Display, Focusing and Full Polarization

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Basic Lambek-Grishin logic

Basic Lambek-Grishin algebra [Moo09]:

- Poset $\mathbb{G} = (G, \leq)$
- 6 operations $\otimes, \oplus, \backslash, \obslash, /, \oslash$ s.t. (1)

$$B \leq A \backslash C \quad \text{iff} \quad A \otimes B \leq C \quad \text{iff} \quad A \leq C / B \qquad C \oslash B \leq A \quad \text{iff} \quad C \leq A \oplus B \quad \text{iff} \quad A \obslash C \leq B \quad (1)$$

Multi-type Display Calculus

If a calculus $\mathcal{K}$ satisfies $C_1 - C_{10}$ [GJL+18, FGK+16]:

- In particular: **Display property**

  for every sequent $\Gamma \vdash \Delta$ *and every substructure* Σ *of either* Γ *or* Δ*, the sequent* $\Gamma \vdash \Delta$ *can be equivalently transformed (via* display postulates*) into* $\Sigma \vdash \Pi$ *or* $\Pi \vdash \Sigma$*, for some structure* Π*.*

  e.g. $X \hat{\otimes} Y \vdash \Delta$ iff $Y \vdash X \check{\backslash} \Delta$

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  e.g. $X \hat{\otimes} Y \vdash \Delta \quad \text{iff} \quad Y \vdash X \check{\backslash} \Delta$

- Canonical cut-elimination theorem, subformula property

Polarization

Classification of connectives / operations

- Left-adjoints ($\otimes, \oslash, \obslash$): **positive formulas** $P, Q, \ldots$
- Right-adjoints ($\oplus, \backslash, /$): **negative formulas** $N, M, \ldots$

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Shifts change the polarity:

- $\downarrow N$ is positive and $\uparrow P$ is negative
- $\uparrow \dashv \downarrow$

Focalization (1/2)

Two kinds of operational (aka. logical) rules:

- **Tonicity rules** (e.g. $\otimes_L$, $\backslash_R$): specifying arity and tonicity per each connective (*Andreoli: synchronous*)
- **Translation rules** (e.g. $\otimes_R$, $\backslash_L$): unary and invertible using Cut (*Andreoli: asynchronous*)

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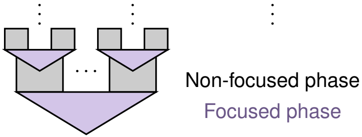
Idea of strong focalization: alternation between

- **Focused phase:** Proof-sections built by tonicity rules, where each sequent has exactly 1 formula in focus (in purple)
- **Non-focused phase:** Proof-sections built by translation rules applied greedily or structural rules (in gray)



Focalization (2/2)

State of the art: **f.LG** by Moortgat and Moot 2011 [MM12]

- Every proof is strongly focalized
- Focus by an extra-linguistic marker $\boxed{A}$
- **Restrictions on the applicability of rules**:
  e.g.

$$\underset{\textit{Axiom}}{\dfrac{}{A \vdash \boxed{A}}\,\mathrm{Ax}} \qquad \underset{\textit{Focusing}}{\dfrac{A \vdash \Delta}{\boxed{A} \vdash \Delta}\,\tilde{\mu}} \qquad \underset{\textit{Defocusing}}{\dfrac{X \vdash \boxed{A}}{X \vdash A}\,\mu^*} \tag{2}$$

  if A is positive

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$$\textit{Axiom} \qquad \textit{Focusing} \qquad \textit{Defocusing}$$

  if A is positive
- Not a Display Calculus according to the usual definition

Goal

General recipe associating to every displayable logic:

1. a **fully-polarized Algebraic Semantics**,
2. a **strongly focalized** and complete **Display Calculus**.

Our contribution: we provided such algebraic and proof-theoretic analysis for the Lambek-Grishin logic.

Full polarization

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- **Shifted** formulas **vs. Pure** formulas

$\Longrightarrow$ Avoids double shift

+ **Sorts** (aka. types) “capture / internalize” restrictions (uniform substitution within each type and no side conditions)

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$$
\begin{array}{llll}
P & ::= & p \mid \mathring{P} \otimes \mathring{P} \mid \mathring{P} \oslash \mathring{N} \mid \mathring{N} \obslash \mathring{P} & \text{Pure positive formulas} \\
N & ::= & n \mid \mathring{N} \oplus \mathring{N} \mid \mathring{P} \backslash \mathring{N} \mid \mathring{N} / \mathring{P} & \text{Pure negative formulas} \\
\dot{P} & ::= & {\color{red}\downarrow} N & \text{Shifted positive formulas} \\
\dot{N} & ::= & {\color{blue}\uparrow} P & \text{Shifted negative formulas}
\end{array}
$$

General positive (resp. negative) formulas to be more compact: $\mathring{P} \in \{P, \dot{P}\}$ (resp. $\mathring{N} \in \{N, \dot{N}\}$)

Structures

Basic structures:

$$
\begin{array}{rcll}
X & ::= & P \mid \mathring{X} \,\hat{\otimes}\, \mathring{X} \mid \mathring{X} \,\hat{\oslash}\, \mathring{\Delta} \mid \mathring{\Delta} \,\hat{\obslash}\, \mathring{X} & \text{Pure positive structures} \\
\Delta & ::= & N \mid \mathring{\Delta} \,\check{\oplus}\, \mathring{\Delta} \mid \mathring{X} \,\check{\backslash}\, \mathring{\Delta} \mid \mathring{\Delta} \,\check{/}\, \mathring{X} & \text{Pure negative structures} \\
\dot{X} & ::= & \dot{P} \mid \check{\downarrow} \Delta & \text{Shifted positive structures} \\
\dot{\Delta} & ::= & \dot{N} \mid \hat{\uparrow} X & \text{Shifted negative structures}
\end{array}
$$

General positive (resp. negative) structures: $\mathring{X} \in \{X, \dot{X}\}$ (resp. $\mathring{\Delta} \in \{\Delta, \dot{\Delta}\}$)

Structures

Basic structures + added adjoints:

$$
\begin{array}{lcll}
X & ::= & P \mid \check{\downarrow}\dot{\Delta} \mid \mathring{X}\,\hat{\otimes}\,\mathring{X} \mid \mathring{X}\,\hat{\oslash}\,\mathring{\Delta} \mid \mathring{\Delta}\,\hat{\obslash}\,\mathring{X} & \text{Pure positive structures} \\
\Delta & ::= & N \mid \hat{\uparrow}\dot{X} \mid \mathring{\Delta}\,\check{\oplus}\,\mathring{\Delta} \mid \mathring{X}\,\check{\backslash}\,\mathring{\Delta} \mid \mathring{\Delta}\,\check{/}\,\mathring{X} & \text{Pure negative structures} \\
\dot{X} & ::= & \dot{P} \mid \check{\downarrow}\Delta \mid \mathring{X}\,\check{\oplus}_\ell\,\mathring{\Delta} \mid \mathring{\Delta}\,\check{\oplus}_r\,\mathring{X} \mid & \text{Shifted positive structures} \\
 & & \mathring{\Delta}\,\check{\backslash}_\ell\,\mathring{\Delta} \mid \mathring{X}\,\check{\backslash}_r\,\mathring{X} \mid \mathring{X}\,\check{/}_\ell\,\mathring{X} \mid \mathring{\Delta}\,\check{/}_r\,\mathring{\Delta} & \\
\dot{\Delta} & ::= & \dot{N} \mid \hat{\uparrow}X \mid \mathring{\Delta}\,\hat{\otimes}_\ell\,\mathring{X} \mid \mathring{X}\,\hat{\otimes}_r\,\mathring{\Delta} \mid & \text{Shifted negative structures} \\
 & & \mathring{\Delta}\,\hat{\oslash}_\ell\,\mathring{\Delta} \mid \mathring{X}\,\hat{\oslash}_r\,\mathring{X} \mid \mathring{X}\,\hat{\obslash}_\ell\,\mathring{X} \mid \mathring{\Delta}\,\hat{\obslash}_r\,\mathring{\Delta} &
\end{array}
$$

General positive (resp. negative) structures: $\mathring{X} \in \{X, \dot{X}\}$ (resp. $\mathring{\Delta} \in \{\Delta, \dot{\Delta}\}$)

Additional structures for the display property:

- $\hat{\uparrow}$ and $\check{\downarrow}$ cancelling the "shiftedness"
- ℓ, r-variants: $\hat{\otimes}_\ell$, $\hat{\otimes}_r$, $\check{\backslash}_\ell$, $\check{\backslash}_r$, ...

  (more explanation later)

Sequents

Well-formed sequents:

Positive sequents	$X \vdash Y$	$\dot{X} \underset{\cdot}{\Vdash} Y$	$X \overset{\cdot}{\Vdash} \dot{Y}$	$\dot{X} \Vvdash \dot{Y}$
Negative sequents	$\Delta \vdash \Gamma$	$\dot{\Delta} \underset{\cdot}{\Vdash} \Gamma$	$\Delta \overset{\cdot}{\Vdash} \dot{\Gamma}$	$\dot{\Delta} \Vvdash \dot{\Gamma}$
Neutral sequents	$X \vdash \Delta$	$\dot{X} \underset{\cdot}{\Vdash} \Delta$	$X \overset{\cdot}{\Vdash} \dot{\Delta}$	$\dot{X} \Vvdash \dot{\Delta}$

(3)

grey cells are underivable (see proposition 14)

Sequents

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Neutral sequents	$X \vdash \Delta$	$\dot{X} \underset{\cdot}{\Vdash} \Delta$	$X \overset{\cdot}{\Vdash} \dot{\Delta}$	$\dot{X} \Vvdash \dot{\Delta}$

(3)

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Notation $\overset{\circ}{\vdash}$, $\overset{\circ}{\vdash}$, $\overset{\circ}{\vdash}$

- If $\mathring{X} = X$ and $\mathring{Y} = \dot{Y}$, then $\mathring{X} \overset{\circ}{\vdash} \mathring{Y}$ is $X \overset{\cdot}{\Vdash} \dot{Y}$

Phases

Definition

A sequent S is in positive (resp. negative) **focused phase** if S is a positive (resp. negative) sequent with a **formula in succedent** (resp. precedent) and containing **no structural shift** (namely $\hat{\uparrow}, \hat{\upharpoonleft}, \check{\downarrow}, \check{\downharpoonleft}$). Otherwise, S is called non-focused.

e.g.

- $p \hat{\otimes} q \vdash p \otimes q$ positive focused (on $p \otimes q$)
- $\hat{\uparrow}(p \hat{\otimes} q) \Vvdash \uparrow(p \otimes q)$ negative non-focused
- $p \hat{\otimes} q \dot{\Vdash} \uparrow(p \otimes q)$ neutral (so, non-focused)

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- $p \mathbin{\hat{\otimes}} q \mathrel{\dot{\Vdash}} \uparrow(p \otimes q)$ neutral (so, non-focused)

Rule sample:

$$\frac{\mathring{P} \mathbin{\hat{\otimes}} \mathring{Q} \mathrel{\mathring{\vdash}} \mathring{\Delta}}{\mathring{P} \otimes \mathring{Q} \mathrel{\mathring{\vdash}} \mathring{\Delta}}\ \otimes_L \qquad \frac{\mathring{X} \mathrel{\mathring{\vdash}} \mathring{P} \qquad \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{Q}}{\mathring{X} \mathbin{\hat{\otimes}} \mathring{Y} \vdash \mathring{P} \otimes \mathring{Q}}\ \otimes_R \qquad \frac{\mathring{Y} \mathrel{\mathring{\vdash}} \mathring{X} \mathbin{\check{\backslash}} \mathring{\Delta}}{\overline{\mathring{X} \mathbin{\hat{\otimes}} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Delta}}}\ \hat{\otimes} \dashv \check{\backslash} \tag{4}$$

Controlling phase transitions

Operational shifts control phase transition:

$$\uparrow_L \frac{\hat{\uparrow} P \mathrel{\mathring{\vdash}} \mathring{\Delta}}{\uparrow P \mathrel{\mathring{\vdash}} \mathring{\Delta}} \qquad \frac{X \vdash P}{\hat{\uparrow} X \Vdash \uparrow P} \uparrow_R \tag{5}$$

Focusing Defocusing

$\Longrightarrow$ $\uparrow$ and $\downarrow$ are "witnesses" of the structure of the proof in the end-sequent

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Focusing   Defocusing

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Structural shifts commute with structural connectives:

- Display postulates in neutral phase

$$\frac{\mathring{X} \mathrel{\mathring{\vdash}} \Delta}{\mathring{X} \mathrel{\mathring{\vdash}} \check{\downarrow} \Delta} \check{\downarrow} \qquad \hat{\uparrow} \frac{X \mathrel{\mathring{\vdash}} \mathring{\Delta}}{\hat{\uparrow} X \mathrel{\mathring{\vdash}} \mathring{\Delta}} \tag{6}$$

Example

Derivation of *Everyone sleeps.*

$$
\begin{array}{r c l}
 & np \vdash np \qquad\qquad s \vdash s & \\ \hline
\backslash L & np \backslash s \vdash np \check{\backslash} s & \\ \hdashline
 & \mathtt{sleeps} \vdash np \check{\backslash} s & \\ \hline
\downarrow L & \downarrow \mathtt{sleeps} \Vvdash \check{\downarrow}(np \check{\backslash} s) & \\ \hline
 & \downarrow \mathtt{sleeps} \Vdash np \check{\backslash} s & \check{\downarrow} \\ \hline
\hat{\otimes} \dashv \check{\backslash} & np \,\hat{\otimes} \downarrow \mathtt{sleeps} \vdash s & \\ \hline
\hat{\otimes} \dashv \check{/} & np \vdash s \check{/} \downarrow \mathtt{sleeps} & \\ \hline
\hat{\uparrow} & \hat{\uparrow} np \Vdash s \check{/} \downarrow \mathtt{sleeps} & \\ \hline
\uparrow L & \uparrow np \Vdash s \check{/} \downarrow \mathtt{sleeps} & \\
 & \vdots &
\end{array}
$$

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\begin{array}{ll}
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\backslash L & np \backslash s \vdash np \check{\backslash} s \\ \hdashline
 & \mathtt{sleeps} \vdash np \check{\backslash} s \\ \hline
\downarrow L & \downarrow\mathtt{sleeps} \Vdash \updownarrow (np \check{\backslash} s) \\ \hline
\updownarrow & \downarrow\mathtt{sleeps} \Vdash np \check{\backslash} s \\ \hline
\hat{\otimes} \dashv \check{\backslash} & np \,\hat{\otimes} \downarrow\mathtt{sleeps} \vdash s \\ \hline
\hat{\otimes} \dashv \check{/} & np \vdash s \,\check{/} \downarrow\mathtt{sleeps} \\ \hline
\hat{\uparrow} & \hat{\uparrow} np \Vdash s \,\check{/} \downarrow\mathtt{sleeps} \\ \hline
\uparrow L & \uparrow np \Vdash s \,\check{/} \downarrow\mathtt{sleeps} \\
 & \vdots
\end{array}
$$

$$
\begin{array}{ll}
 & \vdots \qquad n \vdash n \\ \hdashline
 & \qquad\;\; \mathtt{one} \vdash n \\ \hline
/ L & \uparrow np / n \vdash (s \,\check{/} \downarrow\mathtt{sleeps}) \,\check{/}\, \mathtt{one} \\ \hdashline
 & \mathtt{every} \vdash (s \,\check{/} \downarrow\mathtt{sleeps}) \,\check{/}\, \mathtt{one} \\ \hline
\downarrow L & \downarrow\mathtt{every} \Vdash \updownarrow((s \,\check{/} \downarrow\mathtt{sleeps}) \,\check{/}\, \mathtt{one}) \\ \hline
\updownarrow & \downarrow\mathtt{every} \Vdash (s \,\check{/} \downarrow\mathtt{sleeps}) \,\check{/}\, \mathtt{one} \\ \hline
\hat{\otimes} \dashv \check{/} & \downarrow\mathtt{every} \,\hat{\otimes}\, \mathtt{one} \vdash s \,\check{/} \downarrow\mathtt{sleeps} \\ \hline
\otimes L & \downarrow\mathtt{every} \otimes \mathtt{one} \vdash s \,\check{/} \downarrow\mathtt{sleeps} \\ \hdashline
 & \mathtt{everyone} \vdash s \,\check{/} \downarrow\mathtt{sleeps} \\ \hline
\hat{\otimes} \dashv \check{/} & \mathtt{everyone} \,\hat{\otimes} \downarrow\mathtt{sleeps} \vdash s \\ \hline
\updownarrow & \mathtt{everyone} \,\hat{\otimes} \downarrow\mathtt{sleeps} \Vdash \updownarrow s \\ \hline
\downarrow R & \mathtt{everyone} \,\hat{\otimes} \downarrow\mathtt{sleeps} \Vdash \downarrow s
\end{array}
$$

Signed generation tree

Every formula can be decomposed into subtrees of

- **Skeleton nodes** (boxed in Fig. 1)
- **PIA nodes** (not boxed)

depending on the position

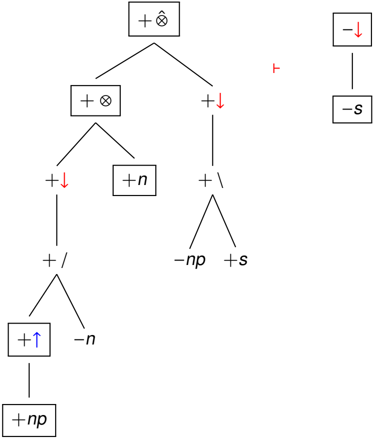

Figure: Signed generation tree of the end-sequent on slide 15

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Operational connectives are introduced:

- by a translation rule if skeleton node
- by a tonicity rule if PIA node

Shifts are roots of these subtrees → **transition nodes**

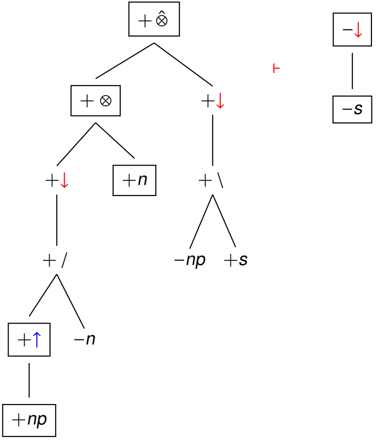

Figure: Signed generation tree of the end-sequent on slide 15

Strong focalization

Andreoli's construction of the formulas inside a focalized proof [And01]:

> *"[...] in a normal proof, each formula is viewed as a succession of layers of asynchronous connectives and of synchronous connectives; each synchronous layer is decomposed in a critical section (i.e. which cannot be interrupted), called a "critical focusing section""*

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Formalisation, adapted and extended from [Lau17]:

Definition (Strong focalization)

A sequent proof π is **strongly focalized** if **cut-free** and, for every formula A occurring in π, every **PIA subtree** of A is **constructed by a proof-section** of π containing **only tonicity rules**.

Phase diagram

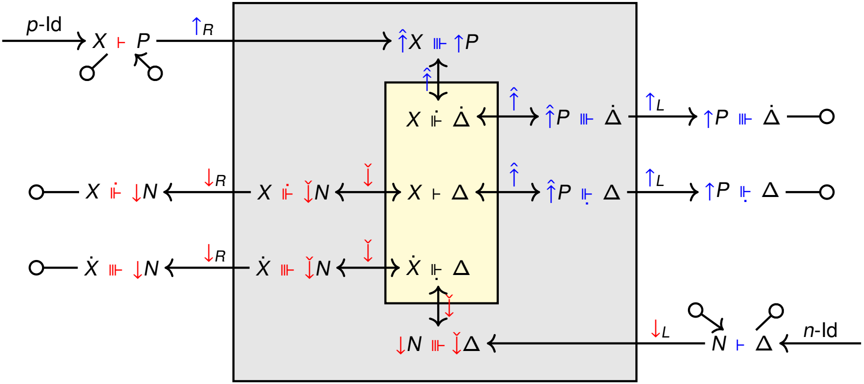

Figure: The topology of **fD.LG**-rules and phase transitions.

Proof of strong focalization

Theorem (thm. 23)

*Every cut-free and ℓ-r-variant-free proof in **fD.LG** is strongly focalized.*

Idea of the proof:

- From a focused sequent $\mathring{X} \mathrel{\mathring{\vdash}} \mathring{P}$ or $\mathring{N} \mathrel{\mathring{\vdash}} \mathring{\Delta}$ we can only

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Derivation translated back to **f.LG**: still strongly focalized

- Thanks to full polarization: PIA are not empty of LG-connectives

Display property

Displsay postulates for shifts:

$$\genfrac{}{}{}{}{\hat{\uparrow} X \Vvdash \dot{\Delta}}{X \vdash \check{\downharpoonright} \dot{\Delta}} \; \hat{\uparrow} \dashv \check{\downharpoonright} \qquad \genfrac{}{}{}{}{\hat{\uparrow} X \underset{\cdot}{\Vdash} \Delta}{X \dot{\Vdash} \check{\downarrow} \Delta} \; \hat{\uparrow} \dashv \check{\downarrow} \qquad \hat{\upharpoonleft} \dashv \check{\downarrow} \; \genfrac{}{}{}{}{\dot{X} \Vvdash \check{\downarrow} \Delta}{\hat{\upharpoonleft} \dot{X} \vdash \Delta} \tag{7}$$

Display property

Displsay postulates for shifts:

$$\begin{array}{c} \hat{\uparrow} X \Vvdash \dot{\Delta} \\ \hline\hline X \vdash \check{\downarrow} \dot{\Delta} \end{array}\ \hat{\uparrow} \dashv \check{\downarrow} \qquad \begin{array}{c} \hat{\uparrow} X \underset{\cdot}{\Vdash} \Delta \\ \hline\hline X \dot{\Vdash} \check{\downarrow} \Delta \end{array}\ \hat{\uparrow} \dashv \check{\downarrow} \qquad \hat{\uparrow} \dashv \check{\downarrow}\ \begin{array}{c} \dot{X} \Vvdash \check{\downarrow} \Delta \\ \hline\hline \hat{\uparrow} \dot{X} \vdash \Delta \end{array} \tag{7}$$

Display postulate on focused sequents with ℓ, r-variants, e.g.

$$\begin{array}{c} \mathring{Y} \mathring{\vdash} \mathring{X} \check{\backslash}_r \mathring{Z} \\ \hline\hline \mathring{X} \hat{\otimes} \mathring{Y} \mathring{\vdash} \mathring{Z} \\ \hline\hline \mathring{X} \mathring{\vdash} \mathring{Z} \check{/}_\ell \mathring{Y} \end{array} \begin{array}{l} \hat{\otimes} \dashv \check{\backslash}_r \\ \\ \hat{\otimes} \dashv \check{\backslash}_\ell \end{array}$$

Translations (not in the paper)

From **f.LG** to **fD.LG**:

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From **fD.LG** to **f.LG**:

- If the end-sequent without ℓ, r-variant or $\hat{\uparrow}$, $\check{\downarrow}$, they can be "removed" from the proof
- By removing shifts on structures
- Slight normalisation
- Identifying proof-sections to translate back as (de)focusing rules

Semantics (1/2)

Fully polarized LG-algebra $\mathbb{FP.LG}$:

- Each sort interpreted as a poset
- Homogeneous sequents interpreted as their orders
- **Heterogeneous sequents interpreted as weakening relations** $\preccurlyeq : \mathcal{A} \nrightarrow \mathcal{B}$

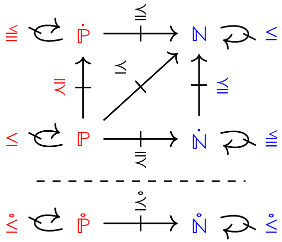

Figure: Weakening relations in $\mathbb{FP.LG}$-algebras.

- $\overline{\preceq}\leqq\ =\ \preceq\ =\ \overline{\preceq}\leqq$

Semantics (2/2)

Heterogeneous operations and their residuation:

$$\begin{array}{llll} \otimes : & \mathring{\mathbb{P}} \times \mathring{\mathbb{P}} \to \mathbb{P} & \oslash : & \mathring{\mathbb{P}} \times \mathring{\mathbb{N}}^{\partial} \to \mathbb{P} & \obslash : & \mathring{\mathbb{N}}^{\partial} \times \mathring{\mathbb{P}} \to \mathbb{P} \\ \oplus : & \mathring{\mathbb{N}} \times \mathring{\mathbb{N}} \to \mathbb{N} & \backslash : & \mathring{\mathbb{P}}^{\partial} \times \mathring{\mathbb{N}} \to \mathbb{N} & / : & \mathring{\mathbb{N}} \times \mathring{\mathbb{P}}^{\partial} \to \mathbb{N} \end{array} \tag{8}$$

$$\begin{array}{ccccc} \mathring{Q} \mathrel{\mathring{\lesssim}} \mathring{P} \backslash \mathring{N} & \text{iff} & \mathring{P} \otimes \mathring{Q} \mathrel{\mathring{\lesssim}} \mathring{N} & \text{iff} & \mathring{P} \mathrel{\mathring{\lesssim}} \mathring{N} / \mathring{Q} \\ \mathring{P} \oslash \mathring{N} \mathrel{\mathring{\lesssim}} \mathring{M} & \text{iff} & \mathring{P} \mathrel{\mathring{\lesssim}} \mathring{M} \oplus \mathring{N} & \text{iff} & \mathring{M} \obslash \mathring{P} \mathrel{\mathring{\lesssim}} \mathring{N} \end{array} \tag{9}$$

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Shift adjunctions

$$\begin{tikzcd} \mathbb{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \dot{\mathbb{N}} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathbb{P}} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathbb{N} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \tag{10}$$

Soundness and completeness

Soundness:

Interpretation of turnstiles:

t	$\vdash$	$\dot{\Vdash}$	$\Vvdash$	$\vdash$	$\underset{\cdot}{\Vdash}$	$\Vvdash$	$\vdash$	$\underset{\cdot}{\Vdash}$	$\dot{\Vdash}$	$\mathring{\vdash}$	$\mathring{\vdash}$	$\mathring{\vdash}$
$t^{\mathbb{A}}$	$\leq$	$\overline{\overline{\prec}}$	$\underline{\underline{\leqq}}$	$\leq$	$\underline{\preceq}$	$\underline{\underline{\leqq}}$	$\preceq$	$\underline{\preceq}$	$\overline{\overline{\prec}}$	$\mathring{\leq}$	$\mathring{\leq}$	$\mathring{\preceq}$

(11)

- Most rules clearly sound by construction
- Soundness of structural shift rules by:
  - $\upharpoonleft \dashv \downarrow$ represents $\leqq$ (i.e. $\upharpoonleft\dot{P} \leq N$ iff $\dot{P} \leqq N$ iff $\dot{P} \leqq \downarrow N$)
  - $\uparrow \dashv \downharpoonright$ represent $\overline{\overline{\prec}}$
  - **heterogeneous adjunction** $\uparrow \dashv^{\overline{\overline{\prec}}}_{\leqq} \downarrow$ represents $\preceq$, i.e. (12) (see section 2.1)

$$\uparrow P \leqq N \quad \text{iff} \quad P \preceq N \quad \text{iff} \quad P \,\overline{\overline{\prec}}\, \downarrow N \tag{12}$$

Soundness and completeness

Soundness:

Interpretation of turnstiles:

t	$\vdash$	$\dot{\Vdash}$	$\Vvdash$	$\vdash$	$\underset{\cdot}{\Vdash}$	$\Vvdash$	$\vdash$	$\underset{\cdot}{\Vdash}$	$\dot{\Vdash}$	$\mathring{\vdash}$	$\mathring{\vdash}$	$\mathring{\vdash}$
$t^{\mathbb{A}}$	$\leq$	$\overset{=}{<}$	$\underset{\equiv}{<}$	$\leq$	$\leqq$	$\underset{\equiv}{<}$	$\preceq$	$\preceqq$	$\overset{=}{\prec}$	$\mathring{\leq}$	$\mathring{\leq}$	$\mathring{\preceq}$

(11)

- Most rules clearly sound by construction
- Soundness of structural shift rules by:
  - $\upharpoonleft \dashv \downarrow$ represents $\leqq$ (i.e. $\upharpoonleft \dot{P} \leq N$ iff $\dot{P} \leqq N$ iff $\dot{P} \underset{\equiv}{<} \downarrow N$)
  - $\uparrow \dashv \downharpoonleft$ represent $\overset{=}{<}$
  - **heterogeneous adjunction** $\uparrow \dashv^{\overset{=}{<}}_{\leqq} \downarrow$ represents $\preceq$, i.e. (12) (see section 2.1)

$$\uparrow P \leqq N \quad \text{iff} \quad P \preceq N \quad \text{iff} \quad P \overset{=}{<} \downarrow N \tag{12}$$

Completeness:

- Routine Lindenbaum-Tarski construction for each subalgebra
- But restriction to **standard sequents** to preserve the shape of focused sequents
  - e.g. $[\![p \mathbin{\hat{\otimes}} q \vdash p \otimes q]\!] = [\![p \otimes q \vdash p \otimes q]\!]$, but only $p \mathbin{\hat{\otimes}} q \vdash p \otimes q$ is derivable due to focalization

Shifts are not necessary isomorphisms

Subtlety: The **behaviour** of an operation **in a subalgebra** should **not be the same** that its behaviour **in the global algebra**, to preserve strong focalization and completeness

- e.g. in **fD.LG** + Comm$_{\otimes}$, $p \hat{\otimes} q \vdash p \otimes q$, $p \hat{\otimes} q \Vdash \uparrow(p \otimes q)$ and $q \hat{\otimes} p \Vdash \uparrow(p \otimes q)$ are derivable
- but not $q \hat{\otimes} p \vdash p \otimes q$!
- so $\hat{\otimes}^{\mathbb{A}}$ is commutative in all models from the point of view of $\overline{\overline{<}}$, but not from the point of view of $\leq$

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- so $\hat{\otimes}^{\mathbb{A}}$ is commutative in all models from the point of view of $\overline{\overline{<}}$, but not from the point of view of $\leq$

- If shifts are isomorphisms in all models, this asymmetry falls down: loss of “semantics focalization” (not in the paper)

Extension to other kinds of semantics

concept lattices, hyperdoctrines for first order,...

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Operational rules

$$
\begin{array}{cccc}
\otimes_L \; \dfrac{\mathring{P} \,\hat{\otimes}\, \mathring{Q} \;\mathring{\vdash}\; \mathring{\Delta}}{\mathring{P} \otimes \mathring{Q} \;\mathring{\vdash}\; \mathring{\Delta}}
&
\dfrac{\mathring{X} \;{\color{red}\mathring{\vdash}}\; \mathring{P} \qquad \mathring{Y} \;{\color{red}\mathring{\vdash}}\; \mathring{Q}}{\mathring{X} \,\hat{\otimes}\, \mathring{Y} \;{\color{red}\vdash}\; \mathring{P} \otimes \mathring{Q}} \; \otimes_R
&
\oplus_L \; \dfrac{\mathring{N} \;{\color{blue}\mathring{\vdash}}\; \mathring{\Gamma} \qquad \mathring{M} \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta}}{\mathring{N} \oplus \mathring{M} \;{\color{blue}\vdash}\; \mathring{\Gamma} \,\check{\oplus}\, \mathring{\Delta}}
&
\dfrac{\mathring{X} \;\mathring{\vdash}\; \mathring{N} \,\check{\oplus}\, \mathring{M}}{\mathring{X} \;\mathring{\vdash}\; \mathring{N} \oplus \mathring{M}} \; \oplus_R
\\[3ex]
\oslash_L \; \dfrac{\mathring{P} \,\hat{\oslash}\, \mathring{N} \;\mathring{\vdash}\; \mathring{\Delta}}{\mathring{P} \oslash \mathring{N} \;\mathring{\vdash}\; \mathring{\Delta}}
&
\dfrac{\mathring{X} \;{\color{red}\mathring{\vdash}}\; \mathring{P} \qquad \mathring{N} \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta}}{\mathring{X} \,\hat{\oslash}\, \mathring{\Delta} \;{\color{red}\vdash}\; \mathring{P} \oslash \mathring{N}} \; \oslash_R
&
\backslash_L \; \dfrac{\mathring{X} \;{\color{red}\mathring{\vdash}}\; \mathring{P} \qquad \mathring{N} \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta}}{\mathring{P} \backslash \mathring{N} \;{\color{blue}\vdash}\; X \,\check{\backslash}\, \Delta}
&
\dfrac{\mathring{X} \;\mathring{\vdash}\; \mathring{P} \,\check{\backslash}\, \mathring{N}}{\mathring{X} \;\mathring{\vdash}\; \mathring{P} \backslash \mathring{N}} \; \backslash_R
\\[3ex]
\obslash_L \; \dfrac{\mathring{N} \,\hat{\obslash}\, \mathring{P} \;\mathring{\vdash}\; \mathring{\Delta}}{\mathring{N} \obslash \mathring{P} \;\mathring{\vdash}\; \mathring{\Delta}}
&
\dfrac{\mathring{N} \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta} \qquad \mathring{X} \;{\color{red}\mathring{\vdash}}\; \mathring{P}}{\mathring{\Delta} \,\hat{\obslash}\, \mathring{X} \;{\color{red}\vdash}\; \mathring{N} \obslash \mathring{P}} \; \obslash_R
&
/_L \; \dfrac{\mathring{N} \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta} \qquad \mathring{X} \;{\color{red}\mathring{\vdash}}\; \mathring{P}}{\mathring{N} / \mathring{P} \;{\color{blue}\vdash}\; \mathring{\Delta} \,\check{/}\, \mathring{X}}
&
\dfrac{\mathring{X} \;\mathring{\vdash}\; \mathring{N} \,\check{/}\, \mathring{P}}{\mathring{X} \;\mathring{\vdash}\; \mathring{N} / \mathring{P}} \; /_R
\\[3ex]
{\color{red}\downarrow}_L \; \dfrac{N \;{\color{blue}\vdash}\; \Delta}{{\color{red}\downarrow} N \;{\color{red}\Vvdash}\; {\color{red}\check{\downarrow}} \Delta}
&
\dfrac{\mathring{X} \;{\color{red}\mathring{\vdash}}\; {\color{red}\check{\downarrow}} N}{\mathring{X} \;{\color{red}\mathring{\vdash}}\; {\color{red}\downarrow} N} \; {\color{red}\downarrow}_R
&
{\color{blue}\uparrow}_L \; \dfrac{{\color{blue}\hat{\uparrow}} P \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta}}{{\color{blue}\uparrow} P \;{\color{blue}\mathring{\vdash}}\; \mathring{\Delta}}
&
\dfrac{X \;{\color{red}\vdash}\; P}{{\color{blue}\hat{\uparrow}} X \;{\color{blue}\Vvdash}\; {\color{blue}\uparrow} P} \; {\color{blue}\uparrow}_R
\end{array}
\tag{13}
$$

Structural rules

$$
\hat{\otimes} \dashv \check{\backslash} \;\;
\begin{array}{c}
\mathring{Y} \mathrel{\mathring{\vdash}} \mathring{X} \mathbin{\check{\backslash}} \mathring{\Delta} \\ \hline\hline
\mathring{X} \mathbin{\hat{\otimes}} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Delta} \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta} \mathbin{\check{/}} \mathring{Y}
\end{array}
\;\; \hat{\otimes} \dashv \check{/}
\qquad
\begin{array}{c}
\mathring{Y} \mathrel{\mathring{\vdash}} \mathring{X} \mathbin{\check{\backslash}_r} \mathring{Z} \\ \hline\hline
\mathring{X} \mathbin{\hat{\otimes}} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{Z} \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \mathring{Z} \mathbin{\check{/}_\ell} \mathring{Y}
\end{array}
\;\; \begin{array}{c} \hat{\otimes} \dashv \check{\backslash}_r \\ \hat{\otimes} \dashv \check{\backslash}_\ell \end{array}
\qquad
\begin{array}{c} \hat{\oslash}_\ell \dashv \check{\oplus} \\ \hat{\obslash}_r \dashv \check{\oplus} \end{array}
\begin{array}{c}
\mathring{\Sigma} \mathbin{\hat{\oslash}_\ell} \mathring{\Delta} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \\ \hline\hline
\mathring{\Sigma} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \mathbin{\check{\oplus}} \mathring{\Delta} \\ \hline\hline
\mathring{\Gamma} \mathbin{\hat{\obslash}_r} \mathring{\Sigma} \mathrel{\mathring{\vdash}} \mathring{\Delta}
\end{array}
\qquad
\begin{array}{c}
\mathring{X} \mathbin{\hat{\oslash}} \mathring{\Delta} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \mathbin{\check{\oplus}} \mathring{\Delta} \\ \hline\hline
\mathring{\Gamma} \mathbin{\hat{\obslash}} \mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta}
\end{array}
\;\; \begin{array}{c} \hat{\oslash} \dashv \check{\oplus} \\ \hat{\obslash} \dashv \check{\oplus} \end{array}
$$

$$
\begin{array}{c} \hat{\obslash} \dashv \check{\oplus}_r \\ \hat{\oslash}_r \dashv \check{\oplus}_r \end{array}
\begin{array}{c}
\mathring{\Gamma} \mathbin{\hat{\obslash}} \mathring{X} \mathrel{\mathring{\vdash}} \mathring{Y} \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \mathbin{\check{\oplus}_r} \mathring{Y} \\ \hline\hline
\mathring{X} \mathbin{\hat{\oslash}_r} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Gamma}
\end{array}
\qquad
\begin{array}{c}
\mathring{X} \mathbin{\hat{\oslash}} \mathring{\Delta} \mathrel{\mathring{\vdash}} \mathring{Y} \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \mathring{Y} \mathbin{\check{\oplus}_\ell} \mathring{\Delta} \\ \hline\hline
\mathring{Y} \mathbin{\hat{\obslash}_\ell} \mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta}
\end{array}
\;\; \begin{array}{c} \hat{\oslash} \dashv \check{\oplus}_\ell \\ \hat{\obslash}_\ell \dashv \check{\oplus}_\ell \end{array}
\qquad
\begin{array}{c} \hat{\otimes}_r \dashv \check{\backslash} \\ \hat{\otimes}_r \dashv \check{/}_r \end{array}
\begin{array}{c}
\mathring{\Gamma} \mathrel{\mathring{\vdash}} \mathring{X} \mathbin{\check{\backslash}} \mathring{\Delta} \\ \hline\hline
\mathring{X} \mathbin{\hat{\otimes}_r} \mathring{\Gamma} \mathrel{\mathring{\vdash}} \mathring{\Delta} \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta} \mathbin{\check{/}_r} \mathring{\Gamma}
\end{array}
\qquad
\begin{array}{c}
\mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \mathbin{\check{\backslash}_\ell} \mathring{\Delta} \\ \hline\hline
\mathring{\Gamma} \mathbin{\hat{\otimes}_\ell} \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Delta} \\ \hline\hline
\mathring{\Gamma} \mathrel{\mathring{\vdash}} \mathring{\Delta} \mathbin{\check{/}} \mathring{Y}
\end{array}
\;\; \begin{array}{c} \hat{\otimes}_\ell \dashv \check{\backslash}_\ell \\ \hat{\otimes}_\ell \dashv \check{/} \end{array}
$$

$$
\begin{array}{c}
\hat{\uparrow} X \Vdash \dot{\Delta} \\ \hline\hline
X \vdash \check{\downarrow} \dot{\Delta}
\end{array}
\;\; \hat{\uparrow} \dashv \check{\downarrow}
\qquad
\begin{array}{c}
\hat{\uparrow} X \mathrel{\dot{\Vdash}} \Delta \\ \hline\hline
X \mathrel{\dot{\Vdash}} \check{\downarrow} \Delta
\end{array}
\;\; \hat{\uparrow} \dashv \check{\downarrow}
\qquad
\hat{\uparrow} \dashv \check{\downarrow} \;\;
\begin{array}{c}
\dot{X} \Vdash \check{\downarrow} \Delta \\ \hline\hline
\hat{\uparrow} \dot{X} \vdash \Delta
\end{array}
$$

$$
\begin{array}{c}
\mathring{X} \mathrel{\mathring{\vdash}} \Delta \\ \hline\hline
\mathring{X} \mathrel{\mathring{\vdash}} \check{\downarrow} \Delta
\end{array}
\;\; \check{\downarrow}
\qquad\qquad
\hat{\uparrow} \;\;
\begin{array}{c}
X \mathrel{\mathring{\vdash}} \mathring{\Delta} \\ \hline\hline
\hat{\uparrow} X \mathrel{\mathring{\vdash}} \mathring{\Delta}
\end{array}
\tag{14}
$$