THE STATIC QUARK MODEL

Outline

- Symmetry and Quantum numbers
- Hadrons : elementary or composite ?
- The eightfold way
- The discovery of the Ω^-
- The static quark model
- The mesons
- Meson quantum numbers
- Meson mixing
- The baryons
- SU(3)
- Color

Simmetry and Conservation Law

In Heisenberg representation the time dependence of the operator *Q(t)* is given by:

$$i\hbar \frac{dQ}{dt} = i\hbar \frac{\partial Q}{\partial t} + [Q, H]$$

An operator with no explicit time dependance is a constant of the motion if it commutes with the hamiltonian operator. In general, conserved quantum numbers are associated to operators commuting with the hamiltonian.

Simmetry and Conservation Law

Example: space translations

$$\psi(r+\delta r)=\psi(r)+\delta r\frac{\partial\psi}{\partial r}=\underbrace{\left(1+\delta r\frac{\partial}{\partial r}\right)}\psi$$

For a finite translation:

$$D=\left(1+\frac{ip}{\hbar}\delta r\right)$$

$$D=\lim_{n\to\infty}\left(1+\frac{ip\Delta r}{n\hbar}\right)^n=e^{\frac{ip\Delta r}{\hbar}}$$

P is the generator of the operator D of space translations.
IF *H* is invariant under transations **[D,H] =0** hence:

[p,H] = 0

The following three statements are equivalent:
•Momentum is conserved for an isolated system.
•The hamiltonian is invariant under space translations.
•The momentum operator commutes with the hamiltonian.

Parity (P)

The operation of spatial inversion of coordinates is produced by the **parity** operator P:

$$P\psi(\vec{r}) = \psi(-\vec{r})$$

Repetition of this operation implies **$P^2=1$** so that P is a unitary operator.

$$\psi(\vec{r}) \xrightarrow{P} \psi(-\vec{r}) \xrightarrow{P} \psi(\vec{r})$$

Therefore if there are parity eigenvalues they must be: **P = ±1**

Examples:

P = +1 $\quad \psi(x) = \cos x \xrightarrow{P} \cos(-x) = \cos x = \psi(x)$

P = -1 $\quad \psi(x) = \sin x \xrightarrow{P} \sin(-x) = -\sin x = -\psi(x)$

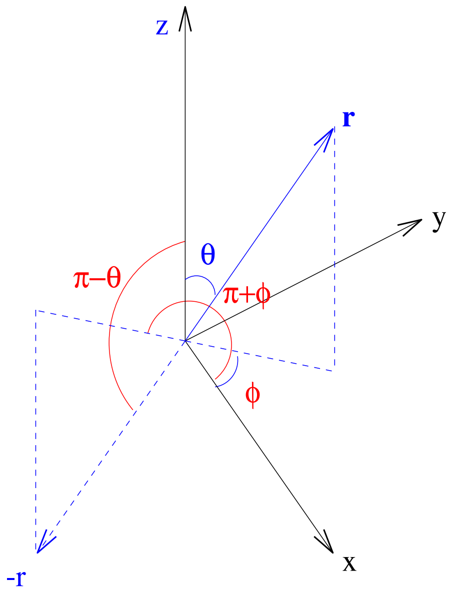
$$\psi(x) = \sin x + \cos x \xrightarrow{P} = -\sin x + \cos x \neq \pm\psi(x)$$

Parity (P): an example

Example: the hydrogen atom

$$\psi(r,\theta,\varphi)=\chi(r)\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_m^l(\cos\vartheta)e^{im\varphi}$$

$$\vec{r}\to-\vec{r}\Leftrightarrow\begin{cases}\theta\to\pi-\theta\\ \varphi\to\pi+\varphi\end{cases}$$



$$e^{im\varphi}\to e^{im(\varphi+\pi)}=(-1)^m e^{im\varphi}$$

$$P_l^m(\cos\theta)\to(-1)^{l+m}P_l^m(\cos\theta)$$

$$Y_l^m\to(-1)^l Y_l^m$$

Parity (P): an example

Hence the spherical harmonics have parity ***$P=(-1)^l$*.**

For example, in electric dipole transitions, which obey the selection rule $\Delta l = \pm 1$, the atomic parity changes. Therefore the parity of the emitted radiation must be negative, in in order to conserve the total parity of the system atom+photon.

$$P(\gamma) = -1$$

P is a multiplicative quantum number. It is conserved in strong and elettromagnetic interactions, but it is not conserved in weak interactions.

Parity conservation law requires the assignment of an intrinsic parity to each particle.

Protons and neutrons are conventionally assigned positive parity

$$P_p = P_n = +1$$

Charged pion parity

The pion (π) is a spin 0 meson. Consider the reaction

$$\pi^- + d \rightarrow n + n$$

(where the deuteron *d* is a *pn* bound state).
In the initial state *l=0*; since $s_\pi=0$, $s_d=1$ the total angular momentum must be J=1 (**J=L+S**). Therefore *also in the final state* we must have J=1. The symmetry of the final state wave function (under interchange of the 2 neutrons) is given by:

$$K = \underbrace{(-1)^{S+1}}_{spin}\underbrace{(-1)^{L}}_{orbitale} = (-1)^{L+S+1}$$

Since we have two identical fermions it must be K=-1, which implies L+S is even.
With the condition J=1 we have the following possibilities:

L=0 S=1 no L=1 S=0 no L=2 S=1 no
L=1 S=1 OK

Therefore the parity of the final state is $P=(-1)^L=-1$. Since the parity of the deuteron is $P_d=+1$ we obtain for the π intrinsic parity $P_\pi = -1$.
The π is therefore a pseudoscalar meson.

Parity of the neutral pion (π^0)

$\pi^0 \rightarrow \gamma\,\gamma$ B.R. = (99.798 ± 0.032) %

Let **k** and **-k** be the momentum vectors of the two γ; $\mathbf{e}_1$ and $\mathbf{e}_2$ their polarization vectors. The simplest linear combinations one can form which satisfy requirements of exchange symmetry for identical bosons are:

$$\psi_1(2\gamma) = A(\vec{e}_1 \cdot \vec{e}_2) \propto \cos\phi$$

$$\psi_2(2\gamma) = B(\vec{e}_1 \times \vec{e}_2) \cdot \vec{k} \propto \sin\phi$$

ψ_1 is a scalar and therefore even under space inversion, ψ_2 is a pseudoscalar and therefore it has odd parity.

$$P_{\pi^0} = +1 \quad |\psi|^2 \propto \cos^2\phi \qquad\qquad P_{\pi^0} = -1 \quad |\psi|^2 \propto \sin^2\phi$$

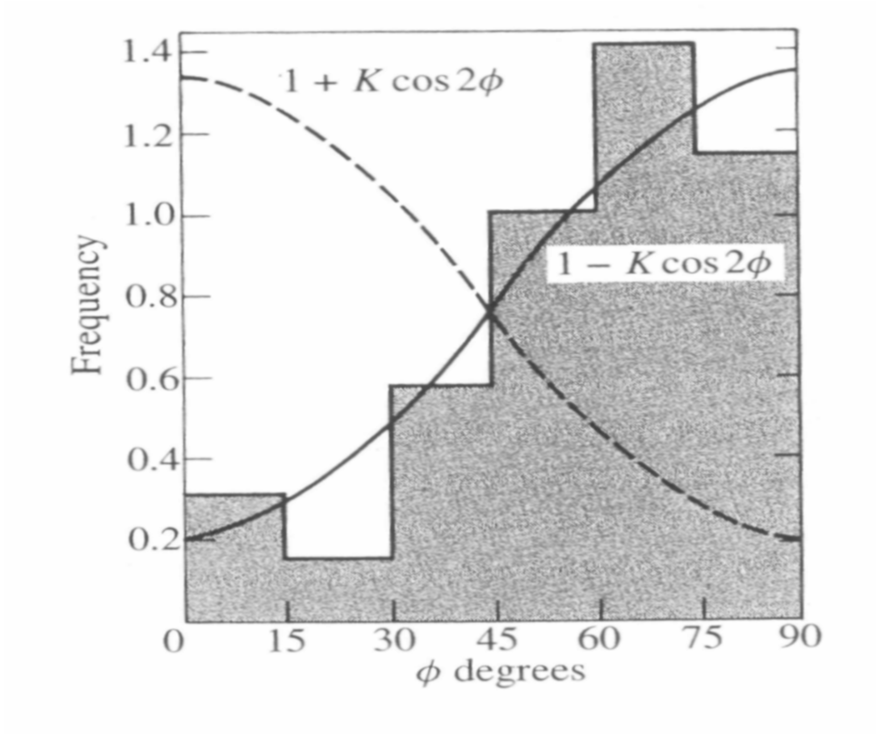
where ϕ is the angle between the polarization planes of the two γ. The experiment was done using the decay:

$$\pi^0 \rightarrow e^+ + e^- + e^+ + e^-$$

(double Dalitz; B.R. = $(3.14 \pm 0.30)\times 10^{-5}$) in which each Dalitz pair lies predominantly in the polarization plane of the “internally converting” photon. The result is $P_{\pi 0}$=-1.

Parity of the neutral pion (π^0)

$$\pi^0 \rightarrow e^+ + e^- + e^+ + e^-$$



Parity

The assignment of an intrinsic parity is meaningful when particles interact with one another (as in the case of electric charge).

The nucleon intrinsic parity is a matter of convention.

The relative parity of particle and antiparticle is not a matter of convention.

Fermions and antifermions are created in pairs, for instance:

$$p + p \to p + p + p + \overline{p}$$

whereas this is not the case for bosons.

Fermions: particle and antiparticle have **opposite parity.**

Bosons: particle and antiparticle have **equal parity.**

$$\left.\begin{array}{l} \vec{r} \to -\vec{r} \\ \vec{p} \to -\vec{p} \end{array}\right\} \quad polar \quad vectors$$

$$\vec{\sigma} \to \vec{\sigma} \qquad axial \quad vector \quad (\vec{r} \times \vec{p})$$

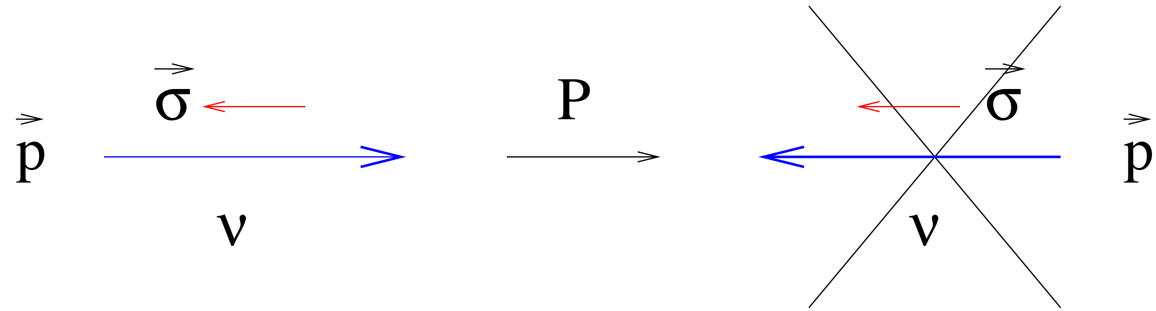
$$\vec{E} \to -\vec{E}$$

$$\vec{B} \to \vec{B}$$

Parity conservation

Parity is conserved in strong and electromagnetic interactions, whereas it is violated in weak interactions. (V-A theory, maximal parity violation)

Example:



In experimental studies of strong and electromagnetic interactions tiny degrees of parity violation are in fact observed, due to contributions from the weak interactions: $H = H_s + H_{em} + H_w$. Atomic transitions:

$$^{16}O^* \longrightarrow {}^{12}C + \alpha$$

$$J^P = 2^- \qquad\qquad J^P = 2^+$$

with total witdth $\Gamma_\alpha = (1.0 \pm 0.3) \times 10^{-10}$ eV, to be compared with $^{16}O^* \rightarrow {}^{16}O + \gamma$ of width 3×10^{-3} eV.

Particle and antiparticles

The relativistic relation between the total energy *E*, momentum *p* and rest mass *m* of a particle is:

$$E^2 = p^2c^2 + m^2c^4$$

The total energy can assume negative as well as positive values:

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

In quantum mechanics we represent the amplitude of an infinite stream of particles, e.g. electrons, travelling along the positive *x*-axis with 3-momentum *p* by the plane wavefunction:

$$\psi = Ae^{-i(Et-px)/\hbar}$$

Formally this expression can also represent particles of **energy *-E* and momentum *-p* travelling in the negative *x*-direction and backwards in time**.

Particle and antiparticles

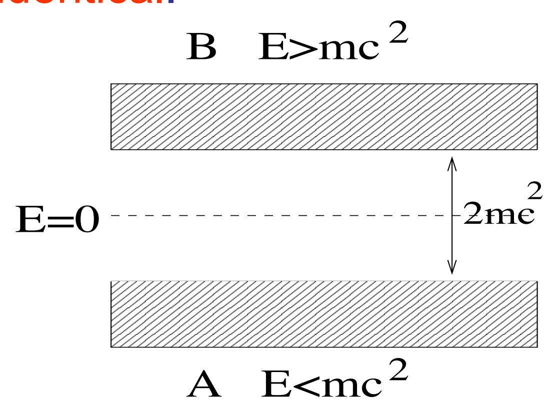
E > 0 $\longrightarrow$ t

E < 0 $\longleftarrow$ t

Such a stream of negative electrons flowing backwards in time is equivalent to positive charges flowing forward, and thus having E>0.

The negative energy particle states are connected with the existence of positive energy antiparticles of exactly equal but opposite electrical charge and magnetic moment, and otherwise identical.

the positron, the antiparticle of the electron, was discovered experimentally in 1932 in cloud chamber experiments with cosmic rays.



Charge Conjugation (c)

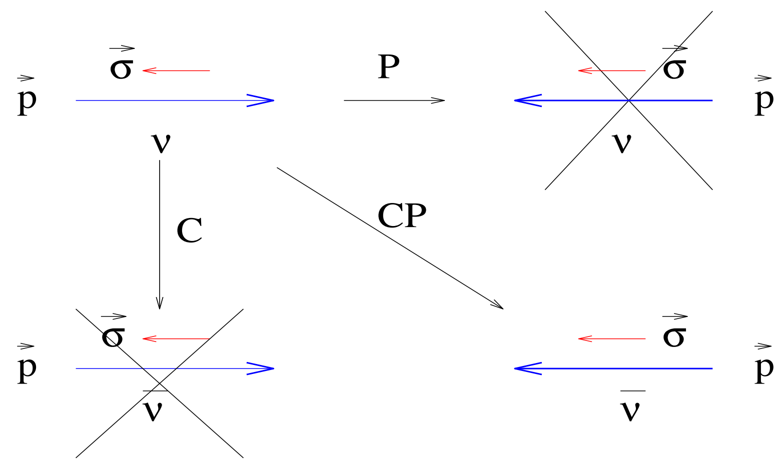
Charge conjugation reverses the charge and magnetic moment of a particle.

In classical physics Maxwell's equations are invariant under:

$$q \to -q \quad \vec{j} \to -\vec{j} \quad \vec{E} \to -\vec{E} \quad \vec{H} \to -\vec{H}$$

In relativistic quantum mechanics: particle $\leftrightarrow$ antiparticle

	p $\xrightarrow{\quad C \quad}$	$\bar{p}$
Q	+e	-e
B	+1	-1
μ	+2.79(e$\hbar$/2mc)	-2.79(e$\hbar$/2mc)
σ	½$\hbar$	½$\hbar$



Eigenstates of the C Operator

Only neutral bosons which are their own antiparticle can be eigenstates of C.

$C|\pi^+\rangle \to |\pi^-\rangle \neq \pm|\pi^+\rangle$ π^+ e π^- are not C eigenstates. For the π°:

$$C|\pi^0\rangle = \eta|\pi^0\rangle$$

$$\eta^2 = 1 \quad \Rightarrow \quad C|\pi^0\rangle = \pm|\pi^0\rangle$$

$$\pi^0 \to \gamma\gamma \quad \Rightarrow \quad C_{\pi^0} = +1$$

Electromagnetic interactions conserve C, therefore the decay

$$\pi^0 \to 3\gamma$$

should be forbidden. Experimentally we find:

$$\frac{BR(\pi^0 \to 3\gamma)}{BR(\pi^0 \to 2\gamma)} < 3.1 \times 10^{-8}$$

Conservation of C

Charge conjugation C is conserved in strong and electromagnetic interactions, but not in weak interaction.

- Spectra of particle and antiparticle, for example:

$$p + \bar{p} \rightarrow \pi^+ + \pi^- + \ldots.$$

$$p + \bar{p} \rightarrow K^+ + K^- + \ldots.$$

- η meson decay ($J^P = 0^-$, $M = 550$ MeV/c^2)

$\eta \rightarrow \gamma\gamma$	B.R. = (39.21 ± 0.34) %
$\eta \rightarrow \pi^+\pi^-\pi^0$	B.R. = (23.1 ± 0.5) %
$\eta \rightarrow \pi^+\pi^-\gamma$	B.R. = (4.77 ± 0.13) %
$\eta \rightarrow \pi^0 e^+ e^-$	B.R. $< 4 \times 10^{-5}$

$$B.R. \equiv \frac{\Gamma_{ch}}{\Gamma_{tot}}$$

Since $\eta \rightarrow \gamma\gamma$ we must have $C_\eta = +1$. Hence the decay $\eta \rightarrow \pi^0 e^+ e^-$ is forbidden by C conservation.

Positronium decay

Positronium is an e^+e^- bound state which possesses energy levels similar to the hydrogen atom (with about half the spacing). Wave function:

$\psi(e^+e^-) = \phi(\text{space})\times\alpha(\text{spin})\times\chi(\text{charge})$

ϕ(space) Particle interchange is equivalent to space inversion introducing a factor $(-1)^L$ where L is orbital angular momentum

α(spin)

$$\begin{cases} \alpha(1,1) = \psi_1(\tfrac{1}{2},\tfrac{1}{2})\psi_2(\tfrac{1}{2},\tfrac{1}{2}) \\ \alpha(1,0) = \tfrac{1}{\sqrt{2}}\left[\psi_1(\tfrac{1}{2},\tfrac{1}{2})\psi_2(\tfrac{1}{2},-\tfrac{1}{2}) + \psi_1(\tfrac{1}{2},-\tfrac{1}{2})\psi_2(\tfrac{1}{2},\tfrac{1}{2})\right] \\ \alpha(1,-1) = \psi_1(\tfrac{1}{2},-\tfrac{1}{2})\psi_2(\tfrac{1}{2},-\tfrac{1}{2}) \end{cases}$$

Triplet S=1
Symmetric

$$\alpha(0,0) = \tfrac{1}{\sqrt{2}}\left[\psi_1(\tfrac{1}{2},\tfrac{1}{2})\psi_2(\tfrac{1}{2},-\tfrac{1}{2}) - \psi_1(\tfrac{1}{2},-\tfrac{1}{2})\psi_2(\tfrac{1}{2},\tfrac{1}{2})\right]$$

Singlet S=0
Antisymmetric

The symmetry of α is therefore $(-1)^{S+1}$

Let the charge wave function acquire a factor C.

The total symmetry of the wave function for the interchange of e^+ and e^- is

$$K = (-1)^L(-1)^{S+1}C$$

Positronium decay

Two decays are observed for positronium annihilation from L=0:

$(e^+e^-) \rightarrow 2\gamma$ $(e^+e^-) \rightarrow 3\gamma$

The two-photon decay must have J=0, so the three-photon decays has to be assigned J=1.

	S=J	L	C	K
2γ	0	0	+1	-1
3γ	1	0	-1	-1

(C = $(-1)^n$ for a system consisting of n photons).

In QED the widths of these states can be calculated very accurately:

	Γ	τ (theory)	τ (experiment)
2γ	$\frac{1}{2}mc^2\alpha^5$	$1.252\times10^{-10}s$	$(1.252\pm0.017)\times10^{-10}s$
3γ	$\frac{2}{9\pi}(\pi^2-9)\alpha^6mc^2$	$1.374\times10^{-7}s$	$(1.377\pm0.004)\times10^{-7}s$

Charge conservation and gauge invariance

Electric charge is known to be very accurately conserved in all processes.

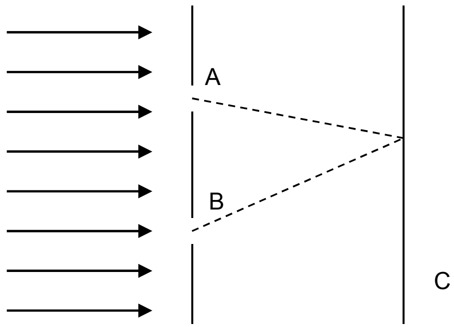
$$\frac{n \to p \nu_e \bar{\nu}_e}{n \to p e^- \bar{\nu}_e} < 9 \times 10^{-24}$$

The conservation of electric charge is related to the gauge invariance of the electromagnetic interaction.

Wigner(1949): Suppose we create a charge Q at a point where the potential is ϕ. Let us now move the charge to a point where the potential is ϕ'. $\Delta W = Q(\phi-\phi')$. Suppose we destroy the charge in this point. If W was the work done to create the charge, this work will be recovered when the charge is destroyed. Therefore we gain a net energy $W-W+\phi-\phi'$ because W does not depend on ϕ.

The conservation of energy implies that we cannot create or destroy charge if the scale of electrostatic potential is arbitrary.

Charge conservation and gauge invariance



Suppose we send a beam of electrons on a screen in which there are two slits *A* and *B* and that we observe the interference on a second screen *C* located at a distance d from the first.

$$\psi = e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\psi = e^{i(\vec{p}\cdot\vec{x}-Et)} = e^{ipx} \qquad p \equiv (E,\vec{p}) \quad x \equiv (t,\vec{x}) \quad \hbar = c = 1$$

Let us redefine ψ by adding a phase -eα.

$$\psi = e^{(ipx-e\alpha)}$$

The interference pattern on *C* depends only on phase differences and it is independent on the global phase ***e***α. If however eα = eα*(x)*:

$$\frac{\partial}{\partial x} i(\ px - e\alpha\ ) = i(\ p - e\frac{\partial \alpha}{\partial x}\ )$$

And the result *would seem to depend on the local phase transformation*.

Charge conservation and gauge invariance

Electrons however are charged and they interact via an electromagnetic potential, which we write as a 4-vector *A*:

$$A \equiv (\phi, \vec{A})$$

The effect of the potential is to change the phase of an electron:

$$p \rightarrow p + eA$$

So the derivative now becomes:

$$\frac{\partial}{\partial x} i\big(px + eAx - e\alpha(x)\big) = i\left(p + eA - e\frac{\partial \alpha}{\partial x}\right)$$

The potential scale is also arbitrary and we can change it by adding to *A* the gradient of any scalar function (Gauge transformation):

$$A \rightarrow A + \frac{\partial \alpha}{\partial x}$$

With this transformation the derivative becomes *ip,* independent of $\alpha(x)$.

The effect of the original local phase transformation is cancelled exactly by the gauge transformation.

$$A \rightarrow A + \frac{\partial \alpha(x)}{\partial x}$$

Time Reversal (T)

$$p + {}^{27}Al \rightleftarrows \alpha + {}^{24}Mg$$

$$t \rightarrow -t$$

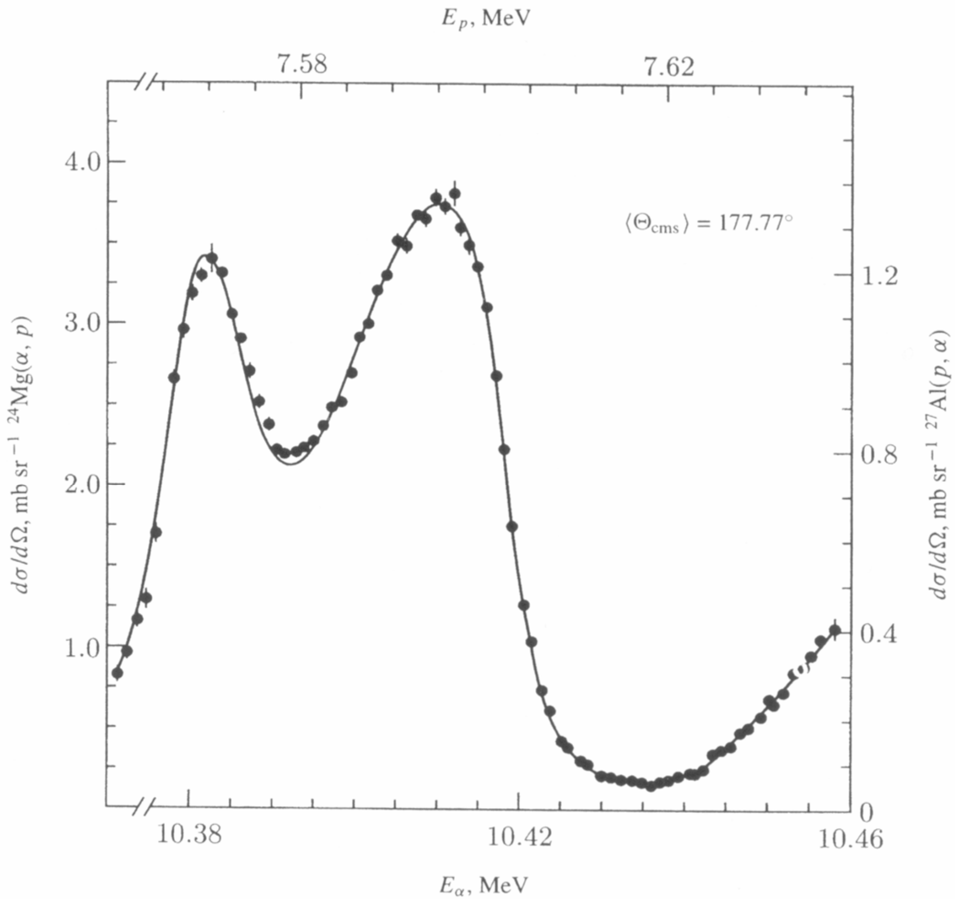
$$\vec{r} \longrightarrow +\vec{r}$$

$$\vec{p} \longrightarrow -\vec{p}$$

$$\vec{\sigma} \longrightarrow -\vec{\sigma}$$

$$\vec{E} \longrightarrow +\vec{E}$$

$$\vec{B} \longrightarrow -\vec{B}$$



T violation < 5×10^{-4}

CPT

CPT theorem:

All interactions are invariant under the succession of the three operation C, P and T taken in any order.

m(particle) = *m*(antiparticle)

$$\frac{m_{K^0} - m_{\overline{K^0}}}{m_{K^0} + m_{\overline{K^0}}} < 10^{-19}$$

mass

τ (particle) = τ (antiparticle)

$$\frac{\tau_{\mu^+} - \tau_{\mu^-}}{\tau_{\mu^+} + \tau_{\mu^-}} < 10^{-4}$$

lifetime

μ (particle)=-μ (antiparticle)

$$\frac{\mu_{e^+} - \mu_{e^-}}{\mu_{e^+} + \mu_{e^-}} < 10^{-12}$$

magnetic moment

CP

In 1964 it was discovered that the long lived K^0_L, which normally decays into three pions (CP = -1), could occasionally decay into two pions (CP=+1). This result represents the discovery of CP violation.

CP violation is at the origin of the asymmetry between matter and antimatter in our universe.

CP violation is equivalent to T violation (via the CPT theorem).

The following observables are sensitive to T violation:

- Transverse polarization $\sigma \cdot (\mathbf{p_1} \times \mathbf{p_2})$ in weak decays such as $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$. Upper limits from these studies $< 10^{-3}$.
- Electric dipole moment $\sigma \cdot \mathbf{E}$. Upper limit for the neutron:

$$\mathrm{EDM}(n) < 1.0 \times 10^{-25}\ e \cdot \mathrm{cm}$$

Spin of the charged pion $\pi^{\pm}$

The spin of the charged pion was determined by applying detailed balance to the reversible reaction:

$$p + p \rightleftarrows \pi^+ + d$$

$$\sigma_{pp\to\pi^+d} = \left|M_{if}\right|^2 \frac{(2s_\pi + 1)(2s_d + 1)}{v_i v_f} p_\pi^2$$

$$\sigma_{\pi^+d\to pp} = \frac{1}{2}\left|M_{fi}\right|^2 \frac{(2s_p + 1)^2}{v_f v_i} p_p^2$$

(the factor ½ comes from the integration over half solid angle, due to the fact that there are two identical bosons in the final state).

$$\frac{\sigma_{pp\to\pi^+d}}{\sigma_{\pi^+d\to pp}} = 2\frac{(2s_\pi + 1)(2s_d + 1)}{(2s_p + 1)^2}\frac{p_\pi^2}{p_p^2}$$

Measuring the cross sections for the direct and reverse reactions one obtains:

$$\boxed{s_\pi = 0}$$

Spin of the neutral pion π^0

For the neutral pion the decay $\pi^0 \to \gamma\gamma$ proves that the spin must be integral and that it cannot be one.

For a photon (zero mass, spin 1) $s_z = \pm 1$. Taking the common line of flight of the photons in the π^0 rest frame as the quantization axis, if S is the total spin of the two photons we can have: $S_z = 0$ oppure $S_z = 2$.

If the π^0 spin is 1, then $S_z = 0$. In this case the two-photon amplitude must behave under spatial rotations like the polynomial $P^0{}_1(\cos\theta)$, which is odd under the interchange of the two photons.

But the wave function must be symmetric under the interchange of the two identical bosons, hence the π^0 spin cannot be 1.

In conclusion $s_\pi = 0$ or $s_\pi \geq 2$.

Isospin

$$m_p = 938.27\,MeV \qquad m_n = 939.57\,MeV$$

$$m_p \approx m_n$$

<u>Heisenberg (1932):</u>

*Proton and neutron considered as different charge substates of one particle, the **Nucleon**.*

A nucleon is ascribed a quantum number, **isospin**, conserved in the strong interaction, not conserved in electromagnetic interactions.

Nucleon is assigned isospin $I = \frac{1}{2}$

$$I_3 = +\tfrac{1}{2} \qquad p$$

$$I_3 = -\tfrac{1}{2} \qquad n$$

$$\frac{Q}{e} = \frac{1}{2} + I_3$$

Isospin

The nucleon has an internal degree of freedom with two allowed states (the proton and the neutron) which are not distinguished by the nuclear force.

Let us write the nucleon states as $|I, I_3\rangle$

$$|p\rangle = \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \quad |n\rangle = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

For a two-nucleon system we have therefore:

Triplet
(symmetric)

$$\begin{cases} \chi(1,1) = \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \\ \chi(1,0) = \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right) \\ \chi(1,-1) = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \end{cases}$$

Singlet
(antisymmetric)

$$\left\{\chi(0,0) = \tfrac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right)\right.$$

Gell-Mann Nishijima formula

The third component of the isospin distinguish the electrical charge within an isospin multiplet

charge

strangeness

$$Q = I_3 + \frac{1}{2}(B+S)$$

Baryonic number

N.B. $B+S = Y$ (hypercharge)

$$\Rightarrow \quad Q = I_3 + \frac{1}{2}Y$$

The electromagnetic interaction breaks the isospin symmetry; as a consequence the masses within a multiplet are different (m_p different from m_n)

Isospin: deuteron

Example: deuteron (S-wave *pn* bound state)

$$\psi = \phi(spazio) \times \alpha(spin) \times \chi(isospin)$$

$$\underbrace{\begin{matrix} (-1)^l = +1 & \quad (-1)^{S+1} = +1 \\ (l = 0) & \quad (S = 1) \end{matrix}} \qquad (-1)^{I+1}$$

$$\boxed{(-1)^{I+1} = -1 \quad \Rightarrow \quad I = 0}$$

ψ is the wave function for *two identical fermions* (two nucleons), hence it must be *globally antisymmetric*. This implies that the deuteron must have **zero isospin**:

$$I_d = 0$$

Isospin: deuteron

Let's see a dynamical consequence of the isospin conservation

- Let's suppose to have two nucleons. From the rule of the addition of angular momentum we know that the total isospin can be 1 or 0.

Symmetric triplet; I = 1

a) $|1,1\rangle = pp$

b) $|1,0\rangle = \frac{1}{\sqrt{2}}(pn+np)$

c) $|1,-1\rangle = nn$

Antisymmetric isosinglet; I = 0

$$|0,0\rangle = \frac{1}{\sqrt{2}}(pn - np)$$

- It exists a bound state proton-neutron (deuteron), but do not exist bound states proton-proton or neutron-neutron, hence the deuteron must be an isospin singlet, otherwise they should exist also the other two states that differ by a rotation in the isospin space.

Isospin

As an example let us consider the two reactions

$$p + p \rightarrow \pi^+ + d$$
$$p + n \rightarrow \pi^0 + d \qquad (I_\pi = 1)$$

Since I_d=0 in each case the final state has isospin 1.

Let us now consider the initial states:

$$pp = |1,1\rangle$$
$$np = \tfrac{1}{\sqrt{2}}\left(|1,0\rangle - |0,0\rangle\right)$$

The cross section

$$\sigma \propto |ampiezza|^2 \approx \sum_I \left|\langle I', I_3' | A | I, I_3 \rangle\right|^2$$

Isospin conservation implies

$$I = I' = 1 \quad I_3 = I_3'$$

The reaction $np \rightarrow \pi^0 d$ proceeds with probability $\left(\tfrac{1}{\sqrt{2}}\right)^2$ with respect to $pp \rightarrow \pi^+ d$ hence:

$$\frac{\sigma\left(pp \rightarrow \pi^+ d\right)}{\sigma\left(np \rightarrow \pi^0 d\right)} = 2$$

Isospin: nucleon-nucleon scattering

Let's consider the following processes:

a) $p + p \rightarrow d + \pi^+$
b) $p + n \rightarrow d + \pi^0$
c) $n + n \rightarrow d + \pi^-$

the π has isospin 1 because it exists in three different states

- Since the deuteron has I=0, for the right hand processes we have:

$$d + \pi^+ = |1,1\rangle \; ; \; d + \pi^0 = |1,0\rangle \; ; \; d + \pi^- = |1,-1\rangle$$

while for the ones on the left we have:

$$p + p = |1,1\rangle \; ; \; p + n = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right) \; ; \; n + n = |1,-1\rangle$$

- Since the total isospin I must be conserved, only the state with I=1 will contribute. The scattering amplitudes have to be in the ratio:

$$1 : \frac{1}{\sqrt{2}} : 1$$

and the σ $\quad 2 : 1 : 2$

- The processes a) and b) have been measured, and once we take into account the e.m. interaction, they are in the predicted ratio.

Isospin in the πN system

The π meson exists in three charge states of roughly the same mass:

$$m_{\pi^{\pm}} = 139.57\, MeV$$

$$m_{\pi^{0}} = 134.98\, MeV$$

Consequently it is assigned $I_{\pi}=1$, with the charge given by $Q/e=I_3$.

$$\left|\pi^{+}\right\rangle = \left|1,1\right\rangle \quad \left|\pi^{0}\right\rangle = \left|1,0\right\rangle \quad \left|\pi^{-}\right\rangle = \left|1,-1\right\rangle$$

For the π $B=0$:

$$\boxed{\frac{Q}{e} = I_3 + \frac{B}{2}}$$

Isospin in the πN system

For the πN system the total isospin can be either I=1/2 or I=3/2

$$\left.\begin{aligned} &\pi^+ p \to \pi^+ p \\ &\pi^- n \to \pi^- n \end{aligned}\right\}$$

pure
I=3/2

$$\left.\begin{aligned} &\pi^- p \to \pi^- p \\ &\pi^- p \to \pi^0 n \\ &\pi^+ n \to \pi^+ n \\ &\pi^+ n \to \pi^0 p \end{aligned}\right\}$$

combination of
I=1/2 and I=3/2

$1\times\frac{1}{2}$	$I = \frac{3}{2}$				$I = \frac{1}{2}$	
I_3	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
$\pi^+ p$	1					
$\pi^+ n$		$\sqrt{\frac{1}{3}}$			$\sqrt{\frac{2}{3}}$	
$\pi^0 p$		$\sqrt{\frac{2}{3}}$			$-\sqrt{\frac{1}{3}}$	
$\pi^0 n$			$\sqrt{\frac{2}{3}}$			$\sqrt{\frac{1}{3}}$
$\pi^- p$			$\sqrt{\frac{1}{3}}$			$-\sqrt{\frac{2}{3}}$
$\pi^- n$				1		

The coefficients in the linear combinations, i.e. the relative weights of the 1/2 and 3/2 amplitudes, are given by *Clebsch-Gordan coefficients*

$$\begin{aligned} \left|\pi^+ n\right\rangle &= \left|1,1\right\rangle \times \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \\ &= \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \end{aligned}$$

$$\begin{aligned} \left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle &= \sqrt{\tfrac{1}{3}}\left|\pi^+ n\right\rangle + \sqrt{\tfrac{2}{3}}\left|\pi^0 p\right\rangle \\ &= \sqrt{\tfrac{1}{3}}\left|1,1\right\rangle \times \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|1,0\right\rangle \times \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \end{aligned}$$

$$\boxed{\begin{aligned} &|\pi^+,p\rangle = |\tfrac{3}{2},+\tfrac{3}{2}\rangle \quad ; \quad |\pi^-,p\rangle = \sqrt{\frac{1}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \\ &|\pi^-,n\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle \quad ; \quad |\pi^0,n\rangle = \sqrt{\frac{2}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{aligned}}$$

Isospin in the πN system

$$(1) \quad \pi^+ p \rightarrow \pi^+ p$$
$$(2) \quad \pi^- p \rightarrow \pi^- p$$
Elastic scattering

$$(3) \quad \pi^- p \rightarrow \pi^0 n$$
Charge exchange

$$\sigma \propto \left|\langle f|H|i\rangle\right|^2 = \left|M_{if}\right|^2$$

H = $\begin{cases} H_1 \text{ if it acts between states of } I=1/2 \\ H_3 \text{ if it acts between states of } I=3/2 \end{cases}$

let $M_1 = \langle I=\tfrac{1}{2}|H_1|I=\tfrac{1}{2}\rangle$

$M_3 = \langle I=\tfrac{3}{2}|H_3|I=\tfrac{3}{2}\rangle$

(1) $\sigma_1 = K|M_3|^2$

$|i\rangle = |f\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$

(2) $\sigma_2 = K\left|\langle f|(H_1+H_3)|i\rangle\right|^2$

$\sigma_2 = K\left|\tfrac{1}{3}M_3 + \tfrac{2}{3}M_1\right|^2$

(3) $|i\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$

$|f\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$

$\sigma_3 = K\left|\sqrt{\tfrac{2}{9}}M_3 - \sqrt{\tfrac{2}{9}}M_1\right|^2$

$$\boxed{\begin{array}{l}\sigma_1 : \sigma_2 : \sigma_3 = \\ |M_3|^2 : \tfrac{1}{9}|M_3+2M_1|^2 : \tfrac{2}{9}|M_3-M_1|^2\end{array}}$$

$M_3 >> M_1 \quad \sigma_1 : \sigma_2 : \sigma_3 = 9:1:2$

$M_1 >> M_3 \quad \sigma_1 : \sigma_2 : \sigma_3 = 0:2:1$

Isospin in the πN system

a) $\pi^+ + p \rightarrow \pi^+ + p$

b) $\pi^- + p \rightarrow \pi^0 + n$

c) $\pi^- + p \rightarrow \pi^- + p$

d) $\pi^- + n \rightarrow \pi^- + n$

$$1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$$

We express the various states in the basis of the total isospin using the Clebsch Gordan coefficients

$$|\pi^+, p\rangle = |\tfrac{3}{2}, +\tfrac{3}{2}\rangle \quad ; \quad |\pi^-, p\rangle = \sqrt{\frac{1}{3}}\, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$|\pi^-, n\rangle = |\tfrac{3}{2}, -\tfrac{3}{2}\rangle \quad ; \quad |\pi^0, n\rangle = \sqrt{\frac{2}{3}}\, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

Isospin in the πN system

- The 4 processes in the new basis

$$a)\ |\tfrac{3}{2},+\tfrac{3}{2}\rangle \to |\tfrac{3}{2},+\tfrac{3}{2}\rangle$$

$$b)\ \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \to \sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$c)\ \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \to \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$d)\ |\tfrac{3}{2},-\tfrac{3}{2}\rangle \to |\tfrac{3}{2},-\tfrac{3}{2}\rangle$$

- To estimate the probability amplitude:

$$\langle\tfrac{3}{2},I_3|S|\tfrac{3}{2},I_3\rangle = A_{3/2}\ ;\ \langle\tfrac{1}{2},I_3|S|\tfrac{1}{2},I_3\rangle = A_{1/2}$$

N.B. $A_{1/2} \neq A_{3/2}$

$$a)\ A_{tot}=A_{3/2}$$

$$b)\ A_{tot}=\left(\frac{\sqrt{2}}{3}A_{3/2}-\frac{\sqrt{2}}{3}A_{1/2}\right)$$

$$c)\ A_{tot}=\left(\frac{1}{3}A_{3/2}+\frac{2}{3}A_{1/2}\right)$$

$$d)\ A_{tot}=A_{3/2}$$

Scattering nucleon-pion

- The cross sections of the 4 processes will be proportional, by means of a K factor equal for all (takes into account the phase space, factors 2, etc ...), to:

$$\text{a)}\quad \sigma\left(\pi^{+} + \text{p} \rightarrow \pi^{+} + \text{p}\right) = K\left|A_{3/2}\right|^2$$

$$\text{b)}\quad \sigma\left(\pi^{-} + \text{p} \rightarrow \pi^{0} + \text{n}\right) = K\left|\frac{\sqrt{2}}{3}\text{A}_{3/2} - \frac{\sqrt{2}}{3}\text{A}_{1/2}\right|^2$$

$$\text{c)}\quad \sigma\left(\pi^{-} + \text{p} \rightarrow \pi^{-} + \text{p}\right) = K\left|\frac{1}{3}\text{A}_{3/2} + \frac{2}{3}\text{A}_{1/2}\right|^2$$

$$\text{d)}\quad \sigma\left(\pi^{-} + \text{n} \rightarrow \pi^{-} + \text{n}\right) = K\left|A_{3/2}\right|^2$$

- Processes a) and d) must have the same cross section at the same energy. This has been verified experimentally.
- For the other processes it is necessary to know $A_{1/2}$ and the relative phase between the amplitudes.

Formation resonance

$$a + b \rightarrow R \rightarrow a + b$$

Elastic Cross-Section

$$a + b \rightarrow R \rightarrow X$$

Total Cross-Section

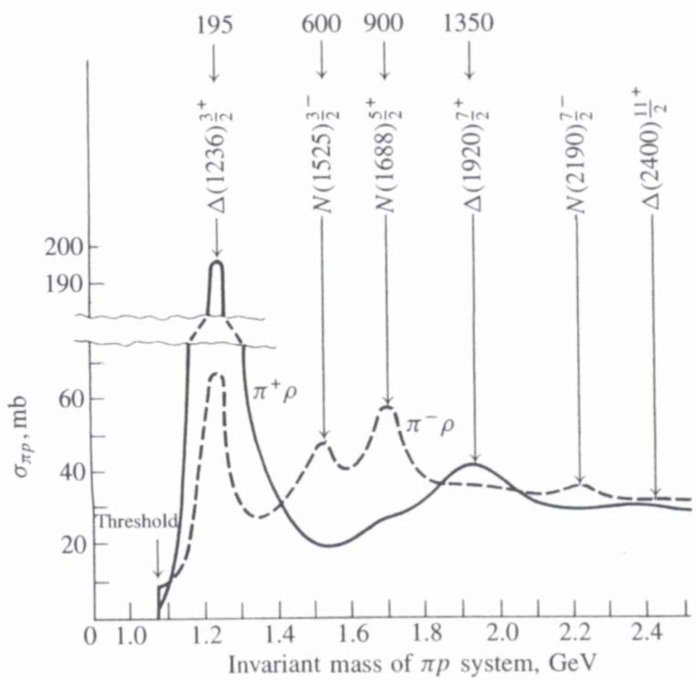
- The scattering process happens through the "formation" of an intermediate resonant state R;
- The resonance can decay in:
  - same particles of the initial state (elastic scattering)
  - other particles (anelastic scattering)
- The resonance is described by the Breit-Wigner formula:

$$\sigma(E) = \frac{4\pi\hbar^2}{p_{cm}^2} \frac{2J+1}{(2S_a+1)\cdot(2S_b+1)} \left[\frac{\Gamma_{in}\cdot\Gamma_{fin}}{(E-M_R)^2+\Gamma^2/4} \right]$$

- P_{cm}: beam momentum in the center of mass reference frame
- E: center of mass energy ($\sqrt{s}$)
- M_R: resonance mass

- S_a, S_b :initial state spins
- J : resonance spin
- Γ, Γ_{in}, Γ_{fin} : resonance total and partial widths

Δ resonance



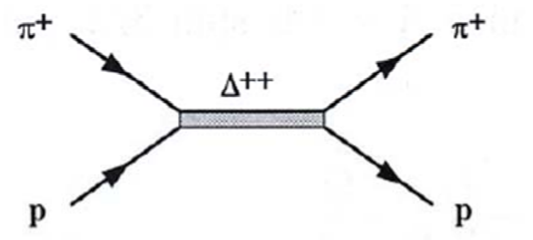
$$\Delta(1236) \quad \Gamma = 120\,MeV$$

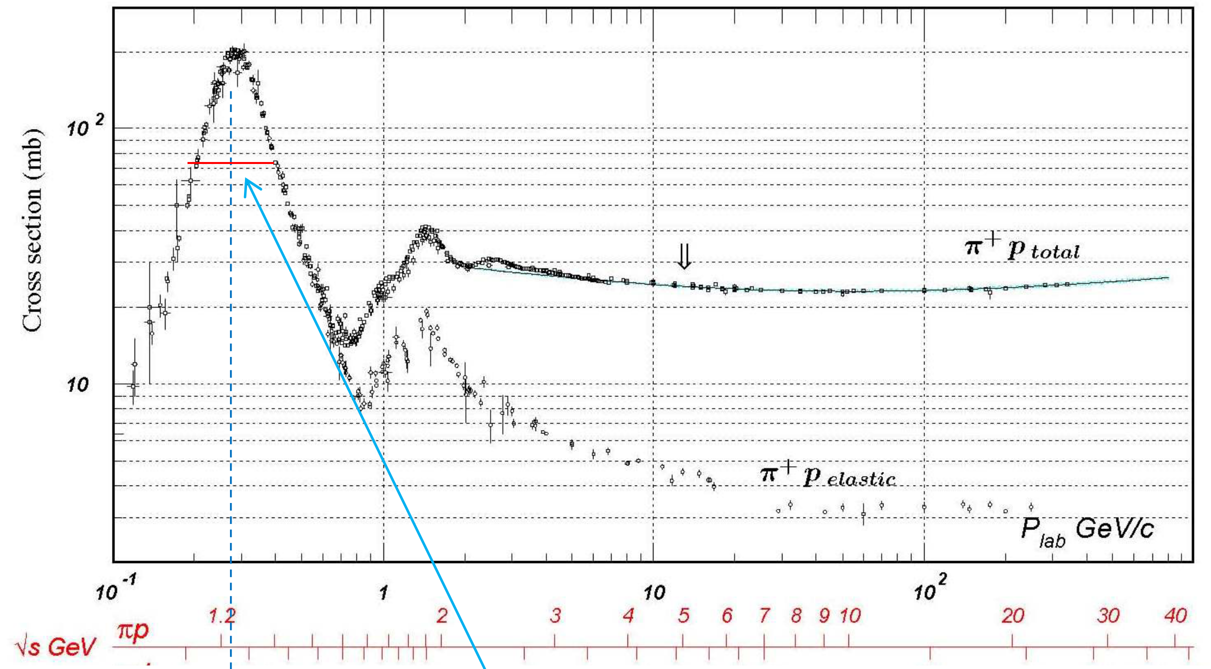
$$J^P = \frac{3}{2}^+ \quad I = \frac{3}{2} \quad (3,3)$$

$$\sigma(E) = \frac{2J+1}{(2s_1+1)(2s_2+1)} \frac{\pi}{k^2} \frac{\Gamma_{ab}\Gamma_{cd}}{(E-M_R)^2 + \frac{\Gamma^2}{4}}$$

a+b $\rightarrow$ R $\rightarrow$ c+d

Δ resonance





From the angular distribution of the decay products it turns out that the spin of the Δ is 3/2

- for √s <1.4 GeV the elastic and the total σ coincide.

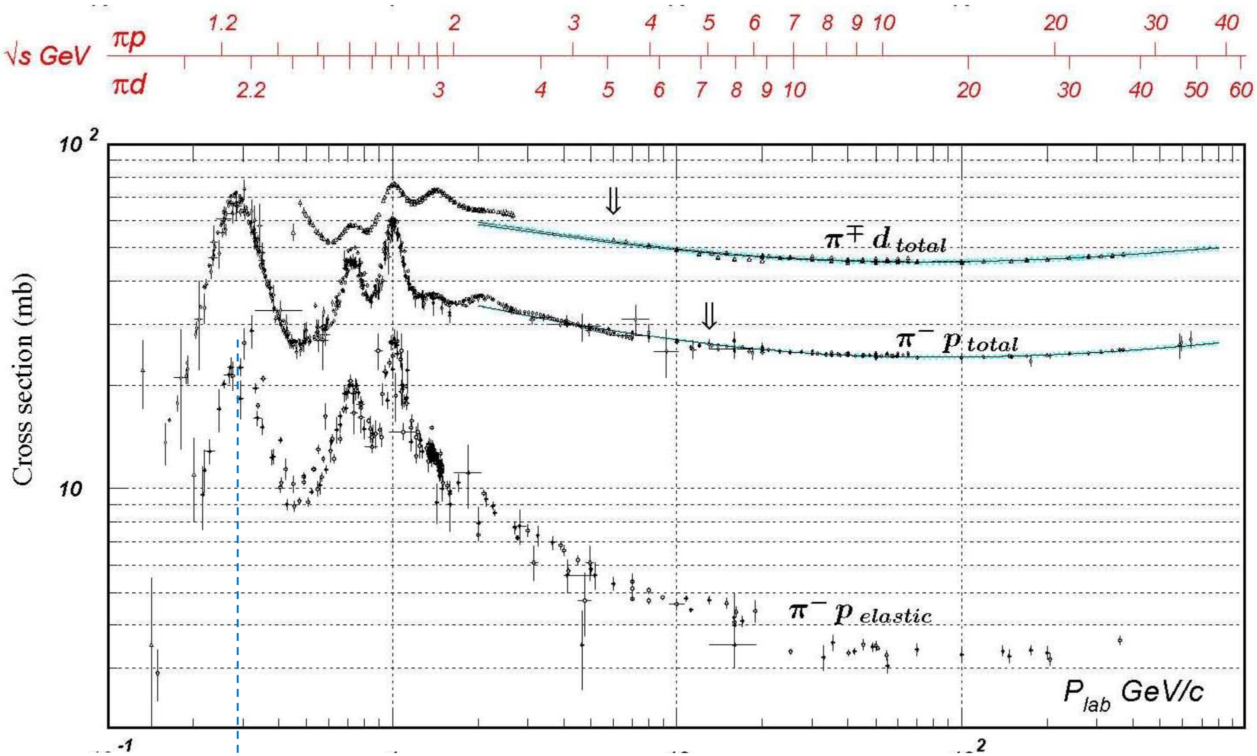
$M = 1232$ MeV

$\Gamma_{tot} = 118$ MeV

$\sigma_{picco} = 195$ mb

$$\tau = \frac{\hbar}{\Gamma_{tot}} = \frac{6.58 \cdot 10^{-16}\ eV \cdot s}{118 \cdot 10^{6}\ eV} = 5.6 \cdot 10^{-24}\ \text{s}$$

Δ resonance



M = 1232 MeV

Same peak position

$\pi^+ p \rightarrow \Delta^{++}$
$\pi^+ n \rightarrow \Delta^{+}$
$\pi^- p \rightarrow \Delta^{0}$
$\pi^- n \rightarrow \Delta^{-}$

... Same Γ_{tot}

$\sigma_{picco}\ (\pi^- p \rightarrow \pi^- p) = 22$ mb

$\sigma_{picco}\ (\pi^- p \rightarrow \pi^0 n) = 45$ mb

Question: why are the elastics σ of $\pi^- p$ and $\pi^+ p$ different? The answer is in the isospin of.

Δ resonance

The resonance Δ has isospin 3/2 (it exists in 4 states of different charge), so the processes in which the Δ appears as resonance of formation can only proceed through the channel with I = 3/2, therefore:

a) $\sigma\left(\pi^{+} + p \rightarrow \pi^{+} + p\right) = K\left|A_{3/2}\right|^{2}$

b) $\sigma\left(\pi^{-} + p \rightarrow \pi^{0} + n\right) = K\left|\frac{\sqrt{2}}{3} A_{3/2}\right|^{2} = \frac{2}{9}\left|A_{3/2}\right|^{2}$

c) $\sigma\left(\pi^{-} + p \rightarrow \pi^{-} + p\right) = K\left|\frac{1}{3} A_{3/2}\right|^{2} = \frac{1}{9}\left|A_{3/2}\right|^{2}$

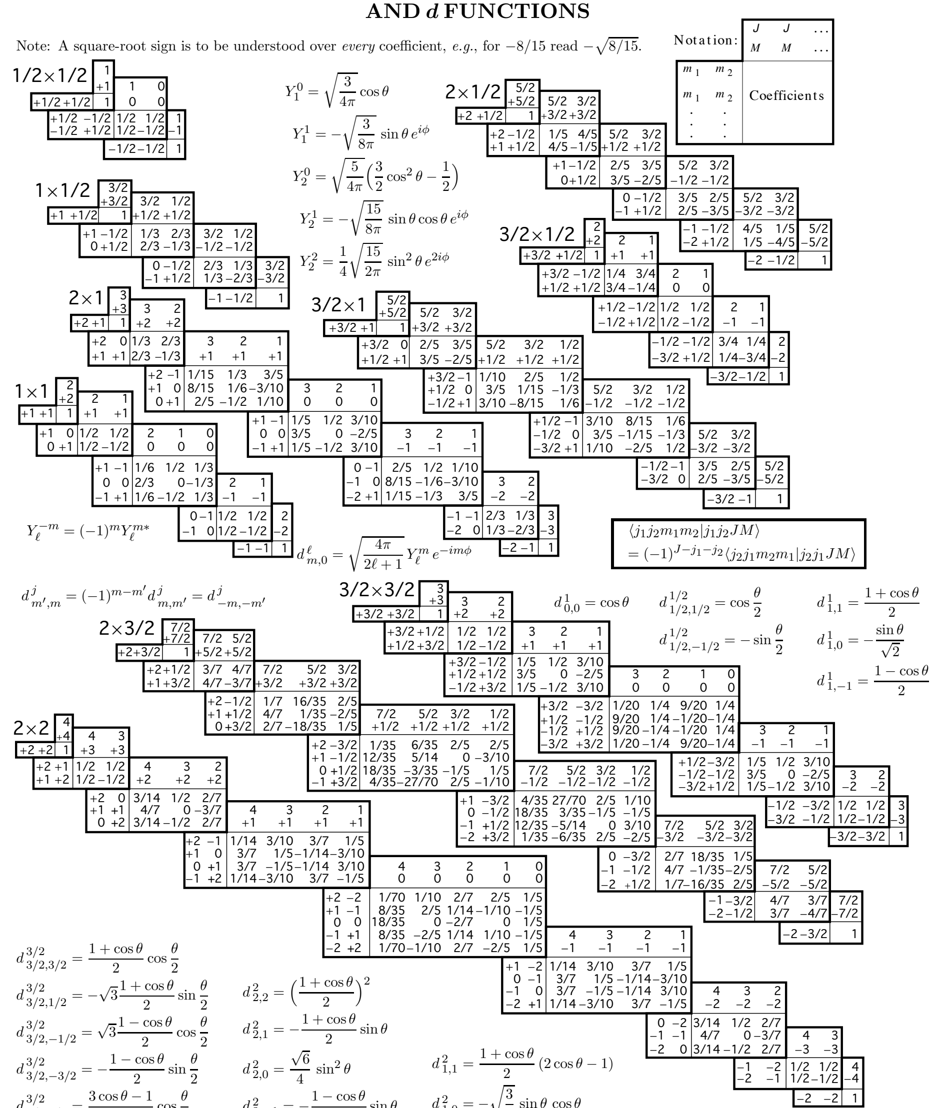
d) $\sigma\left(\pi^{-} + n \rightarrow \pi^{-} + n\right) = K\left|A_{3/2}\right|^{2}$

$$\frac{\sigma\left(\pi^{+} + p \rightarrow \pi^{+} + p\right)}{\sigma\left(\pi^{-} + p \rightarrow \pi^{-} + p\right)} = 9 \quad ; \quad \frac{\sigma\left(\pi^{-} + p \rightarrow \pi^{0} + n\right)}{\sigma\left(\pi^{-} + p \rightarrow \pi^{-} + p\right)} = 2$$

Clebsch-Gordan Coefficient

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

Exercises

- We have a system composed of one Σ^- and one proton. Write the wave function of the system in terms of the total isospin states of the system and calculate the probability of finding the system in a total isotope spin state 1/2

  - The Σ^- has I=1 e I_3=-1 (see Gell-mann-Nishijima formula) , while the proton has I=1/2 e I_3 = +1/2 . Combining together the two states we can have has total isospin ½ or 3/2 with the third component I_3 equal to -1/2.

$$|\Sigma^- p\rangle = \sqrt{\frac{1}{3}}\,|\frac{3}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|\frac{1}{2},-\frac{1}{2}\rangle$$

  - The probability to find the system in a state of total isospin ½ is 2/3.

Exercises

- The baryon Λ decays into proton - π^- or into neutron - π^0 In the decay the s quark of the Λ changes into a u quark of the nucleon, so its strong isospin varies by ½. Assuming that in the Λ decay this selection rule is respected and neglecting other corrections, what is the relationship that would be expected between the B.R. in p- π^- compared to that in n- π^0?

Exercises

- The barion Λ decays in proton– π^- or neutron-π^0. In the decay a s-quark of the Λ transforms into a u-quark of the nucleon, therefore its strong isospin changes by ½. Assuming that in the Λ decays this selection rule is retained, and ignoring other small correction, deduce the ratio of the B.R. of the p- π^- decay with respect to the n- π^0 decay.

- The nucleone has isospin ½ while the pion has isospin 1, therefore a nucleon plus a pion can give total isospin equal to ½ or 3/2. The Λ has isospin zero, so in the total wave function of the system nucleon-pion we need to take into account only the component with isospin ½ , due to the selection rule $\Delta I = ½$.

$$p+\pi^- = \left|\frac{1}{2};\frac{1}{2}\right\rangle + \left|1;-1\right\rangle = -\sqrt{\frac{2}{3}}\left|\frac{1}{2};-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{3}{2};-\frac{1}{2}\right\rangle \qquad n+\pi^0 = \left|\frac{1}{2};-\frac{1}{2}\right\rangle + \left|1;0\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{1}{2};-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{3}{2};-\frac{1}{2}\right\rangle$$

- The transition probability is equal to the square of the w.f.:

$$\frac{B.R.\left(\Lambda \to p+\pi^-\right)}{B.R.\left(\Lambda \to n+\pi^0\right)} = \frac{\left|\left\langle p+\pi^- \mid \frac{1}{2};-\frac{1}{2}\right\rangle\right|^2}{\left|\left\langle n+\pi^0 \mid \frac{1}{2};-\frac{1}{2}\right\rangle\right|^2} = \frac{2/3}{1/3} = 2$$

- The experimental values are:

$$B.R.\left(\Lambda \to p+\pi^-\right) = 63.9\% \quad ; B.R.\left(\Lambda \to n+\pi^0\right) = 35.8\%$$

- $$\frac{B.R.\left(\Lambda \to p+\pi^-\right)}{B.R.\left(\Lambda \to n+\pi^0\right)} = \frac{63.9}{35.8} = 1.78$$

(most likely there is a higher order contribution with ΔI=3/2)

Exercises

- The K^0_S can decay into two charged pions or into two neutral pions. Find the relationship between the B.R. of the decay in neutral pions compared to that in charged pions. Remember that for symmetry reasons the final state must have zero total isospin

- Deduce through which isospin channels the following two reactions may occur:

$$\text{a) } K^- + p \to \Sigma^0 + \pi^0 \quad ; \quad \text{b) } K^- + p \to \Sigma^+ + \pi^-$$

- In the event that the dominant channel is the one with isospin 0 for both reactions, find the ratio between the cross sections σ_a / σ_b

Exercises

Il K^0_S può decadere in due pioni carichi oppure in due pioni neutri. Trovare il rapporto tra il B.R. del decadimento in pioni neutri rispetto a quello in pioni carichi. Si ricorda che per ragioni di simmetria lo stato finale deve avere isospin totale zero

Nei decadimento deboli con $\Delta S=1$ si ha $\Delta I=1/2$, quindi dato che il K ha $I=1/2$, lo stato finale dei due pioni deve avere $I=0$ oppure $I=1$. La funzione d'onda dei due pioni deve essere simmetrica rispetto allo scambio delle due particelle, quindi dato che essi hanno spin zero e si trovano in uno stato di momento angolare $l=0$, anche la parte di isospin deve essere simmetrica, quindi $I=0$.

Utilizzando i coefficienti di Clebsh-Gordan si ha:

$$|0;0\rangle = +\sqrt{\frac{1}{3}}|1,+1;1-1\rangle - \sqrt{\frac{1}{3}}|1,0;1,0\rangle + \sqrt{\frac{1}{3}}|1,-1;1+1\rangle = \; = +\sqrt{\frac{1}{3}}\pi^+\pi^- - \sqrt{\frac{1}{3}}\pi^0\pi^0 + \sqrt{\frac{1}{3}}\pi^-\pi^+$$

Di conseguenza abbiamo:

$$\frac{B.R.\left(K^0_S \rightarrow \pi^0 + \pi^0\right)}{B.R.\left(K^0_S \rightarrow \pi^+ + \pi^-\right)} = \frac{\left|\langle \pi^0\pi^0 \mid 0;0\rangle\right|^2}{\left|\langle \pi^+\pi^- \mid 0;0\rangle\right|^2} = \frac{1}{2}$$

I valori sperimentali sono:

$$B.R.\left(K^0_S \rightarrow \pi^0 + \pi^0\right) = 30.7\% \;\; ; \; B.R.\left(K^0_S \rightarrow \pi^+ + \pi^-\right) = 69.2\%$$

$$\frac{B.R.\left(K^0_S \rightarrow \pi^0 + \pi^0\right)}{B.R.\left(K^0_S \rightarrow \pi^+ + \pi^-\right)} = \frac{30.7}{69.2} = 0.44$$

Probabilmente vi è un contributo di ordine superiore con $\Delta I=3/2$

Exercises

Dedurre attraverso quali canali di isospin possono avvenire le seguenti due reazioni:

a) $K^- + p \rightarrow \Sigma^0 + \pi^0$; b) $K^- + p \rightarrow \Sigma^+ + \pi^-$

Nel caso in cui il canale dominante sia quello con isospin 0 per entrambe le reazioni, trovare il rapporto tra le sezioni d'urto σ_a/σ_b

Ricordiamo l'isospin totale e la terza componente delle particelle coinvolte nella reazione e scriviamo lo stato iniziale ed i due stati finali in termini degli autostati di isospin utilizzando i coefficienti di Clebsh-Gordan.

$$K^- = \left|I = \tfrac{1}{2}; I_3 = -\tfrac{1}{2}\right\rangle \;;\; p = \left|I = \tfrac{1}{2}; I_3 = \tfrac{1}{2}\right\rangle \;\Rightarrow\; K^- + p = +\sqrt{\frac{1}{2}}|1;0\rangle - \sqrt{\frac{1}{2}}|0;0\rangle$$

$$\Sigma^0 = |I = 1; I_3 = 0\rangle \;;\; \pi^0 = |I = 1; I_3 = 0\rangle \;\Rightarrow\; \Sigma^0 + \pi^0 = +\sqrt{\frac{2}{3}}|2;0\rangle - \sqrt{\frac{1}{3}}|0;0\rangle$$

$$\Sigma^+ = |I = 1; I_3 = 1\rangle \;;\; \pi^- = |I = 1; I_3 = -1\rangle \;\Rightarrow\; \Sigma^+ + \pi^- = +\sqrt{\frac{1}{6}}|2;0\rangle + \sqrt{\frac{1}{2}}|1;0\rangle + \sqrt{\frac{1}{3}}|0;0\rangle$$

Di conseguenza la reazione a) può avvenire soltanto attraverso il canale di isospin totale 0, mentre la reazione b) può avvenire attraverso il canale con isospin 0 ed anche con isospin 1.

Nel caso in cui il canale dominante sia quello con isospin 0 per entrambe le reazioni, allora il rapporto tra le sezioni d'urto è pari al rapporto dei quadrati dei coefficienti di C.G. dell'autostato di isospin 0 nei due stati finali:

$$\frac{\sigma_a}{\sigma_b} = \left|\frac{\langle \Sigma^0 + \pi^0 | 0;0\rangle}{\langle \Sigma^+ + \pi^- | 0;0\rangle}\right|^2 = \left|\frac{-\sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}}}\right|^2 = 1$$

Exercises

Verify the conserved quantities in the reaction $\pi^- + p \to \pi^0 + n$. I s the process allowed?

Does the reaction $d + d \to {}^4\mathrm{He} + \pi^0$ conserve isospin?

Compute the isospin balance for $\Sigma^0 \to \Lambda + \gamma$.

Strangeness S

Strange particles are copiously produced in strong interactions

They have a long lifetime, typical of a weak decay.

S quantum number: strangeness *conserved in strong and electromagnetic interactions*, *not conserved in weak interactions*.

Example: $\pi^- + p \rightarrow \Lambda + K^0$

$$\Lambda \rightarrow p + \pi^- \qquad \tau = 2.6 \times 10^{-10}\, s$$

$$\begin{array}{cccc} & \Lambda \rightarrow & p + & \pi^- \\ I & 0 & \frac{1}{2} & 1 \\ I_3 & 0 & \frac{1}{2} & -1 \end{array}$$

I = 0, because the Λ has no charged counterparts

$$\begin{array}{ccccc} & \pi^- + & p \rightarrow & \Lambda + & K^0 \\ I & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ I_3 & -1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}$$

Strangeness S

$$\left.\begin{array}{ll} K^0, K^+ & \dfrac{Q}{e} = I_3 + \dfrac{1}{2} \\ \overline{K}^0, K^- & \dfrac{Q}{e} = I_3 - \dfrac{1}{2} \end{array}\right\} \qquad \frac{Q}{e} = I_3 + \frac{B+S}{2}$$

(Gell-Mann Nishijima)

Y = B+S hypercharge

Using the Gell-Mann Nishijima formula strangeness is assigned together with isospin.

Example.:

n, p	$S = 0$	$I = \frac{1}{2}$
Λ	$S = -1$	$I = 0$
K^0, K^+	$S = 1$	$I = \frac{1}{2}$
$K^-, \overline{K}^0$	$S = -1$	$I = \frac{1}{2}$

Example of strangeness conservation:

	K^-	$+\, p$	$\rightarrow \Lambda$	$+\, \pi^0$
S	-1	0	-1	0
I_3	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0

	$\pi^\pm$	$+\, p$	$\rightarrow \Sigma^\pm$	$+\, K^+$
S	0	0	-1	$+1$
I	1	$\frac{1}{2}$	1	$\frac{1}{2}$
I_3	± 1	$\frac{1}{2}$	± 1	$\frac{1}{2}$

$\Sigma^0 \rightarrow \Lambda + \gamma \quad e.m.$

	Σ^0	Λ	γ
S	-1	-1	0

$\Sigma^+ \rightarrow n + \pi^+ \quad weak$

	Σ^+	n	π^+
S	-1	0	0

$\Xi^- \rightarrow \Lambda + \pi^- \quad weak$

	Ξ^-	Λ	π^-
S	-2	-1	0

G-parity G

$$G = Ce^{i\pi I_2}$$

Rotation of π around the 2 axis in isospin space followed by charge conjugation.

$$I_3 \xrightarrow{e^{i\pi I_2}} -I_3 \xrightarrow{C} I_3$$

Consider an isospin state $\chi(I, I_3=0)$: under isospin rotations this state behaves like $Y^0_l(\theta,\varphi)$ (under rotations in ordinary space)

The rotation around the 2 axis implies:

$$\vartheta \to \pi - \vartheta \quad \varphi \to \pi - \varphi$$

$$Y_l^0 \to (-1)^l Y_l^0$$

therefore

$$\chi(I,0) \to (-1)^I \chi(I,0)$$

G-parity G

Example: for a nucleon-antinucleon state the effect of C is to give a factor $(-1)^{l+s}$ (just is in the case of positronium). Therefore:

$$G\left|\psi(N\overline{N})\right\rangle = (-1)^{l+s+I}\left|\psi(N\overline{N})\right\rangle$$

This formula has **general validity**, not limited to the I_3=0 case.

For the π

$$G\left|\pi^+\right\rangle = \pm\left|\pi^+\right\rangle$$

$$G\left|\pi^-\right\rangle = \pm\left|\pi^-\right\rangle$$

$$G\left|\pi^0\right\rangle = \pm\left|\pi^0\right\rangle$$

For the π^0 C=+1 ($\pi^0 \to \gamma\gamma$), the rotation gives $(-1)^I=-1$(I=1) so that G= -1.

$$G_{\pi^0} = -1$$

It is the practice to assign the phases so that *all members of an isospin triplet have the same G-parity as the neutral member*.

$$G\left|\pi^\pm\right\rangle = -\left|\pi^\mp\right\rangle \qquad \text{with } C\left|\pi^\pm\right\rangle = -\left|\pi^\mp\right\rangle$$

G-parity G

Since the C operation reverses the sign of the baryon number B, **the eigenstates of G-parity must have baryon number zero B=0**.

G is a multiplicative quantum number, so for a system of n π

$$G=(-1)^n$$

$$\rho \to \pi\pi \qquad G_\rho = +1$$

$$\omega \to \pi\pi\pi \qquad G_\omega = -1 \qquad B.R. = 89\%$$

$$\omega \to \pi\pi \qquad G_f = +1 \qquad B.R. = 2.2\%$$

$\eta \to \gamma\gamma$ C=+1 which, with I=0, yields G=+1.

$$\eta \not\to \pi\pi \qquad viola\ P$$

$$\eta \to \pi\pi\pi \qquad viola\ G \Rightarrow e.m.$$

Conservation Laws

	Strong	E.M.	Weak
Energy/Momentum	✓	✓	✓
Electric Charge	✓	✓	✓
Baryon Number	✓	✓	✓
Lepton Number	✓	✓	✓
Isospin (I)	✓	✗	✗
Strangeness (S)	✓	✓	✗
Charm (C)	✓	✓	✗
Parity (P)	✓	✓	✗
Charge Conjugation (C)	✓	✓	✗
CP (or T)	✓	✓	✗
CPT	✓	✓	✓

Exercises

For each of the following reactions (a) establish whether it is allowed
(b) if it is not, give the reasons (there may be more than one), (c) establish the types of interaction that allow it: (1) $\pi^{-}p \rightarrow \pi^{0}+n$; (2) $\pi^{+} \rightarrow \mu^{+}+\nu_{\mu}$;(3) $\pi^{+} \rightarrow \mu^{+}+\nu^{-}{}_{\mu}$; (4) $\pi^{0} \rightarrow 2\gamma$; (5) $\pi^{0} \rightarrow 3\gamma$; (6) $e^{+}+e^{-} \rightarrow \gamma$; (7) $p+{}^{-}p \rightarrow \Lambda+\Lambda$; (8) $p+p \rightarrow \Sigma^{+}+\pi^{+}$; (9) $n \rightarrow p+e^{-}$; (10) $n \rightarrow p+\pi^{-}$.

(1) OK, S; (2) OK, W; (3) Violates $\mathcal{L}_{\mu}$; (4) OK, EM; (5) Violates $\mathcal{C}$; (6) Cannot conserve both energy and momentum; (7) Violates $\mathcal{B}$ and S (8) Violates $\mathcal{B}$ and S; (9) Violates J and Le; (10) Violates energy conservation.