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Exercises

Exercise n°1

Show that the language $L = \{a^n \mid n \text{ is a prime number}\}$ is not context-free.

Exercise n°2

Show that $L = \{ww^Rw \mid w \in \{a,b\}^*\}$is not a context-free language.

Exercise n°3

Is the language $L = \{a^nb^m \mid n = 2m\}$ context-free?

Exercise n°4

Determine whether the language $L = \{a^nb^ja^jb^n \mid n \geq 0, j \geq 0\}$ is context-free.

Exercise n°5

Is the complement of the language$L = \{w \in \{a,b,c\}^* \mid n_a(w) = n_b(w) = n_c(w)\}$ context-free?

Exercise n°6

Show that the family of deterministic context-free lan- guages is not closed under union and intersection.

Exercise n°7

Show that there exists an algorithm to determine whether the language generated by some context-free grammar contains any words of length less than some given number n.

Exercise n°8

Let L_1 be a context-free language and L_2 be regular. Show that there exists an algorithm to determine whether or not L_1 and L_2 have a common element.

Solution n°1

$$L = \{a^n \ \mid \ n \text{ is a prime number}\}$$

Assume L is context free.
By the pumping lemma, $\exists m > 0$, $\forall w \in L$ with $|w| \geq m$ such that $\exists u, v, x, y, z$ — $w = uvxyz$ with $|vxy| \leq m$, $|vy| \geq 1$ and $\forall i \in \mathbb{N}$ $uv^i xy^i z \in L$.

In particular if we take the word $w = a^p$ where p is a prime number such that $p > m + 1$.

Since $|vxy| \leq m$, $v = a^j$, $x = a^k$, $y = a^l$ where $j + l \geq 1$ because $|vy| \geq 1$.

Take $i = p + 1$:

$$uv^{1+p}xy^{1+p}z = uvxyz(vy)^p = a^p(a^{j+l})^p = a^{p(j+l+1)} =$$

The number $p(j + l + 1)$ is divible by $p, 1$ and $j + l + 1$.

However

$$j + l + 1 \neq 1 : \ \ j + l \geq 1 \Rightarrow j + l + 1 \geq 2 > 1$$

$$j + l + 1 \neq p : \ \ j + l + 1 < j + k + l + 1 = m + 1 < p$$

Therefore $p(j + l + 1)$ has at least 3 divisors: $p(j + l + 1)$ is not prime hence $w \notin L$.

This contradicts the pumping lemma. Therefore, our assumption that L is regular cannot be true: L is not context-free.

Solution n°2

$$L = \{ww^R w \mid w \in \{a,b\}^*\}$$

Assume L is context free.
By the pumping lemma, $\exists m > 0$, $\forall w \in L$ with $|w| \geq m$ such that $\exists u, v, x, y, z$ — $w = uvxyz$ with $|vxy| \leq m$, $|vy| \geq 1$ and $\forall i \in \mathbb{N}$ $uv^ixy^iz \in L$.

In particular, we can take the word $w = (ab)^m \cdot (ba)^m \cdot (ab)^m$

- $|vxy| \leq m \Rightarrow vxy = a^j b^k \vee vxy = b^k a^j$ with $k + j \leq m$
- $|vy| \geq 1 \Rightarrow k \geq 1$ or $j \geq 1$ therefore $1 \leq k + j \leq m$

Take $i = 0$:

$uvxyz = uxz$ and we have 3 possibilities for uxz:

- $(ab)^{m-(k+j)}(ba)^m(ab)^m$
- $(ab)^m(ba)^m(ab)^{m-(k+j)}$
- $(ab)^m(ba)^{m-(k+j)}(ab)^m$

And $m-(k+j) \neq m$: $m-(k+j) \leq m-1 < m$ so none of the possibilities belong to L: $w \neq L$ for $i = 0$.

This contradicts the pumping lemma. Therefore, our assumption that L is regular cannot be true: L is not context-free.

Solution n°3

$$L = \{a^n b^m \ \mid \ n = 2m\}$$

Assume L is context free.
By the pumping lemma, $\exists m > 0$, $\forall w \in L$ with $|w| \geq m$ such that $\exists u, v, x, y, z$ — $w = uvxyz$ with $|vxy| \leq m$, $|vy| \geq 1$ and $\forall i \in \mathbb{N}$ $uv^ixy^iz \in L$.

In particular, we can take the word $w = a^n b^m$.

- $|vxy| \leq m \Rightarrow vxy = a^j b^k \vee vxy = a^l \vee vxy = b^l$ with $k + j \leq m \vee l \leq m$
- $|vy| \geq 1 \Rightarrow 1 \leq k + j \leq m \vee 1 \leq l \leq m$

Take $i = 0$:

We have 3 possibilities for $uxyz$:

- $a^{n-l}b^m$ and $n - l \neq 2m :\ \ n - l < n = 2m$ hence $a^{n+l}b^m \notin L$
- $a^n b^{m-l}$ and $2(m - l) \neq n :\ \ n = 2m > (2m - l)$ hence $a^n b^{m-l} \notin L$
- $a^{n+\alpha-j} b^{m+\beta-k}$ where:
  - $\alpha = n_a(x) \wedge \beta = n_b(x)$
  - $(0 \leq \alpha < j \wedge 0 \leq \beta \leq k) \vee (0 \leq \alpha \leq j \wedge 0 \leq \beta < k)$

$$\begin{aligned} n + \alpha - j = 2(m + \beta - k) &\iff n + \alpha - j = n - 2(k - \beta) \\ &\iff j - \alpha = 2(k - \beta) \end{aligned}$$

In the case $0 \leq \alpha < j \wedge 0 \leq \beta \leq k$, it's absurd for $\beta = k$ since $j < \alpha$.

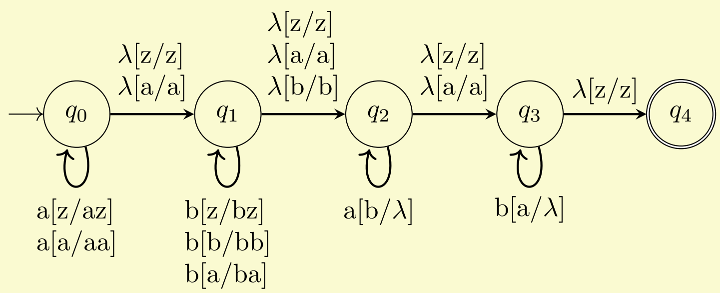
In the case $0 \leq \alpha \leq j \wedge 0 \leq \beta < k$, it's absurd for $\alpha = j$ since $\beta < k$.

As for all cases we have $uv^0xy^0z \neq L$, we can conclude that L is not a context-free language.

Solution n°4

The language is context-free, which can be shown by providing an NPDA that accepts the language:

$$L = \{a^n b^j a^j b^n \ \mid \ n \geq 0, j \geq 0\}$$



Solution n°5

Solution n°6

First we show that the family is not closed under union.

Let $L_1 = \{a^n b^{2n} \ |an \geq 0\}$, and $L_2 = \{a^n b^n \ |an \geq 0\}$. These two languages are deterministic context-free, as you can deterministically count the a's and then discharge it with b's. Any DPDA M with $L(M) = L_1 \cup L_2$would need to decide whether to count the a's twice beforehand. This means it is required to be non-deterministic, and hence $L_1 \cup L_2$ cannot be a deterministic context-free language.

Now for the intersection. Let $L_1 = \{a^n b^n c^m \ \mid \ n \geq 0, m \geq 0\}$, and $L_2 = \{a^n b^m c^m \ \mid \ n \geq 0, m \geq 0\}$. These two languages are deterministic context-free, as there is only one variable that needs to be recorded. However, $L_1 \cap L_2 = \{a^n b^n c^n \ \mid \ n \geq 0\}$, which has already been shown to not be context-free. So it cannot be deterministic context-free either. Hence the family of deterministic context-free languages is not closed under union and intersection.

Solution n°7

Solution n°8

We know that $L_1 \cap L_2$ is context-free. There is an algorithm to decide if$L_1 \cap L_2$ is empty and reversing the result of that algorithm determines whether the two languages have a common element. Thus, there is an algorithm to determine whether $L_1 and L_2$ have a common element.