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RELATIVELY OPTIMAL CONTROL: A STATIC PIECEWISE-AFFINE SOLUTION*

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Abstract. A relatively optimal control is a stabilizing controller that, without initialization nor feedforwarding and tracking the optimal trajectory, produces the optimal (constrained) behavior for the nominal initial condition of the plant. In a previous work, for discrete-time linear systems, we presented a linear *dynamic* relatively optimal control. Here we provide a *static* solution, namely a deadbeat piecewise-affine state-feedback controller based on a suitable partition of the state space into polyhedral sets. The vertices of the polyhedra are the states of the optimal trajectory; hence a bound for the complexity of the controller is known in advance. We also show how to obtain a controller that is not deadbeat by removing the zero terminal constraint while guaranteeing stability. Finally, we compare the proposed static compensator with the dynamic one.

Key words. optimal control, linear systems, discrete-time systems, invariance

AMS subject classifications. 93B52, 93B51, 93C05, 49N35

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1. Introduction.

It is known that, unless for very special cases, determining an optimal control in a feedback form under output or input constraints is a computationally hard task. The problem can be addressed in a receding horizon fashion, but in this case an optimization problem must be solved online at each time interval. Explicit (piecewise-affine) solutions exist [1, 2] but are limited to quadratic or 1-norm cost and linear constraints. However, for those systems that are explicitly built to perform specific operation through a specific trajectory with known initial and final states, the request for optimality from *any* initial state can be relaxed, requiring optimality only from a *specific* initial condition. The relatively optimal control (ROC) [5] is defined as *a stabilizing controller that guarantees optimality of the trajectory and constraint satisfaction from a given (or a set of given) initial condition(s) without the involvement of any feedforward action.* In [5] it has been proved that a controller enjoying these properties is linear dynamic and its order is equal to the length of the optimal trajectory minus the order of the plant. In [6] the zero terminal constraint was removed in order to assign a characteristic polynomial to the closed-loop system, and the problem of output feedback was addressed. Here, a *static* ROC is constructed by partitioning the state space into polyhedral sets whose vertices are the states of the optimal trajectory and their opposite.

The main contribution of the paper can be summarized in the following points.

- It is shown that for discrete-time linear systems with convex constraints and cost, it is always possible to construct a static ROC by means of a proper partition of the state space into polyhedral sets (a procedure to construct it is provided).

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- If the constraints and/or the cost are not convex, a sufficient condition on the optimal trajectory that guarantees that the static ROC can be constructed is provided.
- The proposed controller is a deadbeat piecewise-affine state-feedback controller. The vertices of each of the polyhedral sets are the states of the optimal trajectory and their opposite. The control at each vertex is the corresponding control of the optimal sequence, while the control at a generic state is given by a convex combination of the controls corresponding to the vertices of the polyhedron to which the state belongs.
- An upper bound on the number of polyhedral sets as a function of the order of the system and the length of the optimal trajectory is provided.
- By removing the zero state terminal constraint and requiring the final state of the optimal trajectory to belong to a controlled invariant set, it is possible to obtain a nondeadbeat controller.
- The proposed static controller is compared with the dynamic one previously introduced.

2. Problem statement. We give the discrete-time reachable system

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k), \\ y(k) &= Cx(k)+Du(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^q$ and A, B, C, D are matrices of appropriate dimensions. For this system we consider the locally bounded convex cost functions of the output

$$g(y), \quad l_i(y), \quad i = 1, 2, \dots, s \tag{2}$$

(we assume they are 0-symmetric, i.e., $g(y) = g(-y)$ and $l_i(y) = l_i(-y)$) with assigned initial condition

$$\bar{x} \neq 0 \tag{3}$$

and the constraint

$$y(k) \in \mathcal{Y}, \tag{4}$$

where $\mathcal{Y}$ is a convex, closed, and 0-symmetric set. Then we consider the following problem (consistently with [5] and with no loss of generality, we assume $k = 1$ is initial time):

$$J_{opt}(\bar{x}) = \min \sum_{k=1}^{N} g(y(k)) \tag{5}$$

subject to

$$x(k+1) = Ax(k) + Bu(k), \ k = 1, \dots, N, \tag{6}$$

$$y(k) = Cx(k) + Du(k), \ k = 1, \dots, N, \tag{7}$$

$$\sum_{k=1}^{N} l_i(y(k)) \leq \mu_i, \quad i = 1, 2, \dots, s, \tag{8}$$

$$y(k) \in \mathcal{Y}, \quad k = 1, \dots, N, \tag{9}$$

$$x(1) = \bar{x}, \tag{10}$$

$$x(N+1) = 0, \tag{11}$$

$$N \geq 1 \quad \text{assigned (or free).} \tag{12}$$

In the extremely general formulation of the problem we have considered the option of N free, in order to also consider the special case of minimum-time control. The choice of N depends on the circumstances and has to guarantee the feasibility of the above open-loop optimal control problem. Note that the cost and the constraint achieved, assuming g and l_i only depending on y, are quite general since we can include cost and pointwise or integral constraints depending on both x and u by suitable choices of C and D. Finding an open-loop solution for the above problem is well known as a convex problem [10] which can be solved by means of standard convex programming algorithms. Here we are interested in a feedback static solution; more precisely the problem we consider is the following.

PROBLEM 1. *Find a static state-feedback compensator of the form $u = \Phi(x)$ which is stabilizing and such that for $x(1) = \bar{x}$ the control and state trajectories are the optimal ones.*

Any solution of the above problem will be referred to as a *static relatively optimal controller*. We stress that in the ROC framework, the constraints (8) and (9) represent *design specification "soft" constraints*. Hence their violation implies a performance loss only and is allowed for nonnominal initial conditions. In the following we will construct a solution to Problem 1 in two steps: First, a relatively optimal controller that is only locally stabilizing (being defined in a convex subset of the state space, containing the origin) will be constructed. We will refer to this controller as the *local relatively optimal controller*. Then the local controller will be extended to the whole state space, obtaining a *global relatively optimal controller*.

Remark 2.1. Any initial state $\bar{x}$ for which the problem is feasible (hence the constraints can be satisfied) is suitable as a nominal initial condition; there are no further restrictions.

3. Main results.

We now assume that the optimal trajectory starting from the assigned initial condition $\bar{x}$ does exist and has been computed (offline). We consider the following assumption.

ASSUMPTION 1. *The optimal trajectory is such that the residual cost is strictly decreasing, i.e.,*

$$\sum_{k=h}^{N} g(y(k)) < \sum_{k=h+1}^{N} g(y(k)) \quad \forall h = 1, \ldots, N-1.$$

Assumption 1 is absolutely reasonable and avoids trivialities (it is obviously true, for instance, if g is positive definite with respect to y). The way we solve Problem 1 can be explained as follows: Based on points of the optimal trajectory and their opposite (connected by the solid line in Figure 1), we partition the state space into disjoint regions. The convex hull of the points of the optimal trajectory and their opposite (the shaded hexagon in Figure 1) represents a region that can be divided into simplices, in each of which the control is affine. This region includes the nominal initial state $\bar{x}$ (possibly in its interior). The external part is divided into cones, centered in the origin, and "truncated to keep the outer part," in each of which the controller is linear. The control is Lipschitz continuous. To formally state the main result we need to introduce some notations. The inequality $p \leq 0$, if p is a vector, has to be interpreted componentwise. Let us denote by $\bar{1}$ the vector (the dimension depending on the context) having all components equal to 1:

$$\bar{1} = [\, 1 \ \ 1 \ \ \ldots \ \ 1 \,]^T \tag{13}$$

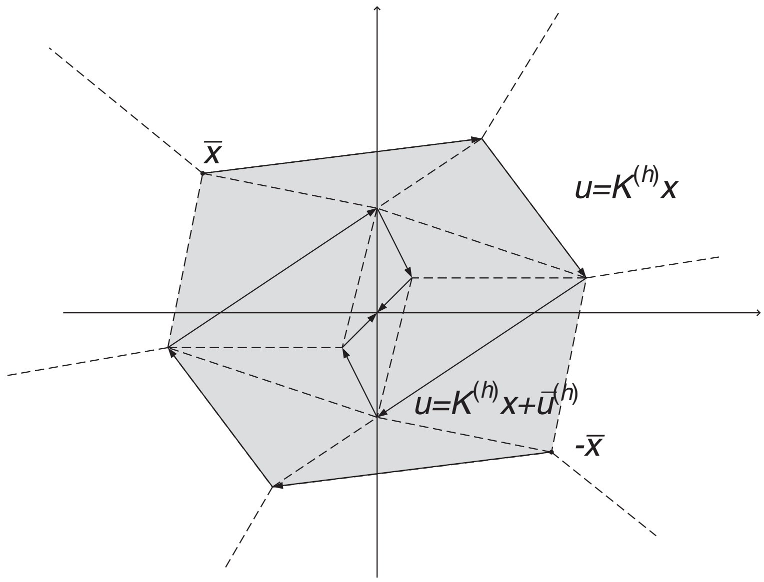

FIG. 1. *The state space partition.*

(note that the expression $\bar{1}^T p$ is the sum of the components of vector p). Given an $n \times (n+1)$ full row rank matrix X, a *simplex* in $\mathbb{R}^n$ is a set of the form

$$\mathcal{S}(X) = \{x = Xp : \quad p \geq 0, \quad \bar{1}^T p = 1\}. \tag{14}$$

Given an $n \times n$ full rank matrix X, a *simplicial cone* in $\mathbb{R}^n$ is a set of the form

$$\mathcal{C}(X) = \{x = Xp : \quad p \geq 0\}. \tag{15}$$

Note that a simplicial cone is always generated by a simplex having the origin among its corners. Together with these standard notations we need to consider the complement (the outer part) of the unit sector in a simplicial cone, which is the closure of the complement in $\mathcal{C}$,

$$\tilde{\mathcal{C}}(X) = \{x = Xp : \quad p \geq 0, \quad \bar{1}^T p \geq 1\}. \tag{16}$$

If $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a function and X is an $n \times m$ matrix, we denote by $\Phi(X)$ the following vector:

$$\Phi(X) = [\ \Phi(x_1) \ \ \Phi(x_2) \ \ \ldots \ \ \Phi(x_m)\ ]^T. \tag{17}$$

The next theorem states that there exists a control which is optimal for $x(1) = \bar{x}$ and locally stabilizing.

THEOREM 3.1. *There exists a convex and compact polyhedron $\mathcal{P}$, including the origin in its interior, which is partitioned into simplices $\mathcal{S}^{(h)}$, each generated by an $n \times (n+1)$ matrix $X^{(h)}$ whose columns are vectors properly chosen from among the states of the optimal trajectory and their opposite:*

$$\mathcal{P} = \bigcup \mathcal{S}^{(h)} = \bigcup \mathcal{S}(X^{(h)}) \tag{18}$$

such that each pair of simplices has an intersection with an empty interior,

$$\text{int}\{\mathcal{S}^{(h)} \bigcap \mathcal{S}^{(k)}\} = \emptyset, \quad h \neq k, \tag{19}$$

and such that $\bar{x} \in \mathcal{P}$. *To each simplex* $\mathcal{S}^{(h)}$ *we can associate an* $m \times (n+1)$ *matrix* $U^{(h)}$ *whose columns are vectors properly chosen from among the inputs of the optimal trajectory and their opposite. The piecewise-affine static controller*

$$u = \Phi_{\mathcal{P}}(x) = K^{(h)}x + \bar{u}^{(h)} = U^{(h)} \begin{bmatrix} X^{(h)} \\ \bar{1}^T \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} \quad \textit{for} \;\; x \in \mathcal{S}^{(h)} \tag{20}$$

is Lipschitz continuous and relatively optimal inside $\mathcal{P}$*; more precisely it is stabilizing with a domain of attraction* $\mathcal{P}$ *and for* $x(1) = \bar{x}$ *produces the optimal trajectory. Moreover, for each* $x(1) \in \mathcal{P}$*, the constraints are satisfied and the transient cost is bounded as*

$$J(x(1)) \leq \max_{i=1,\ldots,n+1} J_{opt}(x_{k_i}), \tag{21}$$

where $x_{k_1},\ x_{k_2}, \ldots, x_{k_{n+1}}$ *are the vertices of the simplex* $\mathcal{S} \ni x(1)$ *and* $J_{opt}(x_{k_i})$ *is the optimal cost associated with the initial condition* x_{k_i}.

The next theorem states that the same control can be globally extended over $\mathbb{R}^n$.

THEOREM 3.2. *The control* (20) *can be extended onto* $\mathbb{R}^n$ *as follows. The complement of the polytope* $\mathcal{P}$ *can be partitioned into complements of simplices inside a cone*

$$\tilde{\mathcal{C}}^{(h)} = \tilde{\mathcal{C}}(X^{(h)}), \tag{22}$$

each generated by a square invertible matrix $X^{(h)}$, *having intersection with empty interior*

$$\text{int}\{\tilde{\mathcal{C}}^{(h)} \bigcap \tilde{\mathcal{C}}^{(k)}\} = \emptyset, \quad h \neq k, \tag{23}$$

and intersection with empty interior with $\mathcal{P}$,

$$\text{int}\{\tilde{\mathcal{C}}^{(h)} \bigcap \mathcal{P}\} = \emptyset, \tag{24}$$

such that

$$\mathcal{P} \bigcup \left[\bigcup_h \tilde{\mathcal{C}}^{(h)} \right] = \mathbb{R}^n. \tag{25}$$

To each set $\tilde{\mathcal{C}}^{(h)}$ *can be associated an* $m \times n$ *matrix* $U^{(h)}$ *whose columns are vectors properly chosen from among the inputs of the optimal trajectory, obtaining a control*

$$u = \Phi(x) = K^{(h)}x = U^{(h)} \left[X^{(h)} \right]^{-1} x \ . \tag{26}$$

The extended control obtained in this way is globally Lipschitz continuous and relatively optimal.

Theorems 3.1 and 3.2 will be proved constructively in sections 4 and 5, respectively.

4. Construction of a local relatively optimal control. Denote by

$$\bar{x}(1), \dots, \bar{x}(N)$$

the optimal state trajectory from the initial condition $\bar{x} = \bar{x}(1)$, obtained by solving (5)–(12). We introduce the notation (basically inverting time)

$$x_1 = \bar{x}(N), \quad x_2 = \bar{x}(N-1), \dots, \quad x_N = \bar{x}(1), \tag{27}$$

and

$$u_1 = \bar{u}(N), \quad u_2 = \bar{u}(N-1), \dots, \quad u_N = \bar{u}(1), \tag{28}$$

and we coherently assume $x_0 = 0$; hence we have that $x_{i-1} = Ax_i + Bu_i$, $i = 1, \dots, N$. We also denote by x_{-i}, $i = 1, \dots, N$, the opposite of x_i. Then we introduce the following assumption, which simplifies considerably the proof of Theorem 3.1 but is not essential (in fact it can be easily removed as we will show later on).

ASSUMPTION 2. *The matrix $X_n = [x_1\ x_2\ \dots\ x_n]$, formed by the last n states of the optimal trajectory, is invertible.*

Let us consider the polyhedral set

$$\mathcal{P}_n = \{x = X_n p : \|p\|_1 \le 1\}. \tag{29}$$

Such a set is the convex hull of the last n states of the optimal trajectory and their opposite. It contains the origin in its interior and is 0-symmetric. An example for $n = 2$ is shown in Figure 2: $\mathcal{P}_n$ (the darkest area) is the convex hull of the last two states of the optimal trajectory (connected by the solid line) and their opposite (connected by the dashed line). Thanks to Assumption 2 the following lemma holds.

LEMMA 4.1. *The linear control*

$$u(x) = U_n X_n^{-1} x, \tag{30}$$

where $U_n = [u_1\ u_2\ \dots\ u_n]$, renders positively invariant the set $\mathcal{P}_n$ satisfying the constraints for all initial conditions inside the set. In particular, it is deadbeat and steers the state to zero in at most n steps.

Proof. The control law $u(x) = U_n X_n^{-1} x$ is a control-at-the-vertices strategy. All $x \in \mathcal{P}_n$ can be written in a unique way as a linear combination of the columns of X_n, namely, the last n states of the optimal trajectory:

$$x = X_n p. \tag{31}$$

Since X_n is invertible, it follows that

$$p(x) = X_n^{-1} x; \tag{32}$$

hence the control law $u(x) = U_n X_n^{-1} x$ basically computes a control which is a linear combination of the controls at the vertices of $\mathcal{P}_n$ according to the coefficients $p(x)$. Positive invariance is a consequence of the fact that, by construction, the control at each vertex keeps the state inside the set [4]. The satisfaction of the constraints is guaranteed for all initial conditions inside the set because the input and state constraints are convex and 0-symmetric. To prove that the control is deadbeat, note that if at time k we have

$$x(k) = x_n p_n + \dots + x_2 p_2 + x_1 p_1, \tag{33}$$

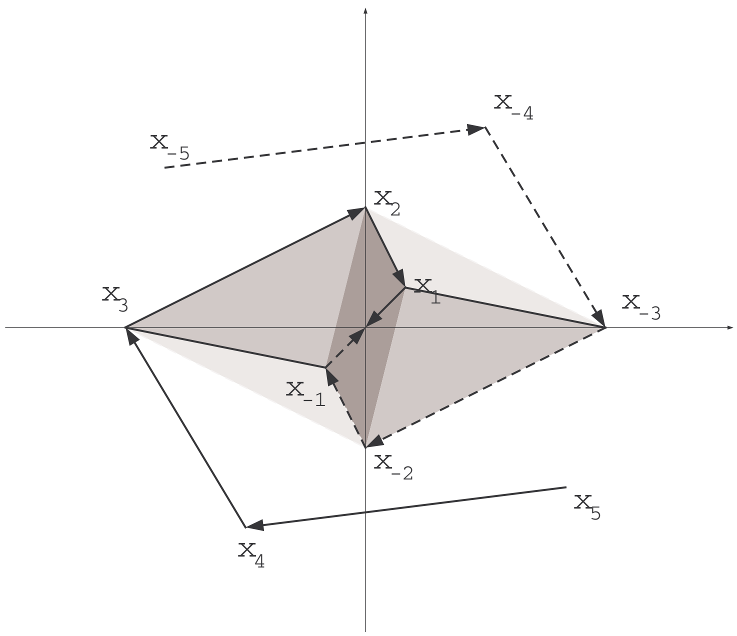

FIG. 2. *Considering* x_3 *and its opposite* x_{-3}*, we can construct four simplices starting from* $\mathcal{P}_2$ *(the darkest area).*

then the computed control will be

$$u(k) = u_n p_n + \cdots + u_2 p_2 + u_1 p_1. \tag{34}$$

Since $x_{i-1} = Ax_i + Bu_i$, we obtain, by linearity,

$$x(k+1) = x_{n-1} p_n + \cdots + x_1 p_2 + 0 p_1, \tag{35}$$

and at the next step we will have, reasoning in the same way,

$$x(k+2) = x_{n-2} p_n + \cdots + x_1 p_3 + 0 p_2 + 0 p_1, \tag{36}$$

and so on; therefore, we immediately verify that after at most n steps the system will reach the origin. ∎

Remark 4.1. The control law defined above is such that inside $\mathcal{P}_n$, at each step, the state is a convex combination of points with decreasing index and 0.

Note that if the system reaches the state $x_n = \bar{x}(N-n+1) \in \mathcal{P}_n$, it starts following the last n steps of the optimal trajectory. Note also that $\mathcal{P}_n$ (which will be the first element of a sequence of sets) is affine to a diamond and thus can be partitioned into simplices. The next sets of the sequence are computed as follows.

Consider the state x_{n+1} (corresponding to the state x_3 in the example of Figure 2). Since x_{n+1} and its opposite $x_{-(n+1)}$ are outside $\mathcal{P}_n$ (as will be shown later), they can be connected with a certain number of vertices of $\mathcal{P}_n$ without crossing such a set; thus simplices are formed by some vertices of $\mathcal{P}_n$ and the two points x_{n+1} and $x_{-(n+1)}$ (in the example of Figure 2, such simplices are the triangles (x_3, x_2, x_{-1})

and (x_3, x_{-1}, x_{-2}) and their symmetric). Denoting by $\mathcal{S}^j_{n+1}$, $j = 1, \ldots, m_{n+1}$, the simplices having x_{n+1} as a vertex and by $\mathcal{S}^j_{n+1}$, $j = -m_{n+1}, \ldots, -1$, those having $x_{-(n+1)}$ as a vertex, we can define the set $\mathcal{P}_{n+1}$ as follows:

$$\mathcal{P}_{n+1} = \bigcup_{j=\pm 1,\ldots,\pm m_{n+1}} \mathcal{S}^j_{n+1} \cup \mathcal{P}_n. \tag{37}$$

The procedure goes on exactly in the same manner to generate the sequence of sets $\mathcal{P}_k$, $k = n+1, n+2, \ldots, N$, ordered by inclusion and the corresponding simplicial partition: If we define the set

$$\mathcal{P}_k = \operatorname{conv}\{x_1, x_2, \ldots, x_k, x_{-1}, x_{-2}, \ldots, x_{-k}\}, \quad k < N, \tag{38}$$

we can consider the vector x_{k+1} and form a new simplicial partition for $\mathcal{P}_{k+1}$ by adding new simplices. It is fundamental to note that each new simplicial partition of $\mathcal{P}_{k+1}$ preserves all the simplices forming the simplicial partition for $\mathcal{P}_k$. To prove that the construction is well defined we need the following lemma.

LEMMA 4.2. *The vector x_{k+1} in the construction is outside $\mathcal{P}_k$.*

Proof. Assume by contradiction that $x_{k+1} \in \mathcal{P}_k$. Then x_{k+1} could be written as a convex combination of the vertices of $\mathcal{P}_k$. So if we take x_{k+1} as an initial state, since we considered convex constraints, then x_{k+1} could be driven to zero in a time not exceeding k at a cost not exceeding the maximum cost of all vertices of $\mathcal{P}_k$. This is in contradiction with Assumption 1. $\square$

Therefore the procedure is such that $\{\mathcal{P}_k\}$ is a strictly increasing (in the sense of inclusion) sequence of sets, each of which preserves the simplicial partition of the former. This construction terminates once $\mathcal{P} \doteq \mathcal{P}_N$ is constructed.

Note that the innermost set $\mathcal{P}_n$ can be partitioned into simplices $\mathcal{S}^j_n$, each having the origin as a vertex. The remaining n vertices are any independent subset of $x_{\pm k}$, $k = 1, \ldots, n$, namely, the last n steps of the optimal trajectory and their opposite. It is easy to recognize that in this case the control law (30) takes the form (20), i.e.,

$$u = \Phi(x) = K^j_n x = U^j_n \left[X^j_n\right]^{-1} x\ , \tag{39}$$

where j denotes the simplex, while X^j_n and U^j_n are matrices whose columns are a subset of the states of the optimal trajectory (and their opposite) and the corresponding optimal control values (and their opposite).

The next step is to show how to associate a control with each simplex. With each of the simplices $\mathcal{S}^j_k$,

1. associate a matrix X^j_k whose columns are the vertices (in arbitrary order).
2. associate a matrix U^j_k whose columns are the controls corresponding to the vertices (in the same order as they appear in X^j_k). If the vertex belongs to the optimal trajectory, take the corresponding control; if it belongs to the opposite of the optimal trajectory, take the opposite of the corresponding control.

Now, the control strategy is as follows: Given $x \in \mathcal{P}$, if $x \in \mathcal{S}^j_k$, then

$$\Phi_{\mathcal{P}}(x) = U^j_k p, \tag{40}$$

where $p \geq 0$ is the (unique) vector such that

$$x = X^j_k\ p, \quad \bar{1}^T p = 1. \tag{41}$$

Note that p is such that

$$\begin{bmatrix} X_k^j \\ \bar{1}^T \end{bmatrix} p = \begin{bmatrix} x \\ 1 \end{bmatrix}, \tag{42}$$

so that u is of the form (20).

Remark 4.2. Given a simplex $\mathcal{S}_k^j$, the vector p of (42) is the vector of the barycentric (normalized) coordinates of x with respect to the vertices of the simplex (the columns of X_k^j). Barycentric coordinates, as well as simplicial partitions, are well known in the context of finite element analysis [7].

To show the properties of the control we need to introduce the index $\mathrm{In}(\mathcal{S})$ of a sector $\mathcal{S}$ as the maximum of the absolute values of the indices of its generating vectors. Formally, if $\mathcal{S}$ is generated by corners $x_{k_1}, x_{k_2}, \ldots, x_{k_n}$, then

$$\mathrm{In}(\mathcal{S}) = \max\{|k_1|, |k_2|, \ldots, |k_n|\}. \tag{43}$$

For reasons that will be clear soon, $\mathrm{In}(\mathcal{S})$ will be referred to as the *distance* of $\mathcal{S}$ from 0.

Remark 4.3. The notion of "sector" deserves some comment. Indeed, we now consider possibly degenerate simplices that can have an empty interior formed by some points x_k and with the origin repeatedly considered. For instance, $\mathcal{S}$ could be generated by $[0\ 0\ x_1\ x_2]$, representing a two-dimensional degenerate simplex in the three-dimensional space. Note also that $\mathrm{In}(\mathcal{S}) \leq k$ for all sectors inside $\mathcal{P}_k$.

The next lemma shows that, with the proposed control, if the system state is inside a sector, then it jumps to another one closer to zero.

LEMMA 4.3. *The proposed strategy is such that if $x \in \mathcal{S}$, a sector of $\mathcal{P}_k$, then $Ax + Bu(x) \in \mathcal{S}'$ with*

$$In(\mathcal{S}') < In(\mathcal{S}), \tag{44}$$

as long as $In(\mathcal{S}) \neq 0$, and therefore if $x(1) \in \mathcal{P}_k$, the control steers the system to zero in at most k steps.

Proof. As a first step we note that, according to Lemma 4.1 and Remark 4.1, the jump to sector closer to 0 occurs for all $x \in \mathcal{P}_n$. Now we proceed by induction. Assume that $x \in \mathcal{P}_{n+1}$. If $x \in \mathcal{P}_n$, there is nothing to prove; otherwise x is necessarily in a sector $\mathcal{S}$ generated by x_{n+1} or its opposite $x_{-(n+1)}$ and other vertices of smaller indices

$$x = \sum_{i=1}^{n+1} x_{k_i} p_i, \quad \sum_{i=1}^{n+1} p_i = 1, \quad p_i \geq 0, \tag{45}$$

with $|k_i| \leq n$, $i = 1, 2, \ldots, n$, and $|k_{n+1}| = n + 1$. Then we have, by construction,

$$Ax + B\Phi_{\mathcal{P}}(x) = A\left[\sum_{i=1}^{n+1} x_{k_i} p_i\right] + B\left[\sum_{i=1}^{n+1} u_{k_i} p_i\right] = \sum_{i=1}^{n+1} p_i \underbrace{[Ax_{k_i} + Bu_{k_i}]}_{\in \mathcal{P}_n} \in \mathcal{P}_n. \tag{46}$$

Therefore, necessarily $Ax + B\Phi_{\mathcal{P}}(x)$ is in a sector with index $\mathrm{In} \leq n$. The rest of the proof proceeds in the same way. Any point x in $\mathcal{P}_{k+1}$, if not in $\mathcal{P}_k$, is included in a sector $\mathcal{S}$ with index $\mathrm{In}(\mathcal{S}) = k + 1$ and, by means of the same machinery, we can show that $Ax + B\Phi_{\mathcal{P}}(x) \in \mathcal{S}'$ with $\mathrm{In}(\mathcal{S}') \leq k$. The fact that if $x(1) \in \mathcal{P}_k$, the state converges to 0 in at most k steps is an immediate consequence. ∎

The procedure for partitioning the state space and constructing the region $\mathcal{P}_N \supset \mathcal{P}_n$ and the associated local controller can be summarized as follows.

PROCEDURE 4.1. *We give the system* (1) *and the optimal open-loop trajectory, computed by solving* (5)–(12)*, which satisfies Assumption* 2.

1. *Let the set* $\mathcal{P}_n = \{x : x = X_n p, \|p\|_1 \le 1\}$*, where* $X_n = [x_1\ x_2\ \dots\ x_n]$*, be the convex hull of the last* n *states of the optimal trajectory and their opposite.*
2. *Let* $U_n = [u_1\ u_2\ \dots\ u_n]$ *be the matrix whose columns are the control vectors corresponding to the last* n *states of the optimal trajectory.*
3. *Take* $i = n + 1$.
4. *Construct the simplices* $\mathcal{S}_i^j,\ j = \pm 1, \dots, \pm m_i$*, by connecting* x_i *and* x_{-i} *to the vertices of* $\mathcal{P}_{i-1}$ *without crossing such a set. This is always possible since* $x_i, x_{-i} \notin \mathcal{P}_{i-1}$.
5. *Let* X_i^j *be the matrix whose columns are the vertices of* $\mathcal{S}_i^j$ *in an arbitrary order and* U_i^j *be the controls corresponding to the vertices in the same order. For vertices belonging to the opposite of the optimal trajectory, take the opposite of the control.*
6. *Let* $\mathcal{P}_i = \bigcup_j \mathcal{S}_i^j \cup \mathcal{P}_{i-1}$.
7. *Increase* i.
8. *If* $i \le N$*, go back to step* 4.

Note that, by construction, the sets $\mathcal{P}_i$, $i = n, \dots, N$, are convex, 0-symmetric, and such that $\mathcal{P}_i \subset \mathcal{P}_{i+1}$. The set $\mathcal{P}_{i+1} \setminus \mathcal{P}_i$, the difference between $\mathcal{P}_{i+1}$ and $\mathcal{P}_i$, is composed of simplices S_i^j, each of which has all vertices but one (precisely x_{i+1} or $x_{-(i+1)}$) belonging to $\mathcal{P}_i$.

In order to prove Theorem 3.1 we must provide the following lemmas.

LEMMA 4.4. *The proposed control* $\Phi_{\mathcal{P}}(x)$ *is Lipschitz continuous inside* $\mathcal{P} = \mathcal{P}_N$.

Proof. Since the cardinality of the partition is finite, it is sufficient to prove continuity. Inside each of the simplices, the control action (40) is a linear combination of the control at each vertex, with the weights being the components of p. Now, the proof follows immediately from a well-known result of finite element analysis, namely, the fact that using barycentric coordinates as weights in a triangular (or, in general, simplicial) mesh guarantees interelement continuity [7]. □

LEMMA 4.5. *For any state* $x(1) = x \in \mathcal{S} \subset \mathcal{P}_N$ *the proposed control* $\Phi_{\mathcal{P}}(x)$ *satisfies the constraints and it ensures a cost* $J(x)$ *bounded as*

$$J(x) \le \max_{i=1,\dots,n+1} J_{opt}(x_{k_i}), \tag{47}$$

where $\mathcal{S}$ *is generated by the points* $x_{k_1},\ x_{k_2}, \dots, x_{k_{n+1}}$.

Proof. The constraints are convex and 0-symmetric and, by construction, they are satisfied by each of the vertices of the convex set $\mathcal{P}_N$. Hence they are satisfied by any state belonging to $\mathcal{P}_N$. Since $\mathcal{P}_N$ is positively invariant under the control law $\Phi_{\mathcal{P}}(x)$, any trajectory originating in $\mathcal{P}_N$ satisfies the constraints. It follows from Lemmas 4.2 and 4.3 that the cost achieved from a given initial condition x is bounded by the maximum cost achieved from the vertices of the sector $\mathcal{S} \ni x$. Consider the cost function $\hat{g}(x) = g(x, \Phi_{\mathcal{P}}(x))$. Since $\hat{g}(x)$ is convex and 0-symmetric, it is easy to recognize that

$$\hat{g}(x) \le \hat{g}(x_{In(\mathcal{S}(x))}), \tag{48}$$

where $\mathcal{S}(x)$ denotes the sector $\mathcal{S} \ni x$ and $x_{In(\mathcal{S}(x))}$ belongs to the optimal trajectory. From Lemma 4.2 and from the fact that $\hat{g}(x)$ is convex and 0-symmetric, it follows that

$$\hat{g}(x_i) \leq \hat{g}(x_j) \tag{49}$$

for $0 < i < j \leq N$. Therefore, the maximum in the right-hand side of (47) is obtained for $k_i = \mathrm{In}(\mathcal{S}(x))$, i.e., it is the (optimal) cost from the vertex $x_{In(\mathcal{S}(x))}$. Let us now compare the cost achieved from x to that achieved from $x_{In(\mathcal{S}(x))}$. We recall that the control law $\Phi_{\mathcal{P}}(x)$ steers the system from $x \in \mathcal{S}(x) \subset \mathcal{P}_N$ to zero in at most $In(\mathcal{S}(x))$ steps. Denoting

$$f(x) = Ax + B\Phi_{\mathcal{P}}(x) \tag{50}$$

and

$$f^i(x) = f(f(\dots f(x) \dots)), \tag{51}$$

we can rewrite (48) as

$$\hat{g}(f^i(x)) \leq \hat{g}(x_{In(\mathcal{S}(f^i(x)))}), \tag{52}$$

for all $i = 0, \dots, \mathrm{In}(\mathcal{S}(x)) - 1$. On the other hand, Lemma 4.3 states that the sequence of indices corresponding to a trajectory originating in $\mathcal{P}_N$ is strictly decreasing. It follows that

$$\mathrm{In}(\mathcal{S}(f^i(x))) \leq \mathrm{In}(\mathcal{S}(x)) - i \tag{53}$$

for all $i = 0, \dots, \mathrm{In}(\mathcal{S}(x)) - 1$. Therefore, from (49) we can write

$$\hat{g}(x_{In(\mathcal{S}(f^i(x)))}) \leq \hat{g}(x_{In(\mathcal{S}(x))-i}) \tag{54}$$

and, by (52),

$$\hat{g}(f^i(x)) \leq \hat{g}(x_{In(\mathcal{S}(x))-i}). \tag{55}$$

Finally, by summing over $i = 0, \dots, \mathrm{In}(\mathcal{S}(x)) - 1$, we obtain

$$J(x) \leq J(x_{In(\mathcal{S}(x))}). \qquad \square \tag{56}$$

Now we show how to remove Assumption 2. If Assumption 2 does not hold, the construction of the regions is basically the same. The only difference is that now we must start the construction from the beginning (i.e., $\mathcal{P}_1$, $\mathcal{P}_2, \dots$) until we construct the region $\mathcal{P}_r$, where $r > n$ is the smallest value for which $[x_1\ x_2\ \dots\ x_r]$ has full rank (and then $\mathcal{P}_r$ is a neighborhood of the origin). In forming the sets $\mathcal{P}_k$, $k < r$, we construct a partition of "degenerate polytopes" in subspaces having the same properties mentioned above. When we add the vertex x_r (and its opposite $-x_r$), we reach full dimension and can construct a (nondegenerate) simplex partition of $\mathcal{P}_r$ in which each simplex has x_r as a vertex. Then the construction proceeds as already mentioned, with the difference that the control is not ultimately linear since, in general, we cannot associate a linear control with $\mathcal{P}_r$. If such an r does not exist (i.e., the whole optimal trajectory belongs to a proper subspace of $\mathbb{R}^n$), we can extend the trajectory backward, i.e., adding points x_{N+1}, x_{N+2} to reach the full rank. Clearly, optimality is ensured only from $\bar{x} = x_N$. Using the same trick of extending the trajectory backward, we can arbitrarily enlarge the domain of attraction.

We are now in the position of proving relative optimality with local stability of the control.

Proof of Theorem 3.1. The constructed simplicial partition and the control are of the form (20), which is Lipschitz as proved in Lemma 4.4. If we assume $x(1) = \bar{x}$, then the trajectory is the optimal constrained one by construction. The fact that the control is stabilizing follows from Lemma 4.3. The satisfaction of the constraints and the cost bound (21) follow easily from Lemma 4.5. ∎

Table 1

Upper bound for the number of simplices given the number N of steps of the optimal trajectory and the order n of the system.

N, n	3	4	8	12	16
4	33	39	-	-	-
8	133	207	1425	-	-
12	297	503	11965	54257	-
16	525	927	47497	592013	2.1 10^6
20	817	1479	132085	3.2 10^6	2.8 10^7

An important question is whether the complexity of the controller (i.e., the number of simplices obtained by partitioning the state space according to Procedure 4.1) is known in advance. For $n = 2$, the number of simplices (triangles) is exactly $4N - k$, where k is the number of the vertices of the convex hull of the optimal trajectory and its opposite [12]. For $n > 2$, since such simplices form a *triangulation* [8] of a point set, their number N_s is bounded according to the expression [13]

$$N_s \leq \binom{2N + 2 - \lceil \frac{n+1}{2} \rceil}{\lfloor \frac{n+1}{2} \rfloor} + \binom{2N + 1 - \lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor} - (n+1), \tag{57}$$

where $\lfloor x \rfloor$ denotes the maximum integer less than or equal to x, $\lceil x \rceil$ denotes the minimum integer greater than or equal to x, and $\binom{a}{b}$ denotes the binomial coefficient. Table 1 reports such an upper bound for some pairs of N and n. Upper bound (57) resembles that provided in [2], in the context of the explicit linear $1/\infty$-norm regulator for constrained systems, with the substantial difference that *for the static ROC the upper bound does not depend on the number of constraints*, since the controller is computed based on the optimal trajectory only. In other words, the number of constraints does not influence directly the complexity of the controller.

Remark 4.4. As shown above, the convexity of the constraints and the cost guarantees that

$$x_i \notin \mathcal{P}_{i-1} \ \forall i = n+1, \ldots, N. \tag{58}$$

However, as long as condition (58) on the optimal trajectory is satisfied, the ROC can be constructed independently of the convexity of the optimization problem. In other words, a sufficient condition for constructing the static ROC is that each of the points of the optimal trajectory does not belong to the convex hull of the subsequent points and their opposite. Obviously, the satisfaction of the constraints is guaranteed for all the trajectories originating in $\mathcal{P} = \mathcal{P}_N$ only if the constraints are convex and 0-symmetric.

5. Construction of a global controller. For $x \in \mathcal{P} = \mathcal{P}_N$, the controller described above is a solution for Problem 1. However, the control law is not defined for $x \notin \mathcal{P}$. A possible way to extend the control outside $\mathcal{P}$ is to "immerse" $\mathcal{P}$ in the maximal invariant set $\mathcal{X}_{max}$, namely, the set of all states which can be brought to the origin in finitely many steps without state or input constraint violations (note that $\mathcal{P} \subseteq \mathcal{X}_{max}$). Then, for $x \notin \mathcal{P}$, one can apply the control law derived from $\mathcal{X}_{max}$ (many algorithms have been proposed to find $\mathcal{X}_{max}$ and an associated control law; see, for example, [9]). By definition, the constraints are satisfied and the convergence is guaranteed if and only if $x(1) \in \mathcal{X}_{max}$.

A different strategy can be derived as the *natural extension of the controller computed within* $\mathcal{P}$. In this way we have basically two advantages:

- the obtained controller is globally Lipschitz;
- the state behavior outside the set $\mathcal{P}$ resembles the internal one and therefore the system performs "reasonably well" outside $\mathcal{P}$.

The set $\mathcal{P}$ is a polytope including the origin in its interior. This means that the state space can be partitioned in simplicial cones, each having a center in the origin and generated by n vertices of $\mathcal{P}$. These cones $\mathcal{C}^{(h)}$ have a nonempty interior, have intersections with empty interior, and cover $\mathbb{R}^n$:

$$\text{int}\{\mathcal{C}^{(h)}\} \neq \emptyset, \tag{59}$$

$$\text{int}\{\mathcal{C}^{(h)} \bigcap \mathcal{C}^{(k)}\} = \emptyset, \quad h \neq k, \tag{60}$$

$$\bigcup_h \mathcal{C}^{(h)} = \mathbb{R}^n. \tag{61}$$

For each cone generated by a square matrix $X^{(h)}$, we consider the complement with respect to $\mathcal{P}$,

$$\tilde{\mathcal{C}}^{(h)}; \tag{62}$$

therefore the union of the complements and the simplices forming $\mathcal{P}$ cover $\mathbb{R}^n$. For each cone generated by an invertible $X^{(h)}$ we consider the corresponding input matrix $U^{(h)}$[1] and the control

$$\tilde{\Phi}(x) = U^{(h)}[X^{(h)}]^{-1}x. \tag{63}$$

Such a control is Lipschitz [3].

In principle, continuity is not an issue in discrete-time systems. In practice, it avoids chattering. Thus we state the next lemma.

LEMMA 5.1. *Consider the following extension outside* $\mathcal{P}$ *of the control* $\Phi_{\mathcal{P}}(x)$*:*

$$\Phi(x) \doteq \begin{cases} \Phi_{\mathcal{P}}(x) & \text{for } x \in \mathcal{P}, \\ \tilde{\Phi}(x) & \text{for } x \notin \mathcal{P}. \end{cases} \tag{64}$$

Such a control is globally Lipschitz.

Proof. Since the control is piecewise affine and the cardinality of the partition is finite, we need only prove global continuity. Then the Lipschitz constant for each component of Φ is given by the maximum value of the norm of its gradient. Note also that $\Phi_{\mathcal{P}}$ is continuous inside $\mathcal{P}$ and $\tilde{\Phi}$ is continuous outside $\mathcal{P}$. As a consequence, we need only prove that the extended control is continuous in $\partial\mathcal{P}$. Consider $\hat{x} \in (\mathcal{P} \cap \tilde{\mathcal{C}}^{(h)}) \subset \partial\mathcal{P}$. By construction, $\tilde{\mathcal{C}}^{(h)}$ and $\mathcal{P}$ have a facet in common, precisely

[1] The matrix whose columns are the (optimal) control vectors associated with the columns in $X^{(h)}$, elements of the optimal trajectory.

the facet whose vertices are the generator vectors of $\tilde{\mathcal{C}}^{(h)}$. Since $\hat{x}$ lies in the common facet, it can be expressed as a linear combination of those vectors in a unique way. From (40) and (63) it follows that $\Phi_{\mathcal{P}}(\hat{x})$ and $\tilde{\Phi}(\hat{x})$ are the linear combination of the controls associated with the same vectors according to the same coefficients; then $\tilde{\Phi}(\hat{x}) = \Phi_{\mathcal{P}}(\hat{x})$ for all $\hat{x} \in \mathcal{P} \cap \tilde{\mathcal{C}}^{(h)}$, i.e., the extended control is continuous in $\partial\mathcal{P}$. ∎

Now the problem is to show that the extended control $\Phi(x)$ is globally stabilizing. Then we consider as a candidate Lyapunov function the Minkowski function of $\mathcal{P}$, that is, the norm whose unit ball is $\mathcal{P}$:

$$\Psi(x) = \min\{\lambda \geq 0 : \quad x \in \lambda\mathcal{P}\}. \tag{65}$$

We have the following preliminary lemma.

LEMMA 5.2. *The function $\Psi(x(k))$ is nonincreasing as long as $x(k) \notin \mathcal{P}$.*

Proof. Since $x(k+1) = Ax + B\Phi(x(k))$, we must prove that

$$\Psi(Ax + B\Phi(x)) \leq \Psi(x) \quad \forall x \notin \mathcal{P}. \tag{66}$$

As shown above, the extended control $\Phi(x)$ is globally continuous. Furthermore, it is linear inside each of the $\tilde{\mathcal{C}}^{(h)}$. As a consequence, outside the interior of $\mathcal{P}$, the control can be expressed as

$$\Phi(x) = \Psi(x)\Phi_{\mathcal{P}}(\bar{x}), \quad x \notin \mathrm{int}(\mathcal{P}), \tag{67}$$

where the vector

$$\bar{x} = \frac{x}{\Psi(x)} \tag{68}$$

belongs to the boundary of $\mathcal{P}$. Consider a generic $x \notin \mathcal{P}$ and its "projection" $\bar{x}$ onto $\partial\mathcal{P}$. Since $\Phi_{\mathcal{P}}$ renders invariant the set $\mathcal{P}$, it follows that

$$A\bar{x} + B\Phi_{\mathcal{P}}(\bar{x}) \in \mathcal{P} \tag{69}$$

and, multiplying by $\Psi(x)$,

$$A\Psi(x)\bar{x} + B\Psi(x)\Phi_{\mathcal{P}}(\bar{x}) \in \Psi(x)\mathcal{P}. \tag{70}$$

From (67) and (70) and by substituting $x = \Psi(x)\bar{x}$ we obtain

$$Ax + B\Phi(x) \in \Psi(x)\mathcal{P}, \tag{71}$$

which implies, by the definition of the Minkowski function, that

$$\Psi(Ax + B\Phi(x)) \leq \Psi(x), \quad x \notin \mathcal{P}. \tag{72}$$ ∎

Lemma 5.2 proves the boundedness of the state but not convergence to 0. To prove convergence, since the function $\Psi(x)$ is only nonincreasing, we must use a trick. Define

$$x(k+1) = f(x) = Ax + B\Phi(x) \tag{73}$$

and consider the N steps forward system defined as the composition of f:

$$x(k+N) = f^N(x(k)) = f(f(\dots f(x)\dots)) \doteq F(x). \tag{74}$$

By means of this system we can show the following.

LEMMA 5.3. *The function $\Psi(x(k))$, as long as $x(k) \notin int(\mathcal{P})$, is strictly decreasing along the trajectory of the system* (74), *precisely*

$$\Delta\Psi(x) \doteq \Psi(F(x)) - \Psi(x) < 0 \quad \forall x \notin \operatorname{int}(\mathcal{P}). \tag{75}$$

Proof. Consider a generic $x \notin \operatorname{int}(\mathcal{P})$ and its projection $\bar{x}$ onto $\partial\mathcal{P}$. As a first step we observe that there exists $1 \le h \le N$ such that

$$\bar{x}, f(\bar{x}), \ldots, f^{h-1}(\bar{x}) \in \partial\mathcal{P}, \tag{76}$$

$$f^h(\bar{x}) \in \operatorname{int}(\mathcal{P}). \tag{77}$$

This is an immediate consequence of the fact that the control steers the system to zero in at most N steps starting from any $x \in \mathcal{P}$ and, in particular, from any $\bar{x} \in \partial\mathcal{P}$. By definition, we can express $x \notin \operatorname{int}(\mathcal{P})$ as the product of its projection $\bar{x}$ onto $\partial\mathcal{P}$ and the Minkowski function $\Psi(x)$:

$$x = \bar{x}\Psi(x). \tag{78}$$

By substituting in $f(x)$ we get

$$f(x) = Ax + B\Phi(x) = A\Psi(x)\bar{x} + B\Psi(x)\Phi_{\mathcal{P}}(\bar{x}) = \Psi(x)(A\bar{x} + B\Phi_{\mathcal{P}}(\bar{x})) = \Psi(x)f(\bar{x}), \tag{79}$$

where the last equality holds since $\Phi_{\mathcal{P}}(\bar{x}) = \Phi(\bar{x})$ $\forall \bar{x} \in \partial\mathcal{P}$. Similarly it can be shown that $f^i(x) = \Psi(x)f^i(\bar{x})$ $\forall i = 2, \ldots, N$. Now, by multiplying (76) and (77) by $\Psi(x)$ it follows that

$$x, f(x), \ldots, f^{h-1}(x) \in \partial(\Psi(x)\mathcal{P}), \tag{80}$$

$$f^h(x) \in \operatorname{int}(\Psi(x)\mathcal{P}); \tag{81}$$

then we obtain

$$\Psi(f^h(x)) < \Psi(x). \tag{82}$$

Thanks to Lemma 5.2, during the next $N - h$ steps, the state cannot escape from the region $\operatorname{int}(\Psi(x)\mathcal{P})$; hence

$$\Psi(F(x)) = \Psi(f^N(x)) \le \Psi(f^h(x)). \tag{83}$$

Finally, from these and (82) it follows that

$$\Psi(F(x)) < \Psi(x). \qquad \square \tag{84}$$

Now we are in the position of proving global stability.

Proof of Theorem 3.2. The considered control is Lipschitz and piecewise affine. We need to prove global asymptotic stability. To prove this fact we show that for any initial state $x(1) = x^* \notin \mathcal{P}$ there exists a finite M such that $x(M) \in \mathcal{P}$. Once $\mathcal{P}$ is reached, the state converges to zero as already proved.

This requires standard Lyapunov arguments. Indeed, the composed function $F(x)$ and the candidate Lyapunov function $\Psi(x)$ are continuous, and thus the function $\Delta\Psi(x)$ is continuous. Consider the compact set

$$\mathcal{H} = \{x : \ 1 \le \Psi(x) \le \Psi(x^*)\}. \tag{85}$$

In such a set, $\Delta\Psi(x)$ admits a negative maximum $-\mu$ with $\mu > 0$. Then we have

$$\Psi(x(k+N)) - \Psi(x(k)) = \Delta\Psi(x(k)) \leq -\mu. \tag{86}$$

This means that

$$\Psi(x(hN)) \leq \Psi(x^*) - h\mu. \tag{87}$$

But this means that, in finite time, the condition $\Psi(x(k)) < 1$ occurs, and therefore $x(k)$ reaches $\mathcal{P}$. □

Remark 5.1. The constraint (11) may be relaxed as follows:

$$x(N+1) \in \mathcal{X}_{fin}, \tag{88}$$

where $\mathcal{X}_{fin}$ is a 0-symmetric controlled-invariant polyhedron (that is, there exists a local control that renders $\mathcal{X}_{fin}$ positively invariant and such that the constraints are satisfied for all initial conditions inside the set). Then one can construct the ROC by positing $\mathcal{P}_n = \mathcal{X}_{fin}$ and following steps 3–7 of Procedure 4.1. As a result, a dual control strategy may be adopted: Apply the control $\Phi(x)$ for $x(k) \notin \mathcal{X}_{fin}$ and switch to the local control as soon as the condition $x(k) \in \mathcal{X}_{fin}$ is satisfied.

6. Example.

Consider the double integrator

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \tag{89}$$

under the constraints $|x(k)| \leq 5, |u(k)| \leq 3$. Given the initial state $x(1) = [-2 \;\; 5]^T$, the horizon $N = 5$, the final state $x(N+1) = 0$, and the cost function $J = \sum_{i=1}^{N} u(k)^2$, the optimal (open-loop) control and trajectory, found by solving a quadratic-programming problem are, respectively,

$$\bar{U} = \begin{bmatrix} -3 & -2.9 & -1.3 & 0.3 & 1.9 \end{bmatrix} \tag{90}$$

and

$$\bar{X} = \begin{bmatrix} -2 & 3 & 5 & 4.1 & 1.9 \\ 5 & 2 & -0.9 & -2.2 & -1.9 \end{bmatrix}. \tag{91}$$

The optimal trajectory is reported in Figure 3. By means of Procedure 4.1, the triangulation reported in Figure 4 is obtained; the number of triangles is 12 (including the four triangles in which the darkest region, i.e., $\mathcal{P}_2$, can be split). The piecewise-affine control law obtained by applying a control-at-the-vertices strategy inside each of the triangles, as stated above, is relatively optimal, and hence is optimal for the nominal initial condition and guarantees convergence and constraint satisfaction for the other initial conditions. In Figure 4, the trajectories from three nonnominal initial conditions are reported. Note that the number of steps required to reach the origin depends on the triangle to which the initial state belongs. Figure 5 shows the effectiveness of the *extended* control, reporting some trajectories starting from outside $\mathcal{P}$; the dash-dotted lines represent the boundaries between the simplicial cones $\mathcal{C}^{(h)}$ in the complement of $\mathcal{P}$.

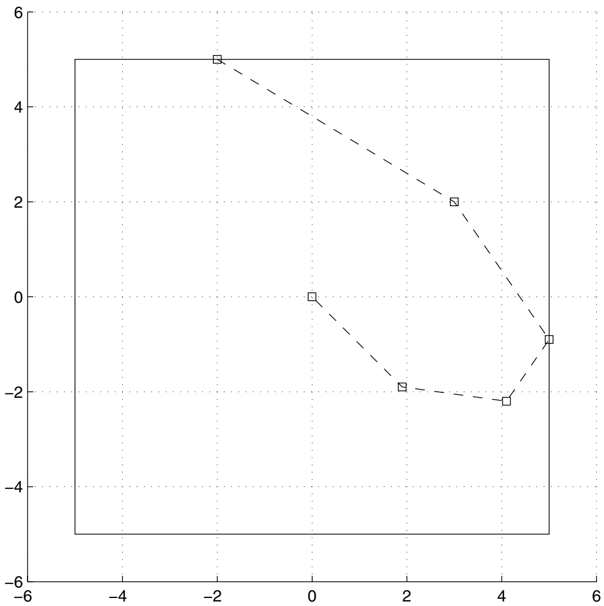

FIG. 3. *The optimal trajectory.*

7. Comparison with the dynamic ROC. Some significant differences between the dynamic ROC [5, 6] and the static one are highlighted in the following points:

1. Since the static ROC is nonlinear, the trajectory originating from $\lambda\bar{x}$ is not proportional, in general, to the one originating from $\bar{x}$ as with the dynamic ROC. However, by construction, opposite initial conditions generate opposite trajectories.
2. The dynamic ROC allows for the optimization from a set of n linearly independent initial conditions, while the static version described in this paper is thought of for a *single initial condition*. Extending the results to more than one initial condition for the static ROC is a matter of further investigation.
3. The dynamic ROC cannot guarantee the satisfaction of the constraints for initial conditions different from the nominal one. Hence it is suitable only in those cases when the constraints are actually *soft* constraints (constraints whose violation causes a performance loss only). On the contrary, by immersing the set $\mathcal{P}$ in the maximal invariant set as outlined in the beginning of section 5, the piecewise-affine solution can deal effectively with *hard* constraints.
4. The dynamic ROC is a linear system of order $N - n$, and hence is of low complexity. By looking at Table 1 it is clear that the complexity of the static ROC is much higher and depends strongly on the dimension of the state space (although the table shows only an upper bound for the number of simplices). As a consequence, the implementation of the static ROC may be difficult for high-order systems. However, almost all the approaches based on partitions of the state space are prone to this problem.

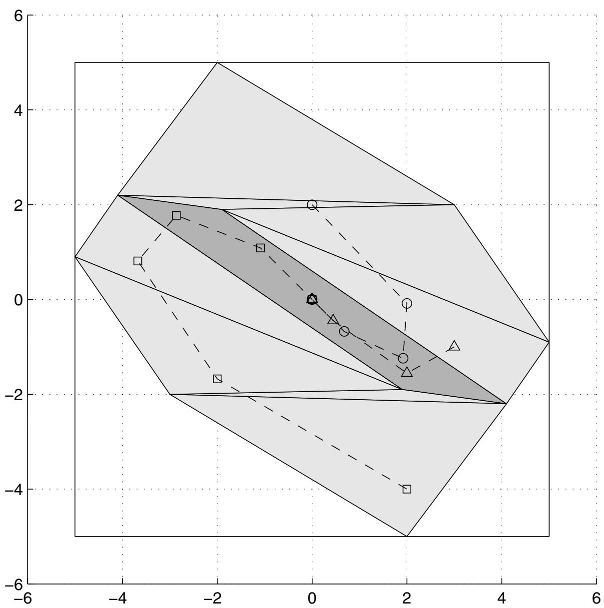

Fig. 4. *The triangulation induced by the optimal trajectory and the trajectories from three nonnominal initial conditions inside $\mathcal{P}$.*

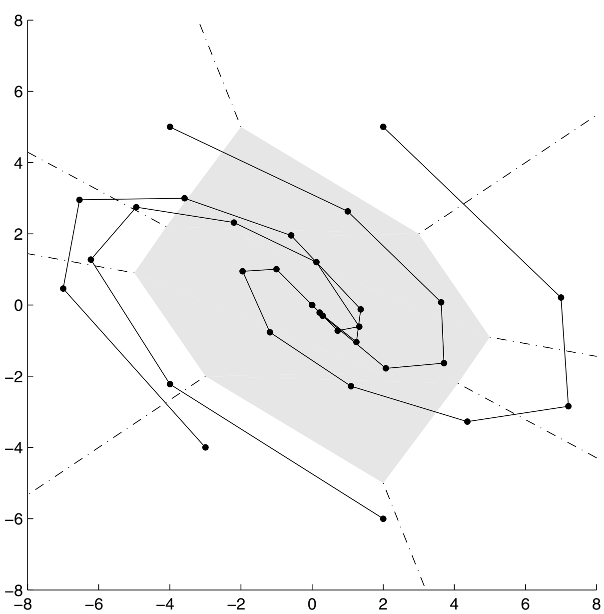

Fig. 5. *Some trajectories starting from outside $\mathcal{P}$.*

8. Conclusions. In this paper, a static version of the relatively optimal control (ROC) [5, 6] is proposed. The proposed controller is a deadbeat piecewise-affine state-feedback controller, based on a triangulation of the points of the optimal trajectory (computed offline). An upper bound on the number of polyhedral sets (whose vertices are the states of the optimal trajectory and their opposite) as a function of the order of the system and the length of the optimal trajectory is provided. The control at each vertex is the corresponding control vector of the optimal sequence, while the control at a generic state is given by a convex combination of the controls corresponding to the vertices of the set to which the state belongs. By removing the zero state terminal constraint and requiring the final state of the optimal trajectory to belong to a controlled invariant set, it is possible to obtain a nondeadbeat controller. The proposed control can deal effectively with hard constraints (a significant advantage with respect to the dynamic one previously introduced). We point out that the 0-symmetry of the constraint set is not necessary for the construction of the ROC, namely, for solving Problem 1; however, it guarantees the additional property that the constraints are satisfied for all initial conditions inside $\mathcal{P}_N$. Further work includes extending the results to more than one initial condition and exploiting the particular structure of the triangulation in order to obtain a tighter bound on the number of simplices.

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