Option pricing through Monte Carlo simulations

Valentina Tonazzo, ID: 2060939 †

(Dated: May 2023)

***Abstract*—This report presents a Monte Carlo simulation approach to model the behavior of an asset price based on geometric Brownian motion. The simulation is applied to different types of options, including Vanilla, Asian, and lookback options, with the aim of recovering their respective prices. Geometric Brownian motion is a widely used stochastic process to describe the movement of financial assets, incorporating drift and volatility parameters. By simulating multiple paths of the asset price, it was possibile to capture the inherent randomness and uncertainty in the market. For each type of option, it was implemented the Monte Carlo simulation method and calculated the corresponding option prices. The Vanilla option represents a standard call or put option, while the Asian option's payoff depends on the average price of the underlying asset over a specified period. The lookback option's value is determined by the maximum or minimum price reached by the asset during its lifetime.**

***Index Terms*—Monte carlo method, Geometric Brownian motion, Asian option, lookback options**

I. Introduction

The origins of the Monte Carlo method can be traced back to the 1940s when it was developed by scientists working on the Manhattan Project, a research project during World War II to develop atomic weapons. Stanislaw Ulam, John von Neumann, and other scientists were faced with complex mathematical calculations involving neutron diffusion in fissile material. To overcome the challenges posed by these calculations, they devised a computational technique based on random sampling. The term "Monte Carlo" was coined by Nicholas Metropolis, a member of the Manhattan Project, who named it after the Monte Carlo casino in Monaco, known for its games of chance. Metropolis used this name to describe the random nature of the method's calculations. The fundamental principle of the Monte Carlo method is to use random sampling to estimate complex systems or solve mathematical problems that cannot be easily solved analytically. It involves simulating numerous random samples or scenarios to approximate the behavior or distribution of a system.

In the context of finance, the Monte Carlo method gained significant attention in the 1970s when Fischer Black and Myron Scholes used it to develop the groundbreaking Black-Scholes-Merton option pricing model. This model revolutionized the field of options pricing and led to a deeper understanding of financial derivatives.

The general Monte Carlo approach could be summarized in the following steps:

1) Define the problem: Clearly specify the problem to be solved or the system to be simulated. For example, in finance, this could involve pricing a complex derivative.
2) Formulate a mathematical model: Develop a mathematical model that describes the behavior of the system or the variables involved. In this case, the latter could be the geometric Brownian motion model for asset prices.
3) Define input parameters: Determine the input parameters required by the model, such as the asset price, volatility, interest rate, and time horizon.
4) Generate random samples: Use random number generators to simulate random samples based on the defined input parameters. The number of samples generated will depend on the desired accuracy and precision of the results.
5) Perform simulations: Apply the mathematical model to each random sample to simulate the behavior of the system.

II. METHODS

As said before, the Geometric brownian Motion can be used as a mathematical model to describe the behaviour of an underlying asset price: in this report, Monte Carlo Simulations were used to sample different GMB paths to obtain the expected payoff in a risk-neutral world and finally discount this payoff at the risk-free rate.

The concept of Brownian motion, named after the botanist Robert Brown, was first observed in the early 19th century. It refers to the random movement exhibited by tiny particles suspended in a fluid. In 1905, Albert Einstein, in his work on the theory of heat, provided a mathematical explanation for Brownian motion, which formed the basis for further developments.
The application of Brownian motion to finance began with the work of Louis Bachelier, a French mathematician, in his doctoral thesis "Théorie de la spéculation" published in 1900. Bachelier proposed a model for stock price movements based on random motion, although his ideas were not widely recognized at the time. In the 1950s, Paul Samuelson, a Nobel laureate economist, rediscovered Bachelier's work and highlighted its significance in the context of financial markets. Later, in the 1960s and 1970s, the Black-Scholes-Merton option pricing model, which relies on GBM, made a groundbreaking impact on the field of finance.

From a practical point of view, Geometric Brownian motion is a continuous-time stochastic process that describes the evolution of asset prices over time. It assumes that the returns of an asset are normally distributed, reflecting the inherent randomness and uncertainty in financial markets. The mathematical formula for GBM can be expressed as follows:

$$dS = \mu S dt + \sigma S dW \tag{1}$$

Here, dS represents the differential change in the asset price, S is the asset price, μ is the average expected return or drift, dt is the infinitesimal time interval, σ is the standard deviation of returns or volatility, and dW is a Wiener process or Brownian motion. The equation states that the change in the asset price, dS, is composed of two terms. The first term, $\mu S dt$, represents the deterministic component, which captures the average rate of return over time. The second term, $\sigma S dz$, represents the stochastic component, which accounts for the random fluctuations in the asset's returns.

To simulate GBM, the first step consist in dividing the desired time period, which in this case is represented by the time to maturity T, into smaller intervals. The length of these intervals will depend on the level of granularity required for the simulation. Common approaches include using daily time steps, so let's denote the length of each time step as $\Delta t = T/n$. The initial price is set equal to S_0, the current market price of the asset, and knowing the drift rate μ and volatility σ, each update of the asset price can be retrieved as:

$$\Delta S = \mu S(t)\Delta t + \sigma S(t)\sqrt{\Delta t}\Delta z, \tag{2}$$

where Δz represents a random increment of the path. The integrated form of 1 is derived from ITO integration:

$$S(t') = S(t)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}, \tag{3}$$

The latter, in discrete time approximation, becomes:

$$S(t + \Delta t) = S(t)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} z_t}. \tag{4}$$

and here z_t is obtained by generating a random number from the standard normal distribution $\mathcal{N}(0, 1)$.

A second goal of the project consists in built a pricer scheme for different type of options: the ones considered are presented below; each of them are characterize by different ways to estimate the final payoff, in particular:

- As can be see from table 1, Asian options [1] are a type of financial derivative whose payoff depends on the average price of the underlying asset over a specific period rather than the asset's price at a single point in time. By using the average price, Asian options help smooth out short-term price fluctuations and should provide a more stable basis for determining the option's value.
- The distinctive feature of a lookback [2] option is that it allows the holder to "look back" and select the most advantageous price for exercising the option. This feature makes lookback options attractive to investors seeking to

capitalize on significant price movements and potential market extremes.

option	call	put
Vanilla	$(S_T - K)^+$	$(K - S_T)^+$
Asian	$\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$	$\left(K - \frac{1}{T}\int_0^T S_t dt\right)^+$
lookback	$(S_T - \min_t S_t)^+$	$(\max_t S_t - S_T)^+$

TABLE 1: Payoff for different options

With Vanilla option it were used, at first, one-step simulations, and for all the other it were considered multi-steps simulations.

Once obtained the corresponding payoff from each Monte Carlo simulated path, they were averaged and multiplied by the discount factor to retrieve the initial price.

$$p_0 = e^{-rT}\frac{1}{M}\sum_{i=0}^{M} payoff_i, \tag{5}$$

where M indicates the total numer of geometric Brownian motions simulated.

III. Results

In table 2 can be seen the fixed parameters taken in consideration:

S_0	100 [$]
σ	20%
r	1%
T	1 [y]
K	99 [$]
n	365
$dt = T/n$	0,00274 [y]

TABLE 2: Generic parameter for the BS market model

where S_0 is the current asset price, σ the volatility, r the interest rate, T represent the time to maturity, K the strike price, n the total number of discrete time step simulations, dt is the time increment.

In Figure 3 are reported $M = 100$ simulated paths of the underlying asset price as a function of time steps. In this case it was not used a VBA code, but just simple excel formulas, according to eq. 4, where the drift μ was setted equal to the interest rate r; in particular the function "=NORMINV(RAND())" has been used to sample random numbers normally distributed as $\mathcal{N}(0, 1)$.

On the other hand, the same eq. 4 was used into a VBA code to simulate $M = 700$ one-step trajectories, so in this case, n, the number of time step, has been setted equal to 1; and applying eq. 5, the price of vanilla options (call and put) were estimated.

p_0^{call} [$]	p_0^{put} [$]
9,59	6,52

TABLE 3: price of Vanilla options through MC 1 step simulations

And the same VBA code has been used to estimate $M = 700$ multi-step trajectories ($n = 365$), in order to price the other exotic option, results are available in table 4

option	p_0^{call} [$]	p_0^{put} [$]
Vanilla	8,14	6,76
Asian	5,53	3,97
Lookback	14,66	15,53

TABLE 4: option's price through MC multi-step simulations

IV. Concluding Remarks

As can be observed from Figure 3, each simulation path exhibits a random pattern due to the stochastic nature of the process. The paths do not follow a straight line or a smooth curve but rather display fluctuations over time. Since 100 simulation are not enough to appreciate the impact of drift parameter μ, in order to accentuate its effect, it was realize another set of simulations imposing $\mu = 1$ which are shown in Figure 4. The drift term represents the average expected growth or decline of the underlying asset. The simulations show an overall upward trend over time, depending on the value of the drift, since it was imposed to be positive, as can be appreciated from Figure 1:

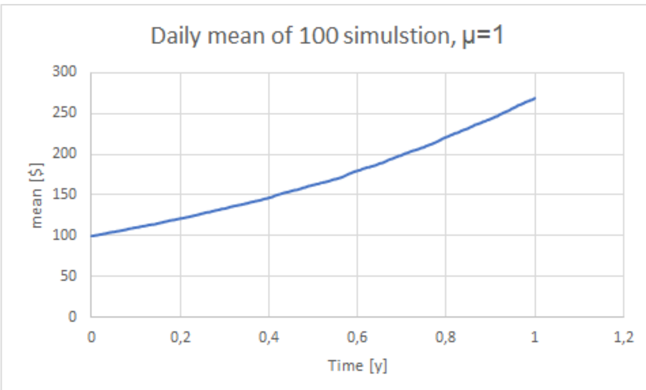

Fig. 1: Daily mean of 100 trajectories for Geometric Brownian motion, with $\mu = 1$

.

Another notation must be made about volatily parameter, imposing $\sigma = 80\%$ the magnitude of price fluctuations becomes large as expected: Some simulation paths have higher volatility, resulting in larger price swings, while others have lower volatility with relatively smoother movements. Since the logarithm of the asset price is directly related to the standard deviation, it can be observeed a logarithmic pattern in the plot of daily standard deviation against time, Figure 2:

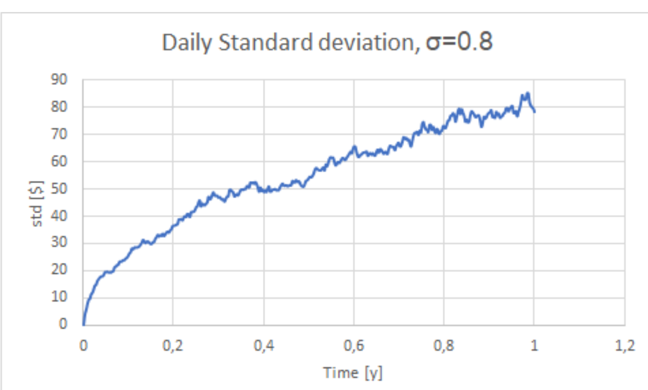

Fig. 2: Daily mean of 100 trajectories for Geometric Brownian motion, with $\sigma = 80\%$

.

Regarding the pricing model, it can be observed that the two ways to price a Vanilla option, through single-step Montecarlo simulations and multi-step Montecarlo simulations give similar results, or at least compatible, suggesting that the additional computational complexity of the multi-step simulation may not be necessary to achieve accurate results in this particular case, but for more accurate results and future improvements a greater number of simulations should improve the convergence of price.

References

[1] J. Hull, *Option, Future, and other Derivatives*. Pearson, 2006.
[2] T. Bjork, *Arbitrage Theory in Continuous Time*. Oxford University Press, 2009.

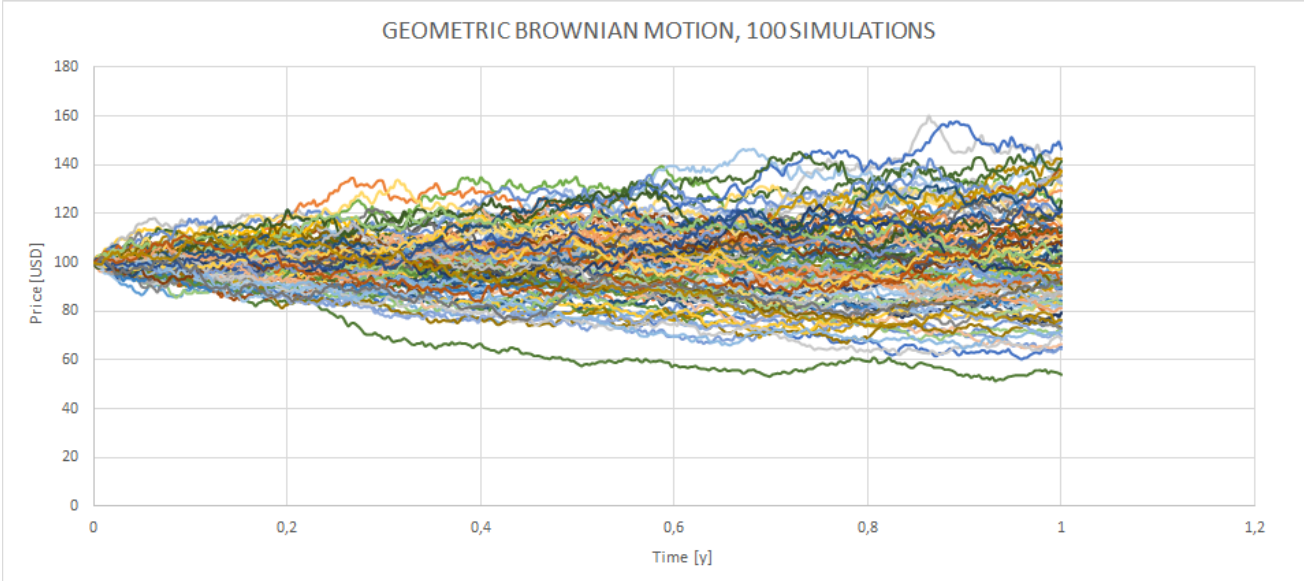

Fig. 3: 100 trajectories for Geometric Brownian motion

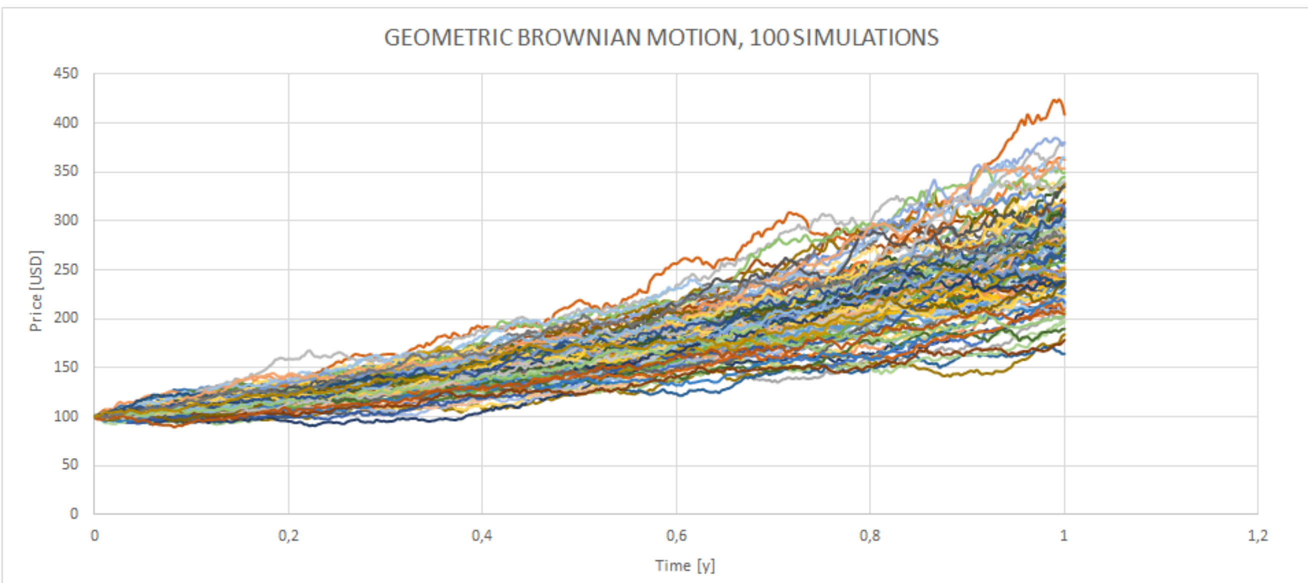

Fig. 4: 100 trajectories for Geometric Brownian motion, $\mu = 1$

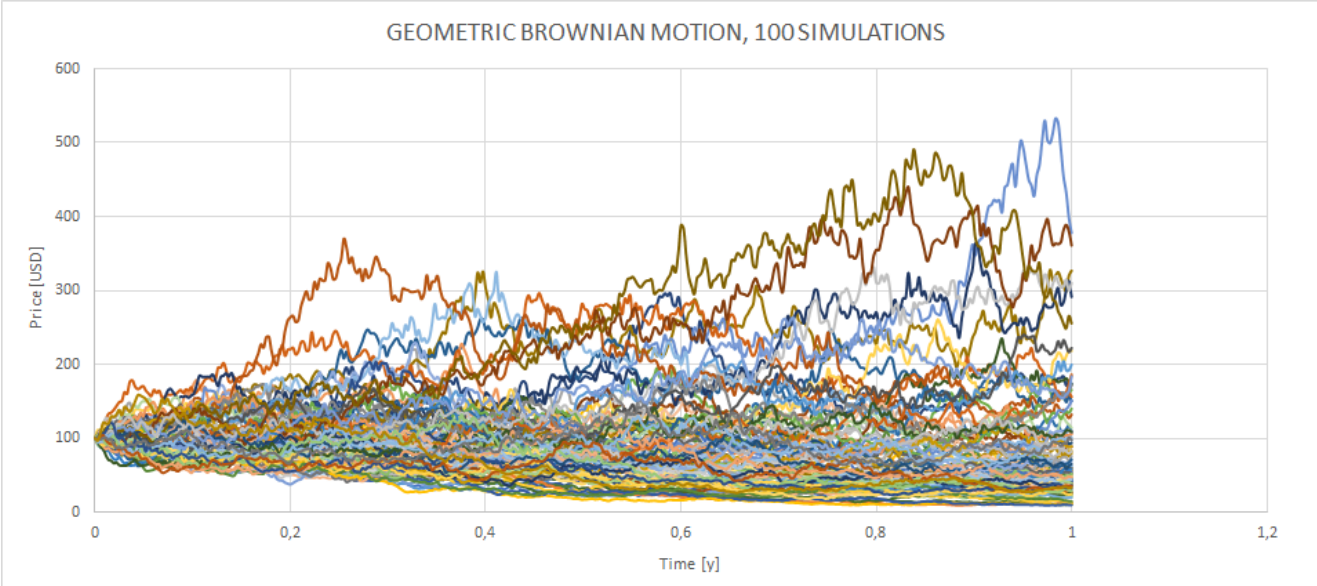

Fig. 5: 100 trajectories for Geometric Brownian motion, $\sigma = 0,8$