1 Dipole-, FKS-Subtraction and MG5-conventions for $gg \to H$

The issue addressed in this note is how to convert between the $\overline{MS}$-scheme and dipole-subtraction used in multi-loop calculation to the renormalization and subtraction scheme used in MG5.

1.1 Renormalization

The α_S renormalization in $\overline{MS}$-amplitudes is:

$$\alpha_S^0 = \alpha_S(\mu^2)\mu^{2\varepsilon}S_\varepsilon^{-1}(1 \underbrace{-\frac{\beta_0}{\varepsilon}}_{:=Z_1 \cdot 2\pi} \frac{\alpha_S}{2\pi} + \mathcal{O}(\alpha_s^2)) \tag{1.1}$$

where $S_\varepsilon = (4\pi)^\varepsilon \exp(-\gamma_E \varepsilon)$. This means the renormalized NLO amplitude in $\overline{MS}$ reads:

$$\begin{aligned} A_{1,MS}^R &= \alpha_S^2 S_\varepsilon^{-2} A_1^0 + S_\varepsilon^{-1}\alpha_S^2 Z_1 A_0 && (1.2)\\ &= \alpha_S^2 S_\varepsilon^{-1}\left(S_\varepsilon^{-1}A_1^0 + Z_1 A_0\right) && (1.3) \end{aligned}$$

where we assume the μ-dependence to be included implicitly included in the A_i.

The α_S renormalization in MG5 (assuming no heavy flavours) can be deduced from (B.5) in [1] as

$$\alpha_S^0 = \alpha_S(\mu^2)(1 + N_\varepsilon Z_1 \alpha_S) \tag{1.4}$$

with

$$N_\varepsilon = (4\pi)^\varepsilon \Gamma(1+\varepsilon) = S_\varepsilon + \mathcal{O}(\varepsilon^2). \tag{1.5}$$

The renormalized amplitude in MG5 is (up to $\mathcal{O}(\varepsilon)$):

$$\begin{aligned} A_{1,MG5}^R &= \alpha_S^2 A_1^0 + S_\varepsilon \alpha_S^2 Z_1 A_0 && (1.6)\\ &= \alpha_S^2 S_\varepsilon\left(S_\varepsilon^{-1}A_1^0 + Z_1 A_0\right). && (1.7) \end{aligned}$$

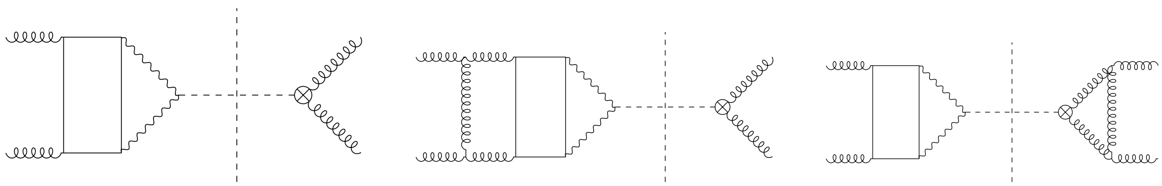
To summarize our findings: We have:

$$A_{1,MG5}^R = S_\varepsilon^2 A_{1,MS}^R \tag{1.8}$$

2 Subtraction Schemes

The name of the game in this section is to match matrix elements obtained from Catani-subtracted $\overline{MS}$-renormalized amplitudes to the ones considered in MadGraph [1] , which are treated in FKS-subtraction. [2] (see eq. (B.1) therein.)

Let us know look at the renormalized matrix elements of the virtual corrections to our Born. We denote A_i as the LO mixed amplitude with i-virtual corrections. B_i is the LO



(a) $M^{0,0} = 2\Re(A_0 B_0^*)$ (b) $M^{1,0} = 2\Re(A_1 B_0^*)$ (c) $M^{0,1} = 2\Re(A_0 B_1^*)$

Figure 1: Relevant matrix elements

HEFT amplitude with i-virtual corrections. The Born matrix element is denoted as $M^{0,0}$ (see fig. 1a), the interference of A_1 with the LO-Heft B_0 is $M^{1,0}$ (see fig. 1b) and the interference of the NLO-Heft with the LO mixed amplitude is is $M^{0,1}$ (see fig. 1c).

2.1 $\overline{MS}$-renormalized and Catani subtracted matrix-elements

The finite amplitude and the renormalized amplitude in $\overline{MS}$ often used in the literature are related by the Catani operator [3]

$$I_1 = \left(\frac{-s - i0}{\mu^2}\right)^{-\varepsilon} \frac{e^{\gamma_E \varepsilon}}{\Gamma(1-\varepsilon)} \left[-\frac{C_A}{\varepsilon^2} - \frac{\beta_0}{\varepsilon}\right] \tag{2.1}$$

$$= \left(\frac{-s - i0}{\mu^2}\right)^{-\varepsilon} \frac{S_\varepsilon^{-1}}{\Gamma(1-\varepsilon)} (4\pi)^\varepsilon \left[-\frac{C_A}{\varepsilon^2} - \frac{\beta_0}{\varepsilon}\right] \tag{2.2}$$

as

$$A^{fin}_{1,MS} = A^R_{1,MS} - \overbrace{\alpha_S^2 I_1 \left(S_\varepsilon^{-1} A_0\right)}^{=:\tilde{I}_1 A_0} . \tag{2.3}$$

We have the relevant NLO-virtual correction matrix-elements:

$$\frac{1}{2} M^{1,0}_{MS,R} = \Re(A^R_{1,MS})\Re(B_0) + \Im(A^R_{1,MS})\Im(B_0) \tag{2.4}$$

$$= \Re(\underline{A^{Fin}_{1,MS}} + \tilde{I}_1 A_0)\Re(B_0) + \Im(\underline{A^{Fin}_{1,MS}} + \tilde{I}_1 A_0)\Im(B_0) \tag{2.5}$$

$$= \underline{\frac{1}{2} M^{1,0}_{fin,MS}} + \left[\left(\Re(\tilde{I}_1)\Re(A_0) - \Im(\tilde{I}_1)\Im(A_0)\right)\right] \Re(B_0) \tag{2.6}$$

$$+ \left[\left(\Re(\tilde{I}_1)\Im(A_0) + \Im(\tilde{I}_1)\Re(A_0)\right)\right] \Im(B_0) \tag{2.7}$$

$$= \frac{1}{2} M^{1,0}_{fin,MS} + \frac{1}{2}\Re(\tilde{I}_1) M^{0,0} + \Im(\tilde{I}_1) \left[\Re(A_0)\Im(B_0) - \Im(A_0)\Re(B_0)\right] , \tag{2.8}$$

and analogously

$$\frac{1}{2} M^{0,1}_{MS,R} = \frac{1}{2} M^{0,1}_{fin,MS} + \frac{1}{2}\Re(\tilde{I}_1) M^{0,0} + \Im(\tilde{I}_1) \left[\Re(B_0)\Im(A_0) - \Im(B_0)\Re(A_0)\right] . \tag{2.9}$$

So we have:

$$M^R_{virt,MS} = M^{0,1}_{MS,R} + M^{1,0}_{MS,R} \tag{2.10}$$
$$= M^{0,1}_{fin,MS} + M^{1,0}_{fin,MS} + 2\Re(\tilde{I}_1)M^{0,0} \tag{2.11}$$
$$=: M_{fin,MS} + 2\Re(\tilde{I}_1)M^{0,0}. \tag{2.12}$$

2.2 MadGraph-renormalized and FKS subtracted matrix-elements

The discussion of these matrix-elements is performed at $Q^2 = \mu^2 = m_H^2$. In eq. (B.1) in the FKS-paper [2] they define (the complete discussion therein is assuming the Born to be a squared amplitude):

$$M^R_{virt,FKS} = \frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\frac{\alpha_S}{2\pi}V^R. \tag{2.13}$$

For our amplitudes we have $n_i = 1$, $n_L = 0$ and $n_H = 0$ which boils there eq. (B.2) down to (see eq. (3.30) and (4.7) in [2]):

$$V^R = -\left(\frac{1}{\varepsilon^2}\sum_{k=1}^{2} C(\mathcal{I}_k) + \frac{1}{\varepsilon}\sum_{k=1}^{2}\gamma(\mathcal{I}_k)\right) M^{0,0}_{FKS} + V_{fin} \tag{2.14}$$
$$= 2\left(-\frac{C_A}{\varepsilon} - \frac{\beta_0}{\varepsilon}\right) M^{0,0} + V_{fin} \tag{2.15}$$

Which means:

$$M^R_{virt,FKS}|_{\mu^2=m_H^2} = 2\frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\left(-\frac{C_A}{\varepsilon} - \frac{\beta_0}{\varepsilon}\right) M^{0,0} + V_{fin}. \tag{2.16}$$

Let us now relate the matrix-elements. The relation for the renormalized matrix-elements follows directly from eq. (1.8) as:

$$M^R_{virt,FKS} = S_\varepsilon^2 M^R_{virt,MS} \tag{2.17}$$
$$= S_\varepsilon^2\left(2\frac{S_\varepsilon^{-2}}{\Gamma(1-\varepsilon)}(4\pi)^\varepsilon\left[-\frac{C_A}{\varepsilon^2} - \frac{\beta_0}{\varepsilon}\right]\Re\left(\left(\frac{-s-i0}{\mu^2}\right)^{-\varepsilon}\right) M^{0,0} + M_{fin,MS}\right) \tag{2.18}$$
$$= 2\frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\left[-\frac{C_A}{\varepsilon^2} - \frac{\beta_0}{\varepsilon}\right]\Re\left(\left(\frac{-s-i0}{\mu^2}\right)^{-\varepsilon}\right) M^{0,0} + M_{fin,MS}. \tag{2.19}$$

In particular at $\mu^2 = m_H^2$ we have:

$$\Re\left(\left(\frac{-s-i0}{\mu^2}\right)^{-\varepsilon}\right) = \cos(\varepsilon\pi) = 1 - \frac{\pi^2\varepsilon^2}{2} + \mathcal{O}\left(\varepsilon^3\right). \tag{2.20}$$

Which means in follow:

$$M^R_{virt,FKS}|_{\mu^2=m_H^2} = 2\frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\left[-\frac{C_A}{\varepsilon^2} - \frac{\beta_0}{\varepsilon}\right] M^{0,0} + C_A\pi^2 M^{0,0} + M_{fin,MS}. \tag{2.21}$$

and more importantly:

$$\boxed{V_{fin} = C_A \pi^2 M^{0,0} + M_{fin,MS}} \tag{2.22}$$

!This relation holds only at $\mu^2 = m_H^2$!

3 Implications for the Implementation

3.1 Implication for the Interference with the Electroweak Amplitudes

In the massless EW-paper we computed we saw that there is no β_0-dependence in the final result[1]. I will now check, that there is indeed no β_0-dependence left, in what will become V_{fin} in our implementation. Let us write the electroweak-amplitudes as:

$$A_{EW} = T^{\mu\nu} i \lambda_{EW} \left(F_{EW}^{(0)} + \frac{\alpha_S}{2\pi} F_{EW}^{(1)} \right) . \tag{3.1}$$

with $\lambda_{EW} \in \mathbb{R}$. The finite part of $F_{EW}^{(1)}$, after Catani subtraction is given in [4] as A_{NLO}^{fin}, which only has a β_0-dependence factorizing the LO[2]:

$$A_{NLO,EW}^{fin} = \tilde{F}_{EW}^{(1)} - \beta_0 \left(\log\left(\frac{m_H^2}{\mu^2} \right) - i\pi \right) F_{EW}^{(0)} \tag{3.2}$$

The HEFT amplitudes are:

$$A_{HEFT} = T^{\mu\nu} C_{QCD} i \left(F_{HEFT}^{(0)} + \frac{\alpha_S}{2\pi} F_{HEFT}^{(1)} \right) \ . \tag{3.3}$$

with $F_{HEFT}^{(0)} = 1$ The $\overline{MS}$-renormalized Catani subtracted HEFT-form factor is:

$$A_{NLO,HEFT}^{fin} = \underbrace{\frac{11}{2}}_{\tilde{F}_{HEFT}^{(1)}} - \beta_0 \left(\log\left(\frac{m_H^2}{\mu^2} \right) - i\pi \right) F_{HEFT}^{(0)} + O\left(\varepsilon^1\right) \ . \tag{3.4}$$

Our Matrix element will basically be:

$$\Re(F_{EW}^{(0)} A_{NLO,HEFT}^{fin}{}^{\star}) + \Re(F_{HEFT}^{(0)} A_{NLO,EW}^{fin}{}^{\star}) \tag{3.5}$$

$$= \Re(F_{EW}^{(0)}) \tilde{F}_{HEFT}^{(1)} + \beta_0 \pi \Im(F_{EW}^{(0)}) F_{HEFT}^{(0)} \tag{3.6}$$

$$+ \Re(\tilde{F}_{EW}^{(1)}) F_{HEFT}^{(0)} - \beta_0 \pi \Im(F_{EW}^{(0)}) F_{HEFT}^{(0)} \tag{3.7}$$

$$= \frac{11}{4} \Re(F_{EW}^{(0)}) + \Re(\tilde{F}_{EW}^{(1)}) \tag{3.8}$$

which is indeed free of β_0. So I am going to construct eq. (2.22) for the finite NLO virtuals and put it in the code. I furthermore checked, that indeed using eq. (2.22) gives the correct result for the pure HEFT.

[1]There was an Mail exchange regarding that, since in the soft approximation paper it was still in what is called virtuals therein.

[2]I checked that by expressing N_f as β_0 and verifying that the coefficient is $\propto F_{EW}^{(0)}$

References

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