# 1

# Exercises

### Exercise n°1

Construct NPDA's that accept the following regular languages:

- $L_1 = L(aaa^*b)$
- $L_2 = L(aab^*aba^*)$
- $L_1 \cup L_2$
- $L_1 \setminus L_2$.

### Exercise n°2

Construct an NPDA that accepts the following language on $\Sigma = \{a, b, c\}$ :

$$L = \{a^n b^m c^{n+m} | n \geq 0, m \geq 0\}$$

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### Exercise n°3

Construct an NPDA that accepts the following language on $\Sigma = \{a, b, c\}$:

$$L = \{w \mid n_a(w) = n_b(w) + 1\}$$

### Exercise n°4

Construct an NPDA that accepts the following language on $\Sigma = \{a, b, c\}$:

$$L = \{a^n b^m \mid n \geq 0, n \neq m\}.$$

### Exercise n°5

What language is accepted by the NPDA
$M = (\{q0, q1, q2\}, \{a, b\}, \{a, b, z\}, \delta, q_0, z, \{q2\})$ with transitions:

- $\delta(q_0, a, z) = \{(q_1, a), (q_2, \lambda)\}$
- $\delta(q_1, b, a) = \{(q_1, b)\}$
- $\delta(q_1, b, b) = \{(q_1, b)\}$
- $\delta(q_1, a, b) = \{(q_2, \lambda)\}$

### Exercise n°6

Construct an NPDA corresponding to the grammar:

$$S \to aABB \mid aAA$$
$$A \to aBB \mid a$$
$$B \to bBB \mid A.$$

### Exercise n°7

Construct an NPDA that will accept the language generated by the grammar $G = (\{S, A\}, \{a, b\}, S, P)$ with productions

$$S \to AA \mid a$$
$$A \to SA \mid b$$

### Exercise n°8

Find a context-free grammar that generates the language accepted by the

NPDA $M = (\{q_0, q_1\}, \{a, b\}, \{A, z\}, \delta, q_0, z, \{q_1\})$, with transitions:

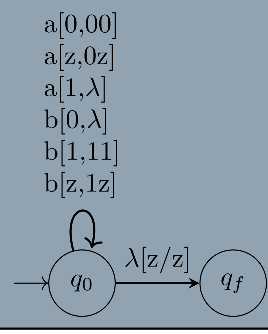
$$\delta(q_0, a, z) = \{(q_0, Az)\}$$
$$\delta(q_0, b, A) = \{(q_0, AA)\}$$
$$\delta(q_0, a, A) = \{(q1, \lambda)\}$$

## Exercise n°9

Is the language $L = \{a^n b^n \mid n \geq 1\} \cup \{b\}$ deterministic?

## Exercise n°10

Show that the following automaton is not deterministic, but that the language $L = \{w \in \{a, b\}^* \mid n_a(w) = n_b(w)\}$ it generates is nevertheless deterministic.




## Exercise n°11

Is the language $L = \{a^n b^m \mid n = m \vee n = m + 2\}$ deterministic?

## Exercise n°12

Show that if $L_1$ is deterministic context-free and L2 is regular, then the language$L_1 \cup L_2$ is deterministic context-free.

## Solution n°1

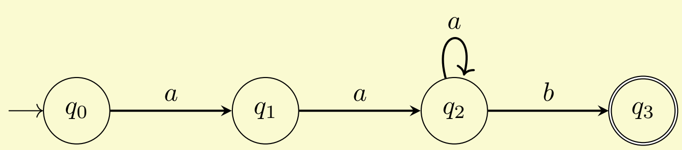
Notice that the languages in this exercise are all regular. This means that there exist NFAs that accept them.
From such an NFA, it is easy to derive an NDPA:

- We simply keep the same set of states, final states, starting state and input alphabet.

- We take the stack starting symbol to be z and the stack alphabet to be the set consisting of only z.

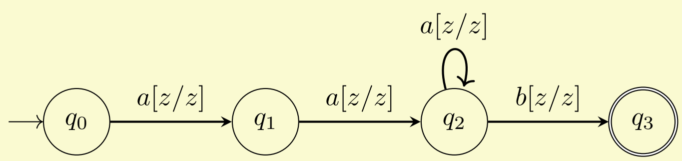
- We transform each transition in the NFA to read z from, and then write it back to the stack, leaving the stack unaltered.

$L(aaa^*b)$ is accepted by the following NFA:



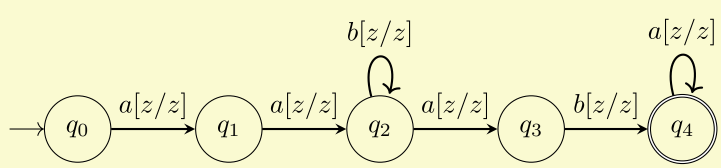

We construct an NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ with $L(M) = L(aaa^*b)$:

$$Q = \{q_0, q_1, q_2, q_3\} \quad \Sigma = \{a, b\} \quad \Gamma = \{z\} \quad F = \{q_3\}$$

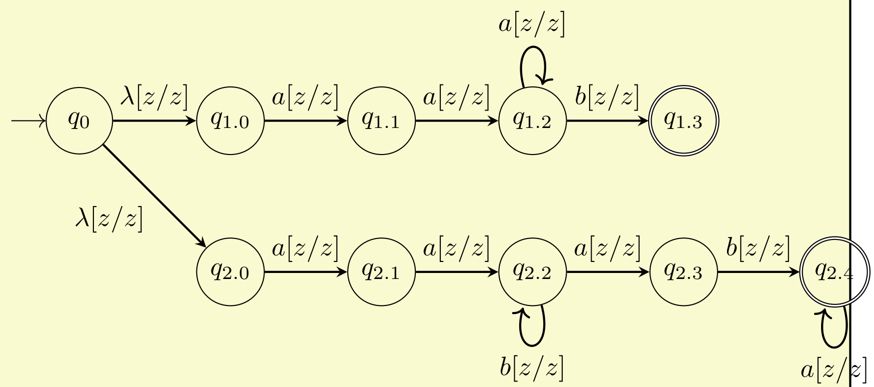



We create an NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ with $L(M) = L(aab^*aba^*)$:

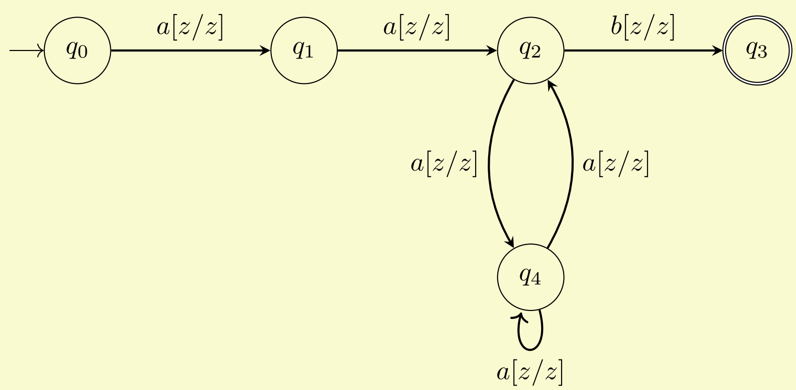
$$Q = \{q_0, q_1, q_2, q_3, q_4\} \quad \Sigma = \{a, b\} \quad \Gamma = \{z\} \quad F = \{q_4\}$$



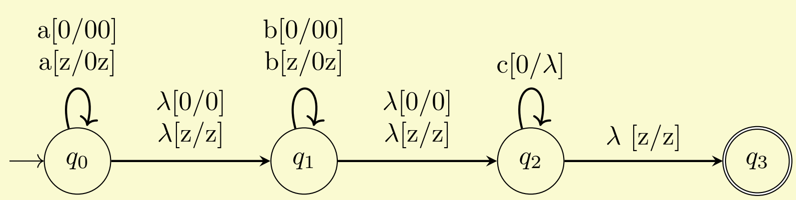

For the union of $L_1$ and $L_2$, we have the following NDPA:




Notice that there is only one word that is present in both $L_1$ and $L_2$: $aaab$. So, for the language $L_1 \setminus L_2$, we can alter the NFA we had for $L_1$ to specifically reject this word. In other words, we force either two or more than three a's. Then we derive the following NPDA:

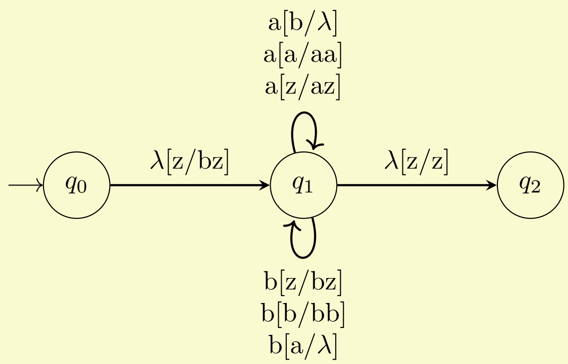



## Solution n°2



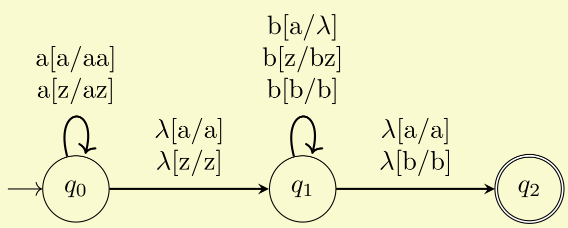
## Solution n°3

$$L = \{w \mid n_a(w) = n_b(w) + 1\}$$



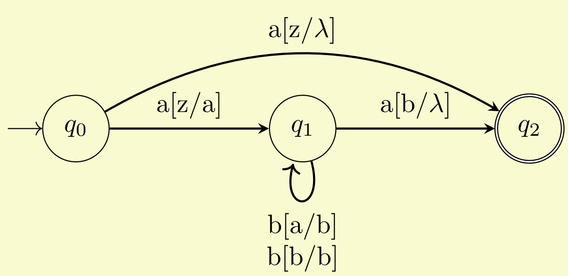

## Solution n°4

$$L = \{a^n b^m \mid n \geq 0, n \neq m\}$$




## Solution n°5

We can represent the NPDA:




The language that is accepted by this NPDA is equal to $L(a + abb^*a)$.

## Solution n°6

Reminder: We construct an NPDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ as follows:

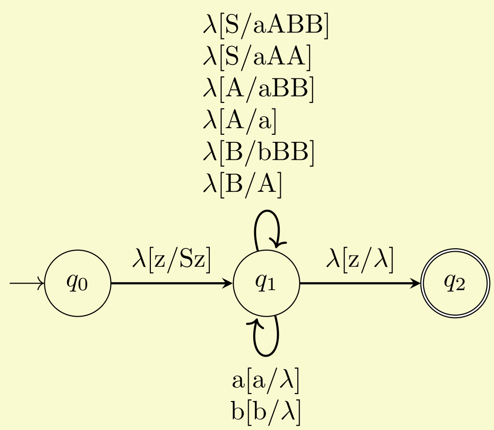
$$Q = \{q_0, q_1, q_2\} \quad \Sigma = T \quad F = \{q_2\} \quad \Gamma = V \cup T \cup \{z\}$$

We add transitions simulating a leftmost derivation:

$$q_0 \xrightarrow{\lambda[z/Sz]} q_1$$
$$q_1 \xrightarrow{\lambda[A/x]} q_1 \text{ for every } A \to x \in P$$
$$q_1 \xrightarrow{a[a/\lambda]} q_1 \text{ for every } a \in T$$
$$q_1 \xrightarrow{\lambda[z/\lambda]} q_2$$

Then the grammar will the following NDPA:

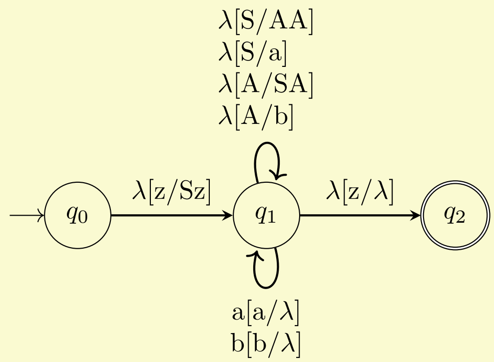
$$S \to aABB \mid aAA$$
$$A \to aBB \mid a$$
$$B \to bBB \mid A.$$




## Solution n°7

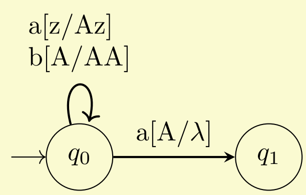
The grammar will the following NDPA:

$S \to AA \mid a$
$A \to SA \mid b$



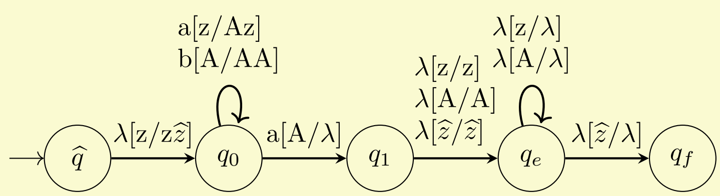

## Solution n°8

We can represent M:




Reminder:

It needs a single final state that is only reachable with an empty stack:




We define a context-free grammar $(V, T, S, P)$ as follows:

$$\mathrm{T} = \Sigma \quad V = \{(qbq_0) | q, q_0 \in Q, b \in \Gamma\} \quad S = q_0 z q_1$$

The set P contains the following rules:

- $q \xrightarrow{\alpha[b/\lambda]} q' \Longrightarrow (qbq') \to \alpha \in P$
- $q \xrightarrow{\alpha[b/c_1 \ldots c_n]} q' \Longrightarrow (qbr_n) \to \alpha(q' c_1 r1) \cdots (r_{n-1} c_n r_n) \in P$

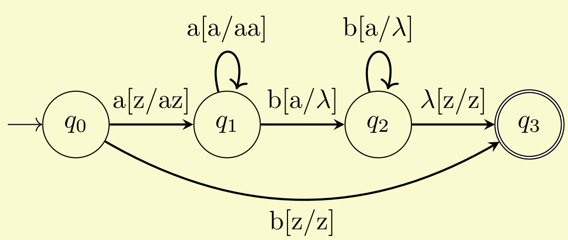
Thus:

$$
\begin{aligned}
\widehat{q} \xrightarrow{\lambda[z/z\widehat{z}]} q_0 &\implies (\widehat{q}zr_2) \to \lambda(q_0zr_1)(r_1\widehat{z}r_2) \\
q_0 \xrightarrow{a[z/Az]} q_0 &\implies (q_0zr_2) \to a(q_0Ar_1)(r_1zr_2) \\
q_0 \xrightarrow{b[A/AA]} q_0 &\implies (q_0Ar_2) \to b(q_0Ar_1)(r_1Ar_2) \\
q_0 \xrightarrow{a[A/\lambda]} q_1 &\implies (q_0Aq_1) \to a \\
q_1 \xrightarrow{\lambda[z/z]} q_e &\implies (q_1zr_1) \to \lambda(q_ezr_1) \\
q_1 \xrightarrow{\lambda[A/A]} q_e &\implies (q_1Ar_1) \to \lambda(q_eAr_1) \\
q_1 \xrightarrow{\lambda[\widehat{z}/\widehat{z}]} q_e &\implies (q_1\widehat{z}r_1) \to \lambda(q_e\widehat{z}r_1) \\
q_e \xrightarrow{\lambda[z/\lambda]} q_e &\implies (q_ezq_e) \to \lambda \\
q_e \xrightarrow{\lambda[A/\lambda]} q_e &\implies (q_eAq_e) \to \lambda \\
q_e \xrightarrow{\lambda[\widehat{z}/\lambda]} q_f &\implies (q_e\widehat{z}q_f) \to \lambda
\end{aligned}
$$

## Solution n°9

A language L is deterministic context-free if there exists a DPDA M with $L(M) = L$.

$$\{a^nb^n \mid n \geq 1\} \cup \{b\}$$



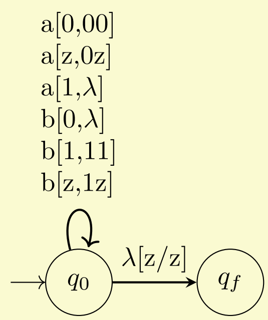

$$
\begin{aligned}
\delta(q_0, a, z) =& \{(q_1, az)\} \\
\delta(q_0, b, z) =& \{q_3, z\} \\
\delta(q_1, a, z) =& \{(q_1, aa)\} \\
\delta(q_1, b, a) =& \{(q_2, \lambda)\} \\
\delta(q_2, b, a) =& \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, z) =& \{(q_3, z\}
\end{aligned}
$$

- $\forall q \in Q, \forall \alpha \in \Sigma, \forall x \in \Gamma,\ |\delta(q, \alpha, x)| \leq 1$

- If $\delta(q, \delta, x) \neq \emptyset$ then $\forall \alpha \in \Sigma$, $\delta(q, \alpha, x) = \emptyset$

Therefore this NPDA is a DPDA and L is deterministic context-free.

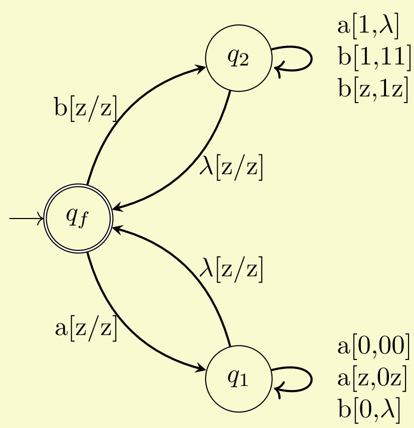
## Solution n°10




Assume NPDA is a DPDA.Then $\delta(q_0, \lambda, z) \neq \emptyset$ so $\delta(q_0, a, z) = \emptyset$.
But $\delta(q_0, a, z) = \{q_0, az\} \neq \emptyset$
Therefore it's not a NPDA.

We have to find a DPDA for this language:

$$L = \{w \in \{a, b\}^* \mid n_a(w) = n_b(w)\}$$



**Solution n°11**

$$L = \{a^n b^m \ | \ n = m \vee n = m + 2\}$$