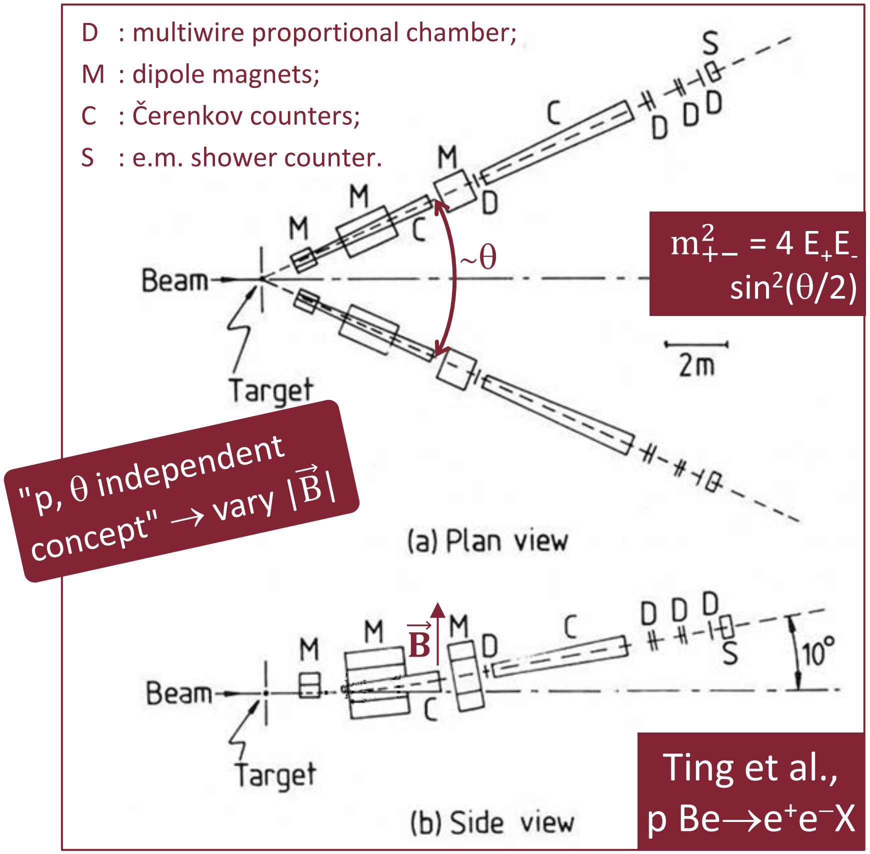


The November revolution: the J experiment

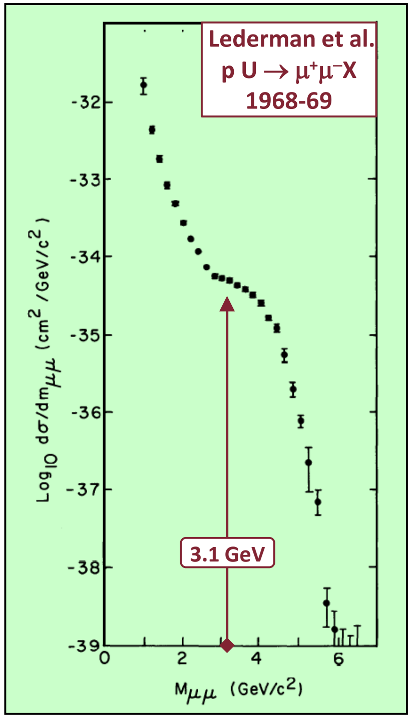
- The Ting experiment used a two arm magnetic spectrometer, to measure separately the electron and the positron.
- Ting (and also Lederman) studied the Drell-Yan process: hadron collisions $\to \gamma^* \to \ell^+\ell^-$ (Ting: e^+e^- / Lederman: $\mu^+\mu^-$).
- Leptonic events are rare $\to$ very intense beams (2×10^{12} ppp $^{(*)}$) $\to$ high rejection power ($\sim10^8$) to discard hadrons, that can fake e^+e^- or $\mu^+\mu^-$.
- Advantage in the $\mu^+\mu^-$ case: μ penetration $\to$ select leptons from hadrons with a thick absorber in a large solid angle $\to$ larger acceptance, higher counting rate.
- Disadvantage : thick absorber $\to$ multiple scattering $\to$ worst mass resolution.

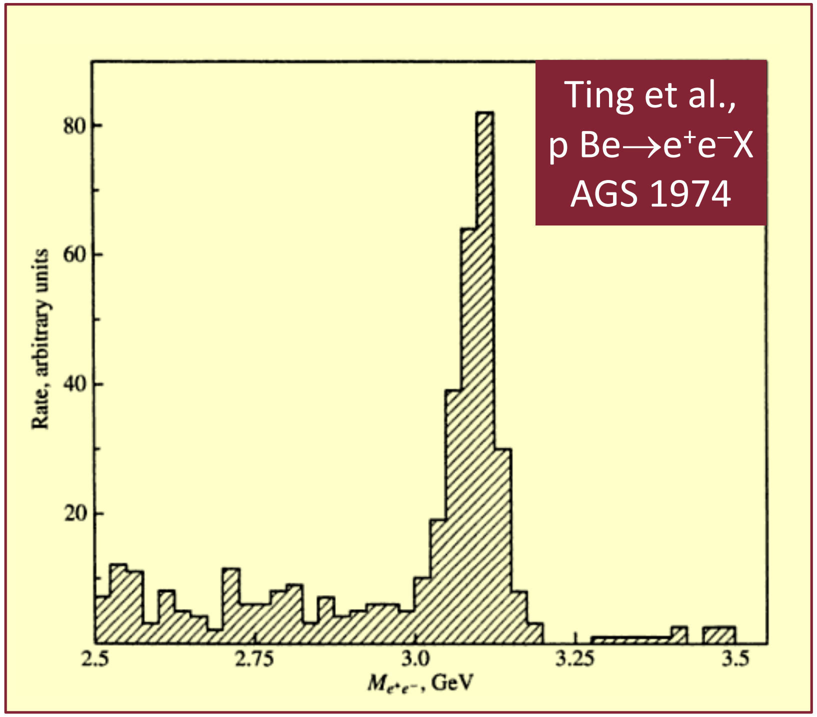
$^{(*)}$ "ppp" : "particles (or protons) per pulse", i.e. once per accelerator cycle every few seconds; it is the typical figure of merit of a beam from an accelerator.

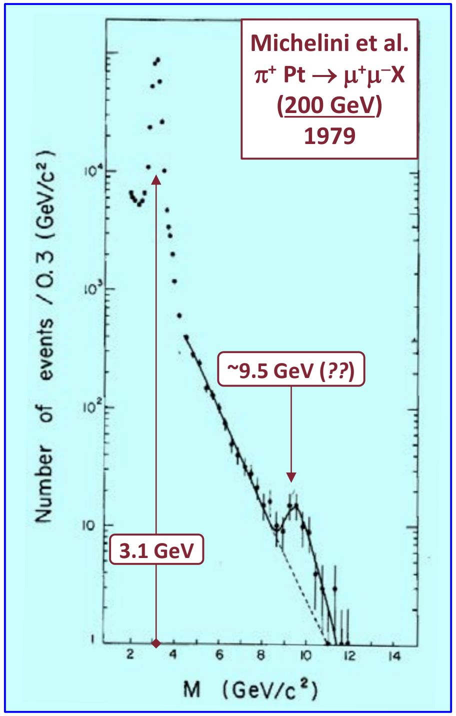
- Benefit in the e^+e^- case: electron identification with Čerenkov counter(s) + calorimeters $\to$ simpler setup.
  Disadvantage : small instrumented solid angle $\to$ smaller yield.



The November revolution: the search for



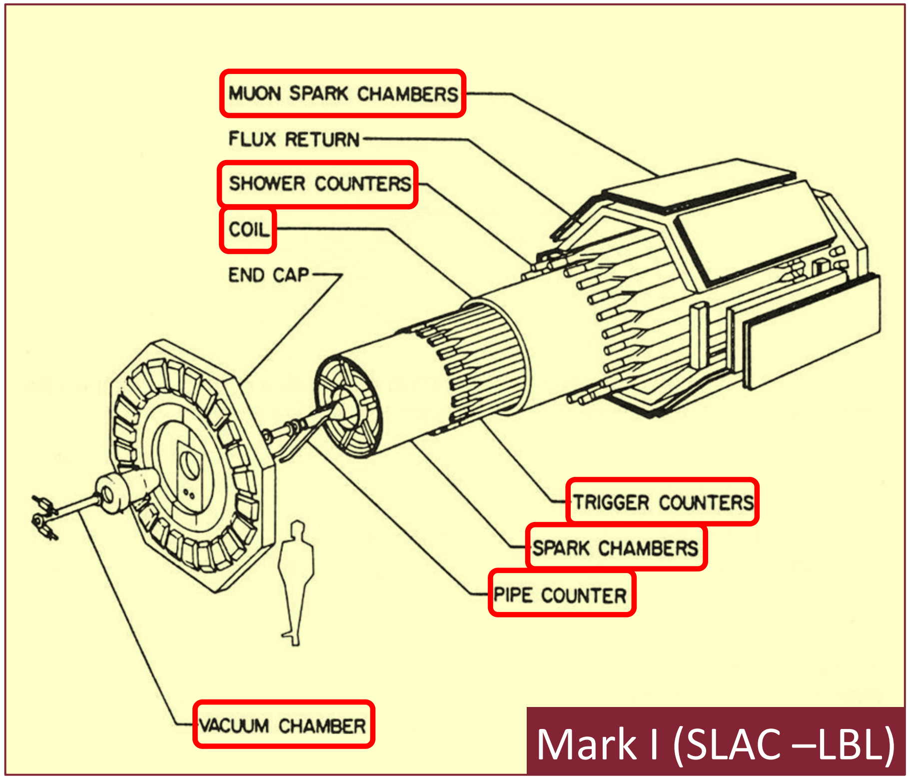




The November revolution: Mark I @ SLAC

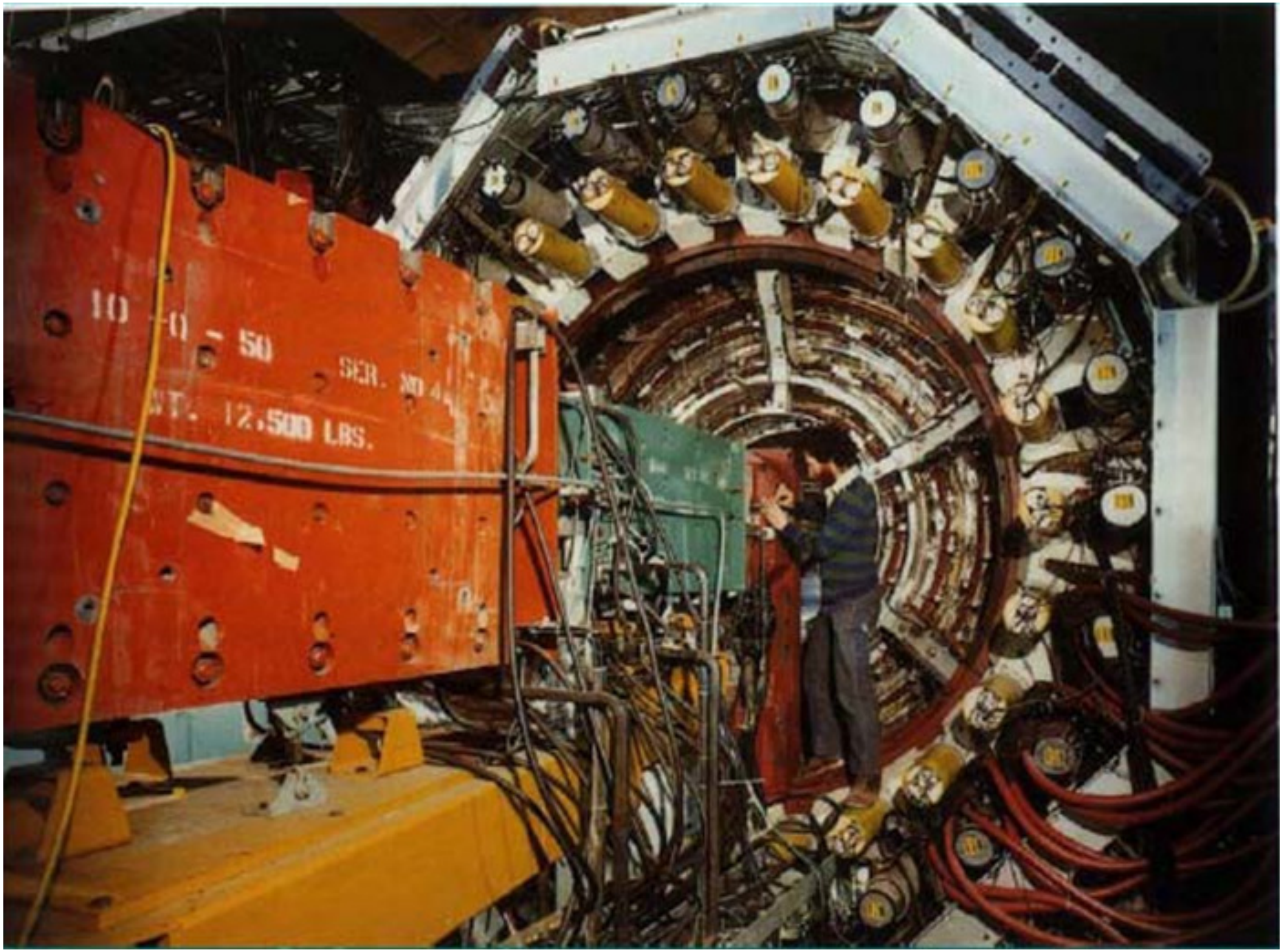
[back to 1974 : they did not know]

- Mark I at the e^+e^- collider SPEAR was studying collisions at $\sqrt{s} = 2.5 \div 7.5$ GeV.
- The detector was made by a series of concentrical layers ("onion shaped").
- Starting from the beam pipe :
  - magnetostrictive spark chamber (tracking),
  - time-of-flight counters (particles' speed + trigger),
  - coil (solenoidal magnetic field, 4.6 kG),
  - electromagnetic calorimeter (energy and identification of γ's and $e^\pm$'s),
  - proportional chambers interlayered with iron plates (identification of $\mu^\pm$'s).



Mark I (SLAC –LBL)

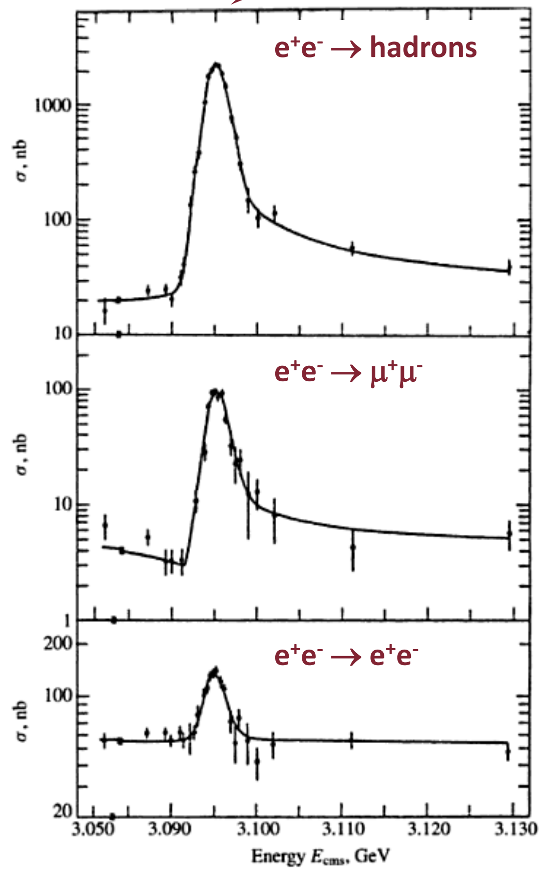
The November revolution: Mark I @ SLAC



The November revolution: ψ

- In 1974, up to the highest available energies, R = $\sigma(e^+e^- \to \text{hadrons}) / \sigma(e^+e^- \to \mu^+\mu^-) \approx 2$.
- Measurements at the Cambridge Electron Accelerator (CEA, Harvard) in the region of energies of SPEAR had found R ≅ 6 (a mixture of continuum and resonances). Also ADONE at LNF, which could reach an energy just sufficient, was not pushed to its max energy [*At the time the large amount of information carried by R was not completely clear*].
- At the novel Collider SPEAR, the scanning in energy was performed in steps of 200 MeV.
- The measured cross-section appeared to be a constant, NOT with expected trend $\propto 1/s$.
- When a drastic reduction in the step (200 $\to$ 2.5 MeV) increased the "resolving power", a resonance appeared, with width compatible with the beam dispersion (even compatible with a δ-Dirac).
- The particle was called "ψ".

inside Mark I acceptance and normalized to Bhabha.



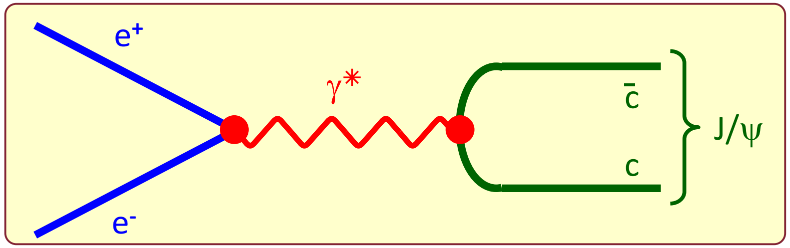
The J/ψ properties

- After some discussion, the correct interpretation emerged :
  - the resonance, now called J/ψ, is a bound state of a new quark, called **charm** (c), and its antiquark;
  - the c had been proposed in 1970 to exclude FCNC [GIM mechanism];
  - the J/ψ has $J^P = 1^-$ [*next slide*];
  - the name "charmonium" is an analogy with positronium ("onium" : bound state particle-antiparticle);
- The cross-section (Breit-Wigner) for the formation of a state (J_R = 1) from e^+e^- ($S_a = S_b$ = ½), followed by a decay into a final state, shows that:

$$\sigma(ab \to J/\psi \to f\bar{f}, \sqrt{s}) = \frac{16\pi}{s}\frac{(2J_R+1)}{(2S_a+1)(2S_b+1)}\left[\frac{\Gamma_{ab}}{\Gamma_R}\right]\left[\frac{\Gamma_{f\bar{f}}}{\Gamma_R}\right]\left[\frac{\Gamma_R^2/4}{\left(\sqrt{s}-M_R\right)^2+\Gamma_R^2/4}\right]$$

- $\sigma(e^+e^- \to J/\psi \to f\bar{f}, \sqrt{s}) = \frac{12\pi}{s}\left[\frac{\Gamma_e}{\Gamma_{tot}}\right]\left[\frac{\Gamma_f}{\Gamma_{tot}}\right]\frac{\Gamma_{tot}^2/4}{\left(m_{J/\psi} - \sqrt{s}\right)^2 + \Gamma_{tot}^2/4}$;
- Γ_f = width for the $\left(J/\psi \leftrightarrow f\bar{f}\right)$ coupling;
- $\Gamma_{tot} = \Gamma_e + \Gamma_\mu + \Gamma_{had}$ = full width of J/ψ;
- $\Gamma_f/\Gamma_{tot} = BR(J/\psi \to f\bar{f})$ [very useful].

- After 1974, many exclusive decays have been precisely measured, all confirming the above picture; the last PDG has 227 decay modes; the present most precise value of the mass and width is
  m(J/ψ) = 3097 MeV, Γ_{tot}(J/ψ) = 93 keV.



The J/ψ quantum numbers

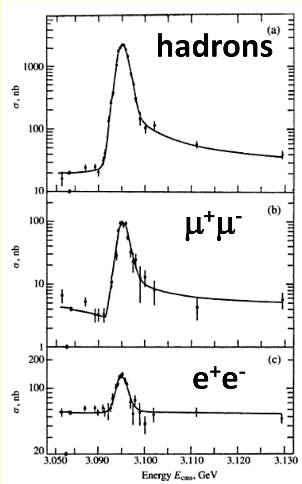
At SPEAR they were able to measure many of the J/ψ quantum numbers :

- the resonance is asymmetric (the right shoulder is higher); therefore there is interference between J/ψ formation and the usual γ^* exchange in the s-channel; therefore the J/ψ and the γ have the same $\boxed{J^P = 1^-;}$
- from the cross section, by measuring σ_{had}, σ_μ and σ_e, they have 3 equations + a constraint (see the box, three σ_f + Γ_{tot}) for the 4 unknowns (three Γ_f + Γ_{tot}); therefore they measured everything, obtaining a $\boxed{\Gamma_{tot}}$ very small (~90 keV, a puzzling results, see next slides);
- the equality of the BR ($J/\psi \to \rho^0\pi^0$) and ($\to \rho^\pm\pi^\mp$) implies isospin $\boxed{I = 0;}$
- the J/ψ decays into an odd (3, 5) number of π, not ~~in an even (2, 4) number~~; this fact has two important consequences :
  - ➢ the G-parity is conserved in the decay (so the $\boxed{\text{J/ψ decays via strong inter.}}$).
  - ➢ $\boxed{\text{G-parity = -1}}$ [also $(-1)^{I+\ell+s} = -1$].

$$\sigma(e^+e^- \to J/\psi \to f\bar{f}) = \frac{3\pi}{s}\,\frac{\Gamma_e\Gamma_f}{\left(m_{q\bar{q}} - \sqrt{s}\right)^2 + \Gamma_{tot}^2/4} = \sigma_f(\Gamma_e, \Gamma_f, \Gamma_{tot}, \sqrt{s});$$

$$\Gamma_{tot} = \Gamma_e + \Gamma_\mu + \Gamma_{had}$$

[see previous slide].



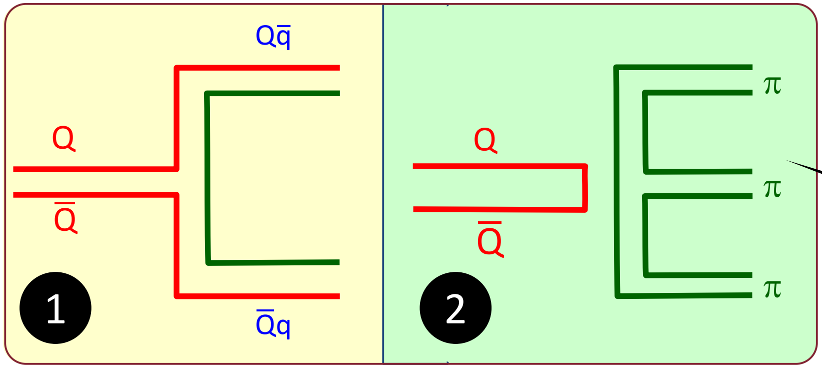
4 equations (f=e,μ,had + Γ_{tot}), 4 unknowns;
NO direct measurement of "width" required.

Charmonium the Zweig Rule (OZI)

The "Zweig rule" was set out empirically in a qualitative way before the advent of QCD :

- compare $(\phi \rightarrow 3\pi) \leftrightarrow (\phi \rightarrow KK) \leftrightarrow (\omega \rightarrow 3\pi)$;
- in the decay of a bound state of heavy quarks Q, the final states without Q's ("decays with disconnected diagrams" 2) have suppressed amplitude wrt "connected decays" 1;
- if only the decays 2 are kinematically allowed (ex. J/ψ or Υ), the total width is small and the bound state is "narrow";

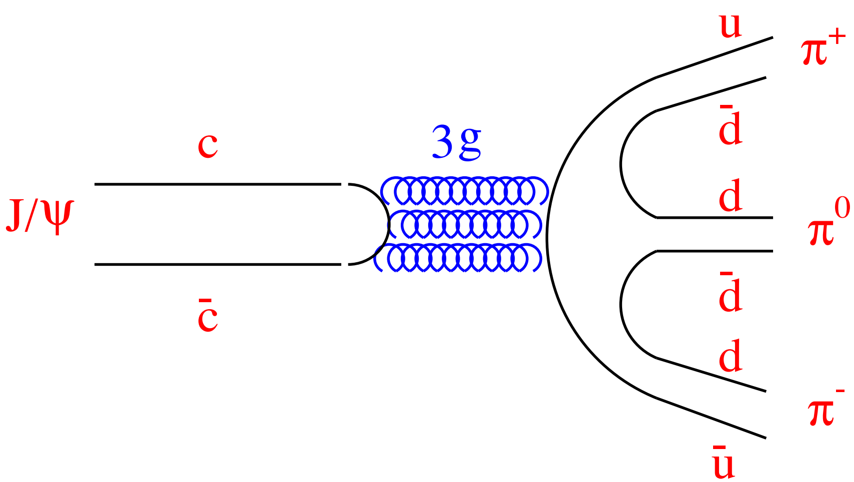
1963-1966 :
Susumu Okubo
(大久保 進
Ōkubo Susumu),
George Zweig,
Jugoro Iizuka (飯塚)

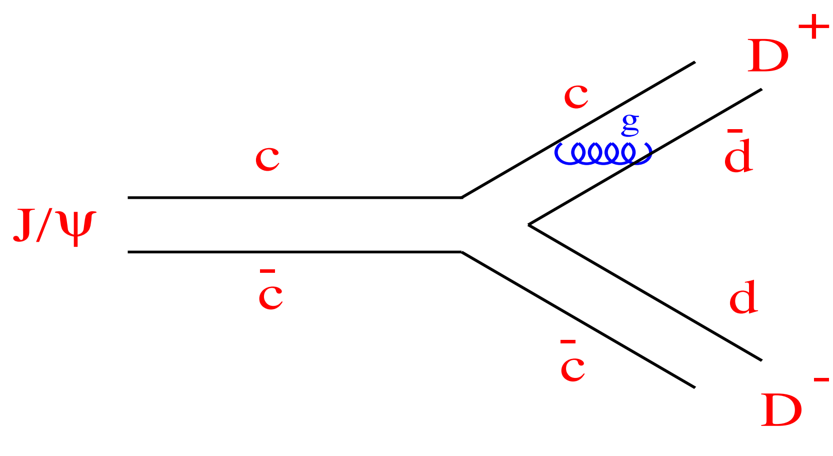


before the QCD advent, gluons were not considered.

Charmonium the Zweig Rule (OZI)

Decay rates described by diagrams with unconnected quark lines are suppressed.





OZI suppressed

OZI allowed
forbidden by
energy conservation

Charmonium: ψ'

- After the discovery of the J/ψ, at SPEAR they performed a systematic energy scanning with a very small step. After ten more days a second narrow resonance was found, called ψ', with the same quantum numbers of the J/ψ.

- The analysis shows that the J/ψ was the 1S state of $c\bar{c}$, while the ψ' is the 2S.

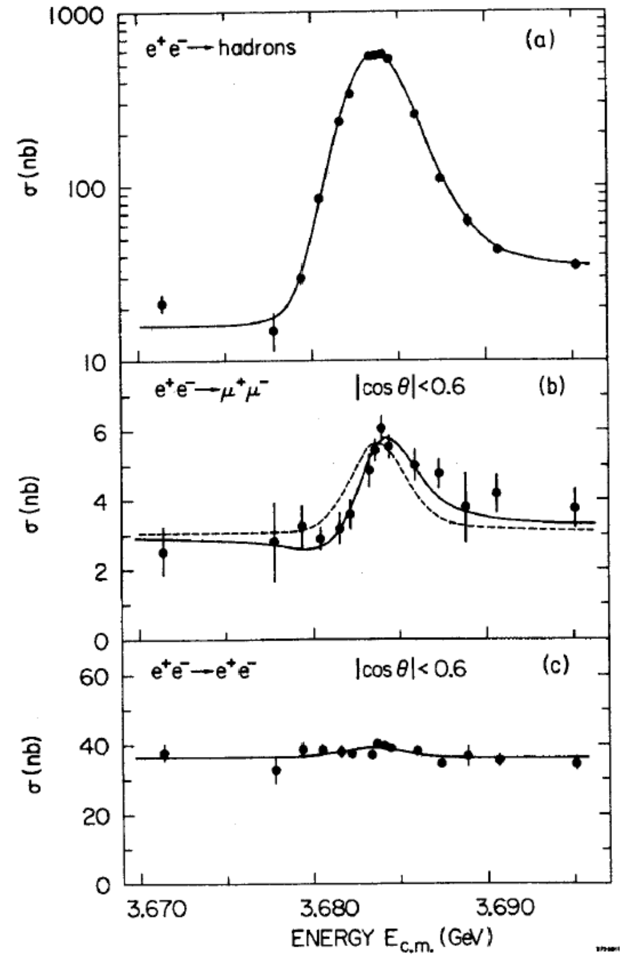
- Both particles have $J^P = 1^-$, I=0.

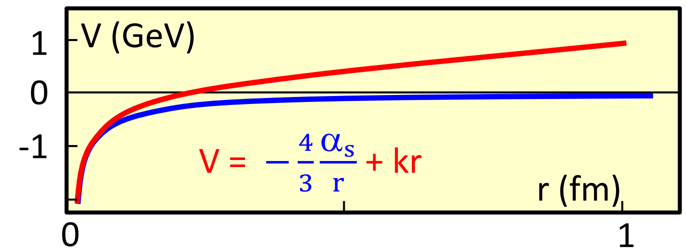
- The next page gives a scheme of the $c\bar{c}$ levels.

- They offer a reasonable agreement with the solution of the Schrödinger equation of a hypothetical QCD potential

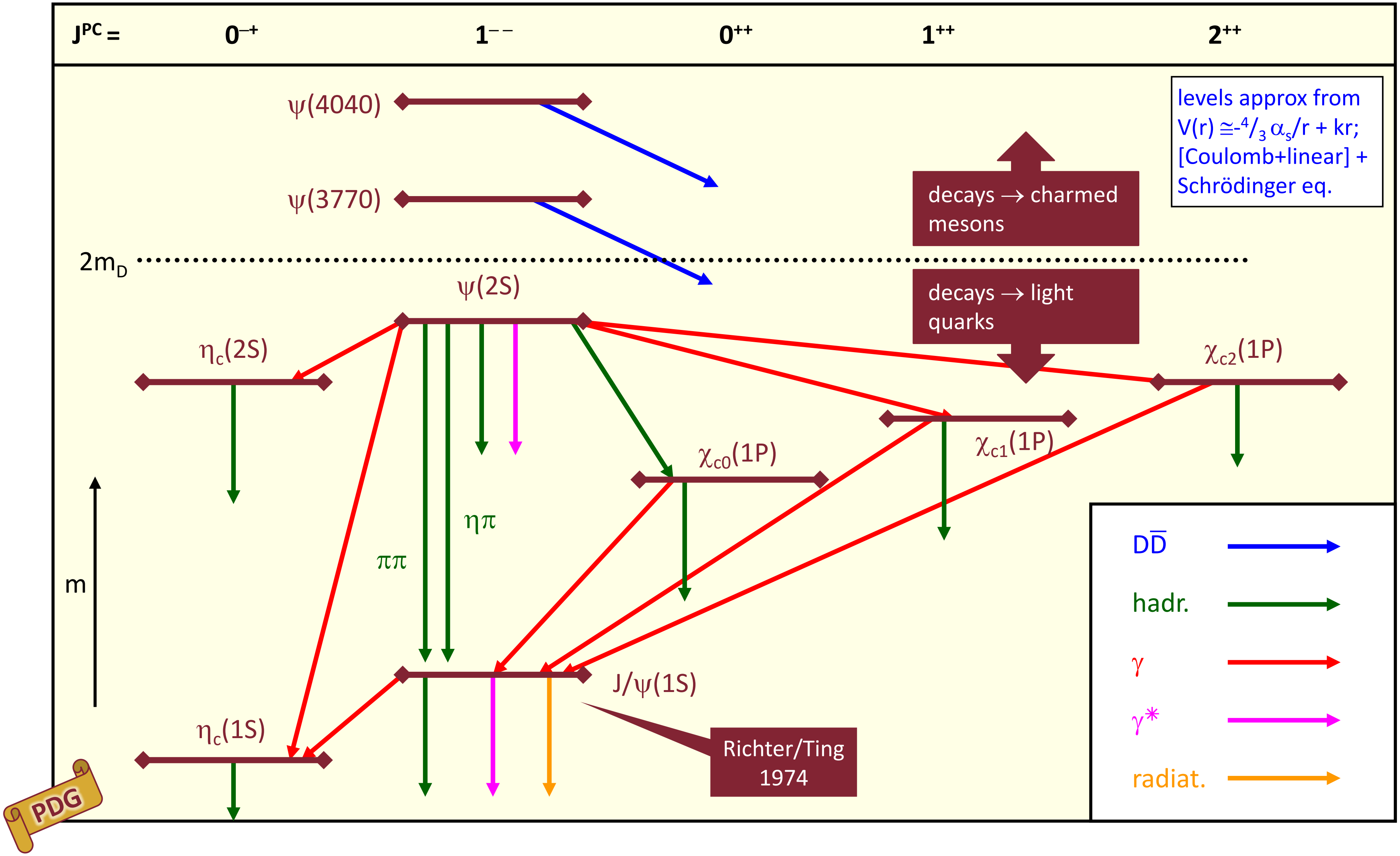
$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + kr = \frac{A}{r} + Br.$$

- Notice that this approximation should become more realistic for heavier quarks, when the non-relativistic limit gets better.





Charmonium states



Charmonium: why it is useful?

Charmonium is a powerful tool for the understanding of the strong interaction. The high mass of the c quark ($m_c \sim 1.5$ GeV/c^2) makes it plausible to attempt a description of the dynamical properties of the ($c\bar{c}$) system in terms of non-relativistic potential models, in which the functional form of the potential is chosen to reproduce the known asymptotic properties of the strong interaction. The free parameters in these models are determined from a comparison with experimental data.

$$\beta^2 \approx 0.2 \quad \alpha_s \approx 0.3$$

Non-relativistic potential models + Relativistic corrections + PQCD

The non-relativistic potential

The functional form of the potential is chosen to reproduce the known asymptotic properties of the strong interaction.

- At small distances asymptotic freedom, the potential is coulomb-like:

$$V(r) \xrightarrow[r \to 0]{} -\frac{4}{3}\frac{\alpha_s(r)}{r}$$

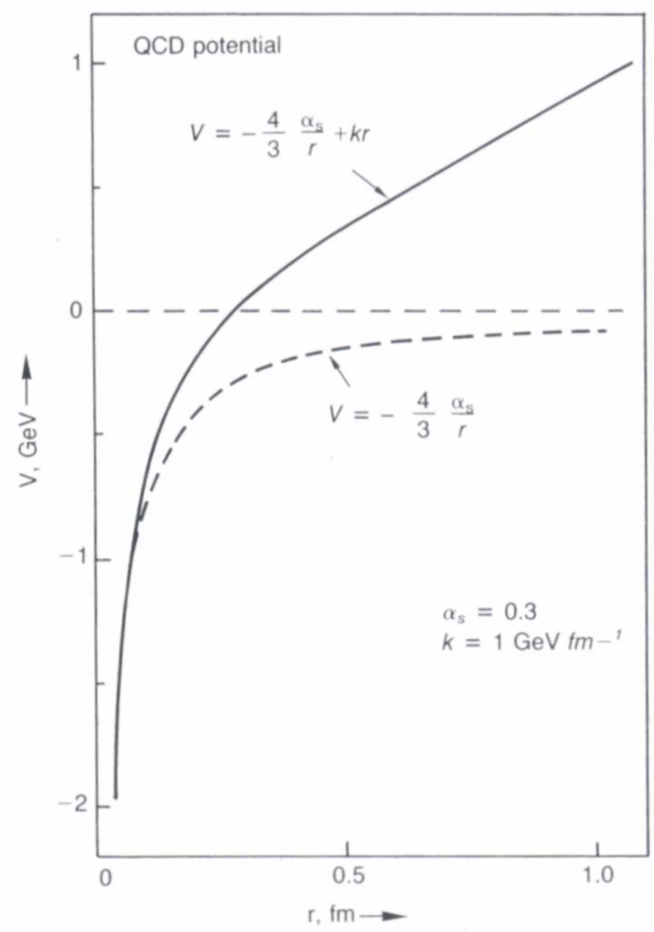
- At large distances confinement:

$$V(r) \xrightarrow[r \to \infty]{} kr$$

The non-relativistic potential

$$\alpha_s(\mu) = \frac{4\pi}{(11 - \frac{2}{3}n_f)\ln(\frac{\mu^2}{\Lambda^2})}$$

n_f = number of flavours
Λ ~ 0.2 GeV QCD scale parameter
k string constant (~ 1 GeV/fm)

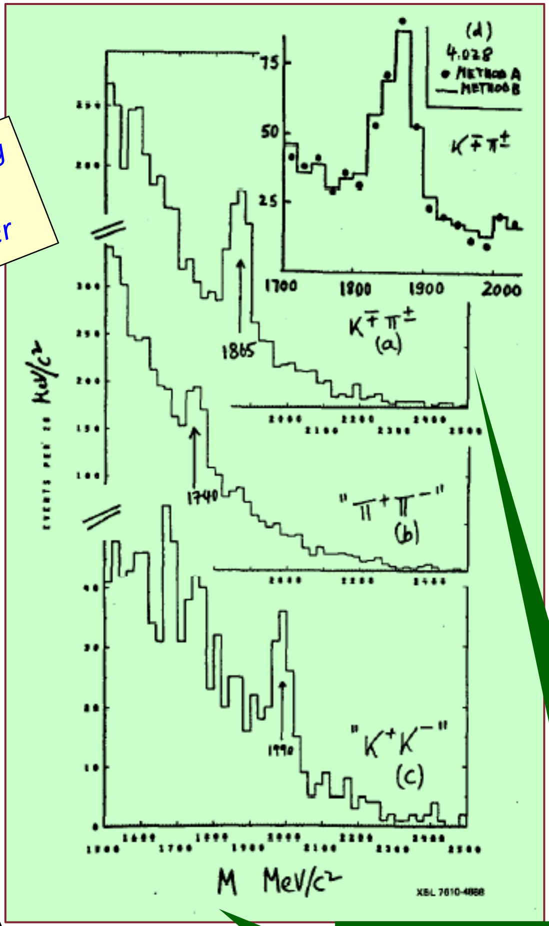


The open charm discovery

- If the J/ψ is a bound $c\bar{c}$ state, then mesons $c\bar{q}$ and $\bar{c}q$ must exist, with a mass $m_{J/\psi}/2$ + 100÷200 MeV [$3690/2 < m_D < 3770/2$ MeV].
- In 1976, the Mark I detector started the search for charmed pseudoscalar mesons, the companions of π's and K's.
- They looked at $\sqrt{s}$ = 4.02 GeV in the channels

  $$e^+e^- \to D^0\,\overline{D}^0\,X^0; \quad \to D^+\,D^-\,X^0.$$
- According to theory, D-mesons lifetimes are small, with a decay vertex not resolved (*with 1976 detectors*) wrt the e^+e^- one.
- Therefore the strategy of selection was the presence of "narrow peaks" in the combined mass of the decay products.
- A first bump at 1865 MeV with a width compatible with the experimental resolution was observed in the combined mass ($K^\pm\pi^\mp$), corresponding to the D^0 and $\overline{D}^0$ decay.

handwriting by Gerson Goldhaber

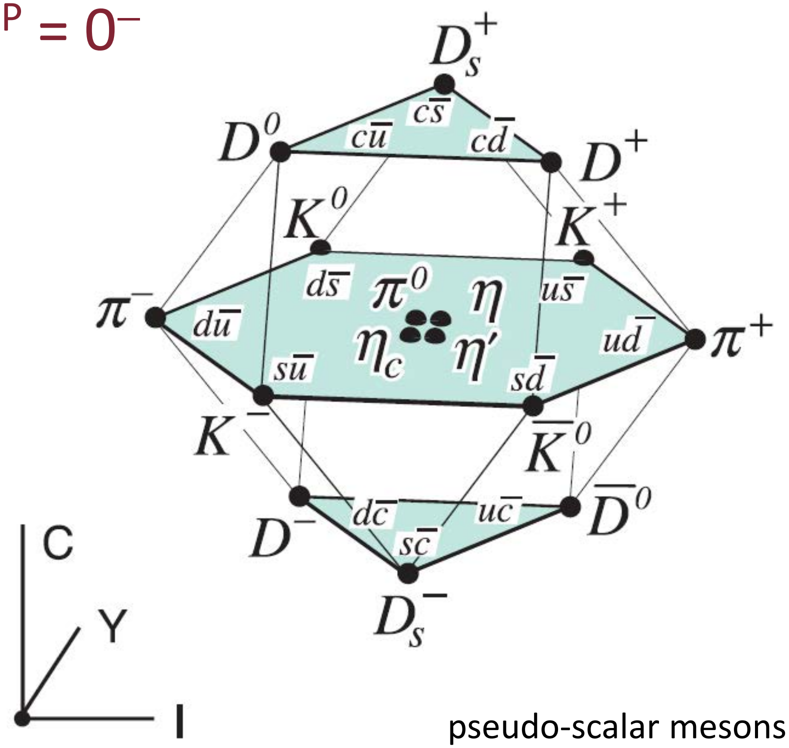


Today's technology allows for it !!!

They were afraid of K/π exp. misidentification $\to$ they computed the mass twice, also with the other particle hypothesis

Open charm: meson multiplets

$J^P = 0^-$

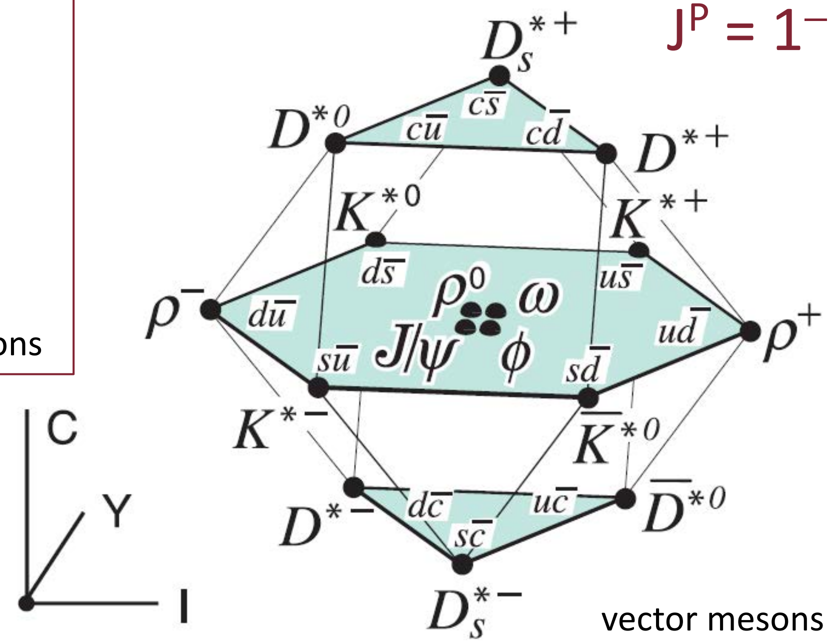


pseudo-scalar mesons

<u>SU(3)$_{\text{flavor}}$</u> $\rightarrow$ <u>SU(4)$_{\text{flavor}}$</u>

With 4 quarks, the SU(3) nonets become multiplets in a 3-D space. However, the c quark has a large mass, so SU(4)$_{\text{flavor}}$ is much more broken that SU(3)$_{\text{flavor}}$.

$J^P = 1^-$

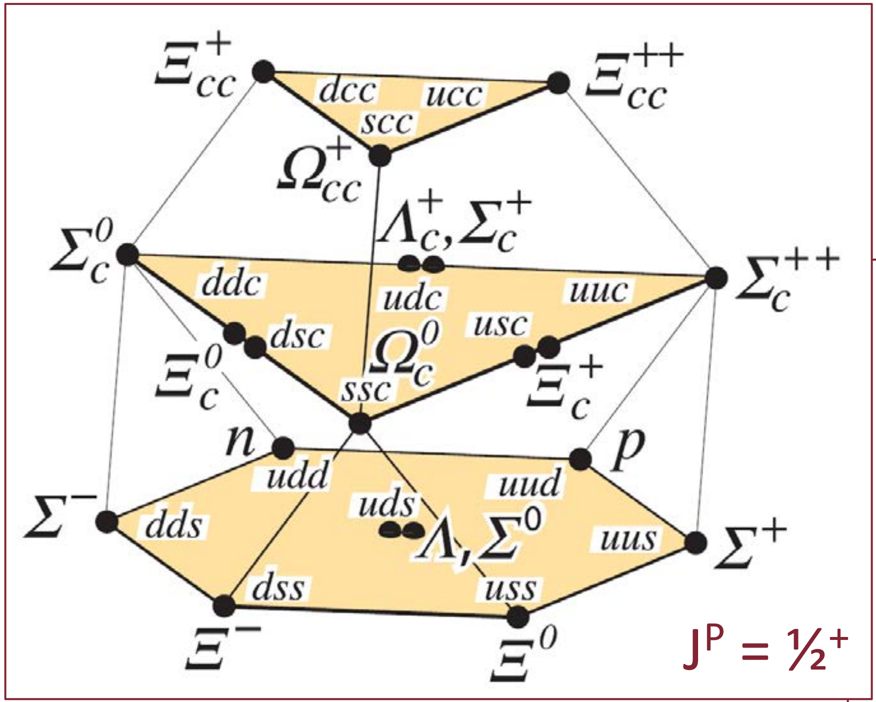


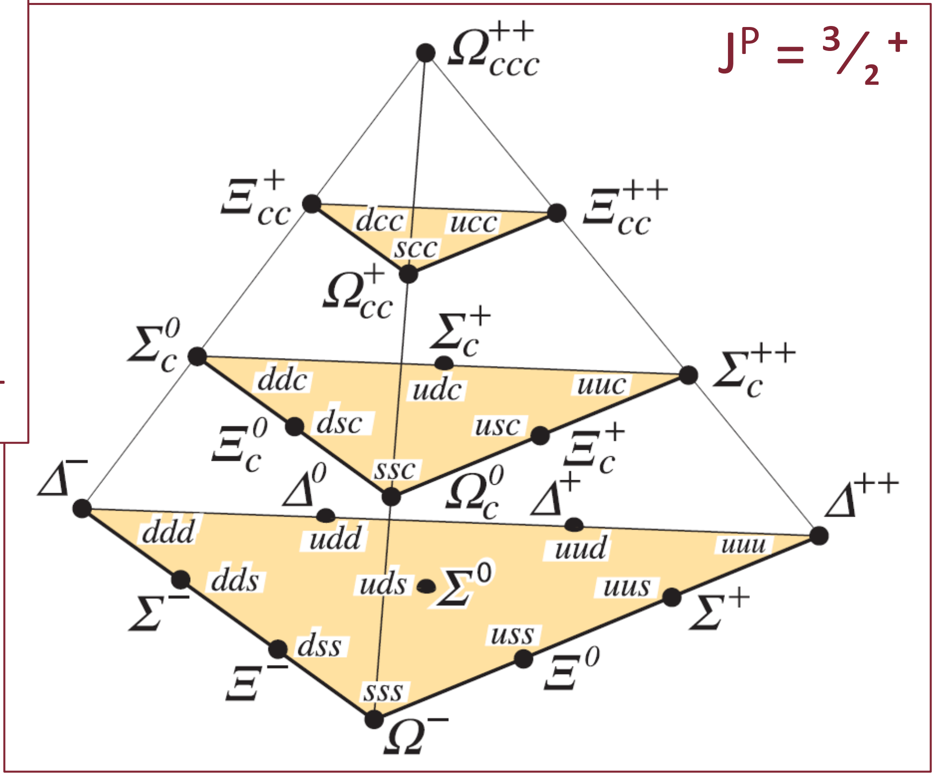
vector mesons

$4 \otimes \bar{4} = 15 \oplus 1.$

Open charm: meson multiplets

SU(4)$_{flavor}$ baryons





The third family: the τ lepton discovery

The analysis of Mark I data produced another beautiful discovery : the τ lepton (M. Perl won the 1995 Nobel Prize):

- the selection followed a method well known, pioneered at LNF-Frascati : the "unbalanced pairs $e^{\pm}\mu^{\mp}$" :

$$e^+e^- \to \tau^+\tau^-$$

$$\left.\begin{array}{l} \tau^- \to \mu^-\bar{\nu}_\mu\nu_\tau \\ \tau^+ \to e^+\nu_e\bar{\nu}_\tau \end{array}\right\} \to \mu^-e^+ \ (\underline{\text{unbalanced}})$$

(+ CC μ^+e^-).

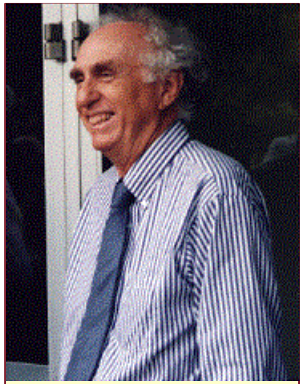
- events from this process are extremely clean and free from background [*see fig.*];
- the e^+e^- / $\mu^+\mu^-$ unbalanced pairs, which have to be present in the correct number

$N_{unb}(e^+e^-) = N_{unb}(\mu^+\mu^-) =$

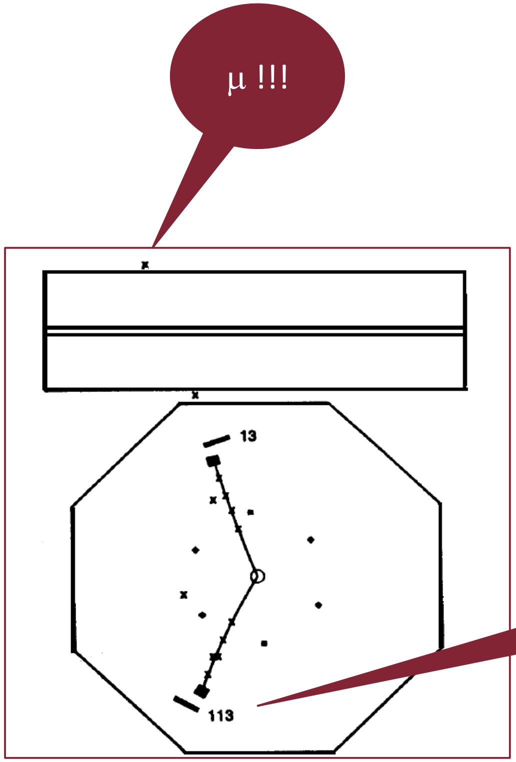
$= N(e^+\mu^-) = N(e^-\mu^+)$,

are only used to cross-check the sample.

In principle the τ lepton has very little to do with the c quark. However collider, detector, energy, selection and analysis are closely linked. Therefore, in experimental reviews, the τ lepton is usually treated together with the charm quark.



Martin Perl

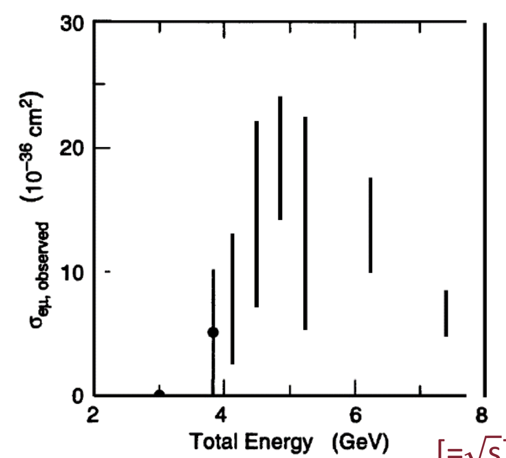


The third family: the τ lepton discovery

Simple method: the yield of $e^\pm\mu^\mp$ pairs vs $\sqrt{s}$: it immediately points to the threshold $\sqrt{s} = 2m_\tau$.

- therefore : $m_\tau \approx 1780$ MeV.
  [best present value 1776.8 MeV]
- why is the $\tau^\pm$ a lepton ?
  - at the time, the evidence came from the lack of any other plausible explanation;
  - today, the evidence is solid :
    - the Z and W decays into ($e\ \mu\ \tau$) with the same BR and angular distribution;
    - the lifetime has been measured and found in agreement with predictions …
- the discovery of the τ started the hunt for the particles of a new (3rd) family, still unknown:
  - the ν_τ (possibly mixed with the others);
  - the pair of quarks q_{up} q_{down}, similar to ud (now called **t**op and **b**ottom).

$$e^+e^- \to \tau^+\tau^-$$
$$\tau^- \to \mu^-\bar{\nu}_\mu\nu_\tau, \quad \tau^+ \to e^+\nu_e\bar{\nu}_\tau \quad \Big\} \to \mu^-e^+$$
(unb.)



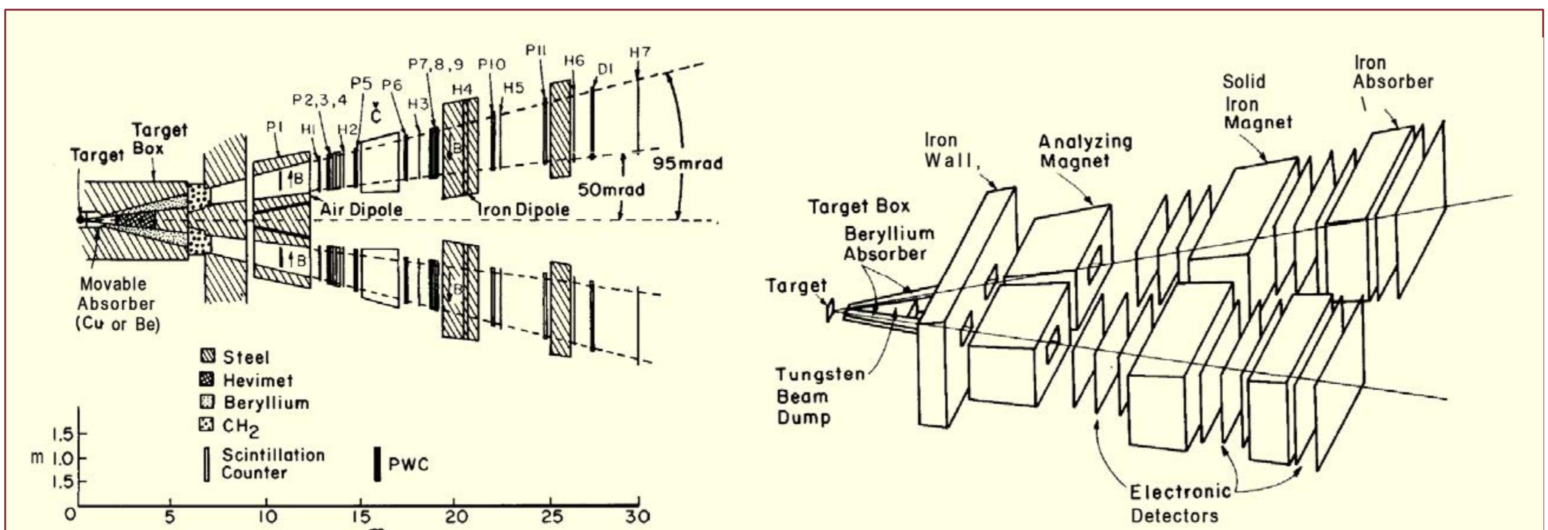
$[=\sqrt{s}]$

The third family: the b quark discovery

- The down quark of the 3rd family was called b (= beauty, bottom).
- In 1977 Leon Lederman and collaborators built at Fermilab a spectrometer with two arms, designed to study $\mu^+\mu^-$ pairs produced by interactions of 400 GeV protons on a copper (or platinum) target.

- The reaction under study was again the Drell-Yan process. As already pointed out, this type of events is rare, therefore requiring intense beams (in this case 10^{11} ppp) and high rejection power against charged hadrons.

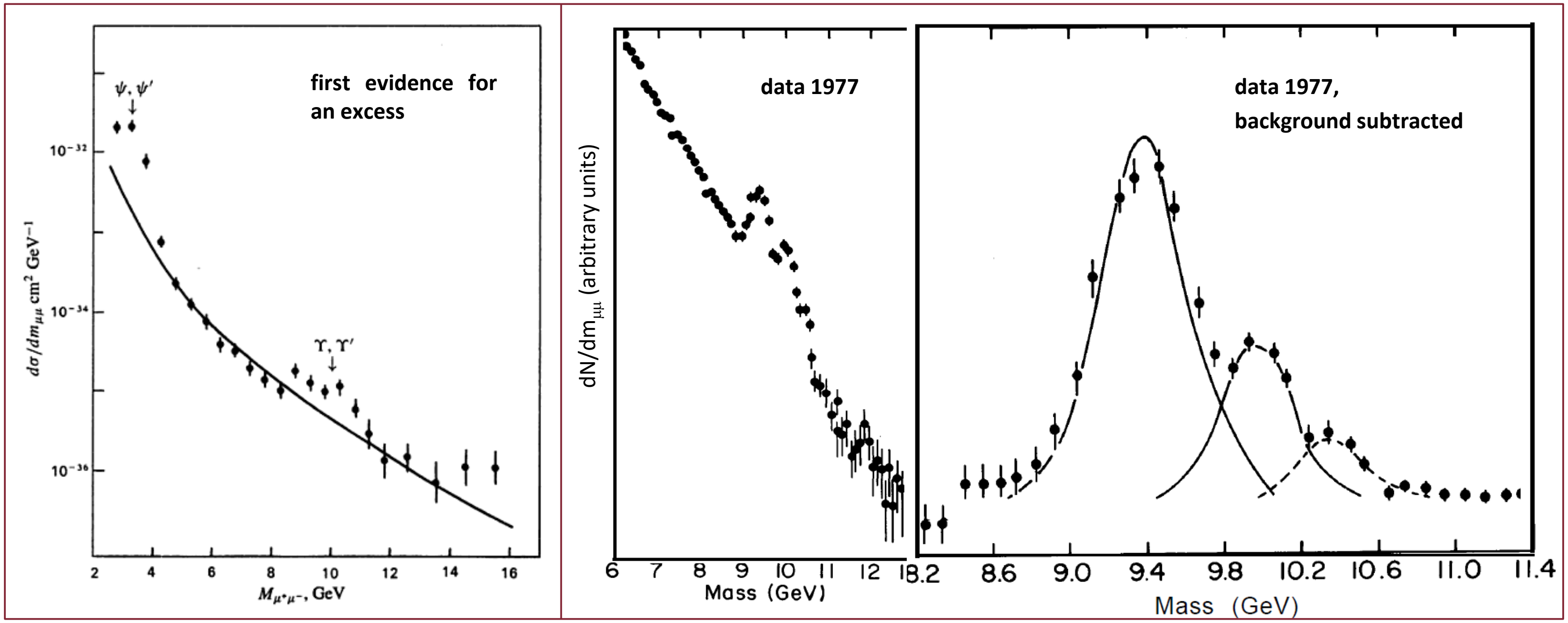
Leon Lederman



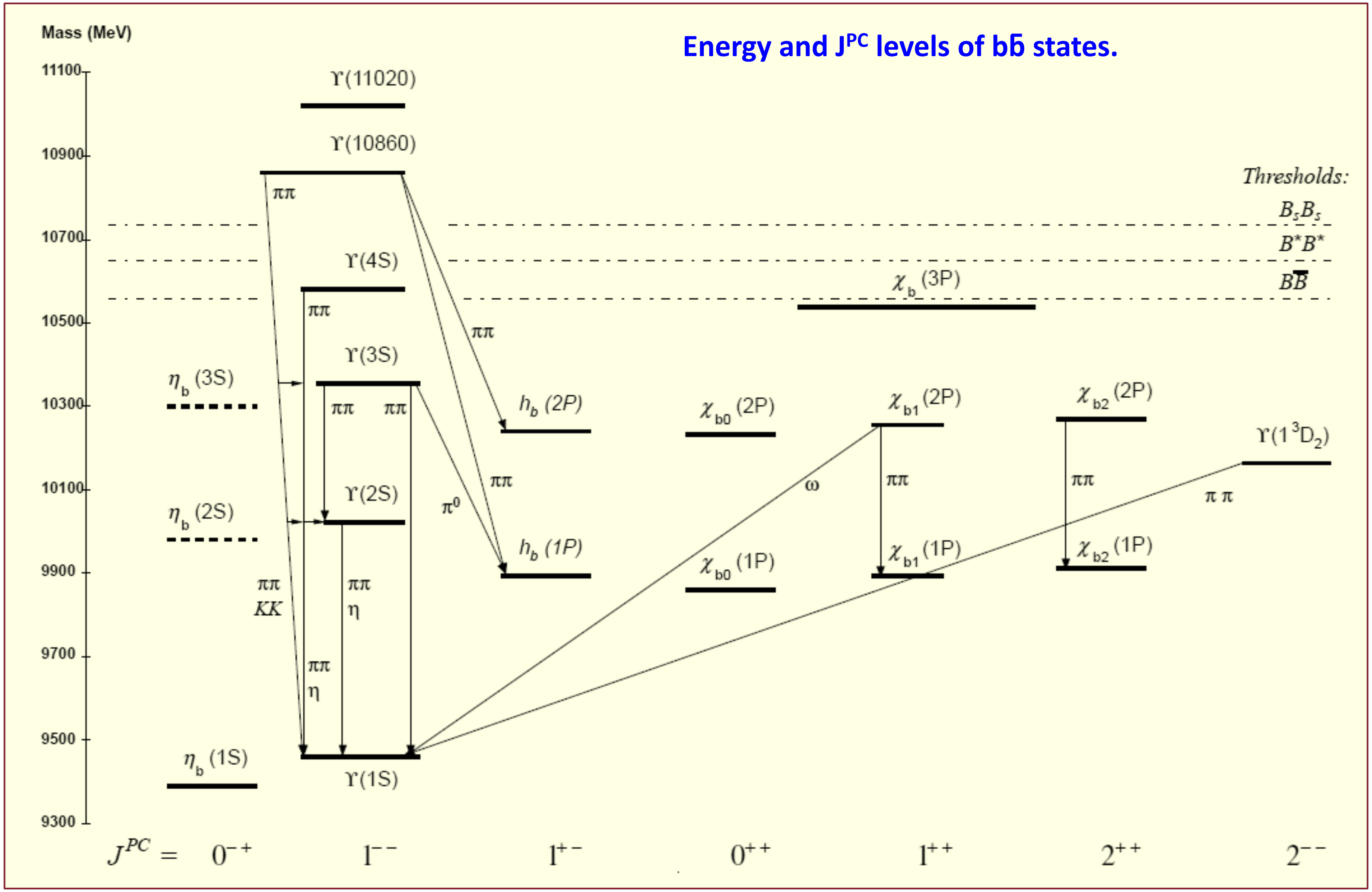
The third family: the b quark spectrum

- The usual price of the absorber technique is a loss of resolution in the muon momenta, which was $\Delta m_{\mu\mu} / m_{\mu\mu} \approx 2\%$.
- The figures show the distribution of $m_{\mu\mu}$. Between 9 and 10 GeV : there is a clearly visible excess.

- When the $\mu\mu$ continuum is subtracted, the excess appears as the superimposition of three separate states.
- The states, called $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ are bound states $b\bar{b}$.

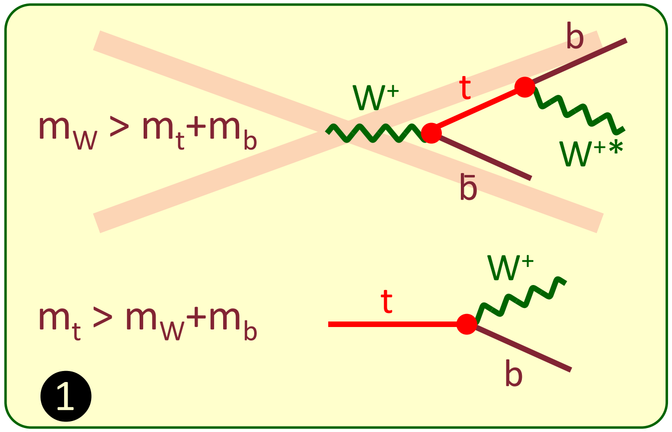


The third family: the b quark spectrum

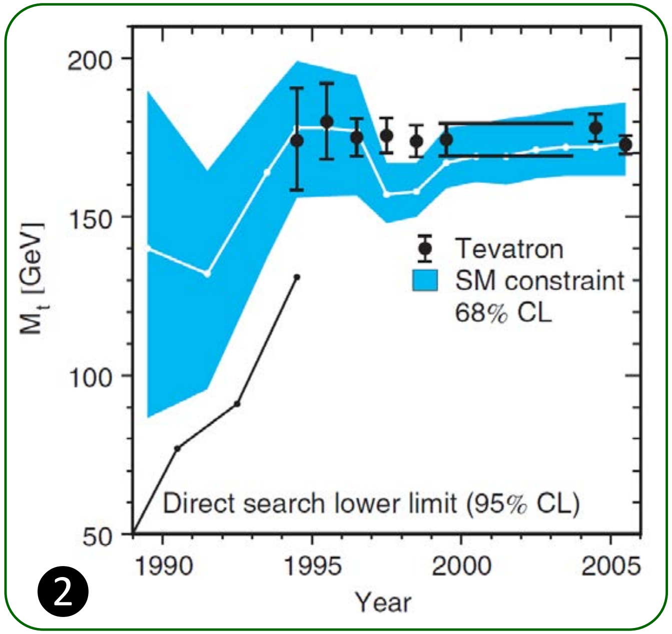


The top quark search

- The top quark was directly searched in hadron ($Sp\bar{p}S$, Fermilab) and lepton (Tristan, LEP) colliders, but was NOT found until 1990's;
- at the time the mass limit was $m_t \geq 90$ GeV;
- at $m_t \approx m_W - m_b$ ($\approx$ 75 GeV), the search changes: the "golden discovery channel" moves from ($W^+ \to t\bar{b} \to W^{+*} b\bar{b}$) to ($t \to W^+ b$) [fig. ❶];
- the mass was first *computed* from the radiative corrections for m_W and m_Z [*see* § *LEP*];
- the LEP data, together with all other e.w. measurements, allowed for a prediction of $m_t \approx 175$ GeV [fig. ❷];
- in the 1990's the search was finally concluded at the Tevatron, by the CDF and D0 experiments.
- At present, we measure $m_t = 173 \pm 0.4$ GeV.

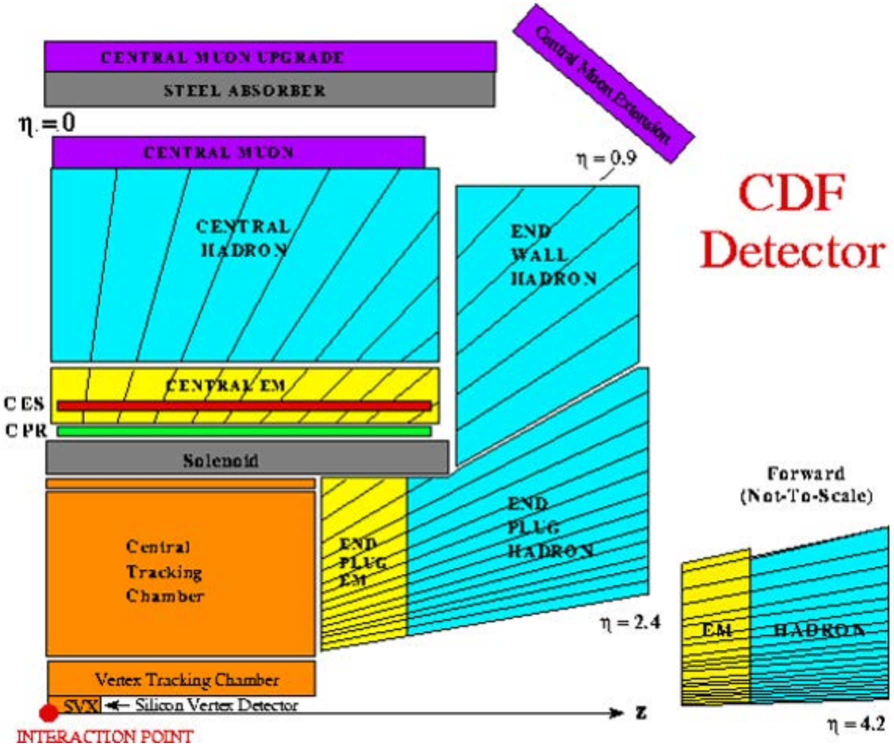


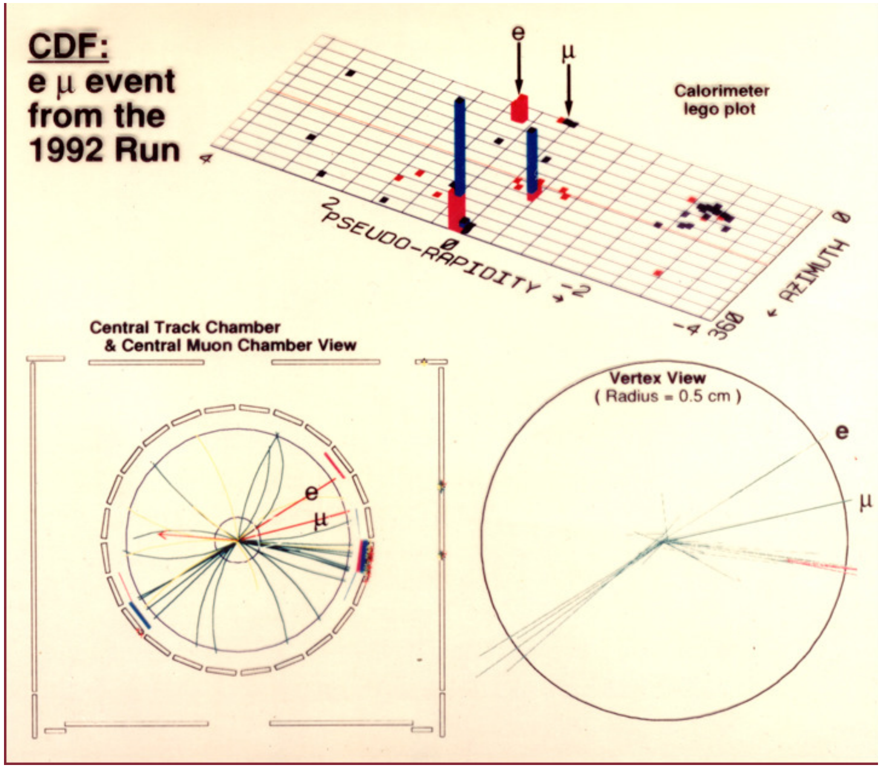
❶



❷

The top quark discovery





main tools for $t\bar{t}$ events at Tevatron (1992-4) :

- multibody final states;
- lepton id ($e^{\pm}$, $\mu^{\pm}$);
- secondary b vertices;
- mass fits.

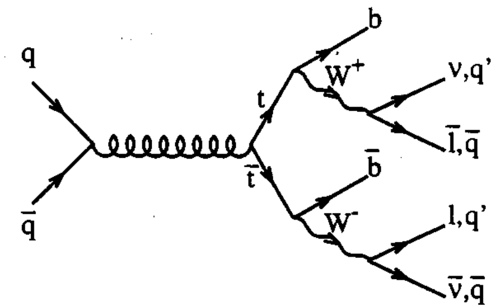
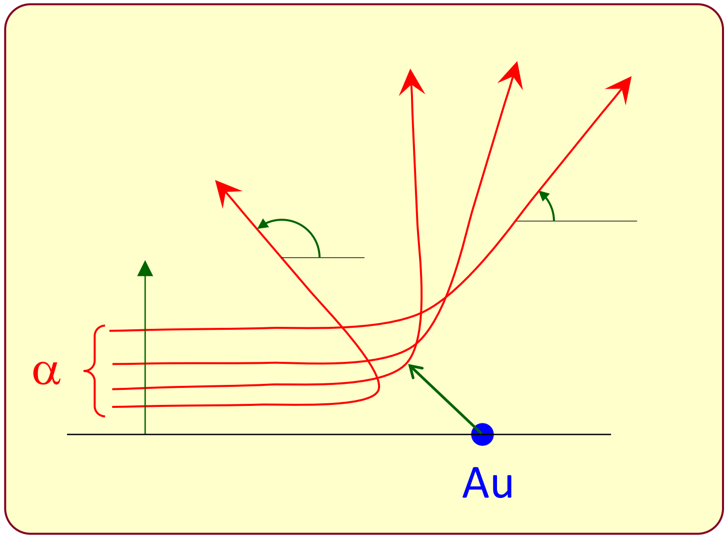


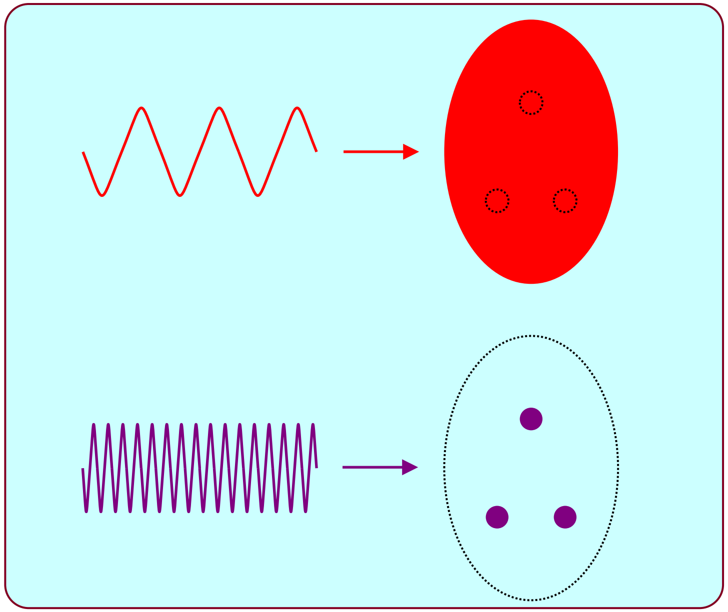
Figure 28: Tree level top quark production by $q\bar{q}$ annihilation followed by the Standard Model top quark decay chain.

THE HADRON STRUCTURE

Outline

- Fermi gas model
- Rutherford scattering
- Kinematics
- Elastic scattering e-Nucleus
- Form factors
- Electron-Nucleon scattering
- Proton structure
- Higher Q^2
- Deep inelastic scattering
- Bjorken scaling
- The parton model
- The quark-parton model
- $F_2(x,Q^2)$
- Summary of cross-sections





The scattering experiment

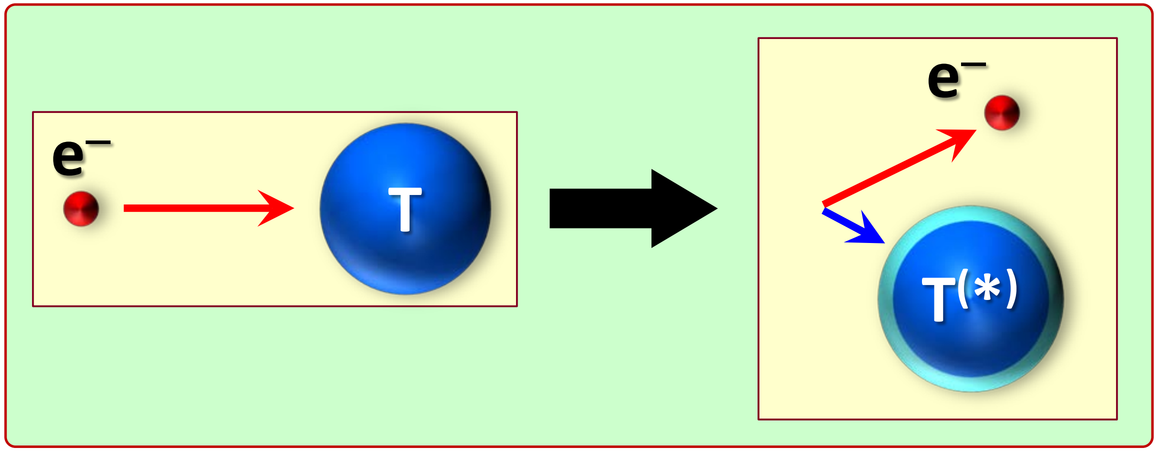
Q : is the target a **pointlike simple object** ? if not, how to probe its shape ?

A : *(à la Rutherford, but (a) he used α particles, (b) he did NOT see the nucleus size)*

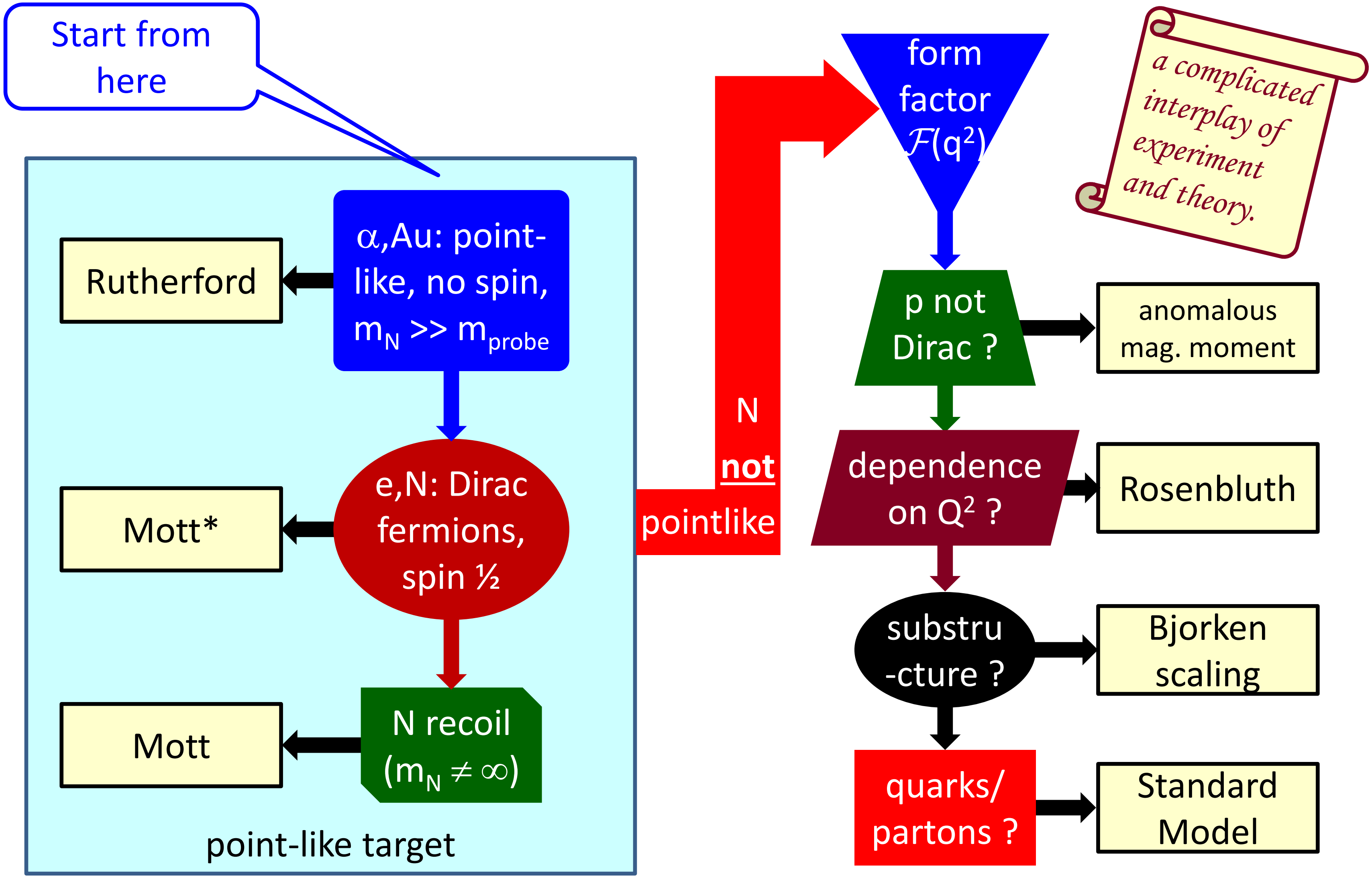
- take a probe: e.g. an electron (e^-),
- study the scattering e^-T, [T=Nucl-eus/on]
- measure the cross section $\sigma(e^-T)$,
- … and the angular distribution of the e^-;
- … and detect the excited states or the final state hadronic system ("inelastic interactions").

Path:

1. study the kinematics ;
2. compute $\sigma(e^-T)$ for pointlike nuclei in classical electrodynamics (Rutherford formula);
3. ditto in QM for spin ½ electrons and pointlike nuclei (Mott formula);
4. detect **deviations** from these models $\rightarrow$ derive informations on nuclear structure;
5. **new theory $\rightarrow$ smaller distance (i.e. higher Q^2) $\rightarrow$ deviations $\rightarrow$ newer theory $\rightarrow$ … $\rightarrow$ … $\rightarrow$ (possibly ad infinitum)**



The treasure map for scattering



The Fermi gas model

- Nuclei are bound states of protons (p) and neutrons (n).
- A simple model: the Fermi gas:

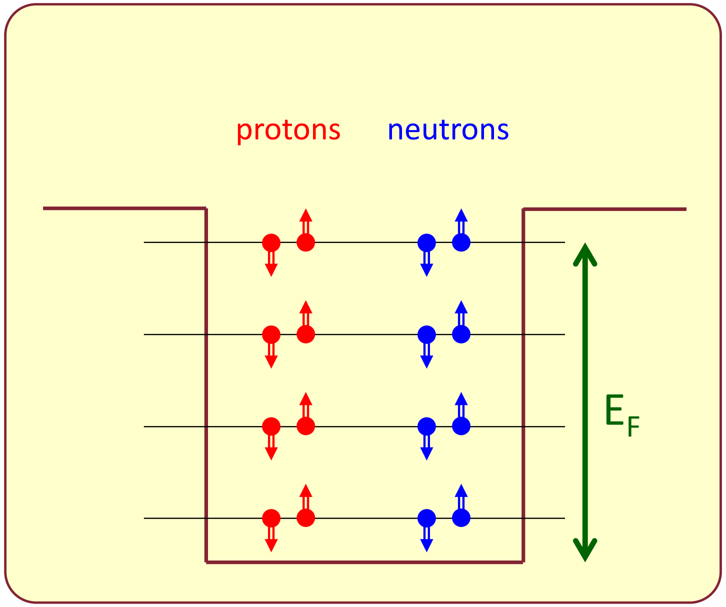
- p, n identical, but charge :
  - little spheres r = r_0, mass = m;
  - spin ½ fermions, pure Dirac-like;
  - bound inside the nucleus, otherwise free to move;
- define:
  - $n_{neutr.}$(= N), $n_{prot.}$ (= Z), A = N + Z,
  - p_{Fermi} (= p_F), E_{Fermi} (= E_F);

  $\to V_{Nucl}$ [$\propto$ A] = $4\pi r_0^3 A/3$;
- no e.m. interactions, only nuclear $\to$ N = Z = A/2, $p_F^p = p_F^n$, $E_F^p = E_F^n$ [better approx (not here):different interactions $\to p_F^p \neq p_F^n$];
- uncertainty principle $\to$ each p/n fills $V_{phase\ space} = [2\pi\hbar]^3$.

Therefore:

- well-shaped potential (⊔), identical for p/n, i.e. only interactions p↔p n↔n;
- Fermi statistics $\to$ two p/n per energy level (spin ⇑⇓);

[...*next page*...]



The Fermi gas model: results

From those approximations, an elementary computation :

$$n^{n,\Uparrow} = n^{n,\Downarrow} = n^{p,\Uparrow} = n^{p,\Downarrow} = \frac{N}{2} = \frac{Z}{2} = \frac{A}{4} =$$

$$= \frac{\left[V_{space} V_{mom}\right]_{TOT}}{\left[V_{space} V_{mom}\right]_{each\ part}} = \frac{4/_3\, \pi r_0^3 A \times 4/_3\, \pi p_F^3}{\left[2\pi\hbar\right]^3} =$$

$$= \frac{2 A r_0^3 p_F^3}{9\pi\hbar^3};$$

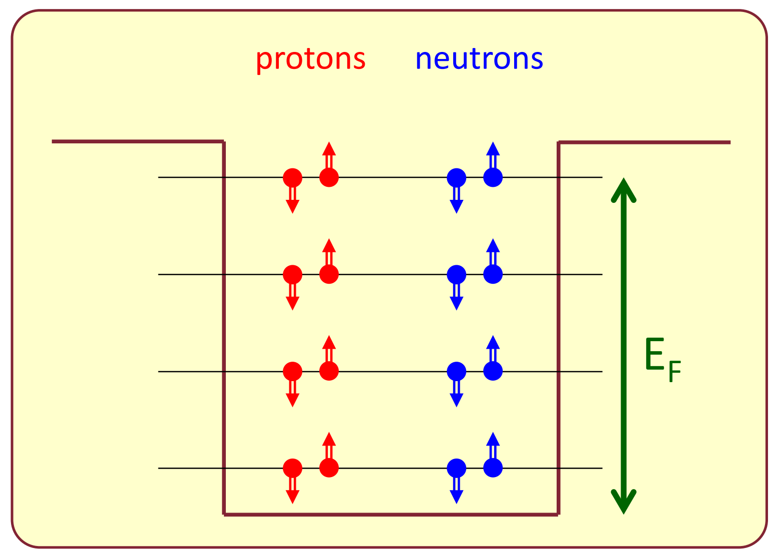
$$N = Z = \frac{\cancel{A}}{2} = \frac{4 \cancel{A} r_0^3 p_F^3}{9\pi\hbar^3}; \qquad p_F = \frac{\hbar}{r_0}\sqrt[3]{9\pi/8};$$

$$r_0 \approx 1.2\text{ fm} \rightarrow \begin{cases} p_F \approx 250\text{ MeV}; \\ E_F^{kin} = p_F^2/2m \approx 33\text{ MeV}. \end{cases}$$

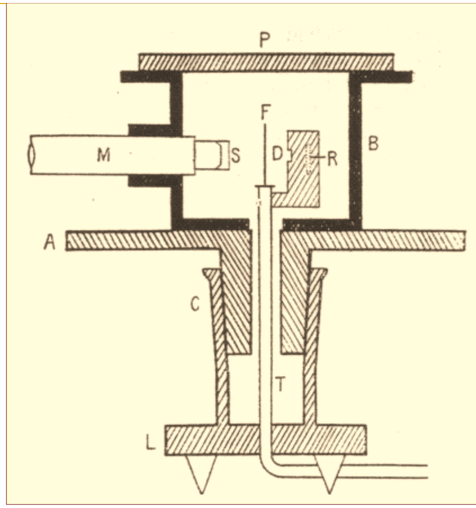
(*) fit from form factors (see later)

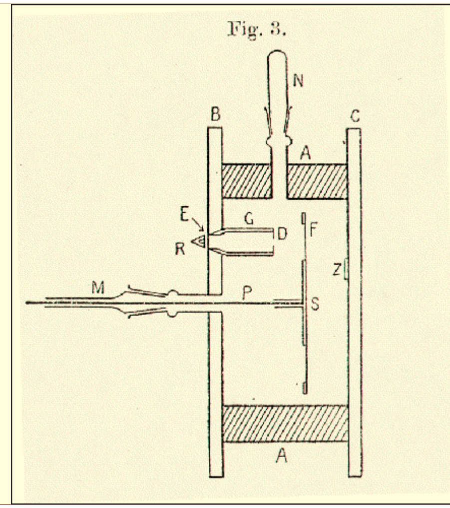
Conclusions :

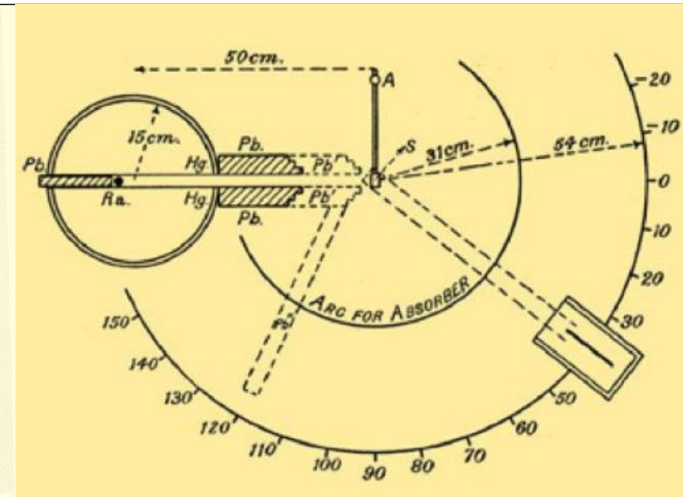
- $V_{space} \approx {}^4/_3 \pi r_0{}^3 A \rightarrow r_{nucl.} \propto A^{1/3}$;
- p_F, E_F not dependent on A (!!!);
- large p_F, small kin. energy;
- when p/n hit by probe ($e^\pm/\nu$), if E_{probe} >> 30 MeV → ignore Fermi motion.
- [*more elaborated model, e.g. add e.m. and spin interactions, etc. – see literature*]



Rutherford scattering (I)







The birth of nuclear physics
(Manchester, 1908-13):

$\alpha(Z_\alpha=2, A_\alpha=4) \rightarrow Au(Z_{Au}=79, A_{Au}=197)$

- actually performed by H.Geiger and E.Marsden [*E.M. was 20 y.o. !*];
- alternative model by J.J.Thompson, with a diffused mass/charge ("soft matter");
- the first "fixed target" scattering experiment.

- *already studied*
- *do NOT repeat the math, simply recall the results;*
- *discussion of the physics;*
- *preparation for further steps.*

Lord Ernest Rutherford

modern simulation (look):
https://phet.colorado.edu/en/

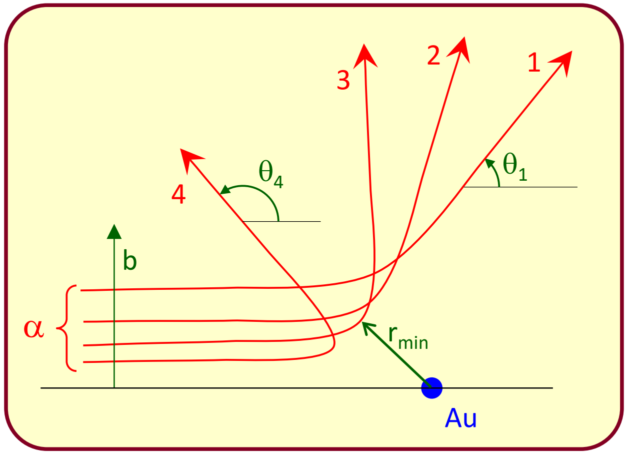
Rutherford scattering (II)

[*an incredible mix of genius, skill and luck*]

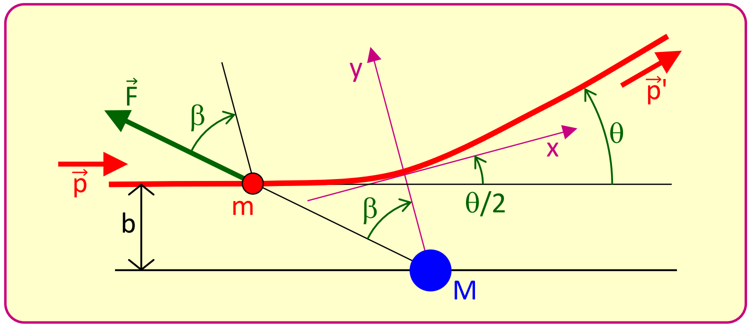
- α-particles (i.e. ionized He) $\rightarrow$ Au foil;
- $E_\alpha^{kin} \approx$ few MeV;
- sometimes, the α was scattered by $\theta >$ 90°; *VERY* rare in reality, but impossible if matter were soft and homogeneous;
- only explanation: "matter" actually concentrated in small heavy bodies ("nuclei");

$\rightarrow$ the "matter" is essentially empty;

- how model the scattering ? Rutherford tried with a two-body scattering with Coulomb (electrostatic) force;
- <u>**success !!!**</u> [*within their limited observation capabilities*]
- a key point: the nucleus is small enough, that the α "sees" always its full charge;
- [*remember the Gauss' theorem: if impact parameter $b > r_{Nucleus}$, only see an effective point-like charge*]
- but the matter is neutral ! yes, but the electrons are so light, that they cannot stop/deflect the α ($m_e/m_\alpha \approx$ 1/8,000).



Rutherford scattering: the math



α (m, z) → nucleus (M, Z):

- $\vec{v}_{\alpha,init} = \vec{v}$, $\vec{v}_{\alpha,final} = \vec{v}'$, $\vec{v}_{nucleus} = 0$;
- $\vec{p} = m\vec{v}$, $\vec{p}' = m\vec{v}'$, m << M;
- Coulomb force only ($\vec{F}$);
- v << c → non-relativistic;
- elastic → $|\vec{p}'| = |\vec{p}|$;
- conserve E, ang. mom $\vec{L}$;
- $\Delta p_x = 0$ because of symmetry, only Δp_y matters;
- integral over β, the angle wrt $\hat{y}$;
- if attractive force (e.g. +–), M → the other focus of the hyperbola.

$$\Delta p = |\vec{p}' - \vec{p}| = 2p\sin(\theta/2);$$

$$|\vec{L}| = pb = |\vec{r} \times m\vec{v}| = |\vec{r} \times m(\frac{dr}{dt}\hat{r} + r\frac{d\beta}{dt}\hat{\beta})| = mr^2\frac{d\beta}{dt};$$

$$\Delta p_y = 2p\sin(\theta/2) = \int_{-\infty}^{+\infty} dt F_y = \int_{-\infty}^{+\infty} dt \frac{zZe^2}{4\pi\varepsilon_0}\frac{\cos\beta}{r(t)^2} =$$

$$= \int_{-(\pi-\theta)/2}^{(\pi-\theta)/2} \frac{zZe^2}{4\pi\varepsilon_0}\frac{\cos\beta}{\cancel{r}^2}\frac{m\cancel{r}^2}{pb}d\beta = \frac{zZe^2}{2\pi\varepsilon_0}\frac{m}{pb}\cos(\theta/2);$$

$$\tan(\theta/2) = \frac{zZe^2}{4\pi\varepsilon_0}\frac{m}{p^2 b} \rightarrow db = -\frac{zZe^2}{4\pi\varepsilon_0}\frac{m}{p^2}\frac{d\theta}{2\sin^2(\theta/2)}.$$

$$d\sigma = 2\pi b\,db = 2\pi\left(\frac{zZe^2 m}{4\pi\varepsilon_0 p^2}\right)^2 \frac{d\theta}{2\tan(\theta/2)\sin^2(\theta/2)};$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2 m}{4\pi\varepsilon_0}\right)^2 \frac{1}{4p^4\sin^4(\theta/2)} = \left(\frac{zZe^2 m}{2\pi\varepsilon_0}\right)^2 \frac{1}{|\vec{p}' - \vec{p}|^4}.$$

$$\boxed{d\Omega = 2\pi\sin\theta d\theta = 4\pi\sin(\theta/2)\cos(\theta/2)d\theta}$$

Rutherford scattering: more math

Useful formulas

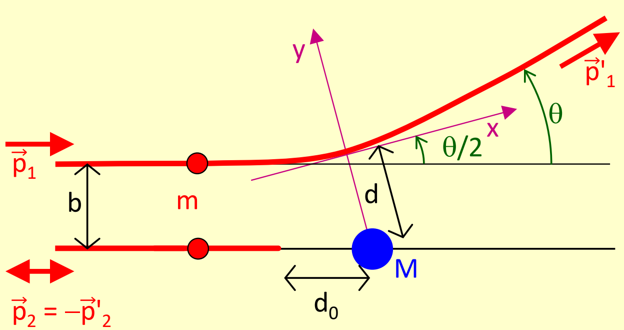
$$d_0 \;=r_{min}(b=0)=\frac{zZe^2}{2\pi\varepsilon_0 mv^2};$$

$$\tan\left(\frac{\theta}{2}\right) \;=\frac{d_0}{2b};$$

$$d\;=r_{min}(b)=\frac{d_0+\sqrt{d_0^2+4b^2}}{2}=$$

$$=\frac{d_0}{2}\left(1+\frac{1}{\sin(\theta/2)}\right);$$

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{d_0^2}{16\sin^4(\theta/2)}}\xrightarrow{\theta\to 0}\frac{d_0^2}{\theta^4}.$$



- [if force attractive (e.g. +–), $\vec{F} \to -\vec{F}$, then $\theta \to -\theta$, but everything else equal, e.g. same $d\sigma/d\Omega$;]
- consider a particle $\vec{p}_2$ with b=0 $\to$ θ_2 = 180°;
- define d_0 = "distance of closest approach" the r_{min} for it (when $r=d_0$, the particle is at rest);
- d_0 is easily computed from energy conservation;
- define $d_0 = (zZe^2)/(2\pi\varepsilon_0 mv^2)$ also for $b\neq 0$;
- write θ and $d\sigma/d\Omega$ as functions of d_0;½
- define d as r_{min}, when $b\neq 0$;
- d is computed from E and $\vec{L}$ conservation [*hint in the box, v_0 is the velocity in d*]:

$$\vec{L}\text{ conserv} \;\to mbv \;= mdv_0 \to v_0/v = b/d$$

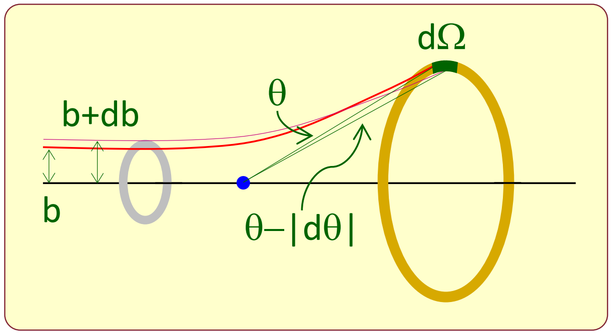
$$\text{E conserv} \;\to \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 + zZe^2/(4\pi\varepsilon_0 d) =$$

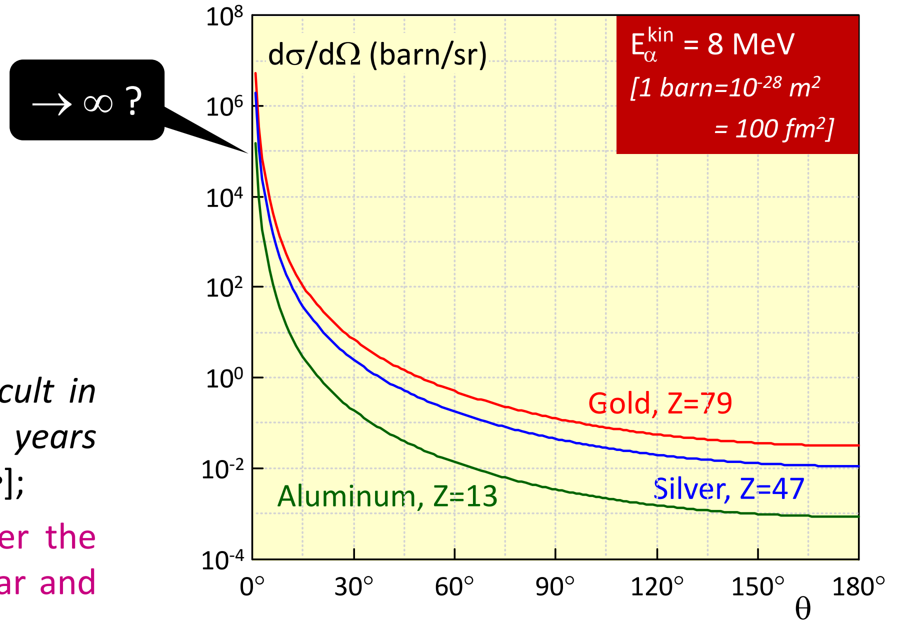
$$= \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}mv^2 d_0/d$$

$$\to (v_0/v)^2 = (b/d)^2 = 1 - d_0/d \to$$

$$\to d^2 - dd_0 - b^2 = 0 \to d = ...$$

Rutherford scattering: $d\sigma/d\Omega$



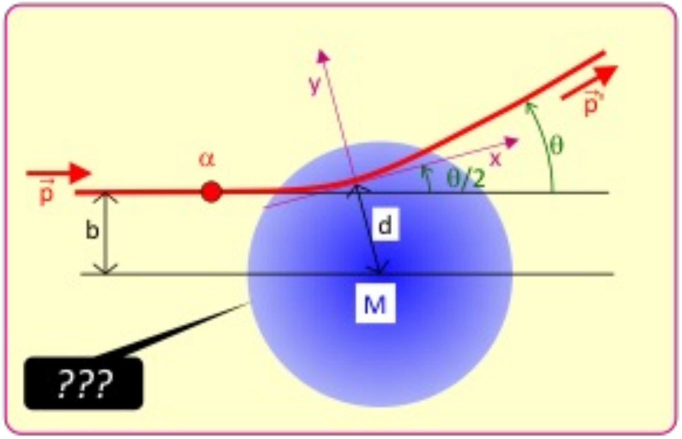


- [*the calculations above are *NOT* difficult in math: Newton could have done all 200 years earlier, had the correct model been made*];
- the real difficulty was to assess whether the matter is soft and continuous or granular and "empty";
- b large → θ small → $d\sigma/d\Omega$ → ∞ [*cutoff provided by other Au nuclei*].

A long and thorough investigation:

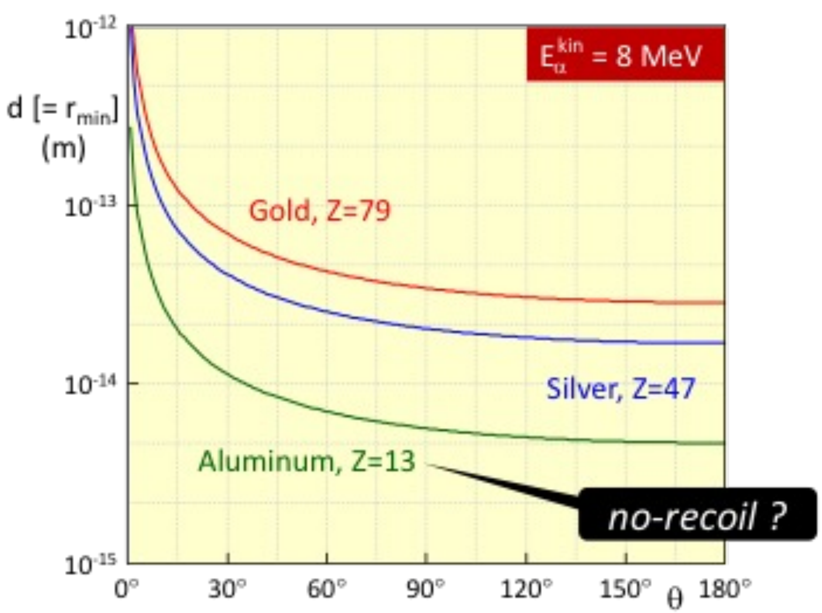
- 1909: found some events $\theta > 90°$: big shock;
- 1911: falsification of the Thomson model, correct assumptions, check of $d\sigma/d\Omega$ in the range 30°–50°;
- 1913: check of $d\sigma/d\Omega$ in the range 5°–150°;
- check that yield ∝ thickness of Au foil;
- other nuclei : check that yield ∝ Z^2 [*roughly*];
- however Rutherford model clearly inconsistent in its "planetary" part: acceleration of charged electrons → radiation → collapse;
- after birth of QM, Rutherford computation redone in Born approx : → same $d\sigma/d\Omega$ [*big luck !*] + no more inconsistency [*next slides*].

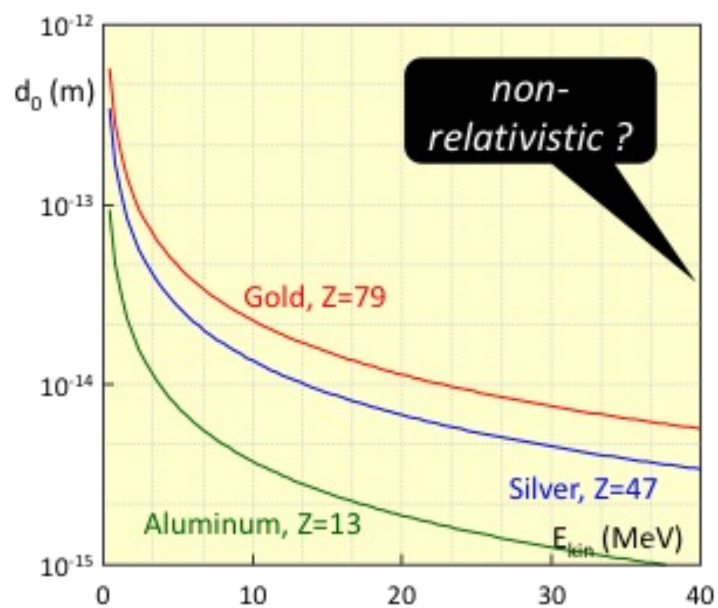
Rutherford scattering: $R_{Nucleus}$



How large is the nucleus ?

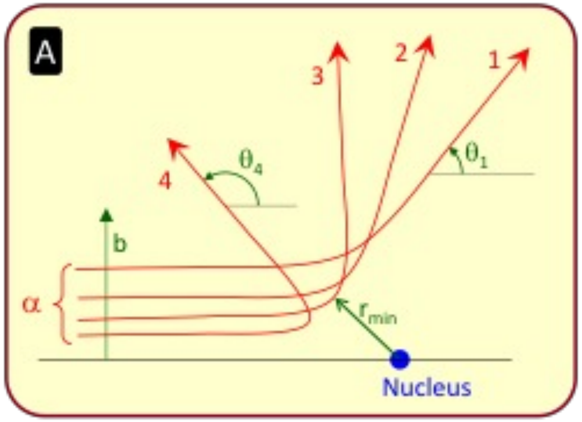
- [*remember the Gauss' theorem*]
- if the α trajectory is completely external to the nucleus, it does *NOT* probe its (*possible*) structure;
- the Rutherford experiment could only limit $R_{nucleus} < 10^{-14}$ m [*still an important result !*];
- to "see" 10^{-15} m → probes with $E_{kin} >$ 20÷30 MeV.

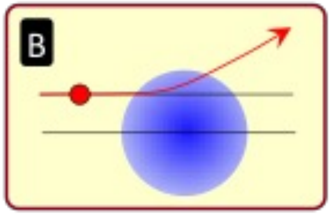


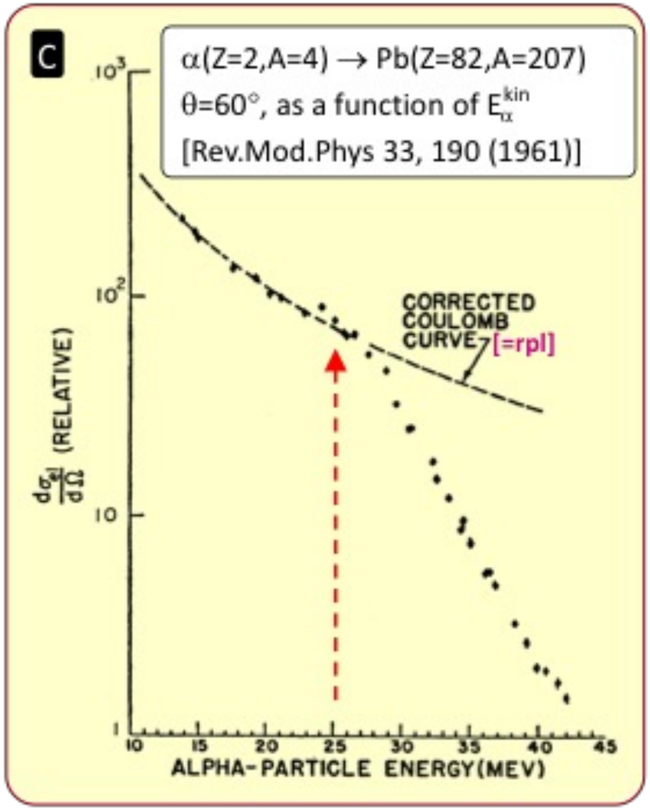


Rutherford scattering: measure $R_{Nucleus}$

- plot [A]: b and r_{min} could *NOT* be measured directly for each event, but Rutherford point-like law (*rpl*) relates $b \leftrightarrow \theta$; in fact $b_{small} \leftrightarrow \theta_{large}$;
- plot [B]: the Gauss' theorem predicts a deviation from rpl, when (E_α^{kin} large) $\rightarrow$ ($r_{min} < R_{nucleus}$) $\rightarrow$ shielding $\rightarrow$ "smaller θ";
- plot [C] (**1961** !!!): a "Rutherford-like" scattering α-Pb; at θ=60°, deviation for E_α^{kin} > 25 MeV;
- at high θ, point-like target $\rightarrow$ larger σ, soft target $\rightarrow$ smaller σ (*deviations from rpl related to size of target*) [*please, remember*].







Q. find r_{min} for Pb, θ = 60°, E_α^{kin} = 25 MeV

A. r_{min} = [☞ formula] = 14 fm.

Hadron structure: kinematics

A "probe", usually <u>assumed point-like</u> (e.g. $e^{\pm}$) hits a hadronic complex system (a nucleus) [*see box*].

In the final state, the probe emerges unchanged, while the nucleus may or may not survive intact:

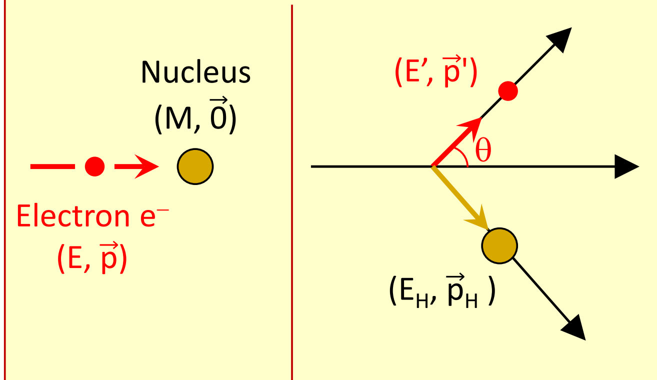
- <u>elastic scattering</u>, when the nucleus is unchanged, i.e. *identical initial and final state particles* (W=M);
- <u>excitation</u>, when the nucleus in the final state is excited, i.e. heavier (W = M* > M);
- a <u>new hadronic system</u>, with n particles (i=1…n):

$$E_H = \sum_{i=1}^{n} E_i; \qquad \vec{p}_H = \sum_{i=1}^{n} \vec{p}_i;$$

$$W = \sqrt{(E_H)^2 - (p_H)^2} = M_{\text{had. sys.}} > M.$$

The underlying idea is to study (*understand ?*) the structure of the hadrons by observing the scattering.

$$e^- N \rightarrow e^- H$$



$$\text{electron } e^{\pm}: \begin{cases} (E, & \vec{p}; & m) & \text{[init.]} \\ (E', & \vec{p}\,'; & m) & \text{[fin.]} \end{cases}$$

$$\text{had. sys.}: \begin{cases} (M, & \vec{0}; & M) & \text{[init.]} \\ (E_H, & \vec{p}_H; & W) & \text{[fin.]} \end{cases}$$

$$\text{4-mom cons.} \rightarrow \begin{cases} \vec{p} + \vec{0} = \vec{p}\,' + \vec{p}_H; \\ E + M = E' + E_H. \end{cases}$$

Hadron structure kinematics: elastic scattering

- To begin with, assume <u>elastic scattering</u>, i.e. "H" = N;
- Define, in the target nucleus ref.sys. :

$$\text{electron } e^{\pm}: \begin{cases} (E, & \vec{p}\,; & m) & [\text{init.}] \\ (E', & \vec{p}\,'; & m) & [\text{fin.}] \end{cases}$$

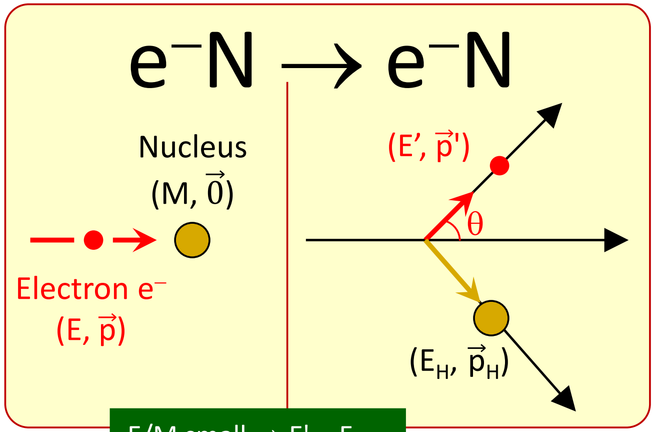
$$\text{nucleus}: \begin{cases} (M, & \vec{0}; & M) & [\text{init.}] \\ (E_H, & \vec{p}_H; & M) & [\text{fin.}] \end{cases}$$

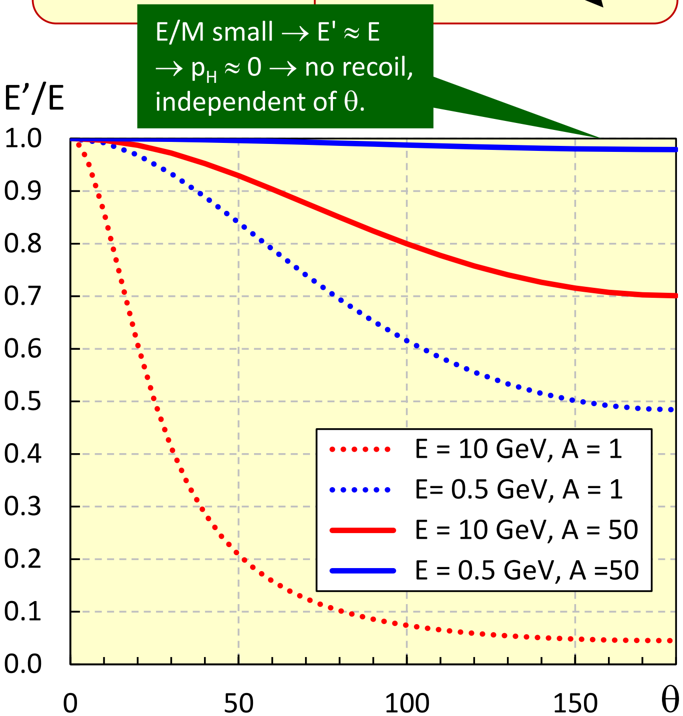
- 4-mom cons. $\rightarrow \begin{cases} \vec{p} + \vec{0} = \vec{p}\,' + \vec{p}_H; \\ E + M = E' + E_H. \end{cases}$

- The relation between the observed quantities (E, E', θ) is [*next slide*] :

$$E' = \frac{E}{1 + \frac{E}{M}(1 - \cos\theta)} = \frac{E}{1 + \frac{2E}{M}\sin^2(\theta/2)} \approx |\vec{p}\,'|;$$

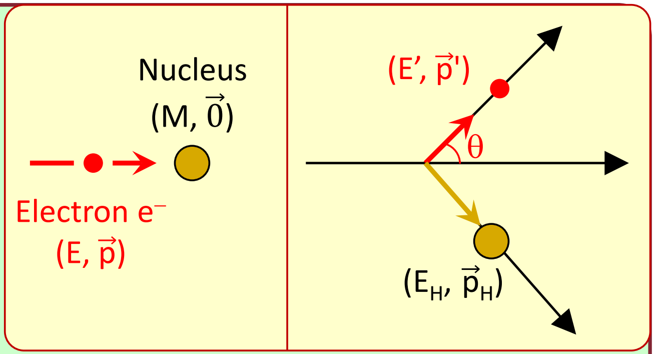
- Therefore, for known initial energy E and fixed M, the final state is defined by <u>one independent variable</u> (E' or θ).





Hadron structure kinematics: elastic scattering E' vσ θ

$$\begin{cases} e^-_{init} & (E,\vec{p};m); \\ N_{init} & (M,\vec{0};M); \end{cases} \qquad \begin{cases} e^-_{fin} & (E',\vec{p}';m); \\ H_{fin} & (E_H,\vec{p}_H;M); \end{cases}$$



4-momentum conservation
$$\begin{cases} E+M=E'+E_H \rightarrow E_H=E+M-E'; \\ \vec{p}+\vec{0}=\vec{p}'+\vec{p}_H \rightarrow \vec{p}_H=\vec{p}-\vec{p}'; \end{cases}$$

Square and subtract
$$\left\{(E_H)^2-(\vec{p}_H)^2=M^2=\left(E^2+M^2+E'^2+2EM-2EE'-2ME'\right)-\left(p^2+p'^2-2pp'\cos\theta\right);\right.$$

Ultra-relativistic approx.
$(m_e \ll E,E') \rightarrow (p \approx E, p' \approx E')$
$$\begin{cases} \cancel{M}^2=\cancel{E}^2+\cancel{M}^2+\cancel{E}'^2+2EM-2EE'-2ME'-\cancel{E}^2-\cancel{E}'^2+2EE'\cos\theta; \\ 0 \quad =EM-EE'-ME'+EE'\cos\theta=EM-E'\left[E\left(1-\cos\theta\right)+M\right]; \end{cases}$$

$$E'=\frac{EM}{M+E(1-\cos\theta)}=\frac{E}{1+\frac{2E}{M}\sin^2\left(\frac{\theta}{2}\right)} \qquad \text{q.e.d.}$$

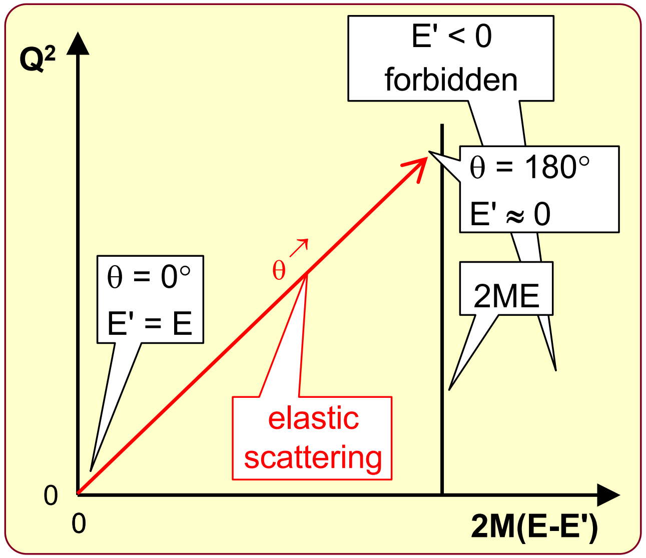
NB – The reaction is planar (why?). The final state is defined by 6 variables. There are 3 (E, $\vec{p}$) conservations and 2 ($m^2=E^2-p^2$) rules. Therefore: 6-5=1 independent variable.

Hadron structure kinematics: Q^2 in elastic scattering

- in the following, $(E, \vec{p}, E', \vec{p}', m, M, \theta)$;
  $[m = m_e \text{ small} \rightarrow E \approx |\vec{p}|,\ E' \approx |\vec{p}'|]$
- new (not independent) variable:
  $\boxed{\vec{q} \equiv \vec{p} - \vec{p}'}$ "momentum transfer";
  $\left[E/M \text{ small} \rightarrow p' = p \rightarrow\ |\vec{q}| = 2|\vec{p}|\sin(\theta/2)\right]$
- relativistic equivalent (p and p' are 4-mom):
  $\boxed{q \ \equiv p - p'} \qquad \left[= (E - E', \vec{p} - \vec{p}')\right]$;
  $Q^2 \equiv -q^2 = -2m_e^2 + 2EE' - 2|\vec{p}||\vec{p}'|\cos\theta$
  $\approx 4EE'\sin^2(\theta/2)$ [defined $\rightarrow$ $Q^2 > 0$];
- $\cancel{E}' = \dfrac{EM}{M + 2E\sin^2(\theta/2)} = \dfrac{EM}{M + Q^2/(2E')} =$
  $= \dfrac{2\cancel{E}'EM}{2E'M + Q^2} \rightarrow 2EM = 2E'M + Q^2$
  $\rightarrow \boxed{Q^2 = 2M(E - E')}$
- [*for elastic scattering one independent variable* $\rightarrow E' = E'(\theta)$, $Q^2 = Q^2(E')$];

Study the kinematical limits:

- $\theta = 0°$: $E' = E$; $Q^2 = 0$;
- $\theta = 180°$: $E - E' = E\dfrac{M+2E}{M+2E} - \dfrac{EM}{M+2E} = \dfrac{2E^2}{M+2E}$
  $(E >> M)$: $E - E' = E \rightarrow E' \approx 0$;
- in conclusion <u>$E > E' > "0"$</u>.
- Plot Q^2 vs $2M(E\text{-}E')$: <u>only a segment allowed</u> [*useless for elastic scatt., but ...*]:



Hadron structure kinematics: importance of |q|, Q^2

an advance of dynamics

The variable $\vec{\mathbf{q}}$ is *very* important:

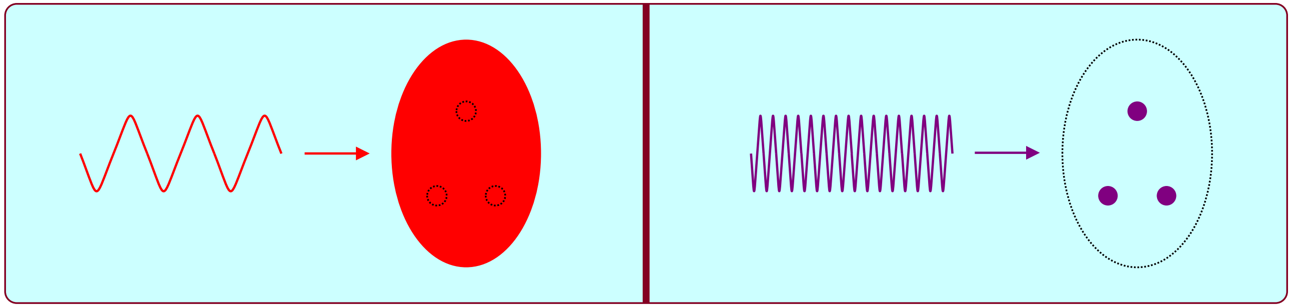
- [if relativistic, use Q^2 or its root $\sqrt{Q^2}$];
- it is related to the deBroglie wavelength of the probe: $ƛ = \hbar/|\vec{q}|$;
- it represents the "scale" of the scattering;
- i.e. structures smaller than $ƛ \sim 1/|\vec{q}|$ are not "visible" to the probe;
- [the uncertainty principle $\Delta p \Delta x \geq \hbar/2$ leads to the same conclusion – *actually it is exactly the same argument*];

Comments:

- large $|\vec{q}| \rightarrow$ large E, but not necessarily the opposite: high-energy & large distance processes do exist;
- the quest for smaller scales leads inevitably to larger Q^2 and therefore to larger E [$\rightarrow$ money and resources...]

____

[as usual] sometimes in the literature the notation is confusing: $Q^2 = -t$, see later;



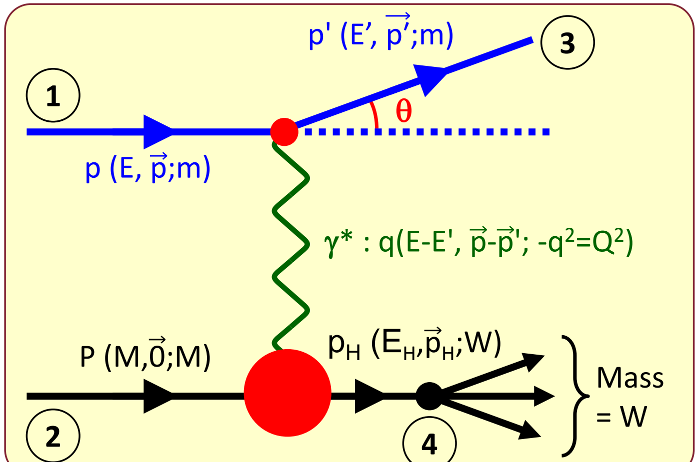
- popular understanding:
  higher Q^2 $\rightarrow$ smaller distance $\rightarrow$
  $\rightarrow$ "better microscope".
- conclusion:
  Q^2 is an important variable, possibly the most important in modern particle physics.

Hadron structure kinematics: inelastic scattering

[in general, $\ell N \to \ell' H$ (ℓ, ℓ' generic leptons); the kinematics is the same, if E_ℓ, $E_{\ell'} >> m_\ell$, $m_{\ell'}$]

Kinematical variables (**ℓ N → ℓ' H**) :

- [$\ell' = \ell$, H=N → elastic];
- 4-mom. in LAB sys (≡ had CM);
- $p_1 = p$, $p_2 = P$, $p_3 = p'$, $p_4 = p_H$;
- q = p' – p [as in previous slides];



Lorentz – invariant variables:

- $\nu = q \cdot P/M = E - E'$ [= energy lost by e^-];
- $Q^2 = -q^2 = 2(EE' - pp'\cos\theta) - m^2 - m^2 \approx 4\ EE'\ \sin^2(\theta/2)$ [= – module of the 4-momentum transfer];
- $x = Q^2 / (2M\nu)$ [later : x-Bjorken x_B, the fraction of the hadron 4-momentum carried by the interacting parton];
- $y = (q \cdot P) / (p \cdot P) = \nu / E$ [= the fraction of the energy lost by the lepton in the target frame];
- $W^2 = (p_H)^2 = (P + q)^2 = M^2 - Q^2 + 2M\nu$ [=(mass)^2 of the hadron system in the final state] : W = M if elastic;
- [with these variables, the (energy)^2 in the CM is $s = (p+P)^2 = (p'+p_H)^2$]

[*next slide*]

Hadron structure kinematics: Q^2, ν, x, y, W^2

$$\begin{cases} e^-_{init} & p\ (E,\vec{p};m_\ell); \\ N_{init} & P\ (M,\vec{0};M); \end{cases} \qquad \begin{cases} e^-_{fin} & p'\ (E',\vec{p}';m_\ell); \\ N_{fin} & p_H\ (E_H,\vec{p}_H;W); \end{cases}$$

p, p', P, P_H, q, Q^2, M, ν, x, y, W^2 Lorentz invariant;

E,E', ... Lab sys (= P at rest).

$$q = p - p' = (E - E', \vec{p} - \vec{p}');$$

m << M (safe approx);

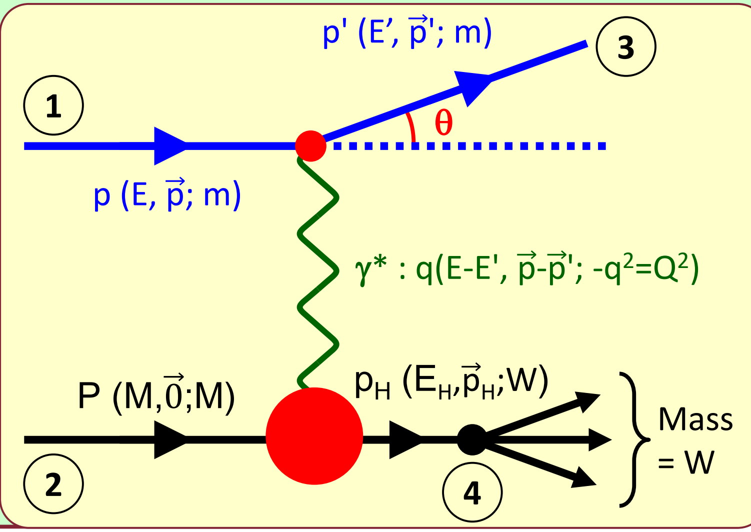
$$q^2 = m^2 + m^2 - 2EE' + 2pp'\cos\theta \approx -2EE'(1-\cos\theta) = -4EE'\sin^2\left(\frac{\theta}{2}\right) \equiv -Q^2;$$

$$\nu \equiv \frac{q\cdot P}{M} = \frac{(E-E')M}{M} = (E-E');$$

warning: x_B is very interesting, *see later*

$$x \equiv \frac{Q^2}{2M\nu}; \qquad y \equiv \frac{q\cdot P}{p\cdot P} = \frac{(E-E')M}{EM} = \frac{E-E'}{E} = \frac{\nu}{E};$$

$$W^2 = p_H^2 = (P+q)^2 = M^2 - Q^2 + 2M\nu.$$



Hadron structure kinematics: the inelastic case

Remarks :

- a lot of kinematical relations, e.g.

  $W^2 = M^2 + 2MEy(1\text{-}x);$

  $Q^2 = 2MExy;$

  $s = M^2 + m^2 + Q^2/(xy);$

- in the elastic case $eN \to eN$ [$ep \to ep$], ν and Q^2 are NOT independent :

  $W^2 = M^2 = (P + q)^2 = M^2 - Q^2 + 2\,M\nu$

  $\to \quad Q^2 = 2M\nu \to Q^2\,/\,(2M\nu) = x = 1;$

- therefore (obviously) in the elastic case, there is only one independent parameter (E' or θ, choice according to the meas.);

- instead, in the inelastic scattering :

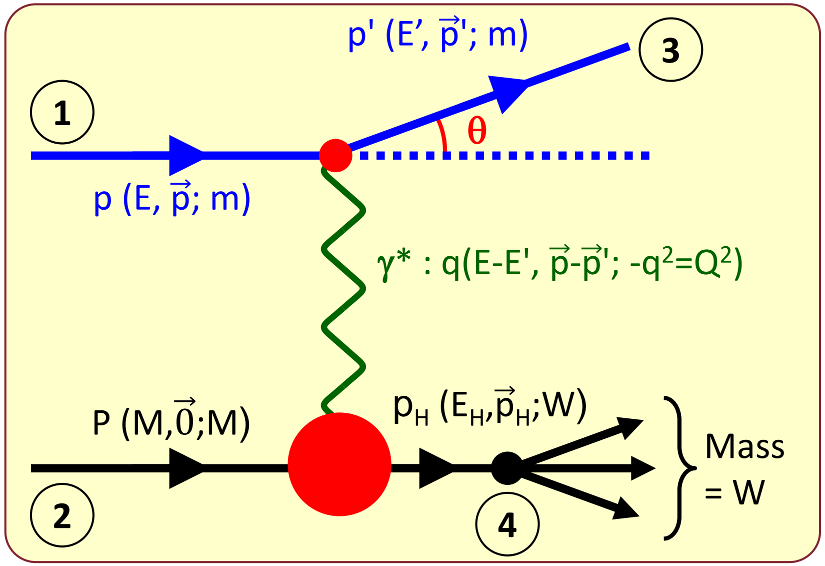
  $Q^2 = M^2 + 2\,M\nu - W^2 =$
  $= 2M\nu - (W^2 - M^2) \leq 2M\nu \to x \leq 1;$

  if W not fixed, Q^2 and ν are independent;

- therefore, in the inelastic case, there are two independent variables;

- in the analysis, choose two among all variables, according to convenience, e.g.:

  (E', θ), (Q^2, ν), (x, y).

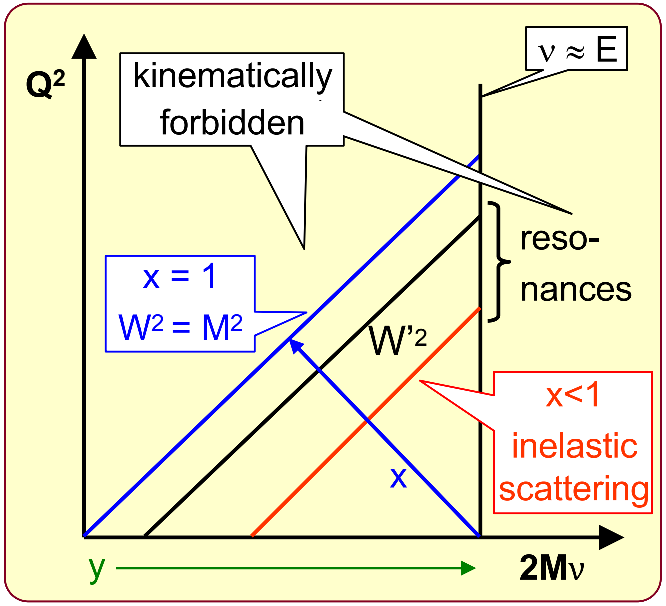


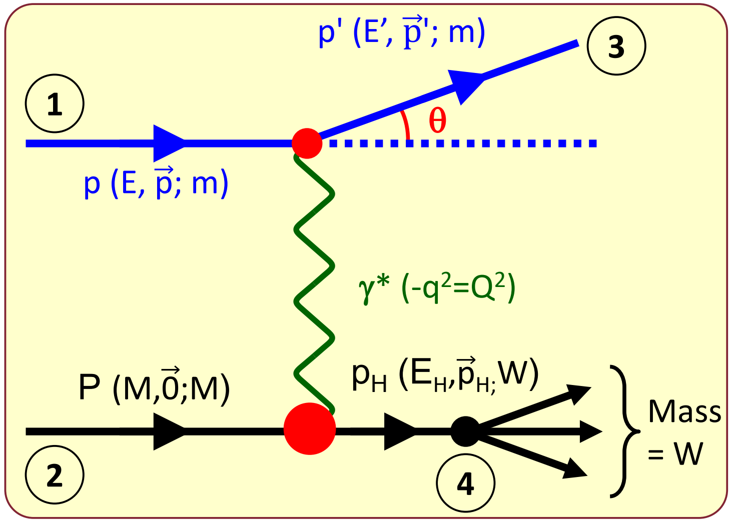
Hadron structure kinematics: DIS

Redefine the kinematics of the scattering process in the plane (Q^2 vs ν) [more precisely (Q^2 vs $2M\nu$)]:

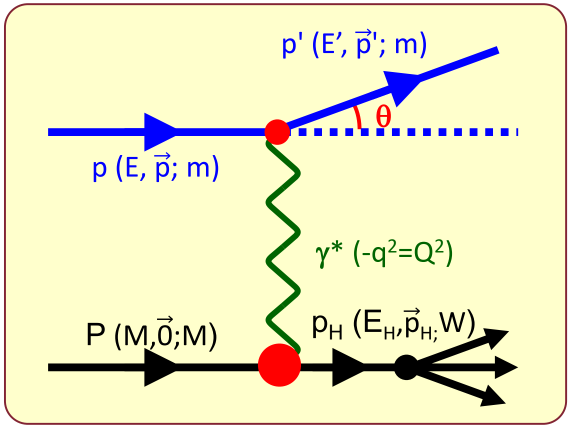
- both are Lorentz-invariant [*but usually used in the lab. frame, where the initial state hadron is at rest*] ;
- $\nu = E - E' \rightarrow 0 \leq \nu \leq E \rightarrow$ only a band is allowed;
- $Q^2 = 4\,EE' \sin^2(\theta/2) \geq 0 \rightarrow$ only the 1st quadrant;
- $x = Q^2 / (2M\nu) \leq 1 \rightarrow 0 \leq x \leq 1 \rightarrow$ only "lower triangle";
- $y = (q \cdot P) / (p \cdot P) = \nu / E \rightarrow 0 \leq y \leq 1$;
- $W^2 = M^2 + 2M\nu - Q^2 \rightarrow$ the bisector x=1 ("/") defines the elastic scattering, where $W^2 = M^2$;
- on the bisector, only θ varies : $\theta = 0 \rightarrow Q^2 = \nu = 0$;
- the loci W'^2 = constant are lines parallel to the bisector $\rightarrow$ some of them define the excited states (one shown in fig.);
- at higher distance from the bisector we have the <u>deep inelastic scattering</u> (*DIS*) and (possibly) new physics.

[*see next slide*]



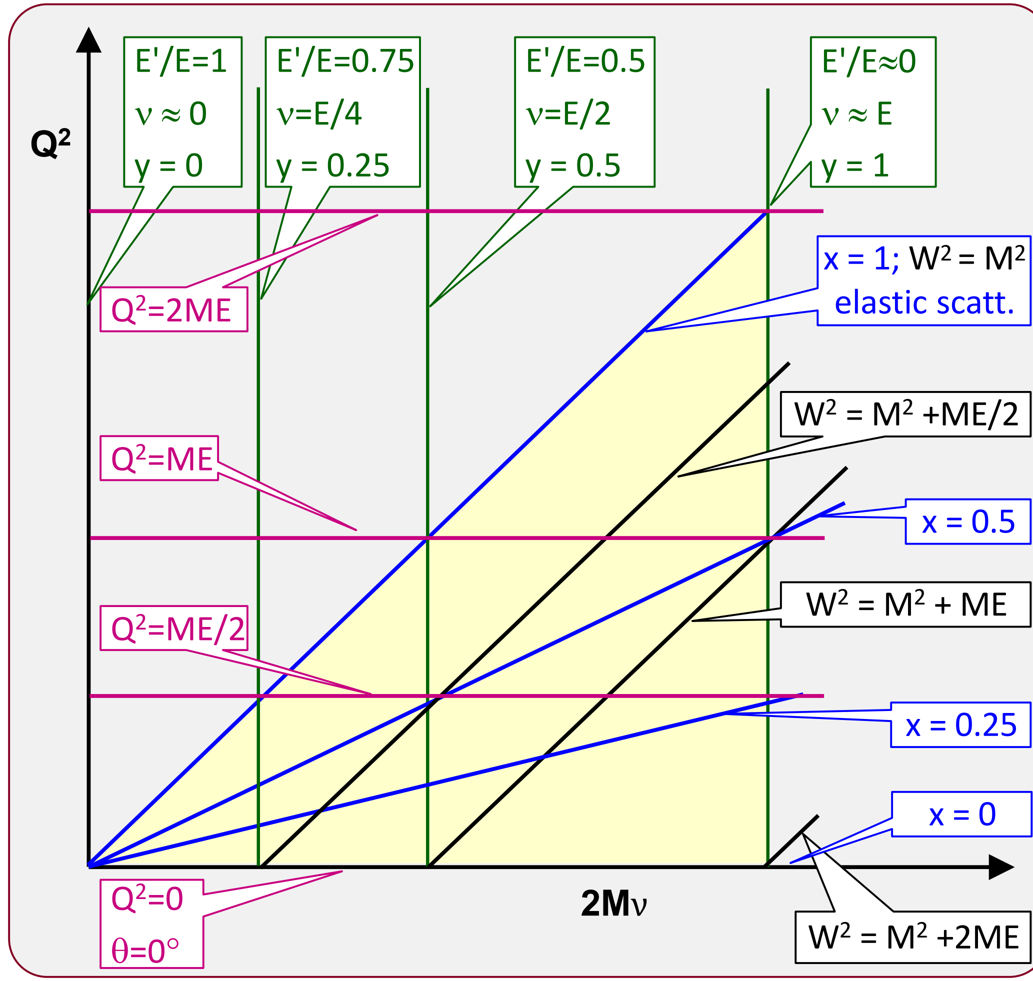


Hadron structure kinematics: summary



0	$<$	$x < 1$
0	$<$	$y < 1$
0	$<$	$\nu < E$
M^2	$<$	$W^2 < M^2+2ME$
0	$<$	$Q^2 < 2ME$
0	$<$	$E' < E$
$0°$	$<$	$\theta < 180°$

limits (some only if E >> M).



"higher Q^2"

Elastic scattering e-N: $\sigma_{\text{Rutherford + Mott}}$

- The scattering α-Nucleus actually takes place between two nuclei (e.g. He^{++}-Au);
- not suitable for measuring a (possible) nucleus structure → replace the α with a more (?) point-like probe: electron (e^-);
- if the process is e.m., the dynamics of the eN scattering can be described by the Rutherford formula (use the momentum transfer $\vec{q}=\vec{p}-\vec{p}'$) [*next slide*]:

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} = \frac{4Z^2\alpha^2E'^2}{|\vec{q}|^4}; \quad |\vec{q}| = 2|\vec{p}|\sin\frac{\theta}{2}.$$

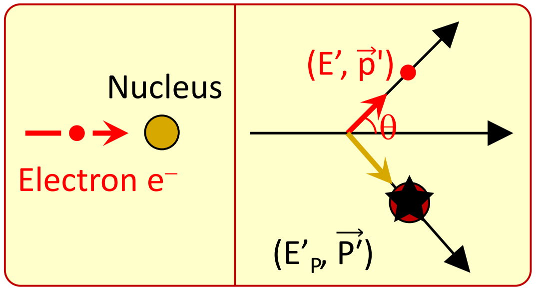
- in relativistic quantum mechanics the elastic scattering cross-section is described by a formula, due to Mott :

$$\left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \times \left(1-\beta^2\sin^2\frac{\theta}{2}\right) \rightarrow$$

$$\xrightarrow{\beta=|\vec{p}|/E\to1} \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \cos^2\frac{\theta}{2} = \frac{4Z^2\alpha^2E'^2}{|\vec{q}|^4}\cos^2\frac{\theta}{2}.$$

- similar to the Rutherford formula, the Mott* cross-section neglects (a) the nucleus dimension and (b) its recoil*.
- unlike Rutherford, Mott takes into account the e^- spin (=½) [*next slide*].

NB The "*" in the definition of Mott* means that the "no-recoil" approximation is used → leave it out when the recoil is considered ("Mott*" → "Mott"].

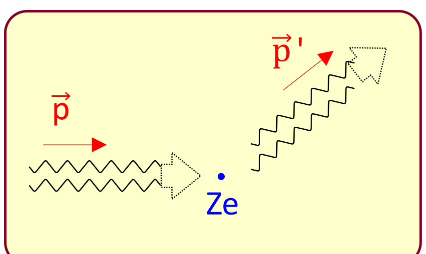


Sir Nevill Francis Mott

Elastic scattering e-N: Rutherford + q.m.

q.m. calculation

- already computed in classical approx.
- non-relativistic q.m. + Born approx.;
- Coulomb potential;
- negligible recoil;
- initial (i) and final (f) particle as plane waves [*see introduction + box*];
- $\vec{q}=\Delta\vec{p}$ (as usual);
- $\hbar$ and c for the last time;
- V(r=∞) does NOT contribute, because of other nuclei → in the last integration, do not use the value at r=∞.



$$V(r) = -\frac{Z\alpha\hbar c}{r}; \quad \vec{q} = \Delta\vec{p} = \vec{p} - \vec{p}'; \quad q = |\vec{q}| = 2p\sin(\theta/2);$$

$$\psi_{i,f} = e^{i\vec{p}\cdot\vec{r}/\hbar}/\sqrt{\mathcal{V}}; \quad \psi_f = e^{i\vec{p}'\cdot\vec{r}/\hbar}/\sqrt{\mathcal{V}}; \quad \frac{dn}{dE'} = \frac{\mathcal{V}4\pi p'^2}{v'(2\pi\hbar)^3};$$

$$\mathcal{M}_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \frac{1}{\mathcal{V}}\int e^{-i\vec{p}'\cdot\vec{r}/\hbar} V(\vec{r}) e^{i\vec{p}\cdot\vec{r}/\hbar} d^3\vec{r} =$$

$$= -\frac{1}{\mathcal{V}}\iiint \frac{Z\alpha\hbar c}{r} e^{i\vec{q}\cdot\vec{r}/\hbar} r^2 dr \sin\theta d\theta d\varphi = -\frac{4\pi}{\mathcal{V}}\frac{Z\alpha\hbar^3 c}{q^2};$$

$$\boxed{\frac{d\sigma}{d\Omega}} = \frac{1}{4\pi}\left[\frac{2\pi}{\hbar}|\mathcal{M}_{fi}|^2 \frac{dn}{dE'}\frac{\mathcal{V}}{v'}\right] \xrightarrow{v'=c,p'=E'/c}$$

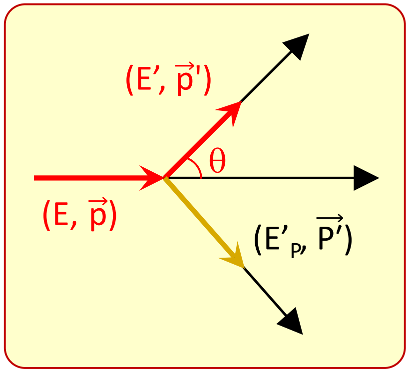
$$= \frac{1}{2\hbar}\left|\frac{4\pi}{\mathcal{V}}\frac{Z\alpha\hbar^3 c}{q^2}\right|^2 \frac{\mathcal{V}E'^2}{2\pi^2 c^3\hbar^3}\frac{\mathcal{V}}{c} = \boxed{\frac{4Z^2\alpha^2\hbar^2E'^2}{q^4c^2}}$$

$$\int_0^{2\pi} d\varphi \int_0^\infty r dr \int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta/\hbar} = 2\pi\int_0^\infty dr \int_{-r}^{r} e^{iqt/\hbar} dt \quad [t = r\cos\theta]$$

$$= \frac{2\pi\hbar}{iq}\int_0^\infty dr\left(e^{iqr/\hbar} - e^{-iqr/\hbar}\right) = \frac{2\pi\hbar}{iq}\frac{\hbar}{iq}\left[e^{iqr/\hbar} + e^{-iqr/\hbar}\right]^{r=0} = -\frac{4\pi\hbar^2}{q^2}.$$

Elastic scattering e-N: helicity

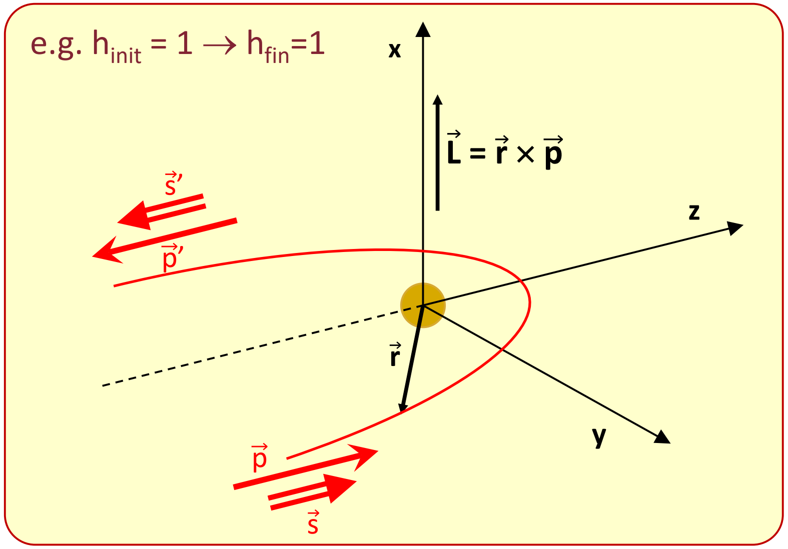
The $\cos^2(\theta/2)$ factor in $[d\sigma/d\Omega]_{Mott}$ comes from Dirac equation; it is understood by considering the extreme case of θ~180°.



For relativistic particles ($\beta \to 1$), the helicity h (the projection of spin along momentum) is conserved :

$$h = \frac{\vec{s} \cdot \vec{p}}{|\vec{s}| \cdot |\vec{p}|}.$$

The conservation requires the "spin flip" of the electron between initial and final state, because the momentum also flips at θ=180°.



In this condition, the angular momentum is NOT conserved, if the nucleus does NOT absorb the spin variation (e.g. because it is spinless). Therefore the scattering for $\theta \approx 180°$ is forbidden.

The factor $\cos^2(\theta/2)$ in the Mott formula is connected to the spin and describes the magnetic part of the interaction.

Elastic scattering e-N: experiment

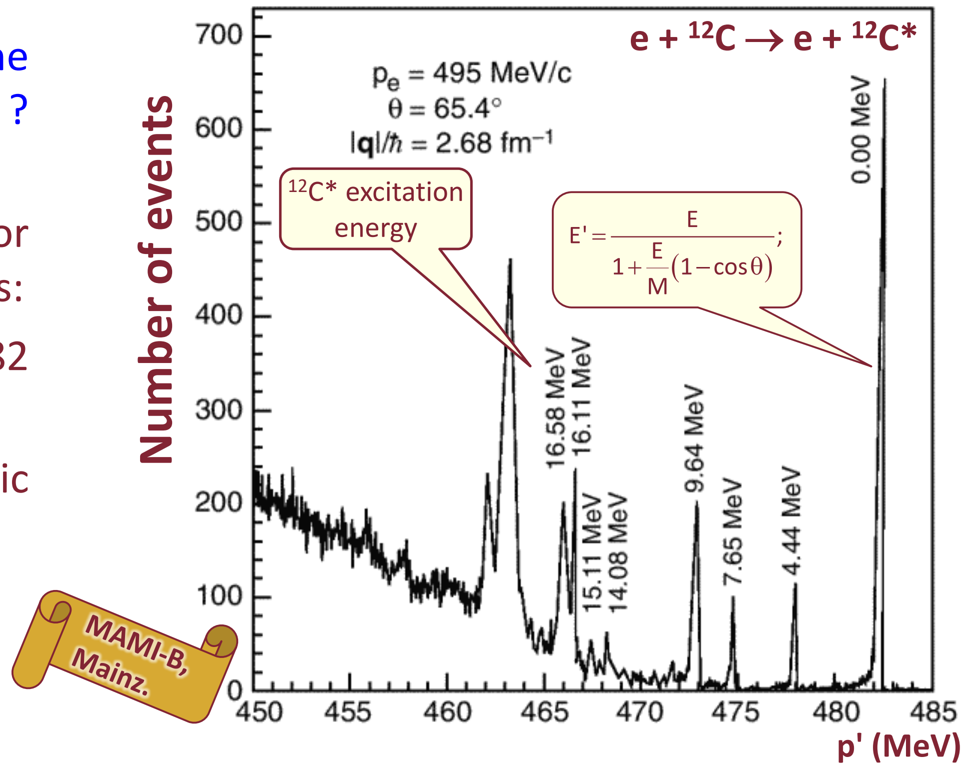
Is the experiment consistent with the kinematics of the elastic scattering ? Get e + ^{12}C data.

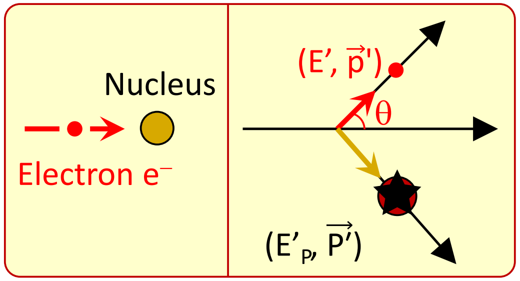
The plot of the number of events, for fixed E_{init} at fixed θ, shows many peaks:

- the expected elastic ($E' \approx p'$ = 482 MeV),
- a rich structure, due to inelastic scattering:

$$e + {}^{12}C \rightarrow e + {}^{12}C^*$$

[^{12}C* = excited carbon, mass M*].





- the expected elastic [e + ^{12}C → e + ^{12}C] is there;
- but "*more things in heaven, than in your philosophy*";
- **back to elastic scattering !**
- kinematics ok, dynamics ?

→ measure $d\sigma/d\Omega$ vs θ !!!

Form factors: definition

- The experimental $d\sigma/d\Omega$ agrees with the Mott cross-section only for small $|\vec{q}|$;
- otherwise, the cross section is smaller;
- possibly the reason is the structure of the nucleus, which results in a smaller effective charge, as seen by the projectile (Gauss' theorem);

→ define the **form factor** [$\mathcal{F}(\vec{q})$], as the **Fourier transform of the charge distribution function** ρ:

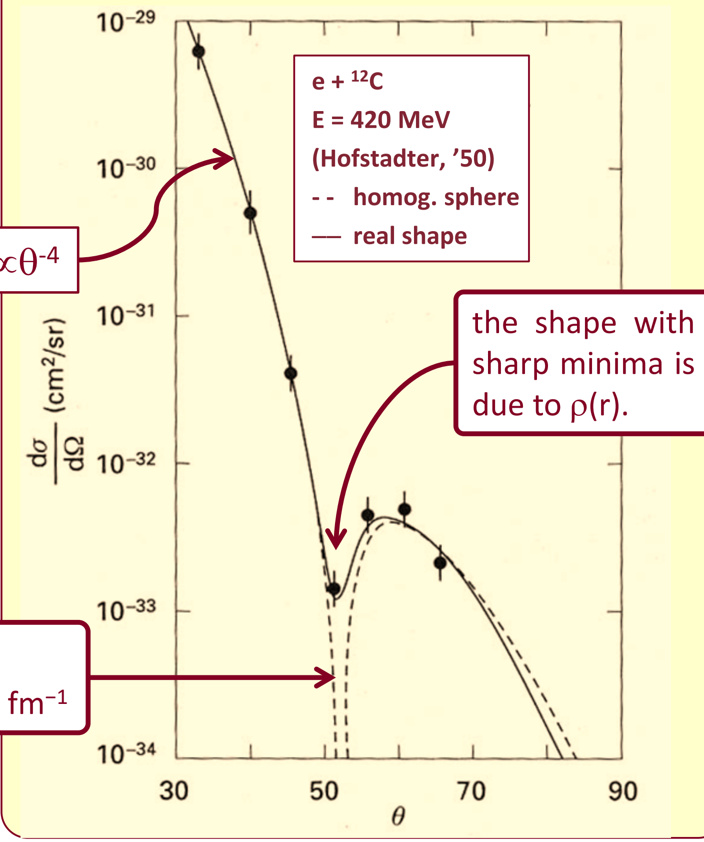
$$\rho(\vec{x}) = Ze f(\vec{x}), \qquad \int f(\vec{x}) d^3x = 1; \qquad \vec{q} = \vec{p} - \vec{p}\,';$$

$$\boxed{\mathcal{F}(\vec{q}) = \int e^{\left(i\frac{\vec{q}\cdot\vec{x}}{\hbar}\right)} f(\vec{x}) d^3x};$$

- if $f(\vec{x}) = \delta(\vec{x}) \quad \rightarrow \quad \mathcal{F}(\vec{q}) = 1$.
- if $\rho(\vec{x})$ depends only on $|\vec{x}|$ [*next slides*]:

$$\left[\frac{d\sigma}{d\Omega}\right]_{exp} = \left[\frac{d\sigma}{d\Omega}\right]^*_{Mott} \times \left|\mathcal{F}(q^2)\right|^2.$$

form factors are measurable, at least in principle.

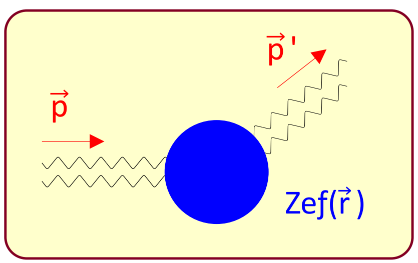


[*in the following, we will discuss only the case with spherical symmetry* $\rho(r)$, *when* $\mathcal{F}(\vec{q})$ depends on $q=|\vec{q}|$].

Form factors: q.m. definition

q.m. calculation

- non-relativistic q.m. + Born approx.;
- Coulomb potential;
- negligible recoil;
- initial (i) and final (f) particle as plane waves with ƛ << nucleus size [*see little box*];
- charge distribution $f(\vec{r})$, normalized to 1;
- $\vec{q} = \vec{p} - \vec{p}\,'$ and $\mathcal{F}(q^2)$ as defined before.



$$V(\vec{r}) = -\int d^3\vec{r}\,' \frac{Z\alpha f(\vec{r}\,')}{4\pi\left|\vec{r}-\vec{r}\,'\right|};$$

$$\psi_i = e^{i(\vec{p}\cdot\vec{x}-Et)}/\sqrt{\mathcal{V}}; \qquad \psi_f = e^{i(\vec{p}\,'\cdot\vec{x}-Et)}/\sqrt{\mathcal{V}};$$

$$\mathcal{M}_{fi} = \left\langle \psi_f \,|\, V(\vec{r}) \,|\, \psi_i \right\rangle = \frac{1}{\mathcal{V}}\int e^{-i\vec{p}\,'\cdot\vec{r}}\, V(\vec{r})\, e^{i\vec{p}\cdot\vec{r}}\, d^3\vec{r} =$$

$$= -\frac{1}{\mathcal{V}}\int\!\!\int e^{i\vec{q}\cdot\vec{r}} \frac{Z\alpha f(\vec{r}\,')}{4\pi\left|\vec{r}-\vec{r}\,'\right|} d^3\vec{r}\,' d^3\vec{r} =$$

$$= -\frac{1}{\mathcal{V}}\int\!\!\int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')} e^{i\vec{q}\cdot\vec{r}\,'} \frac{Z\alpha f(\vec{r}\,')}{4\pi\left|\vec{r}-\vec{r}\,'\right|} d^3\vec{r}\,' d^3\vec{r} =$$

$$= \left[-\frac{1}{\mathcal{V}}\int e^{i\vec{q}\cdot\vec{R}} \frac{Z\alpha}{4\pi\left|\vec{R}\right|} d^3\left|\vec{R}\right|\right] \times \left[\int f(\vec{r}\,') e^{i\vec{q}\cdot\vec{r}\,'} d^3\vec{r}\,'\right] =$$

$$= \mathcal{M}_{fi}^{point} \times \mathcal{F}(q^2)$$

$$\rightarrow \boxed{\left[\frac{d\sigma}{d\Omega}\right]_{\text{non-point}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{point}} \times \left|\mathcal{F}(q^2)\right|^2}.$$

$\vec{R} = \vec{r} - \vec{r}\,'$

$\mathcal{V}$ = volume

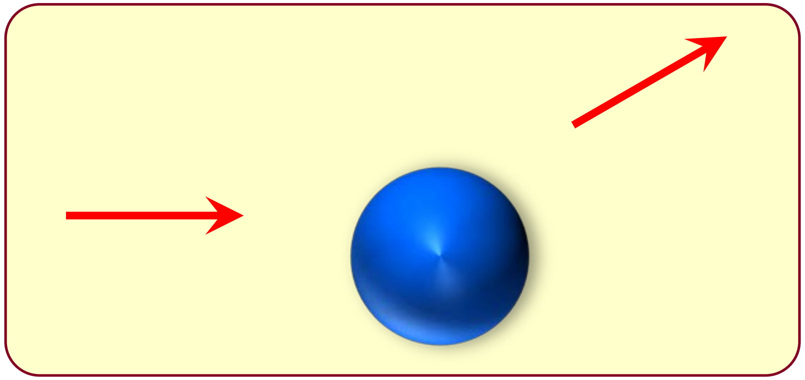
Form factors: radial symmetry

In principle, the function $\rho(r)$ may be computed by measuring $\mathcal{F}(q^2)$ and then, e.g. numerically:

$$\rho(r) = \frac{Ze}{(2\pi)^3} \int \mathcal{F}(q^2) e^{-i\frac{qr}{\hbar}} d^3q$$

However, the range of q accessible to experiments is limited; therefore, the behavior of $\mathcal{F}(q^2)$ for q^2 large (i.e. <u>r small, *the interesting region*</u>) has to be extrapolated with reasonable assumptions.

In the next slides, examples of $\rho(r)$ and $\mathcal{F}(q^2)$ are computed (e.g. the case of a homogeneous sphere of radius R).



Compute the symmetrical case[1]; neglect the nuclear recoil :

$$\mathcal{F}(q^2) = \frac{1}{S}\int e^{i\frac{\vec{q}\cdot\vec{x}}{\hbar}} f(\vec{x}) d^3x = \qquad [f(\vec{x}) = f(r) \rightarrow]$$

$$= \frac{2\pi}{S}\int_0^\infty f(r)\, r^2 dr \int_{-1}^{1} e^{i\frac{qr\cos\theta}{\hbar}} d\cos\theta =$$

$$= \frac{2\pi}{S}\int_0^\infty f(r)\, r^2 \frac{2}{2}\frac{\hbar}{iqr}\left[e^{i\frac{qr}{\hbar}} - e^{-i\frac{qr}{\hbar}}\right] dr =$$

$$= \frac{4\pi}{S}\int_0^\infty f(r)\, r^2 \frac{\sin(qr/\hbar)}{qr/\hbar} dr;$$

$$S = 4\pi\int_0^\infty f(r)\, r^2 dr \qquad [=1 \text{ if normalized}];$$

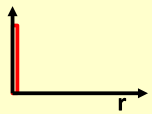
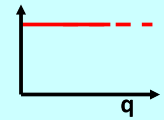
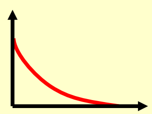
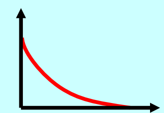
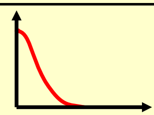
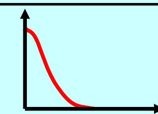
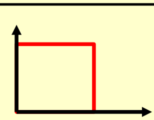
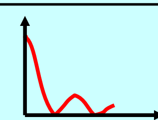
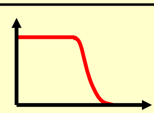
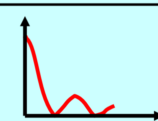
---

[1] $d\sigma/d\Omega$, both Rutherford and Mott, is scale-independent. However, if $\rho(r)$ depends on a scale (e.g. by a sphere radius), form factors break the scale invariance of the dynamics.

Form factors: examples

$$f(\vec{r}) = f(|\vec{r}|)$$

$$f(r) = \frac{1}{(2\pi)^3} \int \mathcal{F}(q^2) e^{-i\frac{qr}{\hbar}} d^3q \qquad \mathcal{F}(q^2) = 4\pi \int_0^\infty f(r)\, r^2 \frac{\sin(qr/\hbar)}{qr/\hbar} dr$$

Charge distribution	f(r)		form factor	$\mathcal{F}(q^2)$		~ example
point-like	$\delta(r)/(4\pi)$		constant	1		$e^{\pm}$
exponential	$(a^3/8\pi)$ $\exp(-ar)$		dipolar	$(1+q^2/a^2\hbar^2)^{-2}$		p
gaussian	$[a^2/(2\pi)^{3/2}]$ $\exp(-a^2r^2/2)$		gaussian	$\exp[-q^2/(2a^2\hbar^2)]$		^6Li
homog. sphere	$3/(4\pi R^3)$ $r \le R$; 0 $r>R$		oscill.	$3\alpha^{-3}(\sin\alpha - \alpha\cos\alpha)$ $\alpha = \lvert q \rvert R/\hbar$		– (see)
sphere with soft surface	$\rho_0 / [1 + e^{(r-c)/a}]$		oscill.			^{40}Ca

Fermi (Woods-Saxon) function

Form factors: homogeneous sphere

Homogeneous sphere with unit charge :

$$\rho(r) = f(r) = \begin{cases} \rho_0 = \dfrac{3}{4\pi R^3} & r \le R \\ 0 & r > R \end{cases}$$

$$\mathcal{F}(q^2) = 4\pi \int_0^\infty f(r)\, r^2 \frac{\sin(qr/\hbar)}{qr/\hbar} dr =$$

$$= \frac{4\pi\hbar\rho_0}{q} \int_0^R r \sin\left(\frac{qr}{\hbar}\right) dr = \qquad \left[w = \frac{qr}{\hbar}; \bar{W} = \frac{qR}{\hbar} \right]$$

$$= \frac{4\pi\hbar^3\rho_0}{q^3} \int_0^{\bar{W}} w \sin w\, dw = \frac{4\pi\hbar^3\rho_0}{q^3} \left[\sin w - w \cos w\right]_0^{\bar{W}} =$$

$$= \frac{4\pi\hbar^3\rho_0}{q^3} \left[\sin\left(\frac{qR}{\hbar}\right) - \frac{qR}{\hbar} \cos\left(\frac{qR}{\hbar}\right) \right] =$$

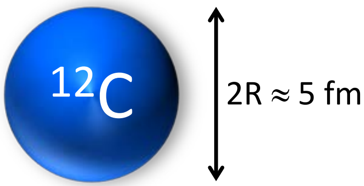
$$= \frac{3\hbar^3}{q^3 R^3} \left[\sin\left(\frac{qR}{\hbar}\right) - \frac{qR}{\hbar} \cos\left(\frac{qR}{\hbar}\right) \right]$$

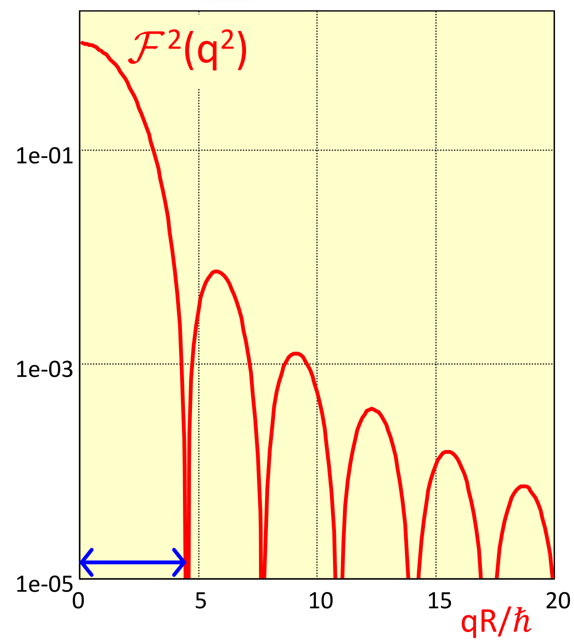
if $qR/\hbar$ [= t] $\to 0$ $\mathcal{F} \approx 3/t^3 [(t - t^3/6) -$ $- t(1 - t^2/2)] = 1.$	first minimum : $qR/\hbar = \tan(qR/\hbar)$ $\to qR/\hbar \approx 4.5$

By comparing the first minimum with the experiment of ^{12}C ($q/\hbar \approx 1.8\ fm^{-1}$), we get :

$$R \approx 4.5\, r_{min} = 4.5/1.8 \approx 2.5\ fm$$

i.e. ^{12}C is approximately a hard sphere with radius of 2.5 fm.





Form factors: $<r^2>$

Study the behavior for $q \to 0$:

$$\mathcal{F}(q^2) = \iiint e^{i\frac{qr\cos\theta}{\hbar}} f(r) r^2 dr\, d\cos\theta\, d\varphi =$$

$$= 2\pi \int_0^\infty f(r) r^2 dr \int_{-1}^{1} \left[1 + i\frac{qr}{\hbar}\cos\theta - \frac{1}{2}\left(\frac{qr}{\hbar}\right)^2 \cos^2\theta + \ldots \right] d\cos\theta =$$

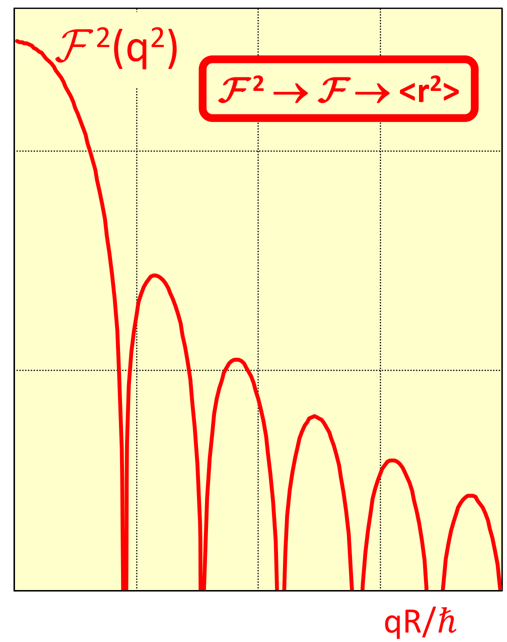
$$= 4\pi \int_0^\infty f(r) r^2 dr + 0 - \frac{4\pi}{6}\frac{q^2}{\hbar^2}\int_0^\infty f(r) r^4 dr + \ldots =$$

$$= 1 - \frac{1}{6}\frac{q^2 < r^2 >}{\hbar^2} + \ldots$$

with $< r^2 > \equiv \iiint r^2 f(\vec{x}) d^3x = 4\pi \int_0^\infty r^2 f(r) r^2 dr.$

i.e. $< r^2 > = -6\hbar^2 \left.\frac{d\mathcal{F}(q^2)}{dq^2}\right|_{q^2=0}.$

The parameter $<r^2>$ contains the information of the charge distribution.



Simple problem : check that for the homogeneous sphere, both directly and from the definition :

$<r^2> = 3R^2/5$.

Form factors: solution

Simple problem : check that for the homogeneous sphere, both directly and from the definition :

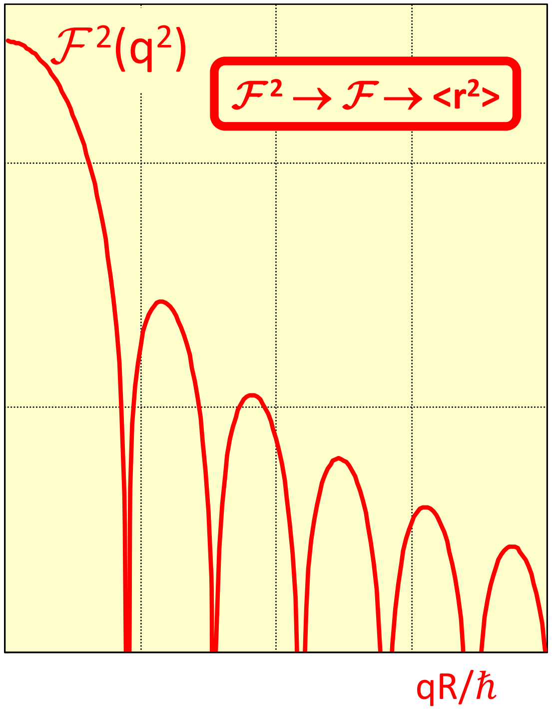
$\langle r^2 \rangle = 3R^2/5$.

$$\left\langle r^n \right\rangle = \frac{1}{V}\iiint r^n d^3x = \frac{4\pi}{V}\int_0^R r^n r^2 dr =$$

$$= \frac{4\pi}{V}\frac{R^{n+3}}{n+3} = \frac{4\pi R^{n+3}}{n+3}\frac{3}{4\pi R^3} =$$

$$= \frac{3}{n+3}R^n$$

$$\xrightarrow{n=2} \left\langle r^2 \right\rangle = \frac{3}{5}R^2$$

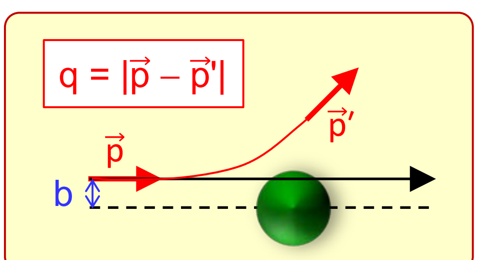


[qed, too easy to enjoy]

Form factors: q→0 vs q→∞

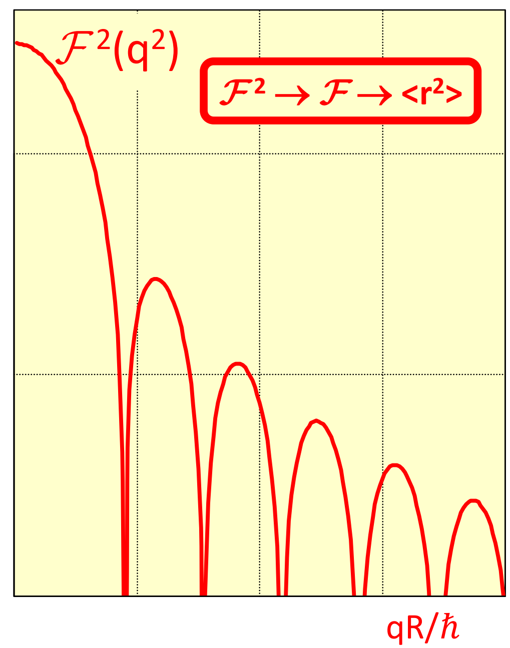
The limits $q \to 0, \to \infty$ have a deep meaning:

- q is (approximately) the conjugate variable of b, the impact parameter of the projectile wrt the target center:
  - → for q very small (i.e. b very large), the target behave as a point-like object;
  - → for q quite small (i.e. b quite large) it behaves as a coherent homogeneous charged sphere with radius $\sqrt{<r^2>}$;
  - → large q probes the nucleus at small b;
- "new physics" (a substructure emerging at very small distance) requires very large q, which in turn is only possible if a large projectile energy is available.



The same story has repeated many times, from Rutherford to the LHC, but at smaller b (i.e. larger q). This fact is the main justification for higher energy accelerators …

… and (unfortunately) larger experiments, larger groups, more expensive detectors, politics, troubles, … [*the usual "laudatio temporis acti", forgive me*]



Form factors: shape of nuclei

Summary of systematic study of the form factors for nuclei [just results, no details]:

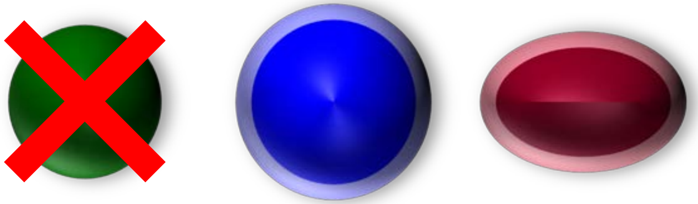
- heavy nuclei :
  - NOT "homogeneous spheres" with a sharp edge;
  - similar to **spheres with a soft edge**;
  - charge distribution is well reproduced by a standard Fermi function :

    $\rho_{charge}(r) = \rho_0 / [1 + e^{(r-c)/a}]$;
  - for large A (see figure) :

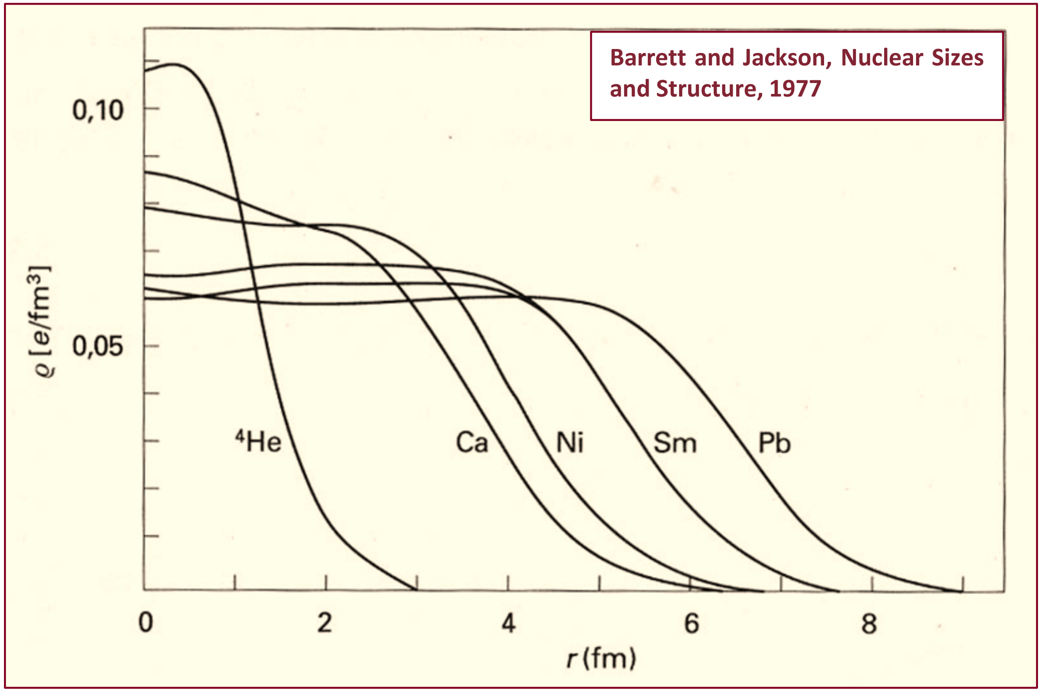
    $c \approx 1.07 \text{ fm} \times A^{1/3}$ ["radius"]

    $a \approx 0.54 \text{ fm}$ ["skin"];

$V_{nucleus} \propto A \rightarrow c \approx r_{nucleus} \propto A^{1/3}$



- light nuclei (^4He, $^{6,7}Li$, ^9Be) more Gaussian-like;
- all these nuclei have spherical symmetry;
- lanthanides (rare earths) are more like ellipsoids [*think to an experiment to show it*].



Form factors: nuclear density

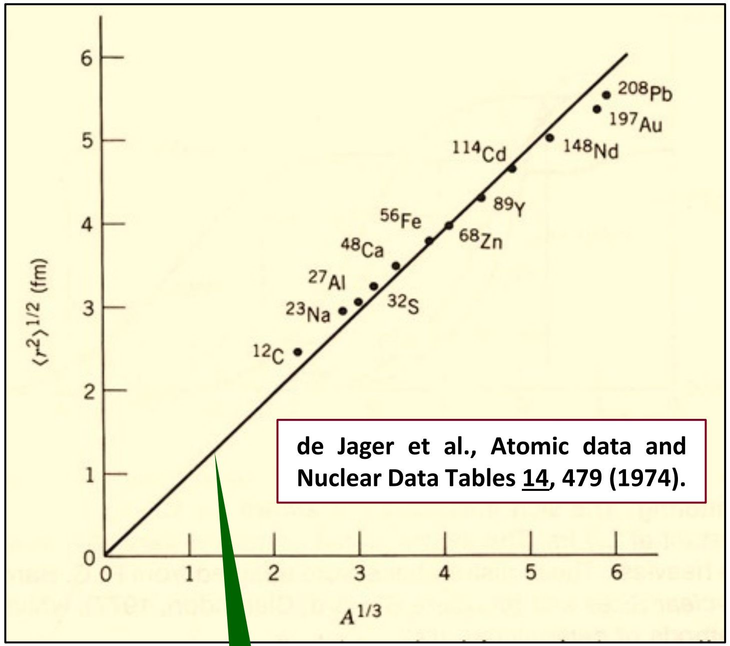
Compute the nuclear densities of p and n
$[q_p \, \rho_Q = dq/dV \, , \, m_p \, \rho_p = dm_p/dV]$:

- assume in the nucleus homogeneous and equal distribution of p and n;
- then:
  - ➢ $\rho_Q = \rho_p$ = proton density;
  - ➢ ρ_n = neutron density = ρ_p;
  - ➢ ρ_T = nuclear density = $\rho_p + \rho_n$;
- compute :
  - ➢ $\rho_T = \rho_p + \rho_n = \rho_p + N \rho_p / Z = A \rho_Q / Z$;
  - ➢ $A = V \rho_T = 4\pi/3 \, R^3 \rho_T$;
  - ➢ ρ_T = 0.17 nucleons / fm^3 (from ρ_0 of previous slide);
- $$\frac{4\pi}{3} R^3 = \frac{4\pi}{3} R_0^3 A \rightarrow$$

$$R_0 = \frac{R}{\sqrt[3]{A}} = \sqrt[3]{\frac{3}{4\pi\rho_T}} \approx 1.12 \text{ fm.}$$

➢ in fair agreement with "c" [*previous slide*] and with the slope of the fig.:

R_0^{exp} = 1.23 fm.



de Jager et al., Atomic data and Nuclear Data Tables 14, 479 (1974).

for light nuclei, the model is NOT valid: do NOT plot them.

e-N scattering: higher energy

Probing smaller space scales requires larger energies, both in the initial and final state [*today experiments work at the TeV scale → ~10^{-18} m = 10^{-3} fm*].

High-energy + q.m. corrections to the Rutherford formula [1st already discussed]:

- consider the electron spin [*Rutherford had only bosons !!!*];
- include the target recoil in the Mott cross section [Perkins-1971, 197];
- use 4-vectors p and p' to describe the scattering [instead of $\vec{p}$ and $\vec{p}'$]:

$$q^2 = (p - p')^2 = 2m^2 - 2\left(EE' - |\vec{p}||\vec{p}'|\cos\theta\right)$$

$$\approx -4EE'\sin^2(\theta/2);$$

$$Q^2 \equiv -q^2 \approx 4EE'\sin^2(\theta/2).$$

- for scattering eN, consider the magnetic moment of the nucleons, by introducing the parameter $\tau = Q^2/(4M^2)$ [*next slide*].

Description of the scattering

$\downarrow$ no electron spin, no recoil, no magn. moment

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} = \frac{4Z^2\alpha^2E'^2}{|\vec{q}|^4};$$

$\downarrow$ + electron spin

$$\left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \times \boxed{1 - \beta^2\sin^2\frac{\theta}{2}};$$

$$\boxed{\approx \cos^2(\theta/2)}$$

$\downarrow$ + recoil

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} \times \frac{E'}{E};$$

$\downarrow$ + magn. moment

$$\boxed{"\tau"}$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{point, spin½}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(1 + 2\boxed{\frac{Q^2}{4M^2}}\tan^2\frac{\theta}{2}\right).$$

e-N scattering: magnetic moments

For particles of mass m, charge e:

- point-like,
- spin ½;

the Dirac equation assigns an intrinsic magnetic dipole moment

$\mu_C = g\, e\, \hbar\, /\, (4\, m)$;

g = "gyromagnetic ratio" = 2;

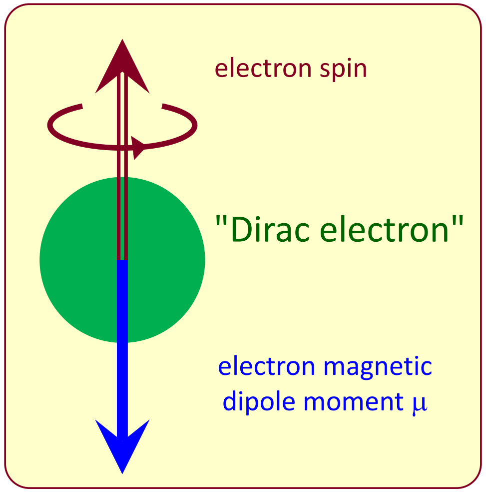
- an ideal "Dirac-electron" has a magnetic dipole moment

  $\mu_e = e\hbar/(2m_e) \approx 5.79 \times 10^{-5}$ eV/T;

- the first measurements roughly confirmed this value.
- for neutral particles (neutron ?) $\mu_N = 0$;
- this effect adds to the cross-section a term, corresponding to the "spin flip" probability, proportional to:

  - $\sin^2(\theta/2)$ [*cfr. the "Mott* factor"*];
  - $1/\cos^2(\theta/2)$ (to remove the non-flip dependence);
  - μ_N^2 ($\propto 1/M^2$);
  - Q^2 (mag field induced by the e $)^2$;
  - $$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{point,}\\ \text{spin½}}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(1 + 2\frac{Q^2}{4M^2}\tan^2\frac{\theta}{2}\right).$$

- Therefore the spin-flip is particularly relevant for large Q^2 and large θ.



e-N scattering: anomalous magnetic moments

In the nuclei and nucleons sector the experiments measured the following quantities :

☺ nuclear magnetism is a combination of the intrinsic magnetic moments of the nucleons and their relative orbital motions;

☺ all nuclei with Z=even and N=even have $\mu_{nuclei} = 0$;

➢ define for the nucleons (proton and neutron) the Dirac value

$$\mu_N = e\hbar/(4m_N) \approx 3.1525\times10^{-14} \text{ MeV/T};$$

➢ if p and n were ideal Dirac particles, they should have

$$\mu_p = 2\mu_N, \qquad \mu_n = 0,$$

i.e. in conventional notation

$$g_p/2 = \mu_p/\mu_N = 1, \qquad g_n/2 = 0;$$

☹ instead, experiments found *anomalies*

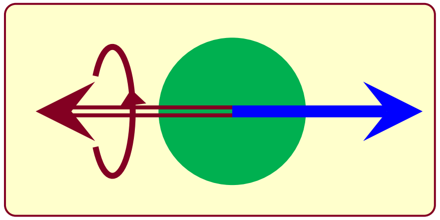
$$g_p/2 = +(2.7928473508 \pm 0.0000000085),$$

$$g_n/2 = -(1.91304273 \quad \pm 0.00000045);$$

☺ therefore, there are other effects which contribute to the magnetic moments, i.e. p and n are NOT ideal spin-½ point-like Dirac particles;

☺ [maybe] they are NOT point-like;

☺ in this case, their "g" is due to their (possibly complicated) internal structure, in analogy with the nuclear case.



e-N scattering: Rosenbluth cross-section

In the eN scattering, the main contribution is from single photon exchange [see fig.].

The **eeγ* vertex** is well under control, with three point-like, well-understood particles.

Instead, the **NN'γ* vertex** is the unknown, due to the internal structure of the proton.

<u>Strategy</u> : assume a simpler process (N = Dirac fermion), compare it with exp., then modify the theory, inserting parameters which model the nucleon structure.

Take also into account the spin and magnetic moment, both of the electron and the nucleon.

"Generalize" the cross section by defining the **<u>Rosenbluth cross-section</u>**, function of TWO form factors, both <u>dependent on Q^2</u>:

- $G_e(Q^2)$ for the electric part (no spin-flip);
- $G_M(Q^2)$ for the magnetic one (spin-flip).

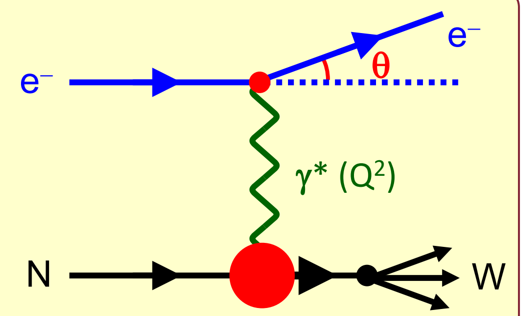
[formerly : $G_e(Q^2) = \mathcal{F}(Q^2)$, no G_M].

For a charged Dirac fermion f_D, proton, neutron :

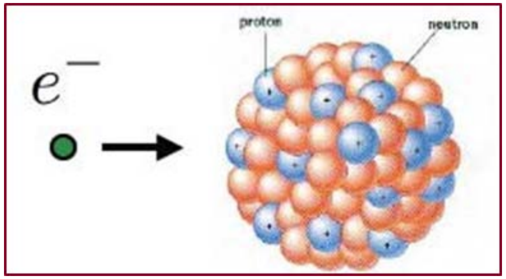
- f_D : $G_E^f(\text{any } Q^2) = 1$, $G_M^f(\text{any } Q^2) = 1$;
- p : $G_E^p(Q^2 = 0) = 1$, $G_M^p(Q^2=0) \approx 2.79$;
- n : $G_E^n(Q^2 = 0) = 0$, $G_M^n(Q^2=0) \approx -1.91$.

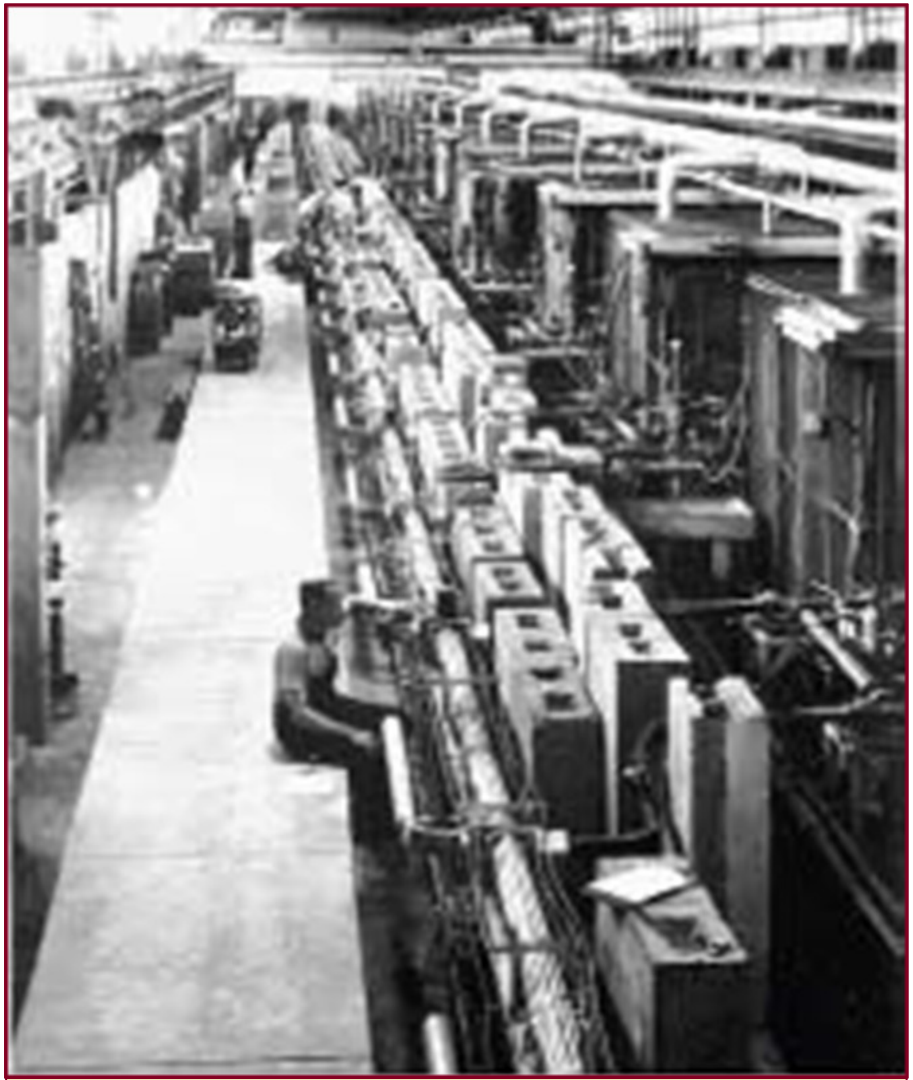
$$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Rosen}\\\text{bluth}}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right);$$

$$\tau = \frac{Q^2}{4M^2}; \qquad G_E = G_E\left(Q^2\right); \qquad G_M = G_M\left(Q^2\right).$$



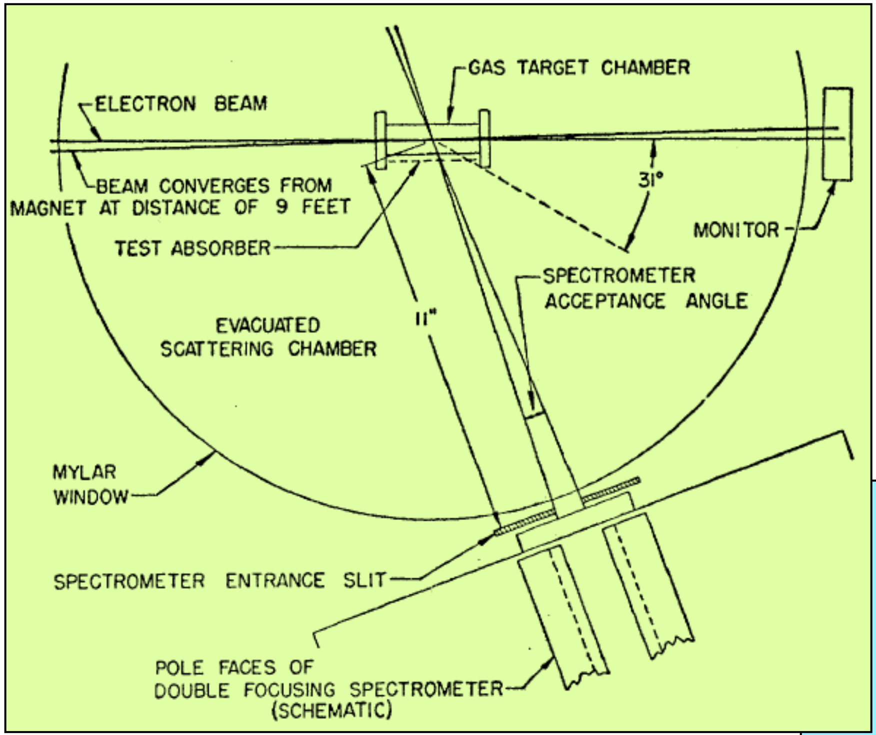
Proton structure: Mark3 Linac





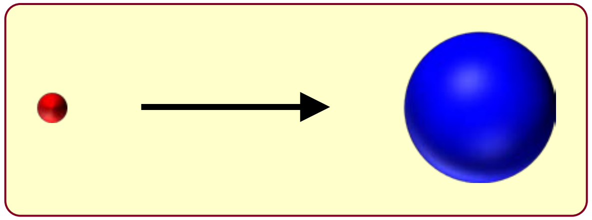
Mark 3 electron Linac – Stanford University – 1953

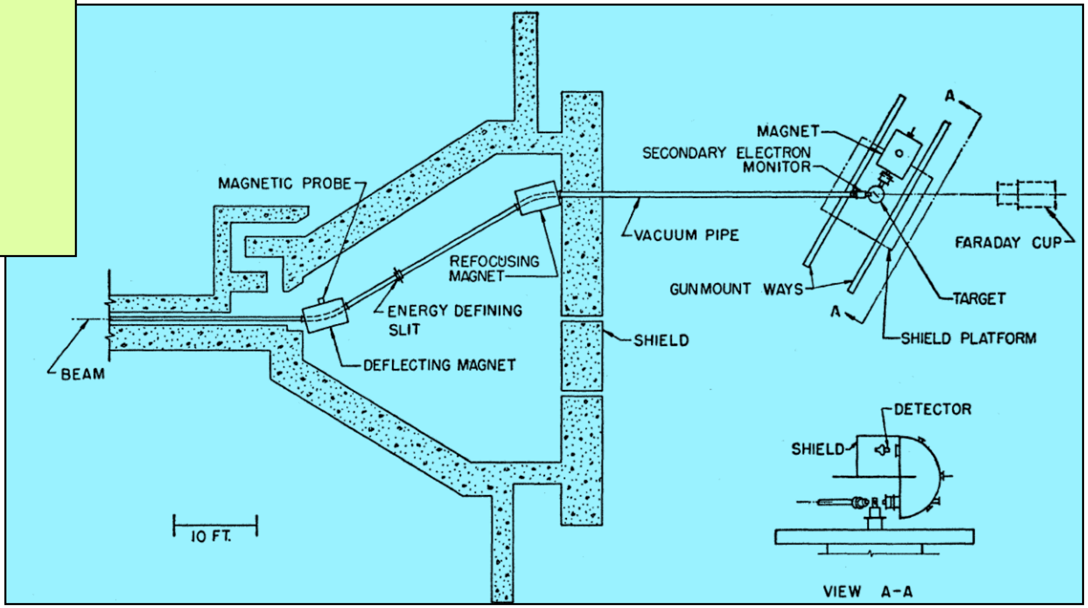
Proton structure: setup



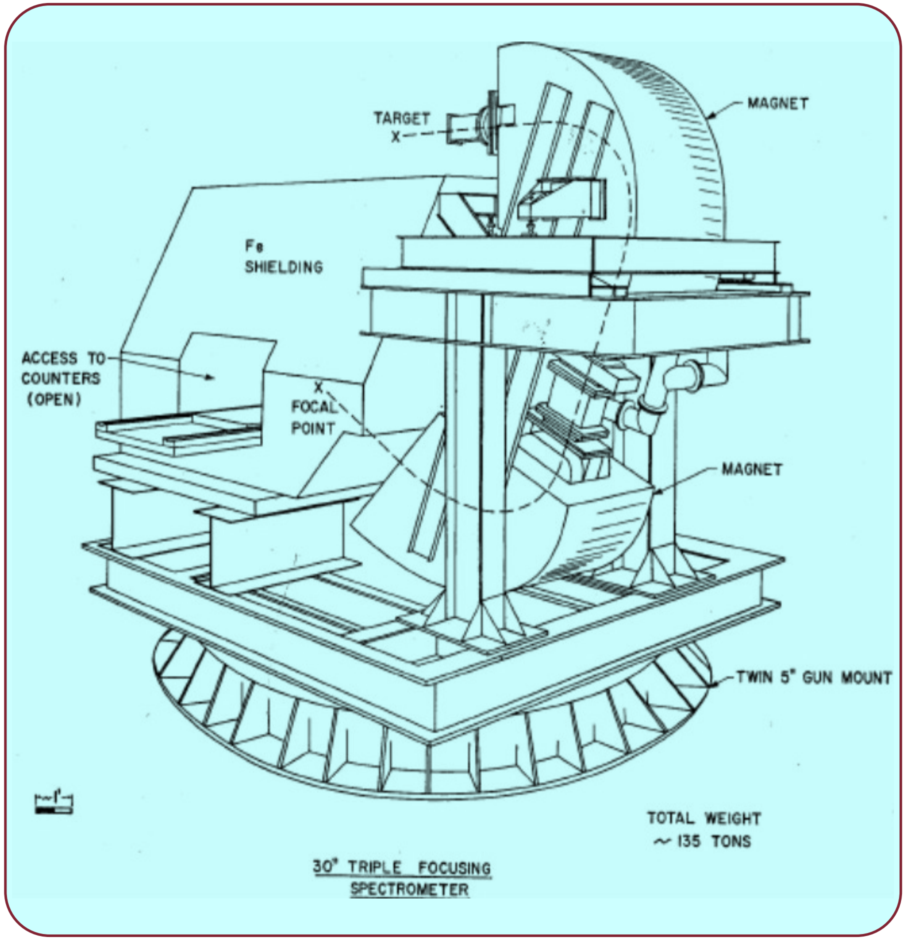
Stanford - 1956

Robert Hofstadter



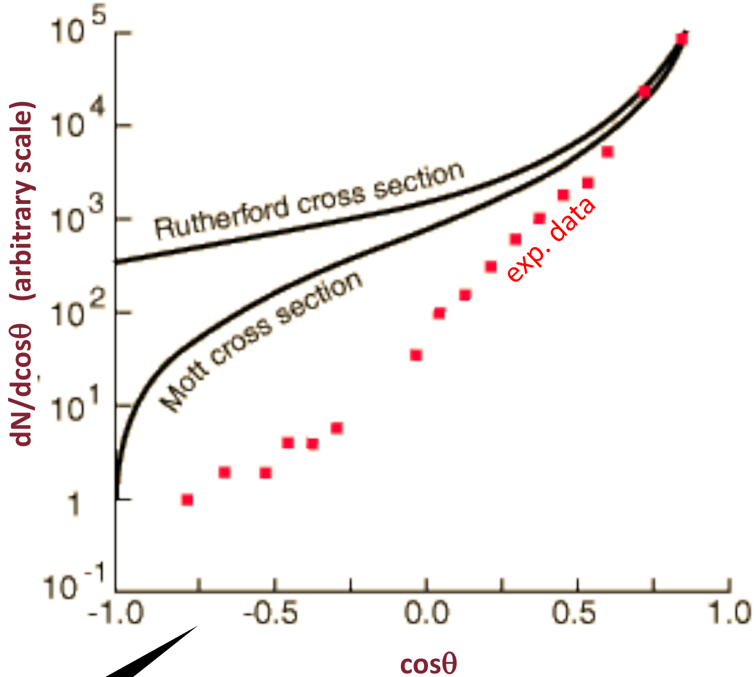


Proton structure: Mark 3 detector



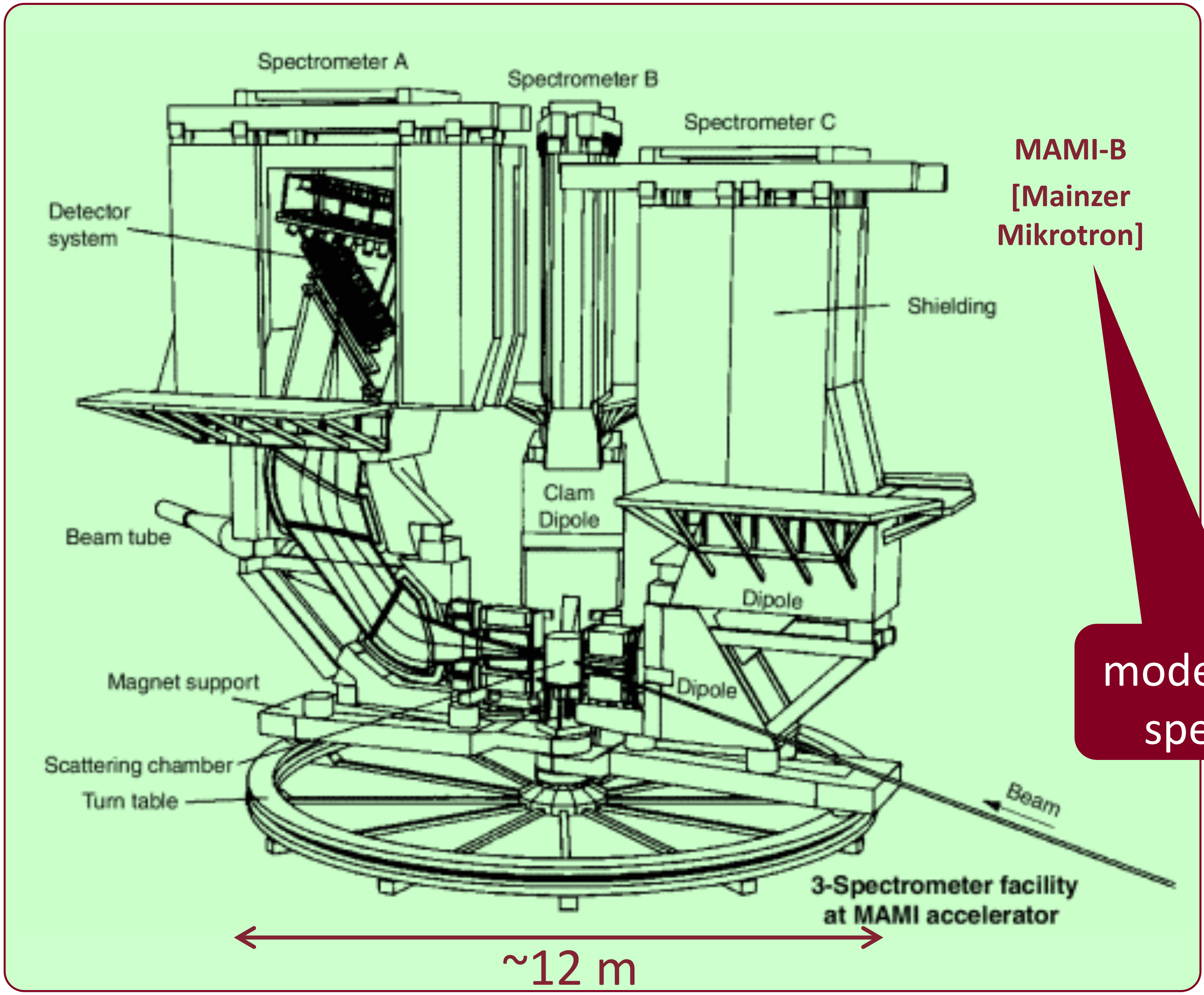
Hofstadter et al., Phys. Rev. 92, 978 (1953)

$p(e^-) = 125$ MeV



A summary of Hofstadter experiments, see later

Proton structure: Mami B



MAMI-B [Mainzer Mikrotron]

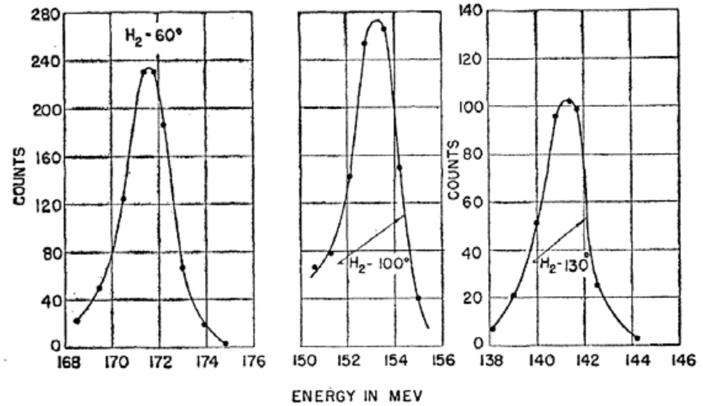
modern magnetic spectrometer

Proton structure: quality check

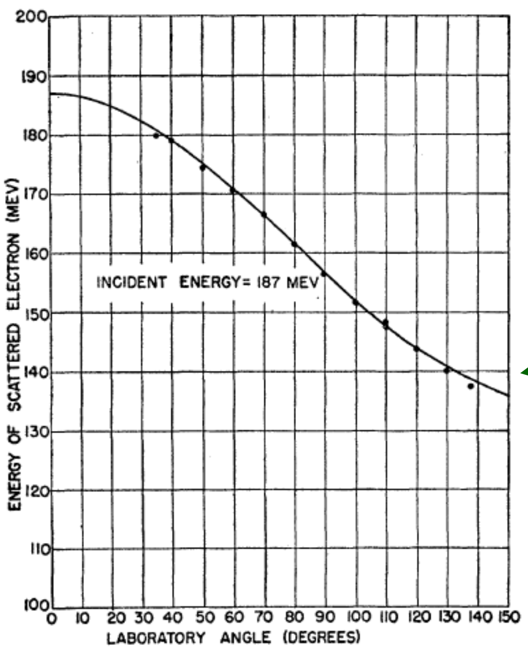
In 1956 the Hofstadter spectrometer measured the elastic ep → ep. It measured θ in the range 35°-138°, and therefore Q^2, using the relations :

$$E' = \frac{E}{1+E(1-\cos\theta)/M};$$

$$Q^2 = 2EE'(1-\cos\theta).$$



Plot E' for E = 185 MeV at fixed θ (60°, 100°, 130°) [*in a perfect experiment, expect* δ_{Dirac}].



Show the plot $\theta = \theta(E')$.

Result:
Kinematics ok. Experiment under control. Study the dynamics.

Proton structure: results

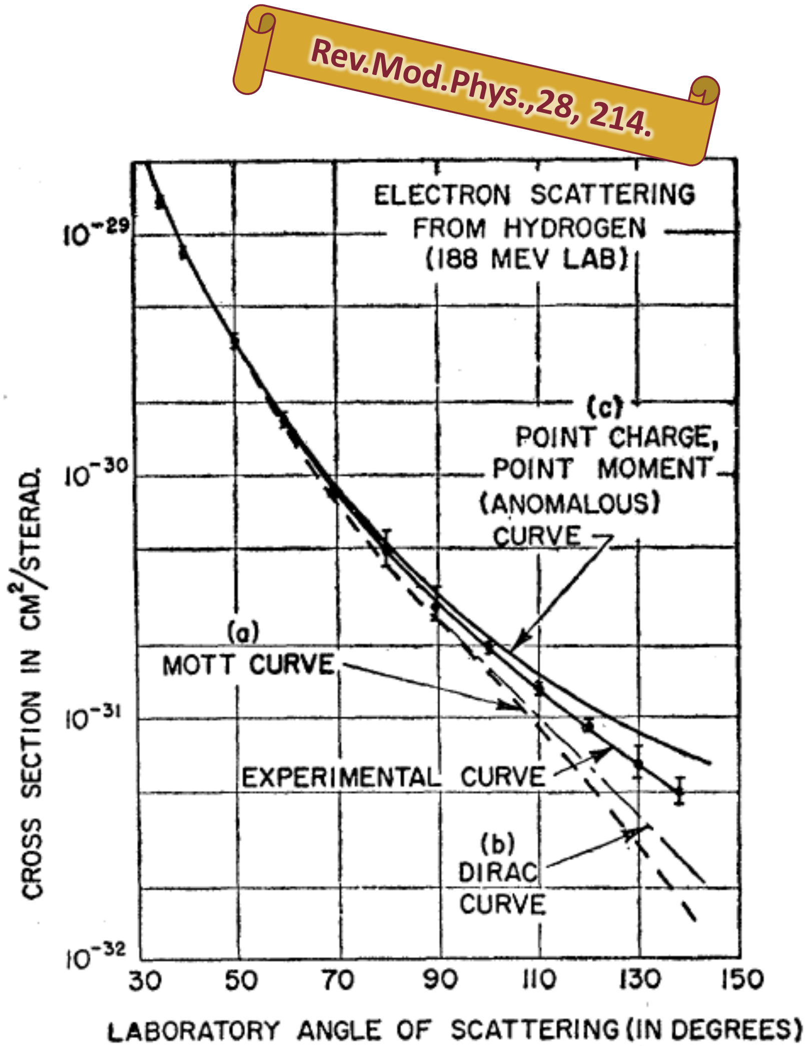
Show the measured cross section:

- at small θ, Mott (a), Dirac (b), Rosenbluth with fixed G_E,G_M (c) and data ("exp. curve") all agree;
- however, for large θ (i.e. large Q^2, small distance), the data do NOT agree with ANY theoretical prediction : they are larger than (a) and (b), but smaller than (c);
- the disagreement with (a) and (b) was foreseen (proton $g_p \neq 2$);
- the one with (c) is more interesting : it shows a dependence on Q^2 (i.e. on scale) $\rightarrow$ the proton is NOT point-like;
- Hofstadter measured ($r_{rms} \equiv \sqrt{<r^2>}$, see) :

  r^p_{rms} = (0.77±0.10) × 10^{-15} m;

  r^α_{rms} = (1.61±0.03) × 10^{-15} m.

… and received the 1961 Nobel Prize in Physics.



Proton structure: $G^{p,n}_{E,M}$ vs Q^2

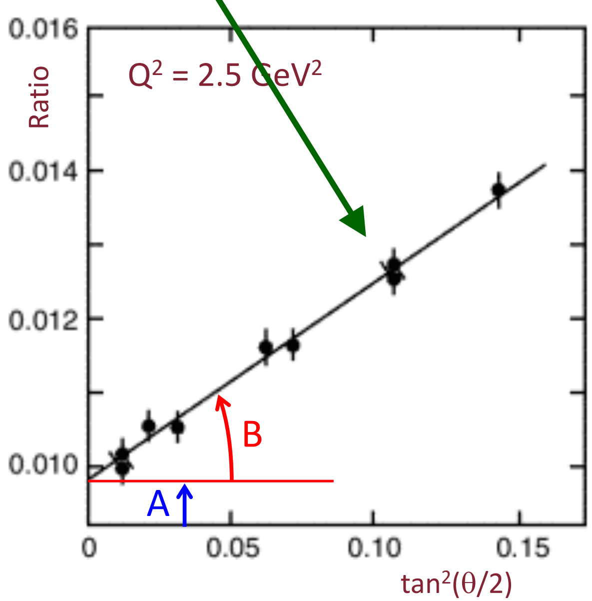
Write the Rosenbluth formula, <u>at fixed Q^2</u>, :

$$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Rosen}\\\text{bluth}}} \Big/ \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right).$$

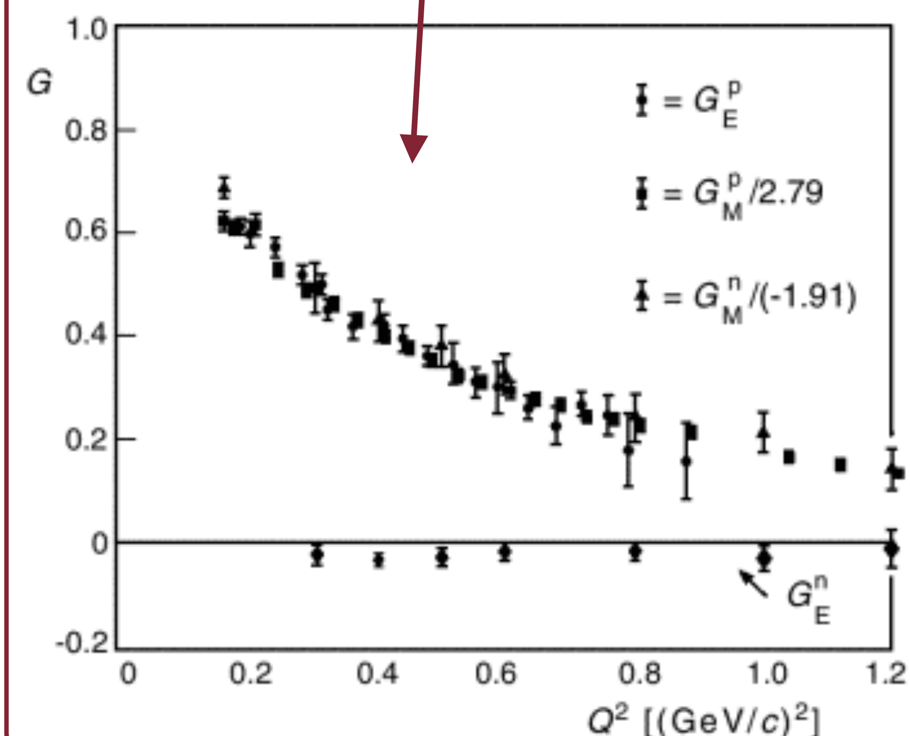
→ Ratio(E, θ, fixed Q^2) = A + B $\tan^2(\theta/2)$;

→ measure (A, B at fixed Q^2) vs $\tan^2(\theta/2)$;

→ get G^p_E, G^p_M, (G^n_E, G^n_M) at fixed Q^2 (example shown)



By repeating it at many Q^2, the full dependence can be measured (SLAC, '60s).



Proton structure: $G^{p,n}_{E,M}$ remarks

- The fig. shows that the electric and magnetic form factors tend to a "universal" function of Q^2, with a **dipolar** shape :

$$G_E^p(Q^2) \approx \frac{G_M^p(Q^2)}{2.79} \approx \frac{G_M^n(Q^2)}{-1.91} \approx G(Q^2) =$$

$$= \left(1 + \frac{Q^2}{A^2}\right)^{-2}; \qquad A^2 = 0.71\ \text{GeV}^2$$

- From the curve, it is possible to derive the function $\rho(r)$, at least where the 3- and 4-momentum coincide, i.e. at small Q^2. It turns out :

  $\rho(r) \approx \rho_0\, e^{-ar}, \;\; a \approx 4.27\ \text{fm}^{-1}$.

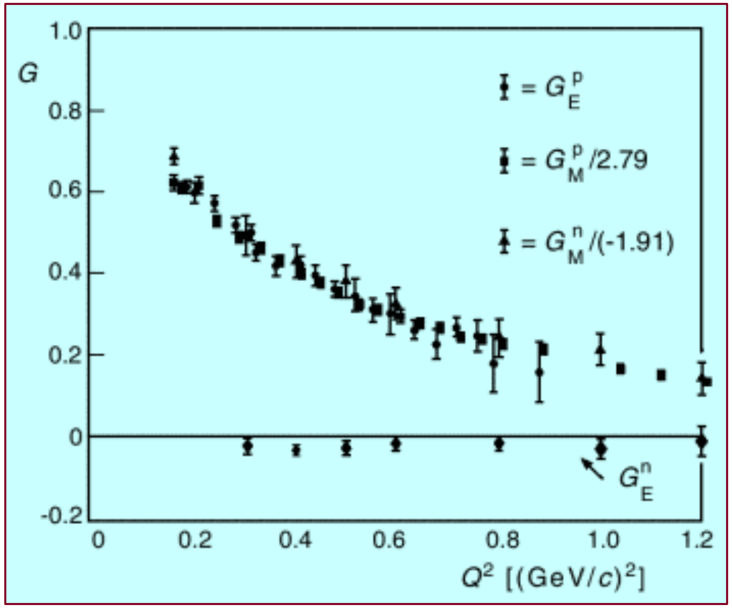
- The nucleons do NOT look like ~~point-like particles~~, nor ~~homogeneous spheres~~, but like diffused non-homogeneous systems.

- From the values at Q^2=0 :

$$< r^2 >_{\text{dipole}} = -6\hbar^2 \left.\frac{dG(q^2)}{dq^2}\right|_{q^2=0} =$$

$$= \frac{12}{a^2} \approx 0.66\ \text{fm}^2;$$

$$\sqrt{< r^2 >_{\text{dipole}}} \approx 0.81\ \text{fm}.$$



Proton structure: comments

$$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Rosen}\\\text{bluth}}} \Big/ \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} =$$

$$= \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2} \right); \quad \left[\tau = \frac{Q^2}{4M^2} \right].$$

therefore $\lim_{Q^2\to 0} \left(\frac{d\sigma}{d\Omega} \right)_{\substack{\text{Rosen}\\\text{bluth}}} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}}$.

Notice also that, if the proton were point-like, one would find :

$G_E^p(Q^2) = G_M^p(Q^2) = 1$, <u>independent of Q^2</u>

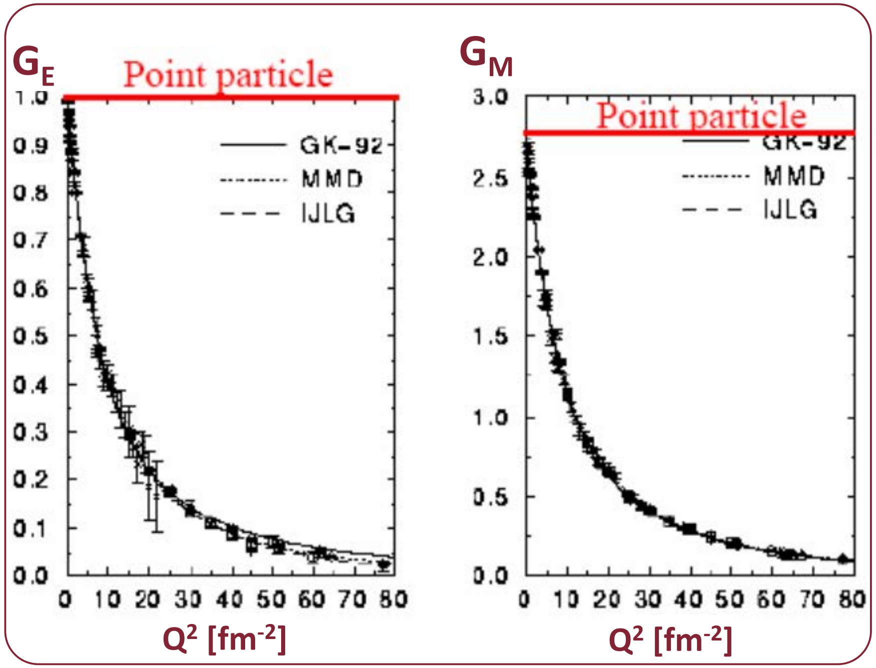
[*and in addition would not understand why "2.79"*].

The form factors of the nucleons show three different ranges :

1. $Q^2 << m_p^2$: τ small, G_E dominates the cross section; in this range we measure the average radius of the electric charge : $\langle r_E \rangle = 0.85 \pm 0.02$ fm;

2. $0.02 \leq Q^2 \leq 3$ GeV^2 :
   G_E and G_M are equally important;

3. $Q^2 > 3$ GeV^2 : G_M dominates.



Proton structure: interpretation

Differences between nuclei and nucleons :

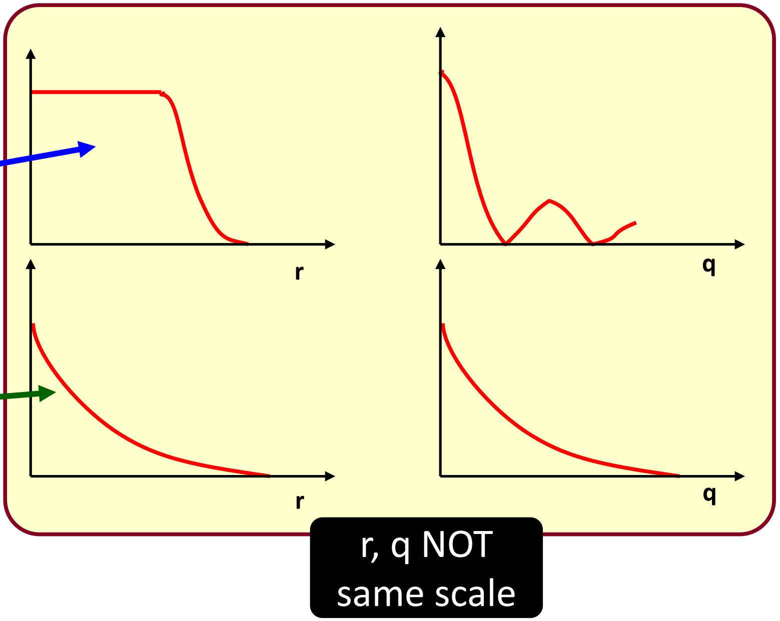
1. nuclei exhibit diffraction maxima/ minima; this fact corresponds to charge distributions similar to homogeneous spheres with thin skin;

2. nucleons have diffused, dipolarly distributed form factors → exp. charge;

3. at this level, it is unclear whether the nucleons have substructure(s) → need experiments at smaller value of distances (i.e. larger values of Q^2);

4. [*hope that*] the structure of the nucleons in the elastic scattering, described by the Rosenbluth formula, is an average with insufficient resolution;

5. at higher Q^2, one can expect a wider variety of phenomena :



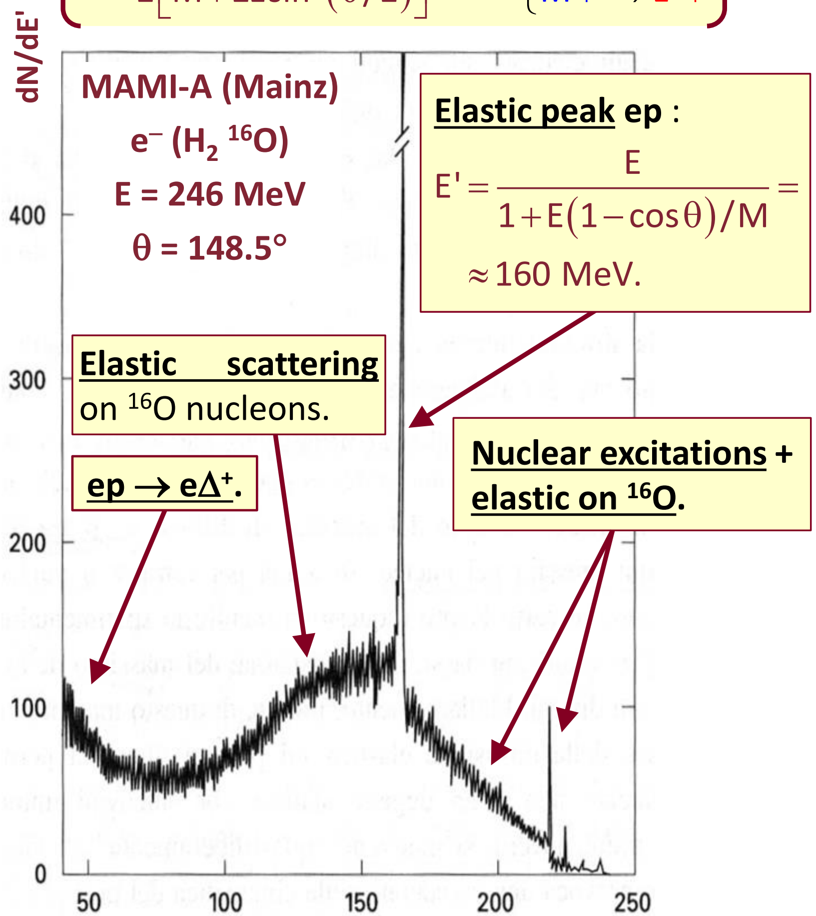
a. elastic scattering : $ep \to ep$;

b. excitation : $ep \to e$ "p^*" (e.g. $ep \to e\Delta^+$, $\Delta^+ \to p\pi^0$);

c. new states : $ep \to eX^+$ (X^+ = system of many particles).

Higher Q^2: H_2O

$$E' = \frac{M^2 + 2ME - W^2}{2\left[M + 2E\sin^2(\theta/2)\right]}, \quad \text{i.e.} \begin{cases} W^2 \uparrow \Rightarrow E' \downarrow \\ M \uparrow \Rightarrow E' \uparrow \end{cases}$$



Send 246 MeV electrons → water vapor.

The scattering shows a complex distribution, with different phenomena in the same plot. At fixed θ of the electron in the final state, with increasing E' :

- e p $\rightarrow$ e Δ^+ (excitation of p from H);
- e p/n $\rightarrow$ e p/n ("elastic" on ^{16}O nucleons);
- e p $\rightarrow$ e p (elastic on H, E'≈160 MeV);
- e p $\rightarrow$ e X^+ (nuclear excitations);
- e ^{16}O $\rightarrow$ e ^{16}O (nucl. exc. / elastic)

The distribution depends also on the electron energy E and the final state angle θ.

Higher Q^2: He4

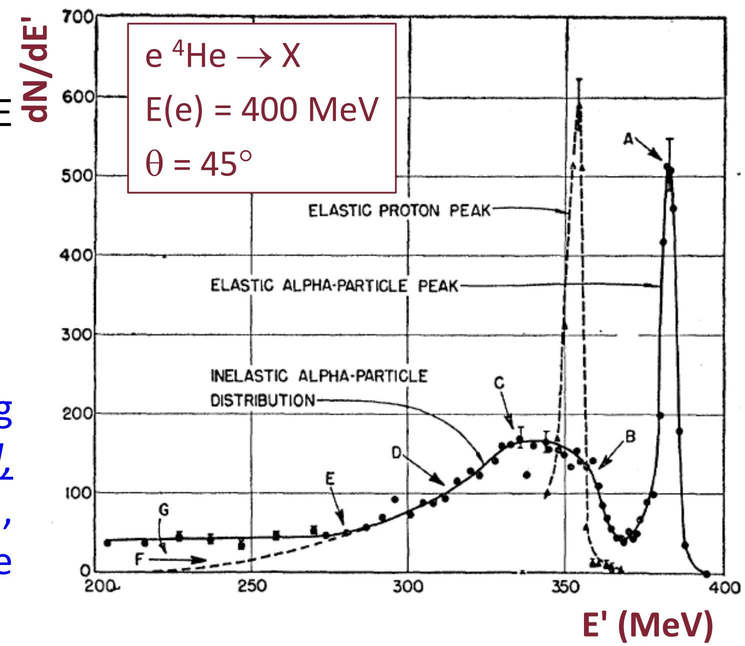
Another of these experiments (Hofstadter 1956, see fig.). Observe :

-- the elastic peak for ep $\to$ ep at the same E and θ, shown for comparison [*no problem*];

A. the elastic scattering e ^{4}He [ok, ***expected***];

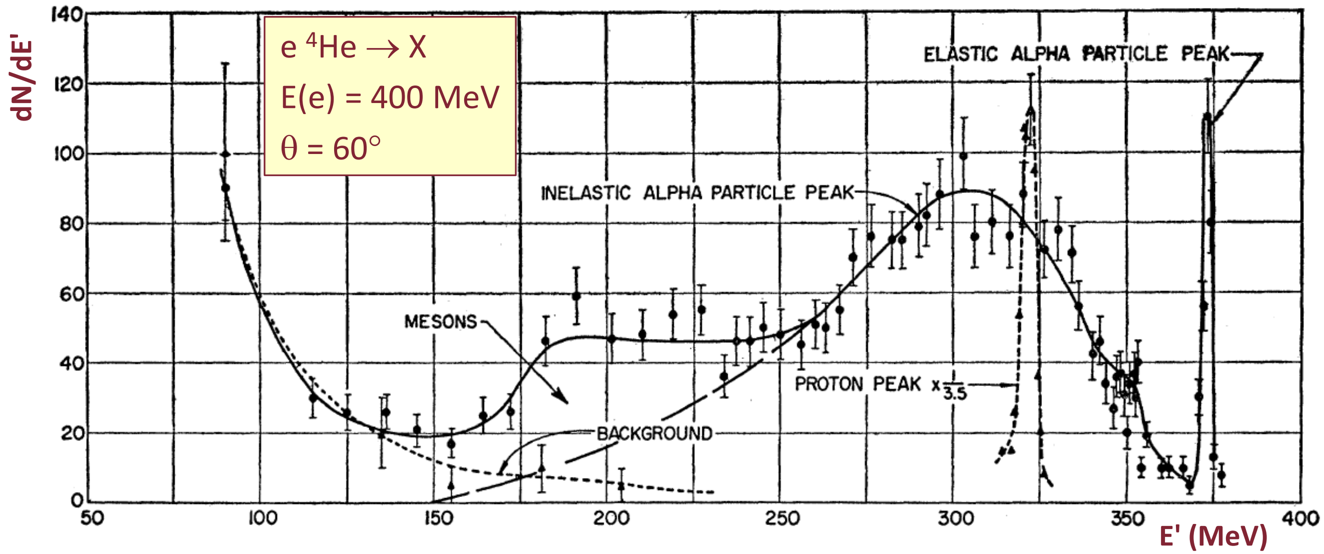
BCDEF. the elastic scattering ep / en (p/n acting as free particles in ^{4}He) [***maybe unexpected, but understandable***]; notice the peak width, due to the Fermi motion of nucleons inside the nucleus;

G. the production of π^- (i.e. of Δ's), which enhances the cross section (otherwise F.); notice : smaller E' $\to$ larger energy transfer [*the new entry in the game*].



$$E' = \frac{M^2 + 2ME - W^2}{2\left[M + 2E\sin^2(\theta/2)\right]}, \quad \text{i.e.} \begin{cases} W^2 \uparrow \Rightarrow E' \downarrow \\ M \uparrow \Rightarrow E' \uparrow \end{cases}$$

Higher Q^2: He4



Same as before, but θ = 60°, i.e. larger Q^2 [$Q^2 \approx 4EE'\sin^2(\theta/2)$]. Notice :

- smaller elastic peak, both for ($e^- {}^4$He) and (e^-p);
- wider ep/en (p/n inside ^{4}He) peak;
- (roughly) constant πproduction (seems independent from Q^2, as expected for point-like (?) partic

Possible conclusions [*possibly wrong*] :

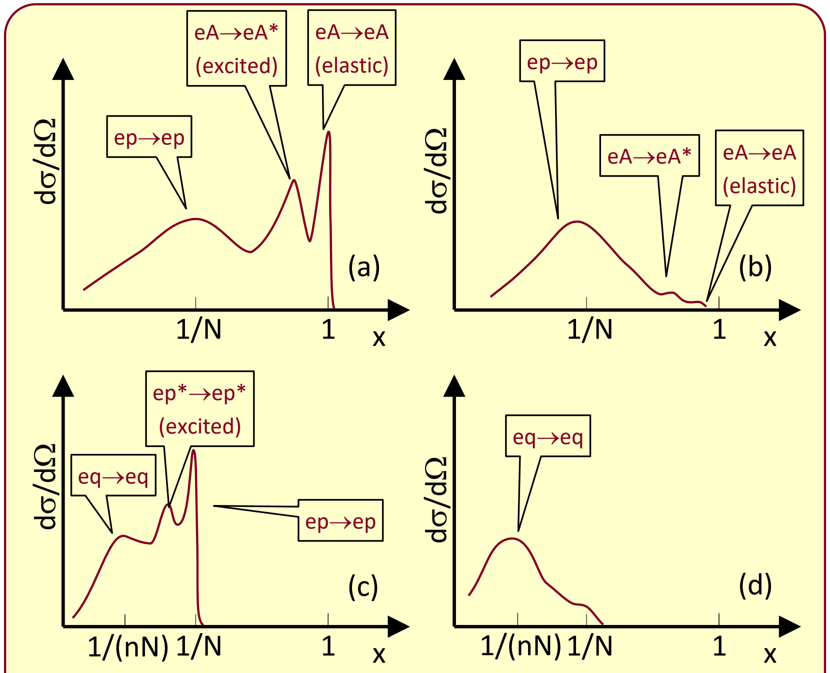
- everything under control for elastic and quasi-elastic data;
- the high-Q^2 part shows no evidence for sub-structures;
- maybe Q^2 is still too small (or maybe there are no substructures … !?);

→ go to even higher Q^2 !!!

Higher Q^2: summary

Follow to understand the dependence of $d\sigma/d\Omega$ on Q^2:

- scattering electron ("e^-") nucleus ("A");
- A with "N" nucleons (use "p", but neutrons similar);
- p with "n" hypothetical components ("q");
- plot vs adimensional variable $x=Q^2/(2M\nu)$, $0 < x < 1$;
- from (a) to (d), Q^2 increases;
  - a) at small Q^2, there are both scatterings with A and p;
  - b) increasing Q^2, the eA scattering disappears, while the ep scattering stays constant;
  - c) increasing Q^2, the constituents (if any) appears as eq $\to$ eq;



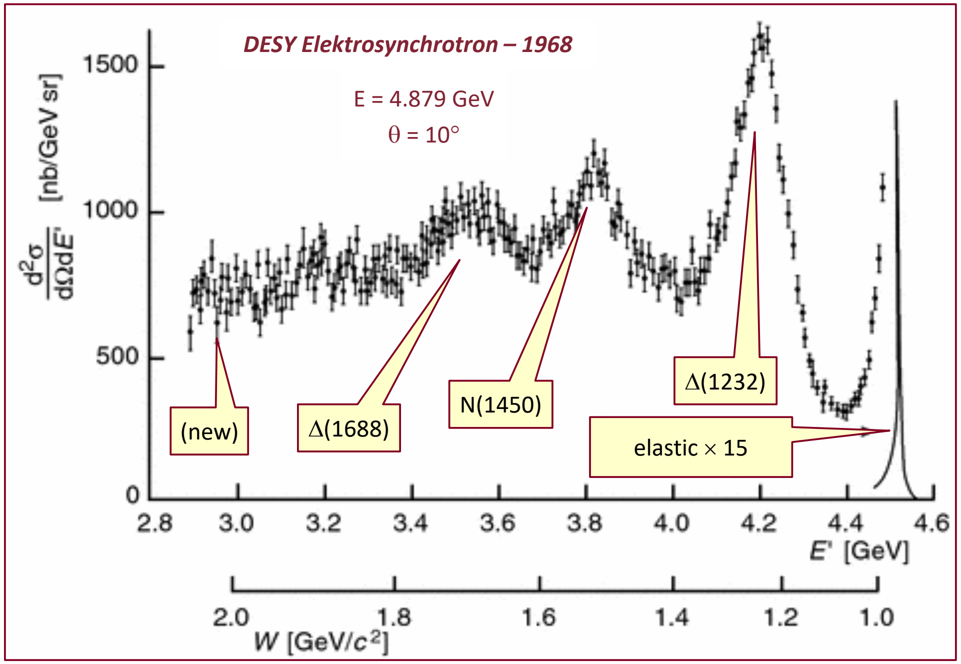
[just a sketch, not a reproduction of real experiments]

- d) finally, at very large Q^2, the most (~ only) important process is eq $\to$ eq (with all the possible inelastic companions).

Higher Q^2: constituents show up

Scattering ep → eX (DESY 1968) :

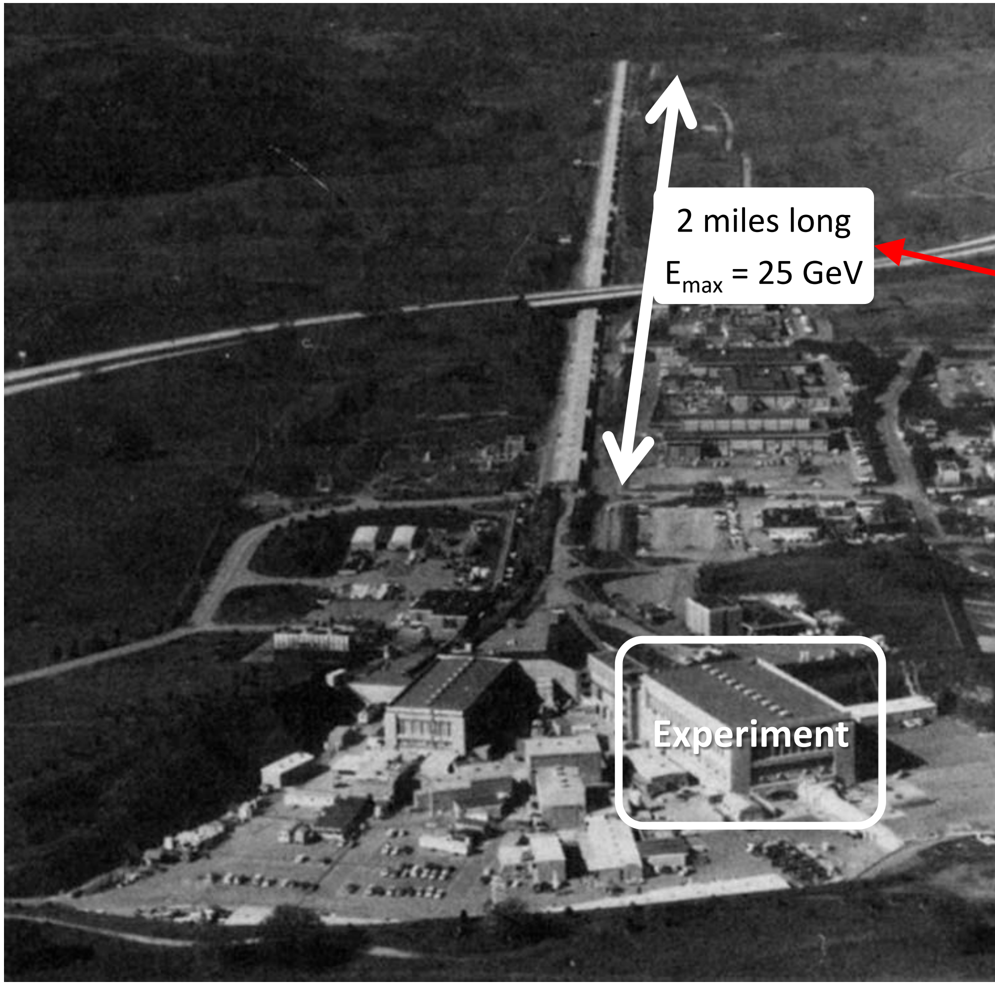
- Electron energy ≈ 5 GeV (higher than SLAC);
- resonances (R) production ep → eR clearly visible;
- new region at small E' (= high W);
- in this "new" region :
  - continuum (NO peaks);
  - rich production of hadrons;
  - NO new particles, only (p n π's); i.e. the proton breaks, but (different from the nucleus) NO constituent appears;
  - the constituents, if any, do not show up as free particles;



→ **Do quarks exist ???**
are <u>they confined</u> ??? why ???

[NB in 1968 color was proposed but not really understood, QCD did not exist]

Deep inelastic scattering : SLAC

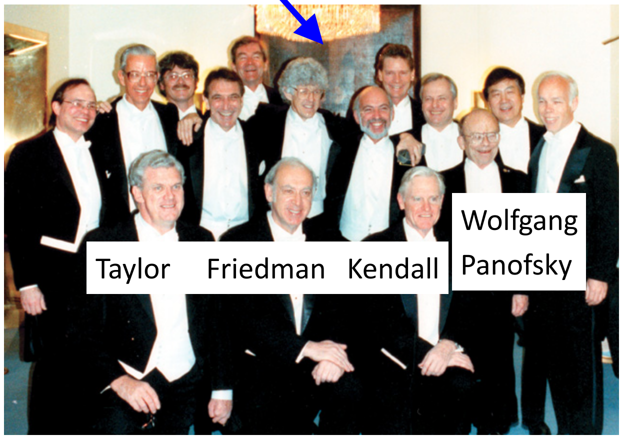


SLAC

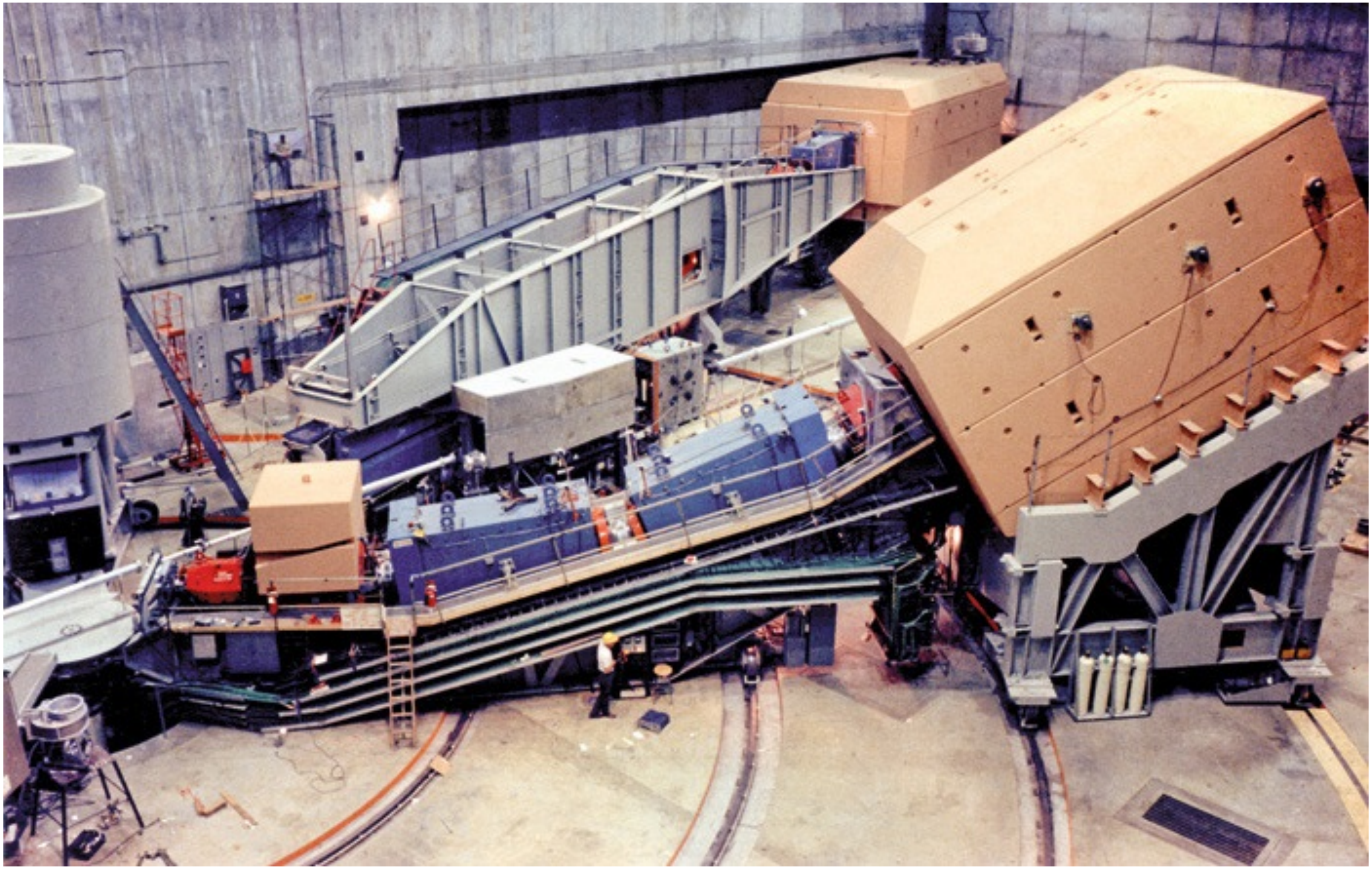
Stanford Linear Accelerator Center

the beginning of the story (1960)

… and this is NOT the end (1990)

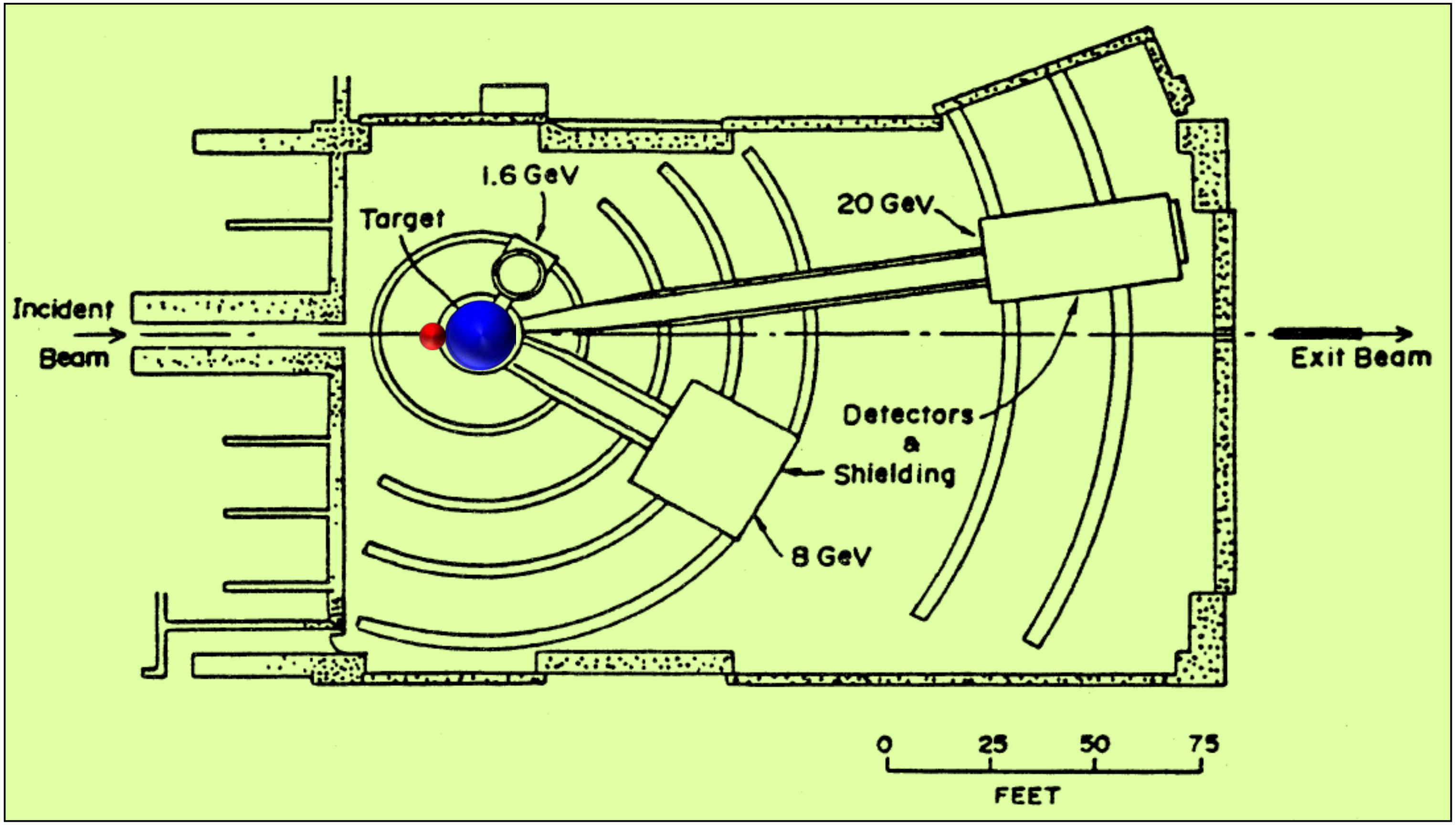


Deep inelastic scattering : SLAC experiment



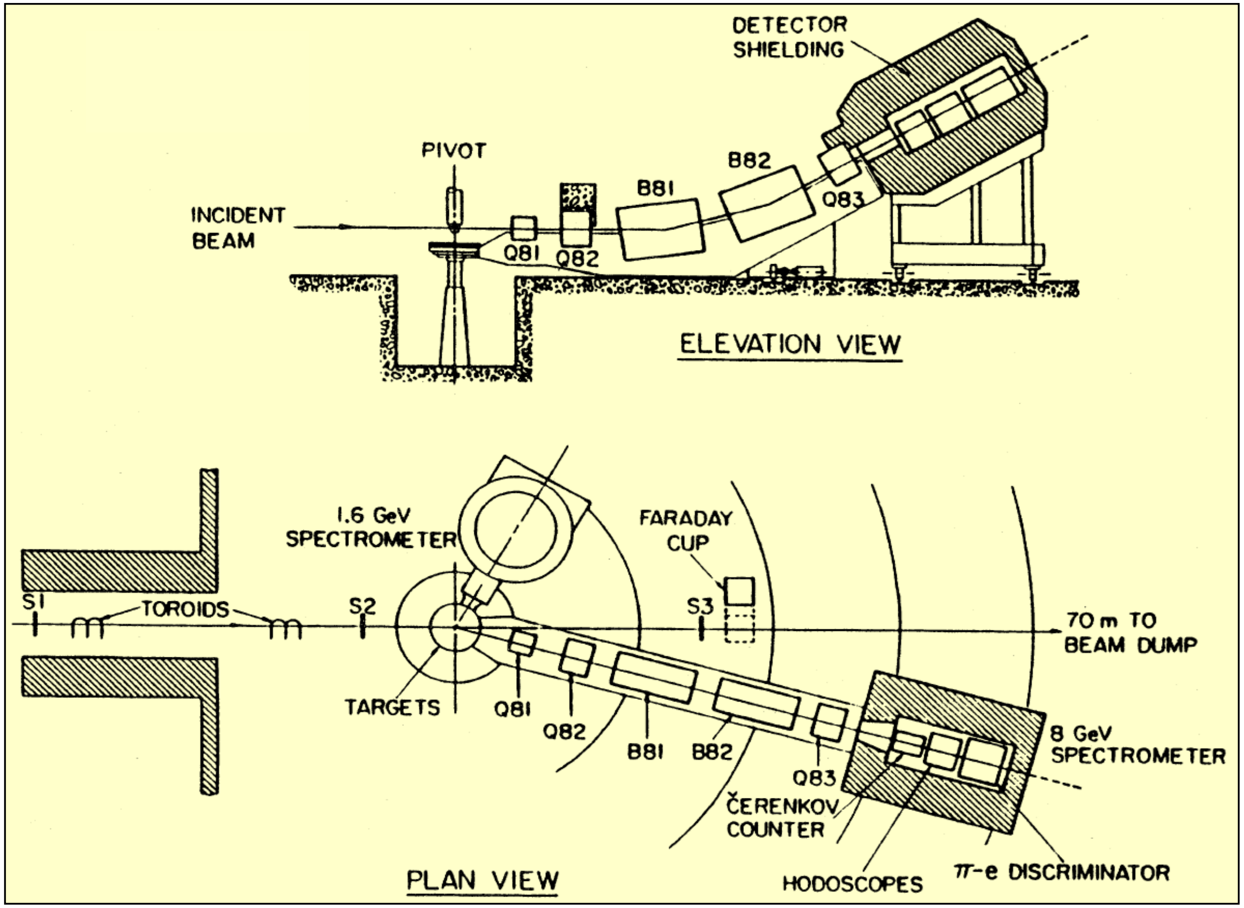
The 8 GeV spectrometer – 1968 (notice the men at the bottom)

Deep inelastic scattering : layout



Layout of the three spectrometers : they can be rotated about their pivot, as shown in the figure. [75 ft $\approx$ 23 m]

Deep inelastic scattering : layout details



Draw of the 8 GeV spectrometer [the 20 GeV is NOT shown]:

B : bending magnets (dipoles);

Q : quadrupoles;

Čerenkov counters;

scintillation hodoscopes,

shower counters for e-π discrimination;

dE/dx counters.

a big effort for physics and engineering of 50 years ago !!!
not to be compared with modern experiments …

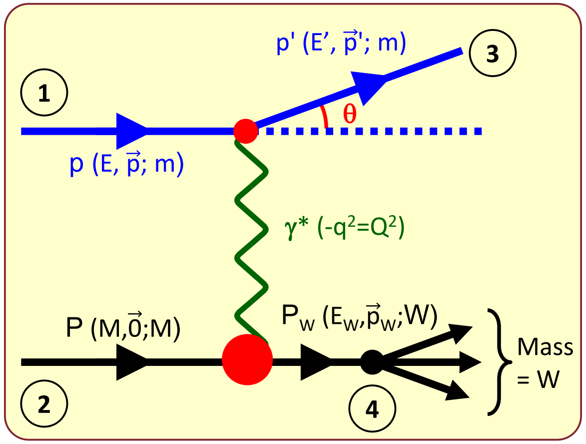
Deep inelastic scattering : functions $W_{1,2}$

The usual parameterization of the cross section in the DIS region is the formula (Z=1 for a proton) :

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS} = \left[\frac{d\sigma}{d\Omega}\right]^*_{Mott}\left[W_2\left(Q^2,\nu\right)+2W_1\left(Q^2,\nu\right)\tan^2\frac{\theta}{2}\right]=$$

$$=\frac{4Z^2\alpha^2\left(\hbar c\right)^2 E'^2}{|qc|^4}\cos^2\frac{\theta}{2}\times\left[W_2\left(Q^2,\nu\right)+2W_1\left(Q^2,\nu\right)\tan^2\frac{\theta}{2}\right]=$$

$$=\frac{4\alpha^2 E'^2}{Q^4}\times\left[W_2\left(Q^2,\nu\right)\cos^2\frac{\theta}{2}+2W_1\left(Q^2,\nu\right)\sin^2\frac{\theta}{2}\right].$$



Remarks :

- the inelastic cross section requires 2 final-state variables, e.g. θ and E'; other choices are equivalent; Q^2 and ν are L-invariant, so more convenient;
- W_1 and W_2 are the equivalent of G_E and G_M for DIS (with dimension 1/E, *see next slide*) : they are called *structure functions* [later they will be a sum of "PDF"];
- W_1 and W_2 reflect the structure of the particles; the formula is general, but contains little information until $W_{1,2}$ are explicitly measured (and/or computed from a deeper theory);
- the dynamics of the scattering depends on the structure of the target; W_1 and W_2 are the real "containers" of this information.

Deep inelastic scattering : $W_{1,2}$ VS $G_{E,M}$

Some algebra, quite boring, to show for the ep (Z=1, M_p):

- the explicit values of Mott and Rosenbluth cross-sections;
- the relation $G_{E,M}$ vs $W_{1,2}$.

Enjoy !!!

$$\left[\frac{d\sigma}{d\Omega}\right]_{Mott} = \left[\frac{4\alpha^2 E'^2}{Q^4}\right]_{Rutherford} \left[\cos^2\frac{\theta}{2}\right]_{\to Mott^*} \left[\frac{E'}{E}\right]_{\to Mott} = \frac{4\alpha^2 E'^3}{EQ^4}\cos^2\frac{\theta}{2};$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{Rosenbluth} = \left[\frac{4\alpha^2 E'^3}{EQ^4}\cos^2\frac{\theta}{2}\right]_{Mott} \left[\frac{G_E^2+\tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right]_{\to Rosenbluth};$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{Rosenbluth} = \frac{12\alpha^2 E'^2}{EQ^4}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right);$$

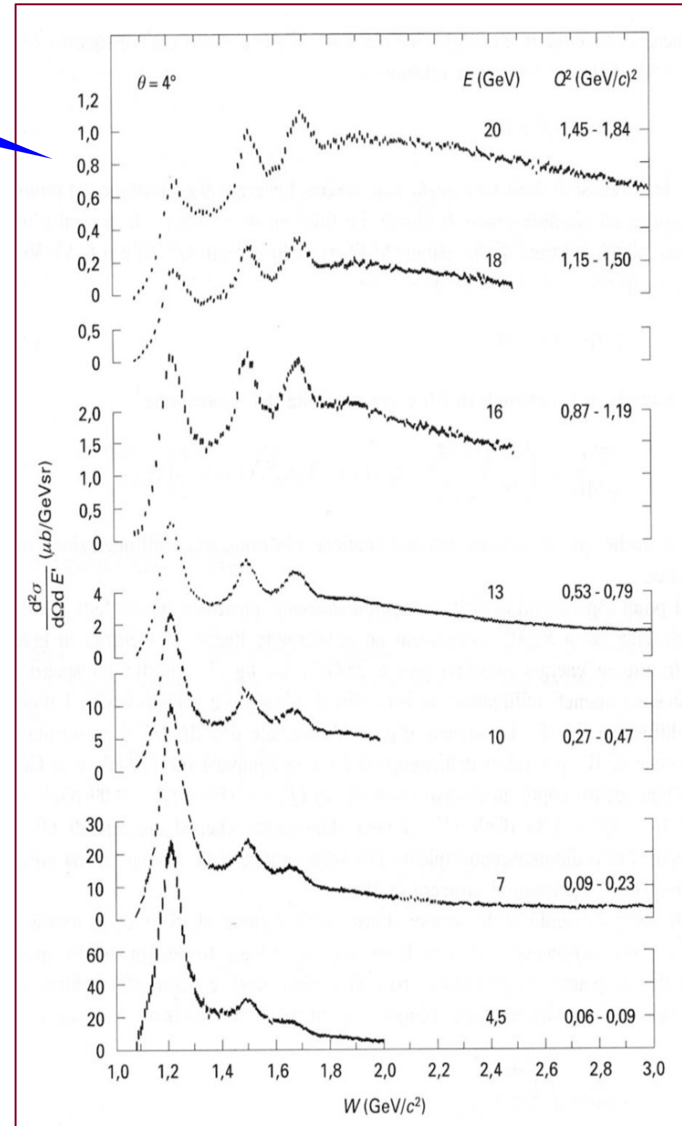
$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS} = \frac{4\alpha^2 E'^2}{Q^4} \times \left[W_2\left(Q^2,\nu\right)\cos^2\frac{\theta}{2} + 2W_1\left(Q^2,\nu\right)\sin^2\frac{\theta}{2}\right];$$

$$W_1\left(Q^2,\nu\right) = \frac{3}{E}\tau G_M^2 = \frac{3Q^2}{4EM_p^2}G_M^2;$$

$$W_2\left(Q^2,\nu\right) = \frac{3}{E}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\right) = \frac{3}{E}\left(\frac{4M_p^2 G_E^2 + Q^2 G_M^2}{4M_p^2+Q^2}\right).$$

Deep inelastic scattering : $d^2\sigma/d\Omega dE'$

ep → eX, θ = 4°, $d^2\sigma/d\Omega dE'$ vs W (= hadr. mass)



Notice :

- the intervals in W and Q^2, due to fixed E and θ;
- the elastic scattering ($W = M_p$) is out of scale;
- the decrease in cross section (the vertical scale) when E increases;
- the presence of excited states of the nucleon (resonances → peaks), e.g. $\Delta^+(1232)$;
- the "fading out" of resonances, when W increases at fixed E and θ;
- the continuum at high W, with ~const σ (1-2 μb / GeV sr, independent from E and Q^2).

Deep inelastic scattering : $d\sigma/d\theta$ vs $d\sigma/d\theta_{Mott}$

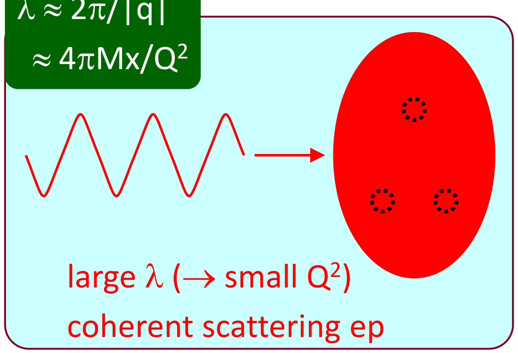
Ratio $R = \text{exp./Mott} = W_2 + 2\, W_1 \tan^2\theta/2 = R(Q^2)$.

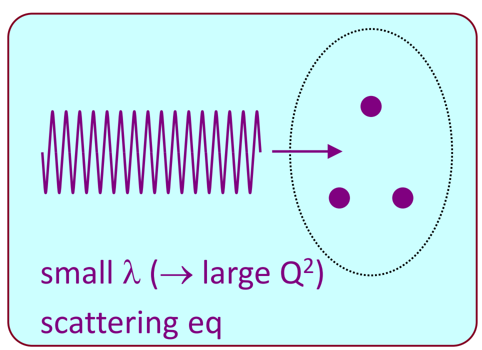
Notice that the structure functions appear to be nearly independent of Q^2. Instead, the elastic scattering for a non-pointlike target has a strong Q^2 dependence !!!

I.e., for DIS, the target (whatever it be), behaves like a point-like particle [$\mathcal{F}(Q^2)$=const] , cfr the Rutherford formula] !!! [NB constant, but << 1 $\rightarrow$ charge < 1]

This Q^2 independence is another confirmation that the DIS "breaks" the proton : the scattering happens with one of its constituents. The constituents looks "quasi-free" and "quasi-pointlike", at least at this scale of Q^2.

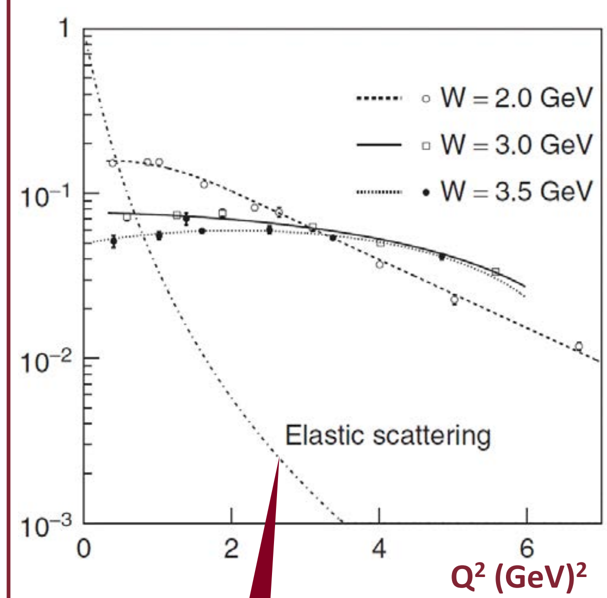
$\lambda \approx 2\pi/|q|$
$\approx 4\pi Mx/Q^2$





M.Breidenbach et al., PRL 23, 935

$R=(d\sigma_{exp}/d\theta)/(d\sigma_{Mott^*}/d\theta)|_{\theta=10°}$



dipole form factor:
- $R(Q^2=0) = 1$;
- $R(Q^2) \propto Q^{-8}$

Bjorken scaling: structure functions F_1, F_2

Define two dimensionless functions F_1 and F_2, instead of W_1 and W_2 [for $d^2\sigma/dxdy$ see later]:

$$F_1(x,Q^2) = MW_1(Q^2,\nu);$$

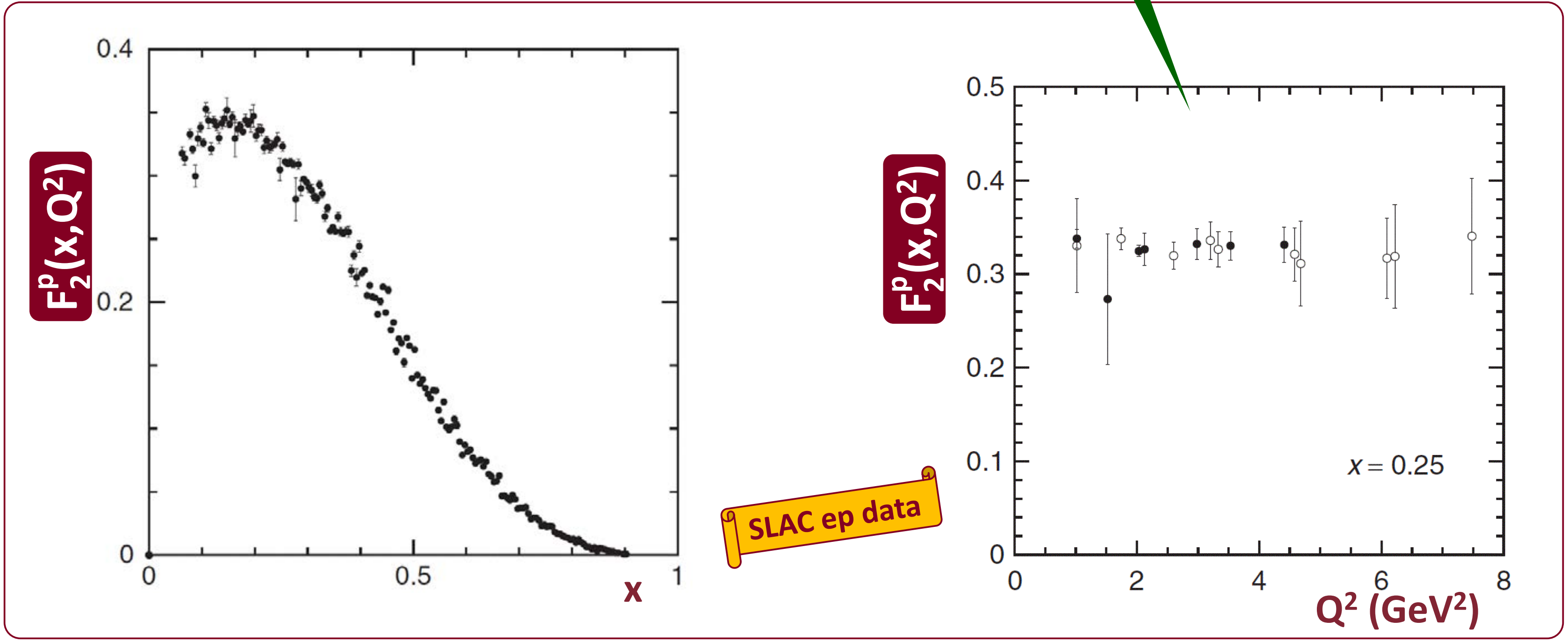
$$F_2(x,Q^2) = \nu W_2(Q^2,\nu).$$

$F_1(x,Q^2)$ and $F_2(x,Q^2)$ are called *structure functions*.

If the nucleons are made by point-like, spin ½ objects, from the DIS formula the **Callan-Gross relation** can be derived [next slide] :

$$2xF_1(x) = F_2(x)$$

Seen as functions of x and Q^2, $F_{1,2}$ appear NOT to depend on Q^2 for a large range of it.



Bjorken scaling: Callan-Gross formula

a) the cross sections of pointlike spin ½ particle of mass m (à la Rosenbluth with $G_E=G_M=1$) :

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{point-like, spin1/2}} = \frac{12\alpha^2 E'^2}{EQ^4}\left[\cos^2\frac{\theta}{2} + 2\tau\sin^2\frac{\theta}{2}\right];$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{DIS}} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right];$$

$$W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2} = \frac{3}{E}\left[\cos^2\frac{\theta}{2} + 2\tau\sin^2\frac{\theta}{2}\right];$$

$$W_1 = \frac{3\tau}{E}; \qquad W_2 = \frac{3}{E}; \qquad \frac{W_1}{W_2} = \frac{F_1(x)}{F_2(x)}\frac{\nu}{M} = \tau = \frac{Q^2}{4m^2};$$

b) from the kinematics of elastic scattering of point-like constituents of mass m :

$$Q^2 = 2m\nu = 2M\nu x \rightarrow m = xM;$$

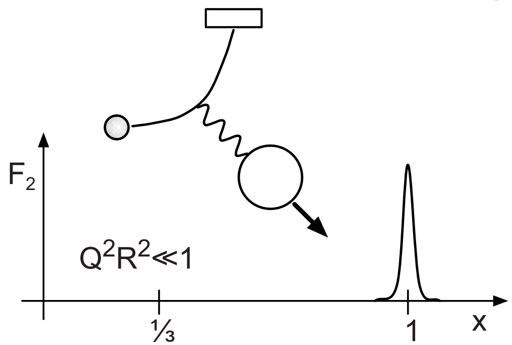
$$\frac{F_1(x)}{F_2(x)} = \frac{Q^2}{4m^2}\frac{M}{\nu} = \frac{2m\nu}{4m^2}\frac{M}{\nu} = \frac{M}{2m} = \frac{1}{2x}; \quad \rightarrow$$

$$\boxed{2xF_1(x) = F_2(x).}$$

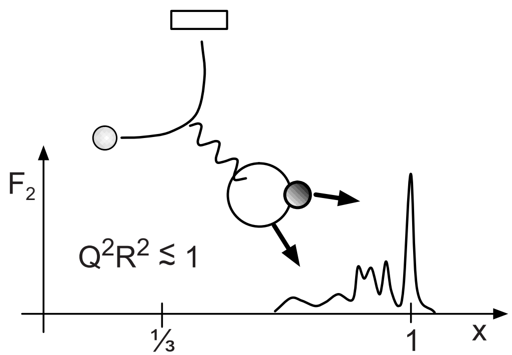
Warnings :

- don't confuse M (the nucleon) with m (the constituent);
- don't confuse the inelastic scattering ep with the elastic scattering eq;
- x refers to the inelastic case;
- an hypothetical [*nobody uses it*] variable ξ, analogous to x but for the constituent scattering; in this case, $Q^2=2m\nu\xi$, ξ = **1;**
- we learn that x = m/M [REMEMBER].

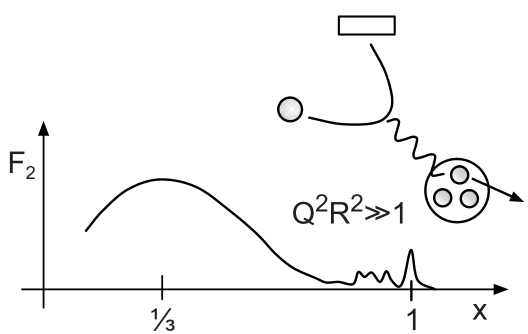
Bjorken scaling: parton model



At small Q^2, where the wavelength of the virtual photon is much larger than the nucleon radius, only elastic scattering is observed.



Once the wavelength becomes comparable to the nucleon radius the transitions to the exited states are seen.



When the photon wavelength is much smaller than the nucleon radius the electrons scatter on the charged constituents of the nucleon.

Bjorken scaling: parton model

Assume that the nucleon be made of ***partons*** (point-like, spin ½, mass m_i), which scatter elastically in the ep process.

Then the DIS cross section

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2 \cos^2\frac{\theta}{2} + 2W_1 \sin^2\frac{\theta}{2}\right];$$

reduces to an incoherent sum of constituent cross sections, $q_{electron} e_i$ being the charge of each of them :

$$\left.\frac{d^2\sigma}{d\Omega dE'}\right|_{m_i} = \frac{4\alpha^2 E'^2}{Q^4}\sum_i \left[e_i^2 \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_i^2}\sin^2\frac{\theta}{2}\right) \delta\left(\nu - \frac{Q^2}{2m_i}\right)\right];$$

where the $\delta()$ means that, at the constituent level, the scattering is elastic, i.e. $Q^2 = 2m_i\nu$.

For such partons [*next 2 slides*]:

$$\begin{cases} F_1\left[x = \dfrac{Q^2}{2m\nu}\right] = MW_1(Q^2,\nu) = \dfrac{1}{2}\sum_j e_j^2 f_j(x) \\ F_2\left[x = \dfrac{Q^2}{2m\nu}\right] = \nu W_2(Q^2,\nu) = x\sum_j e_j^2 f_j(x) \end{cases}$$

i.e. F_1 and F_2 do NOT depend on Q^2 and ν separately, but only on their ratio. F_1 and F_2 are also related by the Callan-Gross equation.

This mechanism (the **Bjorken scaling**) was interpreted by Feynman in 1969 as the dominance of partons in the nucleon dynamics (the **parton model**).


Richard Feynman James Bjorken

Bjorken scaling: $\sigma_{DIS} \to W_{1,2}$

here ONLY ONE parton "q", with m, e, x=m/M. ❶

if $B(x=x_0)=0 \rightarrow$ ❷

$$\int A(x)\delta[B(x)]dx = A(x_0)/|B'(x_0)|,$$

$$B(x) = \nu - \frac{Q^2}{2Mx} \rightarrow x_0 = \frac{Q^2}{2M\nu};$$

$$\rightarrow B'(x_0) = \left.\frac{Q^2}{2Mx^2}\right|_{x=x_0} = \frac{2M\nu^2}{Q^2}.$$

DIS formula for ep, p NOT pointlike, mass=M: ❸

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS} = \frac{4\alpha^2E'^2}{Q^4}\left[W_2\left(Q^2,\nu\right)\cos^2\left(\frac{\theta}{2}\right) + 2W_1\left(Q^2,\nu\right)\sin^2\left(\frac{\theta}{2}\right)\right]$$

Elastic scattering e"q", pointlike, spin ½, charge e, mass m=Mx:

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{e"q"} = \frac{4\alpha^2E'^2}{Q^4}\left[e^2\cos^2\left(\frac{\theta}{2}\right) + e^2\frac{Q^2}{2m^2}\sin^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu - \frac{Q^2}{2m}\right)$$

$$W_1\big|_x = \frac{e^2Q^2}{4m^2}\delta\left(\nu - \frac{Q^2}{2m}\right) = \frac{e^2Q^2}{4M^2x^2}\delta\left(\nu - \frac{Q^2}{2Mx}\right); \qquad \text{[at fixed x]}$$

f(x) : x-distribution of (a single) substructure;

$$W_1 = \int\frac{e^2Q^2}{4M^2x^2}\delta\left(\nu - \frac{Q^2}{2Mx}\right)f(x)dx = \frac{e^2Q^2}{4M^2}\int\frac{f(x)dx}{x^2}\delta\left(\nu - \frac{Q^2}{2Mx}\right) =$$

$$= \frac{e^2Q^2}{4M^2}f(x)\big|_{x=\frac{Q^2}{2M\nu}}\left(\frac{Q^2}{2M\nu}\right)^{-2}\frac{Q^2}{2M\nu^2} =$$

$$= e^2f(x)\left(2^{-2+2-1}\right)\left(M^{-2+2-1}\right)\left(Q^{2-4+2}\right)\left(\nu^{2-2}\right) = \frac{e^2f(x)}{2M}.$$ ❹

[similarly:] ❺

$$W_2\big|_x = e^2\delta\left(\nu - \frac{Q^2}{2Mx}\right);$$

$$W_2 = \int e^2\delta\left(\nu - \frac{Q^2}{2Mx}\right)f(x)dx =$$

$$= e^2f(x)\big|_{x=\frac{Q^2}{2M\nu}}\frac{Q^2}{2M\nu^2} = \frac{e^2xf(x)}{\nu}.$$

Bjorken scaling: $W_{1,2} \rightarrow F_{1,2}$

previous page

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2\left(Q^2,\nu\right)\cos^2\left(\frac{\theta}{2}\right) + 2W_1\left(Q^2,\nu\right)\sin^2\left(\frac{\theta}{2}\right)\right];$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{e"q"} = \frac{4\alpha^2 E'^2}{Q^4}\left[e^2\cos^2\left(\frac{\theta}{2}\right) + e^2\frac{Q^2}{2m^2}\sin^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu - \frac{Q^2}{2m}\right);$$

a single substructure $\{e, m=Mx\}$ $\rightarrow W_1 = \frac{e^2 f(x)}{2M}$; $\quad W_2 = \frac{e^2 x f(x)}{\nu}$.

this form ("Σ...") is actually very important (why ?)

Many sub-structures: for each $\{e, x, f(x)\} \rightarrow \{e_j, x_j, f_j(x)\}$:

$$W_1 = \frac{e^2 f(x)}{2M} \quad \rightarrow W_1 = \sum_j \frac{e_j^2 f_j(x)}{2M} \quad \rightarrow \boxed{MW_1 = F_1(x) = \frac{1}{2}\sum_j e_j^2 f_j(x)};$$

$$W_2 = \frac{e^2 x f(x)}{\nu} \quad \rightarrow W_2 = \sum_j \frac{e_j^2 x f_j(x)}{\nu} \quad \rightarrow \boxed{\nu W_2 = F_2(x) = x\sum_j e_j^2 f_j(x)};$$

$$\rightarrow \boxed{\text{Callan}-\text{Gross} \;:\; 2xF_1(x) = F_2(x)}.$$

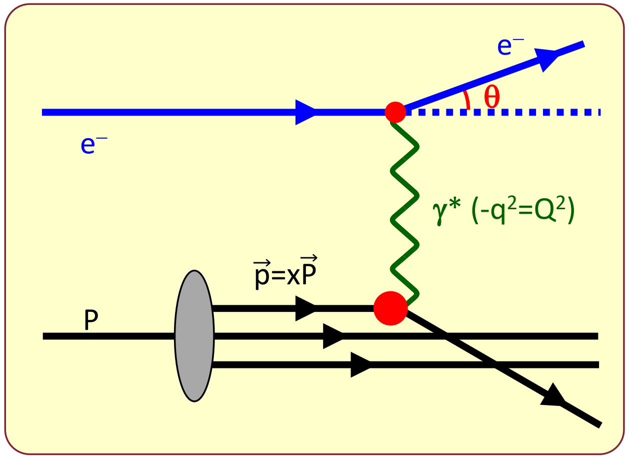
The parton model

Summary: the nucleons are made by partons (later identified with quarks, but at the time there was no reason) :

- point-like (at least at the scale of Q^2 accessible to the experiments, *both then and now*);
- spin ½ fermions;
- define the ratio $|\vec{p}(\text{parton})| \,/\, |\vec{p}(\text{nucleon})|$:

  $$x_{Feynman} = x_F = |\vec{p}_{parton}| \,/\, |\vec{p}_{nucleon}|$$

  (cfr. $x_{Bjorken} = x_B = m/M$);
- the interaction e-parton is so fast, that they behave like free particles (similar, *mutatis mutandis*, to the collision approximation in classical mechanics);
- the other partons [*at least in 1st approx.*] do NOT take part in the interaction ("*spectators*");
- it follows $x_F = x_B$ [next slide];
- the DIS is an incoherent sum of the processes on the partons; at high Q^2 the nucleons as such are mere containers, with no role [$F_{1,2} = \Sigma$...].



Despite the formal identity between x_F and x_B, they have a different dynamical origin :

- x_F is defined in the hadronic system (= fraction of the proton momentum);
- x_B comes from the lepton part (momentum transfer and lepton energies).

The parton model: $x_F \leftrightarrow x_B$

Show : $x_{Feynman} \equiv x_F = x_{Bjorken} \equiv x_B$

In the "infinite momentum frame" (IMF), where all the masses are negligible :

$$p_{proton}^{init}\Big|_{IMF} = (p, \vec{p});$$

$$p_{parton}^{init}\Big|_{IMF} = x_F p_{proton}^{init} = (x_F p, x_F \vec{p});$$

$$p_{parton}^{fin}\Big|_{IMF} = p_{parton}^{init} + q_{transf};$$

$$\left(p_{parton}^{fin}\right)^2 = 0 = \left(p_{parton}^{init} + q_{transf}\right)^2 =$$

$$= 0 + q_{transf}^2 + 2\left(p_{parton}^{init} \cdot q_{transf}\right);$$

$(p_{parton}^{init} \cdot q_{transf})$ is Lorentz-invariant; let's compute it in the lab frame:

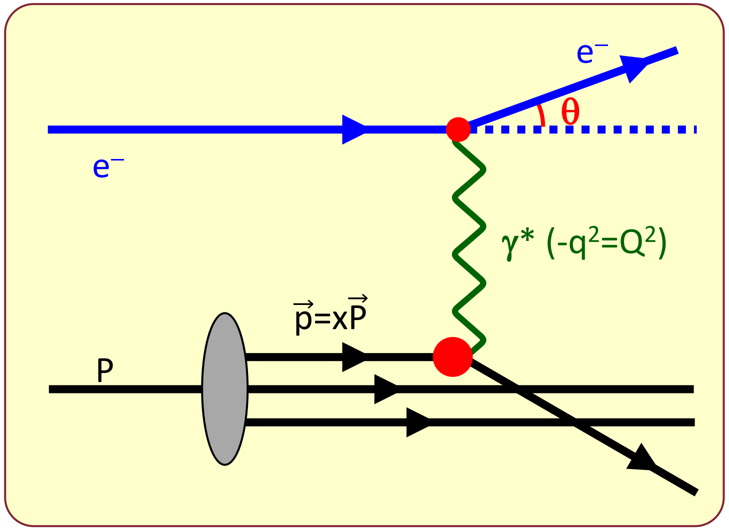
$$p_{proton}^{init}\Big|_{LAB} = (M, \vec{0}); \quad p_{parton}^{init}\Big|_{LAB} = (Mx_F, \vec{0});$$

$$q_{transf}\Big|_{LAB} = (E - E' = \nu, q_x, q_y, q_z);$$

$$2Mx_F\nu = -q_{transf}^2 = Q^2 \rightarrow$$

$$x_F = Q^2 / (2M\nu) \equiv x_B.$$

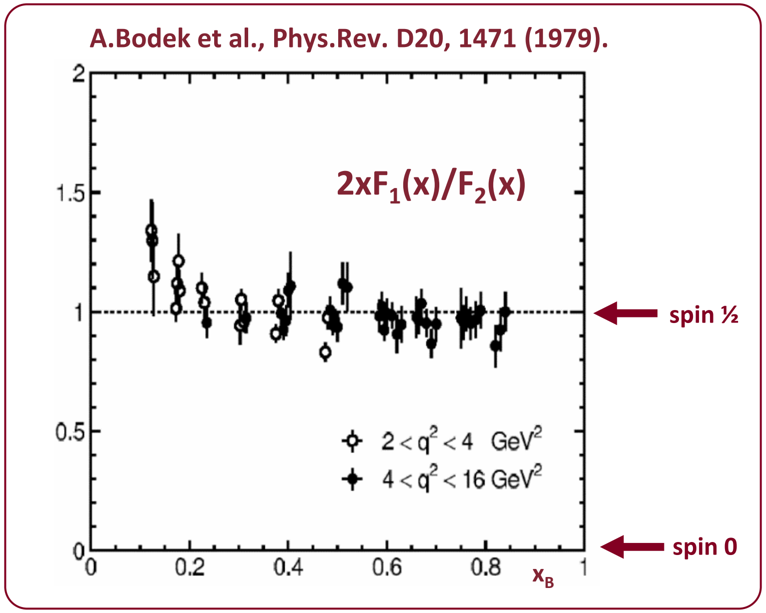
Warning : the equality holds only in the IMF. It is also a reasonable approx. in the "ultra-relativistic" case, when the masses are negligible wrt momenta.



The parton model: sum rule

Remarks and comments (discuss the proton, the neutron is similar):

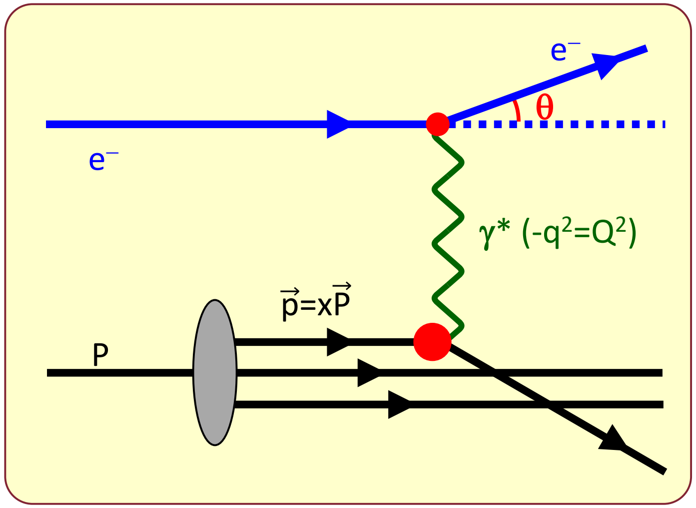
- experimentally, it is enough to control the initial state (E_{e-}, M_p) + measure the leptonic final state (E', θ);
- the model seems to imply that
  $$\Sigma_i \, x_i = 1,$$
  when the sum runs over ALL the partons;
- at the time there was no clue about the nature of the partons, nor if they are charged or neutral (i.e. not interacting with the electrons); therefore:
  $$\Sigma'_i \, x_i \leq 1$$
  (the sum is only over those partons, which interact with the electron);
- given the intrinsic q.m. structure of the nucleon, the values x_i are not fixed, but described by a distribution $f_j^p(x)$ for partons of type "j" in the proton:
  $$f_j^p(x) = dP / dx; \quad \Sigma_j \int dx \, [x f_j^p(x)] \leq 1,$$
  with the same caveats over the sum.
- if partons are spin ½, then the Callan-Gross relation $2xF_1(x) = F_2(x)$ holds;
- instead, spin = 0 $\rightarrow \tau = 0 \rightarrow F_1(x) = 0$;
- but … can we measure it ? YES, it's OK !!!



The parton model: summary

A summary of the model, with final formulæ (shown below and in the next slide):

- at high Q^2, a hadron (p/n) behaves as a mixture of small components, the partons.
- partons are pointlike, spin ½;
- each parton in each interaction is described by its fraction x_i of the 4-momentum of the hadron;



- the x_i are qm variables, described by their distribution functions $f_i^p(x)$ [called "***PDF***"];
- in principle the PDF are different for each parton and each hadron;
- $\sum_j \int dx\, x\, f_j^p(x) \leq 1$;
- parton spin → Callan-Gross $2xF_1(x) = F_2(x)$.

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2\left(Q^2,\nu\right)\cos^2\frac{\theta}{2} + 2W_1\left(Q^2,\nu\right)\sin^2\frac{\theta}{2}\right];$$

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 s}{Q^4}\left[xy^2F_1\left(x,Q^2\right) + \left(1-y\right)F_2\left(x,Q^2\right)\right];$$

$$F_1\left(x,Q^2\right) = MW_1(Q^2,\nu) = \frac{1}{2}\sum_j e_j^2 f_j(x);$$

$$F_2\left(x,Q^2\right) = \nu W_2(Q^2,\nu) = x\sum_j e_j^2 f_j(x).$$

The parton model: $d^2\sigma/dxdy$

$$s=2EM;\quad \nu=E-E';\quad y=\frac{\nu}{E}=1-\frac{E'}{E};\quad E'=E(1-y)=\frac{s}{M}\frac{1-y}{2};\quad Q^2=4EE'\sin^2\frac{\theta}{2}=\left(\frac{s}{M}\right)^2(1-y)\sin^2\frac{\theta}{2}.$$

$$x=\frac{Q^2}{2M\nu}=\left(\frac{s}{M}\right)^2(1-y)\sin^2\left(\frac{\theta}{2}\right)\frac{1\,E}{\boxed{2M}\,\boxed{E}}\frac{1}{\boxed{E-E'}}=\frac{s}{M^2}\frac{1-y}{y}\sin^2\left(\frac{\theta}{2}\right);$$

$$\sin^2\left(\frac{\theta}{2}\right)=\frac{M^2xy}{s(1-y)};\quad \cos^2\left(\frac{\theta}{2}\right)=1-\sin^2\left(\frac{\theta}{2}\right)\approx 1.$$

Kinematics

$$\frac{d^2\sigma}{d\Omega dE'}=\frac{4\alpha^2E'^2}{Q^4}\left(W_2\cos^2\frac{\theta}{2}+2W_1\sin^2\frac{\theta}{2}\right).$$

$$\frac{\partial x}{\partial\cos\theta}=\frac{\partial x}{\partial\sin^2(\theta/2)}\frac{\partial\sin^2(\theta/2)}{\partial\cos\theta}=\frac{s}{M^2}\frac{1-y}{y}\left(-\frac{1}{2}\right)=-\frac{E}{M}\frac{1-y}{y};\quad \frac{\partial y}{\partial E'}=-\frac{1}{E};$$

$$J=\begin{vmatrix}\dfrac{\partial x}{\partial\cos\theta} & \dfrac{\partial x}{\partial E'}\\ \dfrac{\partial y}{\partial\cos\theta} & \dfrac{\partial y}{\partial E'}\end{vmatrix}=\frac{1-y}{My}.$$

$$\frac{\partial}{\partial\cos\theta}\left(\sin^2\frac{\theta}{2}\right)=\frac{\partial}{\partial\cos\theta}\left(\frac{1-\cos\theta}{2}\right)=-\frac{1}{2}$$

Jacobian $\cos\theta, E' \to x, y$

L-inv : s, M, ν, x, y, Q^2.
Labo : E, E', θ, Ω.

result

$$\frac{d^2\sigma}{dxdy}=\frac{2\pi}{|J|}\frac{d^2\sigma}{d\cos\theta dE'}=\frac{2\pi My}{1-y}\frac{4\alpha^2E^2(1-y)^2}{Q^4}\times\left[\frac{F_2(x,y)}{\nu}\cos^2\left(\frac{\theta}{2}\right)+\frac{2F_1(x,y)}{M}\frac{M^2xy}{s(1-y)}\right]=$$

$$=\frac{s\pi y4\alpha^2\cancel{E}(1-y)}{Q^4}\left[\frac{F_2(x,y)}{\cancel{E}y}+\cancel{2}F_1(x,y)\frac{\cancel{M}xy}{\cancel{s}(1-y)}\frac{\cancel{E}}{\cancel{E}}\right]=\frac{4\pi\alpha^2}{Q^4}s\left[(1-y)F_2(x,y)+xy^2F_1(x,y)\right].$$

The quark parton model

partons = quarks ???

In the SM the answer is **YES** :

the **quark-parton model**;

Which is the <u>dynamical meaning</u> of $F_{1,2}$? Can we measure them ? [*yes, of course*]

- in principle the proton and the neutron have different structure functions;
- also a given process could result in a different structure [*e.g. the electron scattering could "see" different* $F_{1,2}$ *from neutrino- or hadron-hadron interactions*];
- in this picture, e.g. we will refer to "$F_1^{ep}(x)$", meaning <u>$F_1(x)$ for the proton, when probed in DIS by an electron</u>;
- similarly "$F_2^{ep}(x)$", "$F_2^{en}(x)$", "$F_2^{\nu p}(x)$", …
- however, these functions are NOT really independent : if they reflect the true dynamics, they must be correlated.

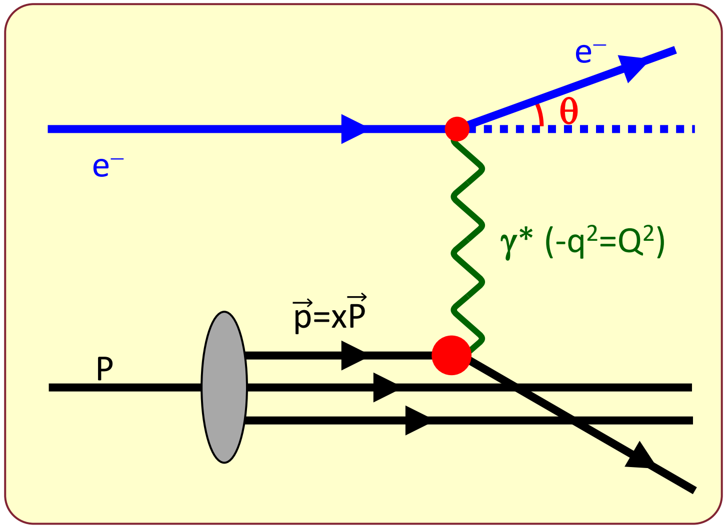
- assume that the nucleons are made by three quarks [*Nature is much more complicated, but wait …*];
- call them "**valence quarks**" [*why ???*];
- each of them is described by a x distribution, identified with "$f_j^p(x)$" [e.g. "$u^p(x)$" = <u>the x distribution for u-quarks in the proton</u>];
- e.g. $u^p(x)dx$ = <u>number</u> of u quarks in the proton, with x in the interval (x, x+dx);
- then $d^p(x)$, $\bar{u}^p(x)$, $\bar{u}^{\bar{p}}(x)$, $u^n(x)$, $\bar{u}^{\bar{n}}(x)$,…;
- (already defined) the functions $q^N(x)$ [q=u,d,ū,…; N=p,n] are called <u>parton distribution functions</u> (**PDF**);

(*continue …*)

The quark parton model: u_p, u_n, d_p, d_n, ...

(… *continue*)

Some obvious relations hold [*the green ones with a* (*) *are provisional, we'll modify them*] :



- from charge conjugation : $u^p(x) = \bar{u}^{\bar{p}}(x)$;
- from quark model and isospin invariance : $u^p(x) \approx d^n(x)$;
- from quark model + isospin $u^p(x) \approx 2\ u^n(x)$;
- from quark model + isospin $d^n(x) \approx 2\ d^p(x)$;
- (*) for valence quarks only, $\bar{u}^p(x) = 0$;
- (*) for valence quarks only, $s^p(x) = 0$;
- (*) therefore, e.g.

$$F_2^{ep}(x) = x \sum\nolimits_j e_j^2 f_j(x) = x \left(\frac{4u^p(x) + d^p(x)}{9} \right);$$

… many more formulæ, all quite intuitive.

The quark parton model: valence and sea

- According to the uncertainty principle, for short intervals q.m. allows <u>quark-antiquark pairs</u> to exist in the nucleons;
- in the hadrons some neutral particles exist, called gluons [*??? ... wait*].

Therefore, let us modify the scheme:

- in the nucleons, 3 types of particles :
  - **valence quarks** [already seen] with distribution $q_V(x)$ *[e.g* $u_V^p(x)$ *[already defined with the simpler notation* $u^p(x)$*]*;
  - **sea quarks**, i.e. the quark-antiquark pairs, described by distributions $q_S(x)$ [e.g $u_S^p(x)$, $s_S^p(x)$, $\bar{u}_S^p(x)$, $\bar{s}_S^p(x)$];
  - **gluons**, described by the distributions $g^p(x)$ and $g^n(x)$.

Obviously only sums can be measured:

$$u^p(x) \equiv u_V^p(x) + u_S^p(x);$$
$$d^p(x) \equiv d_V^p(x) + d_S^p(x);$$
$$\bar{u}^p(x) \equiv \bar{u}_V^p(x) + \bar{u}_S^p(x) = \bar{u}_S^p(x);$$
$$s^p(x) \equiv s_V^p(x) + s_S^p(x) = s_S^p(x);$$

Relations (*final, no further refinement*) :

- charge conjugation constraint :
  $$u^p(x) = \bar{u}^{\bar{p}}(x);$$
- from quark model + isospin invariance :
  $$u_V^p(x) \approx d_V^n(x) \equiv u_V(x);$$
  $$d_V^p(x) \approx u_V^n(x) \equiv d_V(x);$$
- from quark model : $u_V^p(x) \approx 2\, u_V^n(x)$;
- from quark model : $d_V^n(x) \approx 2\, d_V^p(x)$;
- from quantum mechanics and isospin invariance [*but neglecting quark masses*] :
  $$u_S^p(x) = \bar{u}_S^p(x) \approx d_S^p(x) = \bar{d}_S^p(x) \approx$$
  $$\approx s_S^p(x) = \bar{s}_S^p(x) \equiv q_S^p(x) \approx q_S^n(x);$$
- ... many more, all quite intuitive.

the "valence-ness" is not an observable, i.e. a u-quark "does not know" whether (s)he is v or s.

The quark parton model: $F_{proton}(x)$ vs $F_{neutron}(x)$

Putting everything together, we have [neglecting heavier quarks] :

$$F_2^{ep}(x) = x\left\{\frac{4}{9}\left[u^p(x)+\bar{u}^p(x)\right]+\frac{1}{9}\left[d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\} =$$

$$= x\left\{\frac{4}{9}\left[u_V(x)+2q_s(x)\right]+\frac{1}{9}\left[d_V(x)+2q_s(x)\right]+\frac{1}{9}\left[2q_s(x)\right]\right\} =$$

$$= x\left\{\frac{4}{9}u_V(x)+\frac{1}{9}d_V(x)+\frac{4}{3}q_s(x)\right\};$$

$$F_2^{en}(x) = x\left\{\frac{1}{9}u_V(x)+\frac{4}{9}d_V(x)+\frac{4}{3}q_s(x)\right\};$$

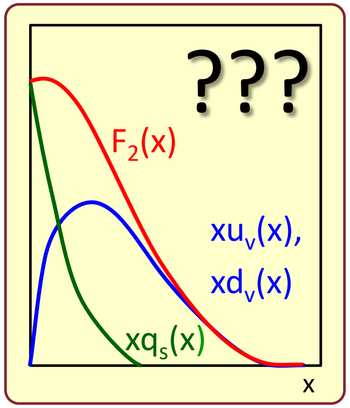
$$F_2^{en}/F_2^{ep} = R_{np} = \begin{cases} 1 & \text{(a);} \\ \left[4d_V(x)+u_V(x)\right]/\left[4u_V(x)+d_V(x)\right] & \text{(b).} \end{cases}$$

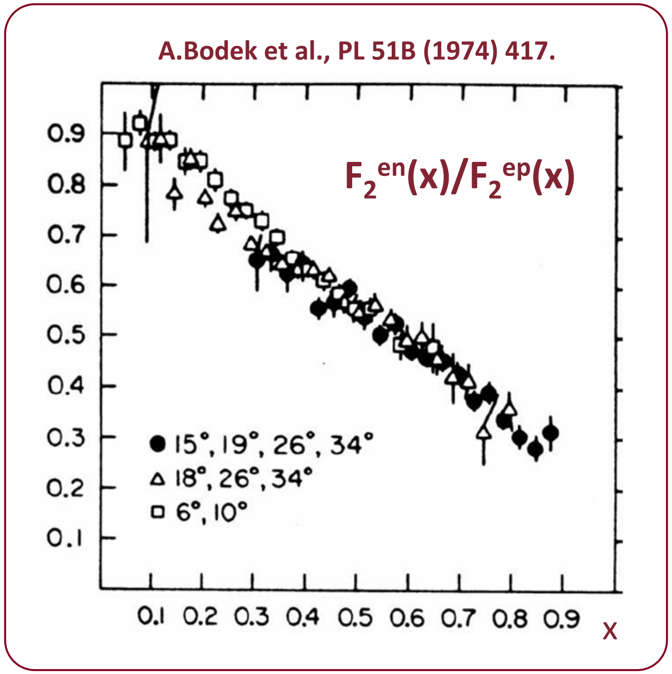
(a) if sea dominates (see little sketch);

(b) if valence dominates [if (u_V>>d_V) $\rightarrow R_{np} \approx$ ¼].

The measurement shows that case (a) happens at low x, while (b) dominates at high x.

In other words, there are plenty of $q\bar{q}$ pairs at small momentum, while valence is important at high x….





The quark parton model: toy model for $F_2(x)$

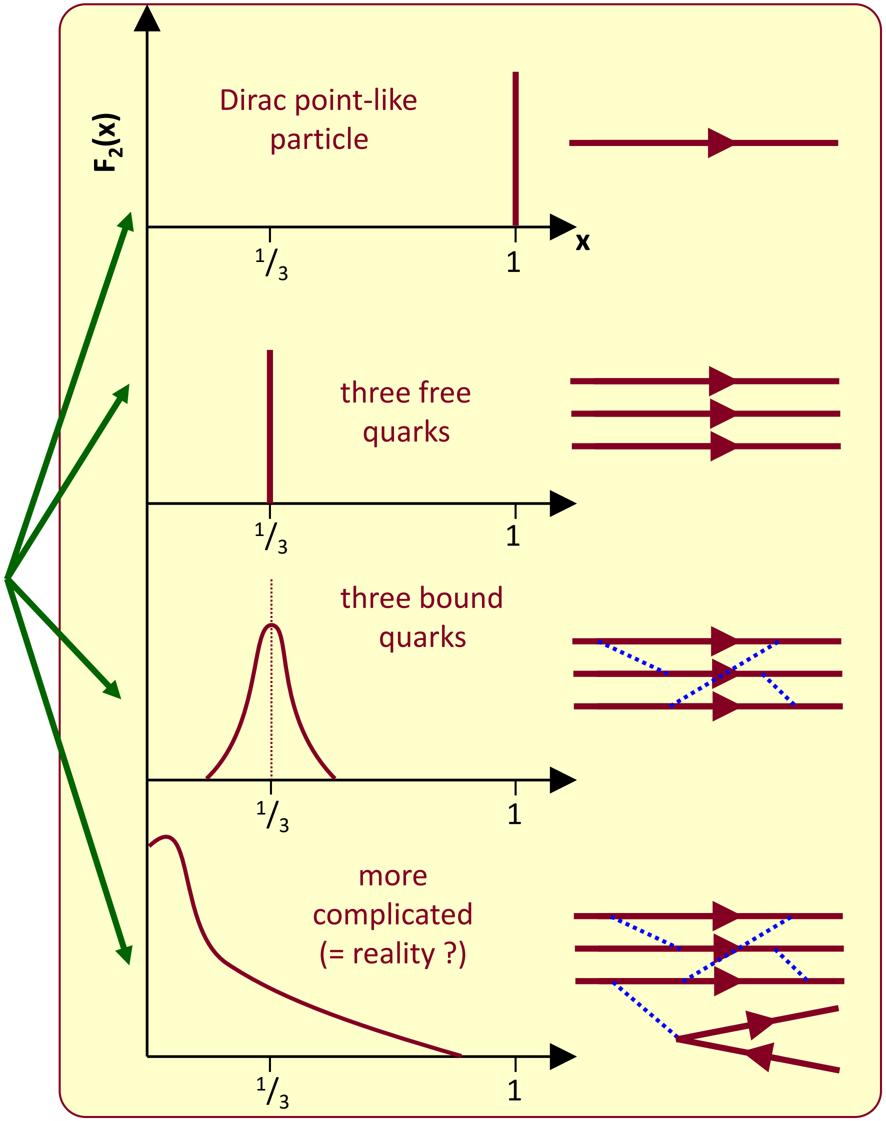
Sum rules (from momentum conservation) :

$$\int_0^1 dx\left[u^p(x)-\bar{u}^p(x)\right]=\int_0^1 dx\,u_V^p(x)=2;$$

$$\int_0^1 dx\left[d^p(x)-\bar{d}^p(x)\right]=\int_0^1 dx\,d_V^p(x)=1;$$

$$\int_0^1 dx\left[s^p(x)-\bar{s}^p(x)\right]=0.$$

Hypothetical (**NOT CORRECT**) shapes of $F_2(x)$ from naïve dynamical models :



The quark parton model: $F_2^{ep}(x)$- $F_2^{en}(x)$

From :

$F_2^{ep}(x) = x\,[4u_V(x) + \;\; d_V(x) + 12\,q_S(x)]\,/\,9;$

$F_2^{en}(x) = x\,[\;\; u_V(x) + 4d_V(x) + 12\,q_S(x)]\,/\,9;$

we get

$F_2^{ep}(x) - F_2^{en}(x) = x\,[u_V(x) - d_V(x)]\,/\,3;$

If, moreover, from the naïve quark model

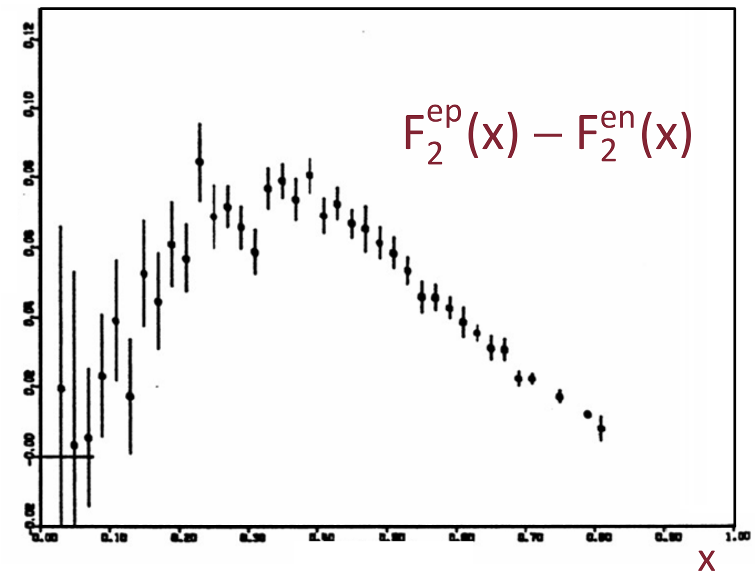
$u_V(x) \approx 2\,d_V(x)$

we get

$F_2^{ep}(x) - F_2^{en}(x) = x\,d_V(x)\,/\,3;$

i.e. this difference, which is an observable, roughly corresponds to the x-distribution of the "lone" valence quark (d_V^p or u_V^n).

Friedman, Kendall - Ann.Rev.Nucl.Sci. 22, 203 (1972)



The quark parton model: the gluon

The integrals of $F_2(x)$ are both calculable and measurable. By neglecting the small contribution of $s\bar{s}$:

$$\int_0^1 dx F_2^{ep}(x) = \frac{4}{9}\int_0^1 x\left[u^p(x)+\bar{u}^p(x)\right]dx + $$
$$+\frac{1}{9}\int_0^1 x\left[d^p(x)+\bar{d}^p(x)\right]dx = \frac{4}{9}f_u + \frac{1}{9}f_d;$$
$$\int_0^1 dx F_2^{en}(x) = \frac{4}{9}\int_0^1 x\left[d^p(x)+\bar{d}^p(x)\right]dx + $$
$$+\frac{1}{9}\int_0^1 x\left[u^p(x)+\bar{u}^p(x)\right]dx = \frac{4}{9}f_d + \frac{1}{9}f_u;$$

where $f_{u,d}$ are the fractions of the proton momentum carried by the quark u,d (and the respective $\bar{q}$).

From direct measurement, we get :

meas.

$$\left.\begin{array}{l} \int_0^1 dx F_2^{ep}(x) = \frac{4}{9}f_u + \frac{1}{9}f_d \approx \boxed{0.18;} \\ \int_0^1 dx F_2^{en}(x) = \frac{4}{9}f_d + \frac{1}{9}f_u \approx \boxed{0.12;} \end{array}\right\} \rightarrow \left\{\begin{array}{l} f_u \approx 0.36; \\ f_d \approx 0.18; \\ f_u + f_d \approx \boxed{0.54.} \end{array}\right.$$

Result (important) :

$$f_u + f_d \approx 50\ \%.$$

Only ≈ ½ of the nucleon momentum is carried by quarks and antiquarks.

The rest is "invisible" in the DIS by a charged lepton.

This was one of the first (and VERY convincing) evidences for the existence of the **gluons**, the carriers of the hadronic force.

The gluons are neutral and do not "see" the e.m. interactions.

The quark parton model: e⁻p vs νp DIS

Compute $F_2^{eN}(x)$ for an *isoscalar target* ***N***, i.e. a target with $n_{protons}$ = $n_{neutrons}$, both *quasi-free* (*Fermi-gas approx*) :

$$F_2^{ep}(x) = x\left\{\frac{4}{9}\left[u^p(x)+\bar{u}^p(x)\right]+\frac{1}{9}\left[d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\};$$

$$F_2^{en}(x) = x\left\{\frac{4}{9}\left[d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[u^p(x)+\bar{u}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\};$$

$$F_2^{eN}(x) \equiv \frac{F_2^{ep}(x)+F_2^{en}(x)}{2} =$$

$$= x\left\{\frac{5}{18}\left[u^p(x)+\bar{u}^p(x)+d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\} \xrightarrow[\text{neglect s}]{}$$

$$\to \frac{5x}{18}\left[u^p(x)+\bar{u}^p(x)+d^p(x)+\bar{d}^p(x)\right].$$

Notice that in neutrino DIS (see) the dynamics is different, but the effective structure function for an isoscalar target **turns out to be very similar**, up to a factor, as in the purely e.m. case :

$$F_2^{\nu N}(x) = x\left[u^p(x)+\bar{u}^p(x)+d^p(x)+\bar{d}^p(x)\right] = F_2^{eN}(x)\Big/\frac{5}{18}.$$

The experimental value (see) is F_2^{eN} / $F_2^{\nu N}$ = 0.29 ± 0.02, very compatible with this prediction (5/18 = 0.278).

why "isoscalar" ?

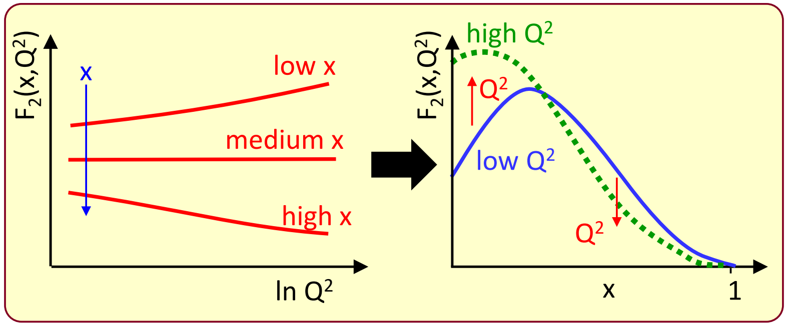
because (especially in ν scattering) the target has to be heavy, i.e. made of heavy nuclei, well reproduced by this approximation.

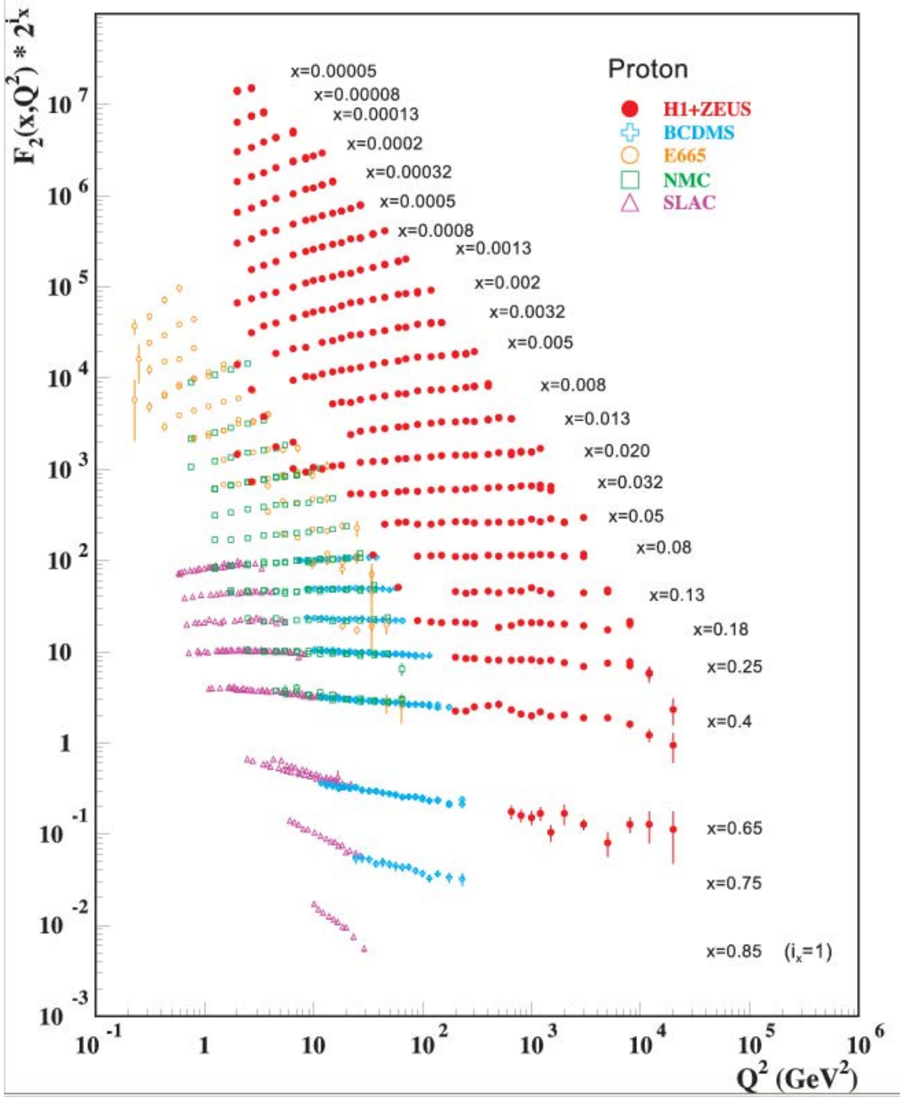
i.e. the structure functions depend on real properties of the nucleon structure, and are not dependent on the interaction.

$F_2(X,Q^2)$: scaling violations

Modern experiments have probed the nucleon to very high values of Q^2. Now electrons are often replaced with muons, which have the advantage of intense beams of higher momenta. Or, even better, the experiments are carried out at e^-p Colliders (HERA).

There are data up to $Q^2 \approx 10^5$ GeV^2: when plotting F_2 as function of Q^2 at fixed x, some Q^2-dependence appears, <u>incompatible with Bjorken scaling</u> [*see plot and sketch, and the next slides*].





$F_2(X,Q^2)$: Q^2 evolution

However, such an effect, known as *__scaling violations__*, is NOT due to sub-structures or other novel effects, but to a dynamical change in F_2, well understood in QCD.

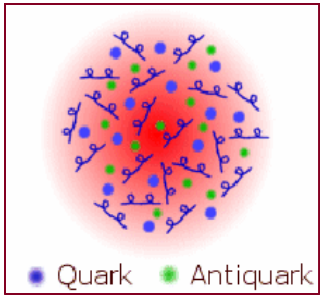
In QCD :

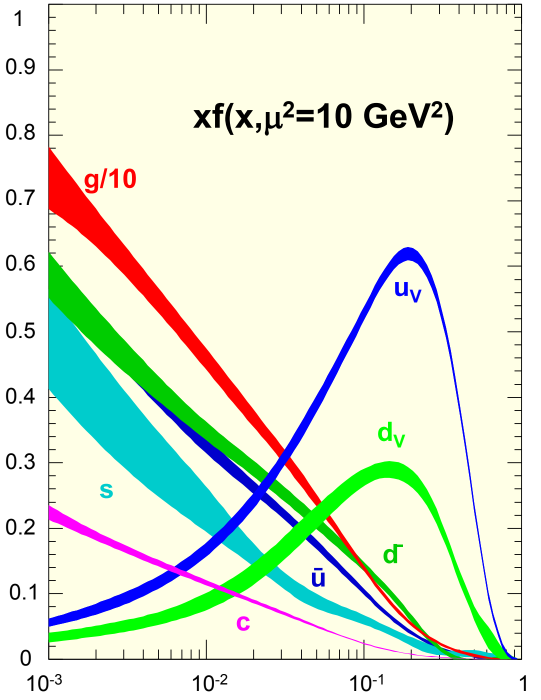
- higher Q^2

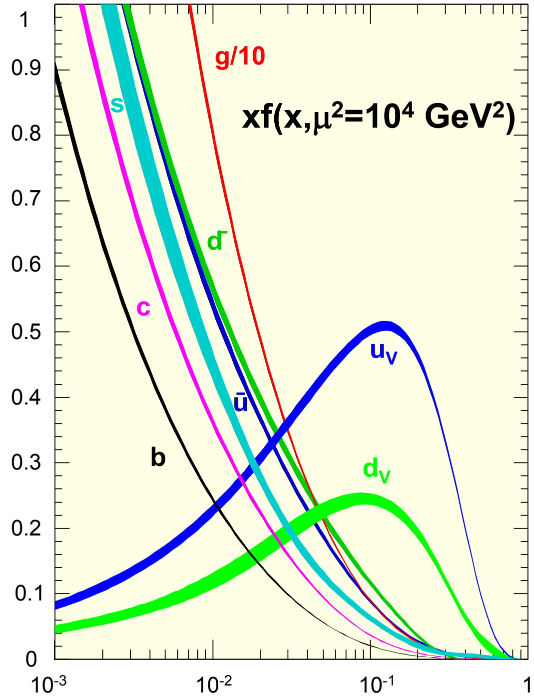
→ smaller size probed

→ more $q\bar{q}$ and gluons

→ less valence quarks.







a modern parameterization of the PDF [*NNPDF3.0-(NNLO)*] shows clearly the difference in the PDF when $Q^2 = 10 \div 10^4$ GeV2:

- u_V, d_V → down;
- $\bar{u}$, $\bar{d}$, [= u_S, d_S,] g → up;
- s, c, b → up (more phase space)

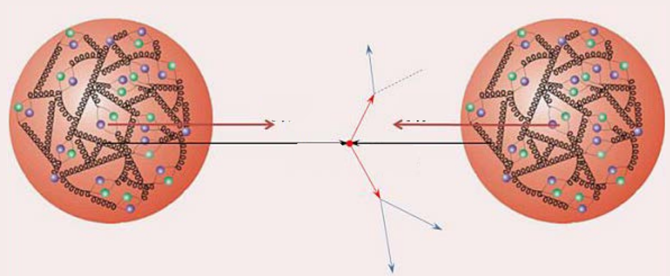
$F_2(X,Q^2)$: parton distribution function

For modern experiments with hadrons the knowledge of $F_2^{p,n}(x)$ is a necessary ingredient of the data analysis.

- The structure functions are an effect of the hadronic forces. However, being a complicated result of an ill-defined number of bodies in non-perturbative regime, they cannot be reliably computed with today's technology (lattice QCD is still a hope).
- *Similar to the chemistry of complicated molecules, which is a difficult subject, although the fundamental interactions are* [*supposed to be*] *well understood*.
- When studying hadron interactions at large Q^2, the initial state is parameterized by its structure function, as an incoherent sum of all the PDF's, including the gluon.

- In practice, all the computations (e.g. the Higgs production) must use a numerical parameterization of the PDF's, and take into account their uncertainties.

- the PDF's are probabilistic, i.e. the value of x is different for each event !!!
- *consequence: the 4-mom conservation at parton level is a difficult constraint in the computation !!! (see later*)



An artist's view of the pp interaction [from the CERN ATLAS www site]

Summary of cross-sections

$$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Ruthe}\\ \text{rford}}} = \frac{4Z^2\alpha^2 E'^2}{|q|^4};$$

$$\left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Ruthe}\\ \text{rford}}} \times \left(1-\beta^2\sin^2\frac{\theta}{2}\right) \rightarrow \left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Ruthe}\\ \text{rford}}} \times \cos^2\frac{\theta}{2};$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} \times \frac{E'}{E}; \qquad \left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{non-}\\ \text{point.}}} = \left[\frac{d\sigma}{d\Omega}\right]^{(*)}_{\text{Mott}} \times \left|F(q^2)\right|^2.$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{point-like}\\ \text{spin1/2}}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(1+2\tau\tan^2\frac{\theta}{2}\right); \qquad \left[\tau = \frac{Q^2}{4M^2c^2}\right];$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\substack{\text{Rosen}\\ \text{bluth}}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2}\right);$$

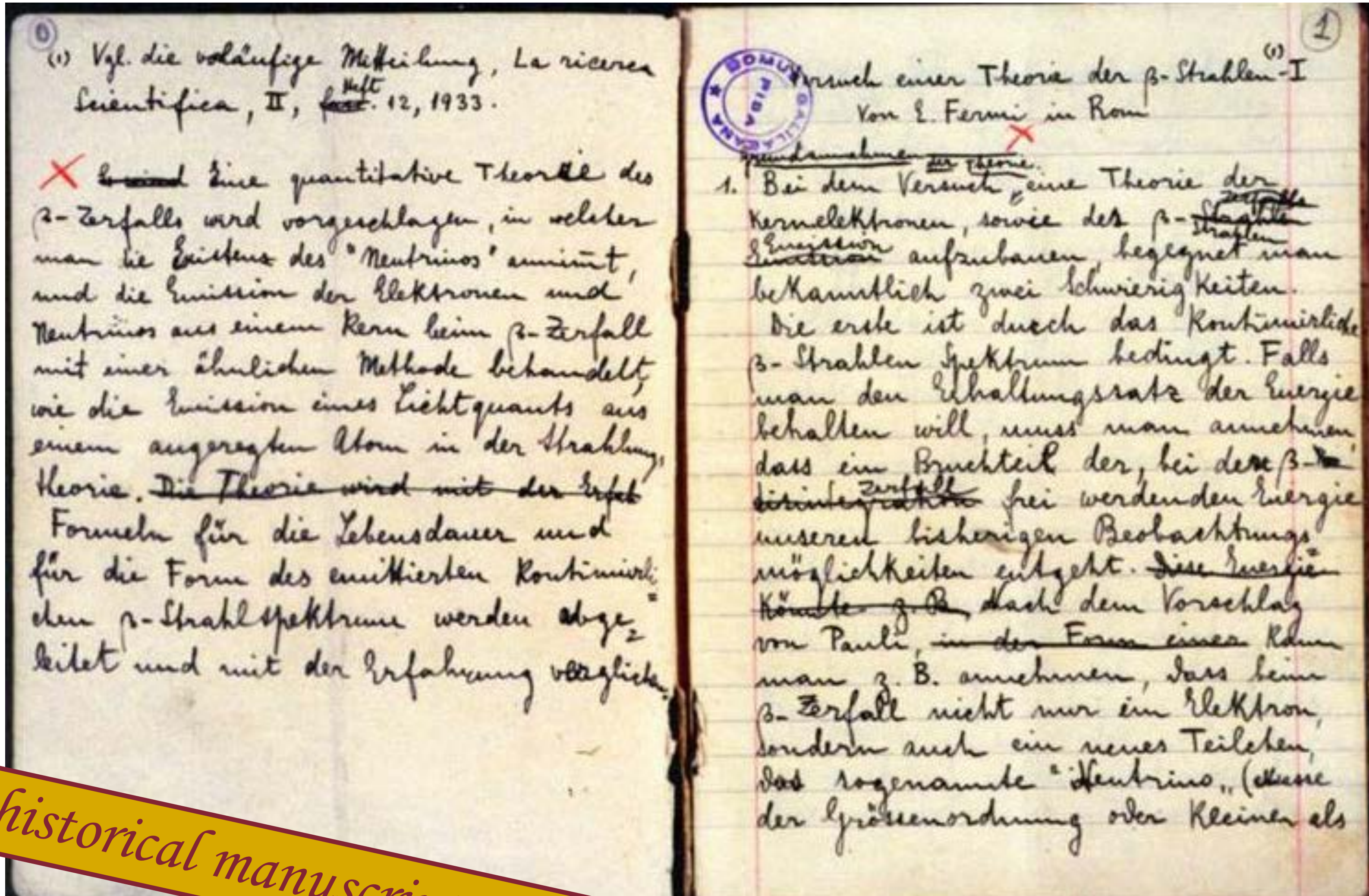
$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{DIS}} = \frac{4\alpha^2 E'^2}{Q^4} \times \left[W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}\right];$$

$$\left[\frac{d^2\sigma}{dxdy}\right]_{\text{DIS}} = \frac{4\pi\alpha^2 s}{Q^4} \times \left[xy^2F_1(x,Q^2) + (1-y)F_2(x,Q^2)\right].$$

Q^2 scale

WEAK INTERACTIONS

The weak interactions: the origins


(1) Vgl. die vorläufige Mitteilung, La ricerca Scientifica, II, Heft 12, 1933.

Eine quantitative Theorie des β-Zerfalls wird vorgeschlagen, in welcher man die Existenz des "Neutrinos" annimmt, und die Emission der Elektronen und Neutrinos aus einem Kern beim β-Zerfall mit einer ähnlichen Methode behandelt, wie die Emission eines Lichtquants aus einem angeregten Atom in der Strahlungstheorie. ~~Die Theorie wird mit der Erfah~~ Formeln für die Lebensdauer und für die Form des emittierten kontinuierlichen β-Strahlspektrums werden abgeleitet und mit der Erfahrung verglichen.

Versuch einer Theorie der β-Strahlen. I (1)

Von E. Fermi in Rom

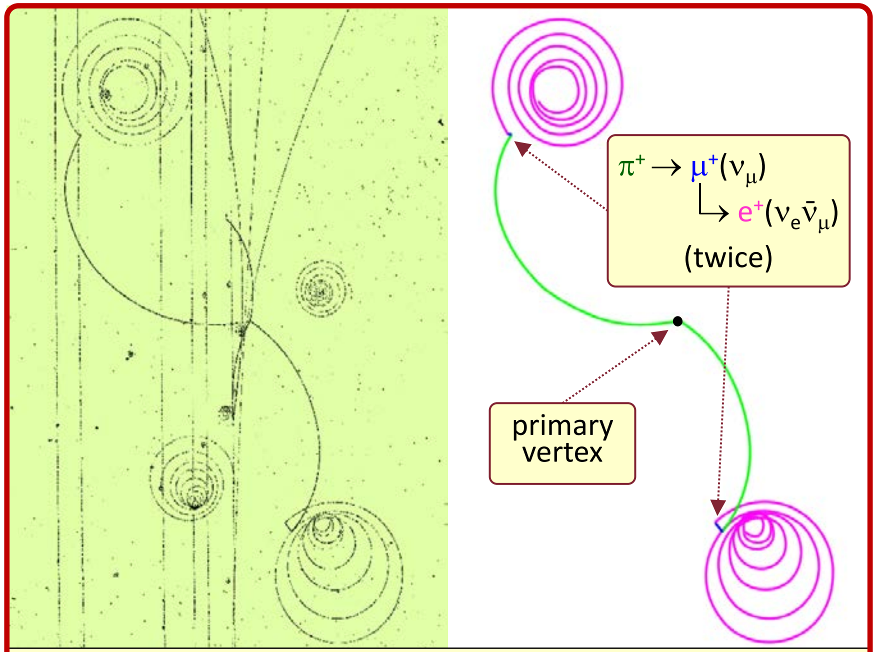
~~Grundannahmen der Theorie.~~

1. Bei dem Versuch, eine Theorie der Kernelektronen, sowie der β-Zerfalle aufzubauen, begegnet man bekanntlich zwei Schwierigkeiten. Die erste ist durch das kontinuierliche β-Strahlen Spektrum bedingt. Falls man den Erhaltungssatz der Energie behalten will, muss man annehmen, dass ein Bruchteil der, bei dem β-Zerfall frei werdenden Energie unseren bisherigen Beobachtungsmöglichkeiten entgeht. ~~Diese Energie könnte z. B.~~ Nach dem Vorschlag von Pauli ~~in der Form eines~~ kann man z. B. annehmen, dass beim β-Zerfall nicht nur ein Elektron, sondern auch ein neues Teilchen, das sogenannte "Neutrino" (Masse der Grössenordnung oder kleiner als

a historical manuscript [thanks to F. Guerra]

The weak interactions: introduction

- Some rare processes, i.e. small coupling, violate the conservation laws, valid for strong and electromagnetic interactions.
- In ordinary matter the **weak interactions** (w.i.) have a negligible effect, except in cases otherwise forbidden (e.g. β decay).
- The w.i. are responsible for the fact that STABLE matter contains only u and d quarks and electrons. Other quarks and leptons are UNSTABLE because of w.i..
- Therefore, in spite of their "weakness" (small range of interaction $\approx 10^{-3}$ fm, tiny cross sections $\approx 10^{-47}$ m^2), the w.i. play a crucial role in the features of our world.
- ALL elementary particles, but ~~gluons~~ and ~~photons~~ (carriers of other interactions), are affected by w.i. : quarks and charged leptons have w.i., ν's have ONLY them.
- In the scattering processes of charged hadrons and leptons, the effects due to the strong and electromagnetic interactions "obscure" those of the w.i..
- Therefore most of our knowledge on this subject, at least until the '70s, has been obtained from the study of the decays of particles and from ν beams.



CERN 2m hydrogen bubble chamber: $K^+p \to \pi^+\pi^+X$

The weak interactions: introduction

1930 Pauli : ν existence to explain β–decay.

1933 Fermi : first theory of β–decay.

1934 Bethe and Peierls : νN and $\bar{\nu} N$ cross sections.

1936 Gamow and Teller : G.-T. transitions.

1947 Powell + Occhialini : decay $\pi^+ \to \mu^+ \to e^+$.

1956 Reines and Cowan : ν's detection from a reactor.

1956 Landè, Lederman and coll. : $K^0{}_L$.

1956 Lee and Yang : parity non-conservation.

1957 Feynman and Gell-Mann, Marshak and Sudarshan : V–A theory.

1958 Goldhaber, Grodzins and Sunyar : ν helicity.

1960 (ca) Pontecorvo and Schwarz : ν beams.

1961 Pais and Piccioni : $K_L \leftrightarrow K_S$ regeneration.

1962 First ν beam from accelerator : Lederman, Schwarz, Steinberger : ν_μ.

1963 Cabibbo theory.

1964 Cronin and Fitch : CP violation in K^0 decay.

1964 Brout, Englert, Higgs : Higgs mechanism.

1968 Weinberg–Salam model.

1968 Bjorken scaling, quark-parton model.

1970 GIM mechanism.

1972 Kobayashi, Maskawa : CKM matrix.

1973-90 ν DIS experiments : Fermilab, CERN.

1973 CERN Gargamelle : neutral currents.

1983 CERN Sp$\bar{\text{p}}$S : $W^\pm$ and Z.

1987 CERN Sp$\bar{\text{p}}$S : B^0 mixing discovery.

1989-95 CERN LEP : Z production + decay.

1997-2000 CERN LEP : W^+W^- production.

1998-2000 ν oscillations.

1999-20xx B^0 mixing detailed studies.

2012 CERN LHC : Higgs boson.

- **only major facts $\geq$ 1930 considered;**

The weak interactions: introduction

- Let's recall the life time of a few decays:

$\Delta^{++} \rightarrow p\pi$	$\sim 10^{-23}$ s	strong int.
$\Sigma^0 \rightarrow \Lambda\gamma$	$\sim 6 \cdot 10^{-20}$ s	1 γ, e.m. int.
$\pi^0 \rightarrow \gamma\gamma$	$\sim 10^{-16}$	2 γ, e.m. int.
$\Sigma \rightarrow n\pi$	$\sim 10^{-10}$ s	weak int.
$\pi^- \rightarrow \mu^- \nu_\mu$	$\sim 10^{-8}$ s	weak int.
$\mu^- \rightarrow e^- \nu_e \nu_\mu$	$\sim 10^{-6}$ s	weak int.
$n \rightarrow p\, e^- \nu_e$	$\sim$ 15 min	weak int.

N.B. we observe the weak interactions only when the strong and e.m. interactions are forbidden.

- We need to explain the enourmos range in the life time going from 10^{-12} s untill 15 min.
- The weak interactions are also characterized by cross-sections extremely small ($\sim 10^{-39}$ cm^2=1 fb)

$$\sigma(\nu_\mu + N \rightarrow N + \pi + \mu) = 10^{-38}\ \mathrm{cm}^2\ (10\ \mathrm{fb})\ \mathrm{a}\ 1\ \mathrm{GeV}$$

$$\sigma(\pi + N \rightarrow N + \pi) = 10^{-26}\ \mathrm{cm}^2\ (10\ \mathrm{mb})\ \mathrm{a}\ 1\ \mathrm{GeV}$$

- The weak interactios violate many conservation rules (parity, charge conjugation, strangeness, etc...)
- Because of their "weakness", the weak interactions can be observed in the "standard" matter only in the beta decay, because they do not give origin to any bound states. However they are the base fuel for the stars functioning , therefore without the weak interactions we coud not exist

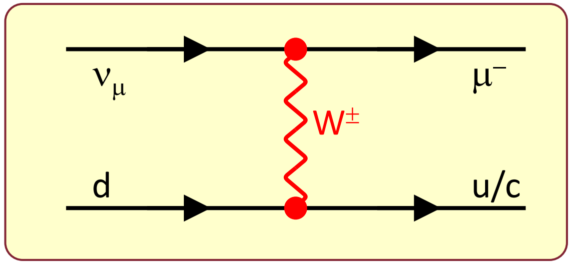
$$p + p \rightarrow d + e^+ + \nu_e$$

The weak interactions: CC and NC

In the SM, weak interactions (w.i.) are classified in two types, according to the charge of their carriers :

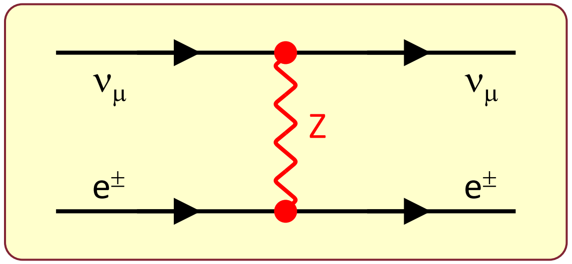
- Charged currents (**CC**), **$W^\pm$** exchange:
  - in the CC processes, the charge of quark and leptons CHANGES by ±1; at the same time there is a variation of their IDENTITY, including FLAVOR, according to the Cabibbo theory (today Cabibbo-Kobayashi-Maskawa)

e.g.



- Neutral currents (**NC**), **Z** exchange:
  - in the NC case, quarks and leptons remain unchanged (no **FCNC**);
  - until 1973 no NC weak process was observed [*but another example of NC was well known, i.e. the e.m. current: γ's carry no charge !*]

e.g.

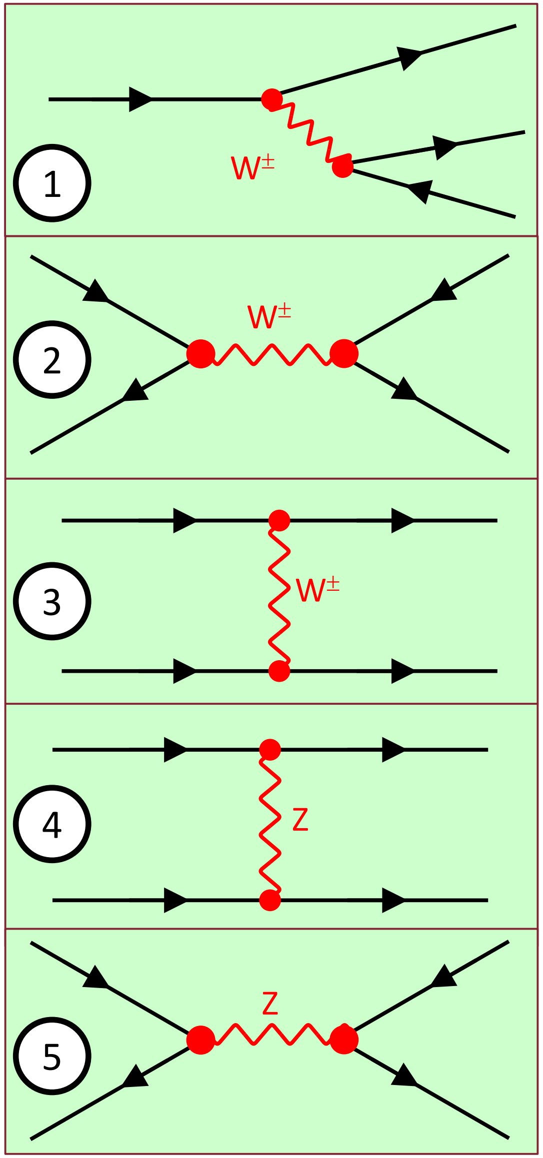


- In the 60's Glashow, Salam and Weinberg (+ many other theoreticians) developed a theory (today known as the "Standard Model", SM), that unifies the w.i. (both CC and NC) and the electromagnetism.

The SM was conceived BEFORE the discovery of NC. So the existence of NC and its carrier (the Z boson), predicted by the SM and observed at CERN in 1973 and 1983 respectively, were among the first great successes of the SM.

The weak interactions: classification

weak interactions	CC	leptonic	$\Delta S = 0$	$\mu \to e\ \nu_e\ \nu_\mu$	1
		semi-leptonic		$\pi^\pm \to \mu^\pm\ \nu_\mu$	2
				$n \to p\ e\ \nu_e$	1*
				$\nu_e\ d \to e^-\ u$	3*
				$d\bar{u} \to W^- \to\ e^-\bar{\nu}_e$	2*
			$\Delta S = \pm 1$	$K^\pm \to \mu^\pm\ \nu_\mu$	2
				$\Lambda \to p\ e\ \nu_e$	1*
		hadronic		$K^\pm \to \pi^\pm\ \pi^0$	2*
				$\Lambda \to p\ \pi^-,\ n\ \pi^0$	1*
	NC	leptonic	$\Delta S = 0$ (only)	$\nu_\mu\ e^\pm \to \nu_\mu\ e^\pm$	4
		semi-leptonic		$\nu\ N \to \nu\ N'$	4*
		hadronic		$u\ \bar{u} \to Z \to q\ \bar{q}$	5*



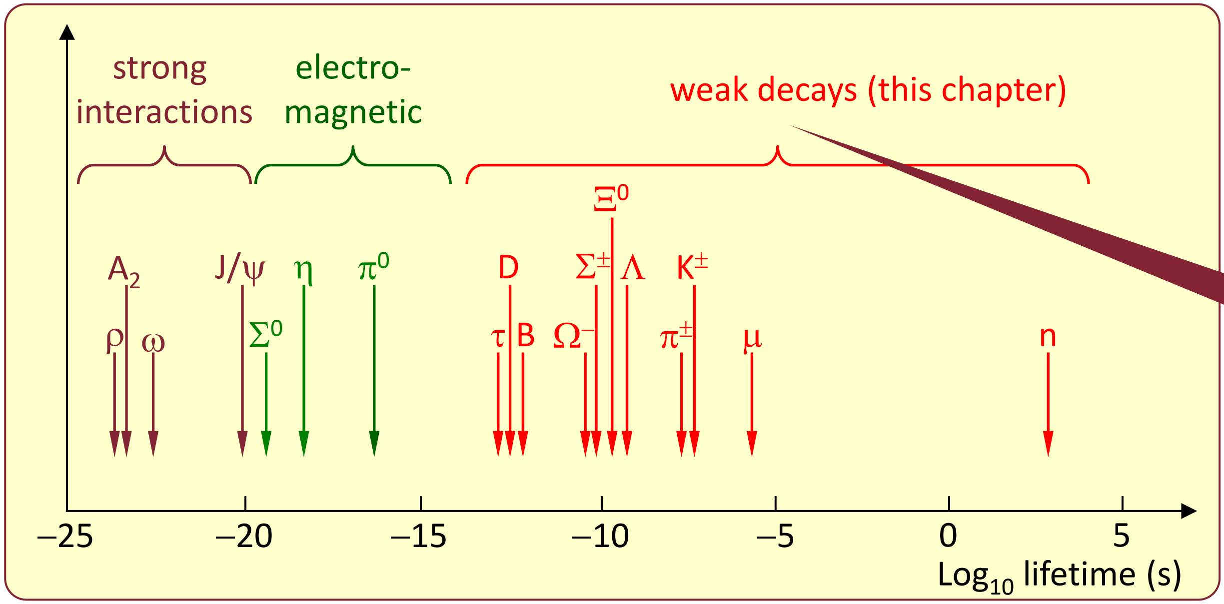
Some processes (list NOT exhaustive), classified in terms of general characteristics and Feynman diagrams.

A "*" in the last column means that the interacting hadron is composite; the diagrams shows only the interacting quark(s); the other partons (the "spectators") do not participate in the interaction, at least in 1st approximation.

In the table, ν means both ν and $\bar{\nu}$ [*only the correct one !*].

Charged Current: decays

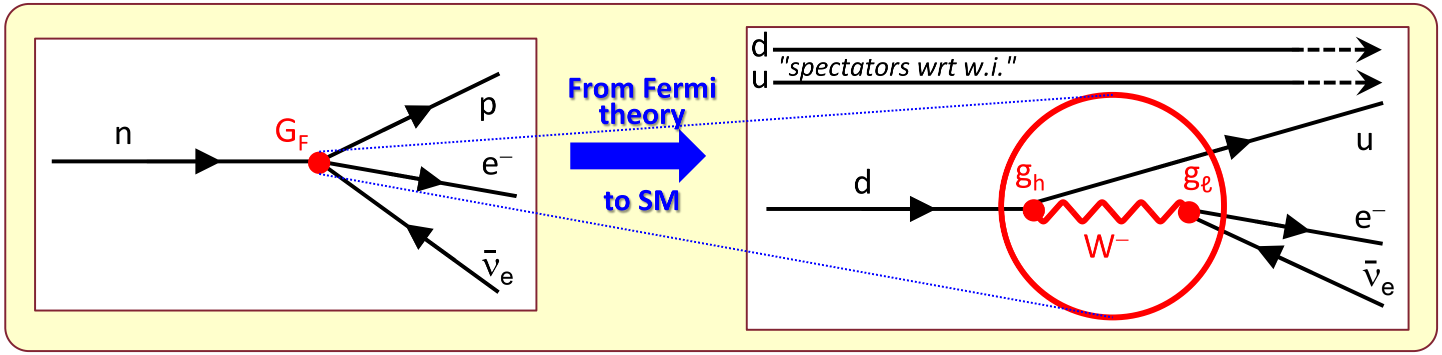
process	Lifetime (s)	comment
$\bar{\nu}_e\, p \to n\, e^+$	(none)	Neutrinos have only weak interactions (not a decay).
$n \to p\, e^-\, \bar{\nu}_e$	$\mathcal{O}(10^3)$	Long lifetime because of small mass difference (p-n).
$\pi^+ \to \mu^+\, \nu_\mu$	$\mathcal{O}(10^{-8})$	The $\pi^\pm$ is the lightest hadron, so it decays $\to$ leptons.
$\Lambda \to p\, \pi^-$	$\mathcal{O}(10^{-10})$	The decay of Λ violates strangeness conservation.



Some of the most interesting weak decays are the neutral heavy mesons of type $Q\bar{Q}$ (K^0, B^0).

Charged Current: Fermi theory

- The modern theory of the CC interactions (i.e. this part of the SM) is a successor of the Fermi theory of β decay.
- The Fermi theory describes a point-like interaction, proportional to the coupling G_F; the theory had intrinsic problems ("not renormalizable" in modern terms, i.e. cross-sections violate unitarity at high energy);
- the SM "expands" the point-like interaction, introducing a heavy charged mediator, called $W^{\pm}$.
- the SM is mathematically consistent (it is "renormalizable");
- *(more important)* it reproduces the experimental data with unprecedented accuracy.

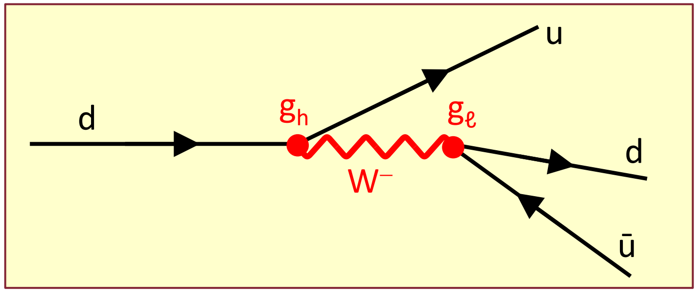


usual comment : to see a smaller scale requires higher $Q^2 \rightarrow$ higher energy

Charged Current: simple problems

Q. why is the decay $n \to p\pi^-$ (similar to $\Delta^0 \to p\pi^-$) forbidden ?

A. write the Feynman diagram



- possible ? forbidden ?

  yes, possible

- then ?

  $m(n) - m(p) \approx 1.3$ MeV

  The only possible pair ff' with q = –1 and baryon/lepton number = 0 is clearly $e^-\bar{\nu}_e$, since $m(e^-) + m(\bar{\nu}_e) \approx m(e^-) \approx 0.5$ MeV.

Q. why $n \to pe^-\bar{\nu}_e$ and not $p \to ne^+\nu_e$?

A. [... left to the reader]

Charged Current: coupling

A simple comparison between the couplings (g is the "charge" of the w.i. and plays a similar role as e):

- Electromagnetism :

  $\alpha \qquad\qquad \propto e^2;$

  amplitude $\propto \alpha \propto e^2;$

  rate $\qquad\quad \propto \alpha^2 \propto e^4.$

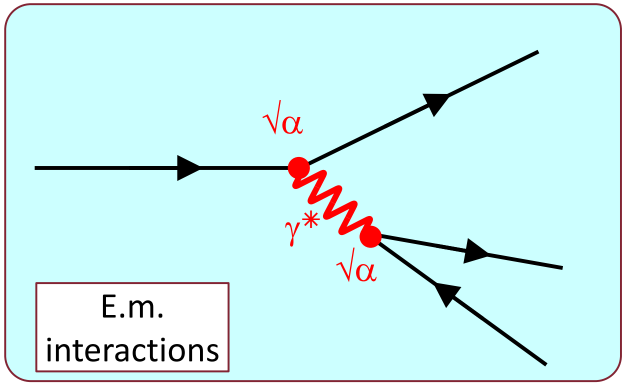
- Weak interactions :

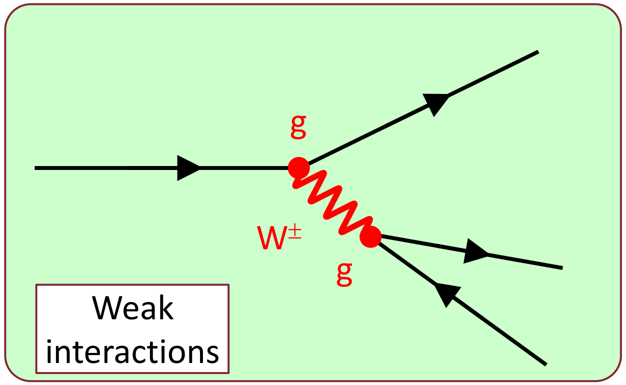
  $G_F \qquad\qquad \propto g^2;$

  amplitude $\propto G_F \propto g^2;$

  rate $\qquad\quad \propto G_F^2 \propto g^4;$

NB. unlike α, G_F is not adimensional (next slide); the <u>similarity</u> electromagnetism $\leftrightarrow$ weak interactions is hidden.





The weak interactions: beta decay

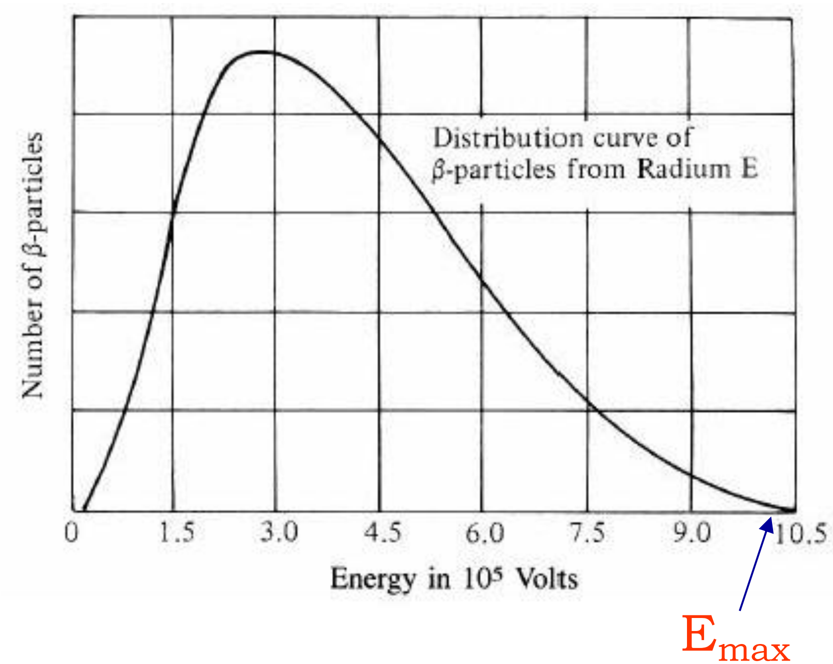
Most of our knowledge about the base principles of beta decay is based upon nuclei beta decays.

$n \rightarrow p + e^- + \bar{\nu}_e \quad \Rightarrow \quad (A,Z) \rightarrow (A,Z+1) + e^- + \bar{\nu}_e$

$p \rightarrow n + e^+ + \nu_e \quad \Rightarrow \quad (A,Z) \rightarrow (A,Z-1) + e^+ + \nu_e$

$e^- + p \rightarrow n + \nu_e \quad \Rightarrow \quad (A,Z) + e^- \rightarrow (A,Z-1) + \nu_e$

- The existence of the β^+ decay was established in 1934 by Curie and Joliot.
- In 1919 Chadwick discovered that the electron in the β decay had a continuos spectrum.



- The maximum energy of the spectrum corresponds fairly well to the Q of the reaction (Q=M(A,Z)-M(A,Z+1)), while for the rest of the spectrum there is a violation of the energy conservation rule.

- Moreover there is also a violation of the momentum and angular momentum conservation rules (without the introduction of the neutrino).

The weak interactions: neutrino

- To re-establish the various conservation laws, in 1930 Pauli made the hypothesis of the existence of a very small neutral particle: the neutron (later renamed neutrino by Fermi).

December 1930: public letter sent by W. Pauli to a physics meeting in Tübingen

Zürich, Dec. 4, 1930

Dear Radioactive Ladies and Gentlemen,
...because of the "wrong" statistics of the N and ^{6}Li nuclei and the continuous β-spectrum, I have hit upon a desperate remedy to save the law of conservation of energy. Namely, the possibility that there could exist in the nuclei electrically neutral particles, that I wish to call neutrons, which have spin ½ and obey the exclusion principle ..... The mass of the neutrons should be of the same order of magnitude as the electron mass and in any event not larger than 0.01 proton masses. The continuous β-spectrum would then become understandable by the assumption that in β-decay a neutron is emitted in addition to the electron such that the sum of the energies of the neutron and electron is constant.
.... For the moment, however, I do not dare to publish anything on this idea ......
So, dear Radioactives, examine and judge it. Unfortunately I cannot appear in Tübingen personally, since I am indispensable here in Zürich because of a ball on the night of 6/7 December. ....

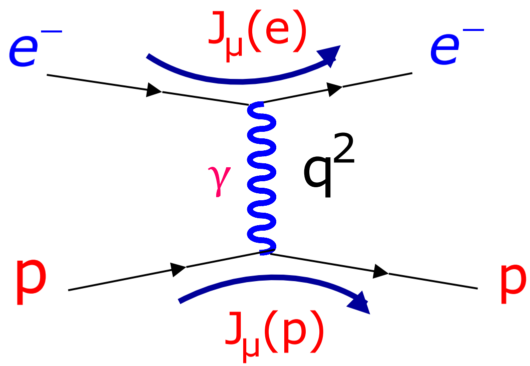
W. Pauli

NOTES
- Pauli's neutron is a light particle ⇒ not the neutron that will be discovered by Chadwick one year later
- As everybody else at that time, Pauli believed that if radioactive nuclei emit particles, these particles must exist in the nuclei before emission

- This letter is very important for Physics … but it is also interesting from a sociological point of view ☺
- The first theory of β decay was done by Fermi in 1934.
- The neutrino was discovered by Reines e Cowan "only" in 1958
- We have three kind of neutrinos; recently have been established the flavour neutrino oscillations that imply that neutrinos have masses different from zero, although they are very small and not yet measured.

Fermi theory of β interactions

- In 1934 Fermi did the first theory of β decay; he took as a model the QED description of the electron-proton scattering:

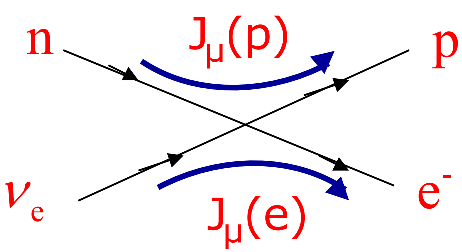


The matrix element is proportional to:

$$\boxed{M_{fi} \approx -\frac{1}{q^2} J_\mu(e) J^\mu(p)}$$

$$M_{fi} \approx \left(\bar{u}_e \sqrt{\alpha}\gamma^\mu u_e\right)\frac{g_{\mu\nu}}{q^2}\left(\bar{u}_p \sqrt{\alpha}\gamma^\nu u_p\right)$$

- Fermi made the hypothesis of a pointlike interaction like: $n+\nu \rightarrow p+e^-$ (that is like $n \rightarrow \bar{\nu}+p+e^-$)



→ There is no propagator

$$\boxed{M_{fi} \approx G\left(\bar{u}_p \gamma^\mu u_n\right)\left(\bar{u}_e \gamma_\mu u_\nu\right)}$$

vector-vector interaction

The G constant is known as Fermi's constant and it is related to the square of the "weak charge".

interaction between two (charged) currents: hadronic and leptonic currents.

$\bar{u}_p$ creates a proton (or destroys an antiproton)
u_n destroys a neutron (or creates an antineutron)
$\bar{u}_e$ creates an electron (or destroys a positron)
u_ν destroys a neutrino (or creates an antineutrino)

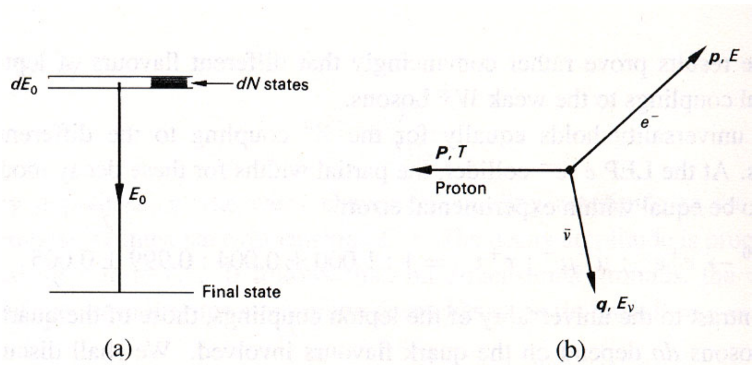
Nuclear β decay

- The transition probability (the decay rate per unit of time) can be found by using the Fermi's golden rule:

$$W=\frac{2\pi}{\hbar}G^2|M|^2\frac{dN}{dE_0}$$

$\boxed{\frac{dN}{dE_0}\text{ : phase space}}$

$|M|^2$ is the matrix element squared. It is computed by integrating over all angles of the final particles, by summing over the final spin states and by averaging on the initial spin states. It is a constant of order one.



Fermi decays: $J_{leptoni}=0 \Rightarrow |M|^2 \approx 1$

Gamow-Teller decays: $J_{leptoni}=1 \Rightarrow |M|^2 \approx 3$

- E_0 is the energy available in the final state (it is equal to the Q of the reaction). The energy spread dE_0 is present because the energy of the initial state is not precisely known due to the finite lifetime (Heisemberg's principle).

$$\vec{P}+\vec{q}+\vec{p}=0 \qquad T+E_\nu+E=E_0$$

- In the nuclear β decays E_0 is of the order of 1 MeV. The proton kinetic energy is of the order of 10^{-3} MeV and can be neglected. The proton is there just to ensure the momentum conservation.

$q_\nu=E_0-E_e$ The energy is shared between the electron and the neutrino

The phase space

- The number of available states for an electron with momentum between p and p+dp, confined in the volume V, within the solid angle dΩ, is:

$$dN = \frac{\mathrm{V}d\Omega}{(2\pi)^3\hbar^3}p^2dp$$

- We normalize the wave function to V=1, we sum over the entire solid angle and we ignore the effect of the spin on the angular distribution. We get the following phase space for the electron and neutrino:

$$dN_e = \frac{4\pi p^2dp}{(2\pi)^3\hbar^3} \quad ; \quad dN_\nu = \frac{4\pi q_\nu^2 dq_\nu}{(2\pi)^3\hbar^3}$$

- The two phase space factors are independent because there is no correlation between q and p, since it is a three bodies decay the proton will absorb the remaining momentum difference. The proton momentum is fixed (given q and p) so there is no proton phase space factor.

- The number of final states is: $dN = dN_e \cdot dN_\nu = \frac{(4\pi)^2}{(2\pi)^6\hbar^6}p^2q_\nu^2dpdq_\nu$

- For a given value of the electron energy E, the neutrino energy E_ν is fixed as well as its momentum:

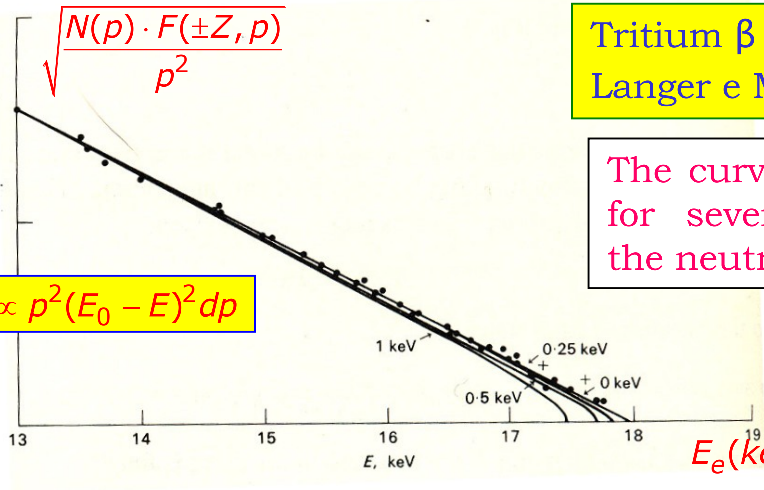
$$q_\nu \equiv E_\nu = (E_0 - E) \quad ; \quad \Rightarrow \mathrm{d}q_\nu = \mathrm{dE}_0 \qquad \Rightarrow \frac{dN}{\mathrm{dE}_0} = \frac{dN}{\mathrm{d}q_\nu} = \frac{1}{4\pi^4\hbar^6}p^2(E_0 - E)^2dp$$

- Once we have integrated the transition probability W over the entire solid angle, M^2 is equal to a constant, therefore the electron energy spectrum is entirely due to the phace space form:

$$\boxed{N(p)dp \propto p^2(E_0 - E)^2dp}$$

Kurie plot

$$\sqrt{\frac{N(p)\cdot F(\pm Z,p)}{p^2}}$$

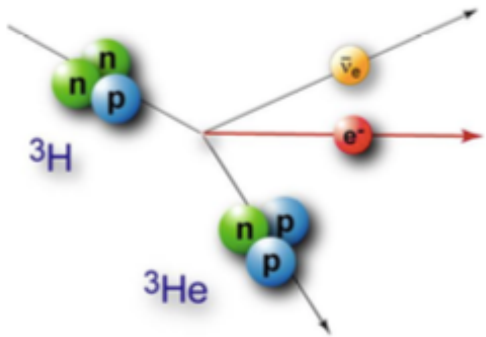


$$N(p)dp \propto p^2(E_0 - E)^2 dp$$

$E_e(keV)$

Tritium β decay.
Langer e Moffatt (1952)

The curves are plotted for several values of the neutrino mass.



$E_0 = 18.6\ keV$ $T_{1/2} = 12.3$ years

- If we plot $(N(p)/p^2)^{1/2}$ versus the electron energy, we get a straight line that crosses the x-axis at $E=E_0$. This graph used to study β decay was developed by Franz N.D. Kurie.
- Experimentally we need to include a correction factor F(Z,p) to take into account the Coulomb interaction between the electron and the nucleus.
- If the neutrino has a mass, its effect would be to modify the distribution in the following way:

$$N(p)dp \propto p^2(E_0 - E)^2 \sqrt{1 - \left(\frac{m_\nu}{E_0 - E}\right)^2}\, dp$$

- The Kurie plot is modified in a way that the curve crosses the x-axis at $E=E_0-m_\nu$. This is how we try to measure the neutrino-e mass. Unfortunately in this region there are very few events and it is very difficult to perform the experiment. At the moment we have only an upper limit.

$m_{\nu_e} \leq 2.2$ eV Mainz exp. ; 2000

The Sargent's rule

- The total decay rate is obtained by integrating the spectrum N(p)dp. It can be done analitically; however in the cases where the electron is relativistic we can use the approximation p≈E and we get a very simple formula:

$$N \propto \int_0^{E_0} E^2 (E_0 - E)^2 dE \propto E_0^5$$

- The decay rate is proportional to the fifth power of the energy available in the process. This is the **Sargent's rule.**

- If we consider all the numerical factors in the process, we get: $W = \dfrac{G^2 |M|^2 E_0^5}{60\pi^3 (\hbar c)^6 \hbar}$ (per $E_0 >> m_e$)

- The Fermi's constant G can be found, as we will see later, from the life time measurements of a few β decays (and with some theoretical speculations, see Cabibbo's angle) or in a more precise way from the muon life time.

- From the PDG we get: $\dfrac{G}{(\hbar c)^3} = 1.16637(1) \cdot 10^{-5}\ \mathrm{GeV}^{-2}$

- Replacing the numerical values in the formula we get: $\dfrac{1}{\tau} = W = \dfrac{1.11}{10^4} |M|^2 E_0^5\ \mathrm{s}^{-1}$ (E_0 in MeV)

- For instance, if E_0≈100 MeV like in the muon decay and M^2=1, we get: $\tau_\mu = \dfrac{1}{W} \approx 10^{-6}\ \mathrm{s}$ (τ_μ=2.2 μs)

N.B. it is the E_0^5 dependence that explains the huge range in the life time of the decays mediated by the weak interactions.

The nuclear β decay

- The nuclear β decays can be discriminated as allowed transitions and forbidden transitions.
- The allowed ones are the most common and are characterized by the fact that the electron and neutrino emitted DOES NOT carry any spatial angular momentum, that is they are in the S-state (L=0). This is justified by the fact that the two leptons have energies of the order a 1 MeV.
- The transitions with L=1 are called first forbidden, the ones with L=2 second forbidden and so on. They have a lifetime considerably longer that the allowed transitions.
- Since e and ν have spin ½ , the nucleus spin change can be either 0 or 1. The transitions with ΔJ=0 are called Fermi transitions while the ones with ΔJ=1 are called Gamow-Teller transitions.

transitions	Δ J nucleus	Leptonic state
Fermi	$\Delta J=0$	singlet
Gamow-Teller	$\Delta J=1$ $\Delta J_z=0,\pm1$	triplet

$\uparrow\downarrow$

$\uparrow\uparrow \quad \uparrow\downarrow \quad \downarrow\downarrow$

- Since e-ν have L=0, there is no change in the spatial angular momentum of the nucleus, therefore its parity will not change. The nuclues undergoes a spin flip for the G.T. transitions.

Fermi: $0^+ \to 0^+, \Delta\vec{J} = 0$	**G.-T. :** $1^+ \to 0^+, \Delta\vec{J} = 1$	**Mixed:** $\frac{1}{2}^+ \to \frac{1}{2}^+, \Delta\vec{J} = 0,1$
${}^{10}C \to {}^{10}B^* e^- \bar{\nu}_e$ ${}^{14}O \to {}^{14}N^* e^+ \nu_e$	${}^{12}B \to {}^{12}C e^- \bar{\nu}_e$	$n \to p e^- \bar{\nu}_e$

The Fermi constant

- The decay rate can be written in a different way with respect to the Sargent rule. We write explicitly the proton mass m, then we include the phase space factors and the Coulombian correction F(±Z,p) in a dimensionless function $f(\pm Z, E_0/m_e)$ that can be computed analitically.

$$\frac{1}{\tau} = W = \frac{(mc^2)^5}{2\pi^3 \hbar (\hbar c)^6} G^2 |M|^2 f(\pm Z, E_0)$$

$$G^2|M|^2 = \frac{\text{constant}}{f \cdot \tau} \qquad (\text{constant} = \frac{2\pi^3}{m^5})$$

E_0 (MeV) ↓ (column W); $\text{MeV}^2 \cdot \text{fm}^6$ ↓ (column $g^2|M_{if}|^2$)

decadimento	*transizione*	τ (s)	W	p_e^{max}	$f\,\tau$	$g^2\,\lvert M_{if}\rvert^2$
$n \to p\, e^- \bar\nu$	$\frac{1}{2}^+ \to \frac{1}{2}^+$	890	1.29	1.18	$1.61\ 10^3$	$4.25\ 10^{-8}$
${}^3_1H \to {}^3_2He\, e^- \bar\nu$	$\frac{1}{2}^+ \to \frac{1}{2}^+$	$5.60\ 10^8$	0.53	0.14	$1.63\ 10^3$	$4.20\ 10^{-8}$
${}^{14}_8O \to {}^{14}_7N^*\, e^+ \nu$	$0^+ \to 0^+$	102	2.32	2.26	$4.51\ 10^3$	$1.52\ 10^{-8}$
${}^{34}_{17}Cl \to {}^{34}_{16}S\, e^+ \nu$	$0^+ \to 0^+$	2.21	4.97	4.94	$4.54\ 10^3$	$1.51\ 10^{-8}$
${}^6_2He \to {}^6_3Li\, e^- \bar\nu$	$0^+ \to 1^+$	1.15	4.02	3.99	$1.17\ 10^3$	$5.85\ 10^{-8}$
${}^{13}_5B \to {}^{13}_6C\, e^- \bar\nu$	$\frac{3}{2}^- \to \frac{1}{2}^-$	$2.51\ 10^{-3}$	13.4	13.4	$1.11\ 10^3$	$6.17\ 10^{-8}$

- In spite of the big variations of lifetime due to the strong dependency of the function f from p_e^{max}, the product G^2M^2 is about the same in all decays.
- However we observe a small difference due to the type of nuclear transition: Fermi, Gamow-Teller or mixed transitions.
- If we consider a pure Fermi transition, we get: $$\frac{G}{(\hbar c)^3} = 1.140(2) \cdot 10^{-5}\ \text{GeV}^{-2}$$

that is slightly different from the one quoted by the PDG taken from the muon decay. We will see later the reason of such a discrepancy (Cabibbo's angle).

Charged Current: effect of m_W on coupling

- The e.m. coupling constant α is proportional to the square of the electric charge e :

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137}.$$

- In a similar way, the intensity of the CC is G_F (Fermi constant), proportional to the square of the "weak charge" g.

- The matrix elements of the transitions are proportional to the square of the "weak charge" g and to the propagator :

$$\mathcal{M}_{fi} \propto g \frac{1}{Q^2 + m_W^2} g \xrightarrow{Q^2 \ll m_W^2} \frac{g^2}{m_W^2} \equiv G_F.$$

- The difference respect to the e.m. case is the mass of the carrier: while the γ is massless, the CC carrier is the $W^\pm$, a massive particle of spin 1. Therefore the range of CC turns out to be small ($1/m_W$).

- Unlike the case of the massless photon, for small Q^2 the propagator term "stays constant".

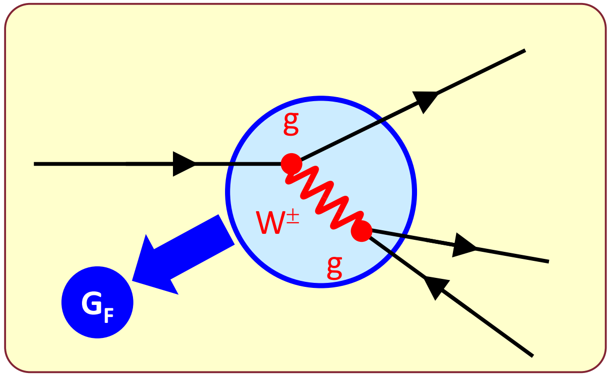
- Therefore the Fermi constant G_F has dimensions :

$$[G_F] = [m_w^{-2}] = [m^{-2}] = [\ell^2],$$

- and a small value, due to m_w :

$$\frac{G_F}{(\hbar c)^3} = O\left(10^{-5}\,\text{GeV}^{-2}\right) = O\left[(10^{-3}\,\text{fm})^2\right].$$

- This effect obscures the similarity of the e.m. and weak charges ($e \leftrightarrow g$), which are indeed of the same order.



Charged Current: G_F

- the most precise value of the Fermi constant G_F is measured by considering the muon decay $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e$:
  - low energy process ($\sqrt{Q^2} \approx m_\mu << m_W$);
  - approximated by a four-fermion point-like process, determined by the Fermi constant ($\approx g^2/m_W^2$);
  - only leptons $\rightarrow$ free from hadronic interactions which affect other processes, e.g. the nuclear β decays.
- if $m_e \approx 0$, m_μ is the only scale of the decay $\rightarrow$ dimensional analysis:

$$\Gamma(\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu) = 1/\tau_\mu \propto G_F^2\, m_\mu^5,$$

- while the correct computation gives :

$$\Gamma\left(\mu^- \rightarrow e^- \overline{\nu}_e \nu_\mu\right) = \frac{G_F^2 m_\mu^5}{192\pi^3}(1+\varepsilon),$$

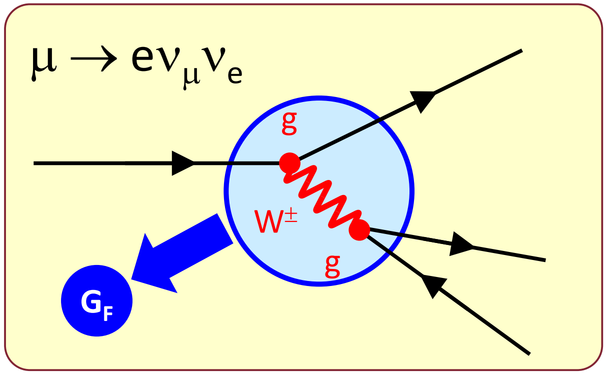
where ε is small and depends on the radiative corrections and on the electron mass.

- the mass of the muon and its average lifetime were measured with great precision:

$$m_\mu = (105.658389 \pm 0.000034)\ \text{MeV};$$
$$\tau_\mu = (2.197035 \pm 0.000040) \times 10^{-6}\ \text{s}.$$

- then the value of the Fermi constant is

$$G_F = (1.16637 \pm 0.00001) \times 10^{-5}\ \text{GeV}^{-2}.$$



lepton universality : (τ⟶e) ↔ (τ⟶μ)

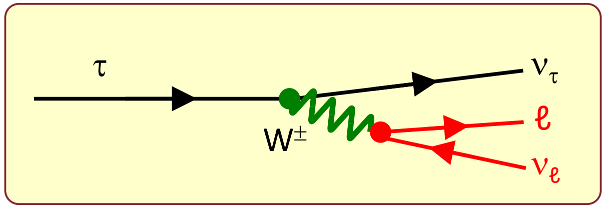
Q. Is the weak CC the same for all leptons and quarks ? Do they share the same coupling constant G_F for all the processes ?

- the **CC universality** has received extensive tests.
- [*absolutely true for leptons, some further refinement – CKM – for quarks*]
- The **e–μ universality** is measured by analyzing the leptonic decays of the $\tau^\pm$ (ℓ^- is the appropriate lepton, e^- / μ^-) :

$$\Gamma\left(\tau^- \to \ell^- \overline{\nu}_\ell \nu_\tau\right) \equiv \Gamma_\ell^\tau = \frac{g_\tau^2 g_\ell^2}{m_W^2 m_W^2} m_\tau^5 \rho_\ell;$$

[where ρ_ℓ is the phase space factor]

$$BR\left(\tau^- \to \ell^- \overline{\nu}_\ell \nu_\tau\right) \equiv BR_\ell^\tau = \frac{\Gamma_\ell^\tau}{\Gamma_{tot}^\tau};$$



- it follows that :

$$\frac{\Gamma_\mu^\tau}{\Gamma_e^\tau} = \frac{BR_\mu^\tau}{BR_e^\tau} = \frac{g_\mu^2 \rho_\mu}{g_e^2 \rho_e} \to$$

$$\left.\frac{BR_\mu^\tau}{BR_e^\tau}\right|_{meas.} = \frac{(17.36 \pm .05)\%}{(17.84 \pm .05)\%} = 0.974 \pm .004,$$

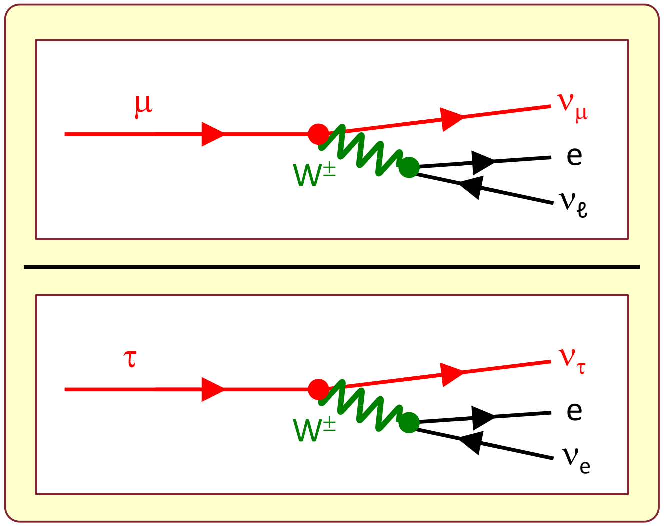
and, taking into account the values of ρ_μ and ρ_e :

$$\boxed{g_\mu / g_e\Big|_{meas.} = 1.001 \pm .002.}$$

!!!

lepton universality : (μ→e) ↔ (τ→e)

The measurement of the **μ–τ universality** is similar $[BR_x = \Gamma_x / \Gamma_{tot} = \tau\,\Gamma_x]$:



$BR\left(\mu^- \to e^- \overline{\nu}_e \nu_\mu\right) \approx 100\%$ (experimentally);

$$\frac{\Gamma\left(\mu^- \to e^- \overline{\nu}_e \nu_\mu\right)}{\Gamma\left(\tau^- \to e^- \overline{\nu}_e \nu_\tau\right)} = \frac{\tau_\tau}{\tau_\mu} \frac{BR\left(\mu^- \to e^- \overline{\nu}_e \nu_\mu\right)}{BR\left(\tau^- \to e^- \overline{\nu}_e \nu_\tau\right)};$$

the prediction is :

$$\frac{\Gamma\left(\mu^- \to e^- \overline{\nu}_e \nu_\mu\right)}{\Gamma\left(\tau^- \to e^- \overline{\nu}_e \nu_\tau\right)} = \frac{g_e^2}{g_e^2} \frac{g_\mu^2}{g_\tau^2} \frac{m_\mu^5}{m_\tau^5} \frac{\rho_\mu}{\rho_\tau} = \frac{g_\mu^2}{g_\tau^2} \frac{m_\mu^5}{m_\tau^5} \frac{\rho_\mu}{\rho_\tau},$$

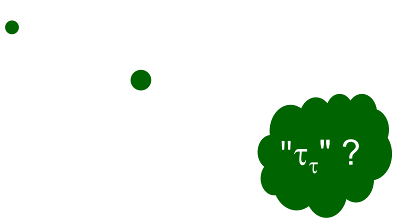
$$\to \frac{g_\mu^2}{g_\tau^2} = \frac{\tau_\tau}{\tau_\mu} \frac{1}{BR\left(\tau^- \to e^- \overline{\nu}_e \nu_\tau\right)} \frac{m_\tau^5 \rho_\tau}{m_\mu^5 \rho_\mu},$$

- from the measured values of m_μ, m_τ, τ_μ, τ_τ

and $BR\left(\tau^- \to e^- \overline{\nu}_e \nu_\tau\right)$, we finally get :

$$\boxed{\left.\frac{g_\mu}{g_\tau}\right|_{meas.} = 1.001 \pm .003.}$$

!!!



lepton universality : τ decays

More ambitious test: extend universality to **τ hadronic decays** :

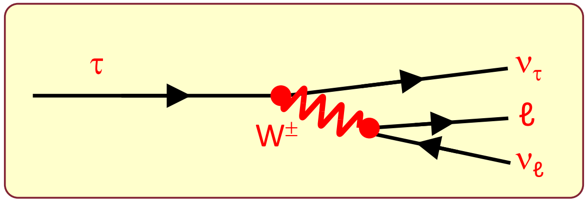
- consider again the leptonic decays of the τ lepton: mainly the following three decay modes :

$$\tau^- \to e^- \bar{\nu}_e \nu_\tau; \quad \tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau; \quad \tau^- \to \bar{u} d \nu_\tau.$$

- from the BR_i ratio, expect (3 for color) :

$$\Gamma^{\text{meas.}}_{\tau\to e} \approx \Gamma^{\text{meas.}}_{\tau\to\mu} \approx \Gamma^{\text{meas.}}_{\tau\to\bar{u}d}/3,$$

in excellent agreement with universality and presence of color in the hadronic sector [*it is the first time we see the color appear in the weak interactions sector*].



Another test is the **τ lifetime** :

$$\Gamma_{\tau\to\mu} \approx \frac{\Gamma^{tot}_\tau}{5} = \frac{m_\tau^5}{m_\mu^5}\Gamma_{\mu\to e} = \frac{m_\tau^5}{m_\mu^5}\frac{1}{\tau_\mu};$$

$$\tau_\tau = 1/\Gamma^{tot}_\tau \approx \frac{\tau_\mu m_\mu^5}{5 m_\tau^5} \approx \boxed{3.1\times10^{-13}\ \text{s};}\ \ \textbf{!}$$

experimentally it is found :

$$\tau^{exp}_\tau = \boxed{(2.956 \pm .031)\times10^{-13}\ \text{s}.}\ \ \textbf{!}$$

- Many other experimental tests [*... but I suppose that you are convinced*].
- At least for CC weak interactions (but also in e.m., and in NC, as in the Z decay) all three leptons have exactly the same interactions.
- The only differences are due to their different mass.
- *Isidor Isaac Rabi said in the 30's about the muon: "who ordered that ?".*

lepton universality : Z decays

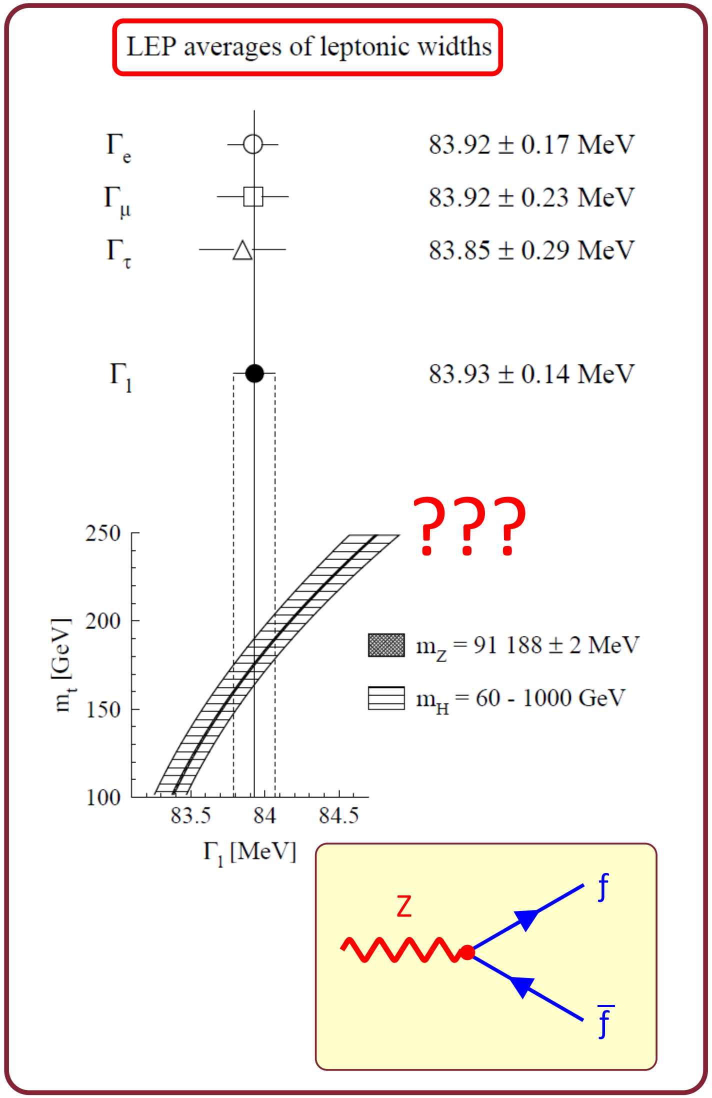
- A similar test on lepton universality has been performed at LEP, in the decay of the Z (a NC process).
- The experiments have measured the decay of the Z into fermion-antifermion pairs.
- They [*well, WE*] have found :

$$Z \rightarrow \quad e^+e^- : \quad \mu^+\mu^- \quad : \quad \tau^+\tau^-$$

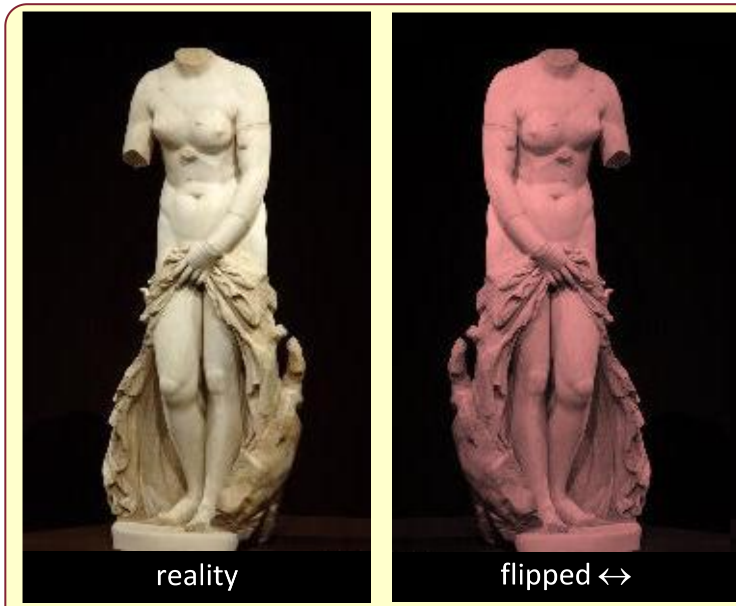
$$1. \quad : 1.000 \pm .004 \; : .999 \pm .005.$$

- Similar – more qualitative – tests can be carried with angular distributions, higher orders, …
- The total amount of information is impressive and essentially no margin is left to any alternative theory.

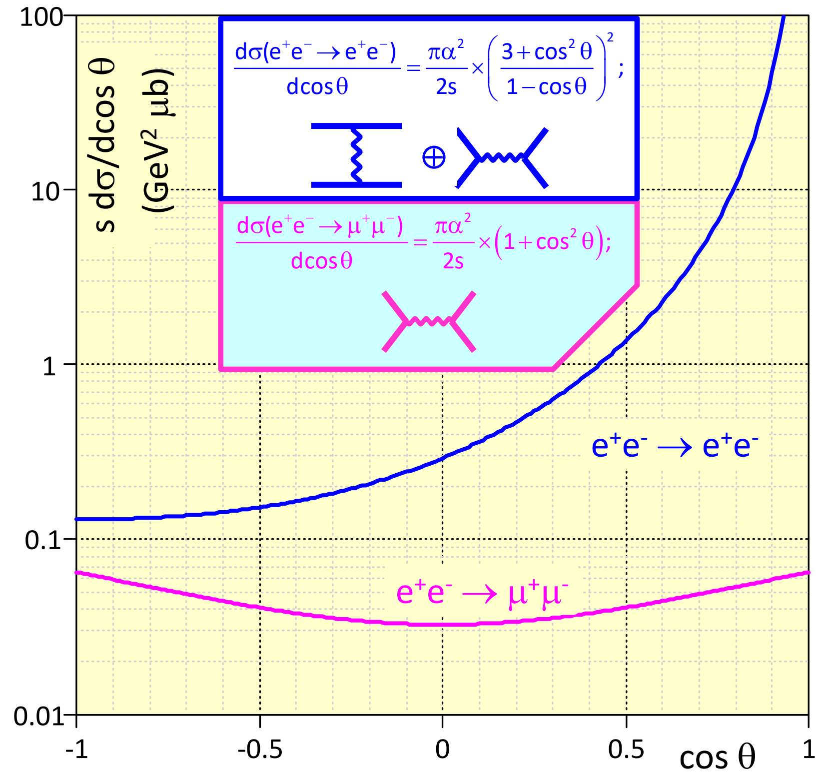
warning – in these pages we mix measurements of different ages, e.g. μ-decay in the '50s, τ-decay in the '80s, Z-decay in the '90s.



Parity violation: meaning



Venus Landolina (Syracuse, Italy)



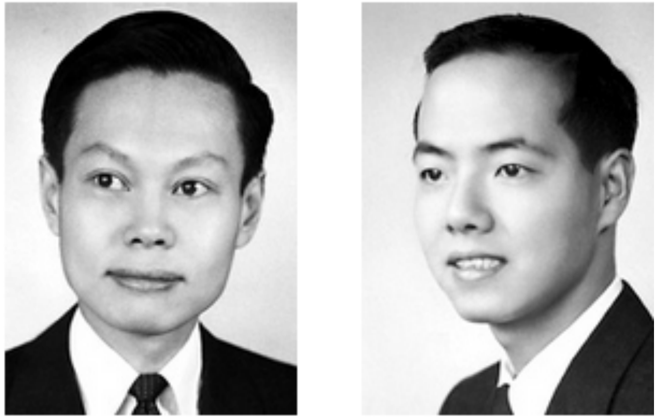
- Look at these two pictures (an ancient sculpture and a modern cross-section);
- one is human-made, the other a law of nature;
- both contain a **symmetry** (left-right legs, forward-backward $\mu^+\mu^-$) and an **asymmetry** (the broken arm, e^+e^-);
- are they examples of ***parity violation*** ?
- Obviously **NO** [*if for no other reason, because p.v. was discovered in the '50s, not in the IV century B.C.*];
- figure out a reasonable explanation
- [*consider flipping the pictures; does it help ?*].

Parity violation: history

- The effect was proposed in 1956 by two young theoreticians in a classical paper and immediately verified in a famous experiment (Mme Wu) [*FNSN 1*] and in the $\pi^{\pm}$- and $\mu^{\pm}$-decays by Lederman and coll.

- The historical reason was a review of weak interaction processes and the explanation of the "θ-τ puzzle", i.e. the K^0 decay into 2π or 3π systems.

particle (ν)	anti-particle ($\bar{\nu}$)
h = −1	h = +1



Nobel Prize 1957
Tsung-Dao Lee (*Lǐ Zhèngdào*, 李政道)

Chen-Ning Franklin Yang (Yáng Zhènníng, 杨振宁 or 楊振寧)

for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles.

- ν only h=–1;
- $\bar{\nu}$ only h=+1;

→ **PARITY VIOLATION**

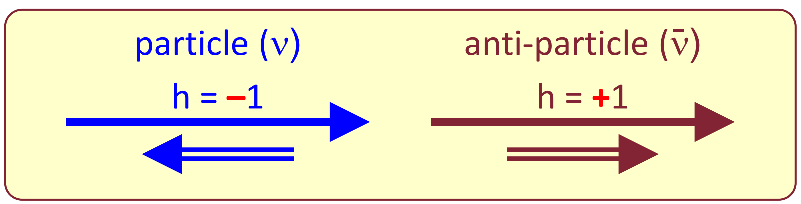
→ vectors & co.

Parity violation: mechanism

- The two authors found that parity conservation in weak decays was NOT really supported by measurements.

[then experiment, and then a new theory]

- The CC current is "V – A", which is an acronym for the factor $\gamma_\mu(1 - \gamma_5)$ in the current; it shows that the CC have a "preference" for <u>left-handed particles</u> and <u>right-handed anti-particles</u>.



- These effects clearly violates the parity : the parity operator $\mathbb{P}$ flips the helicity:

$$\mathbb{P}\ |\nu, h = -1> = |\nu, h = +1>$$

$\rightarrow$ it changes ν's with a –ve helicity into ν's with +ve helicity, which DO NOT EXIST (or do not interact).

- Few comments :
  - V or A alone would NOT violate the parity. The violation is produced by the simultaneous presence of the two, technically by their interference.
  - The conservation is restored, applying also $\mathbb{C}$, the charge conjugation:

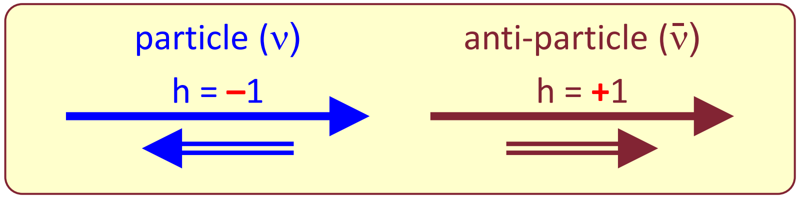
    $$\mathbb{CP}|\nu, h=-1> = \mathbb{C}|\nu, h=+1> = |\bar{\nu}, h=+1>,$$

    i.e. $\nu_{h=-1} \rightarrow \bar{\nu}_{h=+1}$, which <u>does exist</u>. Therefore, "<u>$\mathbb{CP}$ is not violated</u>" [*not by ν's in these experiments, at least*].

  - *the above discussion holds only if* m_ν = 0 *(NOT TRUE), or* $m_\nu << E_\nu$ *(ultra-relativistic approximation -* ***<u>u.r.a.</u>****); the u.r.a. for* ν*'s is used in this chapter.*

Parity violation: ν helicity

- For massless ν's or in the u.r.a. approximation(*), V–A implies :



- Therefore in the "forbidden" amplitudes, there is a factor [$\propto$ (1 − β)] for massive particles, which vanishes when $\beta \to 1$.

- If we assume a factor (1 ± β) for the production of (h = ∓ 1) particles (the opposite for anti-particles), we get :

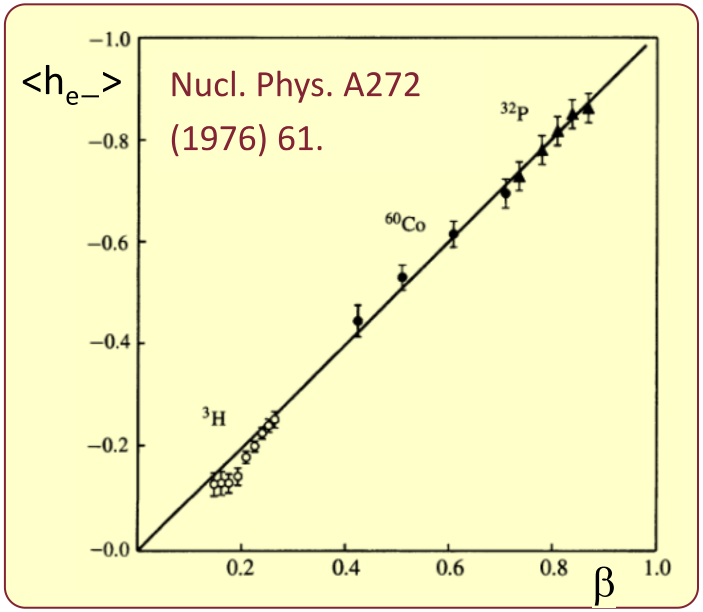
  $<h>_{part}$ = ½ [(1 + β) (-1) + (1 - β)(+1)]= – β;

  $<h>_{\overline{part}}$ = ½ [(1 + β) (+1)+ (1 - β)(-1)] = + β;

  i.e., when produced in CC interactions, particles in average have –ve helicity, while anti-particles have +ve helicity.

- The effect is maximal for ν's ($\beta_\nu \approx 1$), which also have no other interactions.

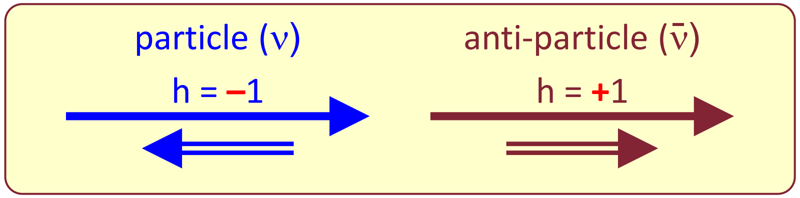
- For e^-, it is also well confirmed by data in β decays [YN1, 570] :



(*) *If $m_\nu > 0 \to \beta_\nu < 1$; a L-transformation can reverse the sign of the momentum, and hence the ν helicity, so the following argument is NOT L-invariant for massive particles [previous slide].*

Parity violation: ν helicity

- For massless ν's or in the u.r.a. approximation(*), V–A implies :



- Therefore in the "forbidden" amplitudes, there is a factor [$\propto$ (1 − β)] for massive particles, which vanishes when $\beta \to 1$.

- If we assume a factor (1 ± β) for the production of (h = ∓ 1) particles (the opposite for anti-particles), we get :

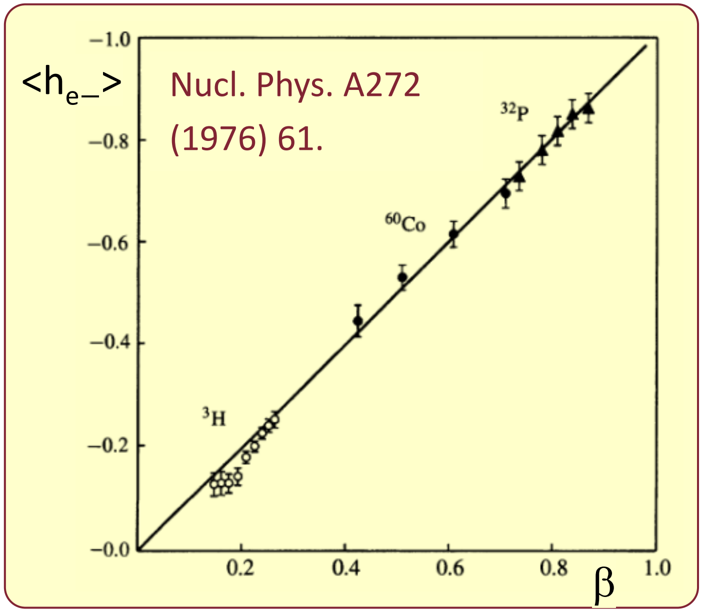
  $\langle h \rangle_{part}$ = ½ [(1 + β) (-1) + (1 - β)(+1)]= − β;

  $\langle h \rangle_{\overline{part}}$ = ½ [(1 + β) (+1)+ (1 - β)(-1)] = + β;

  i.e., when produced in CC interactions, particles in average have –ve helicity, while anti-particles have +ve helicity.

- The effect is maximal for ν's ($\beta_\nu \approx 1$), which also have no other interactions.

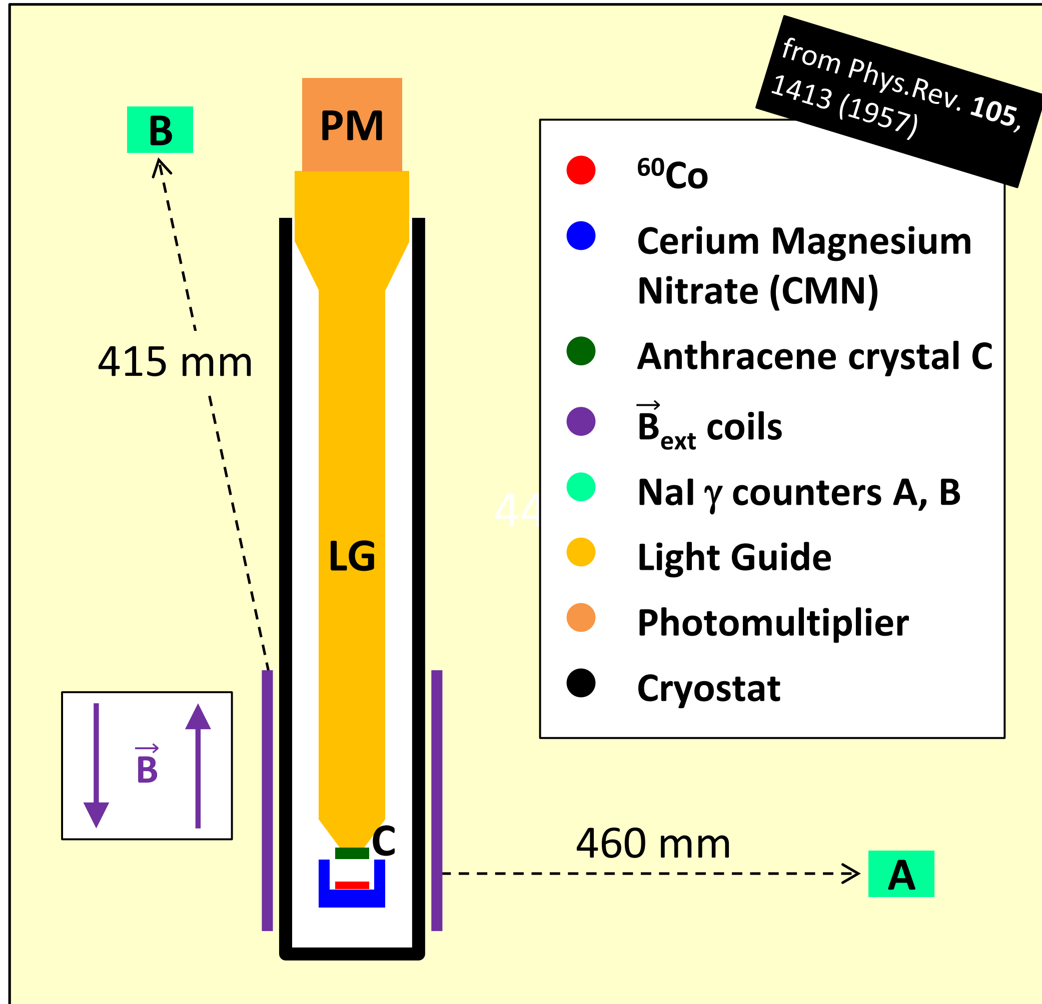
- For e^-, it is also well confirmed by data in β decays [YN1, 570] :



(*) *If $m_\nu > 0 \to \beta_\nu < 1$; a L-transformation can reverse the sign of the momentum, and hence the ν helicity, so the following argument is NOT L-invariant for massive particles [previous slide].*

Parity violation: Wu experiment -1

Chien-Shiung Wu
吴健雄 1912 – 1997



The "Madam Wu" experiment (1957) discovered the parity violation in ^{60}Co decay.

A difficult elegant application of state-of-the-art technologies in nuclear physics and cryogenics.

a very important part of the 3rd year course – repeated here just for completeness

Parity violation: Wu experiment -2

Technicalities:

Align the nuclear spins with an external $\vec{B}$:

- at a given value of T, $E_T = k_B T$ (k_B : Boltzmann constant);
- the magnetic field $E_B = \vec{\mu} \cdot \vec{B}$;
- good alignment if $E_B \geq E_T$ (e.g. $T \approx 10^{-2}$ K, $B \approx 20$ T [*see box*]);

such a large $|\vec{B}|$?

- use external $|\vec{B}_{ext}|$ of few $\times 10^{-2}$ T;
- it polarizes the electrons in the CMN;
- since (μ_e / μ_N = m_N / m_e ≈ 2,000) → it produces a strong $|\vec{B}|$ of few T; ☺☺☺

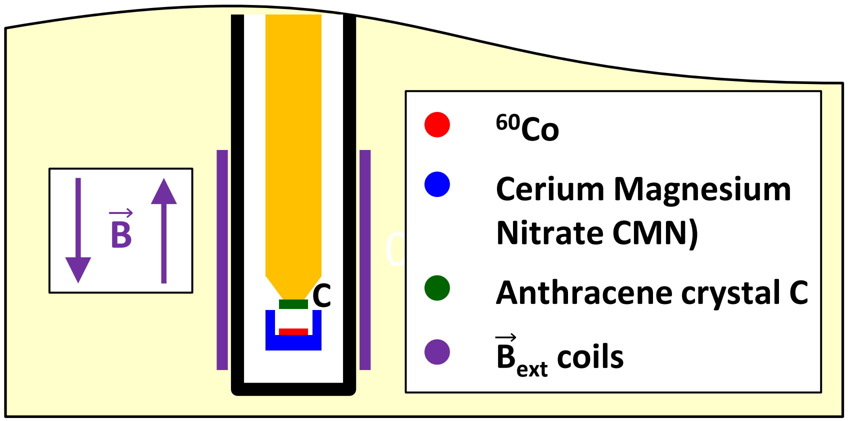
such a small T ?

- everything in a cryostat;
- produce $T \approx 10^{-2}$ K using **adiabatic depolarization**;

how to operate ?

- switch the field off (→ "t_0");
- start counting as a function of time;
- the polarization goes away in few minutes and the effect disappears.

k_B = 8.62×10^{-5} eV / K;
μ_N = 3.15×10^{-8} eV / T;
T = 10^{-2} K → $E_T \approx 8 \times 10^{-7}$ eV;
B = 20 T → $E_B \approx 6 \times 10^{-7}$ eV. ☺☺☺



Parity violation: Wu experiment -3

a) ${}^{60}_{27}\mathrm{Co}(J^P=5^+)^{\Uparrow} \to {}^{60}_{28}\mathrm{Ni}^{**}(J^P=4^+)^{\Uparrow}$ $\boxed{e^-}$ $\bar{\nu}_e$;

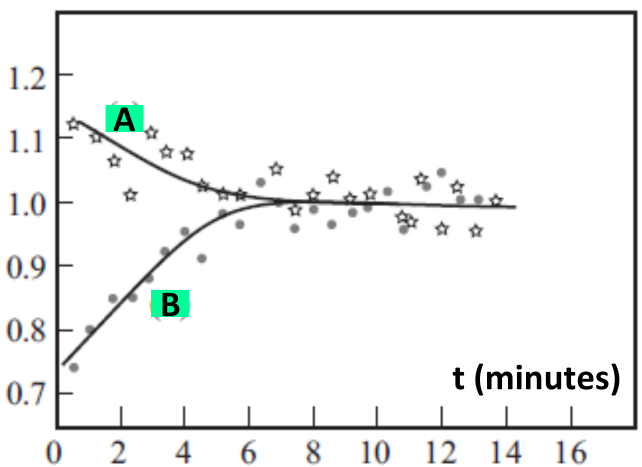
b) ${}^{60}_{28}\mathrm{Ni}^{**}(J^P=4^+)^{\Uparrow} \to {}^{60}_{28}\mathrm{Ni}^{*}(J^P=2^+)^{\Uparrow}$ $\boxed{\gamma_1^{[1.173\ \mathrm{MeV}]}}$;

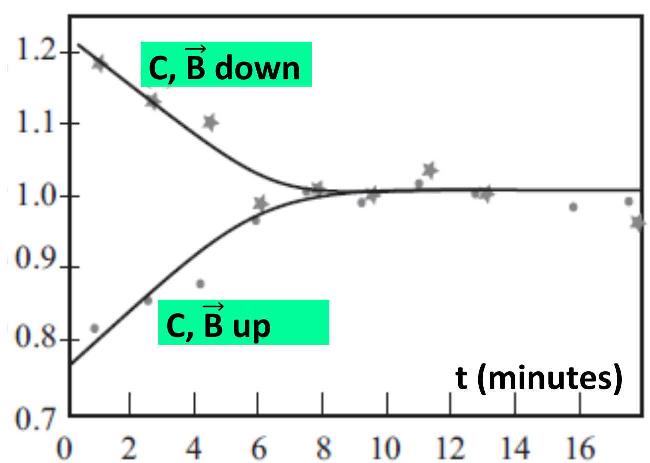
c) ${}^{60}_{28}\mathrm{Ni}^{*}(J^P=2^+)^{\Uparrow} \to {}^{60}_{28}\mathrm{Ni}(J^P=0^+)^{\Uparrow}$ $\boxed{\gamma_2^{[1.332\ \mathrm{MeV}]}}$;

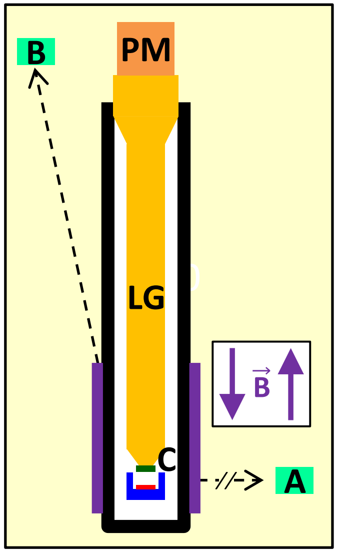
- the chain decay [*box above*];
- decay (a) is **weak** [*interesting*];
- decays (b), (c) are e.m. $\to \mathbb{P}$ conserved;
- both (a) (b) (c) conserve angular mom.;
- in A : see $\gamma_{1,2}$ if $\perp$ to $\vec{B}$;
- in B : see $\gamma_{1,2}$ if // to $\vec{B}$ [or anti-// to $\vec{B}$];
- in C : see e^- if // to $\vec{B}$ [or anti-// to $\vec{B}$].

Plots (=normalized counts in ABC, for $\pm\vec{B}$) :

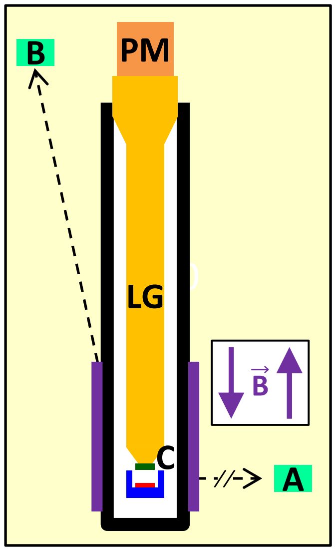
- asymmetries at $t=t_0$, then go away;
- A > B because of polarization $\mathcal{P}$
  $\to$ measure $\mathcal{P}$, to be used later;
- A and B do not depend on $\vec{B}$ direction [e.m. conserves $\mathbb{P}$];
- C does depend on $\vec{B}$ direction, with a rate equal to $\mathcal{P} \to$ **$\mathbb{P}$ is violated**.

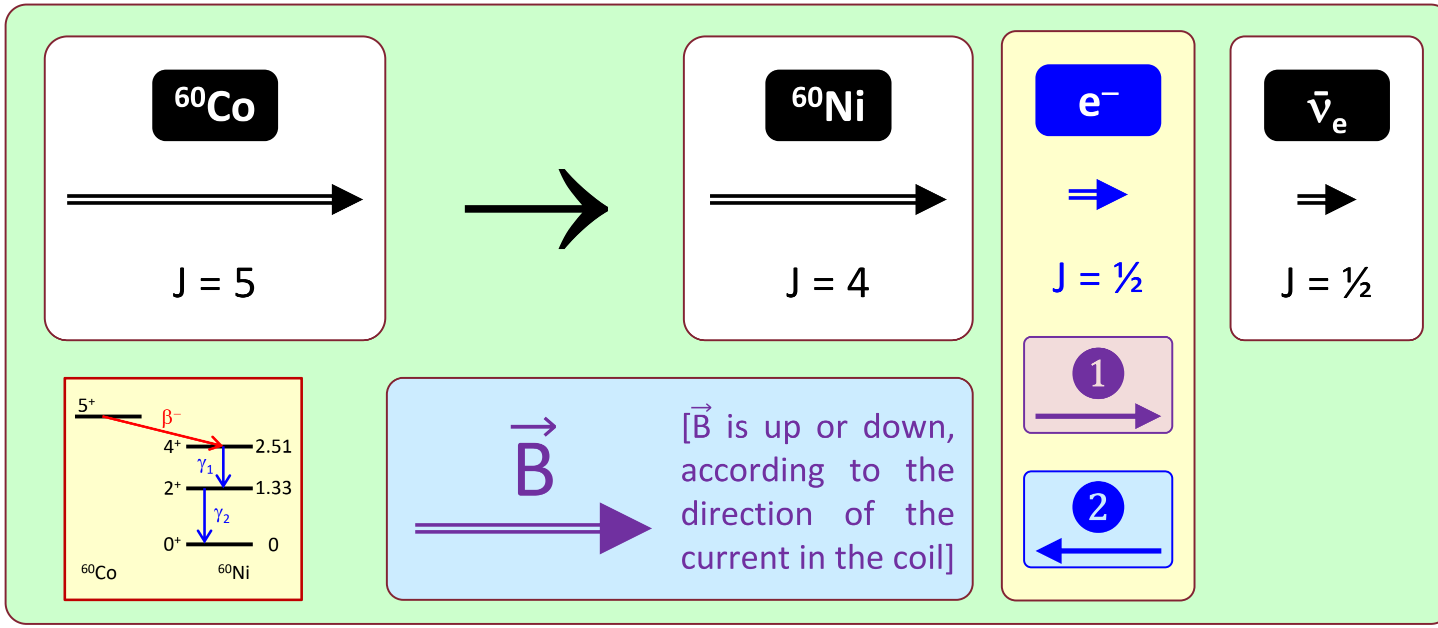






Parity violation: Wu experiment -4





reinterpret the exp. with V – A theory:

- J conservation + $\mathcal{P}$olarization → force spin direction of e^-;
- case 1:
  - ➢ $h_e = +1 \to$ forbidden ($\propto 1 - \beta_e$);
- case 2
  - ➢ $h_e = -1 \to$ allowed;

≈ 0.6 (computed)

- conclusion:
  - ➢ direction opposite to $\vec{B}$ preferred;
  - ➢ electron rate W depends on cos θ, the angle $\vec{B}$ - $\vec{v}_e$:

$$\boxed{W(\cos\theta) \propto 1 - \mathcal{P}\,\beta_e \cos\theta.}$$

≈ 0.65 (from counters A,B)

Parity violation: Feynman view

[... *I*]magine that we were talking to a Martian, or someone very far away, by telephone. We are not allowed to send him any actual samples to inspect; for instance, if we could send light, we could send him right-hand circularly polarized light. [...] But we cannot give him anything, we can only talk to him.

[*Feynman explains how to communicate: math, classical physics, chemistry, biology are simple*]

[...] "Now put the heart on the left side." He says, "Duhhh - the left side?" [...] We can tell a Martian where to put the heart: we say, "Listen, build yourself a magnet, [... *repeat the mme Wu exp ...;*] then the direction in which the current goes through the coils is the direction that goes in on what we call the right.

[... *However,*] does the right-handed matter behave the same way as the right-handed antimatter? Or does the right-handed matter behave the same as the left-handed antimatter? Beta-decay experiments, using positron decay instead of electron decay, indicate that this is the interconnection: matter to the "right" works the same way as antimatter to the "left."

[... *We then*] make a new rule, which says that matter to the right is symmetrical with antimatter to the left.

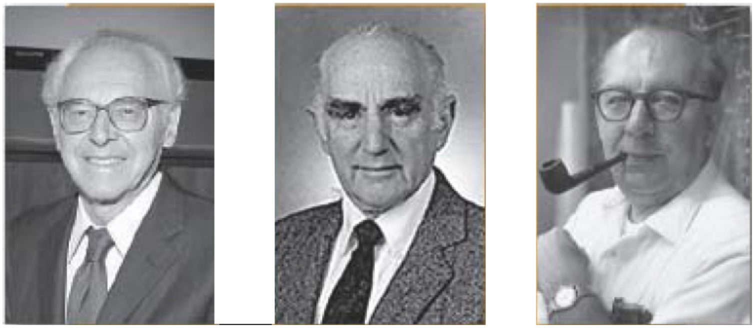
So if our Martian is made of antimatter and we give him instructions to make this "right" handed model like us, it will, of course, come out the other way around. What would happen when, after much conversation back and forth, we each have taught the other to make space ships and we meet halfway in empty space? [...] Well, if he puts out his left hand, watch out!

From Feynman Lectures on Physics, 1, 52: "Symmetry in Physical Laws".

Quite amusing and great physics :

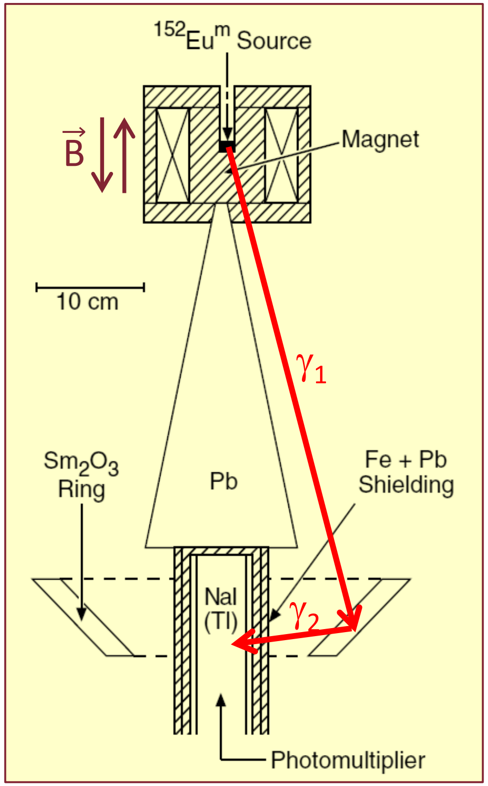
- the symmetry he is talking about is "$\mathbb{CP}$" and NOT simply "$\mathbb{P}$" or "$\mathbb{C}$" !!!
- but $\mathbb{CP}$ is also violated [see § K^0].

Parity violation: the ν_e helicity



In 1958, Goldhaber, Grodzins and Sunyar measured the **helicity of the electron neutrino** ν_e with an ingenious experiment.

- A crucial confirmation of the V–A theory; pure V or A had been ruled out, but V+A was still in agreement with data.
- Metastable Europium (Eu) decays via K-capture → excited Samarium (Sm*) + $\boldsymbol{\nu_e}$, whose helicity is the result of the exp.;
- the Sm* decays again into more stable Samarium (Sm), emitting a γ [γ_1 in fig.].
- For such a γ the transmission in matter depends on the e^- spins; therefore a large B-field is applied to polarize the iron.
- The γ's are used to excite again another Sm; only γ's from the previous chain may do it; another γ is produced [γ_2 in fig.].
- The resultant γ's are detected.



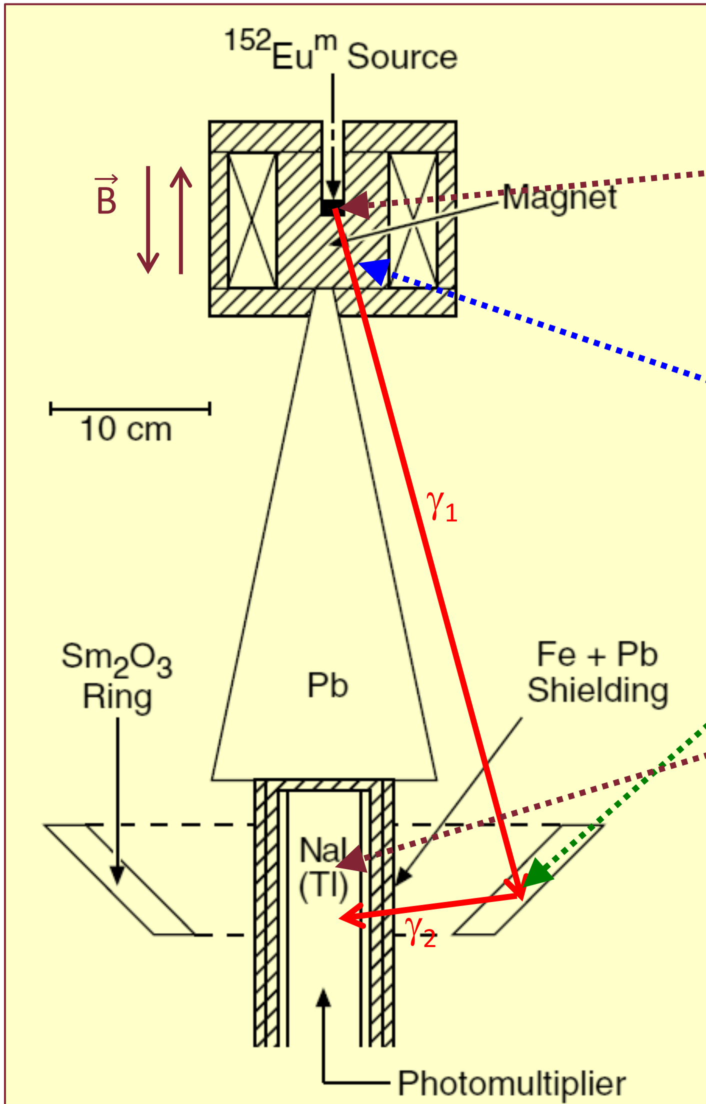
- Final result :

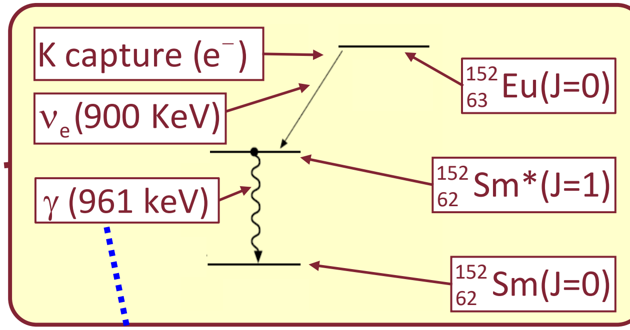
$$\boxed{h(\nu_e) = -1.0 \pm 0.3}$$

consistent with V–A only.

[the experiment is ingenuous and complex: it is discussed step by step.]

Parity violation: summary of the experiment





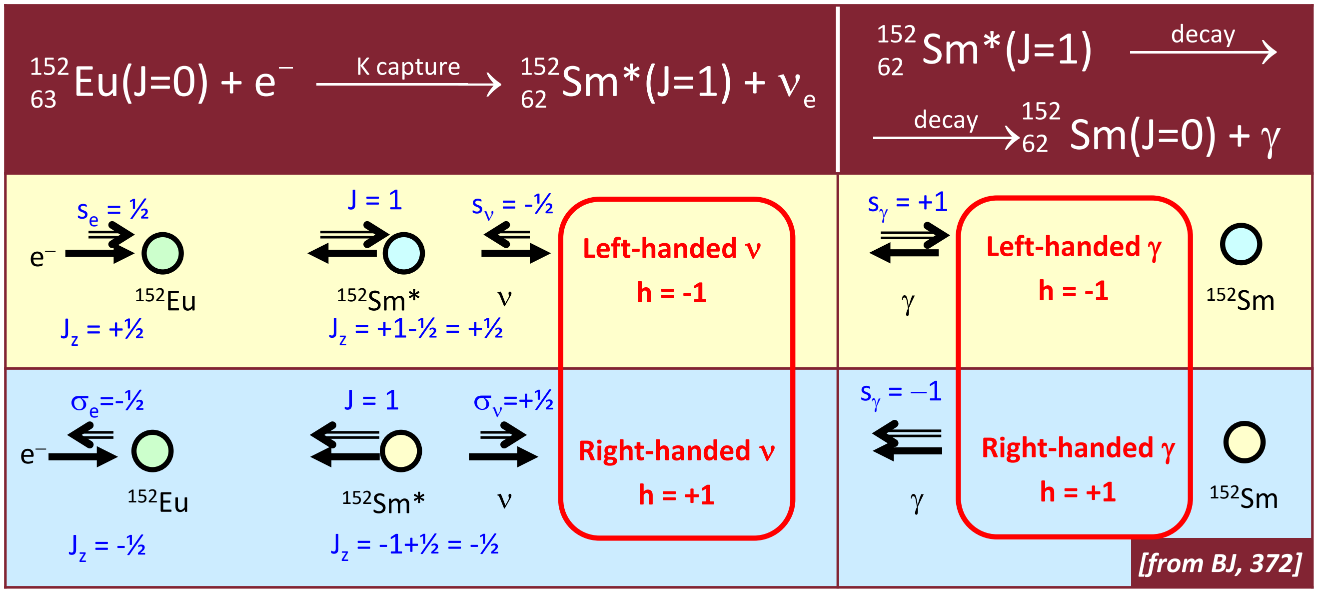
Compton effect does depend on the γ_1-spin wrt $\vec{B}$ (NB γ_1 in the figure escapes Compton effect).

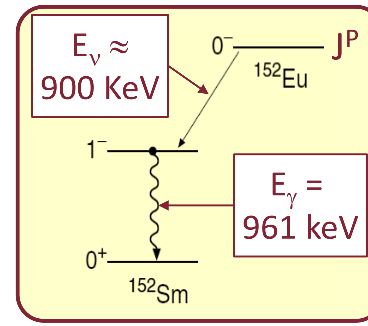
$\gamma_1 + {}^{152}Sm \rightarrow {}^{152}Sm^* \rightarrow {}^{152}Sm + \gamma_2$.

γ_2 detection via photomultiplier.

The experiment detects the number of γ_2 when $\vec{B}$ is (anti-)parallel to γ_1. The asymmetry depends on the (ν_e-helicity $\rightarrow$) γ_1-spin.

Parity violation: Europium → Samarium → γ





- ν_e monochromatic, $E_\nu \approx$ 900 keV;
- Sm* lifetime = ~10^{-14} s, short enough to neglect all other interactions;
- Sm* excitation energy = 961 KeV ($\approx E_\nu$);
- *only* for γ in the direction of Sm* recoil, angular momentum conservation implies Sm* helicity = ν_e helicity = γ helicity = ±1 [*see box with 2 alternative hypotheses*].

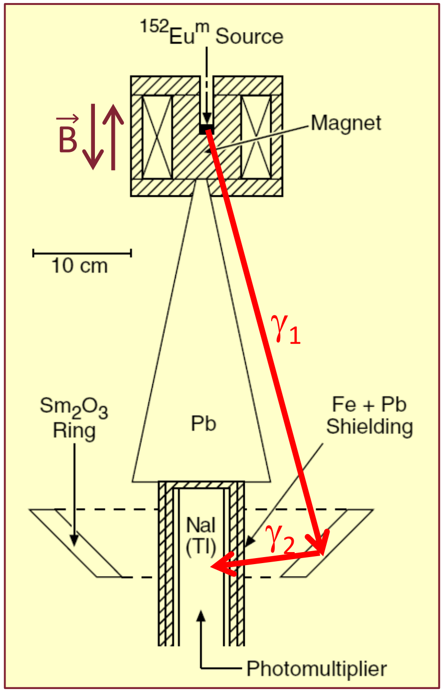
- Therefore, the method is:
  - [cannot measure directly the ν_e spin]
  - select and measure the γ's emitted anti-parallel to the ν_e's, i.e. in the same direction of the (^{152}Sm*);
  - measure their spin;
  - reconstruct the **ν_e helicity**.

Parity violation: the resonant scattering

- For γ of 961 keV, the dominant interaction with matter is the Compton effect; the Compton cross section is spin-dependent: the transmission is larger when the γ and e^- spin are parallel.
- Therefore, a strong and reversible $\vec{B}$ (saturated iron) selects the polarized γ's, producing an asimmetry between the two $\vec{B}$ orientations.
- Need also to select only the γ's polarized according to the ν_e spin, i.e. produced opposite to the ν_e's $\rightarrow$ use the method of *resonant scattering* in the Sm_2O_3 ring:

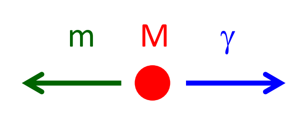
$$\gamma_1 + {}^{152}Sm \rightarrow {}^{152}Sm^* \rightarrow {}^{152}Sm + \gamma_2.$$

- [kinematics (next slide) : a nucleus at rest, excited by an energy E_0, decays with a γ emission; the γ energy in the lab. is reduced by a factor $E_0/(2M)$].
- In general, γ_1 energy is degraded and NOT sufficient for Sm excitation (i.e. to produce γ_2).
- But, if γ_1 is anti-parallel to ν_e, the Sm* recoils against ν_e. The resultant Doppler effect in the correct direction provides γ_1 of the necessary amount of extra energy ($E_\nu \approx E_\gamma$).
- In conclusion, only the γ's anti-parallel to ν_e's are detected, but those γ's carry the information about ν_e helicity.



Parity violation: kinematics

<u>Kinematics</u>



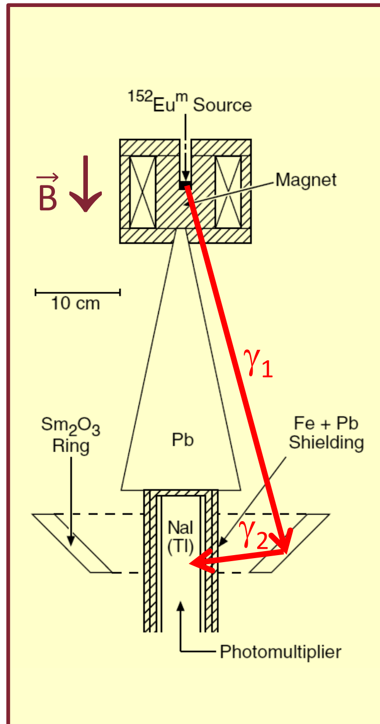
$$M \rightarrow m\,\gamma; \qquad E_0 = M - m;$$

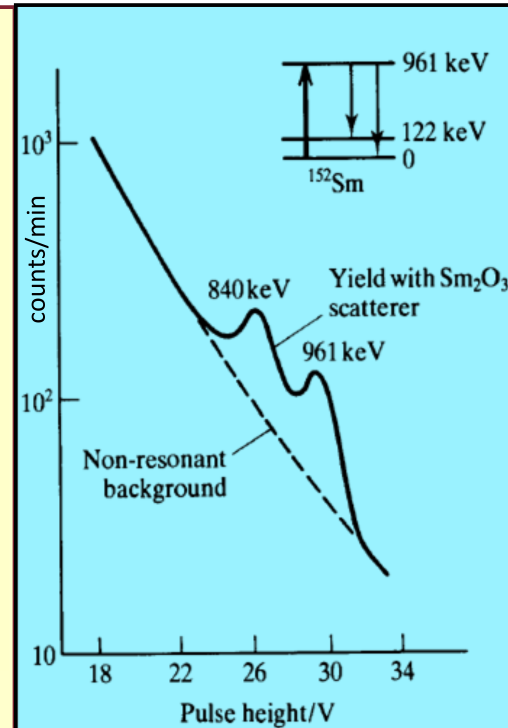
$$\text{M sys.}\ \begin{cases} M = [M, & 0, & 0,0]; \\ \gamma = [E_\gamma, & E_\gamma, & 0,0]; \\ m = [M - E_\gamma, & -E_\gamma, & 0,0]; \end{cases}$$

$$m^2 = (M - E_\gamma)^2 - E_\gamma^2 = M^2 + \cancel{E_\gamma^2} - 2ME_\gamma - \cancel{E_\gamma^2};$$

$$E_\gamma = \frac{M^2 - m^2}{2M} = \frac{M+m}{2M} E_0 =$$

$$= \frac{M + M - E_0}{2M} E_0 = E_0 \left(1 - \frac{E_0}{2M} \right).$$

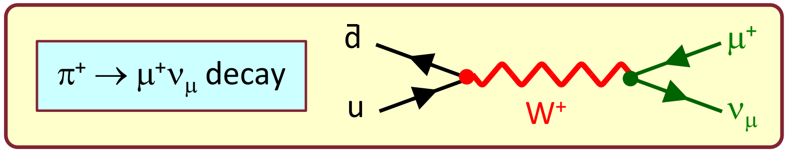




$h_\gamma = -1 \rightarrow \boxed{h_\nu = -1}$

→ if the excited nucleus (M) is <u>at rest</u>, the energy of the γ in the lab. is smaller than the excitation energy E_0; therefore it is insufficient to excite another nucleus at rest; for this to happen, the excited nucleus has to move in the right direction with the <u>appropriate energy</u>.

Weak decays : $\pi^{\pm}$



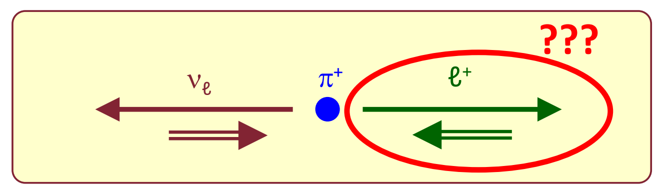
- The $\pi^{\pm}$ is the lightest hadron; therefore it may only decay through semileptonic CC weak processes, like (consider only π^+, for π^-, apply $\mathbb{C}$) :

$$\pi^+ \to \mu^+ \nu_\mu; \quad \pi^+ \to e^+ \nu_e.$$

- In reality, it almost decays only into μ's: the electron decay is suppressed by a factor $\approx$ 8,000, NOT understandable, also because the ($\pi \to e$) decay is favored by space phase.
- The reason is the **helicity**:

ℓ =lepton, i.e. e/μ

  - in the π^+ reference frame, the momenta of the ℓ^+ and the ν_ℓ must be opposite;
  - since the π^+ has spin 0, the spins of the ℓ^+ and the ν must also be opposite;
  - therefore the two particles must have the same helicity;
  - since the ν (a ~massless particle) must have negative helicity, the ℓ^+ (a non-massless antiparticle) is also forced to have negative helicity;
  - therefore the transition is suppressed by a factor $(1 - \beta_\ell)$;
  - the e^+ is ultrarelativistic ($p_e \approx m_\pi / 2 >> m_e$), while the μ^+ has small β [*compute it !!!*];
  - therefore the decay $\pi \to e$ is strongly suppressed respect to $\pi \to \mu$.



Kinematics (next slide) :

- $p_\ell = [(m_\pi^2 - m_\ell^2) / (2\, m_\pi)]$;
- $\beta_e = (1 - 2.6 \times 10^{-5})$;
- $\beta_\mu = 0.38$.

Weak decays : kinematics

SOLUTION : (more general)

Decay M $\rightarrow$ a b. Compute p = $|\vec{p}_a|$ = $|\vec{p}_b|$

in the CM system, i.e. the system of M:

$$\text{CM} \begin{cases} (M, & 0, & 0,0) \\ (\sqrt{m_a^2+p^2}, & p, & 0,0); \\ (\sqrt{m_b^2+p^2}, & -p, & 0,0) \end{cases}$$

$$p^2 = \frac{\left[M^2-(m_a-m_b)^2\right]\left[M^2-(m_a+m_b)^2\right]}{4M^2}.$$

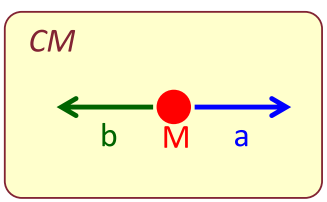
energy conservation : $M = \sqrt{m_a^2+p^2} + \sqrt{m_b^2+p^2}$;

$2\sqrt{m_a^2+p^2}\sqrt{m_b^2+p^2} = M^2 - m_a^2 - m_b^2 - 2p^2$;

$4\left[m_a^2 m_b^2 + p^2\left(m_a^2+m_b^2\right) + \cancel{p^4}\right] = \left(M^2-m_a^2-m_b^2\right)^2 + 4\cancel{p^4} - 4p^2\left(M^2-m_a^2-m_b^2\right)$;

$4p^2\left[\left(\cancel{m_a^2}+\cancel{m_b^2}\right)+\left(M^2-\cancel{m_a^2}-\cancel{m_b^2}\right)\right] = -4m_a^2m_b^2 + \left(M^2-m_a^2-m_b^2\right)^2$;

$4p^2M^2 = \left[\left(M^2-m_a^2-m_b^2\right)+2m_am_b\right]\left[\left(M^2-m_a^2-m_b^2\right)-2m_am_b\right] = $ (see above)



a) $m_a = m_b = m$; e.g. $K^0 \rightarrow \pi^0\pi^0$;

$$p^2 = \frac{M^2-4m^2}{4} = \frac{(M+2m)(M-2m)}{4};$$

b) $m_a = m_b = 0$; e.g. $\pi^0 \rightarrow \gamma\gamma$, $H \rightarrow \gamma\gamma$;

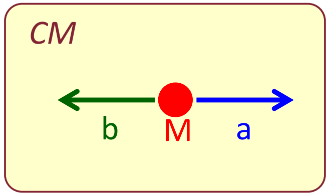
$$p^2 = \frac{M^2}{4}; \quad p = \frac{M}{2};$$

c) $m_a = m$; $m_b = 0$; e.g. $\pi^+ \rightarrow \mu^+\nu_\mu$, $W^* \rightarrow W\gamma$;

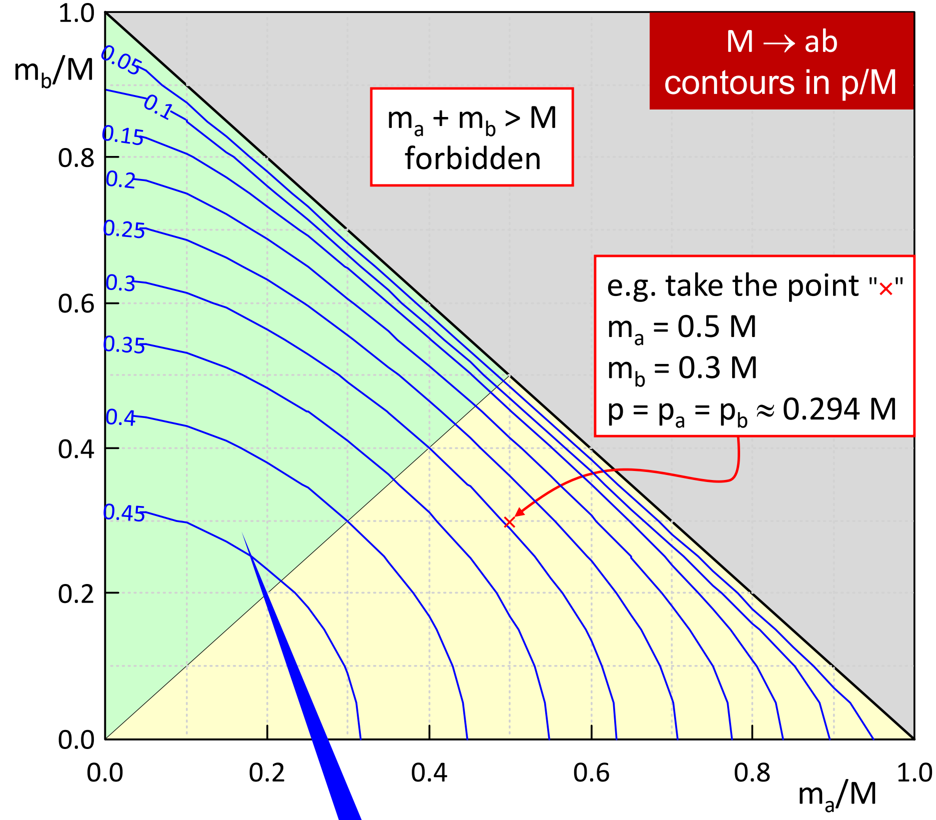
$$p = \frac{M^2-m^2}{2M} = \frac{M}{2}\left[1-\left(\frac{m}{M}\right)^2\right].$$

Weak decays : contour plot

same info as in previous slide, only "easier" to see

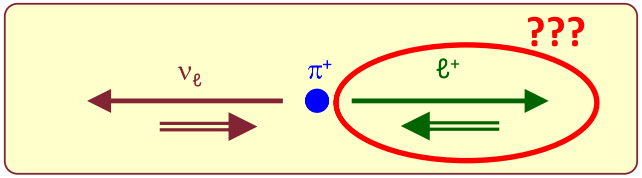


the plot is only here to show you how easy it is to produce an apparently sophisticated and professional plot.



symmetric for m_a vs m_b, plot only $m_a > m_b$.

Weak decays : $\pi^\pm \to (e^\pm / \mu^\pm)$



Problem: compute the factor in the $\pi^\pm$ decay between μ and e.

Assume for the decay $\pi \to \ell$ [$\ell = \mu$ or e] :

p = decay product momentum;

ρ_ℓ = dN/dE_{tot} = phase space factor;

dN = $Vp^2dpd\Omega/(2\pi)^3$;

$(1-\beta_\ell)$ = helicity suppression;

BR_ℓ = $\Gamma_\ell / \Gamma_{tot} \propto \rho_\ell \times (1-\beta_\ell)$.

only factors different for μ/e (ℓ-universality)

In this case the decay is isotropic. Then :

$\rho_\ell \propto p^2dp/dE_{tot}$;

4-momentum conservation [use previous slide and keep only terms ℓ-dependent]:

$$p_\ell = p_\nu = E_\nu \equiv p; \qquad E_{tot} = m_\pi; \qquad E_\ell = m_\pi - E_\nu = m_\pi - p;$$

$$p = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} = \frac{E_{tot}}{2} - \frac{m_\ell^2}{2E_{tot}}; \qquad \frac{dp}{dE_{tot}} = \frac{1}{2} + \frac{m_\ell^2}{2m_\pi^2} = \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2};$$

$$\rho_\ell \propto \left(\frac{m_\pi^2 - m_\ell^2}{2m_\pi}\right)^2 \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2} = \frac{\left(m_\pi^2 + m_\ell^2\right)\left(m_\pi^2 - m_\ell^2\right)^2}{\boxed{8m_\pi^4}};$$

irrelevant

$\rho_e > \rho_\mu$

$$1-\beta_\ell = 1 - \frac{p_\ell}{E_\ell} = 1 - \frac{p}{m_\pi - p} = \frac{m_\pi - 2p}{m_\pi - p} =$$

$$= \frac{m_\pi - 2(m_\pi^2 - m_\ell^2)/(2m_\pi)}{m_\pi - (m_\pi^2 - m_\ell^2)/(2m_\pi)} = \frac{2m_\ell^2}{m_\pi^2 + m_\ell^2};$$

$$BR_\ell \propto \left(m_\pi^2 + m_\ell^2\right)\left(m_\pi^2 - m_\ell^2\right)^2 \frac{\cancel{2}m_\ell^2}{m_\pi^2 + m_\ell^2} =$$

$$\propto m_\ell^2 \left(m_\pi^2 - m_\ell^2\right)^2.$$

$1-\beta_e <<< 1-\beta_\mu$

For electrons, $m_e << m_\pi$, so :

$$\frac{BR\left(\pi^+ \to e^+\nu_e\right)}{BR\left(\pi^+ \to \mu^+\nu_\mu\right)} = \left(\frac{m_e}{m_\mu}\frac{m_\pi^2 - \cancel{m_e^2}}{m_\pi^2 - m_\mu^2}\right)^2 \approx 1.28 \times 10^{-4}.$$

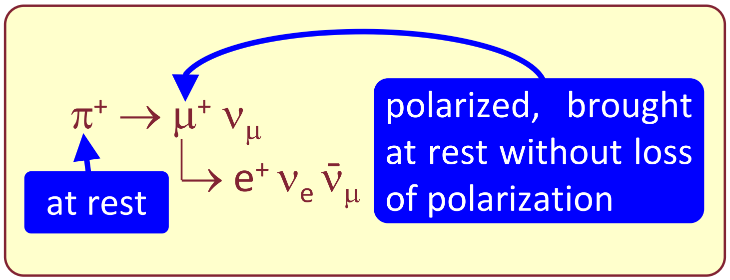
Experimentally, it is measured

$$\frac{BR\left(\pi^+ \to e^+\nu_e\right)}{BR\left(\pi^+ \to \mu^+\nu_\mu\right)} = 1.23 \times 10^{-4}.$$

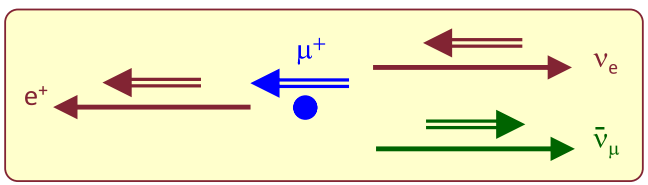
i.e. $N(\pi \to \mu) \approx$ 8,000 $N(\pi \to e)$

Weak decays : $\pi^\pm \rightarrow \mu^\pm$

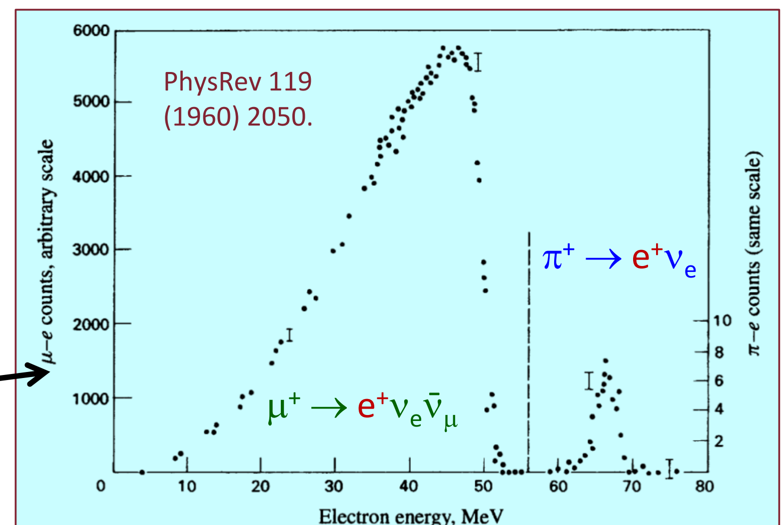
- Consider a famous experiment (Anderson et al., 1960) :



- In the μ^+ ref. frame (=LAB), this configuration is clearly preferred :

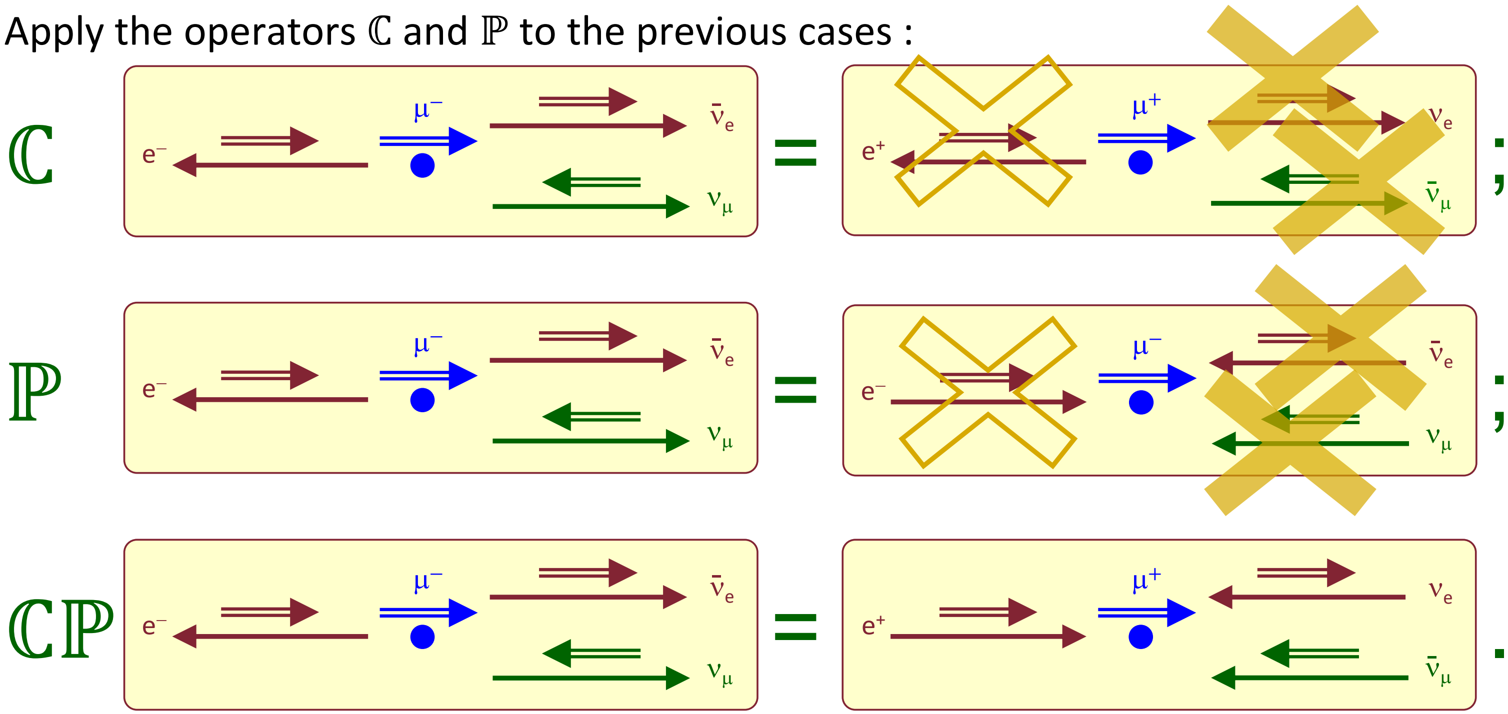


- In this angular configuration, both space and angular momentum are conserved, the particles are left- and the anti-particles right-handed.
- From the figure :
  - few e^+ directly from π^+ decay, shown in the right part ($\int\mu$ / $\int e \approx$ 8,000);
  - the electron energy is the only measurable variable;
  - kinematical considerations show that it is correlated with the angular variables, and that the value $E_e \approx m_\mu$ / 2 is possible only for parallel ν's.
  - the distribution clearly shows the parity violation in muon decay.



Weak decays: C, P in μ decays

Apply the operators $\mathbb{C}$ and $\mathbb{P}$ to the previous cases :



- [the "×" shows the forbidden – not existent – particles]
- both $\mathbb{C}$ and $\mathbb{P}$ alone transforms the decay into non-existent processes (we say "**both $\mathbb{C}$ and $\mathbb{P}$ separately are not conserved in this process**");
- instead, the application of $\mathbb{CP}$ turns a μ^- decay (**which does exist**) into a μ^+ decay (**which also exists**) $\rightarrow$ "$\mathbb{CP}$ is conserved in this process".

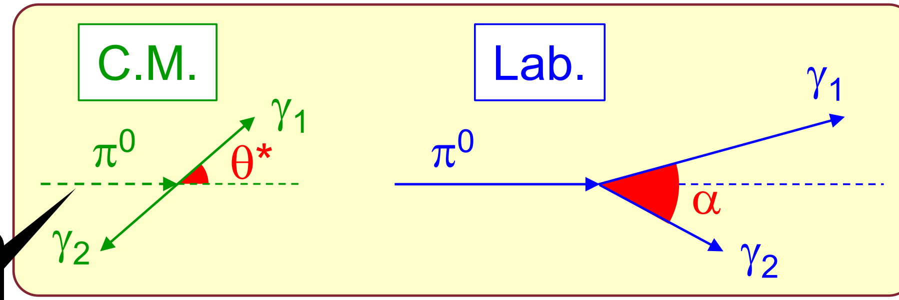
decay $\pi^0 \to \gamma\gamma$: L-transf.

L-transf $\begin{cases} E = \gamma(E^* + \beta p_\ell^*); \\ p_\ell = \gamma(p_\ell^* + \beta E^*); \\ p_T = p_T^*; \end{cases}$

NB: L-transf. CM → Lab.

$$m \equiv m_{\pi^0}; \qquad \beta \equiv \frac{p_{\pi^0}}{E_{\pi^0}}; \qquad \gamma \equiv \frac{E_{\pi^0}}{m_{\pi^0}}.$$

In CM, π^0 at rest.



	C.M.	Lab.
π^0	$m\{1,0,0,0\}$	$m\{\gamma,\beta\gamma,0,0\}$
γ_1	$\frac{m}{2}\{1,\cos\theta^*,\sin\theta^*,0\}$	$\frac{m}{2}\{\gamma(1+\beta\cos\theta^*),\gamma(\cos\theta^*+\beta),\sin\theta^*,0\}$
γ_2	$\frac{m}{2}\{1,-\cos\theta^*,-\sin\theta^*,0\}$	$\frac{m}{2}\{\gamma(1-\beta\cos\theta^*),\gamma(-\cos\theta^*+\beta),-\sin\theta^*,0\}$

- the $\pi^0 \to \gamma\gamma$ decay is an e.m. process; it is here just for convenience;
- see also FNSN1, § Cinematica, 22-26.

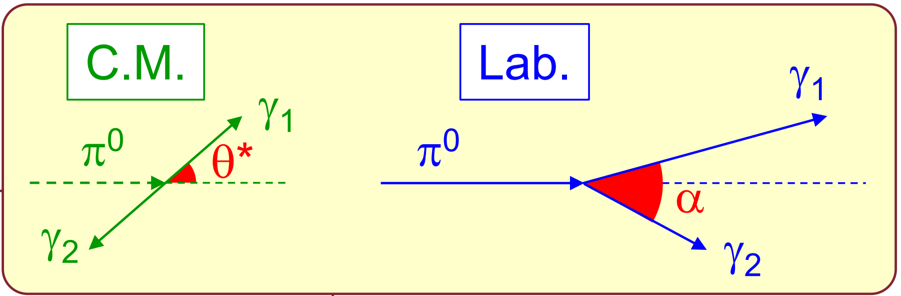
[...] = 1

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = \frac{\vec{p}_1^{Lab}\cdot\vec{p}_2^{Lab}}{E_1^{Lab}E_2^{Lab}} = \frac{\gamma^2\left(\beta^2-\cos^2\theta^*\right)-\sin^2\theta^*\left[\gamma^2(1-\beta^2)\right]}{\gamma^2\left(1-\beta^2\cos^2\theta^*\right)} = \frac{\beta^2\left(1+\sin^2\theta^*\right)-1}{1-\beta^2\cos^2\theta^*};$$

for a γ : $|\vec{p}| = E$

$$\sin^2\frac{\alpha}{2} = -\frac{1}{2}\left(\frac{\beta^2\left(1+\sin^2\theta^*\right)-1}{1-\beta^2\cos^2\theta^*} - \frac{1-\beta^2\cos^2\theta^*}{1-\beta^2\cos^2\theta^*}\right) = \frac{\boxed{\beta^2+\beta^2-2}}{\boxed{-2}\left(1-\beta^2\cos^2\theta^*\right)} = \frac{1}{\boxed{\gamma^2}\left(1-\beta^2\cos^2\theta^*\right)}.$$

decay $\pi^0 \rightarrow \gamma\gamma$: angle α



$$\sin^2\frac{\alpha}{2} = \frac{1}{\gamma^2\left(1-\beta^2\cos^2\theta^*\right)};$$

$$\xrightarrow[\theta^*=90°,\ \cos\theta^*=0]{} \quad [\mapsto\updownarrow] \quad \sin^2\frac{\alpha}{2}\bigg|_{min} = \frac{1}{\gamma^2} = \left(\frac{m_{\pi^0}}{E_{\pi^0}}\right)^2 \quad \rightarrow \alpha_{min} \cong \frac{2m_{\pi^0}}{E_{\pi^0}};$$

$$\xrightarrow[\theta^*=0°,\ \cos\theta^*=1]{} \quad [\mapsto\leftrightarrow] \quad \sin^2\frac{\alpha}{2}\bigg|_{max} = \frac{1}{\gamma^2\left(1-\beta^2\right)} = 1 \quad \rightarrow \alpha_{max} = 180°;$$

	$f(\theta^*)$	$\alpha\big\vert_{min}\left[\cos\theta^*=0\right]$	$\alpha\big\vert_{max}\left[\cos\theta^*=1\right]$
π^0	$m\{\gamma,\beta\gamma,0;1\}$	$m\{\gamma,\beta\gamma,0;1\}$	$m\{\gamma,\beta\gamma,0;1\}$
γ_1	$\frac{m}{2}\{\gamma(1+\beta\cos\theta^*),\gamma(\cos\theta^*+\beta),\sin\theta^*;0\}$	$\frac{m}{2}\{\gamma,\beta\gamma,1;0\}$	$\frac{m}{2}\{\gamma(1+\beta),\gamma(1+\beta),0;0\}$
γ_2	$\frac{m}{2}\{\gamma(1-\beta\cos\theta^*),\gamma(-\cos\theta^*+\beta),-\sin\theta^*;0\}$	$\frac{m}{2}\{\gamma,\beta\gamma,-1;0\}$	$\frac{m}{2}\{\gamma(1-\beta),\gamma(-1+\beta),0;0\}$

decay $\pi^0 \rightarrow \gamma\gamma$: P(α)

$\mathrm{spin}(\pi^0) = 0 \rightarrow \mathcal{P}(\cos\theta^*) = \mathrm{flat} = 1/2.$

Therefore :

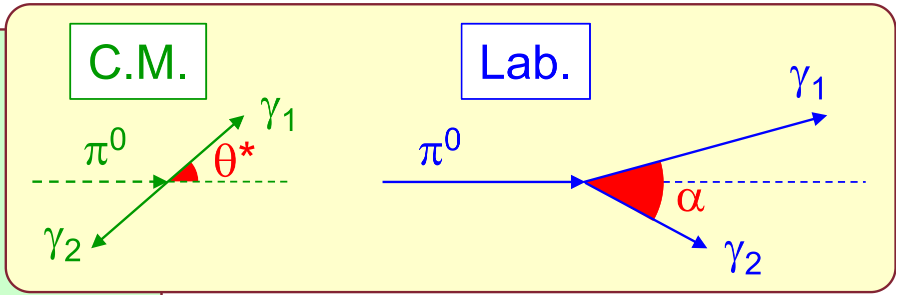
$$E_\gamma^{1,2} = \frac{m\gamma}{2}(1 \pm \beta\cos\theta^*) \rightarrow \frac{dE_\gamma^{1,2}}{d\cos\theta^*} = \pm\frac{m\beta\gamma}{2} \rightarrow$$

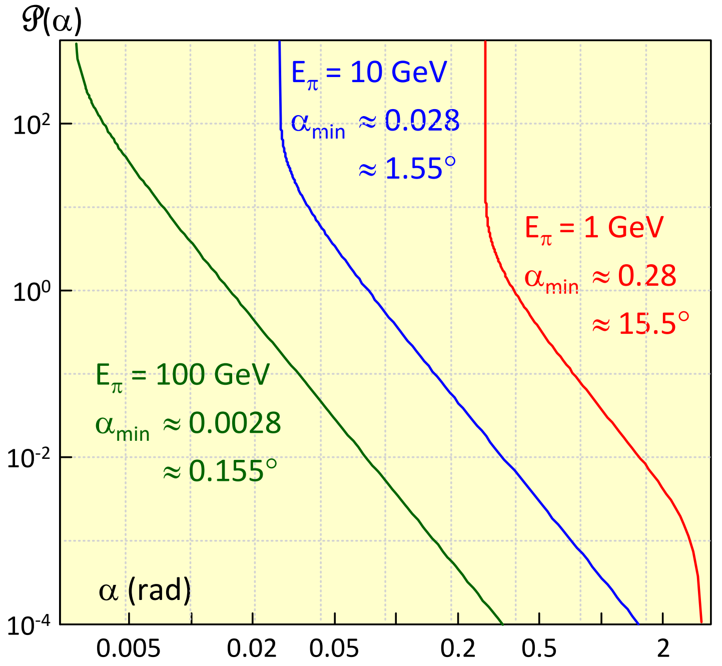
$$\mathcal{P}\left(E_\gamma^{1,2}\right) = \mathcal{P}(\cos\theta^*) \Bigg/ \left|\frac{dE_\gamma^{1,2}}{d\cos\theta^*}\right| = \frac{1}{2}\frac{2}{m\beta\gamma} = \frac{1}{m\beta\gamma} = \frac{1}{p_{\pi^0}}$$

flat in $\left[\frac{m\gamma}{2}(1-\beta), \frac{m\gamma}{2}(1+\beta)\right].$

$$\mathcal{P}(\alpha) = \frac{1}{4\beta\gamma}\frac{\cos(\alpha/2)}{\sin^2(\alpha/2)\sqrt{\gamma^2\sin^2(\alpha/2) - 1}}$$

[no proof, $\rightarrow$ FNSN1, §cinematica, 26].



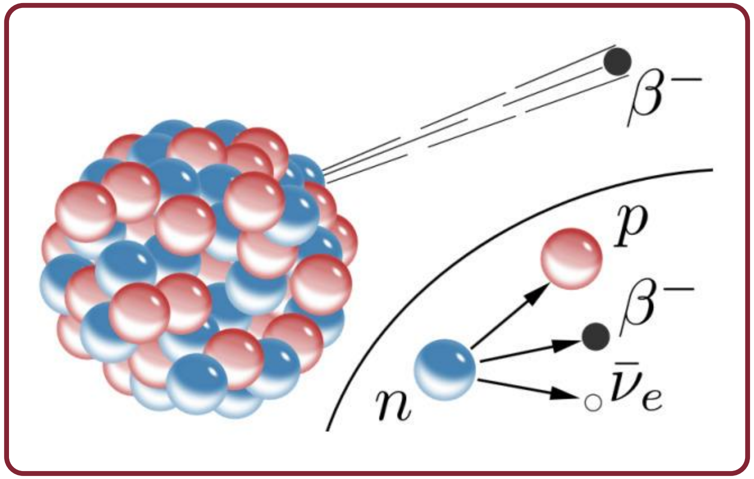


nota bene –
mutatis mutandis, similar kinematics also for $H \rightarrow \gamma\gamma$ [$\mathrm{spin}(\pi^0) = \mathrm{spin}(H) = 0$].

β decay introduction

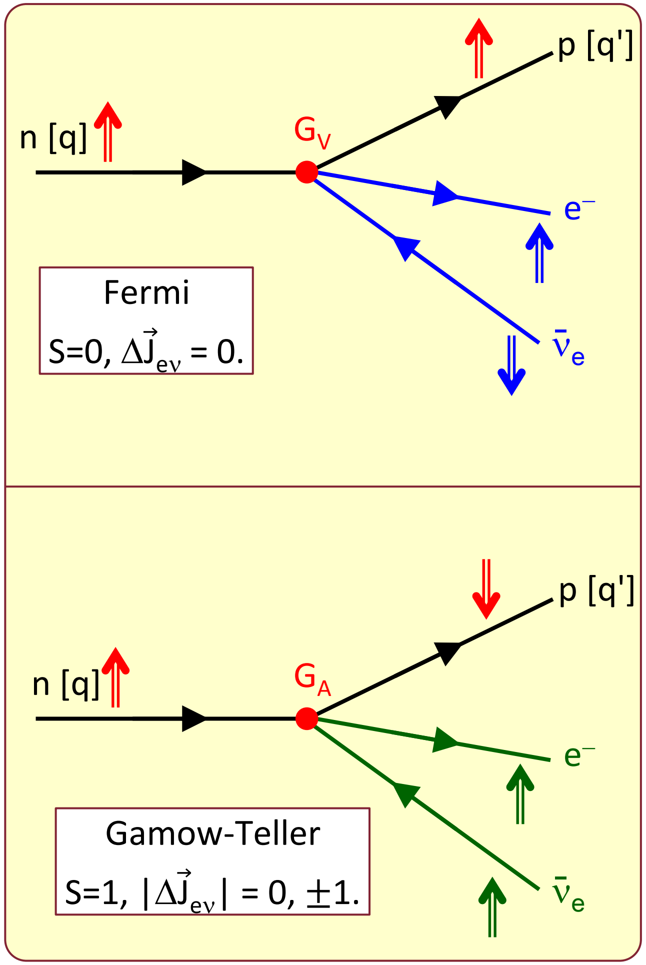
- For **point-like fermions**, CC is "V – A", both for leptons and quarks [*the only difference for hadrons being the CKM "rotation", see later*];
- however, nucleons and hyperons (p, n, Λ, Σ, Ξ, Ω) are **bound states of non-free quarks**;
- for low Q^2 processes, the "spectator model" (in this case the free quark decay) is an **unrealistic** approximation;
- **strong interaction corrections** are important → modify V – A dynamics;
- the standard approach, due to Fermi, is to produce a parameterization, based on the vector properties of the current (S-P-V-A-T, see) and then compute ↔ measure the coefficients;
- pros : quantitative theory, which reproduces the experiments well;
- cons : lack of deep understanding of the parameters.

the simple and successful approach, used for point-like decays, is not valid here, because of strong interaction corrections; those are (possibly understood, but) non-perturbative and impossible to master with present-day math; same as chemistry ↔ electromagnetism.



β decay: Fermi/Gamow-Teller

- In Fermi theory, CC currents were classified according to the properties of the transition operator.
- In neutron β-decay, the e-ν pair may be created as a spin singlet (S=0) or triplet (S=1). In case of NO orbital angular momentum, there are two possibilities to conserve the total angular momentum :
  - Fermi transitions [F], S=0, $\Delta J_{e\nu}$=0 : the direction of the spin of the nucleon remains unchanged; in modern language, [*it can be shown that*] the interaction takes place with vector coupling G_V;
  - Gamow-Teller transitions [G-T], S=1, $\Delta J_{e\nu}$= 0, $\pm$1 : the direction of the spin of the nucleon is turned upside down (it "flips"); […] the transition happens with axial-vector coupling G_A.
- In principle, F and G-T processes are completely different : there is no a-priori reason why the coupling should be similar or even related.



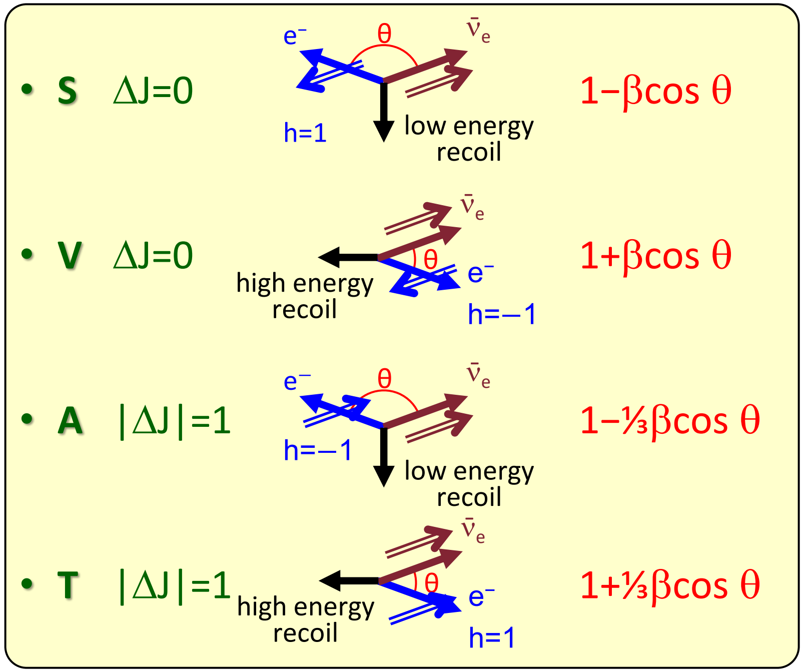
β decay: S, P, V, A, T

- Study the **neutron β decay**; assume :
  - p and n are spin-½ fermions;
  - $e^\pm$ and ν are spin-½ fermions, but only ν's with "– helicity" exist [interact].
- Then, the most general matrix element for the four-body interaction is

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}} \sum_j C_j \left[\bar{u}_p O_j u_n \right] \left[\bar{u}_e O_j \left(\frac{1-\gamma_5}{2} \right) u_\nu \right],$$

  - G_F : the overall coupling;
  - $\bar{u}_{p,n,e,\nu}$ $(u_{p,n,e,\nu})$: creation (destruction) operators for p, n, e, ν;
  - $(1-\gamma_5)/2$: projector of –ve ν helicity;
  - C_j : sum coefficients (adimensional free parameters, *possibly of order 1*);
  - O_j : current operators with given vector properties : **S** = scalar, **P** = pseudo-scalar, **V** = vector, **A** = axial-vector, **T** = tensor.

- For β-decay, the pseudo-scalar term is irrelevant : P can only be built from the proton velocity v_p in the neutron rest frame, which are depressed by v_p/c;
- For the other four terms, the angular distributions are [*BJ 399, YN1 561*] (1, ⅓ for singlet and triplet, β=electron velocity) :



β decay: V, A

- From comparison with data, some terms can be excluded:
  - (S and V) are Fermi transitions : they cannot be both present, due to the lack of observed interference between them;
  - (A and T) are G-T transitions : same argument holds;
  - the angular distributions of the electrons are only consistent with V for F and A for G-T.
- So the matrix element becomes :

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}}\left[\bar{u}_p\gamma^\mu\left(C_V + C_A\gamma_5\right)u_n\right]\left[\bar{u}_e\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)u_\nu\right],$$

- the value of C_V can be measured by comparing (<u>composite</u>) hadrons with (<u>free, pure V–A</u>) leptons; it turns out

$$C_V \approx 1.$$

- The value of $(C_A)^2$ can be measured from the relative strength of F and G-T, by comparing neutron β–decay with a pure Fermi ($^{14}O \rightarrow {}^{14}N\ e^+\nu$); for β decay:

$$|C_A| \cong 1.267.$$

- The sign of C_A could be measured from the polarization of the protons (a very difficult measurement); in practice from the interference between F and G-T in polarized neutrons decays :

$$C_A \cong -1.267.$$

Fermi did not know about parity violation, and would have written different matrix elements for his ("Fermi") transitions.

However, the final result for leptons and free quarks is very similar to his original proposal, but the factor $(1-\gamma_5)/2$ *:*

$$\mathcal{M}_{fi} = \frac{G}{\sqrt{2}}\left[\bar{u}_p\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)u_n\right]\left[\bar{u}_e\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)u_\nu\right].$$

β decay: CVC, PCAC

Focus on the hadron current $\propto [C_V + C_A\gamma_5]$:

- for leptons $C_A = - C_V$, i.e. "V−A" [much simpler, because leptons are free];
- for quarks, when no spectators are present, as in $\pi^\pm$ decays, similar picture (but CKM corrections);
- for composite hadrons, the picture works when their partons (quarks) interact as "quasi-free" particles;
- e.g. the "spectator approximation" works well in ν DIS and in hadron colliders, where the CC looks "V−A" as well;
- however, **at low Q^2** hadrons behave as coherent particles and not as parton containers → "V−A" is **not** valid.

$$\boxed{\mathcal{M}_{fi} \propto \left[\bar{u}_p\gamma^\mu\left(1+\frac{C_A}{C_V}\gamma_5\right)u_n\right]\left[\bar{u}_e\gamma^\mu(1-\gamma_5)u_\nu\right]}$$

- In low Q^2 processes, [*it can be shown that*] the vector part of the hadronic current stays constant (**CVC**, conserved vector current), while the axial part is broken (**PCAC**(*), "*partially conserved axial current*").
- In baryon β-decays, it is measured :
  - ➢ $n \to p\, e\, \bar{\nu}_e$, $C_A/C_V = -1.267$
  - ➢ $\Lambda \to p\, \pi^-, n\, \pi^0$ $= -0.718$
  - ➢ $\Sigma^- \to n\, e\, \bar{\nu}_e$ $= +0.340$
  - ➢ $\Xi^- \to \Lambda e^- \bar{\nu}_e$ $= -0.25$
  - ➢ [high Q^2 (free quarks) $= -1$].

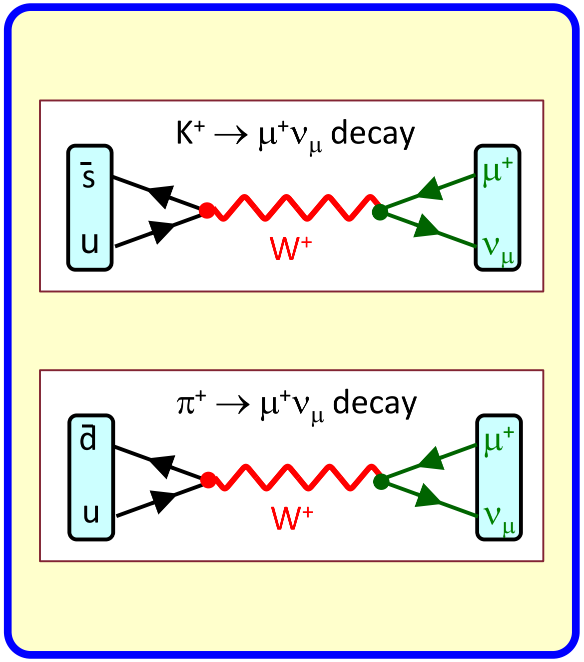
(*) *at the time, they preferred to say "partially conserved" instead of "badly broken"; it now seems that the acronym "PCAC" is slowly disappearing from the texts : you are kindly requested to forget the term "PCAC" forever.*

Quark decays: the puzzle

- For high mass quarks and at high Q^2, the structure "V−A" seems restored: quarks behave as free, point-like particles, exactly like the leptons [*Coll.Phys.*] .

- However, with more accurate data, some discrepancies appear, not due to strong interactions (see boxes).
- An apparent violation of CC universality ? A mistake ?

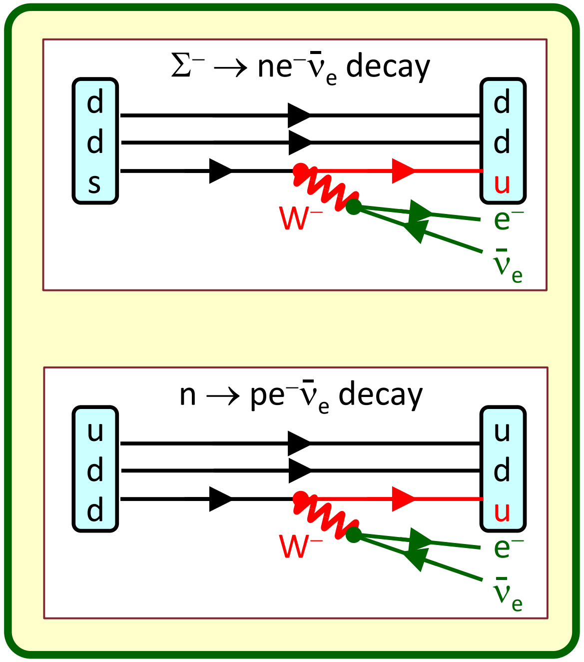
(*continue…*)



It is measured :

$$\frac{G_F^{2'}\left[\begin{array}{l}K^+ \to \mu^+\nu_\mu,\\ \Delta S = 1\end{array}\right]}{G_F^{2''}\left[\begin{array}{l}\pi^+ \to \mu^+\nu_\mu,\\ \Delta S = 0\end{array}\right]} \approx 0.05;$$

$$\frac{\Gamma\left[\begin{array}{l}\Sigma^- \to ne^-\bar{\nu}_e,\\ \Delta S = 1\end{array}\right]}{\Gamma\left[\begin{array}{l}n \to pe^-\bar{\nu}_e,\\ \Delta S = 0\end{array}\right]} \approx 0.05.$$



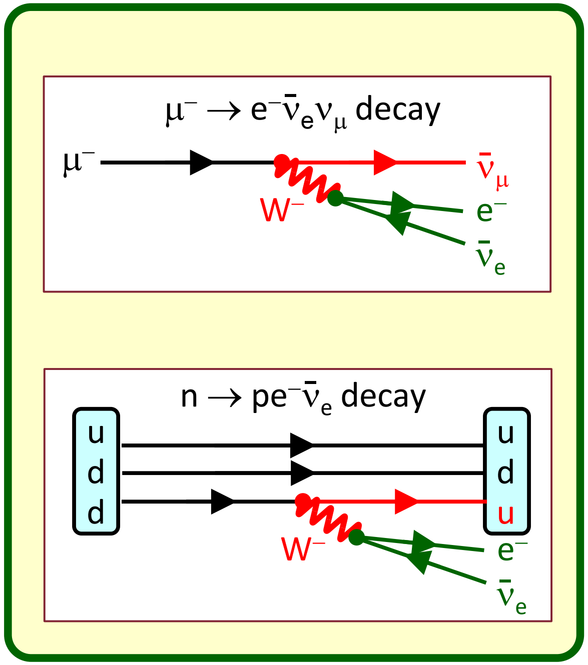
Quark decays: Cabibbo theory

(*… continue …*)

Even tiny, but well measured effects seem to contradict the universality; "G_F" is slightly larger for leptons :

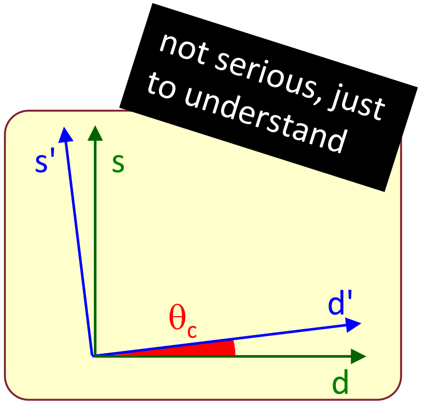
$$G_F\left[\mu^- \to e^-\bar{\nu}_e\nu_\mu\right] \approx 1.166\times10^{-5}\ \mathrm{GeV}^{-2};$$

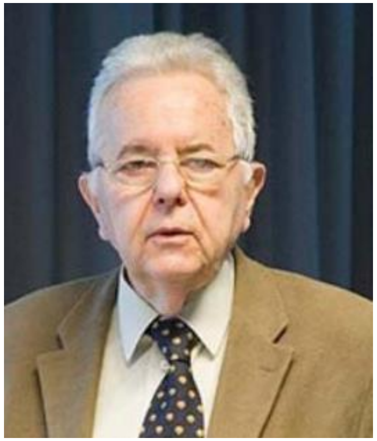
$$G_F\left[\begin{array}{r} n \to pe^-\bar{\nu}_e, \\ \text{i.e. } d \to ue^-\bar{\nu}_e \end{array}\right] \approx 1.136\times10^{-5}\ \mathrm{GeV}^{-2}.$$



In 1963 N. Cabibbo [*at the time much younger than in the image*], invented a theory to explain the effect : the "Cabibbo angle" θ_c :

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix}.$$





Quark decays: Cabibbo rotation

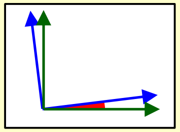
The idea was the following :

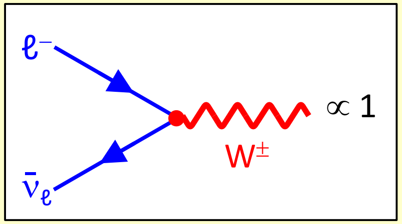
- the hadrons are built up with quarks **u d s** (**c b t** not yet discovered);
- however, in the CC processes, the quarks (d s) – same quantum numbers but S – mix together (= "rotate" by an angle θ_c), in such a way that the CC processes see "rotated" quarks (d' s') :

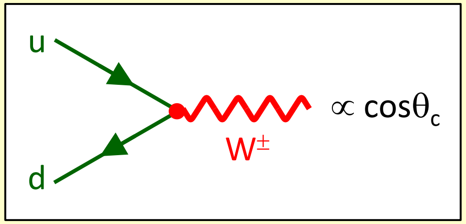
$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}.$$

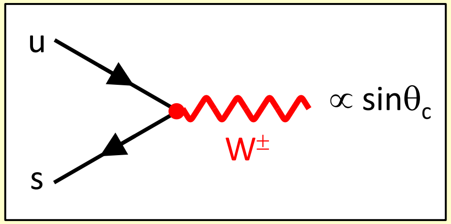
- therefore, respect to the strength of the leptonic processes (no mix), the ud coupling is decreased by $\cos\theta_c$ and the us coupling by $\sin\theta_c$, since the real process in ud', not ud or us.
- therefore the processes with $\Delta S = 0$ happen $\propto \cos^2\theta_c$ and those with $\Delta S = 1$ $\propto \sin^2\theta_c$;
- even processes $\propto \sin^4\theta_c$ may happen (e.g. in the charm sector, see §3), when two "Cabibbo suppressed" couplings are present in the same process;
- all the anomalies come back under control if

$$\sin^2\theta_c \approx .03,\ \cos^2\theta_c \approx .97.$$









Quark decays: GIM mechanism

In this context the GIM mechanism was invented to explain the absence of FCNC:

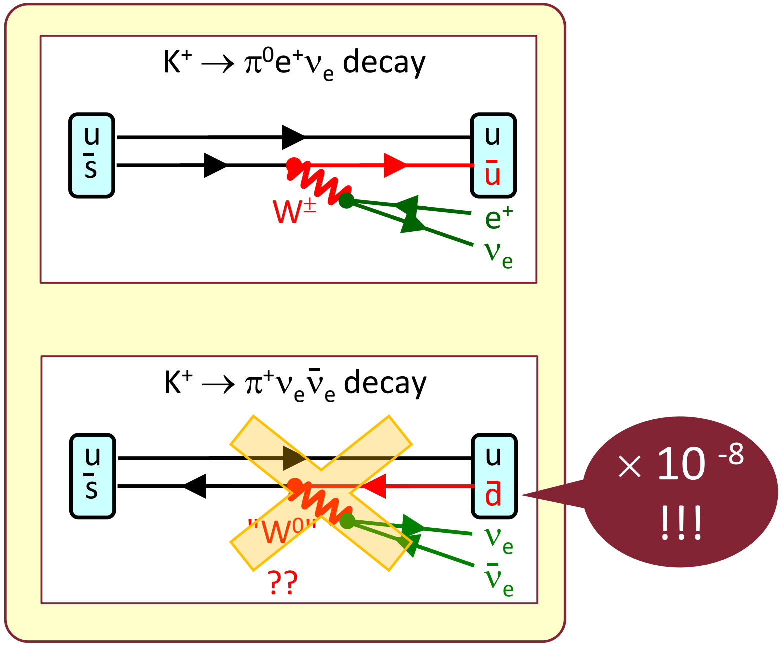
- data, at the time not understandable :

$$\left.\begin{aligned} BR\left(K^0 \to \mu^+\mu^-\right) &= 7\times10^{-9} \\ BR\left(K^+ \to \mu^+\nu_\mu\right) &= 0.64 \end{aligned}\right\} \left[\begin{array}{c}\text{already}\\ \text{mentioned}\end{array}\right];$$

$$\left.\begin{aligned} BR\left(K^+ \to \pi^+\nu\bar{\nu}\right) &= \left(1.5^{+1.3}_{-0.9}\right)\times10^{-10} \\ BR\left(K^+ \to \pi^0 e^+\nu_e\right) &= \left(4.98\pm0.07\right)\times10^{-2} \end{aligned}\right\}.$$

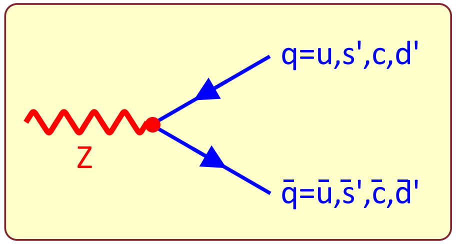
  i.e. a factor ~10^{-8} between NC and CC decays;

- if the Z, carrier of NC, see the same quark mixture as the $W^\pm$ in CC, then the NC decay would be suppressed only by a factor 5%;
- the idea was to introduce a fourth quark, called c (<u>charm</u>), with charge ⅔, as the u quark; this solves the FCNC problem;
- the c quark was discovered in 1974 [*see* § *3*].



Quark decays: no FCNC

In the GIM mechanism, NC contain four hadronic terms, coupled with the Z.



Assume Cabibbo theory and sum all terms:

$u\bar{u} + d'\bar{d}' + c\bar{c} + s'\bar{s}' =$

$= u\bar{u} + (d\cos\theta_c + s\sin\theta_c)(\bar{d}\cos\theta_c + \bar{s}\sin\theta_c) +$

$+ c\bar{c} + (s\cos\theta_c - d\sin\theta_c)(\bar{s}\cos\theta_c - \bar{d}\sin\theta_c) =$

$= u\bar{u} + c\bar{c} + d\bar{d} + s\bar{s} +$ "0". (!!!)

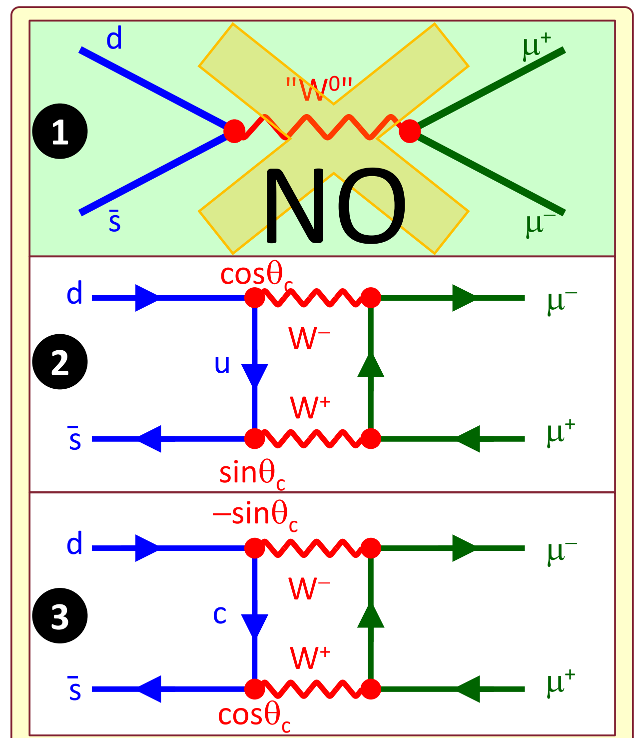
the "non-diagonal" terms, which induce FCNC, **disappear**.

Why $(K^0 \to \mu^+\mu^-)$ is small, but NOT = 0 ?

Look at the "box diagrams" ❷;

- technically a 2nd order ($\propto g^4 \sin\theta_c \cos\theta_c$) CC;
- same final state as a 1st order FCNC ❶;
- incompatible with data (BR too large);
- cured by the diagram ❸ with a c quark, whose contribution cancels the first in the limit $m_c \to m_u$.

The cancellation depends on m_c. The data on $(K^0 \to \mu^+\mu^-)$ put limits on m_c between 1 and 3 GeV [$J/\psi \to 2m_c \approx 3.1$ GeV, *see*].



Quark decays: third family

In 1973, Kobayashi and Maskawa extended the Cabibbo scheme to a new generation of quarks : the new mixing matrix (analogous to the Euler matrix in ordinary space) is a three-dimension unitary matrix, with three real parameters ("Euler angles") and one imaginary phase :

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L \quad \updownarrow W^\pm$$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

The matrix is known as **CKM** (*Cabibbo-Kobayashi-Maskawa*) matrix.

K-M observed that the $\mathbb{CP}$ violation, already discovered, is automatically generated by the matrix, when the imaginary phase is non-zero.

In addition to the $\mathbb{CP}$-violation, the nine elements of the CKM matrix govern the flavor changes in CC processes.

The measurement of the elements and the check of the unitarity relations is an important subject of physics studies : e.g. if some element is too small, this could be an indication of term(s) missing in the sum, i.e. the presence of a next generation of quarks.

[*A discussion of the CKM matrix in* §*5.*]

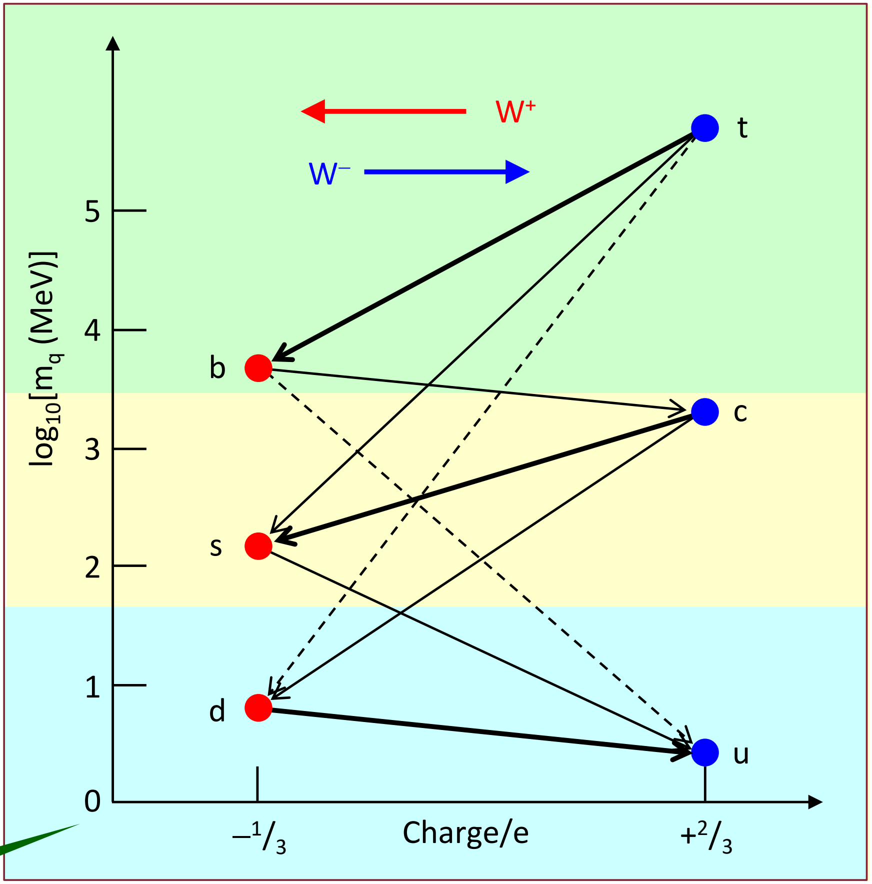
Makoto Kobayashi

Toshihide Maskawa

Summary: charged current decays

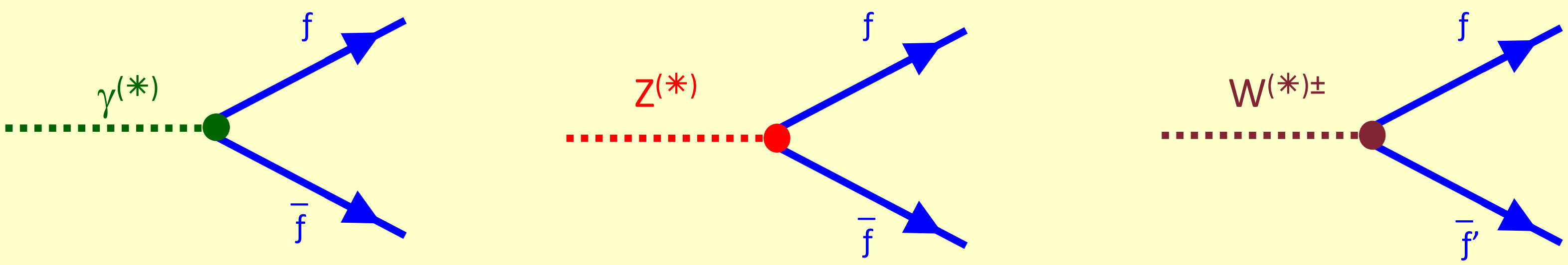
- The quark flavor changes only as a consequence of a weak CC interaction (*).
- Each type of quark can convert into each other with charge ±1, emitting or absorbing a W boson.
- The coupling is modulated by the strength of the mixing (the width of the line in fig.); in the SM it is described by the V_{CKM} matrix [§5].

(*) since FCNC do NOT [seem to] exist, NC processes – with Z mediators – do NOT play any role in flavor decays.



+ the equivalent table for q̄'s.

Summary: EM, NC, CC



photon (γ) (electromagnetism)	neutral IVB (Z) (neutral current)	charged IVB ($W^{\pm}$) (charged current)
$\mathcal{L}_F = -eJ^{\mu}_{e.m.}A_{\mu};$ $J^{\mu}_{e.m.} = Q_f \bar{\Psi}_f \gamma^{\mu} \Psi.$	$\mathcal{L}_F = \frac{-e}{\sin\theta_W \cos\theta_W} J^{\mu}_{nc} Z_{\mu};$ $J^{\mu}_{nc} = \bar{\Psi}_f \gamma^{\mu} \frac{g^f_V - g^f_A \gamma^5}{2} \Psi_f.$	$\mathcal{L}_F = \frac{-e}{\sqrt{2}\sin\theta_W} J^{\mu}_{cc} \tau^{\pm} W^{\pm}_{\mu};$ $J^{\mu}_{cc} = \bar{\Psi}_f \gamma^{\mu} \frac{1-\gamma^5}{2} \Psi_f.$
[V]	[combination g^f_V V + g^f_A A]	[V - A]

Vectors & Co

vector properties of physical quantities :

- a 4-vector $\vec{\mathbf{v}}$ is the well-known quantity, which transforms canonically under a L-transformation $\mathbb{L}$ (both boosts and rotations), and Parity $\mathbb{P}$ in space :
  - ➢ space-time, 4-momentum, electric field, …
- an axial vector $\vec{\mathbf{a}}$ transforms like a vector under $\mathbb{L}$, but gains an additional sign flip under $\mathbb{P}$:
  - ➢ cross-products $\vec{v}\times\vec{v}'$, magnetic field, angular momentum, spin, …
- a scalar **s** is invariant both under $\mathbb{L}$ and $\mathbb{P}$:
  - ➢ [4-]dot-products $\vec{v}\cdot\vec{v}'$ or $\vec{a}\cdot\vec{a}'$, module of a vector, mass, charge, …
- a pseudoscalar **p** is invariant under $\mathbb{L}$, but changes its sign under $\mathbb{P}$:
  - ➢ a triple product $\vec{v}\cdot\vec{v}'\times\vec{v}''$;
  - ➢ a scalar product $\vec{a}\cdot\vec{v}$ between a vector and an axial vector, e.g. the helicity(*);
- a tensor **t** is a quantity which also transforms canonically under $\mathbb{L}$ and $\mathbb{P}$, with ≥ 2 dimensions :
  - ➢ the electro-magnetic tensor $F^{\mu\nu}$.

____

(*) the helicity h is the projection of the spin $\vec{s}$ along the momentum $\vec{p}$:

$$h=\frac{\vec{s}\cdot\vec{p}}{|\vec{s}|\cdot|\vec{p}|}.$$

Q. : this "parity violation" does NOT happen. Why ?

Helicity vs Chirality

Two different concepts:

- h for a particle is defined from its spin and momentum[1];
- χ is a spinor property[2], related to the eigenstates of γ_5.

➢ The χ operator γ_5 does NOT commute with the mass term of the free Hamiltonian, so χ is NOT conserved for a massive particle;

➢ a massive particle with definite spin and momentum has a definite h, but is a mixture of the two eigenstates of χ;

➢ for a massless particle (or in the u.r.a. approximation) χ is conserved and its value reduces to h;

➢ this approximation is generally valid in this chapter, so the slides do not stress the difference h $\leftrightarrow$ χ.

---

(1) $h = \vec{s}\cdot\vec{p}/(|\vec{s}|\ |\vec{p}|)$; sometimes $h = \vec{s}\cdot\vec{p}/|\vec{p}|$; however, the different definition does not affect the difference h $\leftrightarrow$ χ.

(2) define the projectors:

$\psi_R = ½(1+\gamma_5)\psi$; $\quad \psi_L = ½(1-\gamma_5)\psi$;

$\gamma_5\psi_R = +\psi_R$; $\quad \gamma_5\psi_L = -\psi_L$;

$\psi_{R,L}$: eigenstates of χ with eigenvalues ±1.

References:

[Povh, 10.5], [Bettini, 7.4], [YN1, 4.3.5]