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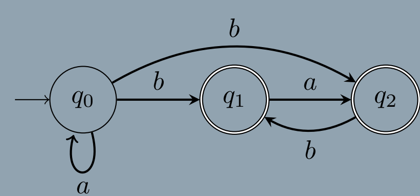
Exercises

Find a regular expression for the language accepted by the following automaton:

Exercise n°1



Exercise n°2

The tail of a language is defined as the set of all suffixes of its strings. That is,

$$\text{tail}(L) = \{y|xy \in L for some x \in \Sigma^*\}$$

Example :

$$tail\,(\{011, 101\}) = \{\lambda, 1, 01, 11, 011, 101\}$$

Show that if L is regular, so is tail(L).

Exercise n°3

For a string $a_1a_2 \ldots a_n$ define the operation shift as

$$shift(a_1a_2 \cdots a_n) = a_2 \cdots a_na_1.$$

From this, we can define the operation on a language as

$$shift(L) = \{v : v = \text{ shift(w) for some w } \in L\}$$

Show that regularity is preserved under the shift operation.

Exercise n°4

Let G_1 and G_2 be two regular grammars. Show how one can define regular grammars for the languages

- $L(G1) \cup L(G2)$
- $L(G1)L(G2)$
- $L(G_1)^*$

Exercise n°5

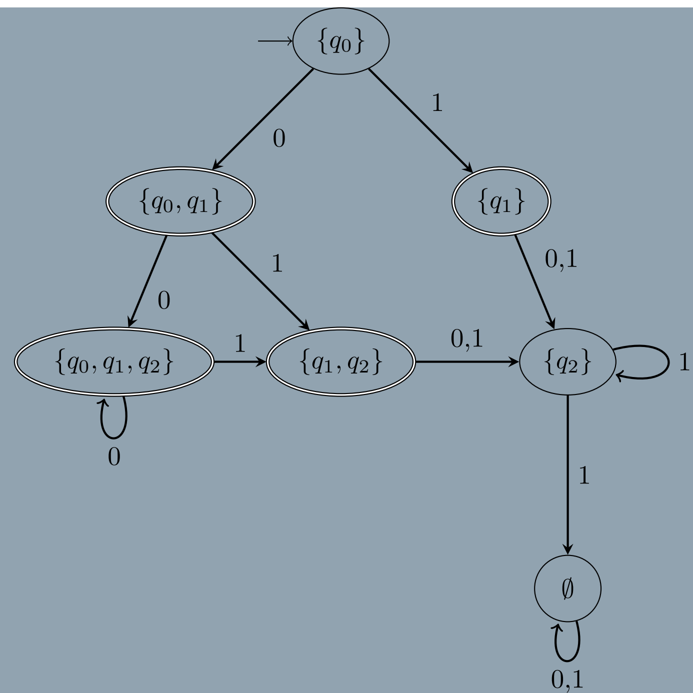
A language is said to be a palindrome language if $L = L^R$. Find an algorithm for determining if a given regular language is a palindrome language.

Exercise n°6

Let L be any regular language on $\Sigma = \{a, b\}$. Show that an algorithm exists for determining if L contains any strings of even length.

Exercise n°7

Minimize the number of states in the following DFA:



Exercise n°8

Prove or disprove: If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language L, then $\hat{M} = (Q, \Sigma, \delta, q_0, Q \setminus F)$ is a minimal DFA for $\overline{L}$.

Solution n°1

The regular expression is a^*ba^*.

Solution n°2

We want to prove that the following language is regular:

$$\text{tail}(L) = \{y \mid xy \in L \text{ for some } x \in \Sigma^*\}$$

Let L be a regular language.

$$L \text{ is regular} \Longleftrightarrow \exists \text{ NFA } M = (Q, \Sigma, \delta, S, F) \text{ with } L(M) = L$$

$$xy \in L = L(M) \Longleftrightarrow \delta(q_s, xy) \vdash \cdots \vdash \delta(q, y) \vdash \cdots \vdash \delta(q_f, \lambda) \tag{1.1}$$

Where q denotes any state: $q \in Q$. In particular q can be q_s or q_f.

Consider the NFA $\widehat{M} = (Q, \Sigma, \delta, Q, F)$

We will show that $tail(L) \subseteq L(\widehat{M})$:

q_f is a final state of M so it's a final state of $\widehat{M}$ and q is a state of M so q is a starting state of $\widehat{M}$ hence y is accepeted by $\widehat{M}$ thus $y \in L(\widehat{M})$. The existence of transitions from q to q_f are guaranteed by the construction of the NFA M as shown by (1.1)

We will show that $L(\widehat{M}) \subseteq tail(L)$:

If y is accepted by $L(\widehat{M})$ it means that there is a path $\delta(q, y) \vdash^* \delta(q_f, \lambda)$ Let x be a word such that $\delta(q_s, x) \vdash^* \delta(q, \lambda)$. It's clear that xy is accepted by M so $xy \in L$ and the existence of x (quaranteed by the construction of the NFA M) shows that $y \in tail(L)$

There is a NFA $\widehat{M}$ such that $L(\widehat{M}) = tail(L)$ hence $tail(L)$ is a regular language.

Solution n°3

$$shift(a_1 a_2 \cdots a_n) = a_2 \cdots a_n a_1.$$

$$shift(L) = \{v : v = \text{ shift(w) for some w} \in L\}$$

Let L be a regular language.

$$L \text{ is regular} \iff \exists \text{ DFA } M = (Q, \Sigma, \delta, q_0, F) \text{ with } L(M) = L$$

For all symbol $a \in \Sigma$ we define M_a by changing the initial state from $q0$ to $q_a \in Q$, where $q_0 \xrightarrow{\text{a}} q_a$. As M is a DFA, this is guaranteed to exist and unique.

Hence $M_a = (Q \setminus \{q_0\}, \Sigma, \delta, q_a, F)$ and each M_a accept the language $\{w \mid aw \in L\}$

We create a new NFA $\widehat{M}$ by combining all the DFAs M_a and defining the set of initial states as the initial states of the M_a.
$\widehat{M}$ accepts the language $\{w \mid aw \in L\}$.

We modify our NFA by adding a state q_f where for every M_a we add a-transitions to q_f for all final states in M_a.Finally, q_f becomes the only final state.

It will now accept the word if it is in the language $\{wa \mid aw \in L\}$, which, if we consider all the $a \in \Sigma$, is exactly $shift(L)$.

We have created an NFA that accepts $shift(L)$ therefore $shift(L)$ is regular.

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Solution n°4

$$L(G_1) \cup L(G_2)$$

Solution n°5

We want to prove the following property:

It is decidable whether a regular language L is a palindrome language: $L = L^R$.

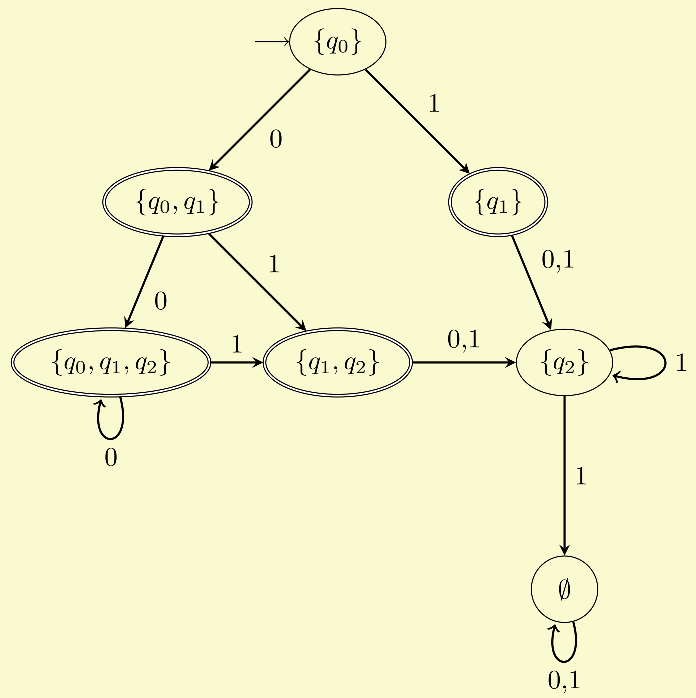
Let L be a regular language. As L is regular, so is LR. Hence, we can create DFAs M and M^R such that $L(M) = L$ and $L(M^R) = L^R$. These two DFAs can then be minimized to $M' = (Q', \Sigma, \delta', q_0', F')$ and $M^{R\prime} = (Q^{R\prime}, \Sigma, \delta^{R\prime}, q_0^{R\prime}, F^{R\prime})$.

If $L = L^R$ then M' and $M^{R\prime}$ must be isomorphic because of the uniqueness of DFAs.

The algorithm, then, must compare M' and $M^{R\prime}$:

Solution n°6

Solution n°7



Initial partitioning:

$$\{\{\{q_0, q_1\}, \{q_0, q_1, q_2\}, \{q_1, q_2\}, \{q_1\}\}, \{\{q_0\}, \{q_2\}, \emptyset\}\}$$

R = { $\{\mathrm{q}_0, q_1\}, \{q_0, q_1, q_2\}, \{q_1, q_2\}, \{q_1\}\}$ S = { $\{\mathrm{q}_0\}, \{q_2\}, \emptyset\}$ 0

New partitioning:

$$\{\{\{q_0, q_1\}, \{q_0, q_1, q_2\}\}, \{\{q_1, q_2\}, \{q_1\}\}, \{\{q_0\}, \{q_2\}, \emptyset\}\}$$

R ={ $\{\mathrm{q}_0\}, \{q_2\}, \emptyset\}$ S = { { $\mathrm{q}_1, q_2\}, \{q_1\}\}$ 1

New partitioning:

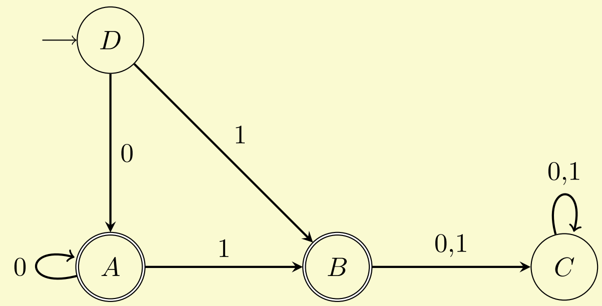
$$\{\{\{q_0, q_1\}, \{q_0, q_1, q_2\}\}, \{\{q_1, q_2\}, \{q_1\}\}, \{\{q_2\}, \emptyset\}, \{\{q_0\}\}\}$$

We introduce a new notation for reasons of readability:

- $A = \{\{q_0, q_1\}, \{q_0, q_1, q_2\}\}$
- $B = \{\{q_1, q_2\}, \{q_1\}\}$
- $C = \{\{q_2\}, \emptyset\}$

- $D = \{\{q_0\}\}$

Reading off the minimal DFA:



Solution n°8

If $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for a regular language L then $\widehat{M} = (Q, \Sigma, \delta, q_0, Q \setminus F)$ is a DFA for $\overline{L}$. But nothing proves that $\widehat{M}$ is minimal.

Assume it's not minimal.

Then there exists a DFA$\widehat{M}' = (Q', \Sigma, \delta', q_0, F')$ with fewer states than M, which accepts the language L.

But then the DFA $M' = (Q', \Sigma, \delta', q_0, Q' \setminus F')$ accepts the language L and has fewer states than M. This means that M is not minimal. This contradicts with M being minimal.

Therefore, our assumption that there is a $\widehat{M}'$ with fewer states cannot be true.

Thus $\widehat{M}$ is minimal DFA which accepts $\overline{L}$.