Investing for the Long Run when Returns are Predictable

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Chapter 1

Introduction

This assignment focuses on an in-depth study and extension of Nicholas Barberis' paper, "Investing for the Long Run When Returns Are Predictable". The paper addresses a fundamental question in financial research: how does the predictability of asset returns affect optimal portfolio allocation for investors with different horizons? Barberis examines this question from both theoretical and empirical perspectives, providing a comprehensive exploration of investment strategies that take into account estimation risk and horizon effects. The findings of the paper are particularly relevant in understanding how investors can balance risk and return while considering uncertainties in predictive models.

The paper opens with an empirical observation that asset returns, such as stock returns, exhibit evidence of predictability based on variables like dividend yields. This challenges classical investment theories, such as those of Samuelson and Merton, which suggest that an investor's horizon does not influence portfolio allocation when returns are independently and identically distributed (i.i.d.). Barberis extends this line of inquiry by considering how time-varying expected returns, coupled with estimation risk, create "horizon effects"—differences in asset allocation strategies depending on the investment horizon.

Barberis divides the study into distinct analytical components:

Theoretical Framework for Portfolio Choice: The paper develops a framework for understanding how predictability and parameter uncertainty influence optimal portfolio decisions. Using a Bayesian approach, Barberis accounts for estimation risk by incorporating the posterior distribution of model parameters into the analysis. This approach provides a "middle ground" between ignoring uncertainty and assuming perfect knowledge of parameters. The model allows for comparing static (buy-and-hold) and dynamic (optimal rebalancing) strategies, offering insights into how investors should adjust allocations over time.

Static Buy-and-Hold Portfolios: The analysis first examines long-term, static buy-and-hold strategies where investors allocate their wealth at the start of the horizon and do not rebalance. Results indicate that long-horizon investors allocate significantly more to equities due to mean reversion in returns, which reduces the perceived risk over time. However, when parameter uncertainty is incorporated, this effect diminishes, highlighting the critical role of estimation risk in moderating equity exposure.

Dynamic Rebalancing Portfolios: For dynamic strategies, where investors periodically rebalance based on updated information, Barberis shows that long-term investors may hold additional equities to hedge against changes in investment opportunities. This "hedging demand," however, depends on investors' risk aversion levels and the degree of predictability in returns. Interestingly, the study finds that incorporating parameter uncertainty can reverse some of the initial findings, leading to lower equity allocations for long-horizon investors.

Impact of Estimation Risk on Allocation Sensitivity: A core contribution of the paper is its treatment of estimation risk, which affects both the perceived variance of returns and the sensitivity of portfolio allocation to predictive variables like dividend yields. The study reveals that accounting for parameter uncertainty not only reduces the allocation to equities but also dampens the variability of portfolio composition over time. This ensures that investors are less prone to extreme shifts in allocation based on fluctuating predictor values.

Predictability and Parameter Uncertainty in VAR Models: Barberis employs a vector autoregression (VAR) framework to model the relationship between asset returns and predictor variables. By comparing scenarios with and without predictability, the paper demonstrates how time-varying expected returns influence both mean and variance dynamics. The analysis highlights that even weak predictability can significantly alter optimal portfolio choices, especially when coupled with high parameter uncertainty.

Building on Barberis' findings, this assignment replicates the key results of the paper using historical financial data and Bayesian methods to construct both static and dynamic portfolios. Additionally, we extend the study by exploring personalized portfolio strategies that incorporate individual risk tendencies. This extension considers behavioral finance perspectives, examining how varying levels of risk aversion and return predictability influence allocation decisions. Specifically, we propose a personalized asset management framework that adapts the Barberis model to individual investors, offering tailored recommendations for equity exposure based on their unique risk profiles.

By combining a detailed replication of Barberis' work with an innovative extension into personalized portfolio management, this assignment aims to provide a comprehensive analysis of the intersection between predictability, parameter uncertainty, and behavioral factors in portfolio optimization.

Chapter 2

Theory

2.1 VAR (vector autoregressive) model

A simple yet effective model for analyzing and modeling asset returns is the *vector autoregressive* (VAR) model. A multivariate time series, denoted by $\mathbf{r}_t$, is said to follow a VAR process of order 1, or VAR(1), if it satisfies the following relationship:

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \Phi \mathbf{r}_{t-1} + \mathbf{a}_t, \tag{2.1}$$

where:

- $\boldsymbol{\phi}_0$ is a k-dimensional vector of intercept terms,
- Φ is a $k \times k$ matrix of coefficients that captures the dependence on lagged values,
- $\mathbf{a}_t$ is a sequence of serially uncorrelated random vectors with mean zero and covariance matrix Σ.

In practical applications, the covariance matrix Σ is required to be *positive definite*; otherwise, the dimensionality of $\mathbf{r}_t$ can be reduced. Additionally, it is common in the literature to assume that $\mathbf{a}_t$ follows a multivariate normal distribution.

Consider the bivariate case ($k = 2$), where $\mathbf{r}_t = (r_{1t}, r_{2t})^\top$ and $\mathbf{a}_t = (a_{1t}, a_{2t})^\top$. The VAR(1) model can then be written as:

$$r_{1t} = \phi_{10} + \phi_{11} r_{1,t-1} + \phi_{12} r_{2,t-1} + a_{1t}, \tag{2.2}$$
$$r_{2t} = \phi_{20} + \phi_{21} r_{1,t-1} + \phi_{22} r_{2,t-1} + a_{2t}, \tag{2.3}$$

where:

- ϕ_{ij} is the (i, j)-th element of Φ,
- ϕ_{i0} is the i-th element of $\boldsymbol{\phi}_0$.

In the first equation, the coefficient ϕ_{12} measures the conditional effect of $r_{2,t-1}$ on r_{1t}, given $r_{1,t-1}$. If $\phi_{12} = 0$, then r_{1t} does not depend on $r_{2,t-1}$, and the model shows that r_{1t} only depends on its own past values. Similarly, if $\phi_{21} = 0$, then the second equation indicates that r_{2t} does not depend on $r_{1,t-1}$ when $r_{2,t-1}$ is given.

When analyzing the two equations jointly:

- If $\phi_{12} \neq 0$ and $\phi_{21} = 0$, there is a unidirectional relationship from r_{1t} to r_{2t}.
- If $\phi_{12} = 0$ and $\phi_{21} = 0$, then r_{1t} and r_{2t} are uncoupled.
- If $\phi_{12} \neq 0$ and $\phi_{21} \neq 0$, there is a feedback relationship between the two series.

The process $\mathbf{r}_t$ is *weakly stationary* if:

- The mean of $\mathbf{r}_t$, denoted $E(\mathbf{r}_t)$, is constant over time,
- The covariance $\text{Cov}(\mathbf{r}_t, \mathbf{r}_{t-\tau})$ depends only on the lag τ, not on time t.

2.1.1 Stationary condition

Taking expectations of the VAR(1) model:

$$E(\mathbf{r}_t) = \boldsymbol{\phi}_0 + \Phi E(\mathbf{r}_{t-1}). \tag{2.4}$$

For stationarity, the mean $E(\mathbf{r}_t) = \mu$ must satisfy:

$$\mu = (I - \Phi)^{-1}\boldsymbol{\phi}_0, \tag{2.5}$$

where I is the $k \times k$ identity matrix, and $(I - \Phi)$ must be invertible.

Define the mean-corrected series:

$$\tilde{\mathbf{r}}_t = \mathbf{r}_t - \mu. \tag{2.6}$$

Substituting into the VAR(1) model:

$$\tilde{\mathbf{r}}_t = \Phi\tilde{\mathbf{r}}_{t-1} + \mathbf{a}_t. \tag{2.7}$$

This equation shows the dynamics of $\mathbf{r}_t$ around its mean.

Expanding the equation for $\tilde{\mathbf{r}}_t$:

$$\tilde{\mathbf{r}}_t = \mathbf{a}_t + \Phi\mathbf{a}_{t-1} + \Phi^2\mathbf{a}_{t-2} + \cdots. \tag{2.8}$$

The covariance matrix of $\tilde{\mathbf{r}}_t$, denoted Γ_0, satisfies:

$$\Gamma_0 = \Phi\Gamma_0\Phi^\top + \Sigma, \tag{2.9}$$

where Σ is the covariance matrix of $\mathbf{a}_t$.

The lag-τ cross-covariance matrix (Γ_τ) is:

$$\Gamma_\tau = \Phi^\tau\Gamma_0, \quad \tau > 0. \tag{2.10}$$

The correlation matrix is obtained by normalizing the covariance matrix. Let $D = \text{diag}(\Gamma_0)$ be the diagonal matrix of variances. The lag-τ correlation matrix is:

$$\rho_\tau = D^{-1/2}\Gamma_\tau D^{-1/2}. \tag{2.11}$$

The series $\mathbf{r}_t$ is stationary if and only if all eigenvalues of Φ have modulus less than 1. This ensures that the powers of Φ (Φ^j) decay to zero as $j \to \infty$, preventing explosive behavior.

2.2 Utility Function and Optimization Problem

The paper models investor preferences using a utility function, specifically the Constant Relative Risk Aversion (CRRA) utility function, which is defined as:

$$U(W) = \begin{cases} \frac{W^{1-\gamma}}{1-\gamma}, & \text{if } \gamma \neq 1, \\ \ln(W), & \text{if } \gamma = 1, \end{cases}$$

where W represents the investor's final wealth and γ is the coefficient of relative risk aversion. This parameter reflects the investor's willingness to trade off risk and return. A higher γ indicates a greater aversion to risk, while $\gamma = 0$ corresponds to risk neutrality. The goal is to maximize the expected utility of final wealth W_T, which is expressed as:

$$\max_{\pi} \mathbb{E}[U(W_T)],$$

where π represents the proportion of wealth invested in risky assets, while the remainder $(1 - \pi)$ is allocated to a risk-free asset. The investor's wealth evolves over time according to the equation:

$$W_{t+1} = W_t \left(1 + \pi_t R_{P,t+1} + (1 - \pi_t) R_F\right),$$

where R_P is the return of the risky portfolio, R_F is the return of the risk-free asset, and π_t is the allocation to the risky portfolio at time t. For a single-period setting, this optimization balances the expected return and the perceived risk of final wealth, determined by the variance of the returns. In multi-period settings, the problem becomes dynamic, as allocation decisions at time t influence the wealth available in subsequent periods.

To solve this optimization problem, numerical methods are often used. First, the distribution of future wealth W_T is simulated using a Monte Carlo approach, which generates trajectories for W_T based on assumed distributions of R_P and R_F. The utility function $U(W_T)$ is then evaluated for each trajectory, and the expected utility is computed by integrating over the simulated outcomes. The optimal portfolio allocation π_t is identified as the value that maximizes the expected utility, either through analytical derivation (if feasible) or numerical optimization methods like gradient descent.

The CRRA utility function is instrumental in determining how risk aversion shapes portfolio decisions. Investors with higher γ allocate a smaller proportion of wealth to risky assets to minimize exposure to volatility. Conversely, risk-neutral investors ($\gamma = 0$) focus solely on maximizing expected returns, irrespective of risk. Over long investment horizons, the presence of mean reversion in returns may reduce the effective risk of equities, encouraging even risk-averse investors to allocate more to equities compared to shorter horizons.

This framework connects risk preferences, return predictability, and portfolio optimization, allowing the paper to analyze both static (buy-and-hold) and dynamic (rebalancing) strategies. The CRRA utility function thus serves as a cornerstone for evaluating the trade-off between risk and return over varying time horizons and under conditions of uncertainty.

2.2.1 Volatility Cone in Asset Management

Definition of Volatility Cone

A *volatility cone* is a graphical representation that depicts the range of implied volatilities across different option expiration dates. It helps investors and portfolio managers visualize how market expectations of volatility evolve over time. The volatility cone is constructed by analyzing historical implied volatilities for options of different maturities and plotting their percentile bands.

Mathematically, the volatility cone is based on the standard deviation of asset returns over different time horizons. Given an asset's return r_t over time, the rolling volatility at a given time horizon T is computed as:

$$\sigma_T = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^2} \tag{2.12}$$

where $\bar{r}$ is the mean return, and n is the number of observations.

Interpretation of the Volatility Cone

The volatility cone is used to assess whether the current implied volatility of an asset or an index is high or low relative to its historical range. Typically, the cone consists of different percentile bands (e.g., 5th, 25th, 50th, 75th, and 95th percentiles) that show the distribution of volatilities for various time horizons.

- If the current implied volatility is near the upper boundary of the cone, the asset may be considered expensive in terms of volatility, implying that options may be overpriced.
- If the implied volatility is near the lower boundary, options may be relatively cheap, making them more attractive for long volatility strategies.

Application in Asset Allocation

Volatility cones play a crucial role in asset allocation by helping investors adjust their portfolio risk dynamically based on market conditions. Some key applications include:

1. **Volatility-Based Portfolio Adjustments:** Investors can allocate capital based on expected future volatility. If the current implied volatility is significantly above historical levels, a more defensive allocation (e.g., shifting towards bonds or defensive stocks) may be warranted.
2. **Hedging Strategies:** When implied volatility is low compared to historical levels, investors might consider hedging with long volatility positions (e.g., buying options) to protect against potential market shocks.
3. **Market Timing for Options Trading:** Traders use volatility cones to determine whether to sell overpriced options or buy underpriced options. A mean-reverting approach to implied volatility can be employed to optimize option trading strategies.
4. **Dynamic Risk Management:** By monitoring the volatility cone, portfolio managers can adjust position sizing and leverage to maintain a balanced risk exposure, ensuring the portfolio remains within acceptable risk parameters.

The volatility cone is a valuable tool in financial markets, particularly for understanding implied volatility trends and making informed asset allocation decisions. By analyzing volatility trends over different time horizons, investors can optimize risk management, identify mispriced options, and enhance portfolio resilience in volatile markets.

2.2.2 Bayes' Theorem: Prior and Posterior

Bayes' Theorem provides a framework for updating the probability of a hypothesis based on new evidence. It is expressed mathematically as:

$$P(H \mid E) = \frac{P(E \mid H) \cdot P(H)}{P(E)},$$

where $P(H)$ represents the prior probability of the hypothesis H, $P(E \mid H)$ is the likelihood of observing the evidence E given H, $P(E)$ is the marginal probability of the evidence, and $P(H \mid E)$ is the posterior probability of the hypothesis after observing E.

The prior probability reflects the initial belief about the hypothesis before observing any evidence. The likelihood quantifies how well the evidence supports the hypothesis. The posterior probability is the updated belief about the hypothesis after incorporating the evidence. The marginal probability of the evidence is the total probability of observing the evidence under all possible hypotheses and ensures that the posterior probabilities sum to one.

Bayes' Theorem is central to probabilistic reasoning and decision-making under uncertainty. It allows for iterative updating of beliefs as new evidence becomes available, making it a foundational tool in fields such as statistics, machine learning, and Bayesian inference.

Chapter 3

Procedure

The primary objective of this paper is to understand the optimal allocation for investing in the long run. The study is divided into two main categories:

1. A Buy-and-Hold Framework, where the investor holds a fixed portfolio allocation over the entire investment horizon.
2. A Dynamic Allocation Framework, where the investor adjusts the portfolio allocation over time based on updated information.

Both frameworks aim to solve the portfolio choice problem for a long-horizon investor under specific assumptions and models of wealth dynamics and preferences.

3.0.1 Wealth Dynamics Assumption

The investor begins with an initial wealth $W_T = 1$ at time T. The portfolio allocation is divided between a risk-free asset and a risky asset (stock index). Let π represent the proportion of wealth allocated to the risky asset. The wealth at the end of the investment horizon $T + \Delta T$ is given by:

$$W_{T+\Delta T} = (1-\pi)\exp(r_f \cdot \Delta T) + \pi \exp\left(r_f \cdot \Delta T + R_{T+1} + R_{T+2} + \cdots + R_{T+\Delta T}\right), \quad (3.1)$$

where:

- r_f is the risk-free rate (continuously compounded),
- $R_{T+1}, R_{T+2}, \ldots, R_{T+\Delta T}$ are the continuously compounded risky asset returns,
- The total excess cumulative stock return over ΔT periods is defined as:

$$\bar{R}_{T+\Delta T} = R_{T+1} + R_{T+2} + \cdots + R_{T+\Delta T}. \quad (3.2)$$

3.0.2 Investor Preferences: Power Utility

The investor's preferences over terminal wealth are described using Constant Relative Risk Aversion (CRRA) power utility functions. The utility function takes the form:

$$V(W) = \frac{W^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1, \quad (3.3)$$

where:

- W is the terminal wealth,
- γ is the coefficient of relative risk aversion.

This function captures the trade-off between expected returns and risk, with γ governing the investor's aversion to risk.

3.0.3 The Buy-and-Hold Investor's Problem

For a buy-and-hold investor, who maintains a fixed portfolio allocation π for the entire investment horizon, the problem is to maximize the expected utility of terminal wealth. The optimization problem can be expressed as:

$$\max_{\pi} \mathbb{E}_T \left[\left((1-\pi) \exp(r_f \cdot \Delta T) + \pi \exp\left(r_f \cdot \Delta T + \bar{R}_{T+\Delta T} \right) \right)^{1-\gamma} \right], \tag{3.4}$$

where:

- $\mathbb{E}_T[\cdot]$ is the expectation operator conditioned on the investor's information set at time T.
- $\bar{R}_{T+\Delta T}$ is the cumulative risky asset return over the investment horizon.

The expectation reflects the investor's beliefs about the distribution of future returns. This formulation assumes that the investor considers both the deterministic component from the risk-free asset and the stochastic returns from the risky asset.

3.0.4 Transition to Dynamic Allocation

In the second part of the study, the analysis shifts from the static buy-and-hold framework to a **dynamic allocation** framework. In this case, the investor adjusts the portfolio allocation π_t at each time step based on new information and updated forecasts of returns. The dynamic strategy accounts for predictability in returns by incorporating information from variables such as historical returns or other predictors into a forecasting model.

The two-phase approach allows the paper to compare the optimal asset allocation decisions of:

- An investor who ignores return predictability and follows a fixed buy-and-hold strategy.
- An investor who accounts for predictability and adjusts the portfolio dynamically over time.

The models are solved under assumptions of wealth dynamics and risk preferences to provide insights into the optimal allocation for long-horizon investors.

3.1 Methodology

The overall methodology involves:

1. **Parameter Estimation**: First, parameters such as mean returns and variances are estimated using historical data under a simple model.

2. **Forecasting Returns**: The estimated parameters are then applied to forecast future returns using either a simple i.i.d. model or a more advanced VAR model.

3. **Optimization**: The investor's problem is solved to maximize utility over the chosen investment horizon, comparing static (buy-and-hold) and dynamic allocation strategies.

3.2 Return Uncertainty and Its Impact on Portfolio Optimization

A key aspect of this paper is to evaluate how treating parameters as fixed versus accounting for their uncertainty affects the results of portfolio maximization. This distinction is particularly important in the **buy-and-hold phase**, where two models for returns are analyzed: a simple model and a more advanced Vector Autoregressive (VAR) model.

3.2.1 Fixed Parameters vs. Uncertainty

The estimation process can take two approaches:

- **Fixed Parameters**: Parameters such as the mean and variance of returns are estimated from historical data and assumed to be known precisely.

- ** Uncertainty**: Instead of thinking of the return as static we think about it as uncertain .

The goal is to determine how these two approaches influence the portfolio allocation decisions and the expected utility of terminal wealth.

3.2.2 The Simple Return Model

The first model assumes that returns follow an independent and identically distributed (i.i.d.) process. Specifically, returns are modeled as:

$$R_t = \mu + e_t, \quad e_t \sim N(0, \sigma^2), \tag{3.5}$$

where:

- R_t is the return at time t,

- μ is the mean return,

- e_t is an innovation term assumed to be normally distributed with mean zero and variance σ^2.

For this model, parameter estimation involves calculating the sample mean and variance as fixed values, or modeling the mean return μ as a random variable with a distribution that incorporates estimation error.

3.2.3 The VAR Model: A Markovian Process

The second model introduces a more advanced structure for returns by assuming that they follow a **Vector Autoregressive (VAR)** process of order one. Under this model, returns are not independent but depend on their past values, reflecting a Markovian structure. The VAR(1) model is expressed as:

$$\mathbf{z}_t = \mathbf{a} + \mathbf{B}\mathbf{z}_{t-1} + \boldsymbol{e}_t, \quad \boldsymbol{e}_t \sim N(\mathbf{0}, \Sigma), \tag{3.6}$$

where:

- $\mathbf{z}_t$ is the state vector, which includes returns and predictor variables,
- $\mathbf{a}$ is the intercept vector,
- $\mathbf{B}$ is the matrix of autoregressive coefficients,
- $\boldsymbol{e}_t$ is a vector of normally distributed innovations with mean zero and covariance matrix Σ.

In this case, the VAR model accounts for the dependence of returns on their immediate past, which captures time variation and predictability in returns.

3.2.4 Parameter Estimation and Uncertainty

For both models (the simple i.i.d. model and the VAR model), the procedure involves two main steps:

1. **Parameter Estimation (Preprocessing)**: The parameters (μ, σ^2 for the simple model and $\mathbf{a}, \mathbf{B}, \Sigma$ for the VAR model) are first estimated using historical data. These parameters can either be treated as:
   - Fixed values with no uncertainty, or
   - Random variables with a normal distribution, where the variance reflects the estimation error.
2. Forecasting: using the estimated parameters, a forecast is run for predicting the return distribution. Finally the latter is used to maxize the utility function.

When parameter uncertainty is incorporated, the additional variance increases the perceived risk, particularly for long-horizon investors, and changes the optimal portfolio allocation.

3.2.5 Impact on Results

The paper [1] compares the outcomes of portfolio optimization under the two approaches (fixed parameters vs. parameter uncertainty) for both the simple i.i.d. model and the VAR model. By explicitly accounting for estimation risk, the analysis highlights how uncertainty in parameter estimates affects the perceived risk-return trade-off and the resulting allocation decisions. This comparison provides insights into the importance of considering estimation error in portfolio construction, particularly for long-term investment strategies.

3.3 Parameter Estimation in the First Simple Model

In the initial phase of the paper, the parameters of a simple model for returns are estimated under the assumption that returns are independently and identically distributed (i.i.d.) over time. The model is specified as:

$$r_t = \mu + e_t, \quad e_t \sim N(0, \sigma^2),$$

where:

- r_t is the return at time t,
- μ is the mean (expected return) of the returns,
- e_t is a random error term (innovation) with mean zero and variance σ^2,
- σ^2 is the variance of the returns, which is assumed constant over time.

This model provides a fundamental framework for estimating returns while ignoring any time variation or predictability. The objective is to estimate the parameters $\theta = (\mu, \sigma^2)$ using historical return data.

The following assumptions are made for the parameter estimation:

- Returns r_t are independently and identically distributed (i.i.d.).
- The innovations e_t follow a normal distribution: $e_t \sim N(0, \sigma^2)$.

At first the parameters μ and σ^2 are estimated using the sample mean and sample variance, respectively. The estimators are as follows:

Sample Mean: The sample mean provides an unbiased estimate of the expected return μ:

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t,$$

where T is the number of observations and r_t is the observed return at time t.

Sample Variance: The sample variance estimates the constant variance σ^2 of the returns:

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} (r_t - \hat{\mu})^2.$$

These estimators are derived by maximizing the log-likelihood function under the assumption of normally distributed returns. The sample mean minimizes the sum of squared deviations from the mean, while the sample variance measures the average squared deviations of returns around the sample mean.

Parameter Uncertainty: Moving to parameter uncertainty we need to estimate the Posterior Distribution $p(\mu, \sigma^2 \mid r)$

To estimate the posterior distribution $p(\mu, \sigma^2 \mid r)$, we follow a Bayesian approach where the parameters μ (mean) and σ^2 (variance) are treated as *non-fixed*. Non-fixed parameters mean that instead of assuming μ and σ^2 as constants, we consider them as random variables with their own probability distributions. This allows us to incorporate parameter uncertainty into the model.

- Define Prior Assumptions

  An uninformative prior is assumed for μ and σ^2, given by:

  $$p(\mu, \sigma^2) \propto \frac{1}{\sigma^2}.$$

  This prior reflects minimal initial information about the parameters.

- Step 2: Posterior Distribution

  Using Bayes' theorem, the joint posterior distribution is given by:

  $$p(\mu, \sigma^2 \mid r) = p(\sigma^2 \mid r) \cdot p(\mu \mid \sigma^2, r),$$

  where:

  $p(\sigma^2 \mid r)$: Marginal posterior for σ^2, which follows an Inverse Gamma distribution:

  $$\sigma^2 \mid r \sim IG\left(\frac{T}{2}, \frac{\sum_{t=1}^{T}(r_t - \bar{r})^2}{2}\right),$$

  where $\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$ is the sample mean. $p(\mu \mid \sigma^2, r)$: Conditional posterior for μ, which follows a Normal distribution:

  $$\mu \mid \sigma^2, r \sim N\left(\bar{r}, \frac{\sigma^2}{T}\right).$$

- Sampling Procedure

  To construct the posterior $p(\mu, \sigma^2 \mid r)$, the following steps are repeated:

  1. Sample σ^2 from the marginal posterior $p(\sigma^2 \mid r)$ (Inverse Gamma).
  2. Sample μ from the conditional posterior $p(\mu \mid \sigma^2, r)$ (Normal), using the sampled value of σ^2.

  By repeating this process many times, we obtain a large sample from the joint posterior distribution $p(\mu, \sigma^2 \mid r)$, which accurately represents the uncertainty in the parameter estimates.

3.3.1 Parameter estimation with VAR model

The evaluation of portfolio allocation for stocks, represented by the parameter ω, is conducted under two scenarios: (i) using **predictability** without accounting for return uncertainty, and (ii) incorporating **return uncertainty**. The procedures for both cases are described below.

Without Accounting for Return Uncertainty: When return uncertainty is not considered, the evaluation proceeds as follows:

1. **VAR Model Estimation**: The VAR model is estimated using historical data from 1988 to 2024.

2. **Future Mean and Variance Calculation**: The future mean and variance of returns are calculated using the formula described below, relying on the last observed value of the historical series (VAR model with lag order $P = 1$).

   Note that since $z_t = a + Bx_{t-1} + \varepsilon_t$, we can write $z_t = a + B_0 z_{t-1} + \varepsilon_t$, where

$$B_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix}.$$

   Therefore:

$$z_{T+1} = a + B_0 z_T + \varepsilon_{T+1}$$

$$z_{T+2} = a + B_0 a + B_0^2 z_T + \varepsilon_{T+2} + B_0 \varepsilon_{T+1}$$

$$\vdots$$

$$z_{T+\hat{\tau}} = a + B_0 a + B_0^2 a + \cdots + B_0^{\hat{\tau}-1} a + B_0^{\hat{\tau}} z_T + \varepsilon_{T+\hat{\tau}} + B_0 \varepsilon_{T+\hat{\tau}-1} + B_0^2 \varepsilon_{T+\hat{\tau}-2} + \cdots + B_0^{\hat{\tau}-2} \varepsilon_{T+2} + B_0^{\hat{\tau}-1} \varepsilon_{T+1}.$$

   Conditional on a, B, and Σ, the sum $Z_{T+\hat{\tau}} = z_{T+1} + z_{T+2} + \cdots + z_{T+\hat{\tau}}$ is Normally distributed with mean and variance given by

$$\mu_{\text{sum}} = \hat{\tau} a + (\hat{\tau} - 1) B_0 a + (\hat{\tau} - 2) B_0^2 a + \cdots + B_0^{\hat{\tau}-1} a + (B_0 + B_0^2 + \cdots + B_0^{\hat{\tau}}) z_T,$$

$$\Sigma_{\text{sum}} = \Sigma + (I + B_0)\Sigma(I + B_0)' + (I + B_0 + B_0^2)\Sigma(I + B_0 + B_0^2)' + \cdots + (I + B_0 + \cdots + B_0^{\hat{\tau}-1})\Sigma(I + B_0 + \cdots + B_0^{\hat{\tau}-1})'.$$

3. **Sampling for Time T'**: A sample of potential future returns is generated for the specified time T', and these returns are added to the historical returns (this addition is valid since the returns are log returns).

With Return Uncertainty:
When return uncertainty is incorporated into the analysis, the following steps are performed:

1. **VAR Model Estimation**: As in the previous case, the VAR model is estimated using historical data from 1988 to 2024.

2. **Forecasting Returns**: Returns are forecasted using the VAR function. Based on these forecasts, a sample of possible returns is generated by taking into account parameter uncertainty.

3. **Cumulative Return Calculation**: The cumulative returns up to time T' are summed to compute μ_{sum} (the mean) and σ_{sum} (the variance).

4. **Excess Return Sampling**: Using multivariate sampling techniques, the total excess return is derived. This excess return is then used in the utility function to evaluate the optimal portfolio allocation.

This distinction highlights the differences in portfolio allocation strategies when predictability is used with and without accounting for parameter uncertainty. Incorporating uncertainty provides a more cautious and realistic assessment of future returns, which influences the allocation parameter ω accordingly.

3.4 From Buy-and-Hold to Dynamic Allocation

In the previous sections, we analyzed the investment problem under a **buy-and-hold strategy**, where the investor chooses an initial allocation and does not rebalance the portfolio over the investment horizon. We now introduce **dynamic allocation**, where the investor optimally rebalances the portfolio at regular intervals based on updated information. In finance, this means adjusting the proportion of wealth invested in different assets periodically to maximize returns while managing risk.)

3.4.1 Incorporating Dividend Yield as a Predictor

To determine the optimal dynamic allocation, we use the same regression model introduced in Section III. The model includes: - **Stock returns** r_t: the excess return of the stock index over risk-free assets. - **Dividend yield** x_t: the ratio of dividends paid by the stock index to its current price, used as a predictive variable.

The model follows the form:

$$z_t = \begin{bmatrix} r_t \\ x_t \end{bmatrix}. \tag{3.7}$$

The investor continuously updates decisions based on changes in these variables.

3.4.2 Solving the Dynamic Allocation Problem

To solve this problem, we employ the standard technique of **discretizing the state space** and using **backward induction**. The following paragraphs formalize this approach.

Defining the Investment Horizon

We define:
- $\hat{T}$ as the total investment horizon in months.
- K as the number of rebalancing intervals, which we choose equal to $\hat{T}$ (i.e., rebalancing occurs monthly).

The wealth at each time step is denoted by W_k, and the dividend yield at each time step is denoted by x_k. The investor's optimization problem is given by:

$$\max E_{t_0}\left[W_K^{1-A}\right], \tag{3.8}$$

where A is the coefficient of relative risk aversion.

Wealth Evolution and Utility Function

The wealth dynamics are governed by the following equation:

$$W_{k+1} = W_k \left[(1 - v_k) e^{r_f \frac{\hat{T}}{K}} + v_k e^{r_f \frac{\hat{T}}{K} + R_{k+1}} \right], \tag{3.9}$$

where v_k represents the fraction of wealth allocated to stocks at time k, and R_{k+1} is the cumulative stock return over the rebalancing interval.

The derived **utility of wealth** is given by:

$$J(W_k, x_k, t_k) = \max E_{t_k} \left[W_K^{1-A} \right]. \tag{3.10}$$

For $A = 1$, this reduces to a log-utility function:

$$J(W_k, x_k, t_k) = \log(W_k) + Q(x_k, t_k), \tag{3.11}$$

where $Q(x_k, t_k)$ is the value function dependent on the dividend yield and time.

Backward Induction and Optimal Allocation

The function $Q(x_k, t_k)$ is found using **backward induction**. The Bellman equation of optimality is:

$$Q(x_k, t_k) = \max_v E_{t_k} \left[(1 - v) e^{r_f \frac{\hat{T}}{K}} + v e^{r_f \frac{\hat{T}}{K} + R_{k+1}} Q(x_{k+1}, t_{k+1}) \right]. \tag{3.12}$$

To compute this expectation:
- We sample R and x from the normal distribution used in the previous chapter.
- We perform approximately **one million simulations** of these random pairs (R, x).
- For each sampled pair, we compute the expected value and determine the optimal v.

At each step, we store $Q(x_k, t_k)$ for each discretized value of x_k, iterating backwards until reaching the initial period. The computed Q values allow us to determine the optimal allocation at each time step.

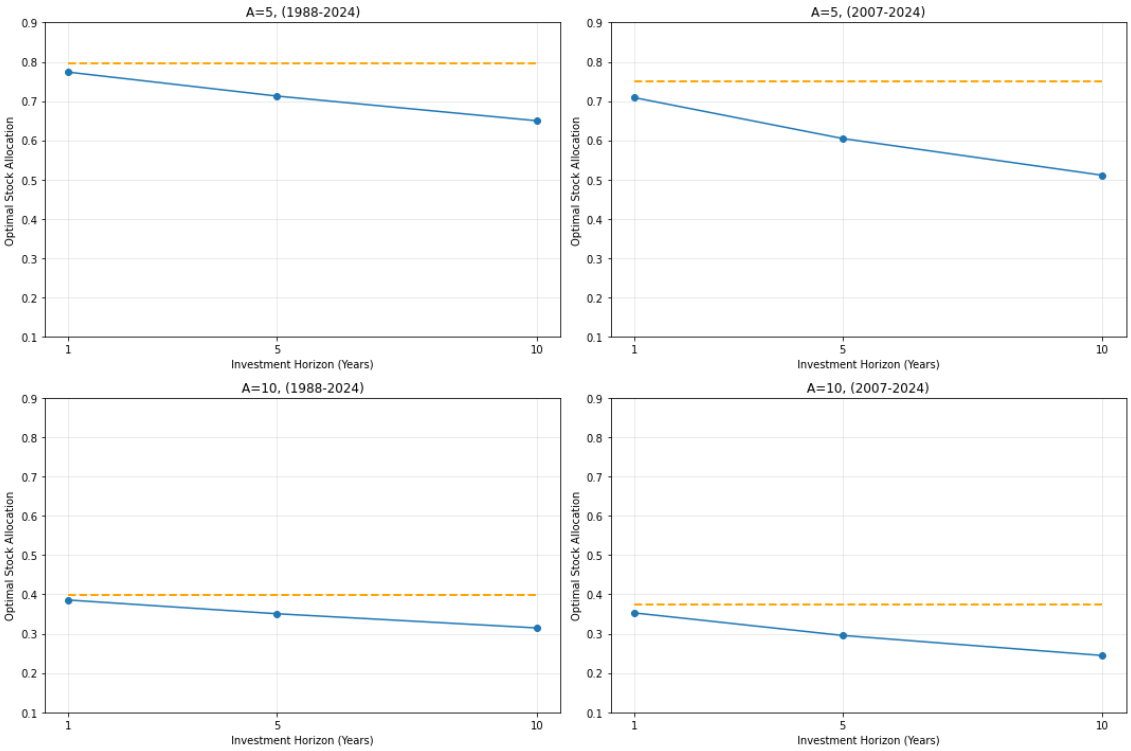
Determining the Optimal Allocation

Using the obtained Q values, we find the optimal wealth allocation decisions:
- The initial wealth value determines the first optimal allocation v_0.
- For each step j, we determine the optimal ω_j, which represents the fraction of wealth allocated to stocks.

This approach ensures that at each time step, the investor makes the best decision given the available information, leading to an optimal rebalancing strategy over the entire investment horizon.

3.5 Results

3.5.1 Simple model



The allocation of investments to stocks varies based on the investment horizon and the extent to which uncertainty about the mean and variance of returns is accounted for. This study examines how parameter uncertainty, which arises from limited data or unpredictable market conditions, influences the optimal percentage of an investor's portfolio allocated to stocks. The findings underscore the critical role of accounting for this uncertainty in investment decisions.

When parameter uncertainty is ignored, the allocation to stocks remains constant regardless of the investment horizon. This reflects a simplistic assumption that the mean and variance of stock returns are precisely known and unchanging. In such a scenario, an investor would allocate the same proportion of their portfolio to equities for short- and long-term horizons, as the perceived risks and returns of the asset remain stable over time. This can be visualized in the graphs where the dashed line, representing fixed parameter estimates, is completely horizontal.

However, when parameter uncertainty is incorporated into the decision-making framework, the allocation to stocks decreases progressively as the investment horizon increases. This is particularly evident in the solid line within the graphs. The decrease in allocation stems from the additional risk introduced by uncertainty in the estimation of stock return parameters, particularly over longer horizons. Stocks are perceived as increasingly risky because uncertainty compounds over time, making long-term forecasts less reliable.

The degree of this effect varies depending on the dataset used to estimate returns. When a comprehensive dataset spanning many decades is used, the effect of parameter uncertainty is moderate, as the larger sample size provides more reliable estimates of return parameters. For instance, using data from 1998 to 2004, an investor with a risk aversion level of 5 would reduce their stock allocation by approximately 10% at a 10-year horizon. In contrast, when using a smaller, more recent dataset—such as one spanning only 2007 to 2024—the effect is more pronounced. Under these circumstances, the same investor would reduce their stock allocation by a full 20% at the 10-year horizon. This highlights how the size and quality of the dataset significantly impact the perception of risk and, consequently, portfolio allocation decisions.

The increased risk perception over longer horizons can be explained by how uncertainty affects the distribution of cumulative returns. Without parameter uncertainty, the mean and variance of cumulative returns grow linearly with the horizon. For instance, if an investor assumes returns are independent and identically distributed, the expected return and its associated risk scale predictably with time. However, in the presence of parameter uncertainty, this linear relationship no longer holds. Instead, the additional uncertainty introduces positive serial correlation in the returns. This means that a high stock return in one period increases the likelihood of high returns in subsequent periods. While this serial correlation can amplify returns under favorable conditions, it also increases the variance of cumulative returns faster than linearly with the horizon. As a result, stocks appear riskier for long-term investors, prompting them to allocate a smaller portion of their portfolios to equities.

From a practical standpoint, this has important implications for portfolio management. It suggests that long-term investors cannot afford to ignore estimation risk, as it materially affects their optimal asset allocation. For investors relying on shorter datasets, the impact of uncertainty is particularly acute, underscoring the importance of using robust estimation techniques and recognizing the limitations of small samples.

Finally, the accuracy of the numerical methods used to derive these results is critical. To minimize simulation error and ensure robust conclusions, the study employs simulations with one million draws from the sampling distribution when calculating expected utility. This high level of computational precision is necessary to accurately capture the impact of parameter uncertainty and provide reliable guidance on optimal portfolio allocations. The results confirm that such large-scale simulations are effective in reflecting the uncertainties faced by investors and the trade-offs they must navigate when making long-term investment decisions.

In summary, this analysis highlights the profound impact of parameter uncertainty on investment strategies. Ignoring uncertainty leads to static stock allocations, which may underestimate the risks associated with longer horizons. Conversely, incorporating uncertainty reveals a need to decrease stock allocations as the horizon grows, with the magnitude of this effect varying based on the size and scope of the dataset used. These findings emphasize the importance of carefully considering estimation risk in portfolio management, particularly for long-term investors.

3.5.2 Var model full

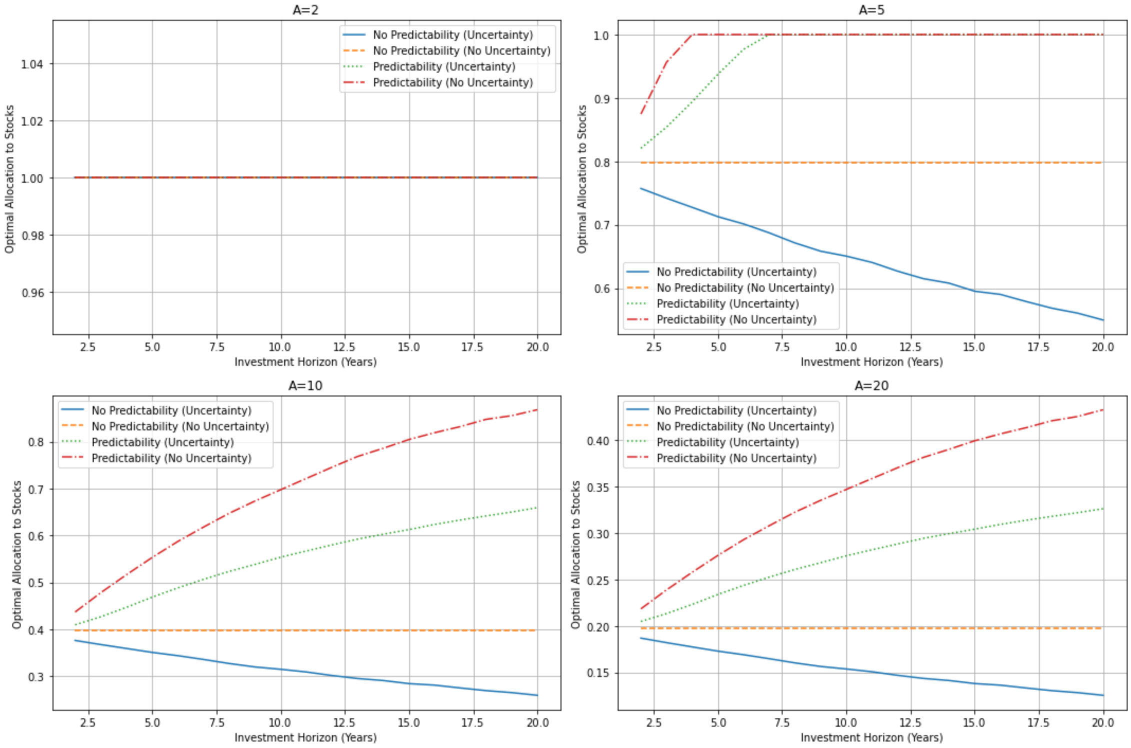


Figure 2 illustrates the optimal allocation to stocks for varying investment horizons, ranging from 1 to 10 years, under different levels of risk aversion ($A = 2, 5, 10, 20$). Each graph contains four lines, which represent different assumptions about the predictability of returns and the incorporation of parameter uncertainty. Specifically, the red and green lines corresponds to the case where predictability is included. The red and the orange lines ignore uncertainty, the green and blu one account for it.

When predictability is included , the horizon effect becomes apparent. these lines show that if predictability is included and parameter uncertainty is ignored, the allocation to stocks increases significantly with the investment horizon. For long-term investors, predictability reduces the perceived risk of stocks, making them more attractive and leading to higher allocations. However, when parameter uncertainty is accounted for, the allocation to stocks is reduced, particularly for longer horizons. This reflects the fact that estimation risk increases the perceived risk of stocks, thereby dampening the horizon effect.

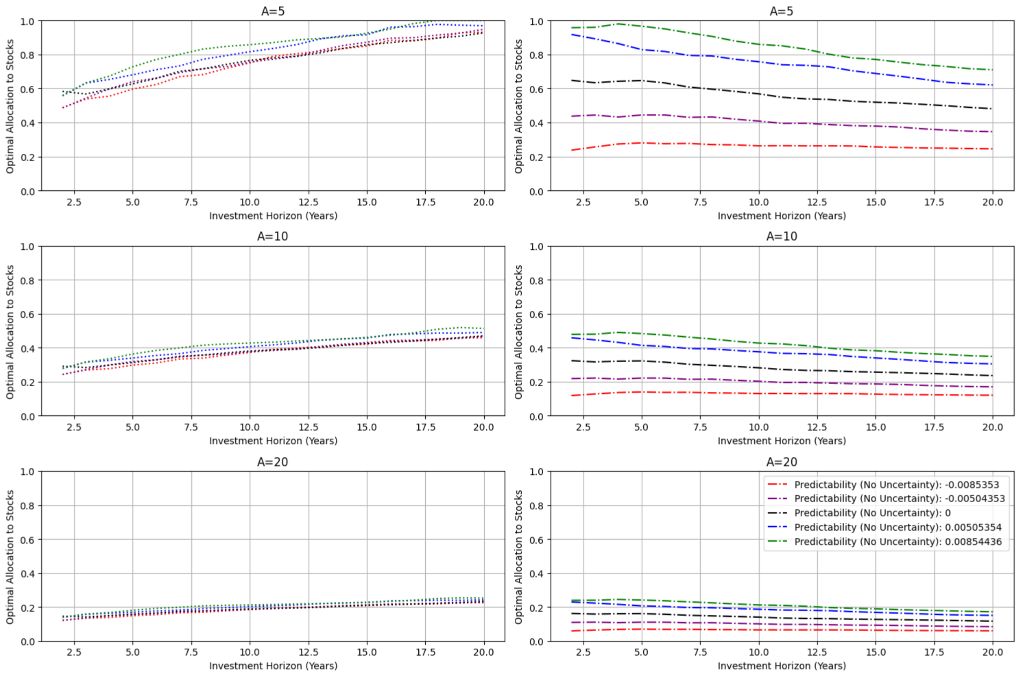
In contrast, when predictability is excluded , the results align with the assumptions of the i.i.d. model, where returns are treated as unpredictable. The orange line shows that the allocation to stocks remains constant across all horizons, as the risk-return trade-off does not change with time. The blu line line indicates that incorporating parameter uncertainty in the i.i.d. model slightly reduces the allocation to stocks, as the increased uncertainty about the mean return leads to a higher perceived risk.

Risk aversion (A) plays a significant role in shaping these results. As A increases, the overall allocation to stocks decreases for all horizons, reflecting the more conservative behavior of highly risk-averse investors. For lower levels of risk aversion ($A = 2, 5$), the horizon effect is more pronounced, with higher allocations to stocks at longer horizons when predictability is included. Conversely, for higher levels of risk aversion ($A = 20$), the allocations to stocks remain modest, even for long-term investors, and the horizon effect is muted.

From a mathematical perspective, the observed horizon effects stem from the predictability of returns modeled using the VAR framework. Predictability reduces the variance of cumulative returns over longer horizons, as the variance grows slower than linearly with the investment horizon. This makes stocks appear less risky for long-term investors, leading to higher allocations. However, parameter uncertainty counteracts this effect by increasing the perceived risk, making stocks less attractive at longer horizons.

3.5.3 Analysis of Dividend Yield and Optimal Portfolio Allocation

The following graphs show the optimal allocation using var model and the longer historical dataset.



The analysis so far has focused on one major effect of including the dividend yield as a predictor in the VAR model: conditioning on the dividend yield reduces the variance of predicted long-horizon cumulative returns, leading to a higher allocation to stocks for long-horizon investors. However, conditioning on the dividend yield also has a direct impact on portfolio allocation due to its role as a predictor. By affecting the mean of the distribution of future returns, the dividend yield influences allocation decisions. Specifically, when the dividend yield is low compared to its historical mean, investors anticipate lower-than-average stock returns and reduce their allocation to equities accordingly. This effect has not been highlighted previously in the analysis because the initial value of the dividend yield was assumed to be fixed at its historical mean over the sample period.

To explore this further, the analysis is repeated for different initial values of the dividend yield. Figure 3 illustrates these results, with each graph corresponding to a different level of investor risk aversion. The graphs on the left show optimal allocations when parameter uncertainty is ignored, while those on the right account for it. Within each graph, the optimal allocation to stocks is plotted against the investment horizon for five different initial values of the dividend yield.

The left-hand graphs demonstrate that, when parameter uncertainty is ignored, the allocation to stocks increases with the investor's horizon for all initial values of the dividend yield. Furthermore, higher initial dividend yields lead to higher stock allocations, as they signal higher expected future returns. Notably, for any fixed initial value of the dividend yield, the allocation for a 10-year horizon is higher than for a one-year horizon. However, the sensitivity of the allocation to the initial value of the dividend yield remains consistent across horizons, with no convergence of the allocation lines as the horizon lengthens.

The situation changes dramatically when parameter uncertainty is incorporated, as shown in the right-hand graphs of Figure 3. For lower initial dividend yields, the allocation to stocks still increases with the investment horizon. However, for higher initial dividend yields, the allocation to stocks declines as the horizon lengthens. This reversal is driven by the increased impact of parameter uncertainty and the associated skewness in the predictive distribution of returns, which makes equities less attractive at high dividend yield levels over long horizons. Importantly, when parameter uncertainty is accounted for, the allocation lines converge. As a result, a 10-year allocation becomes less sensitive to the initial value of the dividend yield than the allocation of a one-year investor. Moreover, the 10-year allocation of an investor who incorporates parameter uncertainty is much less sensitive to the initial value of the dividend yield compared to one who ignores it. This outcome is intuitive: when the true predictive power of the dividend yield is uncertain, it is prudent for a long-horizon investor to reduce reliance on the initial value of the predictor.

The following graphs (figure 4) show the optimal allocation using var model and half of the previous historical dataset.

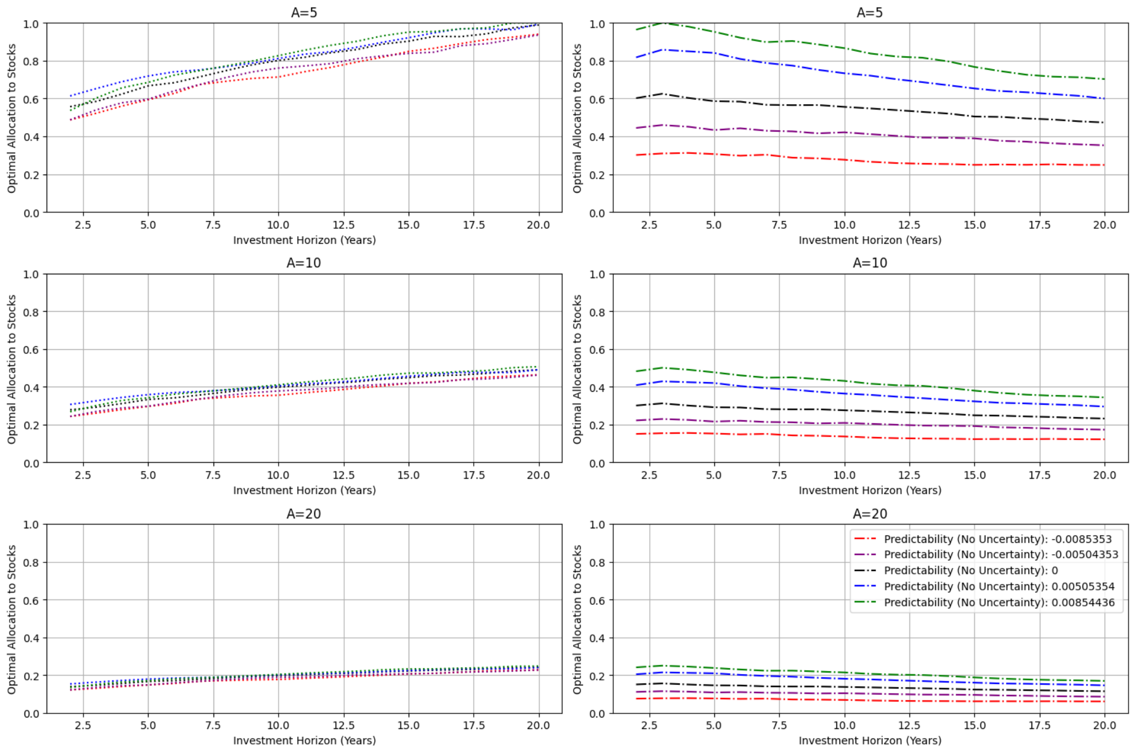


Figure 4 extends the analysis of Figure 3, using a more recent subsample for the investor's decision-making process. When parameter uncertainty is ignored, the optimal allocation to stocks still increases sharply with the investment horizon. However, incorporating parameter uncertainty leads to vastly different portfolio recommendations. Notably, for 10-year horizons, the allocations become largely insensitive to the initial dividend yield, with an even stronger convergence of allocation lines than in Figure 3. In some cases, estimation risk is so significant that the stock allocation for a 10-year horizon is lower than that for a one-year horizon.

A surprising finding in Figures 3 and 4 is that the optimal stock allocation is not always increasing with the initial dividend yield, contrary to expectations. Higher dividend yields, while associated with higher expected returns, also introduce negative skewness in the predictive distribution, making stocks less attractive. This effect explains the non-monotonic relationship between the initial value of the predictor and stock allocation. Skewness, rather than variance, is the key factor driving this result: low dividend yields generate positive skewness, while high dividend yields introduce negative skewness, reducing stock desirability at higher predictor values.

3.5.4 Dynamic allocation

Following the method outlined in the paper, we did not achieve acceptable results, which we believe may be attributed to several factors, including the use of different datasets compared to those originally employed in the paper, an unclear description of the numerical steps in the code, and the necessity of advanced tools and frameworks to solve dynamic programming problems effectively. Despite these challenges, we tested several alternatives, such as solving the dynamic problem using brute force methods, approaching the problem without relying on the Bellman equation, and discretizing the possible values of omega. As a result, we considered the omega values derived from the buy-and-hold models to develop a dynamic allocation strategy, where omega values are updated annually for the period T+k. The outcomes of these tests are presented through the pension fund development graphs, which illustrate the results obtained.

3.6 Pension Fund Growth Model

In this section, we present a model to simulate the growth of a pension fund over a given investment horizon, considering a passive investment strategy known as *buy-and-hold*. The model incorporates several key factors, including asset allocation, annual contributions, and expected returns from stocks and bonds, to compute the fund's wealth year by year.

The wealth at the end of year k, denoted as W_k, is calculated iteratively starting from an initial wealth W_0. At the end of each year, the wealth is updated by taking into account both the annual contributions and the returns generated from the portfolio's investments. Specifically, the wealth at year k is given by:

$$W_{k+1} = (W_k) \cdot [w \cdot (1 + r_s) + (1 - w) \cdot (1 + r_b)] + \text{Contributions},$$

where w is the proportion of the portfolio allocated to stocks, r_s is the average annual return of stocks, and r_b is the average annual return of bonds. The remaining portion of the portfolio, $1 - w$, is allocated to bonds.

The *buy-and-hold* strategy assumes that the asset allocation remains fixed throughout the investment horizon. This passive approach minimizes trading activity and focuses on maintaining the predefined proportion of stocks and bonds, allowing the portfolio to grow steadily over time.

To better understand the implications of different risk preferences, we apply the model to various investor profiles:

- **Risk-loving investors**, who favor higher allocations to stocks, accepting greater volatility in pursuit of potentially higher returns.

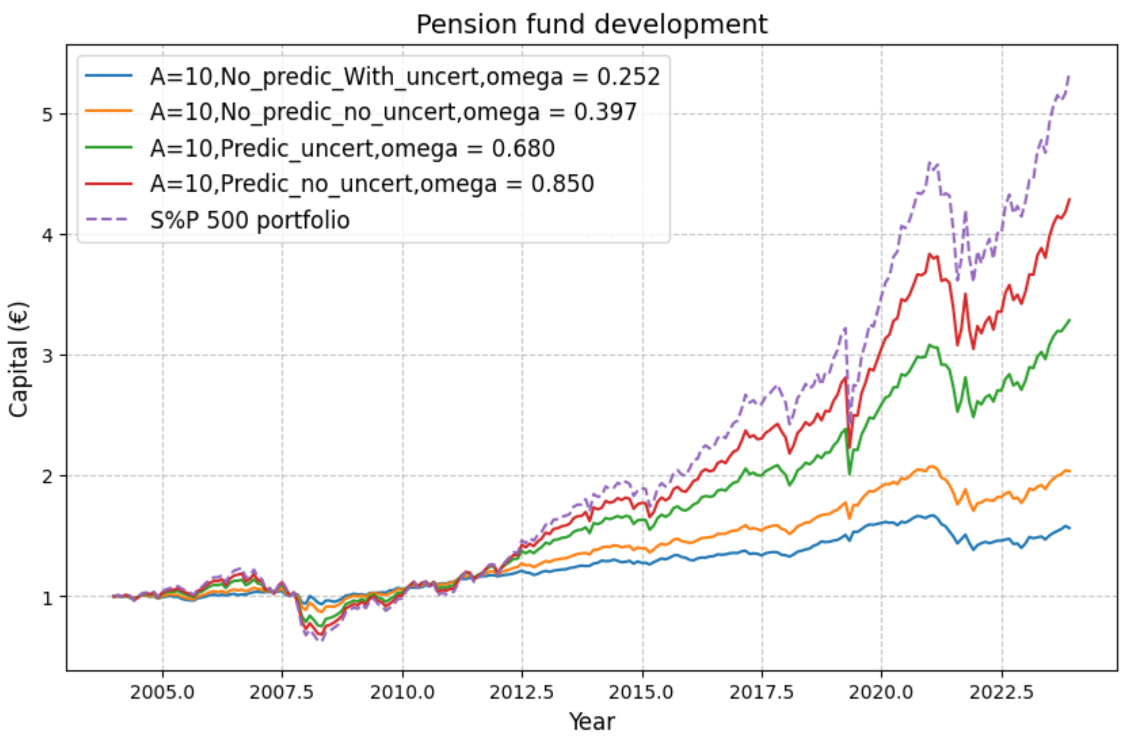
- **Risk-averse investors**, who prioritize stability and opt for lower stock allocations, favoring bonds to reduce risk.

- **Balanced profiles**, where allocations are diversified to strike a balance between growth potential and risk mitigation.

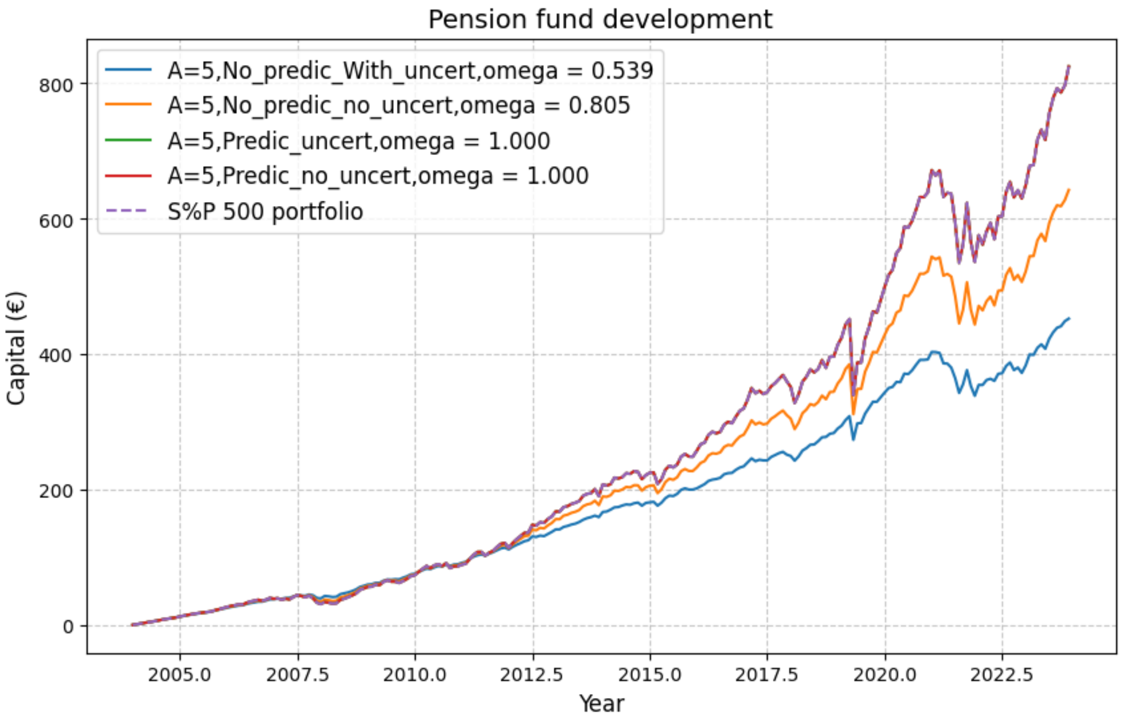
The model simulates wealth trajectories for these profiles under different scenarios, providing valuable insights into the impact of asset allocation, risk tolerance, and investment horizons on pension fund growth.

3.6.1 Results on Pension Fund

Now we will focus on the pension fund simulation. We consider basically only the buy and hold strategy for the two main risk profiles that we considered before (Risk adverse and neutral-adverse clients) and consider for illustrative reasons same initial wealth contribution (1) and same monthly contributions (1) . So we will see now how the previous estimations regarding predictability vs uncertainty varies our resulting fund growth compared only to the S&P 500. Such simulations are computed over 20 years and accounts for the variability of returns of bonds and the returns of stocks.

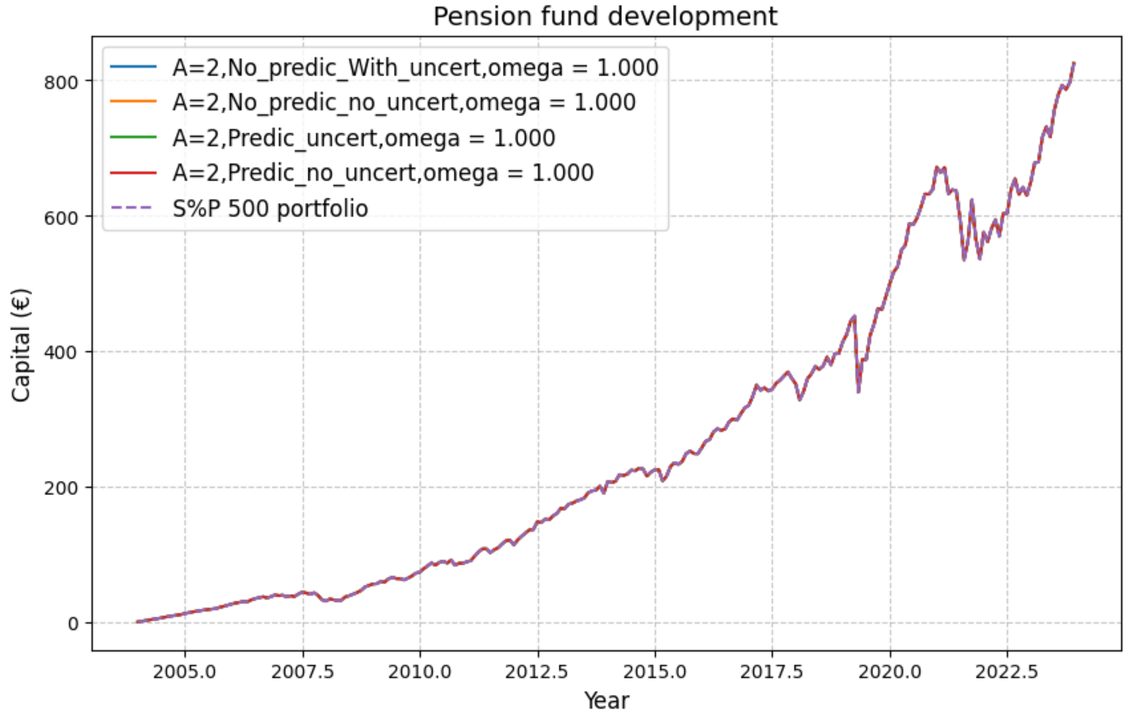


As we can observe now and ongoing , we could not beat the Benchmark index. Such result does not surprise us since our alternative portfolios contain less risky treasury bills that cannot keep the pace with the SnP growth in the bull period , but they have the advantage of soothing out the variance during the recession and shock moments (like the covid impact).For a risk adverse profile, such trade-off negatively impacts on the long run wealth construction, where the best Barberi's result come from an estimation considering predictability of regressors and not accounting for uncertainty in the estimation.

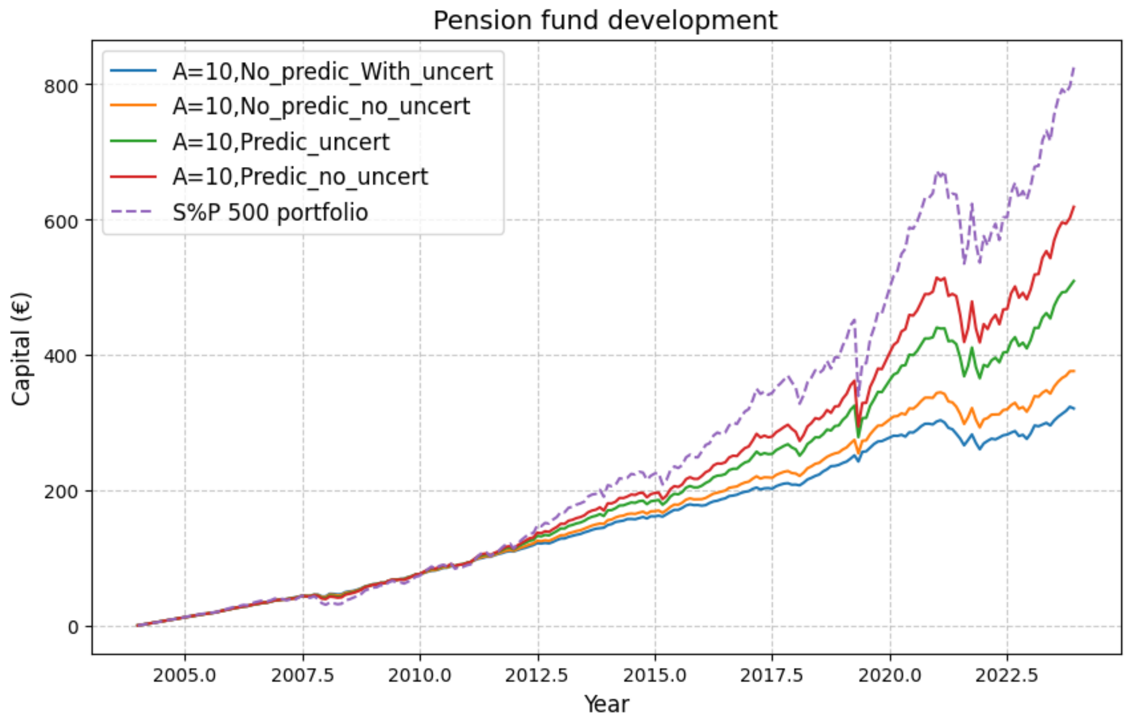


For the case of adverse -neutral risk profile we can observe much more interesting results. Firstly we observe that in the period (2008-09) , the portfolio composed by half stocks-half obligations surpassed the benchmark, but going on the wealth accumulation diverges from the benchmark, failing to keep un with the higher returns of the SnP. Secondly we can observe that during pandemic the optimal portfolio evaluated without uncertainty and predicatability reached the benchmark, but reaching in it's peak even lower performance. Here'out the main difference was determined in the intrinsic risk aversion that determined the optimal asset allocation.

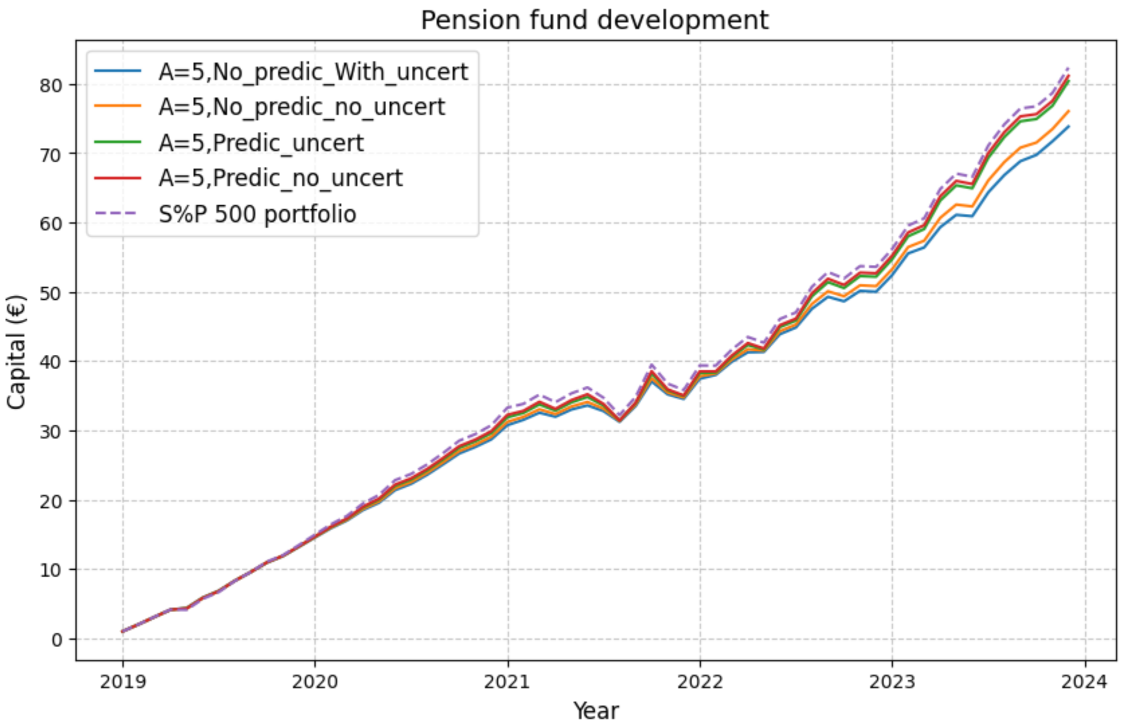
For the below graph, looking to the risk neutral profile, we basically observe that whatever model we consider, the optimal portfolio composition will result in a full allocation in the SnP 500 index. Hereout we reached trivially the benchmark indicator. This last simulation put's on attention how much is important the concept of utility function of various risk profiles. This finding suggest a more deep need of calibration of the risk profile that can be implement by behavioural finance instruments.



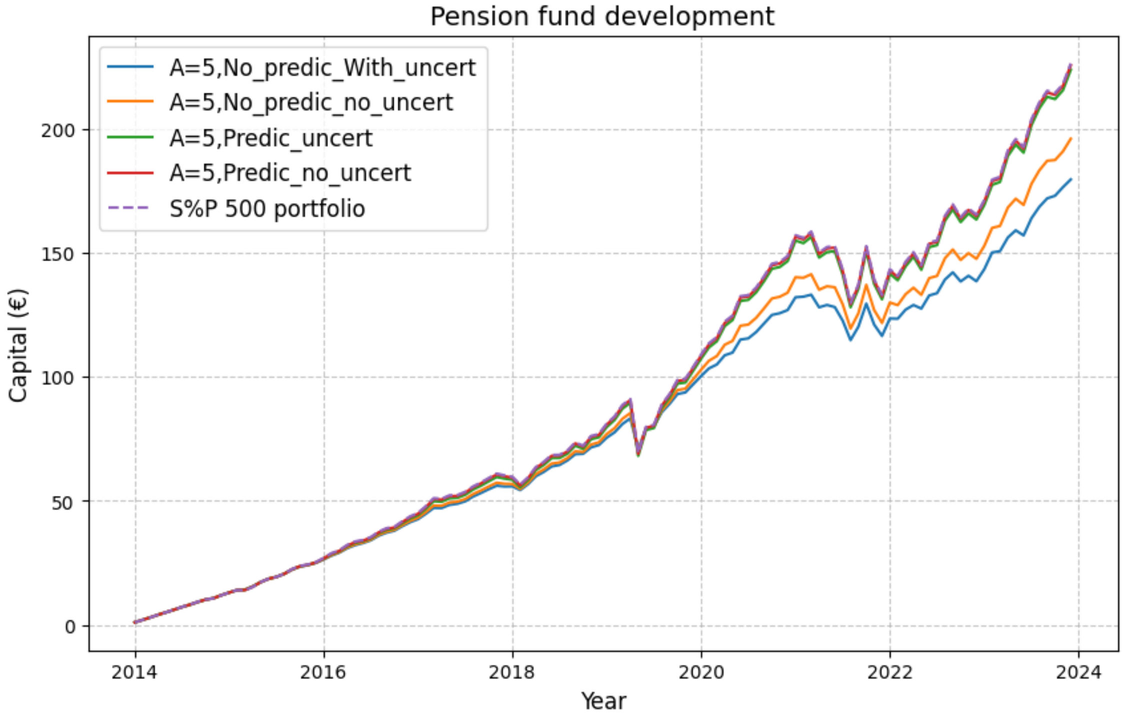
3.6.2 Dynamic allocation P.F.G



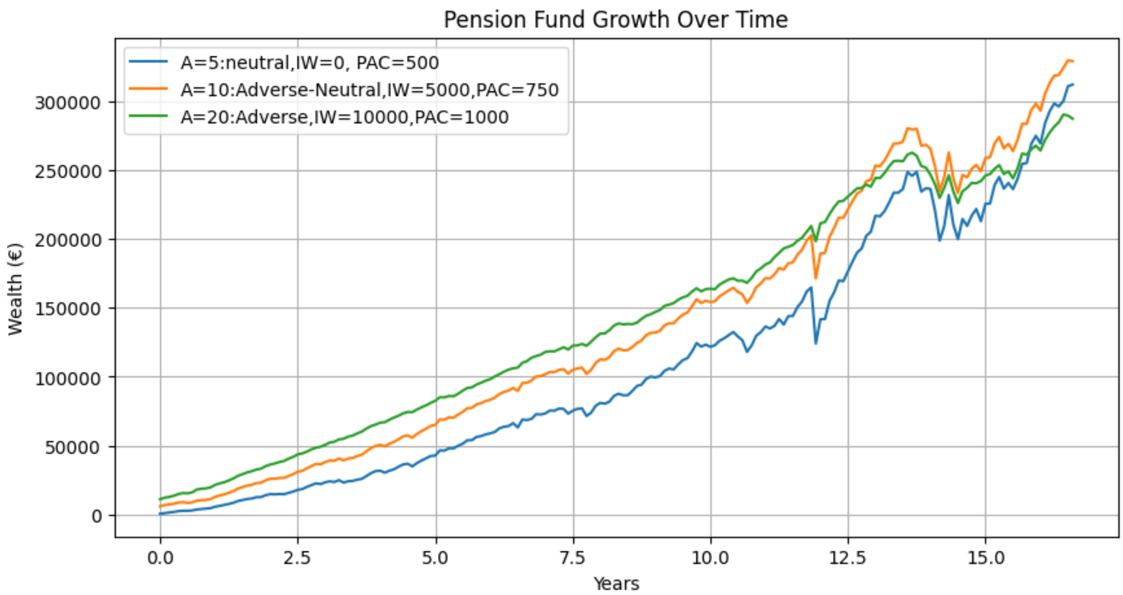
Looking for the main results of the dynamic allocation we can observe a better overall investment evolution in particular for the cases where we don't consider predictability on returns. Other interesting fact derives from the observation of the first years development where we have a stable increase that follows the benchmark and with similar hedging potential as with the buy and hold strategy. The overall development of the P.F presented higher growth in all scenarios that didn't accounted for predictability but with less remarkable growth in scenarios that accounted for predictability (compared to the buy and hold strategy).



At first sight we can observe that the dynamic allocation on a a pension fund in a period of 5 years present a very narrow stick to the benchmark, resulting from a less significant portfolio diversity. This result shows how the inclusion of the predictability and uncertaincy have a insufficient contribution in the short period asset allocation strategies. Even more important is here the consideration of the adverse-neutral profile that plays a determining role in the convergence of the portfolios to the SnP benchmark.



Comparing the same adverse-neutral profile in a 10 years Pension fund development we can have a clear picture of the importance of the predictability and uncertaincy at the long run. Still we observe an insufficient hedging contribution at the long run of the portfolios that did not account for predictability, opposite to the two ones that did (Predictability with and without uncertaincy) where we observe an almost perfect replication of the SnP portfolio. When comparing to the Buy and Hold strategy we can observe a similar results as in the previous simulations, conducting to the narrowing of the possible portfolio outcomes to the benchmark.

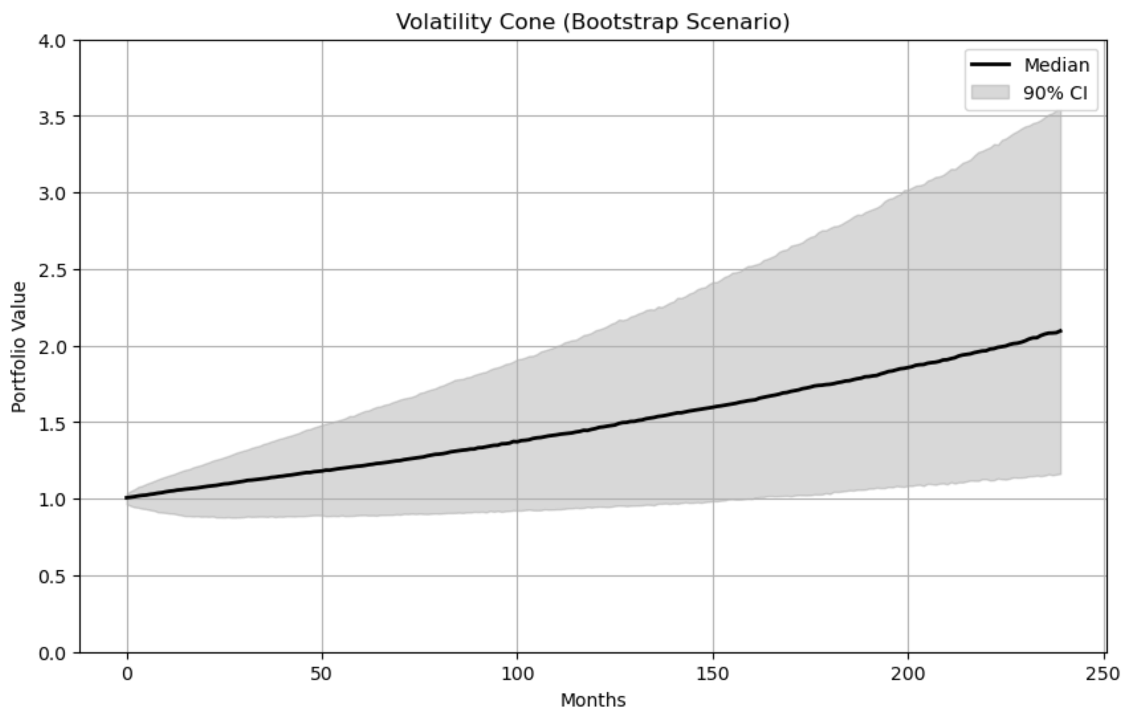


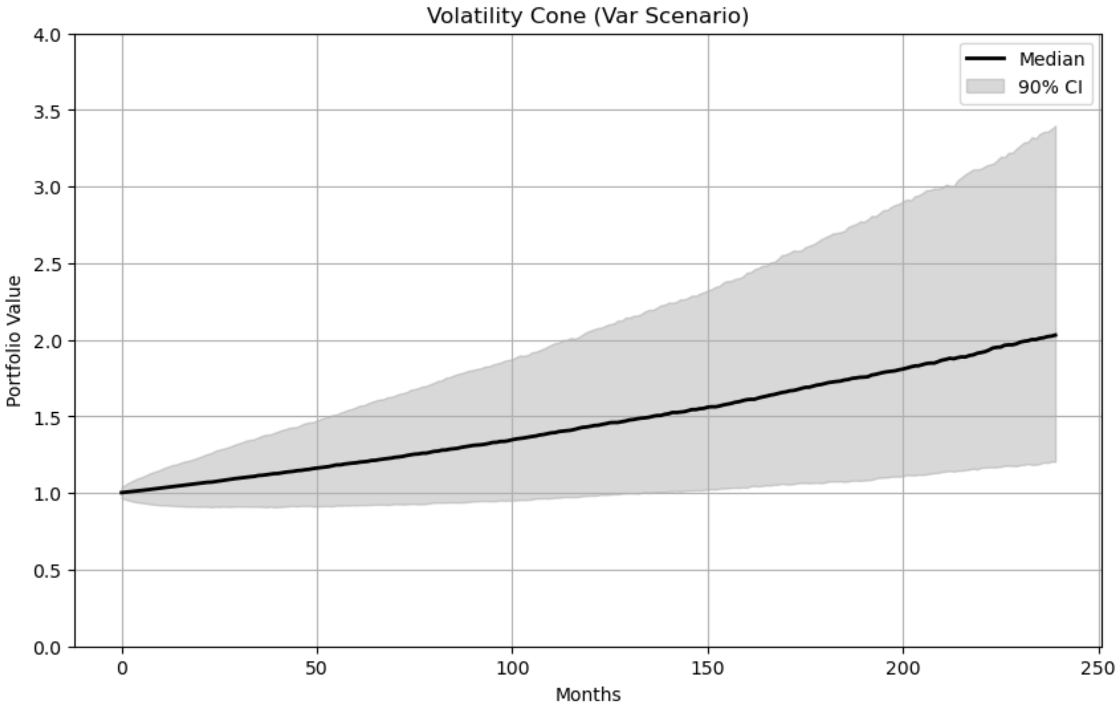
Other interesting results comes from the comparative analysis of the 3 pension funds growth that considers predictability and uncertaincy over an accumulation process of 20 years. In first place we can observe that the risk neutral profile that started with no initial capital and with less mountly contribution was the second by final wealth. This outcome enphasize the importance of the Risk profile as determining the growth. In second place we can observe that the risk adverse profile that presented the beggest initial capital and the highest montly contributions resulted the weakest resulting fund. Still it managed to smothen the most of the volatility in the period, showing very important hedging properties. The last but most important result comes the adverse neutral profile that presented started with average initial values of capital and mounthly contribution, and that finished as the most consistent in the wealth accumulation. This result highlights the implications of the Barberi's work in in the search of the perfect middle estimation that optimises wealth accumulation over time.

3.7 Volatility cones

The first figure depicts the simulated range of outcomes for a 50% stock and 50% bond portfolio over a 250-month horizon using the stationary bootstrap method. The x-axis represents time in months, spanning from 0 to 250 (approximately 20 years), while the y-axis shows the cumulative portfolio value normalized to an initial value of 1. This restrained growth stems from the inclusion of bonds, which dampen both upside potential and downside volatility compared to an all-equity allocation. Surrounding the median is a 90% confidence interval, visualized as a shaded band spanning the 5th to 95th

The second figure presents results for the same 50% stock and 50% bond portfolio but employs a VAR model that incorporates dividend yield predictability. The x-axis mirrors the 250-month horizon, and the y-axis similarly tracks cumulative portfolio growth from an initial value of 1. This improvement arises from the VAR model's ability to leverage lagged relationships between dividend yield changes and asset returns, refining return estimates. The 90% confidence interval, while still widening over time, is notably narrower than in Figure 1, particularly in later periods. This reduction in dispersion reflects the model's capacity to anchor expectations using fundamental predictors, thereby tempering extreme outcomes. Even with the stabilizing influence of bonds, the VAR framework captures mean-reversion dynamics in equity returns, further compressing tail risks.





3.8 Conclusion

The time variation in expected returns is one of the most intriguing empirical findings in finance, yet few studies have explored its implications for portfolio allocation decisions. This study addresses this gap by analyzing how the predictability of returns influences allocation choices, particularly by examining the sensitivity of optimal allocations to an investor's horizon.

If the model parameters (e.g., the expected mean return and dividend yield) are assumed to be known with precision, long-horizon investors tend to allocate significantly more to risky equities compared to short-horizon investors. This occurs because time variation in expected returns generates mean-reversion in realized returns, slowing the growth of conditional variances for long-horizon returns. As a result, equities appear less risky to long-term investors, making them more attractive. In a dynamic setting, where portfolios are rebalanced, more risk-averse investors (with utility functions different from logarithmic utility) also allocate more to equities as their investment horizon increases. In this case, a higher allocation to equities provides a hedge against changes in investment opportunities.

While financial advisors often recommend that long-horizon investors allocate more aggressively to equities, this view finds little support in a world where returns are independent and identically distributed (i.i.d.), as noted by Samuelson (1969). However, the results of this study suggest that time-variation in expected returns may provide a rationale for these practical recommendations.

Nevertheless, this conclusion may be premature. Investors face significant uncertainty about model parameters, such as the intercept and the coefficient on the state variable (e.g., the dividend yield), which are estimated with considerable imprecision. Ignoring this uncertainty can lead to misleading portfolio calculations. Long-horizon investors, while still allocating more to equities than short-horizon investors, may do so to a lesser extent than suggested by models that neglect estimation risk. In some cases, the estimation risk may even lead long-horizon investors to reduce their equity allocation as the horizon increases. Furthermore, parameter uncertainty makes the optimal allocation much less sensitive to the current value of the predictor (e.g., the dividend yield), reducing the risk of drastic and overly sensitive portfolio adjustments.

The framework used in this study is intentionally simple to illustrate the findings, but it can be extended to address additional questions. Future work could include more asset classes, such as long-term government bonds, incorporate additional predictor variables, and account for variations in both expected returns and conditional volatilities. Allowing for time-variation in parameters could further amplify the effects of estimation risk. Finally, the methodology could be applied to other contexts, such as analyzing cross-sectional differences in asset returns based on firm size or book-to-market ratios.

This paper highlights the importance of considering estimation risk and the predictability of returns to avoid overly aggressive or sensitive portfolio decisions. It demonstrates how long-horizon investors can benefit from more realistic optimal strategies that incorporate parameter uncertainty, leading to more robust portfolio recommendations.

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[1] Nicholas Barberis. Investing for the long run when returns are predictable. *The Journal of Finance*, 55(1):225–264, 2000.