Binomial model method vs Leisen-Reimer method comparison and convergence to Black-Scholes formula

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***Abstract*—The purpose of this report is to implement two different discrete time methods to price an European Call option: the Multi-step Binomial tree and the Leisen-Reimer method. A second goal consist in compare the obtained results with the value obtained from the so called Black-Scholes formula, which instead works in a continuous time framework. The results confirms the Leisen-Reimer conclusions achieved in the 1995 paper [1]: the latter outperforms the traditional Binomial tree method providing a faster convergence to the Black-Scholes formula.**

***Index Terms*—Multi-step binomial tree, Leisen-Reimer method, Black-Scholes formula**

I. Methods

A. Multi-step binomial tree

The binomial model previously presented [2], [3] is unrealistically simple. Clearly, an analyst can expect to obtain only a very rough approximation to an option price by assuming that stock price movements during the life of the option consist of one or two binomial steps, but fortunately this method provides a generalizable numerical model for the valuation of options: the multi-step binomial model. Essentially, this kind of model uses a "discrete-time" pattern of the varying price over time of the underlying financial instrument; it was first proposed by William Sharpe in the 1978 edition of Investments, and formalized by Cox, Ross and Rubinstein in 1979 and by Rendleman and Bartter in that same year. The multi-step Binomial options pricing approach has been widely used since it is able to handle a variety of conditions for which other models cannot easily be applied. This is largely because it is based on the description of an underlying instrument over a period of time rather than a single point. As a consequence, it is used to value American options that are exercisable at any time in a given interval as well as Bermudan options that are exercisable at specific instances of time. Being relatively simple, the model is readily implementable in computer software. As said, it simply consist in a generalization of the one-step binomial tree, infact, the approach consists to divide time between now and the option's expiration into N discrete periods. The stock price is initially S_0 and during each time step, it either moves up to u times its initial value or moves down to d times its initial value, in such a way a binomial lattice (Tree) is obtained as shown in fig. 1.

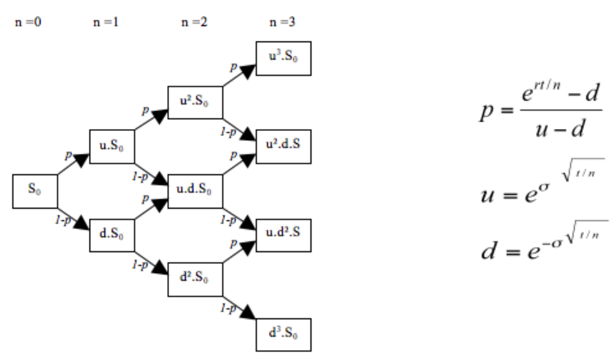

Fig. 1: Stock and option prices in a multi-step tree.

At each time step it is possible to apply the priciples developed for the one-step method, and combining them it is possible to recover the option price at the initial node of the tree:

$$p_0^{Call} = (1+r\Delta t)\sum_{j=0}^{n}\binom{n}{j} f^{u^j d^{n-j}} q^j (1-q)^{n-j} \quad (1)$$

Where r represents the risk-free interest rate, Δt is the length of each time step measured in years, n is the total number of time steps, q represents the risk-neutral probability and f is the current stock price. When binomial trees are used in practice, the life of the option is typically divided into 30 or more time steps.

B. Black-Scholes method

This formula synthetically expresses the theoretical price of European options on a financial asset, typically a share, which does not pay dividends and whose price evolves over time according to a suitable random model. The formula takes its name from the two scholars who proposed it (F. Black and M. Scholes [4], but it was also independently derived by R. Merton [5]. This joint birthright earned Merton and Scholes the 1997 Nobel Prize in economics, but not Black, who died prematurely. Considered a cornerstone of finance theory, the formula has had a formidable impact on the spread of derivatives markets.

In the formula of B.-S. the price is expressed as the difference between an appropriate adjustment of the current value of the underlying and the current value of the exercise price, also adjusted. The adjustments are obtained by applying multiplicative coefficients to the respective base values. Indicating respectively with C_t and S the current values of the option and of the underlying, with K the current value of the strike and with $N(d1)$ and $N(d2)$ the respective multiplicative coefficients, we have:

$$C_t = S \cdot N(d_1) - K \cdot N(d_2). \tag{2}$$

The coefficients $N(d)$ between 0 and 1 are the probabilities that a standard normal distribution assumes determination lower than d. In detail:

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}) \cdot T}{\sigma\sqrt{T}}, \tag{3}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{4}$$

C. Leisen-Reimer method

This model was introduced by Dietmar Leisen and Matthias Reimer in 1995, its main benefit is greater precision with smaller number of steps, compared to earlier models such as Cox-Ross-Rubinstein (CRR). Its main idea is that the underlying price binomial tree is centered around the option's strike price at expiration (not around the current underlying price like CRR). The logic and calculation of tree nodes and option price is the same as in other binomial models, the difference is only in the calculation of tree parameters: up and down move sizes and probabilities. As a first step it is necessary to calculate the probabilities p and $(1-p)$ of an up or a down move: the function used to do so is the so called Peizer-Pratt inversion function, $h^{-1}(x)$: it provides (discrete) binomial estimates for the (continuous) normal cumulative distribution function, taking as inputs the parameters d_1 and d_2, which are the same as in the Black-Scholes formula.

$$\begin{aligned} p_1 &= h^{-1}(d_1) \\ &= \frac{1}{2} + \frac{sign(d_1)}{2}\sqrt{1 - e^{\left[-\left(\frac{d_1}{n+\frac{1}{3}}\right)^2\left(n+\frac{1}{6}\right)\right]}}, \end{aligned} \tag{5}$$

$$\begin{aligned} p_2 &= h^{-1}(d_2) \\ &= \frac{1}{2} + \frac{sign(d_2)}{2}\sqrt{1 - e^{\left[-\left(\frac{d_2}{n+\frac{1}{3}}\right)^2\left(n+\frac{1}{6}\right)\right]}}; \end{aligned}$$

and at this point it is possible to recover the up and down move sizes as follow:

$$u = e^{r\Delta t} \cdot \frac{p_1}{p_2}, \tag{6}$$

$$d = e^{r\Delta t} \cdot \frac{1-p_1}{1-p_2},$$

where exponent term can be interpreted as net cost of holding the underlying security over one step, as Δt is the duration of one step in years, calculated as $\frac{T}{n}$. Now, having the up and down move sizes and probabilities, they can be used to calculate the option price, and this part is the same as in other binomial models.

As already mentioned, one of the main difference between the three methods concern the fact that the multi-step binomial tree and the Leiser-Reimen are discrete time methods, while the Black-Sholes formula works in a continuous time framework. From a theoretical point of view it is possible to demonstrate that as the number of time-step increases (namely the length of each step reduces) the first two methods converge to the latter. The aim of this report is to computationally verify this relation in order to predict a theoretical call option price.

II. Results

In order to have a fair evaluation and then compare the three methods, it were used some fixed values for the needed parameters to characterize the call option, these values are reported in Table 1:

S [$]	r	vol	T[y]	K[$]
101	1%	2,2%	1	101

TABLE 1: Set of fixed parameters for computations

Using eq.1 and eq.5 implemented as VBA codes, it have been calculated the call option prices for different time steps as reported in Table 2. At this point the results were compared with the option price calculated using the Black-Scholes formula 2:

$$P_{BS}^{call} = 9,3142\$.$$

steps	Leisen-Reimer	Binomial
2	9,280792636	8,553944035
3	9,280792636	10,05045476
4	9,300436143	8,964353921
5	9,300436143	9,749759189
6	9,306689196	9,103658043
7	9,306689196	9,621469921
8	9,309465829	9,171620487
9	9,309465829	9,550524802
10	9,310939948	9,211108197
12	9,311816045	9,236546429
15	9,312379056	9,45174793
18	9,3130349	9,276432592
20	9,313235742	9,283845389
25	9,313506102	9,392847781
30	9,313736409	9,304258668
40	9,313923032	9,312918737
50	9,3140124	9,317306359
100	9,314135933	9,322895933
250	9,314172012	9,321486269
500	9,314177285	9,318543245
750	9,314178269	9,316732294
1000	9,314178614	9,315549234

TABLE 2: Time-step results for discrete time methods

III. Concluding Remarks

Generally, as can be seen both models, the binomial and the Leisen-Reimer, tend to converge to the result obtained by Black-Scholes formula. A further consideration that may be done is that both show to be more precise as the number of time step increases, as expected from a theoretical point of view. In the Binomial model case the convergence seems to be not as smooth as in the Leisen-Reimer case: indeed while in the latter case, the results gradually approach the goal price, in the first case, the calculated prices oscillate till the convergence. Another difference between the two methods concern the number of time step to convergence: the Liesen-Reimer seems to outperform the binomial one confirming results from the original paper of Leisen-Reimer.

References

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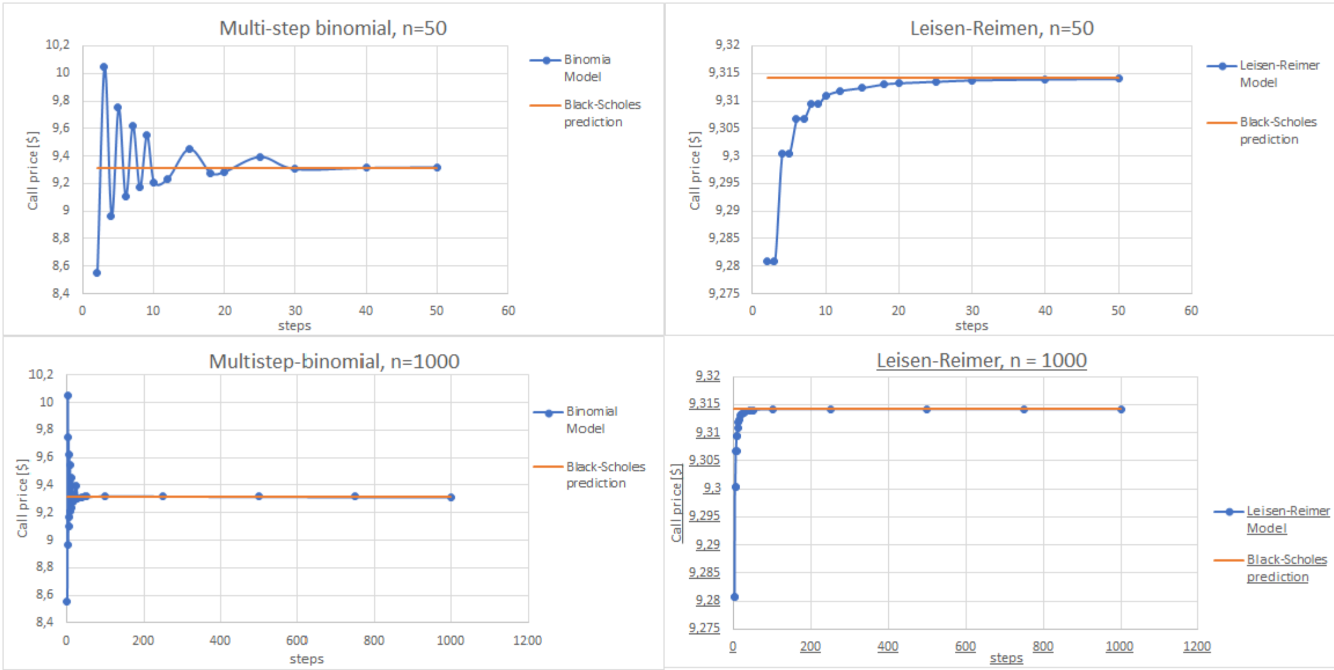

Fig. 2: Comparison between different methods for price an European Call option.