Functions : Visual Notes

Don't forget to always specify domain & codomain when you define functions!

Function Definition:

domain

$f : A \to B$

codomain

uniqueness

$\underbrace{\forall a \in A}_{\text{totality}},\ \underbrace{\exists}_{\text{existence}}! \, b \in B \text{ s.t. } f(a) = b$

3 conditions for a well-defined function:

① $\forall a \in A$, $f(a)$ is defined.
totality : $f(a)$ is defined <u>for all</u> a in the domain A.

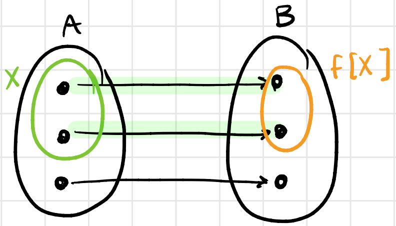
② $\forall a \in A$, $f(a) \in B$
existence : $f(a)$ <u>exists in the codomain</u> B.

③ $\forall a, a' \in A$, if $a = a'$ then $f(a) = f(a')$
uniqueness : everything in the domain only maps to <u>one</u> value in the codomain.

Image Consider a function $f: A \to B$
Let set $X \subseteq A$. Define the <u>image</u> of X under f.

Formally: $f[X] = \{ b \in B \mid \exists x \in X,\ f(x) = b \} \subseteq B$

Intuitively:

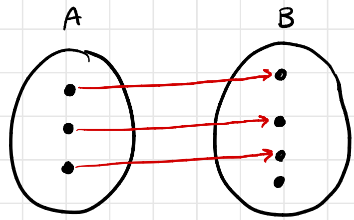


$f[X]$ is everything in B that has an element of X that maps to it.

Note: It's important to understand both the intuitive definition and the formal definition because the intuitive definition helps you visualize but the formal definition is what you use in proofs.

Injections, Surjections, Bijections

For all examples, let function $f: A \to B$

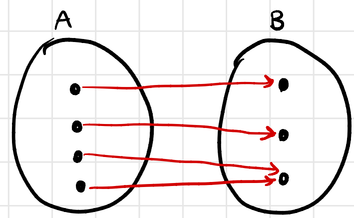


Injections

f is injective (one-to-one) iff

$\forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y$

f injective $\Rightarrow |A| \leq |B|$

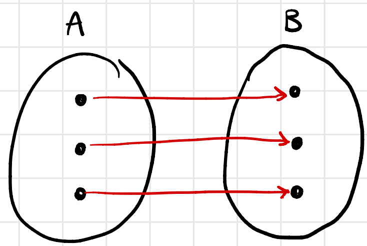


Surjections

f is surjective (onto) iff $\mathrm{Im}(f) = B$,

meaning $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$

f surjective $\Rightarrow |A| \geq |B|$



Bijections

f is bijective iff

f is both injective and surjective

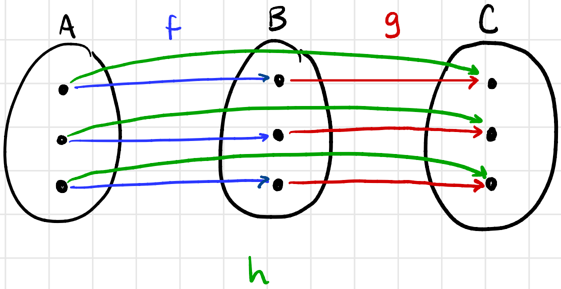
f bijective $\Rightarrow |A| = |B|$

Function Composition

Let A, B, C be sets and $f: A \to B$ and $g: B \to C$ be functions.

The function $h: A \to C$ defined $h(a) = g(f(a))\ \forall a \in A$ is the composition of g and f, written $h = g \circ f$

Visually:

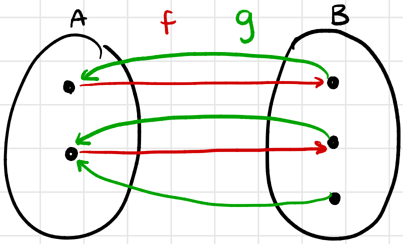


Inverses

Let $f: A \to B$ and $g: B \to C$

- g is left inverse for f $\Leftrightarrow$ $g \circ f = id_A$
- g is right inverse for f $\Leftrightarrow$ $f \circ g = id_B$
- g is 2-sided inverse for f $\Leftrightarrow$ g is both left & right inverse
- f is invertible $\Leftrightarrow$ f has a 2-sided inverse

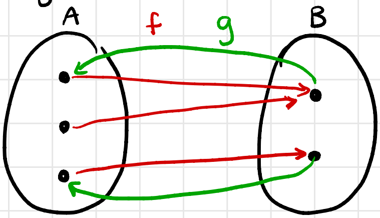
Left inverse: takes you back exactly where you came from



f has left inverse $\Leftrightarrow$ f is injective

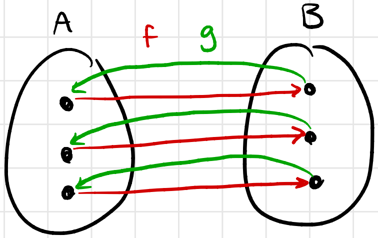
Some elements of b may not be reachable.

Right inverse: takes you back to one of the places you might have come from, but every end point is reachable.



f has right inverse $\Leftrightarrow$ f is surjective

2-sided inverse : every element of B is reachable and there is a path taking you back where you came from.



f has 2 sided inverse $\Rightarrow$ f is bijective

Proof strategies:

- to prove f injective, find a left inverse
- to prove f surjective, find a right inverse
- to prove f bijective, find a 2-sided inverse