LTD-friendly expression for the fermion self-energy

March 24, 2020

a On-shell renormalisation

We intend do derive explicitly the LTD friendly expression for the self-energy correction of a massless quark. In this section, we start by reviewing the general *on-shell* renormalisation conditions for a Dirac fermion.

In ref. [1], we find a very detailed expression of the renormalisation process in the Standard Model, for both QCD and EW corrections. There, the general expression of the renormalised self-energy $\hat{\Gamma}^f_{ij}(p)$ involving two fermion species i and j reads:

$$\hat{\Gamma}^f_{ij}(p) = i\delta_{ij}(\not{p} - m_i) + i\left[\not{p}w_-\hat{\Sigma}^{f,L}_{ij}(p^2) + \not{p}w_+\hat{\Sigma}^{f,R}_{ij}(p^2) + (m_{f,i}w_- m_{f,j}w_+\hat{\Sigma}^{f,S}_{ij}(p^2))\right], \tag{a.1}$$

where $w_\pm = \frac{\mathbb{1}\pm\gamma^5}{2}$ and where the hat denotes renormalised (i.e. non-bare) quantities. The renormalisation conditions then read:

$$\tilde{\Re}[\hat{\Gamma}^f_{ij}(p)]u_j(p)\Big|_{p^2=m^2_{f,j}} = 0, \tag{a.2}$$

$$\bar{u}_j(p')\tilde{\Re}[\hat{\Gamma}^f_{ij}(p')]\Big|_{p'^2=m^2_{f,i}} = 0, \tag{a.3}$$

$$\lim_{p^2\to m^2_{f,i}} \frac{\not{p} + m_{f,i}}{p^2 - m^2_{f,i}}\tilde{\Re}[\hat{\Gamma}^f_{ii}(p)]u_i(p) = iu_i(p), \tag{a.4}$$

$$\lim_{p'^2\to m^2_{f,i}} \bar{u}_i(p')\tilde{\Re}[\hat{\Gamma}^f_{ii}(p')]\frac{\not{p}' + m_{f,i}}{p'^2 - m^2_{f,i}} = i\bar{u}_i(p'), \tag{a.5}$$

where $\tilde{\Re}$ corresponds to *removing* the complex absorptive part of the renormalised self-energies which is not the same as taking its real part when in presence of complex-valued couplings.

The situation is often quite simpler than the general case however, since when both fermion species are identical and massless, we have $i = j$ and $m^2_{f,i} = m^2_{f,j} = 0$. Also, when only in presence of vector couplings, we have $\hat{\Sigma}^{f,S}_{ij}(p^2) = 0$ and $w_-\hat{\Sigma}^{f,L}_{ij}(p^2) + w_+\hat{\Sigma}^{f,R}_{ij}(p^2) \equiv \hat{\Sigma}_f(p^2)$. Moreover, since the self-energy function of massless fermions cannot have an imaginary part (i.e. it does not have classically allowed decay kinematic configurations), we can safely drop the $\tilde{\Re}$ operator.

b Writing the problem

In this section, we demonstrate how applying the subtracted dispersion relation allows to provide an expression of the self-energy that is amenable to numerical computations in four dimensions and reproduces its physical contribution in the on-shell renormalisation scheme discussed in sect. a.

In order to be definite, we will be looking at the specific two Cutkosky cut contributions of a super-graph involving a self-energy insertion:

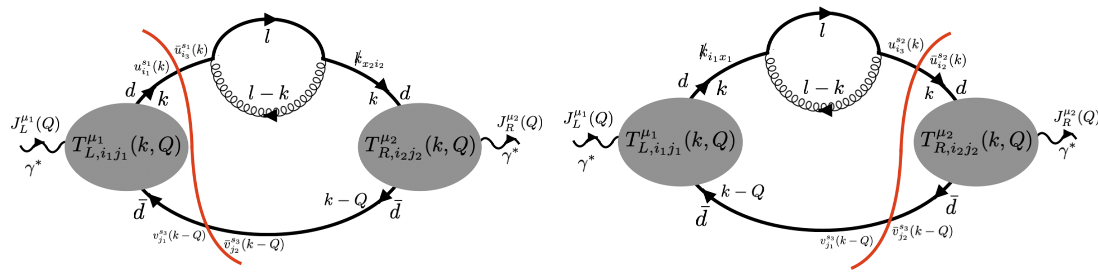

Figure b.1: Cutkosky contributions $\mathrm{C_A}$ (left) and $\mathrm{C_B}$ (right).

where each Cutkoski cut of a four-momentum q translates into the expression $(-2\pi i)\delta^+\left(q^2\right)$. We can now write explicitly the contributions $\mathrm{C_A}$ and $\mathrm{C_B}$ as:

$$
\begin{aligned}
\mathrm{C_A} &= (-2\pi i)^2\delta^+\left((k-Q)^2\right)\delta^+\left(k^2\right)\sum_{s_1\in\pm}\sum_{s_3\in\pm}\left(T^L_{i_1j_1}v^{s_3}_{j_1}u^{s_1}_{i_1}\right)\left(\bar{u}^{s_1}_{i_3}\Sigma^R_{i_3x_2}(k,l)\frac{(-i)\not{k}_{x_2i_2}}{k^2+i\delta}T^R_{i_2j_2}\bar{v}^{s_3}_{j_2}\right)\\
\mathrm{C_B} &= (-2\pi i)^2\delta^+\left((k-Q)^2\right)\delta^+\left(k^2\right)\sum_{s_2\in\pm}\sum_{s_3\in\pm}\left(T^L_{i_1j_1}\frac{(i)\not{k}_{i_1x_1}}{k^2-i\delta}\Sigma^L_{x_1i_3}(k,l)u^{s_2}_{i_3}v^{s_3}_{j_1}\right)\left(\bar{u}^{s_2}_{i_2}T^R_{i_2j_2}\bar{v}^{s_3}_{j_2}\right)
\end{aligned}
\tag{b.1}
$$

with $T_{ij}\equiv J^\mu(k,Q)T^\mu_{ij}(k,Q)$ and where we kept implicit the dependences of the polarization vectors in the loop momenta. The self-energy factors $\Sigma^{L/R}$ are defined as:

$$
\begin{aligned}
\Gamma^R_{ij}(k) &\equiv (-i)(\not{k}_{ij})+\left(\int\frac{\tilde{d}l}{(2\pi)^4}\Sigma^R_{ix}(k,l)\right)(-i)(\not{k}_{xj})\\
\Gamma^L_{ij}(k) &\equiv (i)(\not{k}_{ij})+(i)(\not{k}_{ix})\left(\int\frac{\tilde{d}l}{(2\pi)^4}\Sigma^L_{xj}(k,l)\right)
\end{aligned}
\tag{b.2}
$$

where $\int\tilde{d}l\ \equiv\left(\frac{\mu^2}{4\pi e^{-\gamma}}\right)^\epsilon\int\frac{d^{4-2\epsilon}l}{(2\pi)^{-2\epsilon}}$. When setting all couplings to 1, the explicit expressions of $\Sigma_{ij}^{R/L}$ then reads:

$$
\begin{aligned}
B^R_{i_1 i_2}(k,l) &= \left(\sum_{s_1\in\pm} u^{s_1}_{i_1}(k)\bar{u}^{s_1}_{i_3}(k)\right)\Sigma^R_{i_3 x_2}(k,l)\frac{(-i)\not{k}_{x_2 i_2}}{k^2-i\delta} \\
&= 2C_F(1-\epsilon)[(i)(-i)(i)(i)](-i)\frac{(-\not{k}\not{l}\not{k})_{i_3 i_2}}{(k^2-i\delta)(l^2-i\delta)((l-k)^2-i\delta)} \\
&= 2C_F(1-\epsilon)(-i)\frac{(\not{k}\not{l}\not{k})_{i_1 i_2}}{(k^2-i\delta)(l^2-i\delta)((l-k)^2-i\delta)} = 2C_F(1-\epsilon)\mathcal{F}^R_{i_1 i_2}(k,l)) \\
B^L_{i_1 i_2}(k,l) &= \frac{(i)\not{k}_{i_1 x_1}}{k^2+i\delta}\Sigma^L_{x_1 i_3}(k,l)\left(\sum_{s_2\in\pm} u^{s_2}_{i_3}(k)\bar{u}^{s_2}_{i_2}(k)\right) \\
&= 2C_F(1-\epsilon)(i)[(-i)(i)(-i)(-i)]\frac{(-\not{k}\not{l}\not{k})_{i_1 i_2}}{(k^2+i\delta)(l^2+i\delta)((l-k)^2+i\delta)} \\
&= 2C_F(1-\epsilon)(i)\frac{(\not{k}\not{l}\not{k})_{i_1 i_2}}{(k^2+i\delta)(l^2+i\delta)((l-k)^2+i\delta)} = 2C_F(1-\epsilon)\mathcal{F}^L_{i_1 i_2}(k,l)),
\end{aligned}
\tag{b.3}
$$

where the C_F terms comes from having performed the colour algebra. Our target expression from eq. b.1 cannot be evaluated explicitly given that it is proportional to $\delta^+\left(k^2\right)/k^2$. To address this issue, we will take advaxntage of the dispersion relation which we introduce in the next section.

c Dispersive representation

Any function $f(x)$ and analytic in $\mathbb{C}\setminus(-\infty,-x^\star)\cup(x^\star,\infty)$ can be written using Cauchy's theorem by performing an integral along the contours indicated in fig. c.1 below with $q^0\equiv x$ and $|\vec{q}|\equiv x^\star$: yielding the following dispersive expression for $f(x)$:

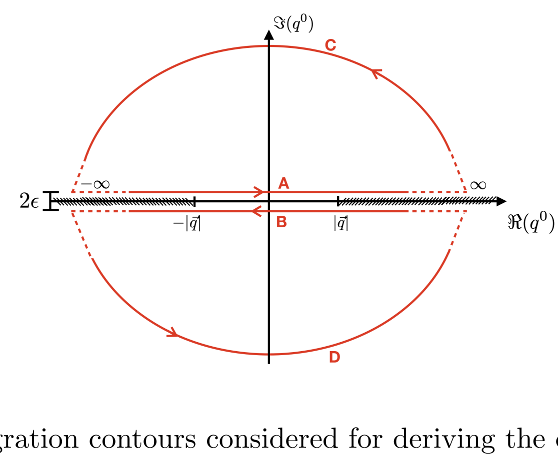

Figure c.1: Integration contours considered for deriving the dispersion relation.

$$\begin{aligned}
f(x) &= \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\left[\underbrace{\int_{-\infty}^{\infty} dz\frac{f(z+i\epsilon)}{z-x+i\epsilon}}_{\text{segment A}}+\underbrace{\int_{\infty}^{-\infty} dz\frac{f(z-i\epsilon)}{z-x-i\epsilon}}_{\text{segment B}}\right]\\
&= \frac{1}{2\pi i}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} dz\frac{f(z+i\epsilon)-f(z-i\epsilon)}{z-x}\\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz\frac{\text{disc}\left[f(z)\right]}{z-x}=\frac{1}{2\pi i}\int_{0}^{\infty} dz\left[\frac{\text{disc}\left[f(z)\right]}{z-x}-\frac{\text{disc}\left[f(-z)\right]}{z+x}\right], \quad \text{(c.1)}
\end{aligned}$$

where we neglected the $\pm i\epsilon$ in the denominators of the second line because we assume that $f(s)$ is analytic everywhere except on $(-\infty,-x^\star)\cup(x^\star,\infty)$ which our contour never crosses. We also used the assumption[1] that $\lim_{|z|\to\infty} f(z)=0$, allowing us to neglect the segments C and D of the contour at infinity. Finally, we find it useful to split the integration domain in the last line of eq. c.1 so as to be able to integrate over only positive z values.

We now consider applying eq. c.1 to the particular case of both functions $\mathcal{F}^L_{i_1 i_2}(k^0,\vec{k},l)$ and $\mathcal{F}^R_{i_1 i_2}(k^0,\vec{k},l)$, and considering $x\equiv k^0$. We will also take advantage of the fact that self-energy functions satisfy (at any order in perturbation theory and also for massive internal lines (?)) the reality condition known as the *Schwartz reflection principle*, namely:

$$\mathcal{F}^{L/R}_{i_1 i_2}(z^*,\vec{k},l)=\mathcal{F}^{L/R}_{i_1 i_2}(z,\vec{k},l)^* \quad \text{(c.2)}$$

which allows us to rewrite the discontinuity as:

$$\begin{aligned}
\text{disc}\left[\mathcal{F}^{L/R}_{i_1 i_2}(z,\vec{k},l)\right] &= \lim_{\epsilon->0^+}\left(\mathcal{F}^{L/R}_{i_1 i_2}(z+i\epsilon,\vec{k},l)-\mathcal{F}^{L/R}_{i_1 i_2}(z-i\epsilon,\vec{k},l)\right)\\
&= \lim_{\epsilon\to 0^+}\left(\mathcal{F}^{L/R}_{i_1 i_2}(z+i\epsilon,\vec{k},l)-\mathcal{F}^{L/R}_{i_1 i_2}(z+i\epsilon,\vec{k},l)^*\right)\\
&= 2i\lim_{\epsilon\to 0^+}\Im\left[\mathcal{F}^{L/R}_{i_1 i_2}(z+i\epsilon,\vec{k},l)\right]\\
&= i\sum_{\mathrm{C}^+_i\in\mathcal{C}^{L/R}_{\mathcal{F}}}\mathrm{C}^+_i\left[\mathcal{F}^{L/R}_{i_1 i_2}(z,\vec{k},l)\right], \quad \text{(c.3)}
\end{aligned}$$

where $\mathcal{C}^{L/R}_{\mathcal{F}}$ denote the list of Cutkosky cuts the integral $\mathcal{F}^{L/R}$ is subject to while the application of the *signed* Cutkosky operator $C^\pm_i$ appearing on the last line reads:

$$C^\pm_i=(\mp 2\pi i)^{|\text{cuts}_i|}\prod_{j\in\text{cuts}_i} q_j^2(z)\delta^\pm\left(q_j^2(z)\right) \quad \text{(c.4)}$$

(**WARNING:** at this stage it seems that the above Cutkosky rule should *not* involve a complex conjugation on the expression of the graph sitting on the right-hand side of the cut!). We note that the last equality in eq. c.3 stems from the Cutkosky rule that expresses the discontinuity of loop integrals, as written in Eq. (55) of ref. [2]. Also, for a complete unknown reason **HELP HERE!**, the discontinuity of $\mathcal{F}^L$ is only non-zero for

[1] Notice that for this assumption to hold, it will be important that we include the denominator $\frac{1}{k^2}$ in the definition of $\mathcal{F}^{L/R}$ in eq. b.3. We stress that this property is then completely independent of whether or not one has performed UV renormalisation at this stage.

positive energy k^0 while $\mathcal{F}^R$ only contributes to the discontinuity at a *negative* k^0. We thus arrive at the following final expression for the re-expression of $\mathcal{F}^{L/R}$ as quantities we will denote $\tilde{\mathcal{F}}^R$ and that are computed through the dispersive relation:

$$
\begin{aligned}
\tilde{\mathcal{F}}^L_{i_1 i_2}(k^0,\vec{k},l) &= \frac{1}{2\pi}\int_0^\infty dz \frac{\sum_{\mathrm{C}_i^+\in\mathcal{C}_{\mathcal{F}}^L} \mathrm{C}_i^+\left[\mathcal{F}^L_{i_1 i_2}(z,\vec{k},l)\right]}{z-k^0} \\
\tilde{\mathcal{F}}^R_{i_1 i_2}(k^0,\vec{k},l) &= -\frac{1}{2\pi}\int_0^\infty dz \frac{\sum_{\mathrm{C}_i^-\in\mathcal{C}_{\mathcal{F}}^R} \mathrm{C}_i^-\left[\mathcal{F}^R_{i_1 i_2}(-z,\vec{k},l)\right]}{z+k^0}
\end{aligned}
\tag{c.5}
$$

PS: Ideally one should be able to ditch the "dispersive" representation and find a way to arrive at the same expressions starting from the two-loop vacuum diagram formed by the one-loop bubble with the external leg closed onto itself: Then maybe one could proceed

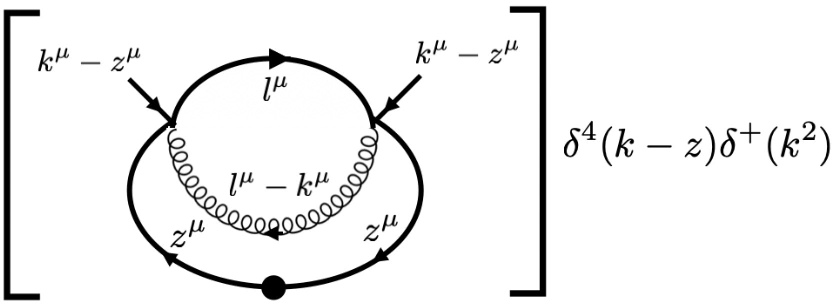

Figure c.2: Possible alternative representation of the dispersive representation that could be entirely derived within our LTD framework.

like for normal two-loop LTD and finally impose $\delta^4(z-k)\delta(k^2)$ on the final expression, although it is not clear how to do so at this stage...

d Direct evaluation

We now turn to directly evaluating the two expressions of Eq. c.5. A first thing to note is that the set of cuts in $\mathcal{C}_{\mathcal{F}}^{L/R}$ are identical, except for their sign, in both the L and R contributions and it reads:

$$
\mathcal{C}_{\mathcal{F}}^{L/R} \equiv \{(2\pi i)^2(l^2)(l-k)^2\delta^{+/-}\left(l^2\right)\delta^{+/-}\left((l-k)^2\right)\} \tag{d.1}
$$

It is also worth noting that it is important that we do not consider further complex conjugation in the application of the Cutkosky operator above, as this would yield an extra incorrect minus sign in contribution $\mathrm{C_B}$ of fig. b.1. The lack of complex conjugation in the application of these *bubble Cutkosky cuts* also implies that the sign of the causal prescription involved in the contributions $\mathcal{F}^{L/R}_{i_1 i_2}$ remains unchanged!

It is useful at this stage to provide a graphical representation of the dispersion relation of eq. c.5 combined with the overall contributions $\mathrm{C_A}$ and $\mathrm{C_B}$ of fig. b.1: Pay attention in particular how the polarization vectors $u^{s_1}_{i_1}(k)$ and $\bar{u}^{s_2}_{i_2}(k)$ have been pulled *inside* the

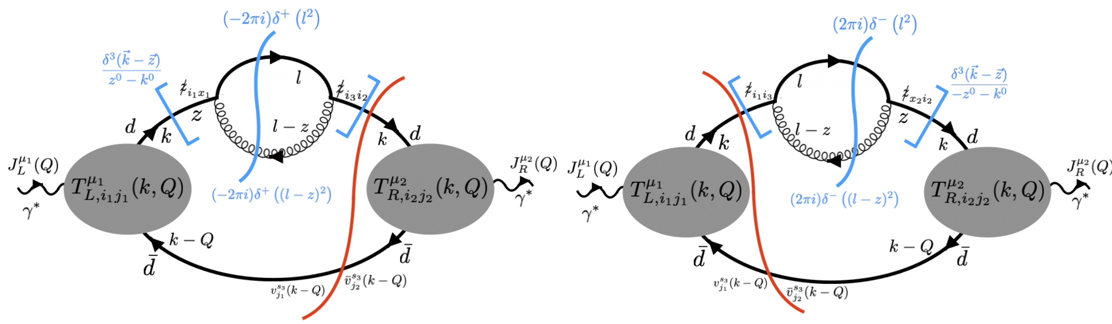

Figure d.1: Contributions $\mathrm{C_A}$ (left) and $\mathrm{C_B}$ (right), including bubble cuts (in blue) stemming from the dispersive representation of eq. c.5. The squared blue cut indicate a break in the energy conservation between $z^\mu = (z, \vec{k})$ and $k^\mu = (k^0, \vec{k})$.

expression of the bubble subject to the dispersive representation. This implies that the momentum dependency in these polarisation vectors changes from k to z, thus allowing us to simplify the polarisation sum into $\not{z}_{i_3 i_2}$ and $\not{z}_{i_1 i_3}$ respectively, as done in eq. b.3.

We are now ready to explicitly compute $\tilde{\mathcal{F}}^L_{i_1 i_2}(k^0, \vec{k}, l)$ as in eq. c.5, while also pulling in the integration measure $\int \frac{\tilde{dl}}{(2\pi)^4}$ and swapping the order of integration with $\int dz$:

$$
\begin{aligned}
\mathrm{C_B}(k) &= \underbrace{T^L_{j_1 i_1}(\not{k} - \not{Q})_{j_2 j_1} T^R_{i_2 j_2} 2 C_F (1-\epsilon)}_{T_{i_2 i_1}(k,Q,\epsilon)} \int \frac{\tilde{dl}}{(2\pi)^4} \tilde{\mathcal{F}}^L_{i_1 i_2}(k^0, \vec{k}, l) \\
&= T_{i_2 i_1} \int \frac{\tilde{dl}}{(2\pi)^4} \frac{1}{2\pi} \int_0^\infty dz^0 \frac{\sum_{\mathrm{C}_i^+ \in \mathcal{C}^L_{\mathcal{F}}} \mathrm{C}_i^+ \left[\mathcal{F}^L_{i_1 i_2}(z^0, \vec{k}, l) \right]}{z^0 - k^0} \\
&= T_{i_2 i_1} \int \frac{\tilde{d}^3 \vec{l}}{(2\pi)^4} \frac{(-2\pi i)^2}{2\pi} \int_0^\infty dz^0 \frac{1}{z^0 - k^0} \frac{(i) \not{z} \not{l} \not{z}}{z^2 (2|\vec{l}|)(2|\vec{l} - \vec{k}|)} \delta\left(z^0 - |\vec{l}| - |\vec{l} - \vec{k}| \right),
\end{aligned} \tag{d.2}
$$

with $z \equiv (z^0, \vec{k})$. For $\mathrm{C_A}$ we get:

$$
\begin{aligned}
\mathrm{C_A}(k) &= T_{i_2 i_1} \int \frac{\tilde{dl}}{(2\pi)^4} \tilde{\mathcal{F}}^R_{i_1 i_2}(k^0, \vec{k}, l) \\
&= T_{i_2 i_1} \int \frac{\tilde{dl}}{(2\pi)^4} \frac{1}{2\pi} \int_0^\infty dz^0 \frac{\sum_{\mathrm{C}_i^- \in \mathcal{C}^R_{\mathcal{F}}} \mathrm{C}_i^- \left[\mathcal{F}^R_{i_1 i_2}(-z^0, \vec{k}, l) \right]}{z^0 - k^0} \\
&= T_{i_2 i_1} \int \frac{\tilde{d}^3 \vec{l}}{(2\pi)^4} \frac{(2\pi i)^2}{2\pi} \int_0^\infty dz^0 \frac{1}{-z^0 - k^0} \frac{(-i) \bar{\not{z}} \not{l} \bar{\not{z}}}{\bar{z}^2 (-2|\vec{l}|)(-2|\vec{l} - \vec{k}|)} \delta\left(-z^0 + |\vec{l}| + |\vec{l} - \vec{k}| \right)
\end{aligned} \tag{d.3}
$$

with $\bar{z} \equiv (-z^0, \vec{k})$. When combining $\mathrm{C_A}$ and $\mathrm{C_B}$ we find:

$$\begin{aligned} \mathrm{C_B} + \mathrm{C_A} &= (-i) T_{i_2 i_1} \int \frac{\tilde{d}^3 \vec{l}}{(2\pi)^3} \int_0^\infty dz^0 \frac{\delta\left(z^0 - |\vec{l}| - |\vec{l} - \vec{k}|\right)}{(z^0)^2 - (k^0)^2} \frac{1}{(2|\vec{l}|)(2|\vec{l} - \vec{k}|)} \\ &\times \left(\frac{z^0 + k^0}{z^2} \not{z} \not{l} \not{z} + \frac{z^0 - k^0}{\bar{z}^2} \not{\bar{z}} \not{l} \not{\bar{z}} \right) \end{aligned} \tag{d.4}$$

We can easily show that $z^2 = \bar{z}^2$ and also that:

$$\begin{aligned} \not{\bar{z}} \not{l} \not{\bar{z}} &= (-z^0 \gamma^0 + \vec{k} \cdot \vec{\gamma}) \not{l} (-z^0 \gamma^0 + \vec{k} \cdot \vec{\gamma}) \\ &= \not{z} \not{l} \not{z} - 2 z^0 \left(\gamma^0 \not{l} (\vec{k} \cdot \vec{\gamma}) + (\vec{k} \cdot \vec{\gamma}) \not{l} \gamma^0 \right) \\ &= -\not{z} \not{l} \not{z} + 2 (z^0)^2 \gamma^0 \not{l} \gamma^0 + 2 (\vec{k} \cdot \vec{\gamma}) \not{l} (\vec{k} \cdot \vec{\gamma}), \end{aligned} \tag{d.5}$$

which allows us to write the final expression for $\mathrm{C_B}+\mathrm{C_A}$ when performing the integration over z^0 with $\delta\left(z^0 - |\vec{l}| - |\vec{l} - \vec{k}|\right)$ and introducing the variable $z^\star \equiv |\vec{l}| + |\vec{l} - \vec{k}|$:

$$\begin{aligned} &\mathrm{C_B} + \mathrm{C_A} = (-i) T_{i_2 i_1} \\ &\times \int \frac{\tilde{d}^3 \vec{l}}{(2\pi)^3} \frac{z^\star \left(2 (z^\star)^2 \gamma^0 \not{l} \gamma^0 + 2 (\vec{k} \cdot \vec{\gamma}) \not{l} (\vec{k} \cdot \vec{\gamma}) \right) + k^0 \left(2 z^\star \left(\gamma^0 \not{l} (\vec{k} \cdot \vec{\gamma}) + (\vec{k} \cdot \vec{\gamma}) \not{l} \gamma^0 \right) \right)}{((z^\star)^2 - (k^0)^2) \left((z^\star)^2 - |\vec{k}|^2 \right) \left(2 |\vec{l}| \, 2 |\vec{l} - \vec{k}| \right)} \end{aligned} \tag{d.6}$$

When considering the case of an external bubble (as in fig. d.1), we can pull in the Cutkosky cut $(-2\pi i) \delta^2 \left(k^2\right)$ which simplifies the above a bit as follows:

$$\begin{aligned} &\left(\mathrm{C_B} + \mathrm{C_A}\right) (-2\pi i) \delta^2 \left(k^2\right) = \frac{(-2\pi i)}{2|\vec{k}|} (-i) T_{i_2 i_1} \\ &\times \int \frac{\tilde{d}^3 \vec{l}}{(2\pi)^3} \frac{z^\star \left(2 (z^\star)^2 \gamma^0 \not{l} \gamma^0 + 2 (\vec{k} \cdot \vec{\gamma}) \not{l} (\vec{k} \cdot \vec{\gamma}) \right) + |\vec{k}| \left(2 z^\star \left(\gamma^0 \not{l} (\vec{k} \cdot \vec{\gamma}) + (\vec{k} \cdot \vec{\gamma}) \not{l} \gamma^0 \right) \right)}{\left((z^\star)^2 - |\vec{k}|^2 \right)^2 \left(2 |\vec{l}| \, 2 |\vec{l} - \vec{k}| \right)} . \end{aligned} \tag{d.7}$$

As we can see, the above expression of the quark self-energy is not as elegant as eq.(336) of Soper's Beowulf notes, because we did not choose a smart momentum routing allowing him to perform the Lorentz algebra shown in his eq.(331). This implies that we could not explicitly cancel the $\left((z^\star)^2 - |\vec{k}|^2\right)$ denominator. However, our expression of eq. d.7 is formally equivalent to Soper's one, as explicitly show in the Mathematica notebook accompanying these notes.

The expression of eq. d.7 has a superficial degree of UV divergence in $|\vec{l}|$ of $7 - 6 = 1$ (as in Soper's case) which therefore calls for local UV counterterms cancelling both the leading *and subleading* UV behaviour and reproducing the on-shell renormalisation conditions of eqs. a.2-a.5. This is the object of the next section.

e UV local counterterms

In this section, we aim at designing local UV counterterms for the contributions $\tilde{\mathcal{F}}^{L}_{i_1 i_2}(k^0,\vec{k},l)$ and $\tilde{\mathcal{F}}^{R}_{i_1 i_2}(k^0,\vec{k},l)$. We will denote these counterterms $\bar{\mathcal{F}}^{L}_{i_1 i_2}(k^0,\vec{k},l)$ and $\bar{\mathcal{F}}^{R}_{i_1 i_2}(k^0,\vec{k},l)$ respectively and they must be designed so as satisfy the following properties (remember that $\not{k}\not{k} = k^2$):

- C1) $\lim_{|\vec{l}|\to\infty} |\vec{l}|^3 \left[\tilde{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l) - \bar{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l)\right] = 0$
- C2) $\lim_{k^0\to|\vec{k}|} \not{k} \int \tilde{d}l \left[\tilde{\mathcal{F}}^{L}_{i_1 i_2}(k^0,\vec{k},l) - \bar{\mathcal{F}}^{L}_{i_1 i_2}(k^0,\vec{k},l)\right] = 0$ (from eq. a.2)
- C3) $\lim_{k^0\to|\vec{k}|} \int \tilde{d}\, l \left[\tilde{\mathcal{F}}^{R}_{i_1 i_2}(k^0,\vec{k},l) - \bar{\mathcal{F}}^{R}_{i_1 i_2}(k^0,\vec{k},l)\right] \not{k} = 0$ (from eq. a.3)
- C4) $\lim_{k^0\to|\vec{k}|} \int \tilde{d} \left[\tilde{\mathcal{F}}^{L}_{i_1 i_2}(k^0,\vec{k},l) - \bar{\mathcal{F}}^{L}_{i_1 i_2}(k^0,\vec{k},l)\right] = 0$ (directly from eq. a.4)
- C5) $\lim_{k^0\to|\vec{k}|} \int \tilde{d} \left[\tilde{\mathcal{F}}^{R}_{i_1 i_2}(k^0,\vec{k},l) - \bar{\mathcal{F}}^{R}_{i_1 i_2}(k^0,\vec{k},l)\right] = 0$ (directly from eq. a.5)

We already see here an apparent difference w.r.t Soper's approach where he insists on enforcing the following on his UV counterterm:

$$\lim_{k^0\to|\vec{k}|} \int \tilde{d}l\bar{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l) = 0, \tag{e.1}$$

which would correspond to the $\overline{\text{MS}}$ renormalisation conditions which I believe are not appropriate in the context of the renormalisation of external self-energies.

In order to separate the task of satisfying condition C1) and that of satisfying eqs. C2)→C5), we will write our counterterms as:

$$\bar{\mathcal{F}}^{L/R}_{i_1 i_2} = \hat{\mathcal{F}}^{L/R}_{i_1 i_2} + \Delta\bar{\mathcal{F}}^{L/R}_{i_1 i_2}, \tag{e.2}$$

where

$$\lim_{|\vec{l}|\to\infty} |\vec{l}|^3 \Delta\bar{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l) = 0 \tag{e.3}$$

$$\lim_{k^0\to|\vec{k}|} \int \tilde{d}\Delta\bar{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l) = \lim_{k^0\to|\vec{k}|} \int \tilde{d} \left[\tilde{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l) - \hat{\mathcal{F}}^{L/R}_{i_1 i_2}(k^0,\vec{k},l)\right]. \tag{e.4}$$

In other terms, the role of $\hat{\mathcal{F}}^{L/R}_{i_1 i_2}$ is to guarantee on its own that C1) is satisfied while the role of $\Delta\bar{\mathcal{F}}^{L/R}_{i_1 i_2}$ is *only* to insure that eqs. C2)→C5) are satisfied.

In this section, we will focus only on designing $\hat{\mathcal{F}}^{L/R}_{i_1 i_2}$. We first note that it would be simple to design such counterterm if we could start from a clear four-dimensional picture similar to that of Fig. c.2. Unfortunately, our current understanding of the relation between LTD and the dispersive formula does not allow us to do this and we must start from the last line of eq. d.2, where the integration over l^0 has already be performed. Within the dispersive approach we can only consider the UV behaviour *after* this energy integration as otherwise the UV divergence due to the apparition of

z^0 in the denominator would not be manifest. Our starting expression to approximate is therefore:

$$\mathrm{C_B}\,(k) = T_{i_2 i_1} \int \frac{\tilde{d}^3\vec{l}}{(2\pi)^3}(-i)\frac{1}{z^\star - k^0}\frac{\not{z}\not{l}\not{z}}{z^2(2|\vec{l}|)(2|\vec{l}-\vec{k}|)}, \tag{e.5}$$

with $z^\star = |\vec{l}| + |\vec{l}-\vec{k}|$ and $z = (z^\star, \vec{k})$. Since the integrand goes like $|\vec{l}|^{-2}$ we will need to subtract *both* the leading and subleading UV limit for condition C1) to be satisfied. The brute force way to proceed is to write $\vec{l} = t\,\hat{\vec{l}}$ thus obtaining:

$$|\vec{l}| \;=\; t\sqrt{\hat{\vec{l}}\cdot\hat{\vec{l}}} \tag{e.6}$$

$$|\vec{l}-\vec{k}| \;=\; \sqrt{t^2\,\hat{\vec{l}}\cdot\hat{\vec{l}} - t\,2\hat{\vec{l}}\cdot\vec{k} + \vec{k}\cdot\vec{k}}. \tag{e.7}$$

We can then expand the resulting expression in t and retain all terms up to (and including) $\mathcal{O}(t^{-3})$. Even though this would certainly be a valid candidate for $\hat{\mathcal{F}}^{L/R}_{i_1 i_2}$, it is not clear how it relates to the one obtained by Soper and also whether we could easily accommodate the resulting denominators as "normal propagators" fitting our Rust implementation of the bubble problem.

f Matching renormalisation conditions

In this section, we carry out the computing of the quantities $\Delta\bar{\mathcal{F}}^{L/R}_{i_1 i_2}$.

TODO: In principle this should "simply" amount to integrating eq. e.4 in dimensional regularisation and reverse-engineering a local expression integrating to that quantity... I don't see a systematic way of doing this however. Maybe contact Soper on the matter...

References

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