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Control theory

Inverted Pendulum

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Control basics

Controlling a system is the act of modifying its dynamics in order to make it reach a wanted state. There are multiple types of control. **Passive control** is a type of control that relies on a system's structure for stabilization. The system returns to a stabilized state without the need for external action. **Active control** can be split in two: **Open loop** and **closed-loop feedback control**. An open loop example is an old speed regulator for a car. The controller sets the engine at a given RPM and expects the car to stabilize around a specific speed using a model. The car could be moving at a different speed than the setpoint without the controller knowing it. On the other hand, a feedback loop uses sensors to know how the system is behaving to adapt its control. Feedback control is useful to control unstable systems. It is also capable of rejecting disturbances in the system.

Operations basics

To describe the controlled system, we use a state-space model of ordinary differential equations:

$$\dot{x} = Ax + Bu \tag{1}$$

In which $\dot{x}$ is a state vector that describes the quantities of interest of the system (position, speed, angles...). A is the state matrix, representing the interaction between all state variables. B is the control matrix and u is the control vector. The vector u represents the inputs to send to the system's actuators to control it, while B shows what impact the control vector has on the system state. This is a system of linear differential equations, but the computations can be extended to non-linear systems of differential equations.

The observation of state variables that come from sensors will be stored in :

$$y = Cx + Du \tag{2}$$

C is the output matrix, describing how the states of the system are mapped to the output y. D is the feedthrough matrix. It represents the direct relationship between the input u and the output y.

Through computations that will not be disclosed here, u can be expressed as such :

$$u = -Kx \tag{3}$$

Which yields :

$$\begin{aligned} \dot{x} &= Ax - BKx \\ &= (A - BK)x \end{aligned} \tag{4}$$

The end goal of these computations is to determine when a system can actually be controlled and how to design K such that the system is controlled the way we want.

Linear Systems of ODEs

A linear system of ODEs is written as :

$$\dot{x} = Ax \tag{5}$$

The basic solution for this system is :

$$x(t) = e^{At}x(0) \tag{6}$$

Since At is a matrix, to take its exponential value, we can compute its Taylor series :

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots \tag{7}$$

However, this matrix isn't practical to compute, so we compute its eigenvalues and eigenvectors to perform a coordinates transformation instead. This makes the computation of e^{At} easier.

Using eigenvalues and eigenvectors

An eigenvector ξ is a vector that satisfies for a matrix A :

$$A\xi = \lambda\xi \tag{8}$$

with λ the eigenvalue related to it. With multiple eigenvectors, we can write :

$$T = [\xi_1\xi_2 \dots \xi_n] \tag{9}$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \tag{10}$$

Such that :

$$AT = TD => T^{-1}AT = D \tag{11}$$

The coordinates transformation goes through T and D. By setting

$$x = Tz \tag{12}$$

Here, z represents the coordinates of x inside the eigenvalues coordinate system T. The z coordinates make computations easier than the x coordinates. We can now rewrite the system in the new coordinates system :

$$\dot{x} = T\dot{z} = Ax \tag{13}$$

So, by substituting $x = Tz$:

$$T\dot{z} = ATz => \dot{z} = T^{-1}ATz \tag{14}$$

And we previously showed that :

$$D = T^{-1}AT \tag{15}$$

So :

$$\dot{z} = Dz \tag{16}$$

Getting expressions with z makes every computation easier because the components of z and $\dot{z}$ are all diagonal. i.e., they are uncoupled from each other. Practically, this implies that when solving the linear system to find $\dot{z}$, every line will be a function of only one variable :

$$\begin{cases} \dot{z}_1 & = \lambda_1.z_1 \\ \dot{z}_2 & = \lambda_2.z_2 \\ & \vdots \\ \dot{z}_n & = \lambda_n.z_n \end{cases} \tag{17}$$

As matrices, this is written :

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{18}$$

To come back to the main goal of the computations, which is to compute the state variables, we can now easily and quickly solve the system using the eigenvectors coordinate system before converting the $\dot{z}$ vector back to the original coordinates system. To solve $\dot{z}$, we have the same solution as before but without requiring to use Taylor series :

$$z(t) = e^{Dt}z(0) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} x(0) \tag{19}$$

Now, by replacing A by TDT^{-1} in the first solution (6) and in the Taylor series, we get :

$$e^{At} = Te^{Dt}T^{-1} \tag{20}$$

This solution is simple, we already computed T and e^{Dt}. To summarize, the final solution we obtain :

$$x(t) = Te^{Dt}T^{-1} \tag{21}$$

Shows that the dynamics of the system rely on the eigenvectors and eigenvalues of A, but the dynamics over time rely solely on the eigenvalues of A.

Stability

The system

$$e^{At} = Te^{Dt}T^{-1} \tag{22}$$

is considered to be stable if the value of $x(t)$ remains tamed as t tends to $+\infty$. To verify this, all eigenvalues simply have to be negative in their real part. As an example :

$$\lambda = -5 + 2i => \text{the system will be stable}$$
$$\lambda = 1 - 8i => \text{the system will be unstable}$$

It happens frequently that the A matrix has positive real eigenvalues. Our goal will be to place these eigenvalues such that they become stable by adding Bu to the system.

Linearization of non-linear systems

A common practice in control theory is to linearize non-linear systems through different assumptions. For $\dot{x} = f(x) => \dot{x} = Ax$ non linear, a few steps can be followed in order to reach its linear equivalent.

Step 1 - Finding fixed points

Fixed points are specific values $\bar{x}$ of some state variables for which $f(\bar{x}) = 0$. In other words, they are points at which the system is stationary. It can be a spring's rest position, a pendulum angle towards the ground or towards the up direction...

Step 2 - Computing the Jacobian of $f(x)$ about $\bar{x}$

For the following state variables :

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = f(x) \tag{23}$$

We compute :

$$\left.\frac{Df}{Dx}\right|_{\bar{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial \dot{x}_n}{\partial x_1} & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \end{bmatrix} \tag{24}$$

In other words, we compute the Jacobian of $\dot{x} = f(x)$ then replace the remaining $x_1, \dots, x_n$ by $\bar{x}_1, \dots \bar{x}_n$ in the derived expressions. The following example will be clearer.

Example: The pendulum

Let's study the motion of a simple pendulum :

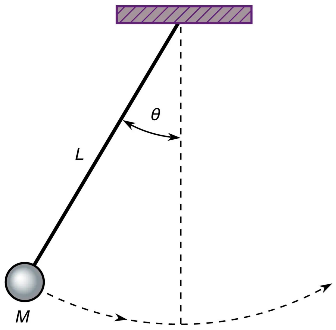


Figure 1: Schematic of a basic pendulum. $L = 9.81 = g$

The equation of motion for this pendulum is

$$\begin{aligned} \ddot{\theta} &= -\frac{g}{L} sin(\theta) - \delta\dot{\theta} \quad (25) \\ &= -sin(\theta) - \delta\dot{\theta} \quad \text{since } g = L \quad (26) \end{aligned}$$

The state variables for a pendulum are its angle θ and its angular velocity $\dot{\theta}$. Therefore we have

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \quad (27)$$

From this, we can compute $\dot{x} = \frac{dx}{dt}$:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -sin(x_1) - \delta x_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \quad (28)$$

Now that we have the expression for $\dot{x}$, we can start applying the linearization process. Let's first express $\bar{x}$, the fixed points of the system. By definition, all the velocities are null at a fixed point, so

$$x_2 = \dot{\theta} = 0 \quad (29)$$

The pendulum is stationary when it is either pointing down or up. Therefore it is when

$$\theta = 0 \quad \text{or} \quad \theta = \pi \quad (30)$$

Which means that :

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} \pi \\ 0 \end{bmatrix} \quad (31)$$

Let's now compute the Jacobian matrix of $f(x)$:

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_n}{\partial x_1} & \frac{\partial \dot{x}_n}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -cos(x_1) & -\delta \end{bmatrix} \quad (32)$$

We can now replace x_1 by $\bar{x}_1$ and x_2 by $\bar{x}_2$ to get the linearized A matrix about the down $(0, 0)$ or up $(\pi, 0)$ position. Note that there isn't any x_2 left in our case. We get for the down position :

$$A_{down} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix} \tag{33}$$

And for the up position :

$$A_{up} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \tag{34}$$

We can see that the eigenvalues of A_{down} are all negative in the real part, so the pendulum is stable in that region. However, that is not the case for A_{up}, which implies that the pendulum won't oscillate back to the upper fixed point if disturbed. This makes sense as gravity will pull the pendulum down.

Reachability

When a system is controllable, these statements are equivalent :

- The system is controllable
- We can arbitrarily place any eigenvalue in order to stabilize an unstable system by defining a specific K :

$$u = -Kx => \dot{x} = (A - BK)x \tag{35}$$

- Any state $x \in \mathbb{R}^n$ is reachable using a specific u.

The concept of controllability will be explained in the following chapter

Controllability

In order to solve the system :

$$\dot{x} = Ax + Bu \tag{36}$$

We need to know when the dynamics of the system can be controlled using the u vector. If the system can be controlled, our next goal will be to find how to design u such that the system behaves the way we want. When a linear system is controllable, the best control to stabilize it in the wanted state is to set :

$$u = -Kx \tag{37}$$

A system is controllable as long as we can find a matrix K such that we can obtain the previous equation. Controllability also means that we can choose any eigenvalue we want for A, using specific values of K. The main factors that determine if a system is controllable are the matrices A and B. If a system is not controllable, the matrices need to be modified. Here is an example of an uncontrollable system :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{38}$$

In this equation, u will only be able to act on $\dot{x}_2$, and $\dot{x}_2$ does not have any influence on $\dot{x}_1$, which means $\dot{x}_1$ is not controlled. This system can be written as such :

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = 2x_2 + u \end{cases} \tag{39}$$

However, the system could be controllable if B and u were defined as such :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{40}$$

As a reminder,

$$x \in \mathbb{R}^n, \quad u \in \mathbb{R}^q, \quad A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times q}, \tag{41}$$

If x_1 and x_2 had been coupled to some extent (i.e. $\dot{x}_1$ is a function of x_1 and x_2), the system may have been controllable using only u_2. This is a strength of control theory. To mathematically check for controllability, we have to compute the controllability matrix C, which is defined as the concatenation of multiple matrices :

$$C = \begin{bmatrix} B & AB & A^2B & ... & A^{n-1}B \end{bmatrix} \tag{42}$$

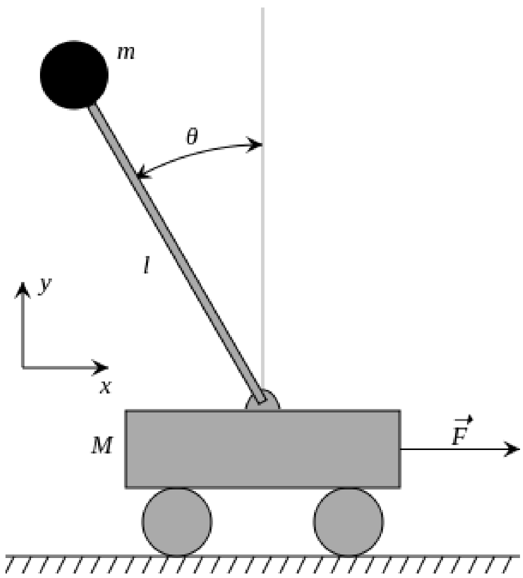
In a single-input case, the system is said to be controllable if C is invertible. In practice, that implies that $det(C) \neq 0$. In general, if C has a rank of n, then the system is controllable.

Control summary

Now that every concept required to control a system has been covered, let's summarize the procedure to create a controller step by step.

Description of the system

Let's control an inverted pendulum that sits on a rolling cart that moves along a single axis. We can control the speed of the cart's motors, which will translate into a force ($\overrightarrow{F}$) acting on the system along the cart's axis. The system is drawn as such :



Determining the state variables

Before making an analysis of the system, let's find what we are looking for. To describe the total movement of the system, we will first need to know the position x of the cart, and its velocity $\dot{x}$. We will also need to know the angle θ between the cart's center and the pendulum's axis, as well as the angular velocity of the pendulum $\dot{\theta}$.

Finding the equations of movement

We now look for the equations of movement of the system. These equations are found using Newton's second law, or Lagrangian/Hamiltonian methods. The goal of these equations is to provide us with an expression of the system's state variables. Here, we look for the second derivative of x and θ, which will allow us to then derive the expression for every state variable.

According to Wikipedia, the state equations of an inverted pendulum are the following :

$$\begin{cases} \ddot{x} = \frac{F - ml\dot{\theta}^2 \sin\theta + ml\ddot{\theta}\cos\theta}{M+m} \\ \ddot{\theta} = \frac{\ddot{x}\cos\theta + g\sin\theta}{l} \end{cases} \tag{43}$$

These equations are not perfect as they are coupled together. These can be solved using tools such as Matlab or sympy. The uncoupled equations of motion we obtain are :

$$\begin{cases} \ddot{x} = \frac{-lm\sin(\theta)\dot{\theta}^2 + F + gm\cos\theta\sin\theta}{-m\cos^2\theta + M + m} \\ \ddot{\theta} = \frac{-lm\cos\theta\sin(\theta)\dot{\theta}^2 + F\cos\theta + gm\sin\theta + Mg\sin\theta}{l(-m\cos^2\theta + M + m)} \end{cases} \tag{44}$$

Linearizing the equations

In order to give the state-space representation of the system, both the equations must be linear. Currently, our equations are incorporated with trigonometric functions and squared terms. To linearize the equations, we need to compute the associated Jacobian matrix by differentiating each equation by each state variable (cf. (24)). Once the Jacobian is found, we can choose a stationary point to simplify our results. At the vertical, we have this stationary point :

$$\dot{x} = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

This gives the simplified Jacobian :

$$\bar{J} = \begin{bmatrix} 0 & 0 & \frac{gm}{M} & 0 \\ 0 & 0 & \frac{Mg+gm}{Ml} & 0 \end{bmatrix} \tag{46}$$

This can be interpreted as $\ddot{x}$ having a $\frac{gm}{M}$ influence on θ, and $\ddot{\theta}$ having a $\frac{Mg+gm}{Ml}$ influence on θ.

Finding the A matrix

From the equations of motion of the system, let's write the expression of $\dot{x}$:

$$\dot{x} = \frac{dx}{dt} = Ax \tag{47}$$

With

$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{48}$$

Using the Jacobian, we can write A :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{gm}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{Mg+gm}{Ml} & 0 \end{bmatrix} \tag{49}$$

Checking for stability

Now that the A matrix was established, we can compute its eigenvalues to verify if the system is passively stable or not. The eigenvalues of the matrix A are the following :

$$Sp(A) = \begin{bmatrix} 0 \\ 0 \\ -\sqrt{\frac{g(M+m)}{Ml}} \\ \sqrt{\frac{g(M+m)}{Ml}} \end{bmatrix} \tag{50}$$

One of the eigenvalues is a negative real number, which implies that the system is unstable. This instability is a sufficient motivation for us to try to implement a feedback controller.

Finding the B matrix

The B matrix describes how the force F applied to the cart will influence the state variables. Solving the coupled equations of motion around the stationary point is a good approach to obtain the influence of F on the system dynamics. Thus, the B matrix is defined as such :

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml} \end{bmatrix} \tag{51}$$

Finding the controllability (C) matrix

MATLAB provides a function to compute the controllability matrix from a matrix A and B. The rank of the resulting matrix is 4, indicating that the system is controllable.

Placing the eigenvalues

Additionally, MATLAB includes a function to determine a matrix K such that the eigenvalues of $A - KB$ are the ones we decide to work with. By placing all eigenvalues as negative real numbers, one can make sure the system will be stable and define the reactivity of the controller's response.

Using the computed matrices

With the matrices A, B, and K, the main equation can be solved :

$$\dot{x} = Ax + Bu = (A - BK)x \tag{52}$$

One can select different behaviors for the controller by setting a setpoint :

$$\dot{x} = Ax - BK(x - setpoint) \tag{53}$$