Introduction to Modern Cryptography
Summary

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Contents

1 Symmetric encryption

Kerkhoffs Principle: The security of a system should only depend on whether the actual key is secret, not on the system itself. The whole system is assumed to be public. No "Security by obscurity".

1.1 Scenario 1

One message with constant length

1.1.1 Cryptosystems

A cryptosystem is a tuple $S = (X, K, Y, e, d)$ with

- X: set of plaintexts
- K: finite set of keys
- Y: set of ciphertexts
- e: encryption function
- d: decryption function

Perfect correctness: $d(e(x,k),k) \forall x \in X, k \in K$

No unnecessary ciphertexts: $Y = \{e(x,k) | x \in X, k \in K\}$

1.1.2 Vernam system

The Vernam cryptosystem of length l is defined as $(\{0,1\}^l, \{0,1\}^l, \{0,1\}^l, e, d)$ where

$e(x,k) = x \oplus k$ and $d(y,k) = y \oplus k$.

A vernam system of length $l > 0$ provides perfect secrecy for every uniform P_K. It is the perfect system for Scenario 1.

1.1.3 Perfect Secrecy

A cryptosystem with key distribution $V = S[P_k]$ provides perfect secrecy if for all plaintext distributions P_X, the probability of every plaintext remains the same, i.e.:

$$P(x) = P(x|y) \quad \forall x \in X, y \in Y, P(y) > 0$$

Example Proof:

We need to show the criteria above for all plaintext distributions P_X. Therefore we use variable probabilities for the plaintexts $P_X(a) = p, P_X(b) = 1 - p$ (for 2 plaintexts, else $p_1, ..., p_n$).

	$K \backslash X$	a	b
$\frac{1}{2}$	k_0	A	B
$\frac{1}{2}$	k_1	B	A

$$\begin{aligned}
P(a|A) &= \frac{P(a,A)}{P(A)} &&= \frac{\frac{1}{2}*p}{\frac{1}{2}*p+\frac{1}{2}*(1-p)} = p = P(a) \\
P(a|B) &= \frac{P(a,B)}{P(B)} &&= \frac{\frac{1}{2}*p}{\frac{1}{2}*p+\frac{1}{2}*(1-p)} = p = P(a) \\
P(b|A) &= \frac{P(b,A)}{P(A)} &&= \frac{\frac{1}{2}*(1-p)}{\frac{1}{2}*(1-p)+\frac{1}{2}*p} = 1-p = P(b) \\
P(b|B) &= \frac{P(b,B)}{P(B)} &&= \frac{\frac{1}{2}*(1-p)}{\frac{1}{2}*(1-p)+\frac{1}{2}*p} = 1-p = P(b)
\end{aligned}$$

Theorem:

Let $S = (X, K, Y, e, d)$ be a cryptosystem providing perfect secrecy, then it holds $|K| \geq |Y| \geq |X|$.

Shannons Theorem:

Let $V = S[P_k]$ be a cryptosystem with key distribution P_K and $|K| = |Y| = |X|$. The system provides perfect secrecy if and only if

1. P_K is a uniform distribution
2. $\forall x \in X, y \in Y \exists k \in K$ with $e(x,k) = y$ (There must be a key for every plaintext/ciphertext pair)

1.2 Scenario 2

Multiple messages with constant length, no repetition

1.2.1 Vernam in Scenario 2

Vernam is not a secure cryptosystem anymore, since from 2 ciphertexts, Eve can learn non-trivial information about the plaintexts:

$$y_0 \oplus y_1 = x_0 \oplus k \oplus x_1 \oplus k = x_0 \oplus x_1$$

Also with 1 plaintext-ciphertext pair (CPA), the key can be calculated as $k = x \oplus y$.

1.2.2 Substitution Cryptosystem

Let X be a non-empty finite set. A substitution cryptosystem over X is a tuple (X, P_X, X, e, d) where P_X is the set of all permutations of X.

$$e(x,\pi) = \pi(x) \quad d(y,\pi) = \pi^{-1}(y) \quad \forall x, y \in X, \pi \in P_X$$

Substitution cryptosystems provide "perfect security" in scenario 2, BUT they are impractical because the substitution table (π) has a size of $2^l * l$.

Therefore, we need a weaker security definition that takes into account, that attackers are resource bound.

1.2.3 l-Block Cipher

Let $l : \mathbb{N} \to \mathbb{N}$ be a polynomial. An l-block cipher B is a cryptosystem of the form

$\left(\{0,1\}^{l(\eta)}_{\eta\in\mathbb{N}},\ Gen(1^\eta),\ \{0,1\}^{l(\eta)}_{\eta\in\mathbb{N}},\ E,\ D\right)$ or simplified: $\left(\{0,1\}^{l},\ Gen(1^\eta),\ \{0,1\}^{l},\ E,\ D\right)$

1.2.4 Substitution-Permutation Cryptosystem (SPCS)

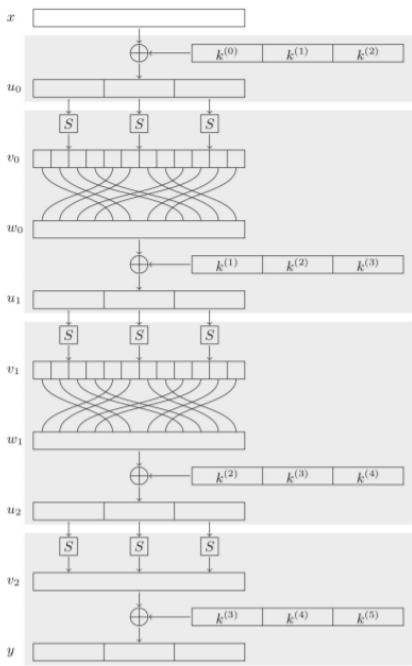
Notation:

- plaintexts are split into m words with length n with $l = m * n$, $x^{(i)}$ denotes the i'th word
- $[r] = \{0, 1, ..., r-1\}$
- $\beta \in P_{[l]}$, then $x^{\beta}(i) = x(\beta(i))$

General Principle: Over r rounds, (round) key additions, word substitutions and bit permutations are applied, including an initial step that just applies key addition and shortened last round without bit permutation.

$E(x : \{0,1\}^{mn}, k : \{0,1\}^{s}) : \{0,1\}^{mn}$

1. *initial white step (round key addition)*
   $u = x \oplus K(k, 0)$
2. $r-1$ *regular rounds*
   for $i = 1$ to $r-1$ do
   a. *word substitutions*
      for $j = 0$ to $m-1$ do
         $v^{(j)} = S(u^{(j)})$
   b. *bit permutation*
      $w = v^{\beta}$
   c. *round key addition*
      $u = w \oplus K(k, i)$
3. *shortened last round (without bit permutation)*
   for $j = 0$ to $m-1$ do
      $v^{(j)} = S(u^{(j)})$
   $y = v \oplus K(k, r)$; return y



Known Attacks:

- Brute Force Attack
- Linear Cryptanalysis
- Differential Cryptanalysis

Linear Cryptanalysis:

- Relies on a set T of plaintext-ciphertext pairs
- Instead of brute forcing the whole key, get small parts of the key at a time

- TODO

AES (Advanced encryption standard): basically SPCS with modifications

1.2.5 Algorithmic Security of Block Ciphers

We consider a block cipher secure if it is almost as good as a substitution cryptosystem w.r.t. resource-bound adversaries. Therefore an adversary U has to be able to distinguish BCS and SCS. Formally, we use the BCS for $b = 1$ (real world) and the SCS for $b = 0$ (random world) in the security game.
The winning probability is $Pr[E(1^n) = 1]$. Since a random guesser already has a probability of 0.5, the advantage is introduced to normalize.

$\mathbb{S}(1^\eta) : \{0, 1\}$

1. *Choose real world or random world.*
   $b \xleftarrow{\$} \{0, 1\}$
   if $b = 1$ then
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$ and $F = E(\cdot, k)$
   else
   $F \xleftarrow{\$} \mathcal{P}_{\{0,1\}^{l(\eta)}}$
2. *Guess phase.*
   $b' \xleftarrow{\$} U(1^\eta, F)$
3. *Output.*
   return b'.

$$Adv_{U,B}(\eta) = 2 * \left(Pr[E_U^B(1^\eta) = 1] - \frac{1}{2}\right) \in [-1, 1] \qquad suc_{U,B}(\eta) = Pr[S_U^B\langle b = 1\rangle(1^\eta) = 1]$$

$$Adv_{U,B}(\eta) = suc_{U,B}(\eta) - fail_{U,B}(\eta) \qquad fail_{U,B}(\eta) = Pr[S_U^B\langle b = 0\rangle(1^\eta) = 1]$$