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Focused sequent calculi [1, 2, 11] make use of syntactic restrictions on the applicability of inference rules achieving three main goals: (i) the proof search space is considerably reduced without losing completeness, (ii) every cut-free proof comes in a special normal form, (iii) a criterion for defining identity of sequent calculi proofs. Being able to identify or tell apart two proofs has far-reaching consequences.

We introduce a novel focused display calculus **fD.LG** and a *fully polarized* algebraic semantics (see the last paragraph of this abstract for more details on this) $\mathbb{FP}.\mathbb{LG}$ for Lambek-Grishin logic [12] by generalising the theory of *multi-type calculi* [5] and their algebraic semantics, admitting not only heterogeneous operators [4], but also *heterogeneous consequence relations* (see [9]) now interpreted as *weakening relations* [10] (i.e. a natural generalisation of partial orders). The calculus **fD.LG** has *strong focalization* and it is *sound and complete* w.r.t. $\mathbb{FP}.\mathbb{LG}$. This completeness result is in a sense stronger than completeness with respect to standard polarized algebraic semantics, insofar we do not need to quotient over proofs with consecutive applications of *shifts operator* over the same formula (see the last paragraph of this abstract for more details on this). We also show a number of additional results. **fD.LG** is sound and complete w.r.t. LG-algebras: this amounts to a semantic proof of the *completeness of focusing*, given that the standard (display) sequent calculus for Lambek-Grishin logic is complete w.r.t. LG-algebras. **fD.LG** and the focused calculus **fLG** of Moortgat and Moot are equivalent with respect to proofs, indeed there is an effective translation from **fLG**-derivations to **fD.LG**-derivations and vice versa: this provides the link with operational semantics, given that every **fLG**-derivation is in a Curry-Howard correspondence with a directional $\overline{\lambda}\mu\widetilde{\mu}$-term.

We conjecture that this approach, here tailored for the signature of the Lambek-Grishin logic, can be extended to a large class of logics, namely all lattice expansions logics extended with *analytic inductive axioms* (see [6]). We conjecture that if a calculus belongs to this class, then it enjoys cut-elimination, aiming at generalizing the cut-elimination meta-theorem in the tradition of display calculi (see [13]). Moreover, we conjecture that any *displayable logic* [6] can be equivalently presented as an instance of this class.

In what follows we summarise the main features of this analysis in general terms, without special reference to Lambek-Grishin logic. In the case of focused sequent calculi, the distinction between *positive* versus *negative* formulas is the key ingredient for organising proofs in *phases*. The distinction is proof-theoretically relevant in that it reflects a fundamental distinction between logical introduction rules. We observe that this distinction is also semantically

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grounded, indeed the main connective of a positive formula (in the original language of the logic) is a left adjoint/residual and the main connective of a negative formula (in the original language of the logic) is a right adjoint/residual. Proofs in *focalized normal form* (see [12]) are cut-free proofs organised in three phases: two focused phases (either positive or negative) and one non-focused phase (also called neutral phase). A focused positive (resp. negative) phase in a derivation is a proof-section where a formula is decomposed as much as possible only by means of non-invertible logical rules for positive (resp. negative) connectives. This formula and all its immediate subformulas in this proof-section are said 'in focus'. All the other rules are applied only in non-focused phases. So, each derivable sequent has at most one formula in focus. Moreover, the interaction between two focused phases is always mediated by a non-focused phase.

Shift operators –usually denoted as $\uparrow$ and $\downarrow$ ([7, 8, 3])– are often considered to polarize a focused sequent calculus, i.e. as a tool to control the interplay between positive and negative formulas and the interaction between phases. Shifts are adjoint unary operators that change the polarity of a formula, where $\uparrow$ goes from positive to negative, $\downarrow$ goes from negative to positive, and $\uparrow \dashv \downarrow$. In this paper, we consider positive and negative formulas as formulas of different sorts. We also distinguish between positive (resp. negative) *pure* formulas and positive (resp. negative) *shifted* formulas, i.e. formulas under the scope of a shift operator. So, we end up considering four different sorts, each of which is interpreted in a different sub-algebra. Therefore, in this setting shifts are heterogeneous operators, where $\uparrow$ gets split into $\uparrow$ (from positive pure formulas into negative shifted formulas) and $\upharpoonleft$ (from positive shifted formulas into negative pure formulas), $\downarrow$ gets split into $\downarrow$ (from negative pure formulas into positive shifted formulas) and $\downharpoonleft$ (from negative shifted formulas into positive pure formulas), $\uparrow \dashv \downharpoonleft$ and $\upharpoonleft \dashv \downarrow$. Moreover, the composition of two shifts is still either a closure or an interior operator (by adjunction), but we do not assume that it is an identity. We call a presentation of a logic with the features described above *full* polarization.

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