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Exercises

Information

The reverse of a word can be defined inductively by

$$\left(\lambda^R = \lambda\right) \quad \wedge \quad (wa)^R = aw^R$$

for some $a \in \Sigma, w \in \Sigma^*$.

Exercise n°1

Prove that

$$\forall u, v \in \Sigma^*, \quad (uv)^R = v^R u^R$$

Exercise n°2

Prove that

$$(L_1 L_2)^R = L_2^R L_1^R$$

Exercise n°3

Let $L = \{ab, aa, baa\}$. Which of the following words are in L^*:

abaabaaabaa aaaabaaaa baaaaabaaaab baaaaabaa

Which of these words are in L^4?

Exercise n°4

Let $L \subseteq \Sigma^*$ for a non empty alphabet Σ. Show that L and L^* cannot both be finite.

Exercise n°5

Prove or disprove that:

$$(L_1 \cup L_2)^R = L_1^R \cup L_2^R \text{ for all languages } L_1, L_2.$$

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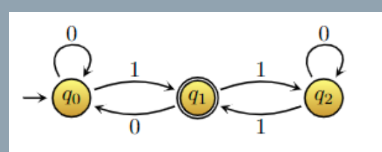
Exercise n°6

Prove that

$$(L^R)^* = (L^*)^R \text{ for all languages L}$$

Exercise n°7

Consider the following DFA:



Which of the words 0001, 01001 and 0000110 does the automaton accept?

Exercise n°8

Construct a DFA that accepts all words over $\Sigma = \{a, b\}$ with at least one a and precisely two b's.

Exercise n°9

Construct a DFA that accepts the language

$$L = \{w \in \Sigma^* \mid |w| \bmod 5 \neq 0\} \text{ where } \Sigma = \{a, b\}$$

Exercise n°10

Construct a DFA that accepts the language of all words over $\Sigma = \{0, 1\}$ that contains 00 but not 000.

Exercise n°11

Construct a DFA that accepts the language

$$L = \{vwv \mid v, w \in \Sigma^* \wedge |v| = 2\} \text{ where } \Sigma = \{a, b\}$$

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Solutions

Solution n°1

By induction on the length of v, we will prove the following

$$P(n): \text{ " } \forall u, v \in \Sigma^*, \ \ (uv)^R = v^R u^R \ \text{ with } \ |v| = n \text{ ".}$$

Base case: When $n = 0$, $(uv)^R = (u\lambda)^R = (u)^R = \lambda u^R = \lambda^R u^R$.

Induction hypothesis: Assume that P(n) is correct for some positive integer n. That means for $|v| = n$ we have $(uv)^R = v^R u^R$.

Induction step: We will now show that P(n + 1) is correct. If $|v| = n+1$, we can write $v = \tilde{v}a$ for some $a \in \Sigma$ and $|\tilde{v}| = n$. Therefore:

$$(uv)^R = (u\tilde{v}a)^R = a(u\tilde{v})^R = a \ \tilde{v}^R u^R = (\tilde{v}^R a)u^R = v^R u^R$$

Hence by mathematical induction P(n) is correct for all positive integers n.

Solution n°2

The concatenation of languages L_1 and L_2 is defined as

$$L_1 L_2 = \{xy \mid x \in L_1, y \in L_2\}$$

The reverse of a language L is defined as

$$L^R := \{x^R \mid x \in L\}$$

Notice $L^R := \{x \mid x \in L^R\}$ since $w \in L^R \iff w^R \in L$

$$\begin{aligned}
(L_1 L_2)^R &= \{xy \mid x \in L_1, y \in L_2\}^R \\
&= \{(xy)^R \mid x \in L_1, y \in L_2\} \\
&= \{y^R x^R \mid x \in L_1, y \in L_2\}^R \\
&= \{yx \mid x \in L_1^R, y \in L_2^R\} \\
&= \{yx \mid y \in L_2^R, x \in L_1^R\} \\
&= L_2^R L_1^R
\end{aligned}$$

Solution n°3

Let $L = \{$ab,aa,baa$\}$. Which of the following words are in L^*:

- abaabaaabaa ✓
- aaaabaaaa ✓
- baaaaabaaaab ✗
- baaaaabaa ✓

The words aaaabaaaa and baaaaabaa are in L^4.

Solution n°4

Let $L \subseteq \Sigma^*$ for a non empty alphabet Σ. We want to show that L and $\overline{L}$ cannot both be finite.
Going by contradiction: Assume that L and $\overline{L}$ are finite.
Then $\overline{L} \cup L$ is finite but $\overline{L} \cup L = \Sigma^*$ which is infinite.
Therefore L and $\overline{L}$ cannot both be finite.

Solution n°5

$$\begin{aligned}(L_1 \cup L_2)^R &= \{w \mid w \in L_1 \vee w \in L_2\}^R \\ &= \{w^R \mid w \in L_1 \vee w \in L_2\} \\ &= \{w \mid w^R \in L_1 \vee w^R \in L_2\} \\ &= \{w \mid w \in L_1^R \vee w \in L_2^R\} \\ &= L_1^R \cup L_2^R\end{aligned}$$

Solution n°6

By induction, we will prove the following

$$P(n) : \text{ " } \forall n \in \mathbb{N},\ (L^n)^R = (L^R)^n \text{ "}.$$

Base case: When $n = 0$,$(L^0)^R = \{\lambda\}^R = \{\lambda\} = (L^R)^0$.

Induction hypothesis: Assume that P(n) is correct for some positive integer n.

Induction step: We will now show that P(n + 1) is correct.

$$(L^{n+1})^R = (L^n L)^R = L^R (L^n)^R = L^R (L^R)^n = (L^R)^{n+1}$$

Hence by mathematical induction P(n) is correct for all positive integers n.

By using the result of the previous exercise:

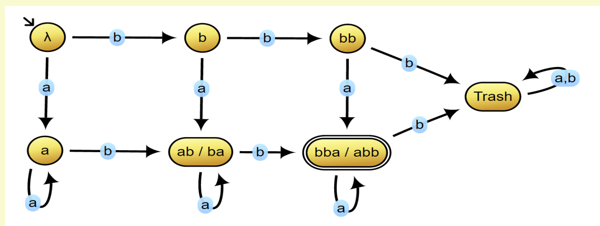
$$(L^*)^R = \left(\bigcup_{n \geq 0} L^n \right)^R = \bigcup_{n \geq 0} (L^n)^R = \bigcup_{n \geq 0} (L^R)^n = (L^R)^*$$

Solution n°7

The words 0001 and 01001 are accepted however 0000110 is rejected.

Solution n°8

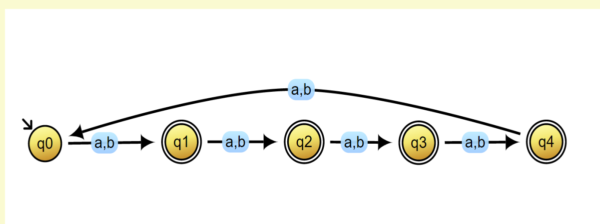
The following DFA accepts the language of all words over $\Sigma = \{a, b\}$ with at least one a and precisely two b's:



Solution n°9

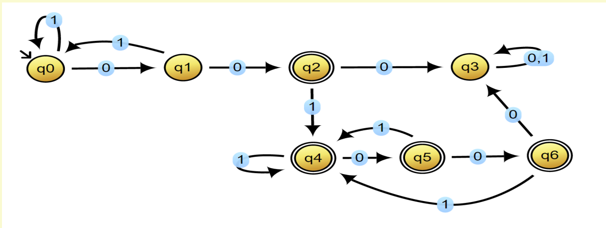
The following DFA accepts the language

$$L = \{w \in \Sigma^* \mid |w| \bmod 5 \neq 0\} \text{ where } \Sigma = \{a, b\}$$

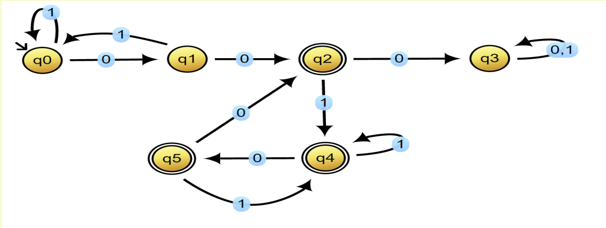


Solution n°10

The following DFA accepts the language of all words that contains 00 but not 000 over $\Sigma = \{0, 1\}$.



A better version:



Solution n°11

The following DFA accepts the language

$$L = \{vwv \ \mid \ v, w \in \Sigma^* \wedge |v| = 2\} \ \text{where} \ \Sigma = \{a, b\}$$