Introduction to Modern Cryptography Summary

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Contents

1 Symmetric encryption

Kerkhoffs Principle: The security of a system should only depend on whether the actual key is secret, not on the system itself. The whole system is assumed to be public. No "Security by obscurity".

1.1 Scenario 1

One message with constant length

1.1.1 Cryptosystems

A cryptosystem is a tuple $\mathcal{S} = (X, K, Y, e, d)$ with

- X: set of plaintexts
- K: finite set of keys
- Y: set of ciphertexts
- e: encryption function
- d: decryption function

Perfect correctness: $d(e(x, k), k) \quad \forall x \in X, k \in K$

No unnecessary ciphertexts: $Y = \{e(x, k) | x \in X, k \in K\}$

1.1.2 Vernam system

The Vernam cryptosystem of length l is defined as $(\{0,1\}^l, \{0,1\}^l, \{0,1\}^l, e, d)$ where

$e(x, k) = x \oplus k$ and $d(y, k) = y \oplus k$.

A vernam system of length $l > 0$ provides perfect secrecy for every uniform P_K. It is the perfect system for Scenario 1.

1.1.3 Perfect Secrecy

A cryptosystem with key distribution $\mathcal{V} = \mathcal{S}[P_k]$ provides perfect secrecy if for all plaintext distributions P_X, the probability of every plaintext remains the same after the ciphertext is seen, i.e.:

$$P(x) = P(x|y) \quad \forall x \in X, y \in Y, P(y) > 0$$

Example Proof:

We need to show the criteria above for all plaintext distributions P_X. Therefore we use variable probabilities for the plaintexts $P_X(a) = p, P_X(b) = 1 - p$ (for 2 plaintexts, else $p_1, ..., p_n$).

	K \ X	a	b
$\frac{1}{2}$	k_0	A	B
$\frac{1}{2}$	k_1	B	A

$$\begin{aligned}
P(a|A) = \frac{P(a,A)}{P(A)} &= \frac{\frac{1}{2}*p}{\frac{1}{2}*p+\frac{1}{2}*(1-p)} = p = P(a) \\
P(a|B) = \frac{P(a,B)}{P(B)} &= \frac{\frac{1}{2}*p}{\frac{1}{2}*p+\frac{1}{2}*(1-p)} = p = P(a) \\
P(b|A) = \frac{P(b,A)}{P(A)} &= \frac{\frac{1}{2}*(1-p)}{\frac{1}{2}*(1-p)+\frac{1}{2}*p} = 1-p = P(b) \\
P(b|B) = \frac{P(b,B)}{P(B)} &= \frac{\frac{1}{2}*(1-p)}{\frac{1}{2}*(1-p)+\frac{1}{2}*p} = 1-p = P(b)
\end{aligned}$$

Theorem:

Let $\mathcal{S} = (X, K, Y, e, d)$ be a cryptosystem providing perfect secrecy, then it holds $|K| \geq |Y| \geq |X|$.

Shannons Theorem:

Let $\mathcal{V} = \mathcal{S}[P_k]$ be a cryptosystem with key distribution P_K and $|K| = |Y| = |X|$. The system provides perfect secrecy if and only if

1. P_K is a uniform distribution
2. $\forall x \in X, y \in Y \exists k \in K$ with $e(x,k) = y$ (There must be a key for every plaintext/ciphertext pair)

1.2 Scenario 2

Multiple messages with constant length, no repetition

1.2.1 Vernam in Scenario 2

Vernam is not a secure cryptosystem anymore, since from 2 ciphertexts, Eve can learn non-trivial information about the plaintexts:

$$y_0 \oplus y_1 = x_0 \oplus k \oplus x_1 \oplus k = x_0 \oplus x_1$$

Also with 1 plaintext-ciphertext pair (CPA), the key can be calculated as $k = x \oplus y$.

1.2.2 Substitution Cryptosystem

Let X be a non-empty finite set. A substitution cryptosystem over X is a tuple (X, P_X, X, e, d) where P_X is the set of all permutations of X.

$$e(x,\pi) = \pi(x) \quad d(y,\pi) = \pi^{-1}(y) \quad \forall x, y \in X, \pi \in P_X$$

Substitution cryptosystems provide “perfect security” in scenario 2, but they are impractical because the substitution table (π) has a size of $2^l * l$.

1.2.3 l-Block Cipher

Let $l : \mathbb{N} \to \mathbb{N}$ be a polynomial. An l-block cipher B is a cryptosystem of the form

$\left(\{0,1\}^{l(\eta)}_{\eta\in\mathbb{N}},\ Gen(1^\eta),\ \{0,1\}^{l(\eta)}_{\eta\in\mathbb{N}},\ E,\ D\right)$ or simplified: $\left(\{0,1\}^{l},\ Gen(1^\eta),\ \{0,1\}^{l},\ E,\ D\right)$

1.2.4 Substitution-Permutation Cryptosystem (SPCS)

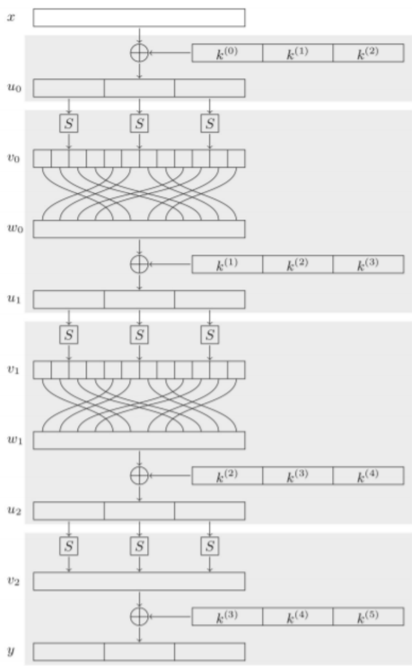
Notation:

- plaintexts are split into m words with length n with $l = m * n$, $x^{(i)}$ denotes the i'th word
- $[r] = \{0, 1, ..., r-1\}$
- $\beta \in \mathcal{P}_{[l]}$, then $x^\beta(i) = x(\beta(i))$

General Principle: Over r rounds, (round) key additions, word substitutions and bit permutations are applied, including an initial step that just applies key addition and shortened last round without bit permutation.

$E(x : \{0,1\}^{mn}, k : \{0,1\}^s) : \{0,1\}^{mn}$

1. *initial white step (round key addition)*
   $u = x \oplus K(k, 0)$
2. $r-1$ *regular rounds*
   for $i = 1$ to $r-1$ do
   a. *word substitutions*
      for $j = 0$ to $m-1$ do
         $v^{(j)} = S(u^{(j)})$
   b. *bit permutation*
      $w = v^\beta$
   c. *round key addition*
      $u = w \oplus K(k, i)$
3. *shortened last round (without bit permutation)*
   for $j = 0$ to $m-1$ do
      $v^{(j)} = S(u^{(j)})$
   $y = v \oplus K(k, r)$; return y



Known Attacks:

- Brute Force Attack
- Linear Cryptanalysis
- Differential Cryptanalysis

Linear Cryptanalysis:

- Relies on a set T of plaintext-ciphertext pairs
- Instead of brute forcing the whole key, get small parts of the key at a time
- Exploit linear dependencies

AES (Advanced encryption standard): basically SPCS with modifications

1.2.5 Algorithmic Security of Block Ciphers

We consider a block cipher secure if it is almost as good as a substitution cryptosystem w.r.t. resource-bound adversaries. Therefore no adversary U should be able to distinguish BCS and SCS. Formally, we use the BCS for $b = 1$ (real world) and the SCS for $b = 0$ (random world) in the security game.
The winning probability is $Pr[\mathbb{E}(1^n) = 1]$. Since a random guesser already has a probability of 0.5, the advantage is normalized.

$\mathbb{S}(1^\eta) : \{0,1\}$

1. *Choose real world or random world.*
   $b \xleftarrow{\$} \{0,1\}$
   if $b = 1$ then
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$ and $F = E(\cdot, k)$
   else
   $F \xleftarrow{\$} \mathcal{P}_{\{0,1\}^{l(\eta)}}$
2. *Guess phase.*
   $b' \xleftarrow{\$} U(1^\eta, F)$
3. *Output.*
   return b'.

$$Adv_{U,B}(\eta) = 2 * \left(Pr[\mathbb{E}_U^B(1^\eta) = 1] - \frac{1}{2} \right) \in [-1,1] \qquad suc_{U,B}(\eta) = Pr[\mathbb{S}_U^B\langle b = 1\rangle(1^\eta) = 1]$$

$$Adv_{U,B}(\eta) = suc_{U,B}(\eta) - fail_{U,B}(\eta) \qquad fail_{U,B}(\eta) = Pr[\mathbb{S}_U^B\langle b = 0\rangle(1^\eta) = 1]$$

1.2.6 PRP/PRF Switching Lemma

Since substitution cryptosystems cannot be distinguished from (secure) l-Block cryptosystems, we can see l-Block cryptosystems as pseudo-random permutations (PRP). Anyway, for proving purposes, it can be easier to see them as pseudo-random functions. The PRP/PRF Switching Lemma says, that we can use them interchangeably, since the difference of advantages is negligible:

Let B be an l-block cipher and U be an l-distinguisher with runtime bound $q(\eta)$ where q is a positive polynomial and $\eta \in \mathbb{N}$. Then the following holds true:

$$|Adv_{U,B}^{PRP}(\eta) - Adv_{U,B}^{PRF}(\eta)| \leq \frac{q(\eta)^2}{2^{l(\eta)+1}}$$

1.3 Scenario 3

Arbitrary messages with any length (possibly with repetition)

1.3.1 Symmetric Encryption Scheme

A symmetric encryption scheme is a tuple $S = (Gen(^\eta), E, D)$ with

- security parameter η
- ppt key generation algorithm $Gen(1^\eta)$
- ppt encryption algorithm $E(x : \{0,1\}^*, k : K) : \{0,1\}^*$
- dpt decryption algorithm $D(y : \{0,1\}^*, k : K) : \{0,1\}^*$

- and $D(E(x,k),k) = x$

E cannot be deterministic, because else we wouldn't be able to send the same message multiple times, i.e. the same plaintext encrypted under the same key should result in a different ciphertext (with a high probability).

1.3.2 Encryption Schemes from Stream Ciphers

Idea: Vernam is safe if we use every key just once. So using the key as seed of a random number generator, that generates a stream of random numbers, enables the usage of the vernam system for arbitrarily long messages.

1.3.2.1 Number generator

A number generator (NG) is a dpt algorithm of the Form $G : (s : \{0,1\}^\eta) : \{0,1\}^{p(\eta)}$ where p is the expansion factor.

1.3.2.2 PRNG-Distinguisher

TODO

1.3.3 Encryption Schemes from Block Ciphers

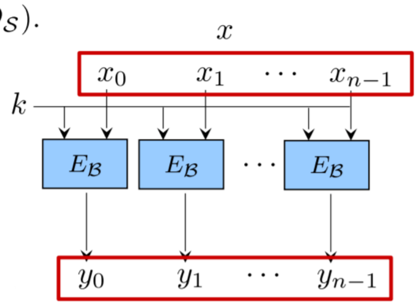
1.3.3.1 ECB Mode

Idea: Split the message in blocks of constant length and encrypt each block under the given key using the underlying block cipher.

$$\mathcal{S} = \mathsf{ECB}\text{-}\mathcal{B} = (\mathsf{Gen}_\mathcal{B}(1^\eta), E_\mathcal{S}, D_\mathcal{S}).$$

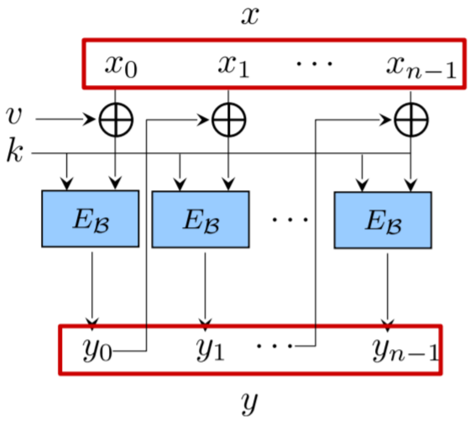
$E_\mathcal{S}(x : \{0,1\}^{l(\eta)+}, k : K_\mathcal{B}) : \{0,1\}^*$:

1. Split x into several blocks of length $l(\eta)$:
   $x =: x_0 || \cdots || x_{n-1}, n \in \mathbb{N}, x_i \in \{0,1\}^{l(\eta)}$
2. $y_i = E_\mathcal{B}(x_i, k) \quad \forall i \in \{0, \ldots, n-1\}$
3. **return** $y := y_0 || \ldots || y_{n-1}$



Security: It's not secure, since the ciphertext carries non-trivial information about the plaintext: for $y = y_0 || y_1$, then $y_0 = y_1$ if $x_0 = x_1$.

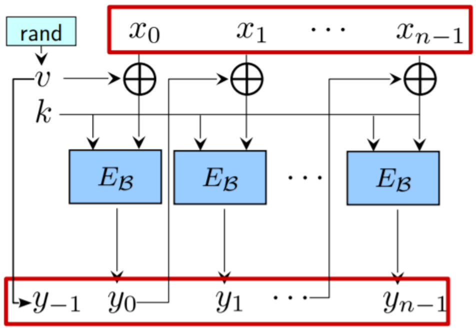
1.3.3.2 CBC Mode



Idea: Add and initialization vector v that is `xor`'ed with the plaintext before encrypting. That v is part of the key.

Problem: Still deterministic, so every plaintext can be sent just once.

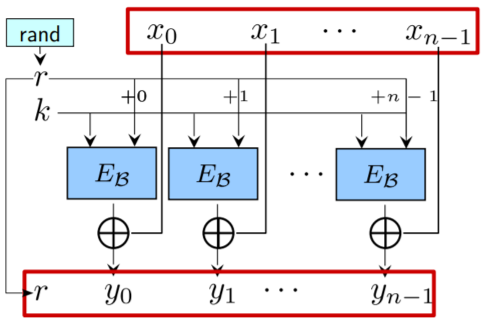
1.3.3.3 R-CBC Mode



Idea: To solve the issues of CBC-Mode, R-CBC moves the initialization vector v out of the key and generates a random one while decryption. The vector is appended as first block of the ciphertext to enable decryption.

Security: Its secure if the underlying block cipher is secure.

1.3.3.4 R-CTR Mode



Idea: Alternative to R-CBC. Generate a random number r (comparable to v of R-CBC), encrypt this random number under the key and xor it with the plaintext. The counter is increased by 1 for each block. The counter r is appended as first block of y to enable decryption.

Security: Its secure if the underlying block cipher is secure.

1.3.4 CPA-Security

CPA: Chosen-Plaintext-Attack

Game: Adversary A consists of finder AF and guesser AG. The finder chooses 2 plaintexts z_0, z_1. One of them is encrypted. The guesser has to determine which of them is the corresponding plaintext.

Advantage, success and failure are defined as for block ciphers.

$\mathbb{E}(1^\eta) : \{0,1\}$

1. *Choose cipher.*
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$; $H = E(\cdot, k)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, H)$
3. *Selection.*
   $b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} H(z_b)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, H, y)$
5. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

1.3.5 CCA-Security

CCA: Chosen-Ciphertext-Attack

Game: In addition to the encryption oracle H from the CPA-game, the adversary also gets a decryption oracle H^{-1}.

Advantage, success and failure are defined as for block ciphers.

$\mathbb{E}(1^\eta) : \{0, 1\}$

1. *Choose cipher.*
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$; $H = E(\cdot, k)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, H)$
3. *Selection.*
   $b \xleftarrow{\$} \{0, 1\}$; $y \xleftarrow{\$} H(z_b)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, H, y)$
5. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

1.3.6 Vaudenay's Padding Attack

- TODO

2 Number Theory

2.1 Fundamental Theorem of Arithmetic

Every natural number $n \in \mathbb{N}, n \geq 2$ has exactly one combination of prime factors.

$$n = p_1 * \dots * p_k \quad \text{with} k \leq log(n)$$

2.2 Modulo

Let $n \in \mathbb{N}\backslash\{0\}, a \in \mathbb{Z}$. Then $\exists! q \in \mathbb{Z}, r \in \{0, \dots, n-1\}$ such that $a = n * q + r$.

$$a \text{ div } n := q \qquad \text{and} \qquad a \bmod n := r$$

2.3 $\mathbb{Z}_n$

Let $n \geq 1$. We define the set $\mathbb{Z}_n := \{0, \dots, n-1\}$ of remainders of divisions by n. Let $a, b \in \mathbb{Z}_n$, then

$$a +_n b := (a + b) \bmod n \qquad \text{and} \qquad a *_n b := (a * b) \bmod n$$

2.4 Group

A tuple $(\mathcal{G}, \cdot)$ is called group if $\mathcal{G}$ is a non-empty set and $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a function such that:

- $(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall x, y, z \in \mathcal{G}$ (associativity)
- $\exists e \in \mathcal{G} : e \cdot x = x \cdot e = x \quad \forall x \in \mathcal{G}$ (neutral element)
- $\forall x \in \mathcal{G} \exists x^{-1} \in \mathcal{G} : x \cdot x^{-1} = e$ (inverse element)

The *order* of a group is the number of elements in $\mathcal{G}$.

The exponentiation is defined as usual. For a finite group $(\mathcal{G}, \cdot)$ with order n and neutral element e, the following holds true:

$$g^n = e \qquad \text{and} \qquad g^a = g^{a \bmod n}$$

2.5 Ring

A Ring is the tuple $(\mathcal{R}, +, \cdot)$ if $(\mathcal{R}, +)$ is an abelian (commutative) group and the function $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is associative, distributive and has a neutral element.

The set of invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^*$. The tuple $(\mathcal{R}^*, \cdot)$ is an abelian group called group of units.

2.6 Greatest common divisor

We say a divides b or $a|b$ if $\exists c \in \mathbb{Z} : b = c \cdot a$. The greatest common divisor is definded as

$$gcd(a, b) = max\{c : c|a \text{ and } c|b\} \text{ where } gcd(0, 0) := 0$$

The set of invertible elements of $\mathbb{Z}_n$ can be determined by the gcd.

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n | gcd(a, n) = 1\}$$

2.7 Eurler's Totient Function

Let $n \geq 2$. The Euler's totient function is defined by

$$\Phi(n) = |\mathbb{Z}_n^*| = (p_0 - 1) + p_0^{\alpha_0 - 1} \ldots (p_{r-1} - 1) + p_{r-1}^{\alpha_{r-1} - 1}$$

where $p_1, \ldots, p_{r-1}$ are primes and $n = p_0^{\alpha_0 - 1} \cdot \cdots \cdot p_{r-1}^{\alpha_{r-1} - 1}$. Let p be a prime, then $\Phi(p) = p - 1$.

2.8 Euclids Algorithm

Algorithm to calculate the gcd. Can be extended to calculate the inverse of an element in $\mathbb{Z}_n^*$.

1. Initialize loop:
   $a' = a, b' = b$

2. Loop: compute gcd by using that gcd(a, b) = gcd$(b, a \bmod b)$ for $b > 0$.
   Invariant: gcd(a, b) = gcd(a', b') and $a' \geq b' \geq 0$.
   while $b' \neq 0$:

   #Do one reduction step.
   $q = a'$ div b', $r = a' \bmod b'$
   $a' = b', b' = r$

2.9 Fast Exponentiation

Algorithm to efficiently compute the exponentiation of a group element. Let $\mathcal{G}$ be a group and $g \in \mathcal{G}, m \in \mathbb{N}$. It uses the fact, that $g^{2k} = (g^k)^2$. Instead of doing $2k$ multiplications, we can do $k+1$. This is applied recursively to minimize the number of exponentiations that need to be computed. To make the algorithm work with any k (not just powers of 2), we use the binary representation of the exponent, e.g.

$$13 = 2^0 + 2^2 + 2^3 = (1101)_2 \quad \Rightarrow \quad g^{13} = g^{2^0} \cdot g^{2^2} \cdot g^{2^3}$$

To compute g^m, the algorithm iterates over the bits of m. If the bit is one, multiply the result with the current factor. In any case, square the current factor.
The algorithm has a complexity of $\mathcal{O}(log(m))$.

1. Initialization:
   $i = l; h = 1; k = g$
2. Iterated squaring:
   while $i \geq 0$:
      if $b(i) = 1$
         $h = kh$
      $k = k^2$
      $i = i - 1$
3. Output:
   return h

Postcondition: $g^m = h$

2.10 Cyclic Groups

A group $\mathcal{G}$ is called cyclic, iff $\exists g \in \mathcal{G}$ such that $\langle g \rangle = \mathcal{G}$.

If $p = |\mathcal{G}|$ is prime, then $\mathcal{G}$ is a cyclic group.

$\mathbb{Z}_p^*$ is a cyclic group if p is prime.

2.10.1 Subgroups

Let $(\mathcal{G}, \cdot)$ be a finite group and $U \subseteq \mathcal{G}$.

Definition: The tuple $(U, \cdot)$ is a subgroup of $\mathcal{G}$ iff U is a group.

Lemma: The tuple $(U, \cdot)$ is a subgroup of $\mathcal{G}$ iff $1 \in U$ and $a \cdot b \in U \quad \forall a, b \in U$

Lagranges Theorem: If U is a subgroup of $\mathcal{G}$, then it holds true that $|U|||\mathcal{G}$.

2.10.2 Generated Groups and Generators

Let $\mathcal{G}$ be a group and $g \in \mathcal{G}$. By $\langle g \rangle$ we denote the smallest subgroup of $\mathcal{G}$ that contains g.

$$\langle g \rangle = \{1, g, g^{-1}, g^2, g^{-2}, \ldots\} \quad \text{and if } \mathcal{G} \text{ if inite: } \langle g \rangle = \{1, g, g^2, \ldots, g^{|\langle g \rangle|-1}\}$$

We call g a generator of $\mathcal{G}$ if $\langle g \rangle = \mathcal{G}$.

2.10.3 Finding Generators

We find generators for a group by guessing a group element and checking whether or not it is a generator. This can be evaluated by the equation

$$g^{n/p} \neq 1 \quad \forall p (\text{prime factors of n}) \text{ and } n = |\mathcal{G}|$$

GeneratorTest$(\mathcal{G}, g, n, P)$
Precondition: $\mathcal{G}$ a finite group, $g \in \mathcal{G}$, $n = |\mathcal{G}|$, $P =$ set of prime factors of $|\mathcal{G}|$.
  For $p \in P$ do
    $h =$ FastExponentiation$(\mathcal{G}, g, n/p)$
    If $h = 1$
      break and **return** "g is not a generator of $\mathcal{G}$."
  return: "g is generator of $\mathcal{G}$".