Kinematic Equations Derivation

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0 Introduction

In classical physics, many natural phenomena can be modeled by a system of uniform acceleration. Physics students are taught *kinematics*, the study of motion, under such constant acceleration. The kinematics unit taught in most introductory physics courses covers five uniformly accelerated motion (UAM) equations, which students are expected to memorize. The derivations of such formulas, however, reinforces students' understanding of physical principles and is a key component of cultivating critical-thinking skills that students need for success in a creativity-based world.

In this article, we derive the five UAM equations, which model objects that exhibit uniform acceleration in one-dimensional motion. The equations are

$$v = v_0 + at \tag{1}$$

$$\Delta x = v_0 t + \frac{1}{2}at^2 \tag{2}$$

$$v^2 = v_0^2 + 2a\Delta x \tag{3}$$

$$\Delta x = \frac{1}{2}t(v_0 + v) \tag{4}$$

$$\Delta x = vt - \frac{1}{2}at^2\,, \tag{5}$$

where

$$\begin{aligned} t &= \text{time} \\ \Delta x &= \text{the displacement (change in position) of the object during the time } t \\ v_0 &= \text{the initial velocity of the object} \\ v &= \text{the final velocity of the object after the time } t \\ a &= \text{the constant acceleration of the object.} \end{aligned}$$

We use the international system of units (SI units) for these quantities. The unit of length is the meter (m), and the unit of time is the second (s). Then the units for velocity v are m/s, and the units for acceleration (a) are m/s^2.

Let us first establish defining expressions to connect position (x), velocity (v), and acceleration (a). Acceleration is the rate at which velocity changes, or the derivative of velocity:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}\,. \tag{6}$$

The velocity is the rate of change of position and the integral of acceleration, given by

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}\,. \tag{7}$$

Position is the derivative of velocity, or the second derivative of position:

$$x = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,. \tag{8}$$

The frame of reference in this article is the object moving in one dimension, starting from time $t = 0$ and initial position $x = x_0$. Displacement, velocity, and acceleration are *vector* quantities, whose sign depends on the direction. Kinematics requires the axis direction to be consistent; that is, the direction of positive x must remain the same.

Throughout this article, we will stick to some notational simplifications. We will drop all subscripts for final quantities—for example, we will use v for final velocity as opposed to v_f. Additionally, we will express displacement, $x - x_0$, as simply Δx. We also will refer to the time interval as t rather than Δt because we assume a frame of reference starting at $t = 0$. Moreover, we will drop vector notation to improve the readability of our equations.

It is important to distinguish *instantaneous rate of change* and *average rate of change*. The instantaneous rate of change of a function $f(x)$ is given by its derivative: $f'(x)$, a function that provides the rate of change of f at given instant x. Conversely, average rate of change represents a quantity of change over a given time interval. For example, Equation (7) can be re-expressed as

$$v = \frac{\Delta x}{\Delta t}\,,$$

which represents the average rate of change of x over the interval Δt.

Sections 1–5 feature calculus-based derivations that involve algebraic manipulations. In Section 6, we present geometric intuitions of the kinematic equations.

1 Derivation of Equation (1)

This section features the derivation of Equation (1):

$$v = v_0 + at\,. \tag{1}$$

We begin by taking Equation (6), $a = \mathrm{d}v/\mathrm{d}t$, and solve for v. Note that a is a constant. Solving this differential equation, we perform separation of variables to get

$$\begin{aligned}\int \mathrm{d}v &= \int a\,\mathrm{d}t\\ v &= at + C\,.\end{aligned}$$

To solve for the constant of integration C, we need a known fact. Initially, when $t = 0$, the velocity is v_0. We therefore have

$$v_0 = 0 + C \implies C = v_0\,.$$

Our solution is then

$$v = v_0 + at\,. \tag{1}$$

Another derivation involves manipulating the *average rate of change* of v :

$$a = \frac{\Delta v}{\Delta t}\,.$$

Note that $\Delta v = v - v_0$ and $\Delta t = t - 0 = t$. Thus,

$$\begin{aligned}a &= \frac{v - v_0}{t}\\ at &= v - v_0\\ v &= v_0 + at\,. \qquad (1)\end{aligned}$$

2 Derivation of Equation (2)

In this section, we derive Equation (2):

$$\Delta x = v_0 t + \frac{1}{2}at^2\,. \tag{2}$$

We begin by integrating Equation (1):

$$\int v\,\mathrm{d}t = \int v_0 + at\,\mathrm{d}t\,. \tag{9}$$

Because $\mathrm{d}x/\mathrm{d}t = v$, it follows that $\int v\,\mathrm{d}t = x$. Moreover, v_0 and a are constants. The integral then becomes

$$x = v_0 t + \frac{1}{2}at^2 + C\,.$$

At $t = 0$, the object is at its initial position, which we call x_0. Thus, at this instant we have

$$x_0 = 0 + 0 + C \implies C = x_0\,.$$

Therefore, the solution to Equation (9) is

$$\begin{aligned} x - x_0 &= v_0 t + \frac{1}{2}at^2 \\ \Delta x &= v_0 t + \frac{1}{2}at^2\,. \end{aligned} \tag{2}$$

3 Derivation of Equation (3)

Let us now derive Equation (3):

$$v^2 = v_0^2 + 2a\Delta x\,. \tag{3}$$

We consider the relationship described by Equation (6) and re-express the equation using the chain rule. We get

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{6}$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\,. \tag{10}$$

Note that $\mathrm{d}x/\mathrm{d}t = v$ (Equation (7)). Therefore, we obtain

$$\begin{aligned} a &= v\frac{\mathrm{d}v}{\mathrm{d}x} \\ a\,\mathrm{d}x &= v\,\mathrm{d}v \\ \int a\,\mathrm{d}x &= \int v\,\mathrm{d}v \\ ax &= \frac{1}{2}v^2 + C\,. \end{aligned} \tag{11}$$

To solve for C, we plug in an inital condition. Initially, we have $x = x_0$ and $v = v_0$. Thus,

$$ax_0 = \frac{1}{2}v_0^2 + C \implies C = ax_0 - \frac{1}{2}v_0^2\,.$$

Plugging this expression into Equation (11) yields the complete solution:

$$\begin{aligned} ax &= \frac{1}{2}v^2 + ax_0 - \frac{1}{2}v_0^2 \\ 2a(x - x_0) &= v^2 - v_0^2 \\ v^2 &= v_0^2 + 2a\Delta x\,. \end{aligned} \tag{3}$$

4 Derivation of Equation (4)

The derivation of

$$\Delta x = \frac{1}{2}t(v_0 + v) \tag{4}$$

only requires algebra. The average velocity on the interval of time t is

$$\overline{v} = \frac{1}{2}(v_0 + v)\,.$$

It then follows that the displacement is the product of average velocity and time:

$$\Delta x = \overline{v}t\,,$$

from which we derive

$$\Delta x = \frac{1}{2}t(v_0 + v)\,. \tag{4}$$

Simple and intuitive!

5 Derivation of Equation (5)

The derivation of

$$\Delta x = vt - \frac{1}{2}at^2 \tag{5}$$

solely requires a substitution. Working with Equation (1), we have

$$v = v_0 + at \implies v_0 = v - at\,,$$

which we substitute into

$$\Delta x = v_0 t + \frac{1}{2}at^2 \tag{2}$$

to get

$$\begin{aligned} \Delta x &= (v - at)t + \frac{1}{2}at^2 \\ &= vt - at^2 + \frac{1}{2}at^2 \\ \Delta x &= vt - \frac{1}{2}at^2\,. \end{aligned} \tag{5}$$

6 Geometric Derivations

In this section, let us present a velocity–time graph to base our derivations. Consider Figure 1, in which the graph of velocity, v, as a function of time is shown. Due to the constant acceleration, velocity increases linearly, so the graph is a straight line of slope a and vertical-axis intercept v_0. Thus, the equation of the line is

$$v = v_0 + at\,. \tag{1}$$

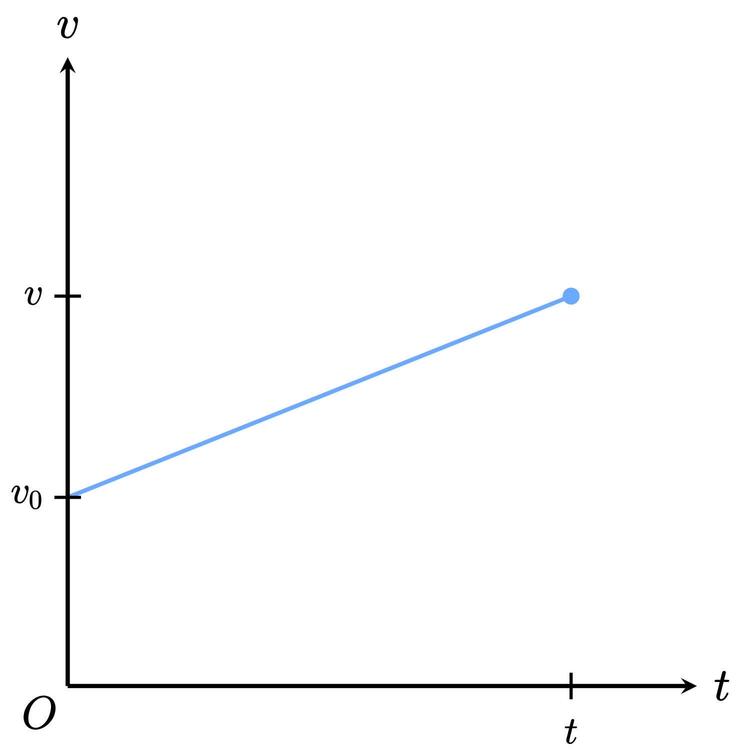

Figure 1: The graph of velocity, v, as a function of time.

Because $x = \int v\,\mathrm{d}t$, displacement is the area under the velocity–time graph. The graph is linear, so we can use areas of geometric figures to derive expressions for Δx. One expression for this area is the sum the areas of the rectangle and triangle under the line. See Figure 2, in which these regions under the graph of the velocity function are shaded.

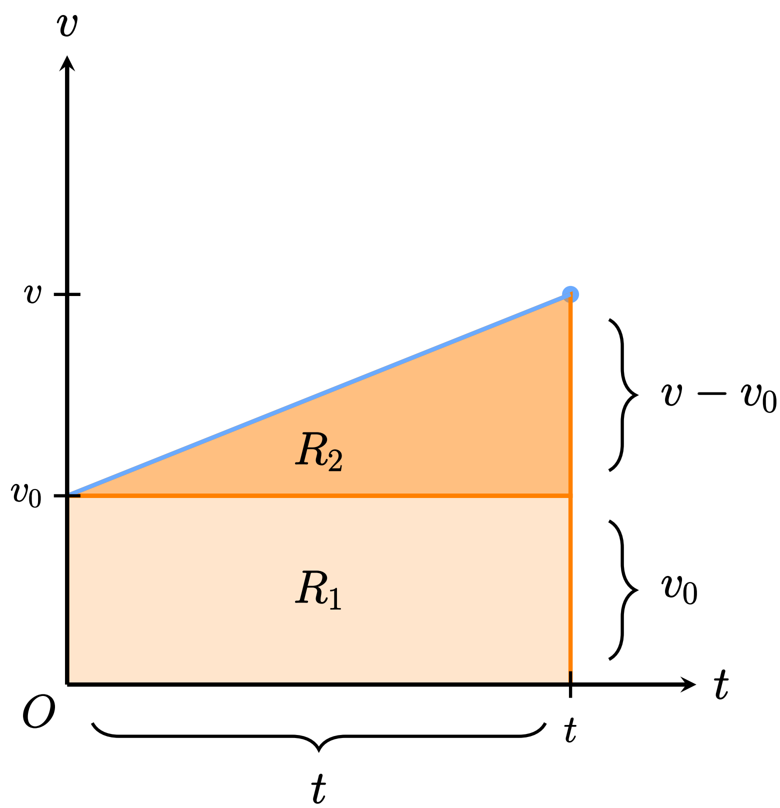

Figure 2: The sum of the areas of R_1 and R_2 yields the displacement, Δx.

The region R_1 is a rectangle of area $v_0 t$, and region R_2 is a triangle of area $\frac{1}{2}t(v - v_0)$. Thus, the sum $R_1 + R_2$ represents Δx. Therefore, we have

$$\begin{aligned} \Delta x &= R_1 + R_2 \\ &= v_0 t + \frac{1}{2}t(v - v_0) \qquad (12) \\ \Delta x &= \frac{1}{2}t(v_0 + v)\,. \qquad (4) \end{aligned}$$

Alternatively, from Equation (1), we have

$$v = v_0 + at \implies v - v_0 = at\,.$$

Thus, substituting at for $v - v_0$ in Equation (12) gives

$$\Delta x = v_0 t + \frac{1}{2} t(at)$$
$$\Delta x = v_0 t + \frac{1}{2} at^2 \,. \tag{2}$$

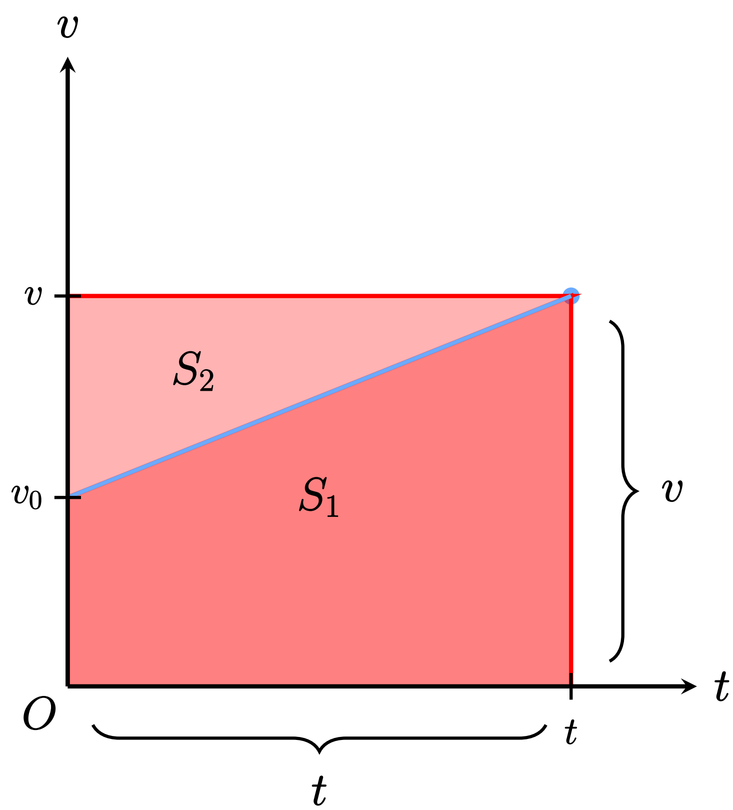

Figure 3: The graph of v is shown, where the region of area S_1 represents Δx and $S_1 = vt - S_2$.

Another expression for Δx is shown in Figure 3, in which two regions of areas S_1 and S_2 are shaded. Notice that $\Delta x = S_1$ and $S_1 = vt - S_2$. Moreover,

$$S_2 = \frac{1}{2}(t)(v - v_0) \,.$$

Thus, we have

$$\begin{aligned} S_1 &= vt - S_2 \\ \Delta x &= vt - \frac{1}{2}t(v - v_0) \end{aligned} \tag{13}$$

Similar to our process in deriving Equation 2, we substitute $at = v - v_0$ to get

$$\begin{aligned} \Delta x &= vt - \frac{1}{2}t(at) \\ \Delta x &= vt - \frac{1}{2}at^2 . \end{aligned} \tag{5}$$