Introduction to Modern Cryptography
Summary

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Contents

1 Symmetric encryption

Kerkhoffs Principle: The security of a system should only depend on whether the actual key is secret, not on the system itself. The whole system is assumed to be public. No "Security by obscurity".

1.1 Scenario 1

One message with constant length

1.1.1 Cryptosystems

A cryptosystem is a tuple $\mathcal{S} = (X, K, Y, e, d)$ with

- X: set of plaintexts
- K: finite set of keys
- Y: set of ciphertexts
- e: encryption function
- d: decryption function

Perfect correctness: $d(e(x,k),k) \quad \forall x \in X, k \in K$

No unnecessary ciphertexts: $Y = \{e(x,k) | x \in X, k \in K\}$

1.1.2 Vernam system

The Vernam cryptosystem of length l is defined as $(\{0,1\}^l, \{0,1\}^l, \{0,1\}^l, e, d)$ where

$e(x,k) = x \oplus k$ and $d(y,k) = y \oplus k$.

A vernam system of length $l > 0$ provides perfect secrecy for every uniform P_K. It is the perfect system for Scenario 1.

1.1.3 Perfect Secrecy

A cryptosystem with key distribution $\mathcal{V} = \mathcal{S}[P_k]$ provides perfect secrecy if for all plaintext distributions P_X, the probability of every plaintext remains the same after the ciphertext is seen, i.e.:

$$P(x) = P(x|y) \quad \forall x \in X, y \in Y, P(y) > 0$$

Example Proof:

We need to show the criteria above for all plaintext distributions P_X. Therefore we use variable probabilities for the plaintexts $P_X(a) = p, P_X(b) = 1 - p$ (for 2 plaintexts, else $p_1, ..., p_n$).

K \ X		a	b
$\frac{1}{2}$	k_0	A	B
$\frac{1}{2}$	k_1	B	A

$$P(a|A) = \frac{P(a,A)}{P(A)} = \frac{\frac{1}{2} * p}{\frac{1}{2} * p + \frac{1}{2} * (1-p)} = p = P(a)$$
$$P(a|B) = \frac{P(a,B)}{P(B)} = \frac{\frac{1}{2} * p}{\frac{1}{2} * p + \frac{1}{2} * (1-p)} = p = P(a)$$
$$P(b|A) = \frac{P(b,A)}{P(A)} = \frac{\frac{1}{2} * (1-p)}{\frac{1}{2} * (1-p) + \frac{1}{2} * p} = 1 - p = P(b)$$
$$P(b|B) = \frac{P(b,B)}{P(B)} = \frac{\frac{1}{2} * (1-p)}{\frac{1}{2} * (1-p) + \frac{1}{2} * p} = 1 - p = P(b)$$

Theorem:

Let $\mathcal{S} = (X, K, Y, e, d)$ be a cryptosystem providing perfect secrecy, then it holds $|K| \geq |Y| \geq |X|$.

Shannons Theorem:

Let $\mathcal{V} = \mathcal{S}[P_k]$ be a cryptosystem with key distribution P_K and $|K| = |Y| = |X|$. The system provides perfect secrecy if and only if

1. P_K is a uniform distribution
2. $\forall x \in X, y \in Y \exists k \in K$ with $e(x,k) = y$ (There must be a key for every plaintext/ciphertext pair)

1.2 Scenario 2

Multiple messages with constant length, no repetition

1.2.1 Vernam in Scenario 2

Vernam is not a secure cryptosystem anymore, since from 2 ciphertexts, Eve can learn non-trivial information about the plaintexts:

$$y_0 \oplus y_1 = x_0 \oplus k \oplus x_1 \oplus k = x_0 \oplus x_1$$

Also with 1 plaintext-ciphertext pair (CPA), the key can be calculated as $k = x \oplus y$.

1.2.2 Substitution Cryptosystem

Let X be a non-empty finite set. A substitution cryptosystem over X is a tuple (X, P_X, X, e, d) where P_X is the set of all permutations of X.

$$e(x, \pi) = \pi(x) \quad d(y, \pi) = \pi^{-1}(y) \quad \forall x, y \in X, \pi \in P_X$$

Substitution cryptosystems provide "perfect security" in scenario 2, but they are impractical because the substitution table (π) has a size of $2^l * l$.

1.2.3 l-Block Cipher

Let $l : \mathbb{N} \to \mathbb{N}$ be a polynomial. An l-block cipher B is a cryptosystem of the form

$\left(\{0,1\}^{l(\eta)}_{\eta\in\mathbb{N}},\ Gen(1^\eta),\ \{0,1\}^{l(\eta)}_{\eta\in\mathbb{N}},\ E,\ D\right)$ or simplified: $\left(\{0,1\}^{l},\ Gen(1^\eta),\ \{0,1\}^{l},\ E,\ D\right)$

1.2.4 Substitution-Permutation Cryptosystem (SPCS)

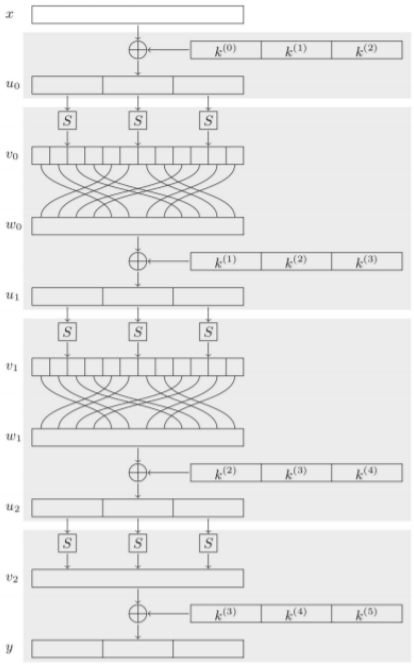
Notation:

- plaintexts are split into m words with length n with $l = m * n$, $x^{(i)}$ denotes the i'th word
- $[r] = \{0, 1, ..., r-1\}$
- $\beta \in \mathcal{P}_{[l]}$, then $x^\beta(i) = x(\beta(i))$

General Principle: Over r rounds, (round) key additions, word substitutions and bit permutations are applied, including an initial step that just applies key addition and shortened last round without bit permutation.

$E(x : \{0,1\}^{mn}, k : \{0,1\}^s) : \{0,1\}^{mn}$

1. *initial white step (round key addition)*
   $u = x \oplus K(k, 0)$
2. $r - 1$ *regular rounds*
   for $i = 1$ to $r - 1$ do
   a. *word substitutions*
      for $j = 0$ to $m - 1$ do
         $v^{(j)} = S(u^{(j)})$
   b. *bit permutation*
      $w = v^\beta$
   c. *round key addition*
      $u = w \oplus K(k, i)$
3. *shortened last round (without bit permutation)*
   for $j = 0$ to $m - 1$ do
      $v^{(j)} = S(u^{(j)})$
   $y = v \oplus K(k, r)$; return y



Importance of S-Box:

Without the S-Box the SPCS is a linear sequence of key additions and since the permutation table is known it is possible to generate a surrogate key based on a known plaintext/ciphertext pair. This is because the system without the S-Box is basically the same as permutating some of the round keys beforehand and then encrypting the plaintext with it. This would result in the same insecurity a Vernam system provides in this scenario.

Importance of bit permutation:

Without the bit permutation the words of the plaintext are encrypted independent of eachother and an adversary is able to construct two plaintexts with equal words at the ending. Since is reveals some non-trivial information to the adversary, by leaking information of some words of the plaintext, the system would be insecure. The advantage of the adversary is given by $adv(U, B) = succ(U, B) - fail(U, B) = 1 - \frac{1}{2^n}$.

Known Attacks:

- Brute Force Attack
- Linear Cryptanalysis
- Differential Cryptanalysis

Linear Cryptanalysis:

- Relies on a set T of plaintext-ciphertext pairs
- Instead of brute forcing the whole key, get small parts of the key at a time
- Exploit linear dependencies
- This can be found through the orientation
- The goal is to gather the best orientation by going through every step
- The parts are
- Parallel composition
- Bit permutation
- Key addition
- Sequential composition

AES (Advanced encryption standard): basically SPCS with modifications

1.2.5 Algorithmic Security of Block Ciphers

We consider a block cipher secure if it is almost as good as a substitution cryptosystem w.r.t. resource-bound adversaries. Therefore no adversary U should be able to distinguish BCS and SCS. Formally, we use the BCS for $b = 1$ (real world) and the SCS for $b = 0$ (random world) in the security game.
The winning probability is $Pr[\mathbb{E}(1^n) = 1]$. Since a random guesser already has a probability of 0.5, the advantage is normalized.

$\mathbb{S}(1^\eta) : \{0,1\}$
1. *Choose real world or random world.*
   $b \xleftarrow{\$} \{0,1\}$
   if $b = 1$ then
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$ and $F = E(\cdot, k)$
   else
   $F \xleftarrow{\$} \mathcal{P}_{\{0,1\}^{l(\eta)}}$
2. *Guess phase.*
   $b' \xleftarrow{\$} U(1^\eta, F)$
3. *Output.*
   return b'.

$$Adv_{U,B}(\eta) = 2 * \left(Pr[\mathbb{E}_U^B(1^\eta) = 1] - \frac{1}{2} \right) \in [-1, 1] \qquad suc_{U,B}(\eta) = Pr[\mathbb{S}_U^B\langle b = 1\rangle(1^\eta) = 1]$$

$$Adv_{U,B}(\eta) = suc_{U,B}(\eta) - fail_{U,B}(\eta) \qquad fail_{U,B}(\eta) = Pr[\mathbb{S}_U^B\langle b = 0\rangle(1^\eta) = 1]$$

1.2.6 PRP/PRF Switching Lemma

Since substitution cryptosystems cannot be distinguished from (secure) l-Block cryptosystems, we can see l-Block cryptosystems as pseudo-random permutations (PRP). Anyway, for proving purposes, it can be easier to see them as pseudo-random functions. The PRP/PRF Switching Lemma says, that we can use them interchangeably, since the difference of advantages is negligible:

Let B be an l-block cipher and U be an l-distinguisher with runtime bound $q(\eta)$ where q is a positive polynomial and $\eta \in \mathbb{N}$. Then the following holds true:

$$|Adv_{U,B}^{PRP}(\eta) - Adv_{U,B}^{PRF}(\eta)| \leq \frac{q(\eta)^2}{2^{l(\eta)+1}}$$

1.3 Scenario 3

Arbitrary messages with any length (possibly with repetition)

1.3.1 Symmetric Encryption Scheme

A symmetric encryption scheme is a tuple $S = (Gen(^\eta), E, D)$ with

- security parameter η
- ppt key generation algorithm $Gen(1^\eta)$
- ppt encryption algorithm $E(x : \{0,1\}^*, k : K) : \{0,1\}^*$
- dpt decryption algorithm $D(y : \{0,1\}^*, k : K) : \{0,1\}^*$
- and $D(E(x,k),k) = x$

E cannot be deterministic, because else we wouldn't be able to send the same message multiple times, i.e. the same plaintext encrypted under the same key should result in a different ciphertext (with a high probability).

1.3.2 Encryption Schemes from Stream Ciphers

Idea: Vernam is safe if we use every key just once. So using the key as seed of a random number generator, that generates a stream of random numbers, enables the usage of the vernam system for arbitrarily long messages.

1.3.2.1 Number generator A number generator (NG) is a dpt algorithm of the Form $G : (s : \{0,1\}^{\eta}) : \{0,1\}^{p(\eta)}$ where p is the expansion factor.

1.3.3 PRNG-Distinguisher

Defined like this: $U(1^{\eta}, x : \{1^{\eta}\}^{p(\eta)}) : \{0,1\}$

$\mathbb{E}_{U,G}^{PRNG}(1^{\eta}) : \{0,1\}$
1. *Choose real world or random world.*
   $b \xleftarrow{\$} \{0,1\}$
   if $b = 1$ then
     $s \xleftarrow{\$} \{0,1\}^{\eta}$ and $x = G(s)$
   else
     $x \xleftarrow{\$} \{0,1\}^{p(\eta)}$
2. *Guess phase.*
   $b' \xleftarrow{\$} U(1^{\eta}, x)$
3. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

With an advantage of:

$$
\begin{aligned}
Adv_{U,G}(\eta) &= 2 \cdot (Pr[\mathbb{E}_{U,G}^{PRNG}(1^{\eta}) = 1] - \frac{1}{2}) \\
suc_{U,G}(\eta) &= Pr[\mathbb{S}_{U,G}^{PRNG}\langle b = 1\rangle(1^{\eta}) = 1] \\
fail_{U,G}(\eta) &= Pr[\mathbb{S}_{U,G}^{PRNG}\langle b = 0\rangle(1^{\eta}) = 1] \\
Adv_{U,G}(\eta) &= suc_{U,G}(\eta) - fail_{U,G}(\eta)
\end{aligned}
$$

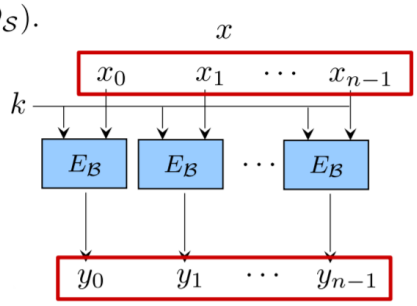
1.3.4 Encryption Schemes from Block Ciphers

1.3.4.1 ECB Mode **Idea**: Split the message in blocks of constant length and encrypt each block under the given key using the underlying block cipher.

$$\mathcal{S} = \text{ECB-}\mathcal{B} = (\text{Gen}_{\mathcal{B}}(1^\eta), E_{\mathcal{S}}, D_{\mathcal{S}}).$$

$E_{\mathcal{S}}(x : \{0,1\}^{l(\eta)+}, k : K_{\mathcal{B}}) : \{0,1\}^*$:

1. Split x into several blocks of length $l(\eta)$: $x =: x_0||\cdots||x_{n-1}, n \in \mathbb{N}, x_i \in \{0,1\}^{l(\eta)}$
2. $y_i = E_{\mathcal{B}}(x_i, k) \quad \forall i \in \{0, \ldots, n-1\}$
3. **return** $y := y_0||\ldots||y_{n-1}$

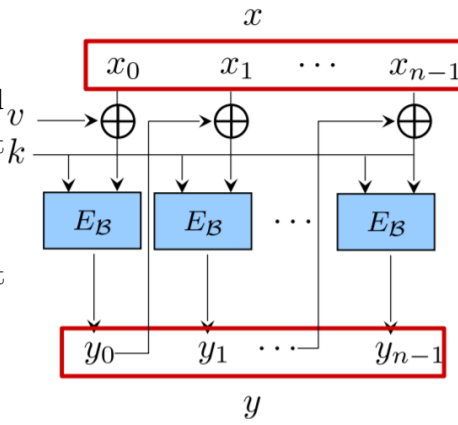


Security: It's not secure, since the ciphertext carries non-trivial information about the plaintext: for $y = y_0||y_1$, then $y_0 = y_1$ if $x_0 = x_1$.

1.3.4.2 CBC Mode

Idea: Add and initialization vector v that is `xor`'ed with the plaintext before encrypting. That v is part of the key.

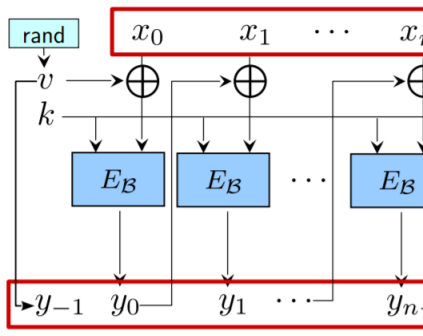
Problem: Still deterministic, so every plaintext can be sent just once.



1.3.4.3 R-CBC Mode

Idea: To solve the issues of CBC-Mode, R-CBC moves the initialization vector v out of the key and generates a random one while decryption. The vector is appended as first block of the ciphertext to enable decryption.

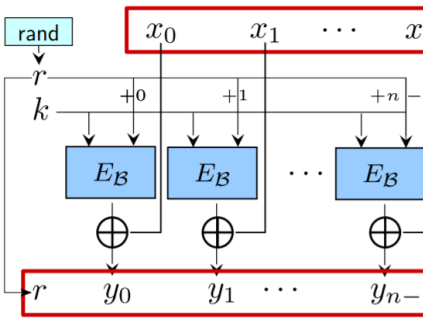
Security: Its secure if the underlying block cipher is secure.



1.3.4.4 R-CTR Mode

Idea: Alternative to R-CBC. Generate a random number r (comparable to v of R-CBC), encrypt this random number under the key and xor it with the plaintext. The counter is increased by 1 for each block. The counter r is appended as first block of y to enable decryption.

Security: Its secure if the underlying block cipher is secure.



1.3.5 CPA-Security

CPA: Chosen-Plaintext-Attack

Game: Adversary A consists of finder AF and guesser AG. The finder chooses 2 plaintexts z_0, z_1. One of them is encrypted. The guesser has to determine which of them is the corresponding plaintext.

Advantage, success and failure are defined as for block ciphers.

$\mathbb{E}(1^\eta) : \{0,1\}$

1. *Choose cipher.*
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$; $H = E(\cdot, k)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, H)$
3. *Selection.*
   $b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} H(z_b)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, H, y)$
5. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

1.3.6 CCA-Security

CCA: Chosen-Ciphertext-Attack

Game: In addition to the encryption oracle H from the CPA-game, the adversary also gets a decryption oracle H^{-1}.

Advantage, success and failure are defined as for block ciphers.

$\mathbb{E}(1^\eta) : \{0,1\}$

1. *Choose cipher.*
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$; $H = E(\cdot, k)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, H)$
3. *Selection.*
   $b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} H(z_b)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, H, y)$
5. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

1.3.7 Vaudenay's Padding Attack

Preconditions - The Vaudenay's attack is based on the fact, that the adversary might have a padding oracle - This padding oracle gives him information if the set padding is correct or not - In reality this can be a server error, delayed answer from server or any feedback from the decryption algorithm. - For this attack the encryption algorithm has blocks of length 16 bytes and the last bytes are always the padding to fill up the last block - Therefore, the padding bytes all represent the number of bytes needed to fill the last block [1, 16]

The attack The attack iterates over the numbers 2 - 16 and tries to pad the ciphertext with the padding based on the numbers. Thereby, it is possible to gather the set padding of the ciphertext and eliminate it right away. Now we get the ciphertext without the padding and can try to gather the plaintext. Therefore, the attacker again iterates over the numbers 1 - 16 in order to get the plaintext byte of the last byte. This process can now be repeated by the attacker for every block and byte in order to gather the whole plaintext.

2 Number Theory

2.1 Fundamental Theorem of Arithmetic

Every natural number $n \in \mathbb{N}, n \geq 2$ has exactly one combination of prime factors.

$$n = p_1 * \cdots * p_k \quad \text{with} k \leq log(n)$$

2.2 Modulo

Let $n \in \mathbb{N}\backslash\{0\}, a \in \mathbb{Z}$. Then $\exists! q \in \mathbb{Z}, r \in \{0, \dots, n-1\}$ such that $a = n * q + r$.

$$a \text{ div } n := q \qquad \text{and} \qquad a \bmod n := r$$

2.3 $\mathbb{Z}_n$

Let $n \geq 1$. We define the set $\mathbb{Z}_n := \{0, \dots, n-1\}$ of remainders of divisions by n. Let $a, b \in \mathbb{Z}_n$, then

$$a +_n b := (a + b) \bmod n \qquad \text{and} \qquad a *_n b := (a * b) \bmod n$$

2.4 Group

A tuple $(\mathcal{G}, \cdot)$ is called group if $\mathcal{G}$ is a non-empty set and $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a function such that:

- $(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \forall x, y, z \in \mathcal{G}$ (associativity)
- $\exists e \in \mathcal{G} : e \cdot x = x \cdot e = x \quad \forall x \in \mathcal{G}$ (neutral element)
- $\forall x \in \mathcal{G} \exists x^{-1} \in \mathcal{G} : x \cdot x^{-1} = e$ (inverse element)

The *order* of a group is the number of elements in $\mathcal{G}$.

The exponentiation is defined as usual. For a finite group $(\mathcal{G}, \cdot)$ with order n and neutral element e, the following holds true:

$$g^n = e \qquad \text{and} \qquad g^a = g^{a \bmod n}$$

2.5 Ring

A Ring is the tuple $(\mathcal{R}, +, \cdot)$ if $(\mathcal{R}, +)$ is an abelian (commutative) group and the function $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is associative, distributive and has a neutral element.

The set of invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^*$. The tuple $(\mathcal{R}^*, \cdot)$ is an abelian group called group of units.

2.6 Greatest common divisor

We say a divides b or $a|b$ if $\exists c \in \mathbb{Z} : b = c \cdot a$. The greatest common divisor is definded as

$$gcd(a, b) = max\{c : c|a \text{ and } c|b\} \text{ where } gcd(0, 0) := 0$$

The set of invertible elements of $\mathbb{Z}_n$ can be determined by the gcd.

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n | gcd(a, n) = 1\}$$

2.7 Eurler's Totient Function

Let $n \geq 2$. The Euler's totient function is defined by

$$\Phi(n) = |\mathbb{Z}_n^*| = (p_0 - 1) \cdot p_0^{\alpha_0 - 1} \dots (p_{r-1} - 1) \cdot p_{r-1}^{\alpha_{r-1} - 1}$$

where $p_1, \dots, p_{r-1}$ are primes and $n = p_0^{\alpha_0} \dots p_{r-1}^{\alpha_{r-1}}$. Let p be a prime, then $\Phi(p) = p - 1$.

It can also be used to calculate the number of generators in a cyclic group as $\Phi(n)$ where $n = |\mathcal{G}|$.

2.8 Euclids Algorithm

Algorithm to calculate the gcd. Can be extended to calculate the inverse of an element in $\mathbb{Z}_n^*$.

1. Initialize loop:
   $a' = a, b' = b$

2. Loop: compute gcd by using that $\gcd(a, b) = \gcd(b, a \bmod b)$ for $b > 0$.
   Invariant: $\gcd(a, b) = \gcd(a', b')$ and $a' \geq b' \geq 0$.
   while $b' \neq 0$:

   #Do one reduction step.
   $q = a' \text{ div } b', r = a' \bmod b'$
   $a' = b', b' = r$

2.9 Fast Exponentiation

Algorithm to efficiently compute the exponentiation of a group element. Let $\mathcal{G}$ be a group and $g \in \mathcal{G}, m \in \mathbb{N}$. It uses the fact, that $g^{2k} = (g^k)^2$. Instead of doing $2k$ multiplications, we can do $k+1$. This is applied recursively to minimize the number of exponentiations that need to be computed. To make the algorithm work with any k (not just powers of 2), we use the binary representation of the exponent, e.g.

$$13 = 2^0 + 2^2 + 2^3 = (1101)_2 \quad \Rightarrow \quad g^{13} = g^{2^0} \cdot g^{2^2} \cdot g^{2^3}$$

To compute g^m, the algorithm iterates over the bits of m. If the bit is one, multiply the result with the current factor. In any case, square the current factor.
The algorithm has a complexity of $\mathcal{O}(log(m))$.

1. Initialization:
   $i = l; h = 1; k = g$
2. Iterated squaring:
   while $i \geq 0$:
      if $b(i) = 1$
         $h = kh$
      $k = k^2$
      $i = i - 1$
3. Output:
   return h

Postcondition: $g^m = h$

2.10 Cyclic Groups

A group $\mathcal{G}$ is called cyclic, iff $\exists g \in \mathcal{G}$ such that $\langle g \rangle = \mathcal{G}$.

If $p = |\mathcal{G}|$ is prime, then $\mathcal{G}$ is a cyclic group.

$\mathbb{Z}_p^*$ is a cyclic group if p is prime.

2.10.1 Subgroups

Let $(\mathcal{G}, \cdot)$ be a finite group and $U \subseteq \mathcal{G}$.

Definition: The tuple $(U, \cdot)$ is a subgroup of $\mathcal{G}$ iff U is a group.

Lemma: The tuple $(U, \cdot)$ is a subgroup of $\mathcal{G}$ iff $1 \in U$ and $a \cdot b \in U \quad \forall a, b \in U$

Lagranges Theorem: If U is a subgroup of $\mathcal{G}$, then it holds true that $|U| \mid |\mathcal{G}|$.

2.10.2 Generated Groups and Generators

Let $\mathcal{G}$ be a group and $g \in \mathcal{G}$. By $\langle g \rangle$ we denote the smallest subgroup of $\mathcal{G}$ that contains g.

$$\langle g \rangle = \{1, g, g^{-1}, g^2, g^{-2}, \dots\} \quad \text{and if } \mathcal{G} \text{ if inite: } \langle g \rangle = \{1, g, g^2, \dots, g^{|\langle g \rangle|-1}\}$$

We call g a generator of $\mathcal{G}$ if $\langle g \rangle = \mathcal{G}$.

2.10.3 Finding Generators

We find generators for a group by guessing a group element and checking whether or not it is a generator. This can be evaluated by the equation

$$g^{n/p} \neq 1 \quad \forall \quad p \in P \text{ (prime factors of } n) \text{ and } n = |\mathcal{G}|$$

FindGenerator($\mathcal{G}$)
Precondition: $\mathcal{G}$ is a cyclic group.
loop
1. Select an element out of $\mathcal{G}$ at random
$g \xleftarrow{\$} \mathcal{G}$
2. Test whether g is a generator of $\mathcal{G}$.
if GeneratorTest(g) outputs "g is a generator of $\mathcal{G}$." **then**
return n
end if
end loop

GeneratorTest($\mathcal{G}, g, n, P$)
Precondition: $\mathcal{G}$ a finite group, $g \in \mathcal{G}$, $n = |\mathcal{G}|$, P = set of prime factors of $|\mathcal{G}|$.
For $p \in P$ do
$h =$ FastExponentiation$(\mathcal{G}, g, n/p)$
If $h = 1$
break and **return** "g is not a generator of $\mathcal{G}$."
return: "g is generator of $\mathcal{G}$".

This only works, because the probability of finding a generator when randomly sampling is relatively high, concretely it is upper-bounded by $Pr[\langle X_g \rangle = \mathcal{G}] \geq \frac{1}{1+log(n)}$ with $X_g \xleftarrow{\$} \mathcal{G}$.

2.10.4 Quadratic Residues

A number $a \in \mathbb{Z}_n^*$ *quadratic residue modulo* n iff there exists $b \in \mathbb{Z}_n$ such that $b^2 \bmod n = a \bmod n$. b is called root of a. All quadratic residues modulo n are denoted by $QR(n)$ and all elements that are *quadratic nonresidue modulo* n $QNR(n)$ with $\mathbb{Z}_n^* = QR(n) \cup QNR(n)$. For prime p, both sets have the same order:

$$|QR(p)| = |QNR(p)| = \frac{p-1}{2}$$

2.10.5 Eulers criterion

Let $p > 2$ be a prime number, $a \in \mathbb{Z}$ and $e = a^{(p-1)/2} \bmod p$

1. $e \in \{0, 1, p-1\}$
2. $a \bmod p = 0 \Leftrightarrow e = 0$
3. $a \bmod p \in QR(p) \Leftrightarrow e = 1$
4. $a \bmod p \in QNR(p) \Leftrightarrow e = -1 \bmod p$

We also call the criterion *Legendre symbol of a and p* $L_p(a) = a^{(p-1)/2}$.

3 Asymmetric Encryption

Symmetric encryption is efficient, but requires a key exchange that makes it impractical for most usecases. Asymmetric encryption works without key exchange, be splitting the key in public and private.

3.1 Asymmetric Encryption Scheme

As with symmetric encryption, the scheme is a tuple $\mathcal{S} = (X, Gen(1^\eta), E, D)$. The key gen algorithm now outputs a tuple $(k, \hat{k})$ (public, private), the encryption E uses the public key k and decrpytion D the private key $\hat{k}$.

3.2 Asymmetric CPA-Security

The security game is defined analogously to symmetric encryption. The key is now a tuple and the adversary knows the public key. Advantage, success and failure are defined as for Block Ciphers.

$\mathbb{E}(1^\eta) : \{0,1\}$

1. *Generate keys.*
   $(k, \hat{k}) \xleftarrow{\$} \mathsf{Gen}(1^\eta)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, k)$
3. *Selection.*
   $b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} E(z_b, k)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, k, y)$
5. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

3.3 RSA

RSA is an asymmetric encryption scheme $\mathcal{S}_{RSA} = (X, Gen(1^\eta), E, D)$. It uses the problem of computing prime factors as oneway function.

- The keygen algorithm randomly selects 2 different primes p and q of binary length η.

$$n = p \cdot q \qquad m = (p-1) \cdot (q-1) = \Phi(n)$$
$$e \xleftarrow{\$} \mathbb{Z}_m^* \qquad d = e^{-1} \mod m \qquad k = ((n,e),(n,d))$$

- $E(x,(n,e)) = x^e \mod n$
- $D(y,(n,d)) = y^d \mod n$

It has not been proven, that the RSA encryption cannot be inverted. An algorithm that attempts that is called inverter I. The advantage is assumed to be negligible, which is called RSA-Assumption.
$|Adv_{I,S}^{RSA}(\eta)| = Pr[\mathbb{E}_{I,S}^{RSA}(1^\eta) = 1]$

$\mathbb{E}(1^\eta) : \{0,1\}$

1. *Generate keys.*
   $((n,e),(n,d)) \xleftarrow{\$} \mathsf{Gen}(1^\eta)$
2. *Message selection.*
   $x \xleftarrow{\$} \mathbb{Z}_n$
   $y = x^e \mod n$
3. *Guess phase.*
   $x' \xleftarrow{\$} I(1^\eta, (n,e), y)$
4. *Evaluation.*
   if $x' = x$, return 1, otherwise 0.

Since the encryption function is deterministic, this textbook RSA is not secure. The scheme can be modified to be secure, by adding randomness. E.g. PKCS#1 v1.5 defines a random padding that fixes that issue:

$$0^{14}||10||r||0^8||x$$

While this is assumed to be sufficient for CPA-Security, RSA does not hold CCA-Security. In fact one doesn't even need a full decryption oracle as demonstrated in the Bleichenbacher Attack.

3.4 ElGamal

The ElGamal is an asymmetric encryption scheme that builds on the fact that it is hard to get g^{ab} from g^a and g^b, but easy to get from either g^a and b or g^b and a.

- The keygen algorithm randomly selects a $b \in \{0, \dots, n-1\}$, where $(\mathcal{G}, n, g)$ is the output of the GroupGen algorithm. The output is the public key $(\mathcal{G}, n, g, g^b)$ and the private key $(\mathcal{G}, n, g, b)$. In a real scenario this represents the keypair of the receiving person.
- The encryption function obtains another private key a randomly and calculates the public key g^a. In practice, this is the senders keypair. E encrypts the plaintext x by calculating $x \cdot (g^b)^a$
- The decryption function takes the senders public key, exponentiates it with the receivers private key and inverts the result to obtain $((g^a)^b)^{-1}$. This can be used to calculate the plaintext as $y \cdot ((g^a)^b)^{-1} = x \cdot (g^b)^a \cdot ((g^a)^b)^{-1} = x$.

Formally:

$E(x : \mathcal{G}, (\mathcal{G}, n, g, h) : K_{pub})$:
  $a \xleftarrow{\$} \{0, \dots, n-1\}$
  return $(g^a, x \cdot h^a)$.

$D((y_0, y_1) : \mathcal{G} \times \mathcal{G}, (\mathcal{G}, n, g, b) : K_{priv})$:
  return $y_1 \cdot \left((y_0)^b\right)^{-1}$.

4 Cheatsheet - 4 Gewinnt

4.1 Block-Cipher

4.1.1 Definition

Let $l : \mathbb{N} \to \mathbb{N}$ be a polynomial. Then the block-cipher is defined like this

$B = (\{0,1\}^l, Gen(1^\eta), E, D)$

Where $Gen(1^\eta)$ = probabilistic key generation

E = deterministic encrpytion

D = deterministic decryption

Only secure for scenario 2 (constant length no duplicated plaintext)

4.1.2 (Shortend) security game

$\mathbb{S}(1^\eta) : \{0,1\}$
1. *Choose real world or random world.*
   $b \xleftarrow{\$} \{0,1\}$
   if $b = 1$ then
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$ and $F = E(\cdot, k)$
   else
   $F \xleftarrow{\$} \mathcal{P}_{\{0,1\}^{l(\eta)}}$
2. *Guess phase.*
   $b' \xleftarrow{\$} U(1^\eta, F)$
3. *Output.*
   return b'.

4.1.3 Advantage

$$\begin{aligned} Adv_{U,B}(\eta) &= 2 \cdot (Pr[\mathbb{E}_U^B(1^\eta) = 1] - \frac{1}{2}) \\ &= Pr[\mathbb{S}_U^B\langle b = 1\rangle(1^\eta) = 1] - Pr[\mathbb{S}_U^B\langle b = 0\rangle(1^\eta) = 1] \end{aligned}$$

4.2 PRNG Game

4.2.1 Number Generator

Let $\eta \in \mathbb{N}$, p a polynomial and G a deterministic polynomial-time algorithm.

$G : (s : \{0,1\}^\eta) : \{0,1\}^{p(\eta)}$

p is expansion factor of G

4.2.2 PRNG-Distinguisher

Let $\eta \in \mathbb{N}$, p a polynomial and U is ppt algorithm.
$U(1^\eta, x : \{0,1\}^{p(\eta)}) : \{0,1\}$

4.2.3 (Shortend) Game

$\mathbb{S}_{U,G}^{PRNG}(1^\eta) : \{0,1\}$
1. *Choose real world or random world.*
   $b \xleftarrow{\$} \{0,1\}$
   if $b = 1$ then
     $s \xleftarrow{\$} \{0,1\}^\eta$ and $x = G(s)$
   else
     $x \xleftarrow{\$} \{0,1\}^{p(\eta)}$
2. *Guess phase.*
   $b' \xleftarrow{\$} U(1^\eta, x)$
3. *Output.*
   return b'.

4.2.4 Advantage

$$\begin{aligned} Adv_{U,G}(\eta) &= 2 \cdot (Pr[\mathbb{E}_{U,G}^{PRNG}(1^\eta) = 1] - \frac{1}{2}) \\ &= Pr[\mathbb{S}_{U,G}^{PRNG}\langle b = 1\rangle(1^\eta) = 1] - Pr[\mathbb{S}_{U,G}^{PRNG}\langle b = 0\rangle(1^\eta) = 1] \end{aligned}$$

4.3 CPA Security

4.3.1 Security game

$\mathbb{E}(1^\eta) : \{0,1\}$
1. *Choose cipher.*
   $k \xleftarrow{\$} \mathsf{Gen}(1^\eta)$; $H = E(\cdot, k)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, H)$
3. *Selection.*
   $b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} H(z_b)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, H, y)$
5. *Evaluation.*
   if $b' = b$, return 1, otherwise 0.

4.3.2 Adversary

Let $\eta \in \mathbb{N}$ and the adversary being a ppt algorithm

$A(1^\eta, H : \{0,1\}^* \leftarrow \{0,1\}^*) : \{0,1\})$

$(AF(1^\eta, H), AG(1^\eta, H, y : \{0,1\}^*))$

AF = finder

AG = guesser

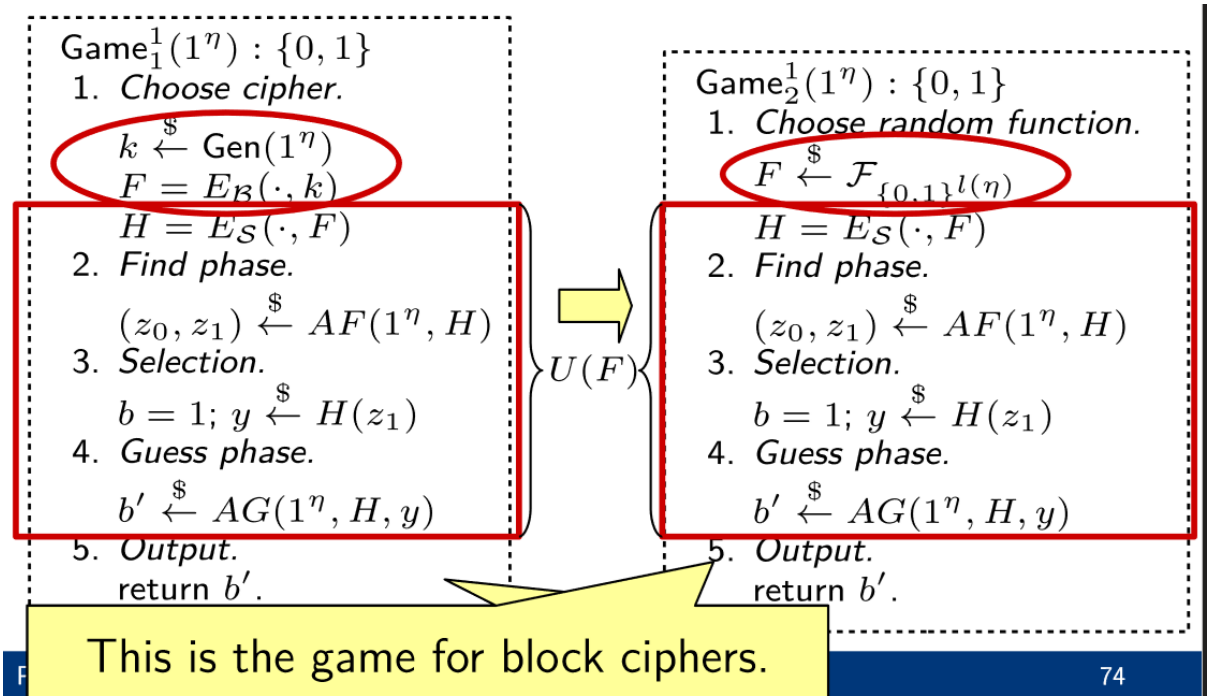
H = encryption oracle

4.3.3 Advantage

$$\begin{aligned} Adv_{U,G}(\eta) &= 2 \cdot (Pr[\mathbb{E}_{A,S}(1^\eta) = 1] - \frac{1}{2}) \\ &= Pr[\mathbb{S}_{A,S}\langle b = 1\rangle(1^\eta) = 1] - Pr[\mathbb{S}_{A,S}\langle b = 0\rangle(1^\eta) = 1] \end{aligned}$$

4.3.4 Proof

By security game switching from success game to failure game proofing that the distingusiher stays the same, but it is possible to change the Gen function to a pseudo-random function.



4.4 CCA

4.4.1 Security game

$\mathbb{E}(1^\eta) : \{0,1\}$
1. Choose cipher.
$k \xleftarrow{\$} \mathsf{Gen}(1^\eta); H = E(\cdot, k); H^{-1} = D(\cdot, k)$
2. Find phase.
$(z_0, z_1) \xleftarrow{\$} AF(1^\eta, H, H^{-1})$
3. Selection.
$b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} H(z_b)$
4. Guess phase.
$b' \xleftarrow{\$} AG(1^\eta, H, H^{-1}, y)$
5. Evaluation.
if $b' = b$ and A did not request $H^{-1}(y)$ in AG,
return 1, otherwise 0.

4.4.2 Adversary

Let $\eta \in \mathbb{N}$ and the adversary being a ppt algorithm

$A(1^\eta, H : \{0,1\}^* \leftarrow \{0,1\}^*) : \{0,1\})$

$(AF(1^\eta, HmH^{-1}), AG(1^\eta, H, y : \{0,1\}^*))$

AF = finder

AG = guesser

H = encryption oracle

H^{-1} = decryption oracle

4.4.3 Advantage

$$\begin{aligned} Adv_{U,G}(\eta) &= 2 \cdot (Pr[\mathbb{E}_{A,S}^{S-CCA}(1^\eta) = 1] - \frac{1}{2}) \\ &= Pr[\mathbb{S}_{A,S}^{S-CCA}\langle b = 1\rangle(1^\eta) = 1] - Pr[\mathbb{S}_{A,S}^{S-CCA}\langle b = 0\rangle(1^\eta) = 1] \end{aligned}$$

4.5 Asymmetric encryption scheme

4.5.1 Definition

A asymmetric encryption scheme $\mathcal{S}$ is a tuple $\mathcal{S} = (X, \text{Gen}(1^\eta), E, D)$ with security parameter $\eta \in \mathbb{N}$,

- Gen(1^η) is a ppt algorithm that outputs a pair of keys $(k, \hat{k})$. We call Gen(1^η) key generation algorithm.
  k is called public key, $\hat{k}$ is called private key. We denote the range of Gen(1^η) by K. We define $K_{pub} := \{k \mid (k, \hat{k}) \in K\}$ to be the set of all public keys, and
  $K_{priv} := \{\hat{k} \mid (k, \hat{k}) \in K\}$ to be the set of all private keys.
- $X = (X_k)_{k \in K_{pub}}$ a family of plaintext sets.
- a ppt encryption algorithm $E(x : \{0,1\}^*, k : K_{pub}) : \{0,1\}^*$,
- a deterministic polynomial-time decryption algorithm $D(y : \{0,1\}^*, \hat{k} : K_{priv}) : \{0,1\}^*$.

4.5.2 Security game

$\mathbb{S}(1^\eta) : \{0,1\}$

1. *Generate keys.*
   $(k, \hat{k}) \xleftarrow{\$} \mathsf{Gen}(1^\eta)$
2. *Find phase.*
   $(z_0, z_1) \xleftarrow{\$} AF(1^\eta, k)$
3. *Selection.*
   $b \xleftarrow{\$} \{0,1\}$; $y \xleftarrow{\$} E(z_b, k)$
4. *Guess phase.*
   $b' \xleftarrow{\$} AG(1^\eta, k, y)$
5. *Output.*
   return b'.

4.5.3 Advantage

$$\begin{aligned} Adv_{U,G}^{A-CPA}(\eta) &= 2 \cdot (Pr[\mathbb{E}_{A,S}^{A-CPA}(1^\eta) = 1] - \frac{1}{2}) \\ &= Pr[\mathbb{S}_{A,S}^{A-CPA}\langle b = 1\rangle(1^\eta) = 1] - Pr[\mathbb{S}_{A,S}^{S-CCA}\langle b = 0\rangle(1^\eta) = 1] \end{aligned}$$

4.6 RSA

4.6.1 Definition

Let $\eta \in \mathbb{N}$. The RSA (asymmetric) encryption scheme is the tuple

$$\mathcal{S}_{RSA} = (X, \mathsf{Gen}(1^\eta), E, D),$$

where

- $\mathsf{Gen}(1^\eta)$ selects two randomly chosen primes $p \neq q, p > 2, q > 2, |p|_2 = |q|_2 = \eta$, sets $n := p \cdot q$, $m := (p-1) \cdot (q-1)$ $[= \Phi(n)]$, chooses an element $e \in \mathbb{Z}_m^*$, computes $d = e^{-1} \mod m$, and outputs $((n,e),(n,d))$.
  We denote the range of $\mathsf{Gen}(1^\eta)$ by $K = \{((n,e),(n,d)) : n = p \cdot q$ for primes $p \neq q$, $e \cdot d \mod m = 1, m = \Phi(n)\}$.
- $X = \{X_{(n,e)}\}_{(n,e) \in K_{pub}}$ where $X_{(n,e)} = \mathbb{Z}_n$
- $E(x,(n,e)) = x^e \mod n, \quad (n,e) \in K_{pub}, x \in \mathbb{Z}_n$
- $D(y,(n,d)) = y^d \mod n, \quad (n,d) \in K_{priv}, y \in \mathbb{Z}_n$ ◇

4.7 ElGamal

4.7.1 Definition

ElGamal is CPA secure.

Let $\eta \in \mathbb{N}$. The ElGamal (asymmetric) encryption scheme based on GroupGen is the tuple $\mathcal{S}_{ElGamal} = (X, \text{Gen}(1^\eta), E, D)$, where

- $\text{Gen}(1^\eta)$ executes $\text{GroupGen}(1^\eta)$ and obtains $(\mathcal{G}, n, g)$. Then, $\text{Gen}(1^\eta)$ chooses $b \in \{0, \ldots, n-1\}$ uniformly at random and outputs $((\mathcal{G}, n, g, g^b), (\mathcal{G}, n, g, b))$. (Public key / Private key)
- $X = \{\mathcal{G}\}_{(\mathcal{G},n,g,h) \in K_{pub}}$. That is, the plaintexts are interpreted as elements of the group $\mathcal{G}$.
- $E(x : \mathcal{G}, (\mathcal{G}, n, g, h) : K_{pub})$:
  $a \xleftarrow{\$} \{0, \ldots, n-1\}$
  return $(g^a, x \cdot h^a)$.
- $D((y_0, y_1) : \mathcal{G} \times \mathcal{G}, (\mathcal{G}, n, g, b) : K_{priv})$:
  return $y_1 \cdot \left((y_0)^b\right)^{-1}$ ◇
  (inverse in $\mathcal{G}$)

4.7.2 Advantage

TODO

4.8 Hashes

4.8.1 Definition

Let $\eta \in \mathbb{N}$, $l : \mathbb{N} \to \mathbb{N}$ be a polynomial. A (cryptographic) hash function is a pair of the form $\mathcal{H} = (\text{Gen}(1^\eta), h)$ where

- $\text{Gen}(1^\eta)$ is a ppt algorithm that outputs a key k. We call $\text{Gen}(1^\eta)$ key generation algorithm. We denote the range of $\text{Gen}(1^\eta)$ by K.
- $h(k : K, x : \{0,1\}^*) : \{0,1\}^{l(\eta)}$ is a ppt algorithm. (We also write $h_k(x)$.)

The output of h is called a hash or a hash value of x.

Let $l' : \mathbb{N} \to \mathbb{N}$ be such that $l'(\eta) > l(\eta)$. If h_k is defined only for inputs $x \in \{0,1\}^{l'(\eta)}$, then we call $\mathcal{H} = (\text{Gen}(1^\eta), h)$ a compression function. ◇

4.8.2 Advantage

TODO

4.9 MAC

4.9.1 Definition

Let $\eta \in \mathbb{N}$ and $l : \mathbb{N} \to \mathbb{N}$ be a polynomial.
A message authentication code (MAC) is a tuple of the form $\mathcal{M} = (\text{Gen}(1^\eta), T, V)$, where

- $\text{Gen}(1^\eta)$ is a ppt algorithm that outputs a key k. We call $\text{Gen}(1^\eta)$ key generation algorithm. We denote the range of $\text{Gen}(1^\eta)$ by K.
- T is a deterministic polynomial time tag-generation algorithm of the form $T(x : \{0,1\}^*, k : K) : \{0,1\}^{l(\eta)}$ that outputs a tag $t \in \{0,1\}^{l(\eta)}$.
- V is a polynomial time verification algorithm of the form $V(x \in \{0,1\}^*, t \in \{0,1\}^{l(\eta)}, k : K) : \{\texttt{valid}, \texttt{invalid}\}$

such that the following holds true:

$\forall x \in X, k \in K : V(x, T(x,k), k) = \texttt{valid}.$

one can generalize this definition to probabilistic tag-generation algorithms.
However, MACs are usually deterministic

4.9.2 Advantage

TODO