MANDELSTAM VARIABLES

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In section 11.3, Coleman introduces an alternative way of looking at scattering processes using *Mandelstam variables*. In this post, we'll see how these variables are defined and cover some of their basic properties. Their application to scattering will be covered later.

We imagine a scattering process where there are two incoming particles that interact in some restricted region, resulting in two outgoing particles. The diagram for the interaction is given as Coleman's Fig. 11.4 which is simply four momentum lines labelled p_1 through p_4, all drawn as arrows pointing inwards towards the interaction region, which is just a 'blob' in the centre (Fig. 1).

This is an unconventional way of drawing the momenta, since usually we have two incoming momenta and another two outgoing momenta, but this new way of drawing the interaction makes sense if we view the two outgoing momenta as negative. In this case, conservation of energy-momentum requires that

$$p_1+p_2+p_3+p_4=0 \tag{1}$$

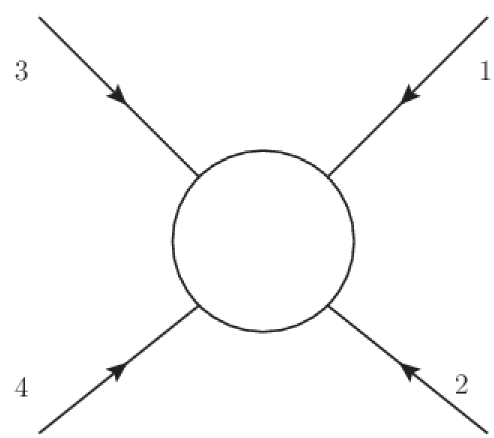

FIGURE 1. Momenta of 4 particles used in Mandelstam variables.

We can interpret such a diagram in several ways. It's easiest to analyze the situation in the centre of momentum frame. We could have p_1 and p_2 as the incoming momenta and p_3 and p_4 as the outgoing momenta. In this case, energy conservation gives us a quantity s:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{2}$$

If the 3-momenta of p_1 and p_2 are taken as positive, then the 3-momenta of p_3 and p_4 are negative. In that case, the momentum transfer is either direct, $p_1 + p_3$, or cross, $p_1 + p_4$. In the first case, we must have

$$t = (p_1 + p_3)^2 = (p_2 + p_4)^2 \tag{3}$$

and in the second case

$$u = (p_1 + p_4)^2 = (p_2 + p_3)^2 \tag{4}$$

The variables s, t and u are the Mandelstam variables. Coleman shows (by using $p_i^2 = m_i^2$) in equations 11.20 to 11.22 that they are related by

$$s + t + u = \sum_{i=1}^{4} m_i^2 \equiv M \tag{5}$$

where m_i is the mass of particle i.

Thus only 2 of the Mandelstam variables are independent, so the condition 5 is the equation of a plane in the 3-d stu space. The region of this plane where all three of s, t and u are positive forms an equilateral triangle, as in Fig. 2.

The boundaries of this triangle are given by the lines $s = 0$, $t = 0$ and $u = 0$ as shown. The top vertex of the triangle corresponds to $t = u = 0$, so from 5 we see that the height of the triangle is M.

Although we need only two unit vectors to specify a position in this plane, it is conventional to define three unit vectors $\boldsymbol{e}_s$, $\boldsymbol{e}_t$ and $\boldsymbol{e}_u$, each of which is perpendicular to the corresponding zero line, as shown in Fig. 2. As the angle between any two of these unit vectors is $\frac{2\pi}{3}$, we have, for $i \neq j$:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \cos\frac{2\pi}{3} = -\frac{1}{2} \tag{6}$$

To express the rectangular unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ in terms of the Mandelstam unit vectors, we observe that $\boldsymbol{e}_s$ points in the $\hat{\boldsymbol{y}}$ direction, so

$$\boldsymbol{e}_s = \hat{\boldsymbol{y}} \tag{7}$$

$\boldsymbol{e}_u$ makes an angle of $\frac{2\pi}{3}$ with $\boldsymbol{e}_s$ so it makes an angle of $\frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$ with the $+x$ axis. It can be written as

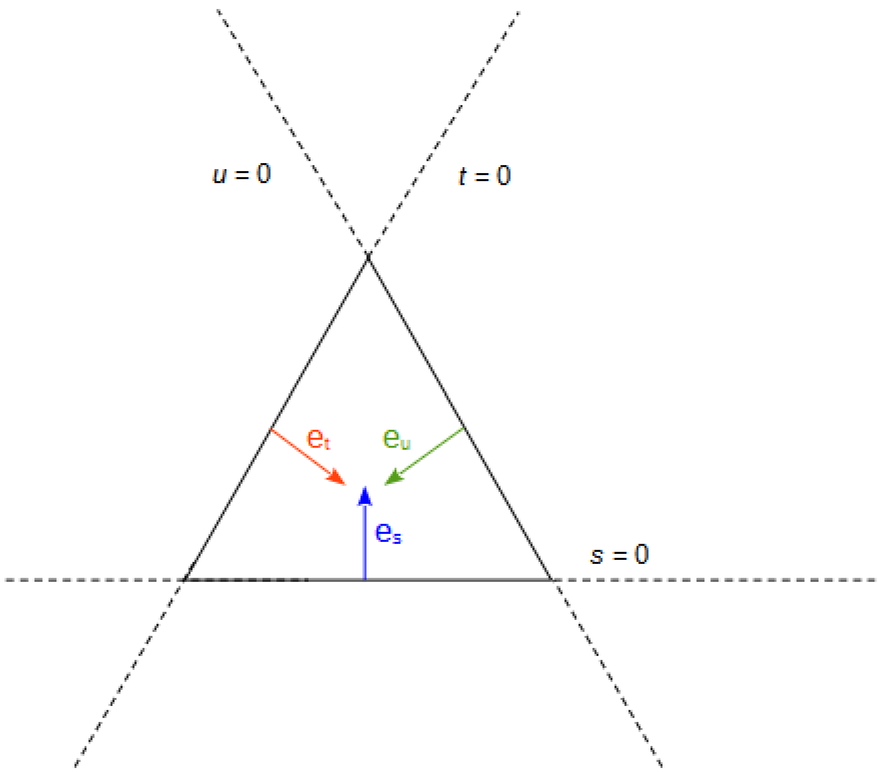

FIGURE 2. The Mandelstam plane.

$$\boldsymbol{e}_u = -\frac{\sqrt{3}}{2}\hat{\boldsymbol{x}} - \frac{1}{2}\hat{\boldsymbol{y}} \tag{8}$$

$\boldsymbol{e}_t$ is a further $\frac{2\pi}{3}$ along from $\boldsymbol{e}_u$, so it makes an angle of $\frac{11\pi}{6}$ or $-\frac{\pi}{6}$ with the $+x$ axis, so we have

$$\boldsymbol{e}_t = \frac{\sqrt{3}}{2}\hat{\boldsymbol{x}} - \frac{1}{2}\hat{\boldsymbol{y}} \tag{9}$$

Inverting these relations gives us

$$\hat{\boldsymbol{x}} = \frac{1}{\sqrt{3}}(\boldsymbol{e}_t - \boldsymbol{e}_u) \tag{10}$$

$$\hat{\boldsymbol{y}} = \boldsymbol{e}_s \tag{11}$$

We now need to see how to represent an arbitrary position vector in terms of the Mandelstam unit vectors. In order to give equal weight to all three

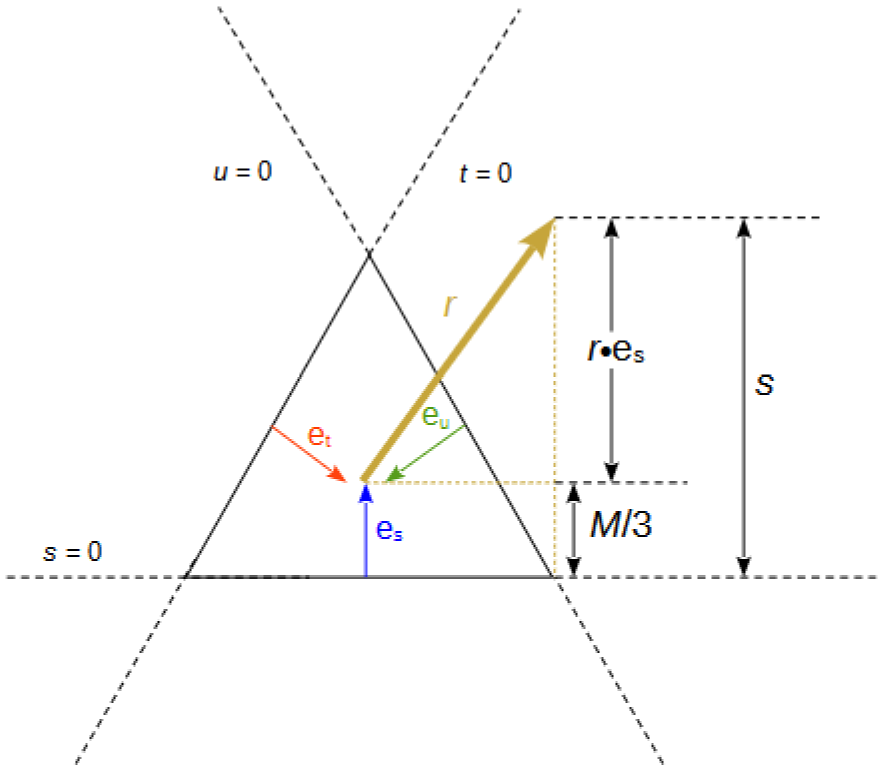

FIGURE 3. A position vector $\boldsymbol{r}$ in the Mandelstam plane.

variables, a position vector $\boldsymbol{r}$ is drawn with its origin at the centre of the triangle, as in Fig. 3.

For our purposes, the 'centre' of the triangle is the point obtained by the intersection of the lines drawn from a vertex to the midpoint of the opposite side. The centre point defined this way is always one third of the distance from the midpoint of the bisected side to the opposite vertex. For our equilateral triangle, all the bisecting lines are the same length M, so the distance from the midpoint of a side to the centre is always $\frac{M}{3}$.

Referring again to Fig. 3, we can see how the basic distances are defined for a position vector $\boldsymbol{r}$. We'll look at the component of $\boldsymbol{r}$ in the $\boldsymbol{e}_s$ direction, as the other two components are defined the same way. The component of $\boldsymbol{r}$ in the $\boldsymbol{e}_s$ direction is given by

$$r_s = \boldsymbol{r} \cdot \boldsymbol{e}_s \tag{12}$$

However, the quantity s is defined as this distance *plus* the distance from the centre of the triangle to the line $s=0$, which is $\frac{M}{3}$. That is, we have

$$s = \boldsymbol{r}\cdot\boldsymbol{e}_s + \frac{M}{3} \tag{13}$$

By the symmetry of the setup, the same relation applies to the other two components, so we have

$$\begin{aligned} s &= \boldsymbol{r}\cdot\boldsymbol{e}_s + \frac{M}{3} \\ t &= \boldsymbol{r}\cdot\boldsymbol{e}_t + \frac{M}{3} \\ u &= \boldsymbol{r}\cdot\boldsymbol{e}_u + \frac{M}{3} \end{aligned} \tag{14}$$

Using the above expressions for the unit vectors, we have

$$s = y + \frac{M}{3} \tag{15}$$

$$t = \frac{\sqrt{3}}{2}x - \frac{1}{2}y + \frac{M}{3} \tag{16}$$

$$u = -\frac{\sqrt{3}}{2}x - \frac{1}{2}y + \frac{M}{3} \tag{17}$$

Inverting these, we get the rectangular components of $\boldsymbol{r}$ in terms of the Mandelstam variables:

$$x = \frac{1}{\sqrt{3}}(t-u) \tag{18}$$

$$y = s - \frac{M}{3} \tag{19}$$

Putting all this together, we have, using the above expressions for the unit vectors and also 5:

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} \tag{20}$$

$$= \frac{1}{\sqrt{3}}(t-u)\frac{1}{\sqrt{3}}(\boldsymbol{e}_t - \boldsymbol{e}_u) + \left(s - \frac{M}{3}\right)\boldsymbol{e}_s \tag{21}$$

$$= \frac{1}{3}\left((t-u)\boldsymbol{e}_t + (u-t)\boldsymbol{e}_u\right) + \left(s - \frac{1}{3}(s+t+u)\right)\boldsymbol{e}_s \tag{22}$$

$$= \frac{1}{3}\left((t-u)\boldsymbol{e}_t + (u-t)\boldsymbol{e}_u\right) + \left(s - \frac{s}{3}\right)\boldsymbol{e}_s - \frac{1}{3}(t+u)\hat{\boldsymbol{y}} \tag{23}$$

Adding 8 and 9 we have

$$\hat{\boldsymbol{y}} = \boldsymbol{e}_t + \boldsymbol{e}_u \tag{24}$$

so we have

$$\boldsymbol{r} = \frac{1}{3}\left(\left(t-u\right)\boldsymbol{e}_t + \left(u-t\right)\boldsymbol{e}_u\right) + \frac{2s}{3}\boldsymbol{e}_s - \frac{1}{3}\left(t+u\right)\left(\boldsymbol{e}_t + \boldsymbol{e}_u\right) \tag{25}$$

$$= \frac{2}{3}\left(s\boldsymbol{e}_s + t\boldsymbol{e}_t + u\boldsymbol{e}_u\right) \tag{26}$$

We can verify this by direct substitution. For example, using 6 and 5, we have

$$\boldsymbol{r}\cdot\boldsymbol{e}_s = \frac{2}{3}\left(s\boldsymbol{e}_s\cdot\boldsymbol{e}_s + t\boldsymbol{e}_s\cdot\boldsymbol{e}_t + u\boldsymbol{e}_s\cdot\boldsymbol{e}_u\right) \tag{27}$$

$$= \frac{2}{3}\left(s - \frac{t}{2} - \frac{u}{2}\right) \tag{28}$$

$$= s - \frac{s}{3} - \frac{t}{3} - \frac{u}{3} \tag{29}$$

$$= s - \frac{1}{3}\left(s+t+u\right) \tag{30}$$

$$= s - \frac{M}{3} \tag{31}$$

which agrees with 13.

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