2-SAT

2-SAT

2-SAT is a special case of *boolean satisfiability* (SAT).

Boolean satisfiability

Given a logical proposition ϕ defined on $x_1, x_2, \ldots, x_n$, can we assign values in $\{\mathbf{T}, \mathbf{F}\}$ to each x_i so that ϕ is true?

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$$
\begin{aligned}
\phi = &((x_{75} \vee x_{100} \wedge \neg x_{67} \vee x_6) \Leftrightarrow \neg x_{86}) \Rightarrow \neg x_{30} \wedge \neg x_{94} \wedge x_{19} \wedge x_{49} \wedge \neg x_{56} \vee \\
& x_{38} \wedge \neg x_{78} \wedge \neg x_{95} \vee \neg x_{76} \vee x_1 \vee \neg x_{49} \wedge (\neg x_{77} \vee x_{22} \Leftrightarrow \neg x_{75}) \wedge x_{44} \vee \\
& x_{90} \Rightarrow x_{79} \Leftrightarrow \neg x_{14} \vee x_{95} \wedge \neg x_7 \Leftrightarrow \neg x_{99} \wedge x_{72} \Leftrightarrow \neg x_{40} \vee x_{85} \vee \neg x_{97} \Rightarrow \\
& x_{76} \wedge \neg x_{33} \vee (((x_{48} \wedge x_{55} \Leftrightarrow \neg x_{97} \Leftrightarrow \neg x_{15}) \vee \neg x_{62} \vee x_7) \Leftrightarrow \neg x_{95} \Leftrightarrow x_{64}) \Rightarrow \\
& x_{18} \vee x_{18} \vee \neg x_{64} \wedge x_{99} \Rightarrow x_{78} \wedge \neg x_{75} \Leftrightarrow x_3 \vee x_{78} \wedge \neg x_8 \Rightarrow \neg x_9 \Leftrightarrow \\
& x_{30} \Rightarrow \neg x_{20} \vee x_{27} \Leftrightarrow x_{43} \wedge \neg x_{22} \wedge x_{21} \wedge \neg x_{55} \Leftrightarrow \neg x_{56} \Leftrightarrow x_{11} \Rightarrow \neg x_{71} \wedge \\
& x_{32} (\Leftrightarrow x_{15} \wedge x_{66} \Leftrightarrow x_{95} \vee x_{60}) \wedge x_{100} \vee \neg x_{47} \Leftrightarrow x_{49} \wedge \neg x_5 \vee x_{93} \wedge \\
& x_{17} \vee x_{73} \wedge x_{32} \Leftrightarrow \neg x_5 \Rightarrow x_{80} \vee \neg x_{12} \Leftrightarrow \neg x_{48} \Rightarrow (x_{99} \Leftrightarrow x_1 \wedge x_{15} \Leftrightarrow \\
& x_{23} \vee \neg x_{99} \Leftrightarrow x_{73} \Rightarrow x_{39} \vee x_{91} \Leftrightarrow \neg x_{22} \Rightarrow x_4 \Rightarrow x_{17}) \Leftrightarrow x_{83} \vee x_{32} \vee \\
& \ldots
\end{aligned}
$$

Forget that... SAT is **NP-complete**.

2-SAT

So what is 2-SAT?...

Same as SAT but with ϕ of the form:

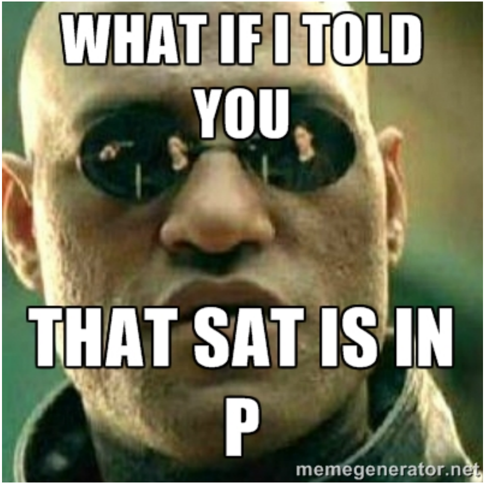
$$\phi = (L_1 \vee L_2) \wedge (L_3 \vee L_4) \wedge \ldots \wedge (L_{k-1} \vee L_k)$$

with $L_i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n \neg x_n\}$

Example:

$$(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_3) \wedge (x_2 \vee x_3)$$

2-SAT



Implication view

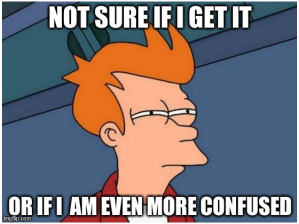
$L_1 \vee L_2$ is equivalent to $\neg L_1 \Rightarrow L_2$ and also to $\neg L_2 \Rightarrow L_1$

L_1	L_2	L_1	$\vee$	L_2	$\neg$ L_1	$\Rightarrow$	L_2	$\neg L_2$	$\Rightarrow$	L_1
T	T	T	T	T	F	T	T	F	T	T
T	F	T	T	F	F	T	F	T	T	T
F	T	F	T	T	T	T	T	F	T	F
F	F	F	F	F	T	F	F	T	F	F

2-SAT

Define a graph G_ϕ with:

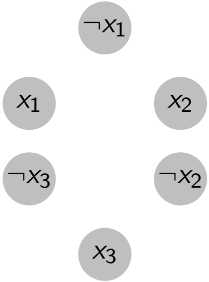
- $V = \{x_1, x_2, \dots, x_n, \neg x_1, \neg x_2, \dots, \neg x_n\}$
- $(\alpha, \beta) \in E$ if there is a clause logically equivalent to $\alpha \Rightarrow \beta$.



2-SAT: Associated graph

$$(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_3) \wedge (x_2 \vee x_3)$$

- For this formula the graph is:



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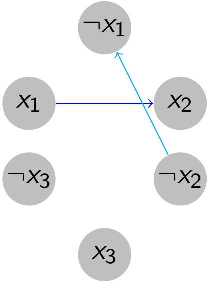

$$\neg x_1 \vee x_2 \Leftrightarrow x_1 \Rightarrow x_2 \Leftrightarrow \neg x_2 \Rightarrow \neg x_1$$

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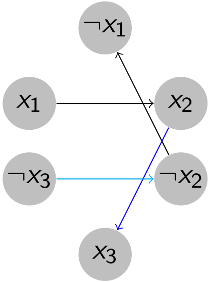

$$\neg x_2 \vee x_3 \Leftrightarrow x_2 \Rightarrow x_3 \Leftrightarrow \neg x_3 \Rightarrow \neg x_2$$

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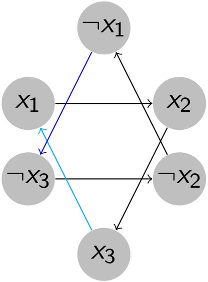

$$x_1 \vee \neg x_3 \Leftrightarrow \neg x_1 \rightarrow \neg x_3 \Leftrightarrow x_3 \rightarrow x_1$$

2-SAT: Associated graph

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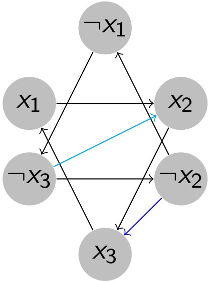

$$x_2 \vee x_3 \Longleftrightarrow \neg x_2 \rightarrow x_3 \Longleftrightarrow \neg x_3 \rightarrow x_2$$

2-SAT: Associated graph

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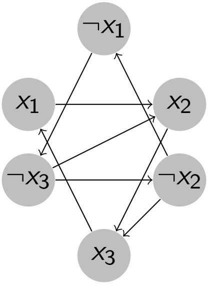


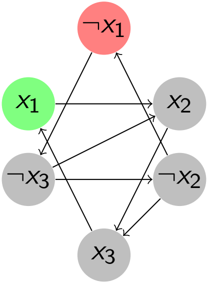
Ok but what do we do with this graph?

2-SAT: Paths and satisfiability

Suppose we assign **T** to vertex x_1 (so $\neg x_1$ is **F**).

Edges are implications and we know that, if A is true, for $A \Rightarrow B$ to be true we need B to be true.

We can thus propagate information!



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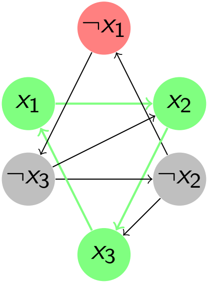

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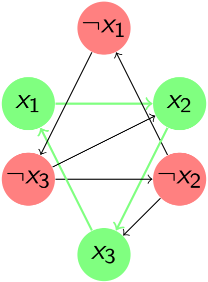


2-SAT: Paths and satisfiability

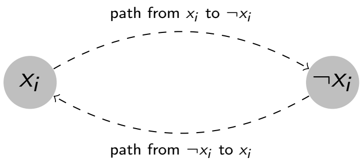
Observation:

In the graph point of view, a truth assignment satisfies all constraints iff there are no edges from a **T** node to a **F** node.

More generaly:

There can be no path from a **T** node to a **F** node.

2-SAT: Paths and satisfiability



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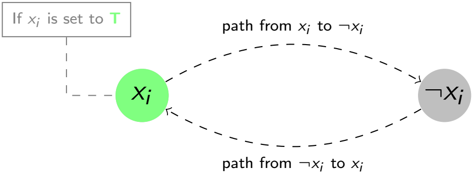

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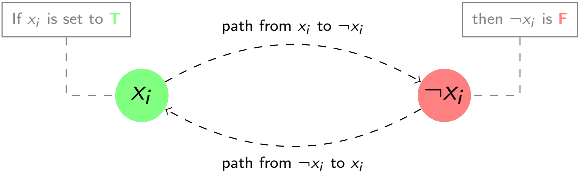

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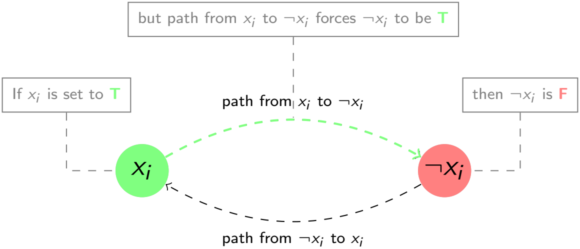

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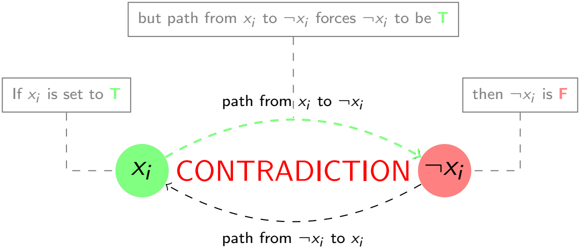

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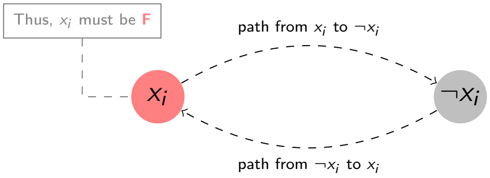

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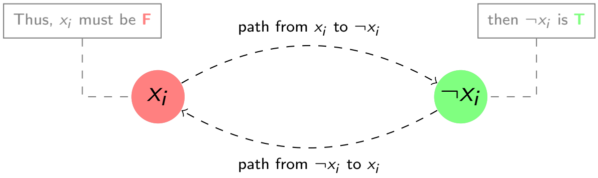

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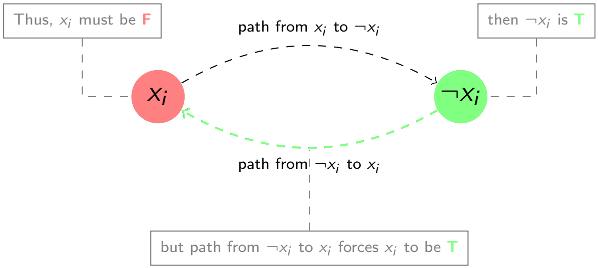

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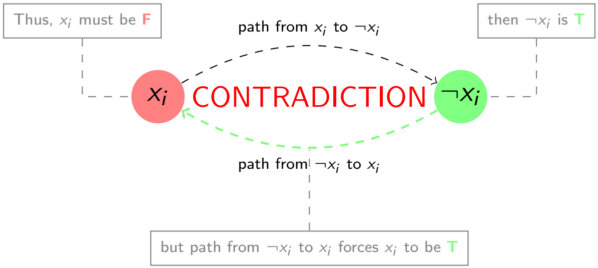

2-SAT: Paths and satisfiability

We have proved the following:

Lemma

If there exists a variable x_i such that there is a path from x_i to $\neg x_i$ and a path from $\neg x_i$ to x_i then ϕ is not satisfiable.

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But... Is the converse true?

2-SAT: Paths and satisfiability

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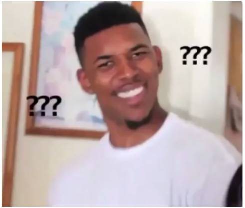
But... Is the converse true? **YES!**

2-SAT: Paths and satisfiability

Lemma

If for every x_i there is no path from x_i to $\neg x_i$ or no path from $\neg x_i$ to x_i then ϕ is satisfiable.

Why?

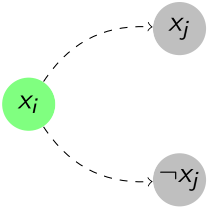


2-SAT: Paths and satisfiability

If there is a no path from x_i to $\neg x_i$:

Set x_i to **T** (and $\neg x_i$ to **F**).

No conflict between x_i and $\neg x_i$. But what about others?...



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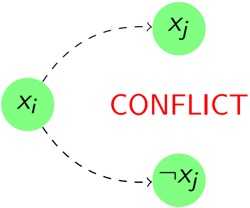

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