Alma Mater Studiorum · Università di Bologna

MASTER'S DEGREE IN QUANTITATIVE FINANCE

Take home exam
Economics of Financial Markets
Optimal Portfolio Allocation

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Sections and Chapters

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15/07/2023

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Preliminary assumptions

Before setting up the structure of this asset allocation simulation it is useful to clarify which are the basic assumptions underlying the investment decisions that will be displayed.

First, we assume a medium-term investment time horizon. This means that the fictional investor we are consulting is willing to hold the portfolio we construct for about three years, which leads us to work under relaxed liquidity constraints. Secondly, we need to make assumptions about the return objective we are pursuing. Throughout this investment analysis we follow a defensive strategy aimed to minimize risk in favor of a well-diversified portfolio.

These mild assumptions are in line with the classical portfolio theory, allowing us to make use of some of the most common optimization techniques employed in the financial environment.

1 Moments of Stock Returns

We start off by computing the percentage stock returns as

$$\widetilde{R}_i = \frac{P_1^i - P_0^i}{P_0^i} \tag{1}$$

This basic formulation has been chosen in spite of the widely used logarithmic version because it is the one presented in the seminal work of Markowitz 1952.

We proceed to compute mean, variance and higher moments by using the following formulas

$$\mu = \frac{1}{n}\sum_{i=1}^{n} \widetilde{R}_i \tag{2}$$

$$\sigma_i^2 = \frac{1}{n}\sum_{i=1}^{n}(\widetilde{R_i} - \mu)^2 \tag{3}$$

$$Kurtosis_i = \mathbb{E}\left[\left(\frac{\widetilde{R_i} - \mu_i}{\sigma_i}\right)^4\right] = \frac{(\mu_i)_4}{\sigma_i^4} \tag{4}$$

$$Skewness_i = \mathbb{E}\left[\left(\frac{\widetilde{R}_i - \mu_i}{\sigma_i}\right)^3\right] = \frac{(\mu_i)_3}{\sigma_i^3} \tag{5}$$

The results for each stock are displayed in tables A (for daily observations) and B (for monthly observations) in the appendix. Notice that, out of the eighty-eight securities that compose our dataset, thirteen have a curtailed time series. Indeed, Pirelli C, Illimity Bank, Aquafil, Equita Group, Italgas, Enav, Poste Italiane, Gambero Rosso have missing values at the beginning of the sample period, while Astaldi, Dea Capital, Banca Intermobiliare, Borgosesia RSP and Cattolica Assicurazioni, were delisted at some point between 01/01/2015 and 04/07/2023 and present missing values from the delisting date on. We exclude these stocks from our analysis, as missing values could undermine the robustness of the models we are going to implement. This reduces our choice by roughly 15%, therefore we assume the trade-off to be fair.

It is worth to spend some words on the distributions of the returns. By looking at the summary statistics in table X it is clear that the aggregated returns do not display a gaussian distribution. Despite the mean seems to be consistently close to 0 in the daily observation sample, variance is far from the unitary value of the normal standard distribution. By looking to higher moments the hypothesis of gaussianity seems to drift even further. Indeed, both the daily and monthly returns exhibit an average significantly positive skewness. This means that the right tail of the distribution of returns is, on average, more pronounced than the left one. Kurtosis, which provides the degree of peakedness, significantly exceeds the normal value of three. As expected, this behavior is more pronounced in the daily sample, as daily observations are more likely to provide extreme values.

Table 1: Summary statistics of daily observations

DAILY	Mean	Standard Deviation	5%	Median	95%
Mean Returns	0,000	0,000	0,000	0,000	0,001
Variance	0,001	0,000	0,000	0,000	0,001
Skewness	0,528	1,144	-0,872	0,244	2,852
Kurtosis	15,298	21,052	3,573	10,651	49,660

Table 2: Summary statistics of monthly observations

MONTHLY	Mean	Standard Deviation	5%	Median	95%
Mean Returns	0,006	0,009	-0,009	0,006	0,020
Variance	0,012	0,012	0,003	0,009	0,034
Skewness	0,701	1,496	-0,623	0,200	4,588
Kurtosis	5,052	10,288	-0,223	2,014	29,821

We tested the null hypothesis of standard normal distribution by performing a Jarque-Bera test for each security, both for the monthly and daily samples.

$$JB = \frac{n}{6}\left(S^2 + \frac{(K-3)^3}{4}\right) \tag{6}$$

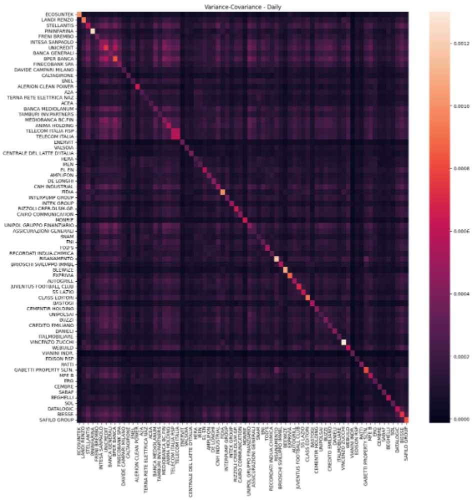
This model tests the null hypothesis of standard normal distribution against the alternative hypothesis of non normal distribution. To do so it compares actual skewness and kurtosis values in the sample data to what is expected under the normal distribution assumption; if the sample data has skewness and kurtosis values that are close to what would be expected, then the p-value associated with the test will be significantly different from zero, meaning that the hypothesis is failed to reject. Converserly, a low p-value indicates that there is strong evidence to reject the null hypothesis, suggesting that the data does not follow a normal distribution. As can be observed from the results (last column of Table 16 and Table 17 in Appendix A) the p-value for the daily returns is equal to 0 for each stock. This means that none of the daily returns of the stocks in the dataset follows a normal distribution. Regarding the monthly returns of stocks, for 39.24% of the stocks, there is not enough statistical evidence to reject the null hypothesis of a Gaussian distribution at a 5% confidence level.

Due to the fact that all of the models we'll be using assume that the returns are normally distributed, we must proceed cautiously in light of this result.

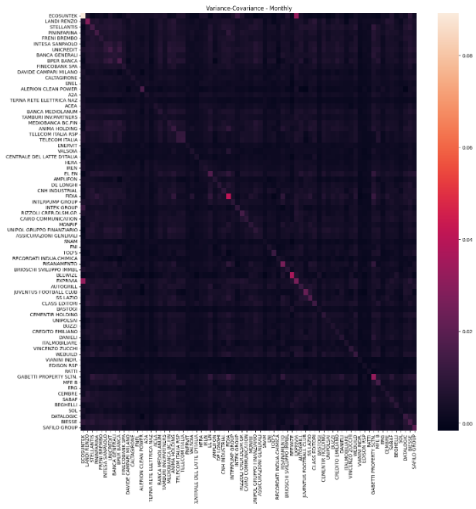
2 Variance-Covariance Matrix and Correlation Matrix

The concept of diversification is of crucial importance in the canonical portfolio optimization problem. This objective can be achieved by analyzing the relationship among the assets to be included in a portfolio. Our goal is to minimize the chances that the stocks we choose move simultaneously in the same direction, leading our strategy to perform either very well or dreadfully. According to the basic assumption of investors preferences this scenario is to be avoided, granting instead the lowest degree of correlation between the assets held in the portfolio.

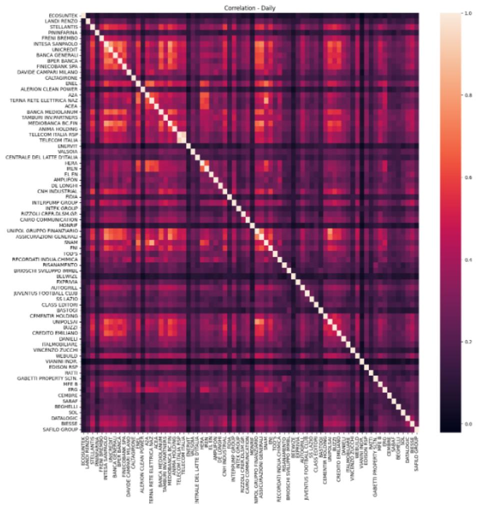
Two useful objects to analyze the correlation structure of the assets in place are the variance-covariance and correlation matrices.



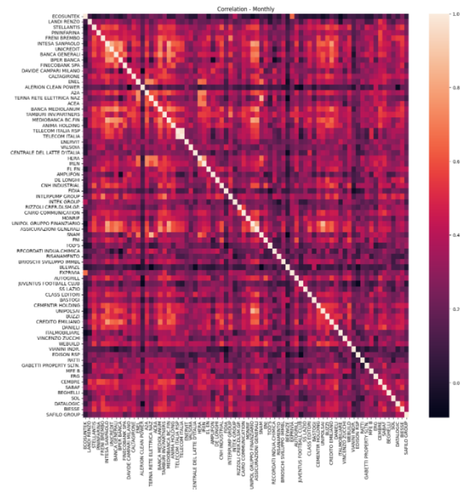
(a) Variance-Covariance Daily



(b) Variance-Covariance Monthly



(c) Correlation Daily



(d) Correlation Monthly

The values inside the matrices are retrieved by applying formulas (7) and (8) for each couple of securities, both for the daily and for the monthly observations. Lighter is the color of the matrix greater the correlation or the variance-covariance.

$$\sigma_{AB} = \mathbb{E}\left[\left(\widetilde{R}_A - \mu_A\right)\left(\widetilde{R}_B - \mu_B\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(\widetilde{R}_A - \mu_A\right)\left(\widetilde{R}_B - \mu_B\right) \tag{7}$$

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} \tag{8}$$

The matrices (a) and (b) illustrate, on daily and monthly basis, how much covariance exists between two stocks, which indicates how much their returns move in sync. Analyzing this matrix enables us to gain insights into the interdependencies and diversification opportunities within the stock portfolio.

Instead the other two matrices represent the correlation among stocks (daily and monthly frequency), ranging from -1 to 1.

- ρ = 1 indicates a perfect positive correlation: the stocks move in the same direction (Lighter is the color of the matrix greater the correlation)
- ρ = -1 indicates a perfect negative correlation: the stocks move in opposite directions
- ρ close to 0 suggests a weak relationship between the stocks

3 Securities Selection

Now we can dive into the selection of eleven securities that are going to compose the portfolio. The financial literature refers to this problem as cardinality constrained portfolio selection and many works focus on different approaches that can be implemented to tackle this kind of issue.

A first strand of literature adopts assumptions in order to relax these conditions by constructing a surrogate constraint (Bienstock [1]) or by eluding them (Bertsimas and Shioda [2]). Xie et al. (2008) ([3]) propose a randomized algorithm with some quality guarantee, while Gao and Li (2013) ([4]) concentrate on the symmetric properties of the polyhedral constraints by implementing a Lagrangian relaxation method.

Other authors consider the moments of the distribution and include them in entropy models (among others, [5], [6]).

All these approaches stem from the mean-variance portfolio model (Markowitz, [7]) and attribute great importance to the correlation structure between the initial pool of assets. This is the reason why we decided to implement an algorithm that minimizes the correlations between the assets. The intuition behind this algorithm is to depart from the couple of assets with the lowest correlation among the whole dataset.

Our algorithm aims to minimize the correlation between the chosen stocks. Given the correlation matrix, we flatten it into a vector where each index represents a pair of stocks (stock1, stock2), and the value indicates the correlation between stock1 and stock2. The vector is sorted in ascending order based on the correlation values.

Next, we define the number of stocks to be selected (in our case, 10) and a threshold that indicates the maximum correlation a candidate stock can have with the already selected stocks. If fewer than eleven candidate stocks are found, a solution the threshold is incremented by 0.01 until eleven stocks are reached. By doing this, we ensure that our selected stocks have a very low correlation with each other. To further improve the algorithm, we can incorporate additional information regarding returns and diversify across sectors and regions.

This procedure is performed both on the daily and monthly correlation matrices and yields to two different portfolios (table 1 and 2).

Table 3: Portfolio based on optimization of daily correlation matrix

VIANINI INDR.	I:VIN
ECOSUNTEK	I:MON
MONRIF	I:MON
BEEWIZE	I:FUL
PININFARINA	I:PINF
VINCENZO ZUCCHI	I:ZUC
ALERION CLEAN POWER	I:ARN
RATTI	RATTI
ENERVIT	I:ENV
SABAF	I:SAB

Table 4: Portfolio based on optimization of monthly correlation matrix

SNAM	I:SRG
BEEWIZE	I:FUL
FIDIA	I:FD
ALERION CLEAN POWER	I:ARN
ECOSUNTEK	I:MON
VIANINI INDR.	I:VIN
LANDI RENZO	I:LRZ
RIZZOLI CRER.DLSM.GP	I:RCS
TOD'S	I:TOD
JUVENTUS FOOTBALL CLUB	I:JUVE

The stocks in the portfolio in table 3 have a correlation that ranges between -0.040 and 0.120, while the portfolio 4 ranges between -0.185 and 0.240. In order to choose between the two portfolios, we compute the average returns both at a monthly and daily frequency. It results that the portfolio optimized according to the daily correlation matrix (Table 3) has a daily average return of 0.032% and a monthly average return of 0.558%. On the other hand, the portfolio optimized through the monthly correlation matrix (Table 4) has a daily average return of 0.038% and a monthly average return of 0.992%. Therefore we decided to opt for the more profitable portfolio (Table 4).

4 Price behavior of the selected securities

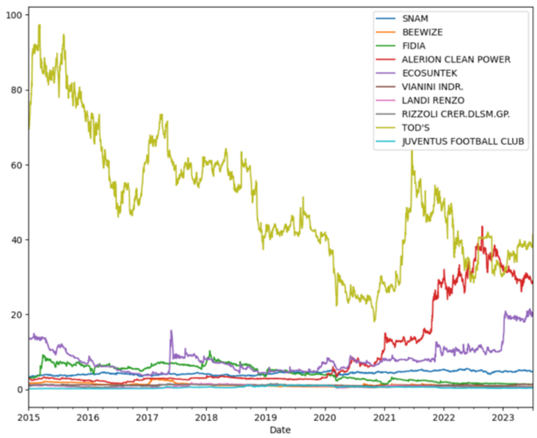


Figure 2: Daily prices of selected securities

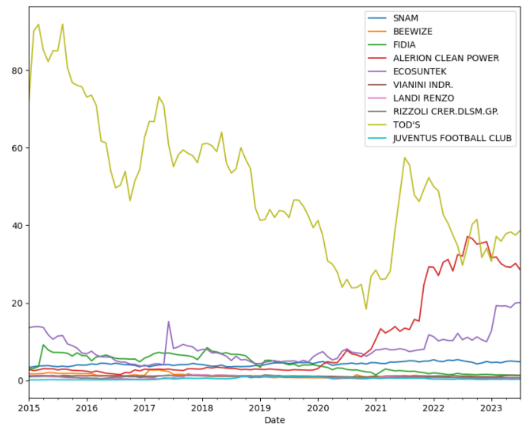


Figure 3: Monthly prices of selected securities

Figures 2 and 3 plot the paths followed by the prices of the ten stocks of our portfolio, for daily and monthly frequencies respectively. As expected, the monthly plot is smoother, whereas the daily is more scattered.

We can see that the low correlation among the assets can be also appreciated graphically. The goal of portfolio diversification is to balance risk by including assets with different return behaviors. Hence, observing low correlation among the returns of some stocks in the chart suggests that these assets can contribute to reducing the overall portfolio risk when combined.

For example, in the second quarter of 2017, TOD'S experienced a steep decrease in stock prices, while in contrast VIANNINI INDR. registered an increase in stock price. Even more striking is the behavior of ALERION and TOD'S in 2023. We can see that as TOD's price rises for the whole year, ALERION proceeds dropping down.

5 Mean Variance Optimal Portfolio Allocation

After choosing the securities to be included in the portfolio, it is now time to decide the relative weights that they will take in the investment strategy that we implement.

In the Mean-Variance setting, such weights can be retrieved by solving a rather simple optimization problem:

$$\begin{gathered}\min_{w} f(w) \\ \text{subject to } \sum_{i=1}^{n} w_{i=1} \\ w_i \in [-1, 1]\end{gathered} \tag{9}$$

The model originally proposed by Markowitz ([7]) suggests that this optimization problem should be translated into the minimization of the variance of the portfolio, as it can be seen as a proxy for risk. This approach will be carried out in section $14.

Another common optimization function that springs from Markowitz work in the Mean Variance classical optimization literature is the Utility function, which is proposed by Merton (1971). Clearly, any kind of utility function that is chosen has to be maximized.

$$\max_{w} Utility$$

where

$$Utility = \omega'\mu - \frac{\gamma}{2}\omega'\Sigma\omega \tag{10}$$

Anyways, this kind of optimization problems have a major caveat, namely they do not account for moments strictly greater than two. This is why the financial literature elaborated models that also consider skewness and kurtosis in the portfolio optimization (see[8] and [9]).

Nevertheless, these kind of models exceed the scope of this work. Therefore, in order to find the optimal portfolio we decided to adopt the maximization of the Sharpe ratio (Sharpe, 1966).

$$SharpeRatio = \frac{R_p}{\sigma_p} \tag{11}$$

This is a risk-weighted return on investment: it indicates the ratio of the extra return of the risky portfolio on the risk-free security, relative to σ_p. We opted for the formulation of the Sharpe ratio that does not take into account the risk free rate, as it would require the elaboration of very strong assumptions either considering it fixed or time varying.

The optimization problem therefore becomes as follows.

$$\max_w \frac{\mathbb{E}\left[R_p\left(w\right)\right]}{\sigma_p} \tag{12}$$

We employ the actual annualized returns and the annualized variance covariance matrices. This yield to the following asset allocation:

Figure 4: Daily allowing short selling - Max Sharpe ratio

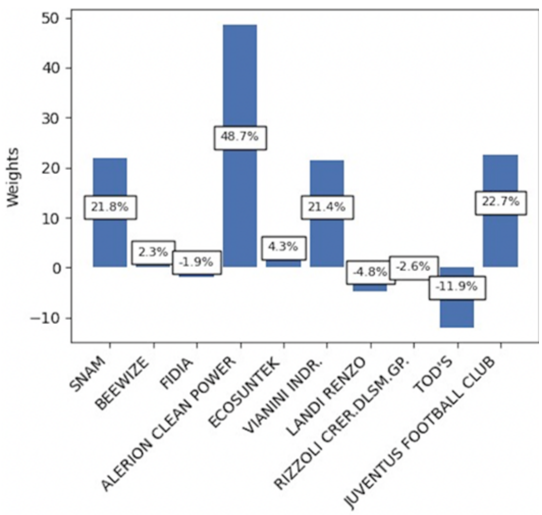
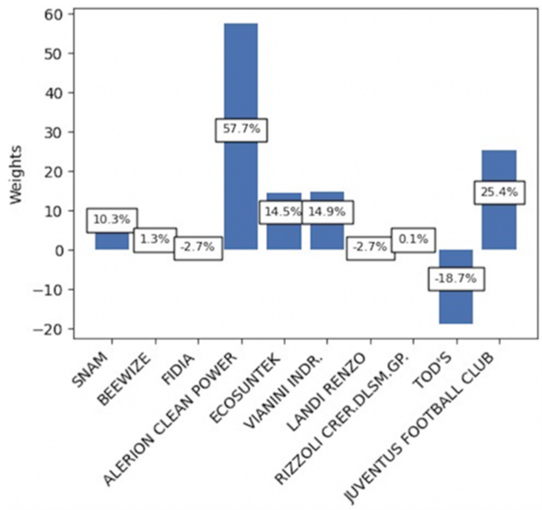


Figure 5: Monthly allowing short selling - Max Sharpe ratio



It can be seen that the weights differ when considering either the daily (Figure 4) or the monthly (Figure 5) frequency. Despite this, it seems that the proportions between the assets hold, as well as the signs of the weights (except for RIZZOLI).

On average, we pose that in order to obtain the mean variance portfolio with the maximum Sharpe ratio, it is necessary to short FIDIA, LANDI RENZO, RIZZOLI (or to hold very little proportion of it) and TOD'S. The two securities that consistently must be held in relatively high proportions are ALERION and JUVENTUS stocks.

It can be useful to display the descriptive statistics of these stocks. From these descriptive statistics it seems that, on average, the investment strategy based on this kind of optimization requires to go long on assets with high sharpe ratio and lower volatility, and to short assets with lower sharpe ratio and higher volatility.

Table 5: Summary statistics for portfolio assets on a daily frequency

	Mean return	SD	Sharpe ratio
Assets to be held			
SNAM	0,067	0,032	1,227
BEEWIZE	0,024	0,031	0,087
ALERION CLEAN POWER	0,344	0,025	2,123
ECOSUNTEK	0,164	0,023	0,64
VIANINI INDR.	0,053	0,000	0,643
JUVENTUS FOOTBALL CLUB	0,178	0,000	1,021
Assets to be shorted			
FIDIA	0,038	0,032	0,148
LANDI RENZO	0,056	0,031	0,237
RIZZOLI CRER.DLSM.GP.	0,052	0,025	0,320
TOD'S	0,001	0,023	0,008

Table 6: Summary statistics for portfolio assets on a monthly frequency

	Mean return	SD	Sharpe ratio
Assets to be held			
SNAM	0,058	0,052	1,787
BEEWIZE	0,034	0,19	0,079
ALERION CLEAN POWER	0,357	0,127	1,859
ECOSUNTEK	0,304	0,299	0,284
VIANINI INDR.	0,030	0,055	0,823
JUVENTUS FOOTBALL CLUB	0,213	0,149	0,795
Assets to be shorted			
FIDIA	0,071	0,201	0,147
LANDI RENZO	0,073	0,169	0,211
RIZZOLI CRER.DLSM.GP.	0,053	0,121	0,302
TOD'S	-0,004	0,110	-0,028

6 Mean Variance Portfolio with non-negativity constraint on portfolio weights

In the optimization problem of the previous point we make a strong assumption, namely that the weights of the stocks included in our portfolio can range from -1 to 1, and that they must sum to 1. This mathematical constraint represents the possibility for economic agents to short an asset. This is not always the case, as some markets or some specific agents are prevented from this possibility by natural or regulatory restraints. For example, quite seldom a non-institutional investor can short a security on regulated markets, this because policy makers are afraid that such practice would induce market agents to speculate. By imposing the new constraint the optimization problem becomes the following.

$$\max_{w} \frac{\mathbb{E}\left[R_p\left(w\right)\right]}{\sigma_p}$$
$$\text{subject to } \sum_{i=1}^{n} w_{i=1}$$
$$w_i \in [0, 1] \tag{13}$$

This leads to the construction of two portfolios (one for the daily frequency and one for the monthly frequency) with two new sets of weights.

Figure 6: Daily not allowing short selling - Max Sharpe ratio

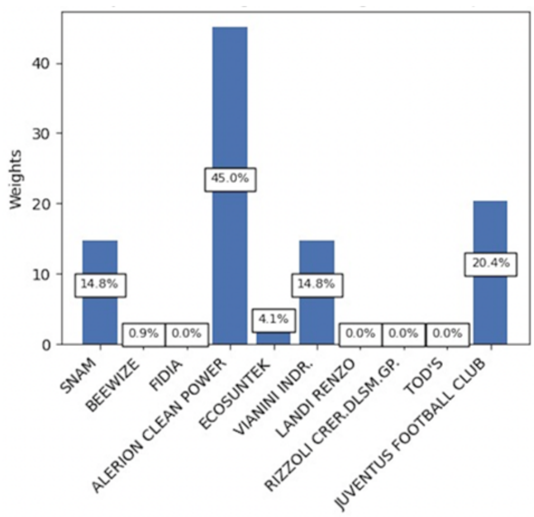
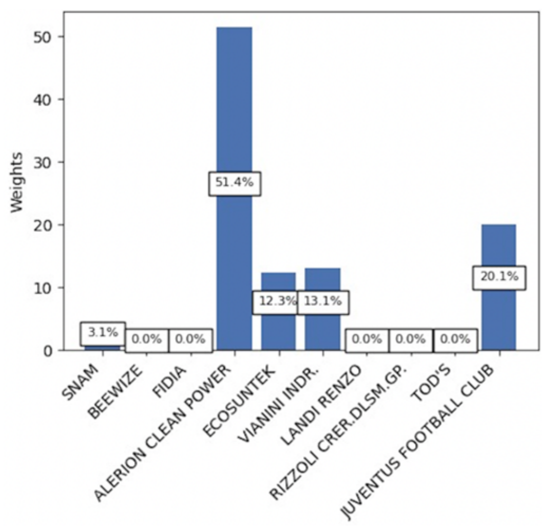


Figure 7: Monthly not allowing short selling - Max Sharpe ratio



7 Moments of the Mean Variance Portfolio

In Table A are displayed the average mean, variance, standard deviation, skewness and kurtosis of the ten stocks composing the

Mean Variance portfolio and for the portfolio itself, both for daily and monthly frequencies.

Table 7: Summary statistics for the portfolio on a monthly and daily basis

	Daily		Monthly	
	Stocks average	Portfolio	Stocks average	Portfolio
Mean return	0,000	0,001	0,010	0,021
Variance	0,001	0,000	0,026	0,005
SD	0,027	0,018	0,147	0,072
Skewness	1,241	0,652	2,691	0,812
Kurtosis	14,857	6,871	17,579	1,428

The daily and monthly returns of the portfolio are computed as the return of the asset times the relative weight computed in the previous steps. The pillar on which the Mean Variance model leans on is the concept of diversification. The main contribution of diversification is to obtain a portfolio that has a mean return equal to the mean of the returns of the assets of which it is composed, yet a lower variance of the mean of the variances of such securities (Marzo, 2022 [10]). In fact, our portfolio has a daily mean return 2.87 times higher than the average of the stocks' returns and a variance equal to 0.44 times the average of the daily variances of the securities. This inverse relation holds in the monthly observations as well. Here, the portfolio return is 0.021 against the mean return of the stocks equal to 0.010, whereas the variance of the portfolio is equal to 0.005 versus the mean variance of the stocks of 0.003. Moving our analysis to higher moments, we can assume that agents dislike negative skewness (and like positive) and dislike kurtosis, as investors are averse to extreme outcomes in a distribution (Dittmar, 2002). In this sense we can pose that neither in the daily nor in the monthly sample the investor gains particular benefits from our portfolio in terms of skewness. On the other hand, the portfolio seems to effectively dampen kurtosis. In point of fact, in the daily frequency the stocks' kurtosis ranges between 22.389 and 3.796, while the portfolio exhibits a mean value of just 6.871.

8 Efficient frontier

Figure 8: Efficient frontier with selected stocks, daily

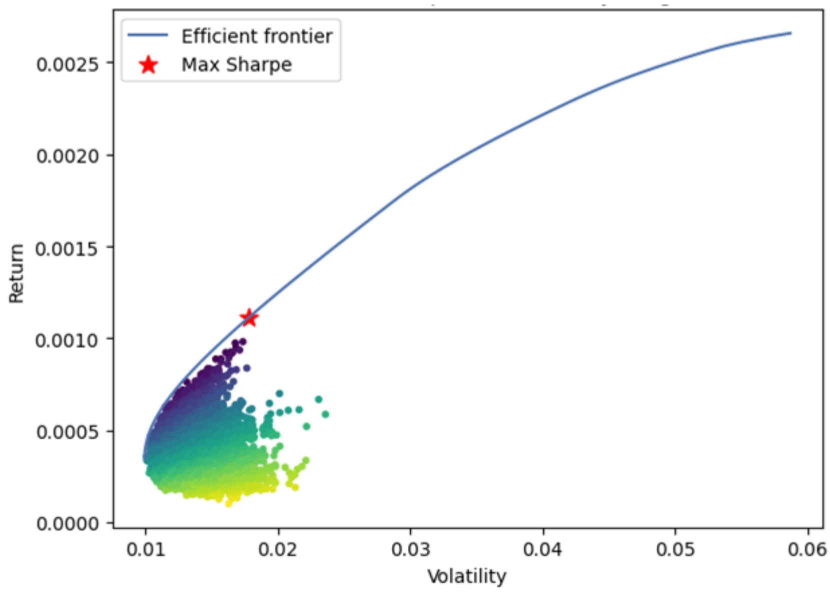
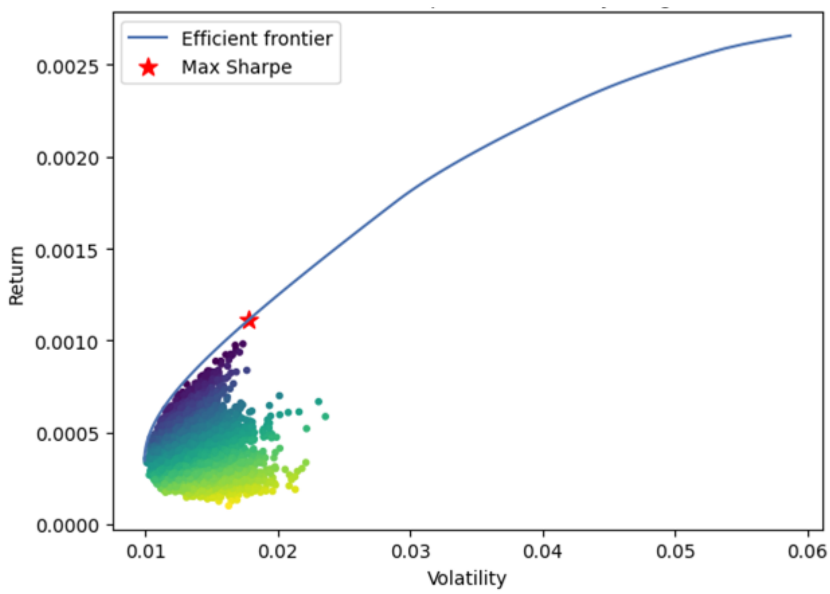


Figure 9: Efficient frontier with selected stocks, monthly



Regardless of the optimization function one chooses, the goal of Mean Variance portfolio optimization results in finding a portfolio located on a tangency point of the efficient frontier. This object is obtained by mapping each level of risk (represented by the standard deviation on the x axis) to the maximum return associated to such risk. All the combination above the frontier cannot be achieved given the assets composing the portfolio. Conversely, all the points below such curve can be obtained, but are not efficient as the investor could obtain the same return by bearing less risk. These portfolios are called dominated portfolios and in the figures their color gets lighter as the level of dominance increases.

In order to select the optimal portfolio the investor must specify her risk preferences. A very risk-averse investor will choose the portfolio with the lowest value of standard deviation, i.e., the Minimum Global Variance Portfolio (which will be discussed in point 14). On the other hand, an investor with a high risk appetite will locate herself on the upper-right extreme of the efficient frontier, gaining the highest level of return while bearing the maximum possible risk.

The maximization carried out in point 5 yields to the two portfolios marked with a red star in Figure A and B. As it can be seen, by opting for the maximization of the portfolio's Sharpe ratio we manage to locate our investment in the between of the two extremes.

If we annualize the mean returns and the standard deviations of the two portfolios, the two series of observations yield to a mean return of 0.28 for daily frequency and 0.25 for monthly observations. As the annualized standard deviations are very close to the mean returns values, the Sharpe ratio of the portfolio amounts to 1.00, which is just above the threshold to consider an investment worthy. Since our preliminary choice of the ten assets in point 3 was driven by diversification rather than profitability, these results are sensible and consistent with the preliminary assumptions.

9 Moments of the FTSE MIB All Share Total Return Index

Table 8: Summary statistics for the FTSE MIB All Share Total Return on a monthly and daily basis

	Daily		Monthly	
	FTSE AS TR	Portfolio	FTSE AS TR	Portfolio
Mean return	0,000	0,001	0,009	0,021
Variance	0,000	0,000	0,003	0,005
SD	0,014	0,018	0,058	0,072
Skewness	-1,332	0,652	-0,559	0,812
Kurtosis	15,221	6,871	2,992	1,428

In order to compute the beta first of all we use an index as a proxy of the market in which we operate. Among the possible choices we have: FTSE MIB Index - Price Index, FTSE MIB Index - Tot Ret Index, FTSE Italia All Share - Price Index and finally FTSE Italia All Share - Tot Ret Index.

We decided to adopt the latter since the the stock prices we are provided with are adjusted for dividends. Furthermore, the choice for an all share index is justified by the fact that only 2 of the 10 stocks of our portfolio (SNAM, ALERION) are considered by the standard FTSE MIB as they are the only ones that have a sufficiently high capitalization.

Table A exhibits the descriptive statistics of the distribution of the FTSE Italis All share-Tot Ret Index (from now on, the Index) together with those of our portfolio. From these statistics we can state that, on average, our portfolio outperforms the market both in daily and monthly frequencies observations.

Indeed, the higher volatility of the portfolio appears to be compensated by higher returns, as the Sharpe ratio of the Index is approximately half the one of the portfolio. Furthermore, the Index presents negative skewness and much higher kurtosis, which stand for a higher probability of extreme values.

10 Beta for Stocks and Portfolio

The Beta is a measure of an asset's volatility related to the market as a whole. It is computed as

$$\beta_{im} = \frac{\sigma_{i,M}}{\sigma_M^2} = \rho_{im}\frac{\sigma_i \sigma_M}{\sigma_M^2} = \rho_{im}\frac{\sigma_i}{\sigma_M^2} \tag{14}$$

As regards the daily betas' displayed in Table 9 it is possible to see that we do not have betas larger than 1. This means that the portfolio is composed only by defensive stocks: for any level of positive correlation between a stock and the market the volatility of the title is lower than that of the market. Applying the same procedure to the monthly returns we find that the the monthly betas are slightly higher for all the stocks. Table 9 shows only one aggressive stock for monthly observations, namely LANDI RENZO has a Beta=1.220, this means that this asset exacerbates the movement of the market.

For what concerns the beta of the portfolio it is computed as the weighted average of the betas of the stocks

$$\beta_{pM} = \sum_{i=0}^{n} w_1\beta_{1M} + ... + w_i\beta_{iM} \tag{15}$$

As we can see from Table 9 we have a defensive portfolio for daily and monthly data. This result is certainly influenced by our initial choice to construct a portfolio with the less correlated stocks. This is likely to lead to a strong reduction of the volatility of the portfolio and of the beta as well.

Table 9: Daily and monthly estimation of beta for selected stocks and portfolio

Stock	Daily Beta	Monthly Beta
ECOSUNTEK	0,298	0,821
LANDI RENZO	0,743	1,220
ALERION CLEAN POWER	0,430	0,529
FIDIA	0,582	0,992
RIZZOLI CRER.DLSM.GP.	0,746	0,858
SNAM	0,689	0,464
TOD'S	0,744	0,749
BEEWIZE	0,542	0,319
JUVENTUS FOOTBALL CLUB	0,736	0,734
VIANINI INDR.	0,123	0,204
Portfolio	0,398	0,423

11 SML

The Security Market Line (SML) is a graphical representation of the relationship between the expected return and risk of an investment and it depicts a linear relationship that plots the expected return of a security or portfolio against its systematic risk, measured by Beta. The following formula defines the level of the mean equilibrium return of the stock compatible with the risk premium of the market for a given risk free asset.

$$R_i = R_f + \beta_{iM} \left(\mu_M - R_F\right) \tag{16}$$

In Figure 9 and Figure 10 we plot the SML and two stocks (JUVENTUS, LANDI RENZO). As we can see JUVENTUS has a beta lower 1, and an annual return of 0.18, which is much greater than the average return of the market. Laying above the SML, the stock is probably undervalued or it generates returns in excess with respect its volatility. For what concerns the other stock (LANDI RENZO) the graph shows that it is overvalued with a beta greater than the beta of the market. These considerations are true for the daily and the monthly SML.

Figure 10: Security market line - daily data

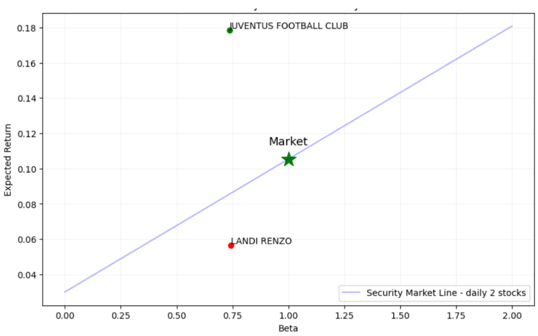
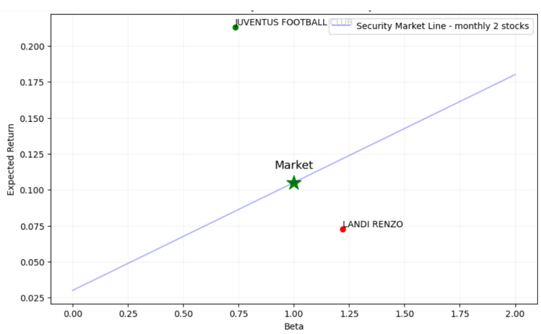


Figure 11: Security market line - monthly data



The Figures 12-13 show where our portfolio is positioned with respect to the SML. When compared to the market portfolio our portfolio exhibits higher mean return and lower beta. This can suggest that our portfolio is well managed and generates outstanding returns relative to its level of risk.

Figure 12: Security market line for the portfolio - daily data

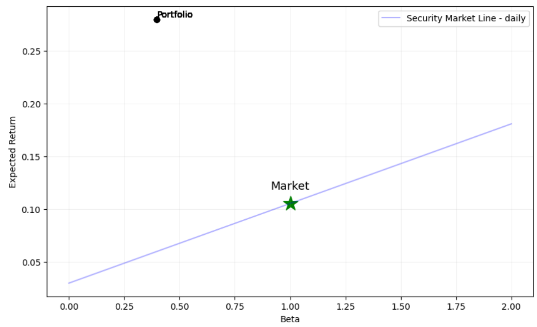
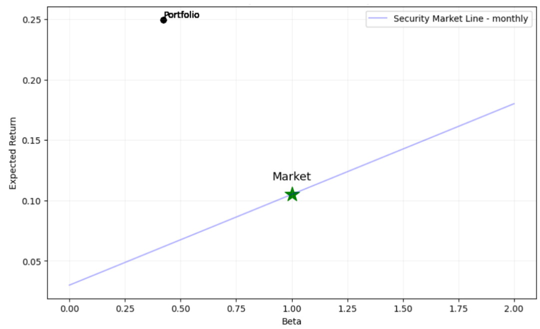


Figure 13: Security market line for the portfolio - monthly data



12 Black Litterman Approach

Even though Mean Variance models show to yield to effective results, they have a crucial caveat: they are purely data driven. The need to take into account investors' forecasts and beliefs based on experience and intuition into the portfolio optimization models led to the formulation of Bayesian models.

This category of asset allocation frameworks is characterized by the elaboration of a prior, or a-priori. This object is a set of hypothesis about the structure of variables involved in the asset allocation. In standard Bayesian models the prior is commonly built on assumptions about the distribution of the variables.

In this class of models, the Black-Litterman model (Black and Litterman, 1992 [11]) stands out of its relative conceptual simplicity.

The Black Litterman formula is represnted by (17), which is the weighted average of the views and the prior, while the weights are the levels of confidence and the scalar constant tau (Thomas M. Idzorek, 2004 [12]). This last parameter equals 1/T, with T equal to the lenght of the returns of our sample.

$$\mathbb{E}(R) = \left[(\tau\Sigma)^{-1} + P^T \Omega^{-1} P \right]^{-1} \left[(\tau\Sigma)^{-1} \Pi + P \Omega^{-1} Q \right] \tag{17}$$

The model poses that the equilibrium returns are based on the market capitalization, indeed the market-implied returns are determined by

$$\Pi = \delta \Sigma \omega_{mkt} \tag{18}$$

where

$$\delta = \frac{R - R_f}{\sigma^2} \tag{19}$$

We estimate the risk aversion coefficient delta, the covariance matrix of the excess returns and then we use the market capitalization of the five stocks to be as adherent as possible to the original model. Market capitalization data are retrieved from Yahoo Finance). Thanks to this data we can built the market prior (Figure 14 and Figure 15), which is a prevision of the future performance of the stocks based on market data.

Figure 14: Market Prior for daily data

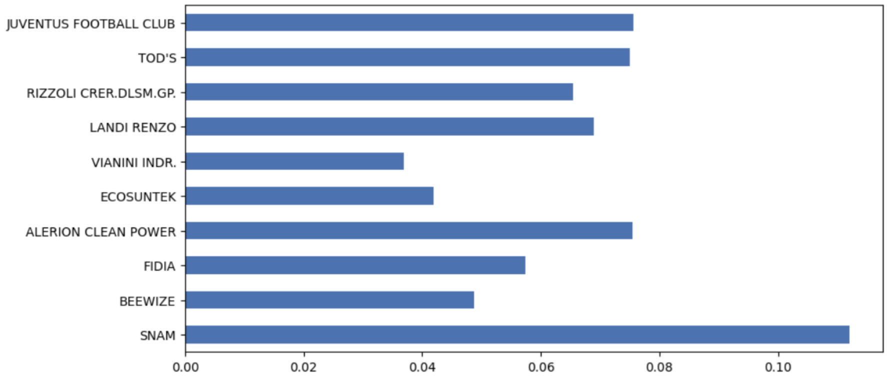
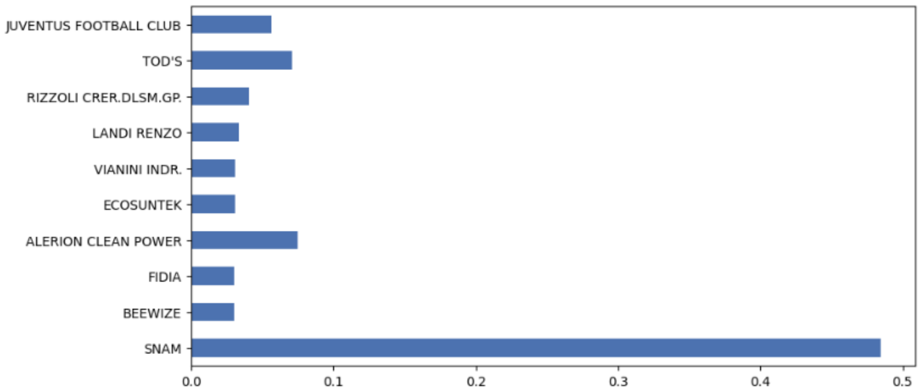


Figure 15: Market Prior for monthly data



12.1 Views

The Black-Litterman model belongs to the category of mixed estimation Bayesian models. Here, the priors are basically future previsions about the performances of the assets held in the portfolio that is to be optimized. Such views can be absolute or relative and each of them has its confidence level.

12.1.1 Absolute Views

Absolute views are forecasts on the future performances of a single asset independently of the behaviors of other variables.

We base our two absolute views on the estimations gathered by the Financial Times about the firms listed on the Euronext Milan Market. The forecasts provide three alternative scenarios (up, medium, down) on the outcomes of a stock in a 12-month time horizon. We choose the medium forecast and we assume that our confidence level is equal to the probability implicitly attributed to each scenario, namely 0.33. This process leads us to elaborate two views on

- Tod's: the medium scenario is a decrease of 4.4% (with a confidence level of 0.33)
- Rizzoli: the medium scenario is a decrease of 22.6% (with a confidence level of 0.33).

12.1.2 Relative Views

Contrarily to absolute views, relative views involve at least two stocks as they are meant to express the performance of an asset in comparison to that of another security held in the portfolio.

We form our relative views on the Fama-French three-factor model ([13]), more specifically on the SMB (small minus big) and the HML (high minus law) factors. This model is widely applied and, being an extension of the CAPM, seems suitable to provide relevant insights. We apply the intuition behind the well-known results by forming this couple of relative views:

- The stock with the second largest market capitalization will outperform the stock with the largest market capitalization
- The stock with the second highest price-to-book (from now on P/B) value will outperform the stock with the highest price to book value

The first set of data about market capitalization is obtained as described above. The updated P/B ratios are taken by Yahoo Finance. By ranking the stocks inside our portfolio, we find that the two stocks with the largest market capitalization are SNAM (15.3 billion euros) and ALERION (1.53 billion euros), while the two most overvalued firms according to the P/B ratio are BEEWIZE (117.17) and JUVENTUS (6.78).

Figure 16: Yearly spreads for SMB couple

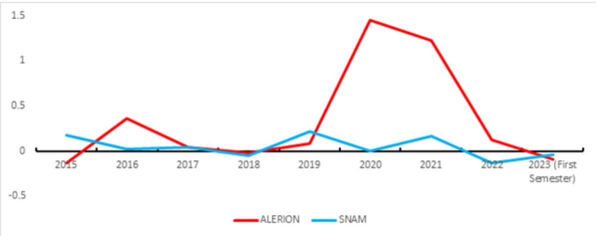
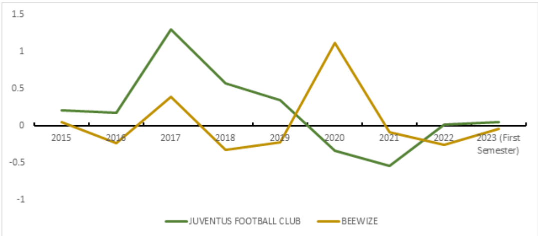


Figure 17: Yearly spreads for HML couple



12.2 Posterior

Once we have introduced the views we can build the posterior, which is the final estimation of the returns after combining the views and the market data. This is the crucial passage of a Bayesian approach: the prior is shaped by specific hypotheses, that is, for the

Black-Litterman model, the views. Indeed it is a new elaboration of the future returns once the investor estimates the previsions.

The posterior is given by

$$f_{po}\left(\mu|v\right) \sim N\left(\mu_{BL}, \Sigma_{BL}\right) \tag{20}$$

$$\mu_{BL} = \left[\left(\tau\Sigma\right)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[\left(\tau\Sigma\right)^{-1}\mu_{eq} + P'\Omega^{-1}v\right] \tag{21}$$

$$\Sigma_{BL} = \left[\left(\tau\Sigma\right)^{-1} + P'\Omega^{-1}P\right]^{-1} \tag{22}$$

After computing the information above is possible to compute the predictive density

$$f\left(R_{t+1}|\mu, \Sigma\right) \sim N\left(\mu_T, \left(\Sigma + \Sigma_T\right)\right) \tag{23}$$

Looking at Figures 18 and 19 we can observe that the distribution of the posterior (green) can differ from that of the prior (blue). This happens due to the effects of the views and the assumption that we made about the future trend on the expectations of the returns.

Figure 18: BL posterior estimates for mu (annualized) - Daily frequency

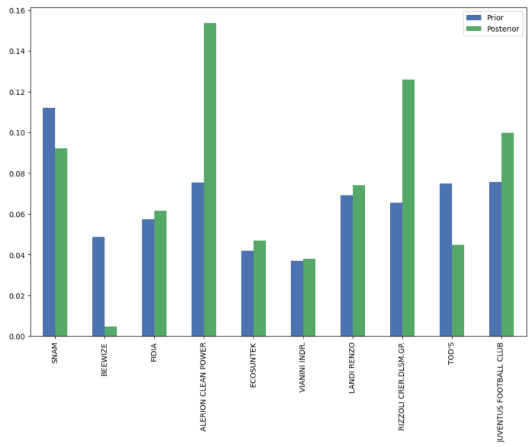
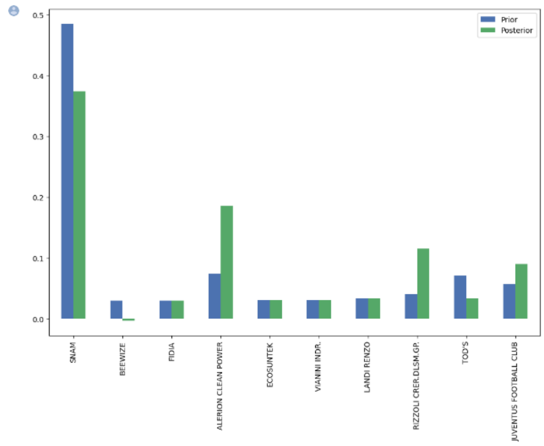


Figure 19: BL posterior estimates for mu (annualized) - Monthly frequency



12.3 Weights and Efficient Frontier of our Portfolio with Black-Litterman Approach

After setting up the preliminary results we can finally find the optimal portfolio. The optimal weights for the portfolio allocation given by the Black-Litterman model can be computed according to (24)

$$w_{BL} = \frac{1}{\gamma}\Sigma_{BL}^{-1}\mu_{BL}. \tag{24}$$

In order to provide consistent and easily comparable results, we use the maximization of the Sharpe ratio to find our optimal portfolio allocation as we do for the Mean Variance model in section 5. The model is run without including the non-negativity constraint, so the weights can assume values (-1,1).

The daily and monthly weights of the assets in our portfolio are represented respectively by Figures 17 and 18. Weights for different frequencies can be slightly different, this occurs because there is more volatility in higher frequency price movements.

Figure 20: Daily allowing short selling - Max Sharpe ratio - BL

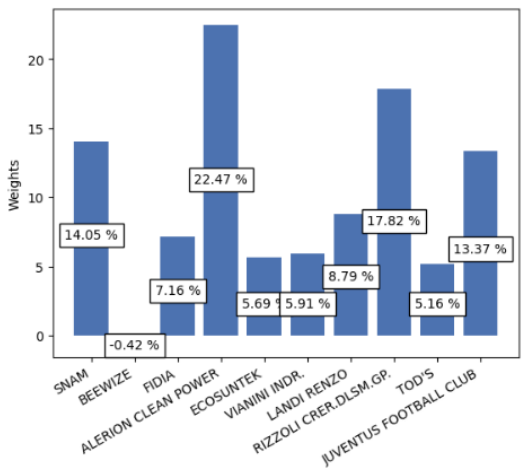


Figure 21: Monthly allowing short selling - Max Sharpe ratio - BL

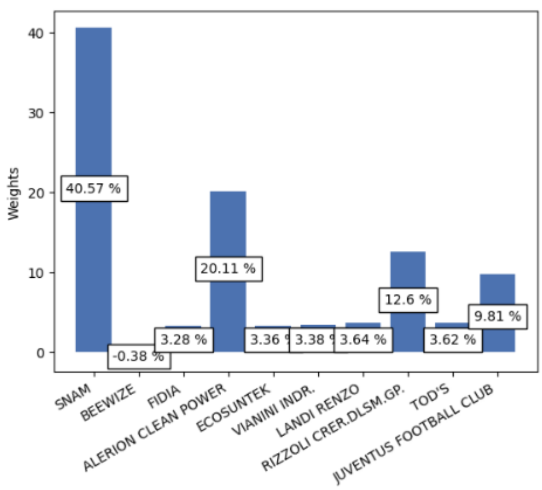


Table 10: Summary statistics for the portfolio (annualized) - BL

	Daily	Monthly
Mean	0,1120	0,2179
Standard Deviation	0,2018	0,2795
Variance	0,0421	0,0736
Skewness	0,3667	2,0149
Kurtosis	3,2475	4,2327

As in previous sections, of all the possible combinations of assets our maximization function identifies the portfolio with the maximum Sharpe ratio.

Figure 22: Efficient frontier with selected stocks - daily -BL

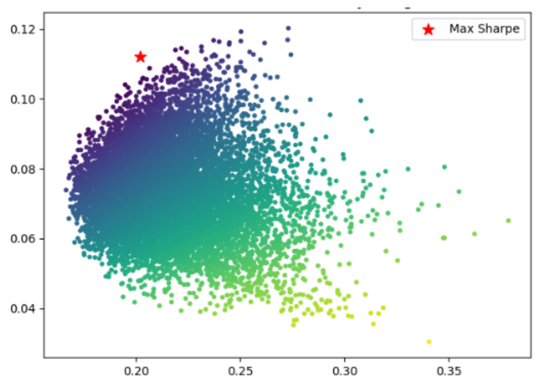
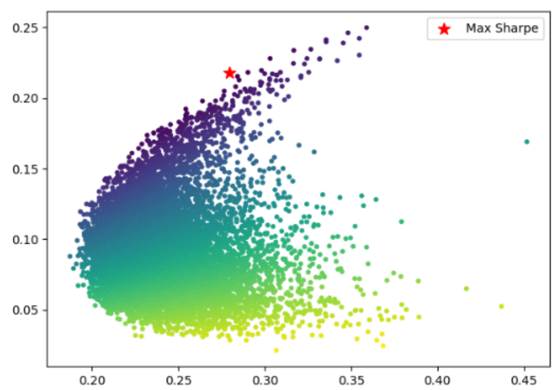


Figure 23: Efficient frontier with selected stocks - monthly -BL



The portfolio's moments (annualized) are summarized in Table 10. The mean of the returns is equal to 11,20% and 21,79% respectively for the daily and the monthly observation and the standard deviation is equal to 20,18% and 27,95% relatively.

The positive Skewness for both data samples indicates a positive tail of the returns distribution which is slightly longer (or "heavier") than the negative tail. This suggests that there may be a slightly greater probability of seeing higher returns than would be expected in a symmetric distribution.

A kurtosis of 3 for the daily statistics indicates a distribution of returns similar to a standard normal distribution. This means that portfolio returns tend to be balanced and are not particularly extreme. Concerning the monthly estimation of the kurtosis, a value of 4 indicates a distribution of yields that is more pointed and has heavier tails than a normal distribution.

The Sharpe ratio of our optimal portfolio in figure 21 and 22 is respectively 0,555 and 0,779 which means that the portfolio is suboptimal since it belongs to the interval (0,1).

These results will be compared to those of alternative portfolio allocations in section 15.

13 Standard Bayesian Asset Allocation

As previously mentioned, the Bayesian approach is a necessary model to impose a sensible structure to the empirical data on which portfolio optimizations are grounded. The Standard Bayesian model allows us to make previsions exploiting not only data and observation given by statistic distributions but also the so called prior information. This instrument includes extra-sample informations, refining the results of the portfolio allocation. Since we assume a conjugate prior distribution the prior and the posterior share the same distribution, namely the standard normal. The mean and the variance of the prior are represented by (26) and (27).

$$f_{pr}(\mu) \sim N(\mu_0, \Upsilon_0) \tag{25}$$

$$\mu_0 = \mu_r + 1 * \sigma_r \tag{26}$$

$$\Upsilon_0 = 2 * \Sigma_r \tag{27}$$

The posterior density is instead

$$f_{po}(\mu, \Sigma|Y) = f(Y|\mu\Sigma) f_{pr}(\mu|\Sigma) f_{pr}(\Sigma) \tag{28}$$

The posterior parameters can be defined as in (29) and (30)

$$\mu_1 = \left[\Upsilon_0^{-1} + T\Sigma^{-1}\right] \left[T\Sigma^{-1}\mu_r + \Upsilon_0^{-1}\mu_r\right] \tag{29}$$

$$\Sigma_1 = \left[T\Sigma^{-1} + \Upsilon_0^{-1}\right]^{-1} \tag{30}$$

After imposing this setting to our inputs we find the weights choosing the portfolio with the maximum value of the Sharpe ratio among all the other portfolio that we can construct. This gives us the results in Figure 23 and 24.

Figure 24: Daily allowing short selling - Max Sharpe ratio - Bayes

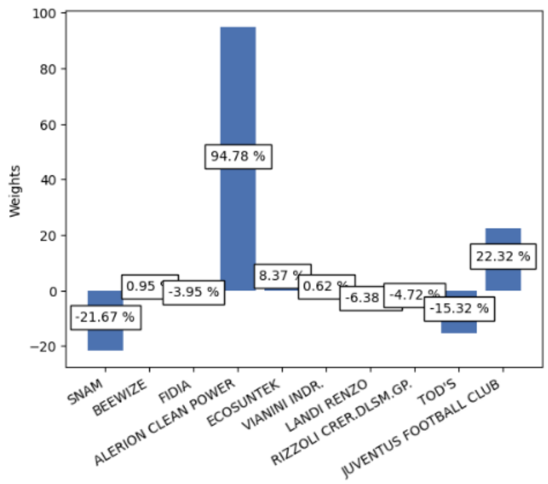


Figure 25: Monthly allowing short selling - Max Sharpe ratio - Bayes

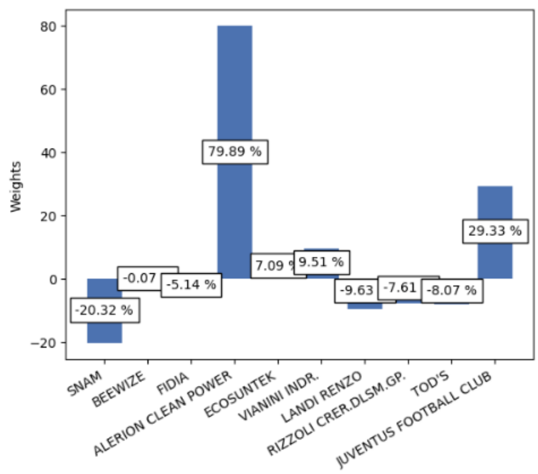


Table 11: Summary statistics for the portfolio (annualized) - Bayes

	Daily	Monthly
Mean	0,279	0,249
Standard Deviation	0,071	0,134
Variance	0,042	0,074
Skewness	1,465	0,988
Kurtosis	1,144	-0,610
Sharpe Ratio	1,127	1,863

This asset allocation seems to perform very well both at daily and monthly frequencies. Annual standard deviations are high but their values are justified by mean returns that more than double the market the assets belong to (see statistics of the FTSE Index above). This results in quite satisfactory Sharpe ratios. For a more insightful and comprehensive interpretation of these results we refer to point 15 of this work.

14 Global Minimum Portfolio Variance

Many asset allocation theoretical studies and manuals state that models based on the maximization of a specific utility function or target value are more efficient. Nevertheless, some empirical express strong support in favor of the global minimum variance portfolio, also called GMVP (Kempf & Memmel [14]).

The GMVP is the combination of stocks that guarantees the lowest return variance for a given covariance matrix Σ.

Considering a portfolio composed of N risky asset (Guillem Bornion Colom & Albert Sala Juvanteny [15]) , with

- Possibility of short selling
- distribution equal to

$$X_t = (X_{1,t}, X_{2,t}, .., X_{N,t})$$

- a vector of N x 1 returns defined as

$$r_t = (r_{1,t}, r_{2,t}, ..., r_{N,t})$$

- a N x 1 vector which represents the weights

$$w = (w_{1,t}, w_{2,t}, ..., w_{N,t})$$

The problem can be generalized as follows

$$\min_{w} w^T \Sigma w$$

$$w^T \mathbf{1} = 1 \text{ and } w^T \mu = u* \tag{31}$$

After solving the minimization problem (25) we find the weights of our stocks which are represented in Figure 23 for daily frequency and in figure 24 for the monthly one.

Figure 26: Daily allowing short selling - GVMP

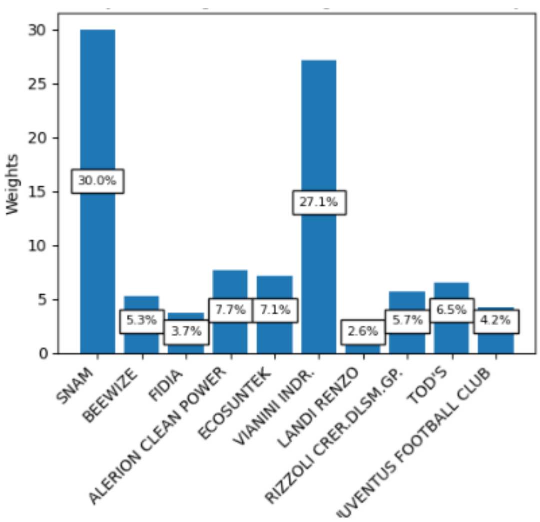


Figure 27: GVMP allowing short selling - GVMP

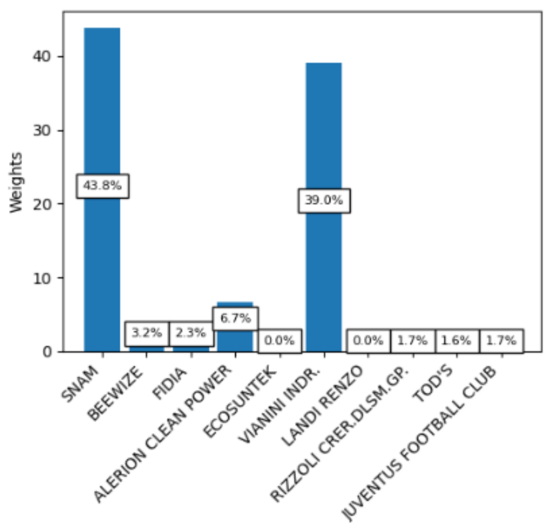


Table 12: Summary annualized statistics for the portfolio on monthly and daily frequencies

	Daily	Monthly
Mean	0,028	0,025
Standard Deviation	0,058	0,057
Variance	0,003	0,0032
Skewness	1,155	0,463
Kurtosis	13,751	17,579
Sharpe ratio	0,478	0,439

Figure 28: Efficient Frontier GMVP - Daily

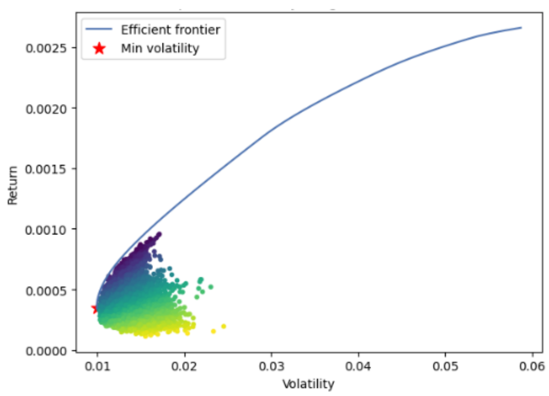
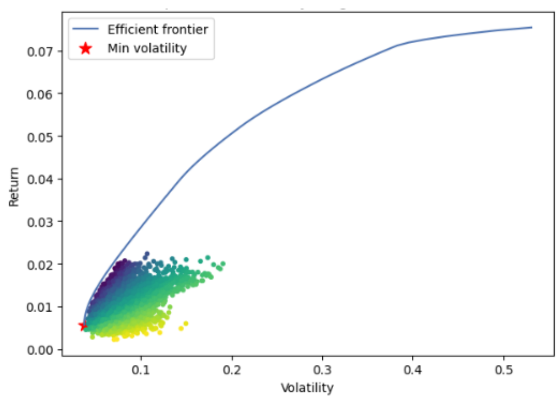


Figure 29: Efficient frontier GMVP - Monthly



As expected, the graphical representation of the Minimum Global Variance Portfolio is the furthest left point of the Efficient Frontier.

15 Linear combination of the four portfolios and final remarks

This section is devoted to the display of a joint view of the four asset allocations that we described above. To do so, we show the annualized mean, standard deviation and sharpe ratio of each of the four portfolios in Table 14. For the sake of clarity we do not present skewness and kurtosis values, as they can be easily retrieved from the above dissertation.

Table 13: Annualized summary statistics for daily and monthly frequency

	Daily			Monthly		
	Mean	SD	Sharpe ratio	Mean	SD	Sharpe ratio
Mean Variance Max Sharpe	0,280	0,282	0,992	0,250	0,249	1,001
Black Litterman	0,112	0,202	0,555	0,218	0,280	0,780
Standard Bayesian	0,080	0,071	1,127	0,250	0,134	1,863
Minimum Global Variance	0,028	0,058	0,478	0,025	0,057	0,439

The evidence that emerges is only partially explained by the general views that the literature expresses about the efficiency of portfolios. We use the Sharpe ratio as an effective proxy for portfolio performance since, as stated above, it can promptly convey the trade-off between risk and return. What appears to be odd with previous authors is that the Black-Litterman model appears to perform poorly when compared to a far less sophisticated model as the Mean-Variance with the maximization of the Sharpe ratio. This could be improved by increasing the number of views and refining the estimation of growth and the confidence levels of such forecasts. Nevertheless, the Black-Litterman allocation still present better results than the Global Minimum Variance Portfolio. The Bayesian approach exhibits outstanding Sharpe ratios, but very low annualized returns. It could benefit from tailor-made specifications of the prior and of the posterior, rather than a simplistic conjugate that approximates with a normal distribution. Probably choosing for a distribution that accounts for heavy tails (such as the Pareto) could improve the efficiency of the model.

We then moved on building a linear combination of the four portfolios, seeking to compound the diversification benefits carried in by each one of the asset allocation sets. We decided to provide four allocations:

- Equal weights
- Maximization of returns: attributing weights equal to 0.5, 0.25, 0.15 and 0.10 to the portfolio with the highest to the lowest return respectively
- Minimization of standard deviation: attributing weights equal to 0.5, 0.25, 0.15 and 0.10 to the portfolio with the lowest to the highest standard deviation respectively
- Maximization of Sharpe ratio: attributing weights equal to 0.5, 0.25, 0.15 and 0.10 to the portfolio with the highest to the lowest Sharpe ratio respectively.

The results are shown in Table 14.

Table 14: Alternative allocations of the four portfolios

	Daily			Monthly		
	Mean	SD	Sharpe	Mean	SD	Sharpe
Equally allocated	0,125	0,153	0,815	0,186	0,180	1,031
Max retrutn	0,183	0,208	0,879	0,219	0,220	0,995
Min SD	0,151	0,177	0,851	0,208	0,204	1,018
Max Sharpe	0,129	0,142	0,912	0,222	0,177	1,257

Each one of these allocations can satisfy the needs of investors pursuing different objectives. In order to be consistent with the choice of the Sharpe ratio as a proxy for portfolio performances we display the values of the latter portfolio in Table 15.

Table 15: Annualized statistic values for the Max Sharpe linear combination

	Daily	Monthly
Mean	0,129	0,222
Variance	0,020	0,031
SD	0,142	0,177
Skewness	1,802	0,592
Kurtosis	5,648	2,901
Sharpe ratio	0,912	1,257

The annualized mean returns, for both daily and monthly frequencies, exhibit satisfactory performance. The observed values of skewness and kurtosis indicate that the distribution deviates from normality. However, these deviations are relatively moderate, implying a scarcity of extreme returns. As anticipated, the distribution of daily returns deflects further from normality compared to the distribution of monthly returns. Lastly, the Sharpe ratio allows us to assert that investing in this combination is worthwhile and merits further pursuit.

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A Appendix A

Table 16: Summary statistics for stocks' daily returns and p-value

	Code	Mean	Variance	Skewness	Kurtosis	P-Value
LEONARDO	I:LDO	0,000	0,001	-0,334	11,760	0,000
ECOSUNTEK	I:ECK	0,001	0,001	2,543	18,735	0,000
LANDI RENZO	I:LRZ	0,000	0,001	1,037	11,645	0,000
STELLANTIS	I:STL	0,001	0,001	-0,451	5,447	0,000
PININFARINA	I:PINF	0,000	0,001	-0,550	95,227	0,000
FRENI BREMBO	I:BRE	0,001	0,000	0,129	3,770	0,000
INTESA SANPAOLO	I:ISP	0,000	0,000	-0,763	12,121	0,000
UNICREDIT	I:UCG	0,000	0,001	-0,034	7,081	0,000
BANCA GENERALI	I:BANC	0,000	0,000	0,024	8,984	0,000
BPER BANCA	I:BPE	0,000	0,001	0,167	7,528	0,000
FINECOBANK SPA	I:FCBK	0,001	0,000	-0,123	3,542	0,000
DAVIDE CAMPARI MILANO	I:CPR	0,001	0,000	-0,251	8,498	0,000
CALTAGIRONE	I:CALT	0,000	0,000	0,129	4,642	0,000
ENEL	I:ENEL	0,000	0,000	-1,123	14,045	0,000
ALERION CLEAN POWER	I:ARN	0,001	0,001	1,668	12,308	0,000
A2A	I:A2A	0,000	0,000	-0,878	10,620	0,000
TERNA RETE ELETTRICA NAZ	I:TRN	0,000	0,000	-0,651	8,273	0,000
ACEA	I:ACE	0,000	0,000	-0,257	6,838	0,000
BANCA MEDIOLANUM	I:BMED	0,000	0,000	-0,436	5,724	0,000
TAMBURI INV.PARTNERS	I:TIPS	0,001	0,000	0,230	10,742	0,000
MEDIOBANCA BC.FIN	I:MB	0,000	0,000	-0,860	12,014	0,000
ANIMA HOLDING	I:ANI	0,000	0,001	0,116	5,091	0,000
TELECOM ITALIA RSP	I:TITR	0,000	0,001	0,137	16,892	0,000
TELECOM ITALIA	I:TIT	0,000	0,001	0,549	16,364	0,000
ENERVIT	I:ENV	0,000	0,000	0,829	15,315	0,000
VALSOIA	I:VAL	0,000	0,000	0,692	6,944	0,000
CENTRALE DEL LATTE D'ITALIA	I:CLT	0,000	0,000	2,786	27,309	0,000
HERA	I:HER	0,000	0,000	-0,514	12,474	0,000
IREN	I:IRE	0,000	0,000	-0,513	6,121	0,000
EL EN	I:ELN	0,001	0,001	0,266	3,644	0,000
AMPLIFON	I:AMP	0,001	0,000	-0,393	6,042	0,000
DE LONGHI	I:DLG	0,000	0,000	0,206	3,936	0,000
CNH INDUSTRIAL	I:CNHI	0,001	0,001	-0,388	4,646	0,000
FIDIA	I:FD	0,000	0,001	2,660	22,389	0,000
INTERPUMP GROUP	I:IP	0,001	0,000	-0,123	2,630	0,000
INTEK GROUP	I:IKG	0,001	0,000	1,073	10,510	0,000
RIZZOLI CRER.DLSM.GP.	I:RCS	0,000	0,001	1,059	11,711	0,000
CAIRO COMMUNICATION	I:CAI	0,000	0,000	0,460	5,103	0,000

	Code	Mean	Variance	Skewness	Kurtosis	P-Value
MONRIF	I:MON	0,000	0,001	2,347	22,730	0,000
UNIPOL GRUPPO FINANZIARIO	I:UNI	0,000	0,000	-0,280	8,992	0,000
ASSICURAZIONI GENERALI	I:G	0,000	0,000	-0,737	12,080	0,000
SNAM	I:SRG	0,000	0,000	-1,190	16,776	0,000
ENI	I:ENI	0,000	0,000	-0,969	17,033	0,000
TOD'S	I:TOD	0,000	0,001	0,825	13,369	0,000
RECORDATI INDUA.CHIMICA	I:REC	0,001	0,000	0,045	12,556	0,000
RISANAMENTO	I:RN	0,001	0,001	1,426	9,653	0,000
BRIOSCHI SVILUPPO IMMBL	I:BRI	0,000	0,001	0,560	7,205	0,000
BEEWIZE	I:FUL	0,000	0,001	2,306	18,726	0,000
EXPRIVIA	I:AISW	0,001	0,001	1,859	14,581	0,000
AUTOGRILL	I:AGL	0,000	0,001	1,328	32,154	0,000
JUVENTUS FOOTBALL CLUB	I:JUVE	0,001	0,001	0,257	8,057	0,000
SS LAZIO	I:SSL	0,001	0,001	0,277	9,665	0,000
CLASS EDITORI	I:CLE	-0,001	0,001	1,529	14,394	0,000
BASTOGI	I:B	0,000	0,001	1,139	10,682	0,000
CEMENTIR HOLDING	I:CEM	0,000	0,000	0,178	2,661	0,000
UNIPOLSAI	I:US	0,000	0,000	-0,172	5,006	0,000
BUZZI	I:BZU	0,001	0,000	-0,035	4,400	0,000
CREDITO EMILIANO	I:CE	0,000	0,000	-0,036	2,903	0,000
DANIELI	I:DAN	0,000	0,000	0,588	9,153	0,000
ITALMOBILIARE	I:ITM	0,001	0,000	2,214	33,356	0,000
VINCENZO ZUCCHI	I:ZUC	0,000	0,001	2,938	48,012	0,000
WEBUILD	I:IPG	0,000	0,001	-0,192	11,050	0,000
VIANINI INDR.	I:VIN	0,000	0,000	0,383	3,796	0,000
EDISON RSP	I:EDNR	0,000	0,000	-0,354	13,489	0,000
RATTI	RATTI	0,000	0,000	0,944	12,832	0,000
GABETTI PROPERTY SLTN.	I:GAB	0,000	0,001	1,229	7,868	0,000
MFE B	I:MS	0,000	0,001	1,165	21,903	0,000
ERG	I:ERG	0,001	0,000	-0,150	12,546	0,000
CEMBRE	I:CMB	0,001	0,000	0,149	3,630	0,000
SABAF	I:SAB	0,000	0,000	0,442	5,075	0,000
BEGHELLI	I:BE	0,000	0,001	1,975	14,328	0,000
SOL	I:SOL	0,001	0,000	0,381	1,790	0,000
DATALOGIC	I:DAL	0,000	0,001	0,424	7,429	0,000
BIESSE	I:BSS	0,000	0,001	-0,151	5,811	0,000
SAFILO GROUP	I:SAFI	0,000	0,001	0,432	10,208	0,000

Table 17: Summary statistics for stocks' monthly returns and p-value

	Code	Mean	Variance	Skewness	Kurtosis	P-Value
LEONARDO	I:LDO	0,009	0,012	0,199	2,784	0,000
ECOSUNTEK	I:ECK	0,025	0,089	7,564	68,295	0,000
LANDI RENZO	I:LRZ	0,006	0,029	3,573	22,042	0,000
STELLANTIS	I:STL	0,019	0,014	-0,209	0,918	0,170
PININFARINA	I:PINF	-0,002	0,014	1,101	4,288	0,000
FRENI BREMBO	I:BRE	0,013	0,009	-0,013	-0,243	0,836
INTESA SANPAOLO	I:ISP	0,005	0,009	-0,398	2,061	0,000
UNICREDIT	I:UCG	0,006	0,015	-0,204	1,736	0,003
BANCA GENERALI	I:BANC	0,008	0,008	-0,650	0,929	0,007
BPER BANCA	I:BPE	0,005	0,017	0,400	0,477	0,191
FINECOBANK SPA	I:FCBK	0,013	0,008	-0,081	-0,021	0,935
DAVIDE CAMPARI MILANO	I:CPR	0,018	0,005	-0,373	0,535	0,206
CALTAGIRONE	I:CALT	0,010	0,006	-0,153	0,850	0,249
ENEL	I:ENEL	0,007	0,004	-0,059	0,813	0,328
ALERION CLEAN POWER	I:ARN	0,030	0,016	1,866	4,569	0,000
A2A	I:A2A	0,010	0,005	-0,950	2,020	0,000
TERNA RETE ELETTRICA NAZ	I:TRN	0,008	0,002	-0,092	-0,480	0,531
ACEA	I:ACE	0,006	0,006	-0,286	0,124	0,505
BANCA MEDIOLANUM	I:BMED	0,009	0,008	-0,456	2,371	0,000
TAMBURI INV.PARTNERS	I:TIPS	0,014	0,004	-0,104	-0,063	0,888
MEDIOBANCA BC.FIN	I:MB	0,009	0,009	-0,710	2,246	0,000
ANIMA HOLDING	I:ANI	0,005	0,012	0,069	0,905	0,244
TELECOM ITALIA RSP	I:TITR	-0,005	0,011	0,700	1,493	0,000
TELECOM ITALIA	I:TIT	-0,006	0,012	1,228	4,179	0,000
ENERVIT	I:ENV	0,001	0,004	0,039	0,942	0,222
VALSOIA	I:VAL	-0,002	0,006	0,984	2,923	0,000
CENTRALE DEL LATTE D'ITALIA	I:CLT	0,002	0,005	1,928	6,266	0,000
HERA	I:HER	0,005	0,004	-0,575	0,943	0,015
IREN	I:IRE	0,009	0,006	-0,509	0,611	0,066
EL EN	I:ELN	0,027	0,017	0,173	0,210	0,748
AMPLIFON	I:AMP	0,022	0,007	-0,694	1,418	0,001
DE LONGHI	I:DLG	0,007	0,009	-0,069	0,101	0,958
CNH INDUSTRIAL	I:CNHI	0,013	0,010	-0,205	2,126	0,000
FIDIA	I:FD	0,006	0,041	5,033	35,719	0,000
INTERPUMP GROUP	I:IP	0,019	0,009	-0,548	0,111	0,084
INTEK GROUP	I:IKG	0,016	0,010	0,940	2,116	0,000
RIZZOLI CRER.DLSM.GP.	I:RCS	0,004	0,015	1,052	2,638	0,000
CAIRO COMMUNICATION	I:CAI	-0,006	0,010	0,347	0,538	0,239

	Code	Mean	Variance	Skewness	Kurtosis	P-Value
MONRIF	I:MON	-0,011	0,009	1,838	9,005	0,000
UNIPOL GRUPPO FINANZIARIO	I:UNI	0,007	0,009	-0,524	1,837	0,000
ASSICURAZIONI GENERALI	I:G	0,004	0,005	-0,199	2,008	0,000
SNAM	I:SRG	0,005	0,003	-0,216	-0,439	0,421
ENI	I:ENI	0,002	0,006	0,638	2,883	0,000
TOD'S	I:TOD	0,000	0,012	1,086	3,286	0,000
RECORDATI INDUA.CHIMICA	I:REC	0,014	0,005	0,049	0,866	0,282
RISANAMENTO	I:RN	0,012	0,026	1,436	3,986	0,000
BRIOSCHI SVILUPPO IMMBL	I:BRI	0,002	0,008	-0,093	1,226	0,069
BEEWIZE	I:FUL	0,003	0,036	5,092	32,397	0,000
EXPRIVIA	I:AISW	0,019	0,026	2,376	13,788	0,000
AUTOGRILL	I:AGL	0,007	0,012	1,477	10,635	0,000
JUVENTUS FOOTBALL CLUB	I:JUVE	0,018	0,022	1,329	5,840	0,000
SS LAZIO	I:SSL	0,016	0,017	0,943	4,600	0,000
CLASS EDITORI	I:CLE	-0,017	0,013	0,750	2,392	0,000
BASTOGI	I:B	-0,011	0,003	1,179	4,901	0,000
CEMENTIR HOLDING	I:CEM	0,009	0,010	0,677	0,837	0,007
UNIPOLSAI	I:US	0,003	0,005	-0,103	0,811	0,311
BUZZI	I:BZU	0,011	0,006	-0,121	-0,303	0,686
CREDITO EMILIANO	I:CE	0,005	0,006	0,505	3,319	0,000
DANIELI	I:DAN	0,004	0,007	-0,117	0,733	0,375
ITALMOBILIARE	I:ITM	0,012	0,005	2,054	9,318	0,000
VINCENZO ZUCCHI	I:ZUC	-0,003	0,010	1,632	5,719	0,000
WEBUILD	I:IPG	0,000	0,012	0,763	1,423	0,000
VIANINI INDR.	I:VIN	0,003	0,003	0,530	1,444	0,002
EDISON RSP	I:EDNR	0,006	0,003	0,229	2,596	0,000
RATTI	RATTI	0,004	0,004	0,218	1,623	0,006
GABETTI PROPERTY SLTN.	I:GAB	0,014	0,032	2,147	7,152	0,000
MFE B	I:MS	-0,004	0,014	3,240	20,451	0,000
ERG	I:ERG	0,013	0,005	0,144	1,594	0,009
CEMBRE	I:CMB	0,014	0,007	-0,084	3,620	0,000
SABAF	I:SAB	0,007	0,009	0,200	0,246	0,673
BEGHELLI	I:BE	0,000	0,009	1,415	4,330	0,000
SOL	I:SOL	0,016	0,004	0,299	-0,461	0,283
DATALOGIC	I:DAL	0,004	0,012	0,273	-0,188	0,480
BIESSE	I:BSS	0,011	0,016	-0,191	0,226	0,703
SAFILO GROUP	I:SAFI	-0,006	0,019	1,006	3,358	0,000