ValCalc

Sequences and Series Tests

Limits of Sequences

If $\lim_{n\to\infty} a_n = L$, then $\{a_n\}$ converges to L; otherwise, the sequence diverges.

Monotonic Sequence Theorem

Any bounded, monotonic sequence is convergent.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Infinite Geometric Series

If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Telescoping Series

Let all terms in a partial sum cancel except the first and last:

$$\begin{aligned} S_N &= \sum_{i=1}^{N} (a_i - a_{i+1}) \\ &= (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{N+1} - a_N) + (a_N - a_{N+1}) \\ &= a_1 + a_{N+1}. \end{aligned}$$

Then $\sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{N\to\infty} S_N$.

p-Series

The series $\sum_{n=1}^{\infty} \frac{1}{x^p}$ converges for $p > 1$ and diverges for $p \leqslant 1$.

Infinite Series with Constants

Suppose that c is any real number. If $\sum_{n=1}^{\infty} a_n$ converges, then so do the following series; if it diverges, then so do the following series:

$$c + \sum_{n=1}^{\infty} a_n \qquad c - \sum_{n=1}^{\infty} a_n \qquad c\sum_{n=1}^{\infty} a_n.$$

Sums and Differences of Series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then

- $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
- $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Integral Test

If f is continuous, positive, and decreasing on $[1, \infty)$ and $f(n) = a_n$, then

- $\sum_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)\,dx$ converges.
- $\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^{\infty} f(x)\,dx$ diverges.

Direct Comparison Test

Let a_n and b_n both be positive.

- If $a_n \leqslant b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $a_n \geqslant b_n$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Limit Comparison Test

Let a_n and b_n both be positive, with $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If L is positive and finite, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating Series Test

The series $\sum_{n=1}^{\infty} (-1)^n b_n$, $b_n > 0$, converges if $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} \leqslant b_n$ for all $n \geqslant 1$.

Absolute and Conditional Convergence

If $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent (and therefore converges). If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

Ratio Test

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \neq 1$.

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.

Root Test

Let $L = \lim_{n\to\infty} |a_n|^{1/n} \neq 1$.

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$, then $\sum a_n$ diverges.