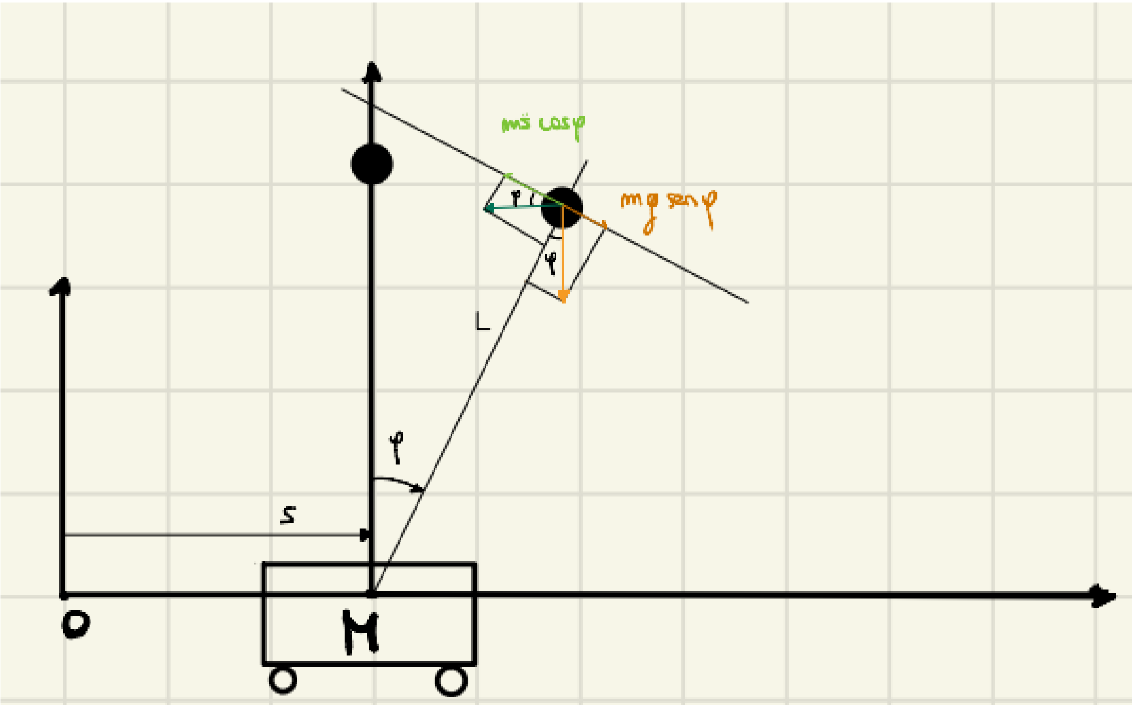
ASSIGNMENT 7

BY

Andrea Marchesi and Valerio Auriemma

Consider the model of an inverted pendulum on a cart described by the equations:

$$\begin{cases} \underbrace{M\ddot{s}}_{\text{inertia}} + \underbrace{F\dot{s}}_{\text{damping}} - \underbrace{\mu}_{\text{control}} = \underbrace{d_1}_{\text{disturbance}} & \leftarrow \text{FORCE EQUILIBRIUM at pivot} \\ \underbrace{\ddot{\phi}}_{\text{inertia}} - \underbrace{\frac{g}{L}\sin\phi}_{\text{gravity}} + \underbrace{\frac{1}{L}\ddot{s}\cos(\phi)}_{\text{base acceleration}} = 0 & \leftarrow \text{TORQUE EQUILIBRIUM at pivot} \end{cases}$$



s(t) is the displacement of the pivot,

φ(t) is the angular rotation of the pendulum,

μ(t) is the external force exerted on the cart

d1(t) is an external disturbance acting on the cart.

M = 1 kg is the mass of the cart,

L = 1 m is the effective pendulum length,

F = 1 [kg/s] is a friction coefficient,

g = 9.81 [m/s^2] is the gravitational acceleration.

In this model the afterwards assumptions have been made:

- no rotational friction on the pivot, it would be $[-\gamma\dot{\phi}]$
- in force equilibrium, pendulum top mass position contribution $[-m\,g\cos(\phi)\sin(\phi)]$ can be ignored
- pendulum (top weight + rod) mass and inertia can be ignored

1 A1)

Compute all the equilibrium points of the system for μ(t) = d1(t) = 0.

the system can be re-written after some substitutions:

$$x=\begin{bmatrix} s(t)\\ \dot{s}(t)\\ \phi(t)\\ \dot{\phi}(t)\end{bmatrix}=\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}\rightarrow \dot{x}=\begin{bmatrix} \dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix} x_2\\ -\frac{x_2\,F-u_1-d_1}{M}\\ x_4\\ \frac{\sin(x_3)\,M\,g+x_2\cos(x_3)\,F-u_1\cos(x_3)-d_1\cos(x_3)}{L\,M}\end{bmatrix}$$

equilibrium points will be found for every x that respects $\dot{x}=0$, with $\mu=0$ and $d_1=0$:

so:

$$\begin{cases} x_2=0\\ -\frac{-d_1+x_2\,F-u(1)}{M}=0\rightarrow x_2=\frac{u(1)+d_1}{F}=0\rightarrow \text{ok}\\ x_4=0\rightarrow \text{ok}\\ \frac{\sin(x_3)\,M\,g-\cos(x_3)\,d_1+x_2\cos(x_3)\,F-u(1)\cos(x_3)}{L\,M}=0\rightarrow \frac{\sin(x_3)\,M\,g}{L\,M}=0\rightarrow x_3=0+k\,\pi\quad \text{with}\, k\in\mathbb{N}\end{cases}$$

the equilibrium will be with $\dot{s}=0$ and $\phi=0+k\,\pi$ with $k\in\mathbb{N}$, there is no boundary on s.

$$\dot{x}_{(x_0)}=F(x_o)=0\quad\rightarrow\quad x_o=\begin{pmatrix}\mathbb{R}\\ 0\\ 0+n\,\pi, n\in\mathbb{N}\\ 0\end{pmatrix}, \mu=0, d_1=0$$

2 A2)

Write the equations of the linearised system around the equilibrium $\phi=s=\dot{\phi}=\dot{s}=0$.

The equilibrium point $x_0=[0,0,0,0]^T$ is where the linearized system will be evaluated.

Since $F(x_0)=0$ then $F(x_0)+J|_{x_o}\delta_x\Rightarrow \delta x'=J|_{x_o}\delta_x$.

$$\nabla(F)\Bigg|_{x_o}=\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4}\\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4}\\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4}\\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_3} & \frac{\partial F_4}{\partial x_4}\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & -\frac{F}{M} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{\cos(x_3)\,F}{L\,M} & \frac{\cos(x_3)\,M\,g+(-x_2\,F+u_1+d_1)\sin(x_3)}{L\,M} & 0\end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & -\frac{F}{M} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0\end{pmatrix}$$

from which the linearized system around the equilibrium point x=0 can be written:

$$\dot{X}_{\text{linearized}}=\nabla(F)|_{x0}\,X+\underbrace{\begin{bmatrix}\frac{\partial F_1}{\partial\mu}\\ \frac{\partial F_2}{\partial\mu}\\ \frac{\partial F_3}{\partial\mu}\\ \frac{\partial F_4}{\partial\mu}\end{bmatrix}\Bigg|_{x0}}_{B}\mu+\underbrace{\begin{bmatrix}\frac{\partial F_1}{\partial d_1}\\ \frac{\partial F_2}{\partial d_1}\\ \frac{\partial F_3}{\partial d_1}\\ \frac{\partial F_4}{\partial d_1}\end{bmatrix}\Bigg|_{x0}}_{P}d_1=\begin{bmatrix} x_2\\ -\frac{x_2\,F}{M}+\frac{u_1}{M}+\frac{d_1}{M}\\ x_4\\ \frac{x_3\,g}{L}+\frac{x_2\,F}{L\,M}-\frac{u_1}{L\,M}-\frac{d_1}{L\,M}\end{bmatrix}$$

3 A3)

Express the obtained linear dynamics in the standard state space form:

$\dot{x}=A\,x+B\,u+P d$

$y = C x$

where $x(t) = \begin{bmatrix} s(t) \\ \dot{s}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix}$, $u(t) = \mu(t)$, $y(t) = \begin{bmatrix} s(t) \\ \phi(t) \end{bmatrix}$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{bmatrix}}_{P} d_1$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} x$$

4 A4)

Show that the pair (A; B) is controllable.

the pair (A; B) is controllable if P rank is full.

$P = (B, \mathrm{AB}, A^2 B, A^3 B)$

$$\mathrm{P} = \begin{bmatrix} 0 & \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} \\ \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} & -\frac{F^3}{M^4} \\ 0 & 0 & \frac{F}{L\,M^2} & -\frac{F^2}{L\,M^3} \\ 0 & \frac{F}{L\,M^2} & -\frac{F^2}{L\,M^3} & \frac{F g}{L^2 M^2} + \frac{F^3}{L\,M^4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 10.81 \end{bmatrix} \rightarrow \mathrm{rank}(P) = 4$$

so around the equilibrium point x=0 the linearized system is controllable.

5 A5)

Consider the linear system in part A3). Assume that d1 is an unknown constant. The control law should be designed in such a way that the effect of the disturbance d1 is asymptotically rejected, and the first output $s(t)$ asymptotically tracks the reference signal $d_2(t) = \alpha \sin(\omega t)$.

Pose this problem as a regulator problem. (Note that this is a non-standard regulation problem, since the regulation condition involves only part of the measured output.)

To pose this problem in the the regulator form the system shall be defined as follows:

$$\begin{cases} \dot{x} = A\,x + B\,u + P d \\ y = C\,x \\ \dot{d} = S\,d \\ e = C_e\,x + Q\,d \end{cases}$$

$$x(t) = \begin{bmatrix} s(t) \\ \dot{s}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix} \qquad u(t) = \mu(t) \qquad y(t) = \begin{bmatrix} s(t) \\ \phi(t) \end{bmatrix}$$

The exogeneous signals have the afterward form:

$$d_1 = \text{const} \rightarrow \dot{d}_1 = 0 \rightarrow \dot{d}_1 = S_1\, d_1 \,\text{with}\, S_1 = [0] \rightarrow d_1(t) = d_1(0) \quad \forall t \geqslant 0$$

$$d_2 = a\sin(\omega\, t) \rightarrow \dot{\bar{d}}_2 = S_2\, \bar{d}_2 \,\text{with}\, S_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \bar{d}_2(0) = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

using the dummy signal $\tilde{d}_2$

$$\dot{d} = S\, d \rightarrow S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & & \\ 0 & & S_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix} \quad d(t) = \begin{bmatrix} d_1 \\ d_2 \\ \tilde{d}_2 \end{bmatrix} \qquad d(0) = \begin{bmatrix} \text{const} \\ 0 \\ \alpha \end{bmatrix}$$

$$\dot{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix} d$$

$$e(t) = x_1(t) - d_2(t) = \underbrace{[\, 1 \;\; 0 \;\; 0 \;\; 0 \,]}_{C_e}\, x + \underbrace{[\, 0 \;\; -1 \;\; 0 \,]}_{Q}\, d$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{L\,M} & 0 & 0 \end{bmatrix}}_{P} d$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} x$$

6 A6)

Consider the regulator problem determined in part A5). Show that the problem is solvable by means of a full information control law.

We want to determine if a full information control law $u = K\, x + L\, d$

can guarantee

- (S) undisturbed system stability

- (R) complete system convergence to zero of error $e(t)$

Firstly hypothesis shall be verified:

1. The matrix S of the exosystem has all eigenvalues with non-negative real part

   $\det(\lambda\, I - S) = 0 \rightarrow \lambda = [-\mathrm{i}\,\omega, \mathrm{i}\,\omega, 0] \leftarrow \text{ok}$

2. The system (1) with d = 0 is reachable.

   verified with controllability in task A4

FBI equation and Theorem:

$\exists K, L$: (S) and (R) are verified $\Leftrightarrow \exists \Pi, \Gamma$: $\begin{cases} \Pi S = A\Pi + B\Gamma + P \\ 0 = C\Pi + Q \end{cases}$

from this we can modify the control law in $u = Kx + (\Gamma - K\Pi)\, d$

Hautus Lemma:

FBI eq. have solution in $\Pi, \Gamma \quad \forall P, Q \Leftrightarrow \operatorname{rank}\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = n + p \qquad \forall s \in \sigma(S)$

$\sigma(S) = [-\mathrm{i}\,\omega, \mathrm{i}\,\omega, 0]$

$$s_1 = -i\omega \rightarrow \operatorname{rank}\begin{bmatrix} -i\omega I - A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{i}\,\omega & -1 & 0 & 0 & 0 \\ 0 & \frac{F}{M} - \mathrm{i}\,\omega & 0 & 0 & \frac{1}{M} \\ 0 & 0 & -\mathrm{i}\,\omega & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & -\mathrm{i}\,\omega & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 5 \leftarrow \text{ok}$$

$$s_2 = i\omega \rightarrow \operatorname{rank}\begin{bmatrix} i\omega I - A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{i}\,\omega & -1 & 0 & 0 & 0 \\ 0 & \mathrm{i}\,\omega + \frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & \mathrm{i}\,\omega & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & \mathrm{i}\,\omega & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 5 \leftarrow \text{ok}$$

$$s_3 = 0 \rightarrow \operatorname{rank}\begin{bmatrix} -A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & 0 & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = 5 \leftarrow \text{ok}$$

So the solution exists.

7 A7)

TASK:

Consider the regulator problem determined in part A5). Show that the problem is solvable by means of an error feedback control law. (Since the problem is not in the standard form discussed in the lectures, you should use the output y(t) to verify the observability condition and the error signal e(t) = s(t) - d2(t) to assess the regulation requirement.).

An error variable is defined as follow:

$$e_0 = \begin{bmatrix} e \\ \phi \end{bmatrix} = \begin{bmatrix} s - d_2 \\ \phi \end{bmatrix} = Cx + Q_0\, d \qquad \text{with } Q_0 = \begin{bmatrix} & Q & \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$e_0(t)$ is a signal that contains both the signals of $y(t)$ and will be used as input for the estimtor, which is designed to evaluate $x(t)$ with the variable $\xi(t)$ and $d(t)$ with the variable $\delta(t)$.

The dynamic controller will have the form that follows:

$$\begin{cases} \dot{\chi}(t) = F\chi(t) + G\underbrace{e_0(t)}_{\text{input}} \\ u(t) = H\chi(t) \end{cases} \quad \text{with } \chi(t) = \begin{bmatrix} \xi(t) \\ \delta(t) \end{bmatrix}$$

the control law has to respect the asymptotical stability and convergence to zero of the error variable.

STABILITY:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ \dot{\chi}(t) = F\chi(t) + G\,e_0(t) \end{cases} \rightarrow \text{with}\, \mathrm{Re}(\lambda) \in \mathbb{R}^+ \,\text{from}\, \sigma(A) \ \text{ and } \ \sigma(F)$$

ERROR CONVERGENCE TO ZERO:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,H\,\chi + P d \\ \dot{\chi}(t) = F\chi(t) + G\,(C\,x + Q_0\,d) \\ y = C\,x \\ e = C_e x + Q\,d \\ \dot{d} = S\,d \end{cases} \rightarrow \text{with}\, \lim_{t\rightarrow\infty} e(t) = 0$$

with F,H,G and L defined as follow:

$$F = \begin{bmatrix} A + G_1 C + B K & P + G_1 Q_0 + B L \\ G_2 C & S + G_2 Q_0 \end{bmatrix} \quad H = [\, K \ \ L \,] \quad G = -\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \qquad L = \Gamma - K\,\Pi$$

to verify estimator effectiveness it is necessary to check the observability of the couple (A_0, C_0)

$$(A_0, C_0) = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, [\, C \ \ Q_0 \,] \right) \rightarrow (A_0, C_0)\,\text{is observable} \Leftrightarrow \text{rank}\left(\begin{bmatrix} C_0 \\ C_0\,A_0 \\ \ldots \\ C_0\,A_0^{n-1} \end{bmatrix} \right) = n$$

$$\text{rank}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^2}{L\,M^2} & 0 & \frac{g}{L} & \frac{F}{L\,M^2} & 0 & 0 \end{pmatrix} = 7 \leftarrow \text{ok}$$

if instead of $e_0 = \begin{bmatrix} e \\ \phi \end{bmatrix}$ it is used as input only $e = C_e\,x + Q\,d$, the rank of the observability matrix will be 5, causing an unfeasible control.

To check the regulation task it is considered the error signal $e = C_e\,x + Q\,d = s - d_2$.

Due to FBI theorem:

$$\exists F, G, H\text{: respecting stability and error tracking} \Leftrightarrow \exists \Gamma, \Pi\text{: } \begin{cases} \Pi\,S = A\,\Pi + B\,\Gamma + P \\ 0 = C_e\,\Pi + Q \end{cases}$$

And due to Hautus Lemma:

$$\exists \Gamma, \Pi\text{: } \begin{cases} \Pi\,S = A\,\Pi + B\,\Gamma + P \\ 0 = C_e\,\Pi + Q \end{cases} \quad \forall P, Q \quad \Leftrightarrow \text{rank}\left(\begin{bmatrix} s\,I - A & B \\ C_e & 0 \end{bmatrix} \right) = n + p \quad \forall s \in \sigma(s)$$

this has already been verified in the previous point, so stability and error tracking conditions are respected.

8 B1)

TASK:

Assume d1(t) is a square wave of amplitude 0.5 and period 50 [s], $\alpha = 1$ and $\omega = 0.1$. Design a full information control law solving the regulator problem posed in part A5).

To design a full information control law a solution to $\begin{cases} \Pi S = A\Pi + B\Gamma + P \\ 0 = C_e \Pi + Q \end{cases}$ shall be found.

Observing matrix sizes it can be said that: $\Pi = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} \end{bmatrix}$ and $\Gamma = [\, \gamma_1 \;\; \gamma_2 \;\; \gamma_3 \,]$

from the second equation:

$$[\, 0 \;\; 0 \;\; 0 \,] = [\, 1 \;\; 0 \;\; 0 \;\; 0 \,] \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} \end{bmatrix} + [\, 0 \;\; -1 \;\; 0 \,] \rightarrow \pi_{1,1} = 0 \quad \pi_{1,2} = 1 \quad \pi_{1,3} = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{bmatrix} [\, \gamma_1 \;\; \gamma_2 \;\; \gamma_3 \,] + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{L\,M} & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\mathrm{pi}_{2,3}\,\omega & \mathrm{pi}_{2,2}\,\omega \\ 0 & -\mathrm{pi}_{3,3}\,\omega & \mathrm{pi}_{3,2}\,\omega \\ 0 & -\mathrm{pi}_{4,3}\,\omega & \mathrm{pi}_{4,2}\,\omega \end{pmatrix} = \begin{pmatrix} \mathrm{pi}_{2,1} & \mathrm{pi}_{2,2} & \mathrm{pi}_{2,3} \\ -\frac{\mathrm{pi}_{2,1}\,F}{M} & -\frac{\mathrm{pi}_{2,2}\,F}{M} & -\frac{\mathrm{pi}_{2,3}\,F}{M} \\ \mathrm{pi}_{4,1} & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L} + \frac{\mathrm{pi}_{2,1}\,F}{L\,M} & \frac{\mathrm{pi}_{3,2}\,g}{L} + \frac{\mathrm{pi}_{2,2}\,F}{L\,M} & \frac{\mathrm{pi}_{3,3}\,g}{L} + \frac{\mathrm{pi}_{2,3}\,F}{L\,M} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \frac{\gamma_1}{M} & \frac{\gamma_2}{M} & \frac{\gamma_3}{M} \\ 0 & 0 & 0 \\ -\frac{\gamma_1}{L\,M} & -\frac{\gamma_2}{L\,M} & -\frac{\gamma_3}{L\,M} \end{pmatrix} + $$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\mathrm{pi}_{2,3}\,\omega & \mathrm{pi}_{2,2}\,\omega \\ 0 & -\mathrm{pi}_{3,3}\,\omega & \mathrm{pi}_{3,2}\,\omega \\ 0 & -\mathrm{pi}_{4,3}\,\omega & \mathrm{pi}_{4,2}\,\omega \end{pmatrix} = \begin{pmatrix} \mathrm{pi}_{2,1} & \mathrm{pi}_{2,2} & \mathrm{pi}_{2,3} \\ -\frac{\mathrm{pi}_{2,1}\,F}{M} + \frac{\gamma_1}{M} + \frac{1}{M} & \frac{\gamma_2}{M} - \frac{\mathrm{pi}_{2,2}\,F}{M} & \frac{\gamma_3}{M} - \frac{\mathrm{pi}_{2,3}\,F}{M} \\ \mathrm{pi}_{4,1} & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L} + \frac{\mathrm{pi}_{2,1}\,F}{L\,M} - \frac{\gamma_1}{L\,M} - \frac{1}{L\,M} & \frac{\mathrm{pi}_{3,2}\,g}{L} + \frac{\mathrm{pi}_{2,2}\,F}{L\,M} - \frac{\gamma_2}{L\,M} & \frac{\mathrm{pi}_{3,3}\,g}{L} + \frac{\mathrm{pi}_{2,3}\,F}{L\,M} - \frac{\gamma_3}{L\,M} \end{pmatrix}$$

from which $\mathrm{pi}_{2,1} = 0 \quad \mathrm{pi}_{2,2} = 0 \quad \mathrm{pi}_{2,3} = \omega \quad \mathrm{pi}_{4,1} = 0$, those values are substituted:

$$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \\ 0 & -\mathrm{pi}_{3,3}\,\omega & \mathrm{pi}_{3,2}\,\omega \\ 0 & -\mathrm{pi}_{4,3}\,\omega & \mathrm{pi}_{4,2}\,\omega \end{pmatrix} = \begin{pmatrix} 0 & 0 & \omega \\ \frac{\gamma_1}{M} + \frac{1}{M} & \frac{\gamma_2}{M} & \frac{\gamma_3}{M} - \frac{F\,\omega}{M} \\ 0 & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L} - \frac{\gamma_1}{L\,M} - \frac{1}{L\,M} & \frac{\mathrm{pi}_{3,2}\,g}{L} - \frac{\gamma_2}{L\,M} & \frac{F\,\omega}{L\,M} + \frac{\mathrm{pi}_{3,3}\,g}{L} - \frac{\gamma_3}{L\,M} \end{pmatrix} \rightarrow$$

$$\frac{\gamma_1}{M} + \frac{1}{M} = 0; \quad \frac{\gamma_2}{M} = -\omega^2 \quad \frac{\gamma_3}{M} - \frac{F\,\omega}{M} = 0 \rightarrow \gamma_1 = -1; \quad \gamma_2 = -M\,\omega^2 \quad \gamma_3 = F\,\omega$$

$$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \\ 0 & -\mathrm{pi}_{3,3}\,\omega & \mathrm{pi}_{3,2}\,\omega \\ 0 & -\mathrm{pi}_{4,3}\,\omega & \mathrm{pi}_{4,2}\,\omega \end{pmatrix} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \\ 0 & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L} & \frac{\omega^2}{L}+\frac{\mathrm{pi}_{3,2}\,g}{L} & \frac{\mathrm{pi}_{3,3}\,g}{L} \end{pmatrix} \to \begin{cases} -\mathrm{pi}_{3,3}\,\omega = \mathrm{pi}_{4,2} \\ \mathrm{pi}_{3,2}\,\omega = \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L} = 0 \to \mathrm{pi}_{3,1} = 0 \\ -\mathrm{pi}_{4,3}\,\omega = \frac{\omega^2}{L} + \frac{\mathrm{pi}_{3,2}\,g}{L} \\ \mathrm{pi}_{4,2}\,\omega = \frac{\mathrm{pi}_{3,3}\,g}{L} \end{cases} \to$$

$$\begin{cases} \mathrm{pi}_{3,3} = \mathrm{pi}_{4,2} = 0 \\ \mathrm{pi}_{3,2}\,\omega = \mathrm{pi}_{4,3} \\ -\mathrm{pi}_{3,2}\,\omega^2 = \frac{\omega^2}{L} + \frac{\mathrm{pi}_{3,2}\,g}{L} \to \mathrm{pi}_{3,2}\left(\omega^2 + \frac{g}{L}\right) = -\frac{\omega^2}{L} \to \mathrm{pi}_{3,2} = -\frac{\omega^2}{\omega^2 L + g} \to \mathrm{pi}_{4,3} = -\frac{\omega^3}{\omega^2 L + g} \end{cases}$$

Afterward Π and Γ:

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -\frac{\omega^2}{L\,\omega^2+g} & 0 \\ 0 & 0 & -\frac{\omega^3}{L\,\omega^2+g} \end{pmatrix}, \Gamma = [-1, -M\omega^2, F\omega]$$

The control law will be: $u = K\,x + (\Gamma - K\,\Pi)\,d$ where K matrix will let $(A + B\,K)$ eigenvalues to be assigned, this is possible since (A,B) is controllable, and K matrix values can be assigned with Ackermann or Mitter.

In this case Ackermann will be applied:

We know that $\sigma(A) = \left[-\sqrt{\frac{g}{L}}, \sqrt{\frac{g}{L}}, -\frac{F}{M}, 0\right] = [-3.13, 3.13, -1, 0]$

$p_A(\lambda) = \lambda\left(\lambda + \frac{F}{M}\right)\left(\lambda - \sqrt{\frac{g}{L}}\right)\left(\lambda + \sqrt{\frac{g}{L}}\right) = \lambda^4 + 1.0\,\lambda^3 - 9.81\,\lambda^2 - 9.81\,\lambda$

the $\sigma(\text{desired}) = [-1, -2, -3, -4]$ (eigenvalues must differ in Ackermann method)

$p_{\text{des}}(\lambda) = (\lambda+1)\,(\lambda+2)\,(\lambda+3)\,(\lambda+4) = \lambda^4 + 10\,\lambda^3 + 35\,\lambda^2 + 50\,\lambda + 24$

the control matrix K will be: $K = -\breve{\pi}^T p_{\text{des}}(A)$

where

$\breve{\pi}^T = [0,0,0,1]P^{-1} = [0,0,0,1]\,(B, \mathrm{AB}, A^2B, A^3B)^{-1} = [\ -0.10\ \ 0\ \ -0.10\ \ 0\ ]$

$\breve{\pi}^T A = (\ 0.0\ \ -0.1019367991845056\ \ 0.0\ \ -0.1019367991845056\ )$

$\breve{\pi}^T A^2 = (\ 0.0\ \ 0.0\ \ -1.0\ \ 0.0\ )$

$\breve{\pi}^T A^3 = (\ 0.0\ \ 0.0\ \ 0.0\ \ -1.0\ )$

$\breve{\pi}^T A^4 = (\ 0.0\ \ -1.0\ \ -9.81\ \ 0.0\ )$

$K = -(\breve{\pi}^T A^4 + 10\,\breve{\pi}^T A^3 + 35\,\breve{\pi}^T A^2 + 50\,\breve{\pi}^T A + 24\,\breve{\pi}^T) = (\ 2.45\ \ 6.10\ \ 47.26\ \ 15.10\ )$

And from calculations it can be observed that $\sigma(A + B\,F) = [-2, -1, -4, -3]$.

$u = K\,x + (\Gamma - K\,\Pi)\,d$

$$u = (\ 2.45\ \ 6.10\ \ 47.26\ \ 15.10\ )\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \left(\ -1.0\ \ \frac{47.26\omega^2}{\omega^2+9.81} - \omega^2 - 2.45\ \ \frac{15.10\,\omega^3}{\omega^2+9.81} - 5.10\,\omega\ \right)\begin{pmatrix} d_1 \\ d_2 \\ \tilde{d}_2 \end{pmatrix}$$

9 B2)

TASK:

B2) Display plots of y(t) and u(t) for x(0) = 0. These plots should show that the regulation goal has been achieved. Discuss why the disturbance d1(t) does not affect the output y(t).

Simulation time: 100 [s]

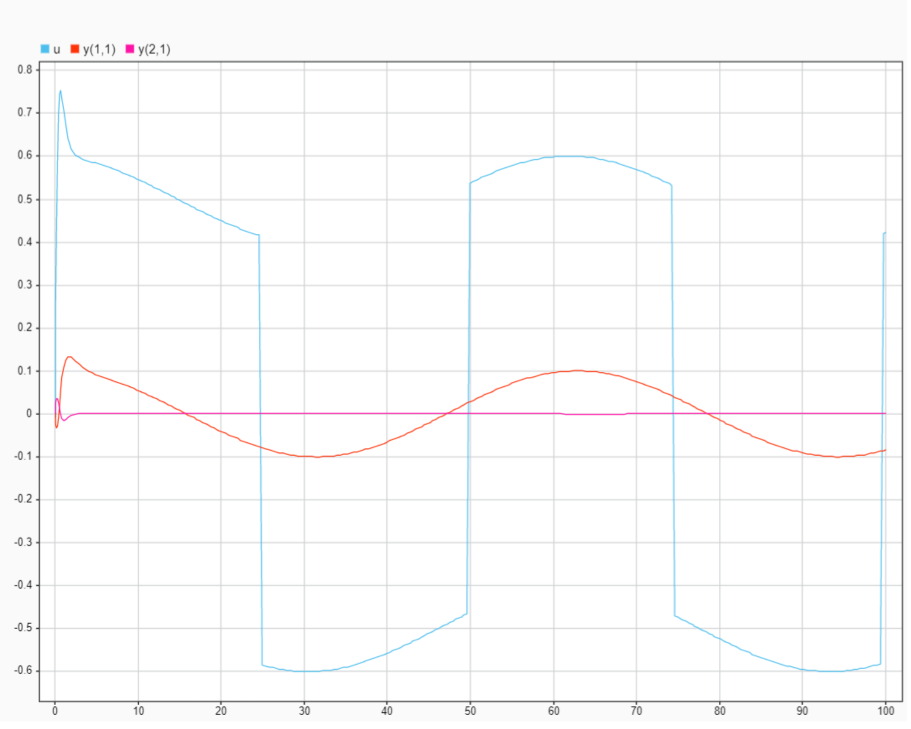


Figure 1. Linearized System, $u(t)$, $y_1(t) = s(t)$, $y_2(t) = \phi(t)$

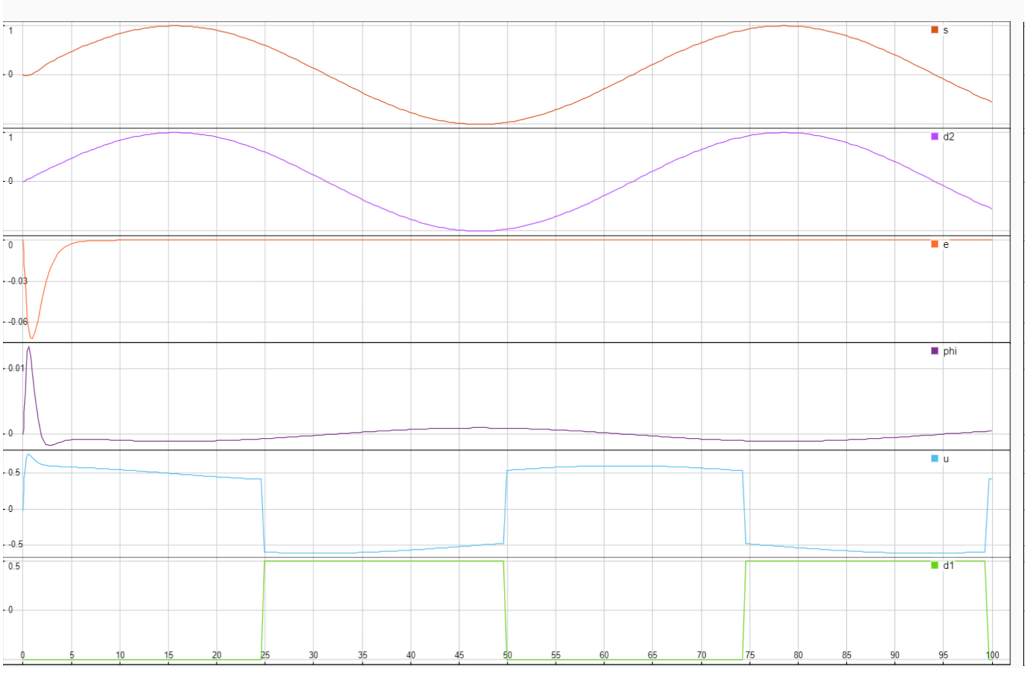


Figure 2. Linearized System, $y_1 = s$; $d_2 = \text{ref. signal}$; $e = \text{track error}$; $y_2 = \phi$; $u = \text{control}$; $d_1 = \text{actuator noise}$

As it can be observed form the images above the tracking error is regulated to zero and is not affected by the noise on the actuator. As a matter of fact the control signal compensate the noise by opposing its value, which is directly measured, and, from the knowledge of the reference signal and of the full status, track to zero the error.

10 B3)

TASK:

Assume that the full information control law designed in part B1) is used to control the nonlinear model of the pendulum. Display plots of y(t) and u(t) for the nonlinear system, from the initial state x(0) = 0. Is the regulation goal achieved?

Discuss why the disturbance d1(t) does not affect the output y(t) and why the regulation goal is only approximately achieved.

The plot represented above with $u(t)$ and $y(t)$ has not been represented, it shows the exact same pattern of the linearized one

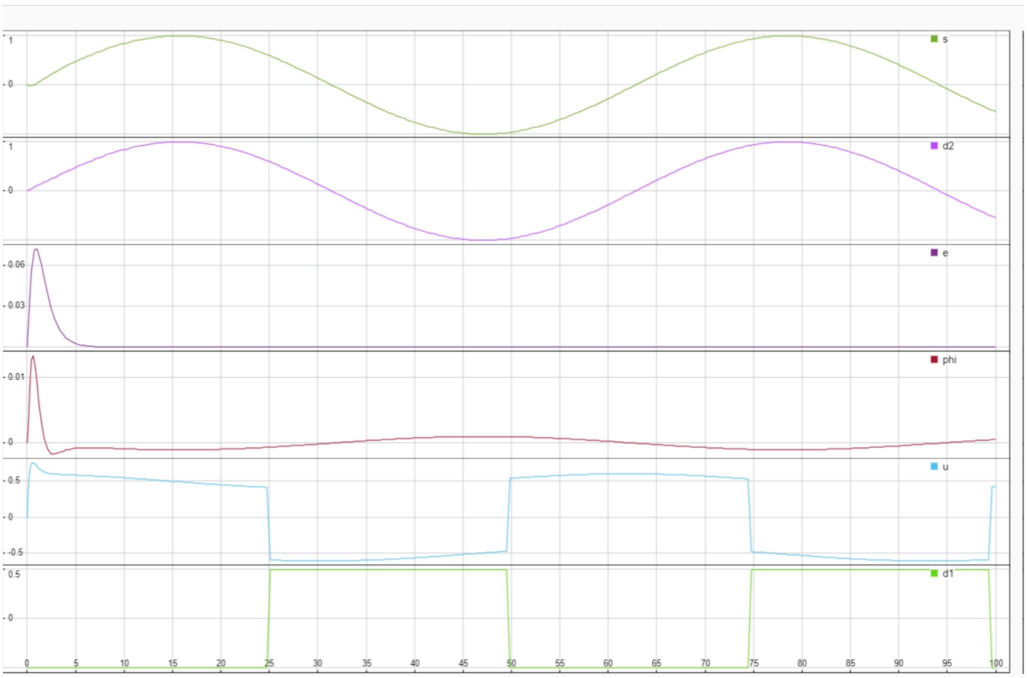


Figure 3. Non Linear System, $y_1 = s$; $d_2 = \text{ref. signal}$; $e = \text{track error}$; $y_2 = \phi$; $u = \text{control}$; $d_1 = \text{actuator noise}$

11 B4)

TASK:

Repeat steps B1) to B3) with ω = 1 and ω = 10. Discuss the obtained results.

afterward the simulations with $\omega = 1\,[\text{rad}/s]$

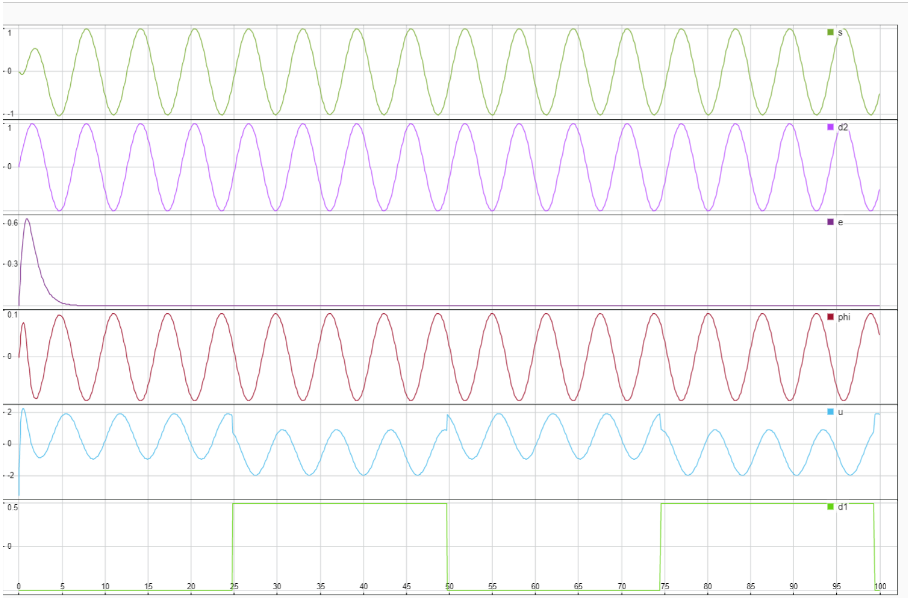


Figure 4. Linearized System, $\omega = 1[\mathrm{rad}/s]$; $y_1 = s$; $d_2 = \text{ref. signal}$; $e = \text{track error}$; $y_2 = \phi$; $u = \text{control}$; $d_1 = \text{actuator noise}$

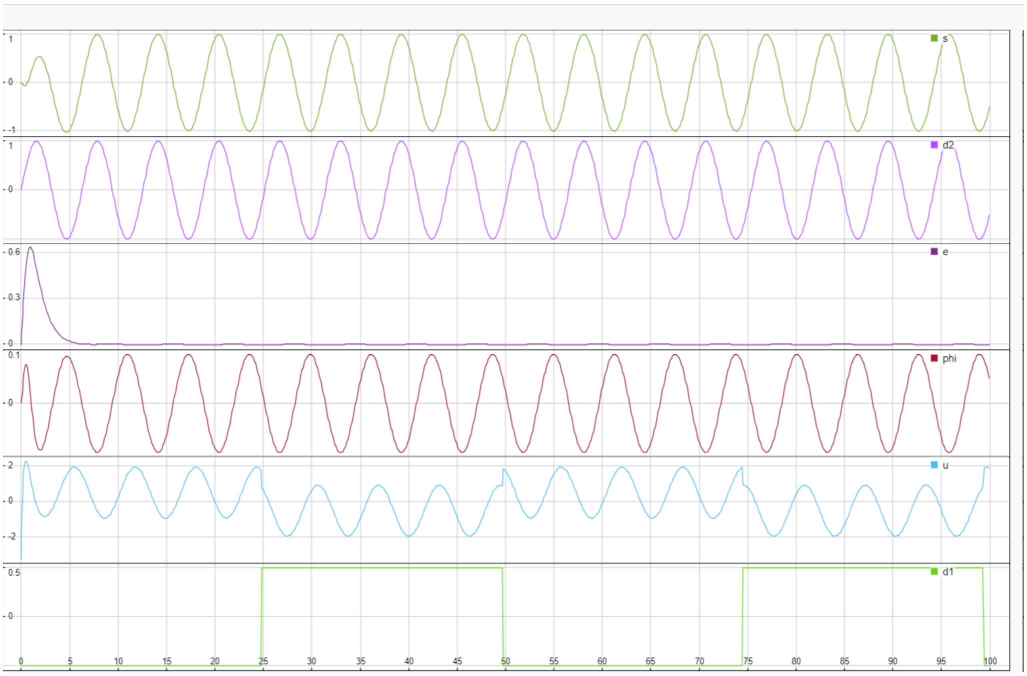
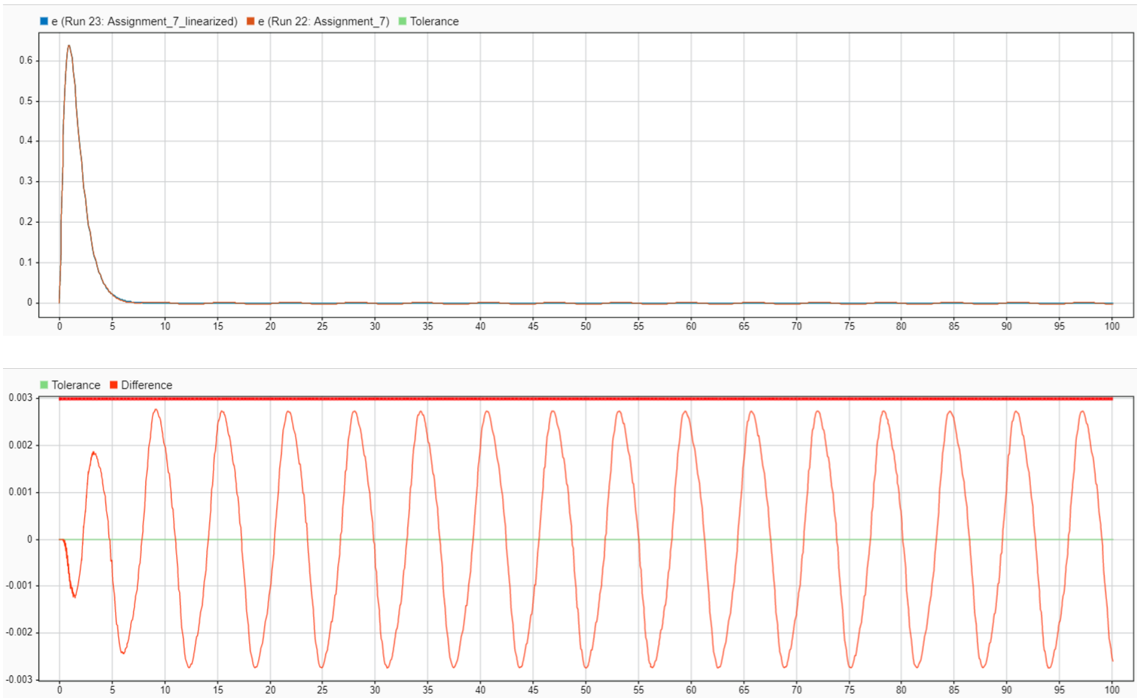
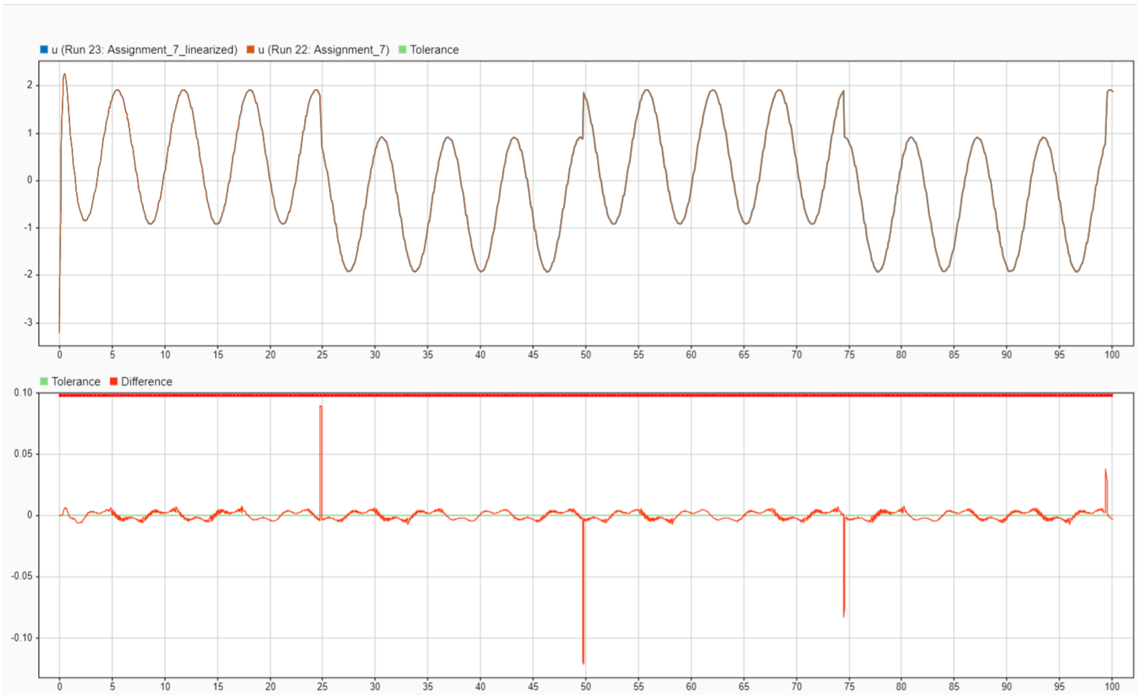


Figure 5. Non Linear System, $\omega = 1[\mathrm{rad}/s]$; $y_1 = s$; $d_2 = \text{ref. signal}$; $e = \text{track error}$; $y_2 = \phi$; $u = \text{control}$; $d_1 = \text{actuator noise}$

errore:





afterward the simulations with $\omega = 10\,[\mathrm{rad}/s]$,

Simulation time has been setted to 5.5 [s] since the non linear system diverges.

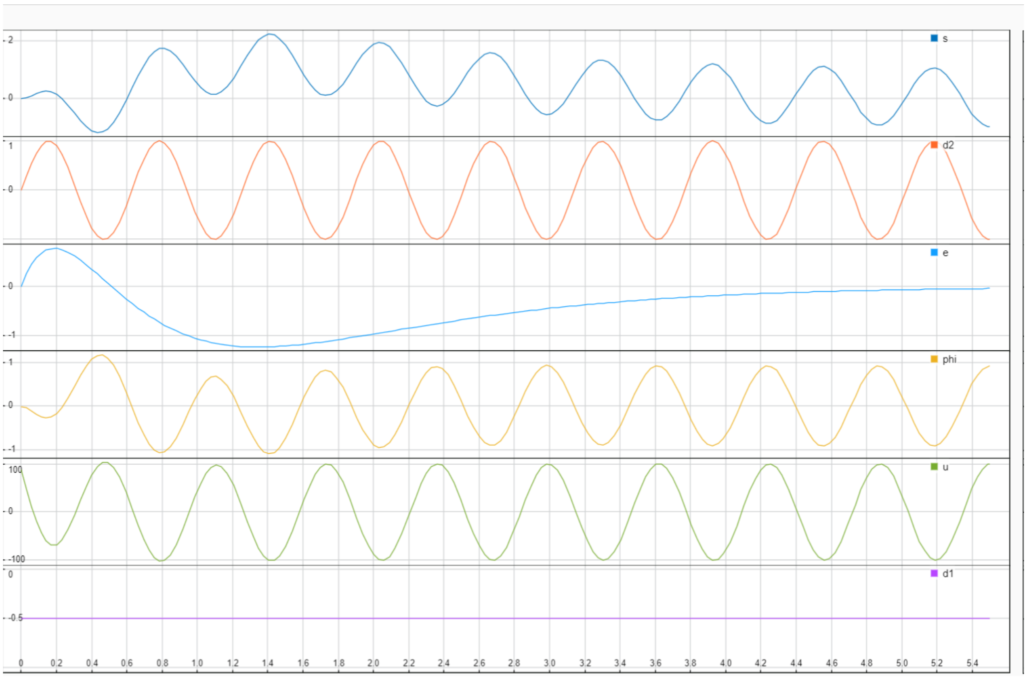


Figure 6. Linearized System, $\omega = 10[\mathrm{rad}/s]$; $y_1 = s$; $d_2 =$ ref. signal; $e =$ track error; $y_2 = \phi$; $u =$ control; $d_1 =$ actuator noise

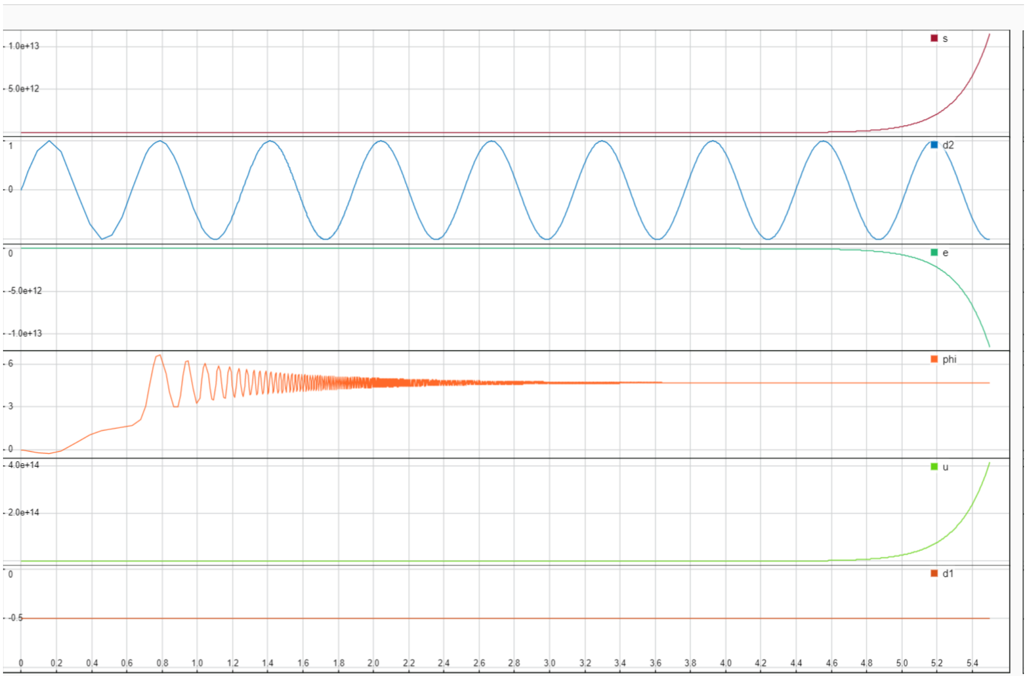


Figure 7. Non Linear System, $\omega = 10[\mathrm{rad}/s]$; $y_1 = s$; $d_2 =$ ref. signal; $e =$ track error; $y_2 = \phi$; $u =$ control; $d_1 =$ actuator noise

Increasing the disturb frequency to $\omega = 10$ will bring the system in a state far from the equilibrium point where linearization has been performed, this reduces controller effectivness, since it is designed on the linearized system. As a matter of fact the system looses its stability and the pendulum sets its position on the lower equilibrium point.

SYMBOLIC CALCULATIONS:

```
Maxima 5.45.1 https://maxima.sourceforge.io
using Lisp SBCL 2.0.0
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
(%i1) S2:matrix([0,omega],[-omega,0])
```

(%o1) $\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$

```
(%i2) eS2:factor(demoivre(expand(matrixexp(S2*t))))
```

Proviso: assuming 4*omega*t # 0(%o2) $\begin{pmatrix} \cos(\omega\, t) & \sin(\omega\, t) \\ -\sin(\omega\, t) & \cos(\omega\, t) \end{pmatrix}$

```
(%i3) d2:eS2.transpose([0,alpha])
```

(%o3) $\begin{pmatrix} \alpha \sin(\omega\, t) \\ \alpha \cos(\omega\, t) \end{pmatrix}$

```
(%i4) eq1:M*s_ddot+F*s_dot-mu-d[1]
```

(%o4) $F\, s\_\mathrm{dot} + M\, s\_\mathrm{ddot} - \mu - d_1$

```
(%i5) eq2:phi_ddot-(g/L)*sin(phi)+1/L*s_ddot*cos(phi)
```

(%o5) $\frac{\cos(\varphi)\, s\_\mathrm{ddot}}{L} + \mathrm{phi\_ddot} - \frac{g \sin(\varphi)}{L}$

```
(%i6) A:solve([eq1,eq2],[s_ddot,phi_ddot])
```

(%o6) $\left[\left[s\_\mathrm{ddot} = -\frac{F\, s\_\mathrm{dot} - \mu - d_1}{M}, \mathrm{phi\_ddot} = \frac{F \cos(\varphi)\, s\_\mathrm{dot} + M\, g \sin(\varphi) - \mu \cos(\varphi) - d_1 \cos(\varphi)}{L\, M}\right]\right]$

```
(%i7) F0:transpose(matrix([s_ddot,phi_ddot]))
```

(%o7) $\begin{pmatrix} s\_\mathrm{ddot} \\ \mathrm{phi\_ddot} \end{pmatrix}$

```
(%i8) F1:subst(A[1],F0)
```

(%o8) $\begin{pmatrix} -\frac{F\, s\_\mathrm{dot} - \mu - d_1}{M} \\ \frac{F \cos(\varphi)\, s\_\mathrm{dot} + M\, g \sin(\varphi) - \mu \cos(\varphi) - d_1 \cos(\varphi)}{L\, M} \end{pmatrix}$

```
(%i9) F2:subst([s_dot=x[2],phi=x[3],mu=u[1]],F1)
```

(%o9) $\begin{pmatrix} -\frac{x_2\, F - u_1 - d_1}{M} \\ \frac{\sin(x_3)\, M\, g + x_2 \cos(x_3)\, F - u_1 \cos(x_3) - d_1 \cos(x_3)}{L\, M} \end{pmatrix}$

```
(%i10) equilibrium1:solve(F2[1],x[2])
```

(%o10) $\left[x_2 = \frac{u_1 + d_1}{F}\right]$

```
(%i11) equilibrium1:subst([u[1]=0,d[1]=0],equilibrium1)
```

(%o11) $[x_2 = 0]$

```
(%i12) equilibrium2:solve(F2[2],x[3])
```

(%o12) $\left[\sin\left(x_3\right)=\frac{\left(-x_2\,F+u_1+d_1\right)\cos\left(x_3\right)}{M\,g}\right]$

```
(%i13) equilibrium2:subst([x[2]=0,u[1]=0,d[1]=0],equilibrium2)
```

(%o13) $[\sin\left(x_3\right)=0]$

```
(%i14) solve(equilibrium2,x[3])

solve: using arc-trig functions to get a solution.
Some solutions will be lost.
```

(%o14) $[x_3=0]$

```
(%i15) Ftot:matrix([x[2]],F2[1],[x[4]],F2[2])
```

(%o15) $$\begin{pmatrix} x_2 \\ -\frac{x_2\,F-u_1-d_1}{M} \\ x_4 \\ \frac{\sin\left(x_3\right)M\,g+x_2\cos\left(x_3\right)F-u_1\cos\left(x_3\right)-d_1\cos\left(x_3\right)}{L\,M} \end{pmatrix}$$

```
(%i16) A:addcol(diff(Ftot,x[1]),diff(Ftot,x[2]),diff(Ftot,x[3]),diff(Ftot,x[4]))
```

(%o16) $$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\cos\left(x_3\right)F}{L\,M} & \frac{\cos\left(x_3\right)M\,g-x_2\sin\left(x_3\right)F+u_1\sin\left(x_3\right)+d_1\sin\left(x_3\right)}{L\,M} & 0 \end{pmatrix}$$

```
(%i17) A:subst([x[2]=0,x[3]=0],A)
```

(%o17) $$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 \end{pmatrix}$$

```
(%i18) Adat:subst([M=1,L=1,F=1,g=9.81],A)
```

(%o18) $$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 9.81 & 0 \end{pmatrix}$$

```
(%i19) B:diff(Ftot,u[1])
```

(%o19) $$\begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{\cos\left(x_3\right)}{L\,M} \end{pmatrix}$$

```
(%i20) P:addcol(diff(Ftot,d[1]),diff(Ftot,d[2]),diff(Ftot,d[3]))
```

(%o20) $$\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\cos\left(x_3\right)}{L\,M} & 0 & 0 \end{pmatrix}$$

(%i21) `P:subst([x[2]=0,x[3]=0],P)`

(%o21) $$\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$$

(%i22) `B:subst([x[2]=0,x[3]=0],B)`

(%o22) $$\begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{pmatrix}$$

(%i23) `Bdat:subst([M=1,L=1,F=1,g=9.81],B)`

(%o23) $$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

(%i24) `linearized:A.matrix([x[1]],[x[2]],[x[3]],[x[4]])+B*u[1]+P*d[1]`

```
fullmap: arguments must have same formal structure.
 -- an error. To debug this try: debugmode(true);
```

(%i25) `P_0:B`

(%o25) $$\begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{pmatrix}$$

(%i26) `P_1:A.P_0`

(%o26) $$\begin{pmatrix} \frac{1}{M} \\ -\frac{F}{M^2} \\ -\frac{1}{L\,M} \\ \frac{F}{L\,M^2} \end{pmatrix}$$

(%i27) `P_2:A.P_1`

(%o27) $$\begin{pmatrix} -\frac{F}{M^2} \\ \frac{F^2}{M^3} \\ \frac{F}{L\,M^2} \\ -\frac{g}{L^2\,M} - \frac{F^2}{L\,M^3} \end{pmatrix}$$

(%i28) `P_3:A.P_2`

(%o28) $$\begin{pmatrix} \frac{F^2}{M^3} \\ -\frac{F^3}{M^4} \\ -\frac{g}{L^2\,M} - \frac{F^2}{L\,M^3} \\ \frac{F\,g}{L^2\,M^2} + \frac{F^3}{L\,M^4} \end{pmatrix}$$

(%i29) P_:addcol(P_0,P_1,P_2,P_3)

$$\text{(\%o29)}\quad \begin{pmatrix} 0 & \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} \\ \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} & -\frac{F^3}{M^4} \\ 0 & -\frac{1}{L\,M} & \frac{F}{L\,M^2} & -\frac{g}{L^2\,M}-\frac{F^2}{L\,M^3} \\ -\frac{1}{L\,M} & \frac{F}{L\,M^2} & -\frac{g}{L^2\,M}-\frac{F^2}{L\,M^3} & \frac{F\,g}{L^2\,M^2}+\frac{F^3}{L\,M^4} \end{pmatrix}$$

(%i30) rank(P_)

(%o30) 4

(%i31) Pdat:subst([M=1,L=1,F=1,g=9.81],P_)

$$\text{(\%o31)}\quad \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -10.81 \\ -1 & 1 & -10.81 & 10.81 \end{pmatrix}$$

(%i32) C:matrix([1, 0, 0, 0], [0, 0, 1, 0])

$$\text{(\%o32)}\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(%i33) eigenvalues(Adat)

```
rat: replaced -9.81 by -981/100 = -9.81
```

$$\text{(\%o33)}\quad \left[\left[-\frac{3\sqrt{109}}{10},\frac{3\sqrt{109}}{10},-1,0\right],[1,1,1,1]\right]$$

(%i34) S:matrix([0,0,0],[0,0,omega],[0,-omega,0])

$$\text{(\%o34)}\quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix}$$

(%i35) s:eigenvalues(S)[1]

(%o35) $[-\mathrm{i}\,\omega,\mathrm{i}\,\omega,0]$

(%i36) I:ident(4)

$$\text{(\%o36)}\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(%i37) H0:s[1]*I-A

$$\text{(\%o37)}\quad \begin{pmatrix} -\mathrm{i}\,\omega & -1 & 0 & 0 \\ 0 & \frac{F}{M}-\mathrm{i}\,\omega & 0 & 0 \\ 0 & 0 & -\mathrm{i}\,\omega & -1 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & -\mathrm{i}\,\omega \end{pmatrix}$$

(%i38) H1:addcol(H0,B)

(%o38) $$\begin{pmatrix} -i\omega & -1 & 0 & 0 & 0 \\ 0 & \frac{F}{M}-i\omega & 0 & 0 & \frac{1}{M} \\ 0 & 0 & -i\omega & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & -i\omega & -\frac{1}{L\,M} \end{pmatrix}$$

(%i46)

(%o46) zero(2)

(%i39) `H2:addcol(C,matrix([0],[0]))`

(%o39) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(%i40) `H:addrow(H1,H2)`

(%o40) $$\begin{pmatrix} -i\omega & -1 & 0 & 0 & 0 \\ 0 & \frac{F}{M}-i\omega & 0 & 0 & \frac{1}{M} \\ 0 & 0 & -i\omega & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & -i\omega & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(%i41) `rank(H)`

(%o41) 5

(%i42) `H0:s[2]*I-A`

(%o42) $$\begin{pmatrix} i\omega & -1 & 0 & 0 \\ 0 & i\omega+\frac{F}{M} & 0 & 0 \\ 0 & 0 & i\omega & -1 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & i\omega \end{pmatrix}$$

(%i43) `H1:addcol(H0,B)`

(%o43) $$\begin{pmatrix} i\omega & -1 & 0 & 0 & 0 \\ 0 & i\omega+\frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & i\omega & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & i\omega & -\frac{1}{L\,M} \end{pmatrix}$$

(%i44) `H2:addcol(C,matrix([0],[0]))`

(%o44) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(%i45) `H:addrow(H1,H2)`

(%o45) $$\begin{pmatrix} i\omega & -1 & 0 & 0 & 0 \\ 0 & i\omega+\frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & i\omega & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & i\omega & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(%i46) `rank(H)`

(%o46) 5

(%i47) `H0:s[3]*I-A`

(%o47) $$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & \frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & 0 \end{pmatrix}$$

(%i48) `H1:addcol(H0,B)`

(%o48) $$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & 0 & -\frac{1}{L\,M} \end{pmatrix}$$

(%i49) `H2:addcol(C,matrix([0],[0]))`

(%o49) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(%i50) `H:addrow(H1,H2)`

(%o50) $$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & 0 & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(%i51) `rank(H)`

(%o51) 5

task A7 -> osservabilità (Ao, Co)

(%i52) `Q:matrix([0,-1,0],[0,0,0])`

(%o52) $$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(%i53) `Co:addcol(C,Q)`

(%o53) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(%i54) `Ao:addcol(A,P)`

(%o54) $$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$$

(%i55) `zeros:matrix([0,0,0,0],[0,0,0,0],[0,0,0,0])`

(%o55) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(%i56) `Ao1:addcol(zeros,S)`

(%o56) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 0 & -\omega & 0 \end{pmatrix}$

(%i57) `Ao:addrow(Ao,Ao1)`

(%o57) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 0 & -\omega & 0 \end{pmatrix}$

(%i58) `OO_0:Co`

(%o58) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(%i59) `OO_1:OO_0.Ao`

(%o59) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

(%i60) `OO_2:OO_1.Ao`

(%o60) $\begin{pmatrix} 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$

(%i61) `OO_3:OO_2.Ao`

(%o61) $\begin{pmatrix} 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^2}{L\,M^2} & 0 & \frac{g}{L} & \frac{F}{L\,M^2} & 0 & 0 \end{pmatrix}$

(%i62) `OO:addrow(OO_0,OO_1,OO_2,OO_3)`

(%o62) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^2}{L\,M^2} & 0 & \frac{g}{L} & \frac{F}{L\,M^2} & 0 & 0 \end{pmatrix}$

(%i63) `rank(OO)`

(%o63) 7

(%i64) `Q`

(%o64) $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(%i65) `Pi:matrix([ pi[1,1], pi[1,2], pi[1,3]], [pi[2,1], pi[2,2], pi[2,3]], [pi[3,1], pi[3,2], pi[3,3]], [pi[4,1], pi[4,2], pi[4,3]])`

(%o65) $$\begin{pmatrix} \mathrm{pi}_{1,1} & \mathrm{pi}_{1,2} & \mathrm{pi}_{1,3} \\ \mathrm{pi}_{2,1} & \mathrm{pi}_{2,2} & \mathrm{pi}_{2,3} \\ \mathrm{pi}_{3,1} & \mathrm{pi}_{3,2} & \mathrm{pi}_{3,3} \\ \mathrm{pi}_{4,1} & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \end{pmatrix}$$

(%i66) `[1,0,0,0].Pi`

(%o66) $$\begin{pmatrix} \mathrm{pi}_{1,1} & \mathrm{pi}_{1,2} & \mathrm{pi}_{1,3} \end{pmatrix}$$

(%i67) `Pi:matrix([ 0, 1, 0], [pi[2,1], pi[2,2], pi[2,3]], [pi[3,1], pi[3,2], pi[3,3]], [pi[4,1], pi[4,2], pi[4,3]])`

(%o67) $$\begin{pmatrix} 0 & 1 & 0 \\ \mathrm{pi}_{2,1} & \mathrm{pi}_{2,2} & \mathrm{pi}_{2,3} \\ \mathrm{pi}_{3,1} & \mathrm{pi}_{3,2} & \mathrm{pi}_{3,3} \\ \mathrm{pi}_{4,1} & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \end{pmatrix}$$

(%i68) `Pi.S`

(%o68) $$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\mathrm{pi}_{2,3}\,\omega & \mathrm{pi}_{2,2}\,\omega \\ 0 & -\mathrm{pi}_{3,3}\,\omega & \mathrm{pi}_{3,2}\,\omega \\ 0 & -\mathrm{pi}_{4,3}\,\omega & \mathrm{pi}_{4,2}\,\omega \end{pmatrix}$$

(%i69) `A.Pi`

(%o69) $$\begin{pmatrix} \mathrm{pi}_{2,1} & \mathrm{pi}_{2,2} & \mathrm{pi}_{2,3} \\ -\frac{\mathrm{pi}_{2,1}\,F}{M} & -\frac{\mathrm{pi}_{2,2}\,F}{M} & -\frac{\mathrm{pi}_{2,3}\,F}{M} \\ \mathrm{pi}_{4,1} & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L}+\frac{\mathrm{pi}_{2,1}\,F}{L\,M} & \frac{\mathrm{pi}_{3,2}\,g}{L}+\frac{\mathrm{pi}_{2,2}\,F}{L\,M} & \frac{\mathrm{pi}_{3,3}\,g}{L}+\frac{\mathrm{pi}_{2,3}\,F}{L\,M} \end{pmatrix}$$

(%i70) `Gamma:[gamma[1],gamma[2],gamma[3]]`

(%o70) $[\gamma_1, \gamma_2, \gamma_3]$

(%i71) `B.Gamma`

(%o71) $$\begin{pmatrix} 0 & 0 & 0 \\ \frac{\gamma_1}{M} & \frac{\gamma_2}{M} & \frac{\gamma_3}{M} \\ 0 & 0 & 0 \\ -\frac{\gamma_1}{L\,M} & -\frac{\gamma_2}{L\,M} & -\frac{\gamma_3}{L\,M} \end{pmatrix}$$

(%i72) `A.Pi+B.Gamma+P`

(%o72) $$\begin{pmatrix} \mathrm{pi}_{2,1} & \mathrm{pi}_{2,2} & \mathrm{pi}_{2,3} \\ -\frac{\mathrm{pi}_{2,1}\,F}{M}+\frac{\gamma_1}{M}+\frac{1}{M} & \frac{\gamma_2}{M}-\frac{\mathrm{pi}_{2,2}\,F}{M} & \frac{\gamma_3}{M}-\frac{\mathrm{pi}_{2,3}\,F}{M} \\ \mathrm{pi}_{4,1} & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L}+\frac{\mathrm{pi}_{2,1}\,F}{L\,M}-\frac{\gamma_1}{L\,M}-\frac{1}{L\,M} & \frac{\mathrm{pi}_{3,2}\,g}{L}+\frac{\mathrm{pi}_{2,2}\,F}{L\,M}-\frac{\gamma_2}{L\,M} & \frac{\mathrm{pi}_{3,3}\,g}{L}+\frac{\mathrm{pi}_{2,3}\,F}{L\,M}-\frac{\gamma_3}{L\,M} \end{pmatrix}$$

we subst:
$\mathrm{pi}_{2,1}=0 \quad \mathrm{pi}_{2,2}=0 \quad \mathrm{pi}_{2,3}=\omega \quad \mathrm{pi}_{4,1}=0$

(%i73) `Pi:subst([pi[2,1]=0, pi[2,2]=0, pi[2,3]=omega, pi[4,1]=0],Pi)`

(%o73) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \mathrm{pi}_{3,1} & \mathrm{pi}_{3,2} & \mathrm{pi}_{3,3} \\ 0 & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \end{pmatrix}$$

(%i74) `A.Pi+B.Gamma+P`

(%o74) $$\begin{pmatrix} 0 & 0 & \omega \\ \frac{\gamma_1}{M}+\frac{1}{M} & \frac{\gamma_2}{M} & \frac{\gamma_3}{M}-\frac{F\omega}{M} \\ 0 & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L}-\frac{\gamma_1}{L\,M}-\frac{1}{L\,M} & \frac{\mathrm{pi}_{3,2}\,g}{L}-\frac{\gamma_2}{L\,M} & \frac{F\omega}{L\,M}+\frac{\mathrm{pi}_{3,3}\,g}{L}-\frac{\gamma_3}{L\,M} \end{pmatrix}$$

(%i75) `Pi.S`

(%o75) $$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \\ 0 & -\mathrm{pi}_{3,3}\,\omega & \mathrm{pi}_{3,2}\,\omega \\ 0 & -\mathrm{pi}_{4,3}\,\omega & \mathrm{pi}_{4,2}\,\omega \end{pmatrix}$$

$\gamma_1=-1;\ \ \gamma_2=-M\omega^2\ \ \gamma_3=F\omega$

(%i76) `Gamma:subst([gamma[1]=-1,gamma[2]=-M*omega^2,gamma[3]=(F*omega)],Gamma)`

(%o76) $[-1,-M\omega^2,F\omega]$

(%i77) `Gamma_:subst([M=1,L=1,F=1,g=9.81],Gamma)`

(%o77) $[-1,-\omega^2,\omega]$

(%i78) `A.Pi+B.Gamma+P`

(%o78) $$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \\ 0 & \mathrm{pi}_{4,2} & \mathrm{pi}_{4,3} \\ \frac{\mathrm{pi}_{3,1}\,g}{L} & \frac{\omega^2}{L}+\frac{\mathrm{pi}_{3,2}\,g}{L} & \frac{\mathrm{pi}_{3,3}\,g}{L} \end{pmatrix}$$

subst:
$\mathrm{pi}_{3,2}=-\frac{\omega^2}{\omega^2L+g}\rightarrow \mathrm{pi}_{4,3}=-\frac{\omega^3}{\omega^2L+g}$

(%i79) `Pi:subst([pi[3,1]=0, pi[3,2]=-((omega^2)/(omega^2*L+g)), pi[3,3]=0, pi[4,2]=0, pi[4,3]=-((omega^3)/(omega^2*L+g))],Pi)`

(%o79) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -\frac{\omega^2}{L\,\omega^2+g} & 0 \\ 0 & 0 & -\frac{\omega^3}{L\,\omega^2+g} \end{pmatrix}$$

(%i80) `Pi_:subst([M=1,L=1,F=1,g=9.81],Pi)`

(%o80) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -\frac{\omega^2}{\omega^2+9.81} & 0 \\ 0 & 0 & -\frac{\omega^3}{\omega^2+9.81} \end{pmatrix}$$

(%i81) `eig:eigenvalues(A)`

(%o81) $$\left[\left[-\sqrt{\frac{g}{L}},\sqrt{\frac{g}{L}},-\frac{F}{M},0\right],[1,1,1,1]\right]$$

(%i82) `subst([M=1,L=1,F=1,g=9.81],eig)`

(%o82) $[[-3.132091952673165, 3.132091952673165, -1, 0], [1, 1, 1, 1]]$

(%i83) `lambda*eig[1]`

(%o83) $$\left[-\sqrt{\frac{g}{L}}\,\lambda,\sqrt{\frac{g}{L}}\,\lambda,-\frac{F\lambda}{M},0\right]$$

(%i84) `polyA:(lambda-eig[1][1])*(lambda-eig[1][2])*(lambda-eig[1][3])*(lambda-eig[1][4])`

(%o84) $$\lambda\left(\lambda+\frac{F}{M}\right)\left(\lambda-\sqrt{\frac{g}{L}}\right)\left(\lambda+\sqrt{\frac{g}{L}}\right)$$

(%i85) `expand(subst([M=1,L=1,F=1,g=9.81],polyA))`

(%o85) $\lambda^4+1.0\,\lambda^3-9.81\,\lambda^2-9.81\,\lambda$

(%i86) `polydes:(lambda+1)*(lambda+2)*(lambda+3)*(lambda+4)`

(%o86) $(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)$

(%i87) `expand(polydes)`

(%o87) $\lambda^4+10\,\lambda^3+35\,\lambda^2+50\,\lambda+24$

(%i88) `P_val:(subst([M=1,L=1,F=1,g=9.81],P_))`

(%o88) $$\begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -10.81 \\ -1 & 1 & -10.81 & 10.81 \end{pmatrix}$$

(%i89) `invert(P_val)`

(%o89) $(1.0, 1.0, 0.0, 0.0; 1.0, -0.1019367991845056, 0.0, -0.1019367991845056; -0.101936799184\backslash 5056, -0.1019367991845056, -0.1019367991845056, -0.1019367991845056; -0.1019367991845056, 0, -0.1019367991845056, 0)$

(%i90) `pih:matrix([-0.1019367991845056, 0, -0.1019367991845056, 0])`

(%o90) $\begin{pmatrix} -0.1019367991845056 & 0 & -0.1019367991845056 & 0 \end{pmatrix}$

(%i91) `pih1:pih.A`

(%o91) $\begin{pmatrix} 0.0 & -0.1019367991845056 & 0.0 & -0.1019367991845056 \end{pmatrix}$

(%i92) `pih2:(subst([M=1,L=1,F=1,g=9.81],pih1.A))`

(%o92) $\begin{pmatrix} 0.0 & 0.0 & -1.0 & 0.0 \end{pmatrix}$

(%i93) `pih3:(subst([M=1,L=1,F=1,g=9.81],pih2.A))`

(%o93) $\begin{pmatrix} 0.0 & 0.0 & 0.0 & -1.0 \end{pmatrix}$

(%i94) `pih4:(subst([M=1,L=1,F=1,g=9.81],pih3.A))`

(%o94) $\begin{pmatrix} 0.0 & -1.0 & -9.81 & 0.0 \end{pmatrix}$

(%i95) `A_:(subst([M=1,L=1,F=1,g=9.81],A))`

(%o95) $$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 9.81 & 0 \end{pmatrix}$$

$F = -(\breve{\pi}^T A^4 + 10\,\breve{\pi}^T A^3 + 35\,\breve{\pi}^T A^2 + 50\,\breve{\pi}^T A + 24\,\breve{\pi}^T)$

(%i96) `K:-(pih4+10*pih3+35*pih2+50*pih1+24*pih)`

(%o96) $(2.446483180428134, 6.09683995922528, 47.25648318042813, 15.09683995922528)$

(%i97) `B_:subst([M=1,L=1,F=1,g=9.81],B)`

(%o97) $$\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

(%i98) `B_.K`

(%o98) $(0.0, 0.0, 0.0, 0.0; 2.446483180428134, 6.09683995922528, 47.25648318042813, 15.09683995\backslash$
$922528; 0.0, 0.0, 0.0, 0.0; -2.446483180428134, -6.09683995922528, -47.25648318042813, -15.096\backslash$
$83995922528)$

(%i99) `eigenvalues(A_+B_.K)`

```
rat: replaced -1.0 by -1/1 = -1.0

rat: replaced -115.6121912671025 by -12362296/106929 = -115.6121912671025

rat: replaced -36.93416503786001 by -11848000/320787 = -36.93416503786001

rat: replaced 2.446483180428134 by 800/327 = 2.446483180428134

rat: replaced 37.44648318042813 by 12245/327 = 37.44648318042813

rat: replaced -15.09683995922528 by -14810/981 = -15.09683995922528

rat: replaced -240.8587318064635 by -77264350/320787 = -240.8587318064635

rat: replaced -76.94617716220834 by -74050000/962361 = -76.94617716220836

rat: replaced 5.09683995922528 by 5000/981 = 5.096839959225281

rat: replaced 37.44648318042813 by 12245/327 = 37.44648318042813

rat: replaced -15.09683995922528 by -14810/981 = -15.09683995922528
```

(%o99) $[[-2, -1, -4, -3], [1, 1, 1, 1]]$

(%i100) `X:transpose([x(1),x(2),x(3),x(4)])`

(%o100) $$\begin{pmatrix} x(1) \\ x(2) \\ x(3) \\ x(4) \end{pmatrix}$$

(%i101) `D:transpose([d(1),d(2),d(3)])`

(%o101) $\begin{pmatrix} d(1) \\ d(2) \\ d(3) \end{pmatrix}$

(%i102) `K.X`

(%o102) $15.09683995922528\, x(4) + 47.25648318042813\, x(3) + 6.09683995922528\, x(2) + 2.446483\backslash$
$180428134\, x(1)$

(%i103) `Gamma_:matrix(Gamma_)`

(%o103) $\begin{pmatrix} -1 & -\omega^2 & \omega \end{pmatrix}$

(%i104) `Lcontrol:Gamma_-K.Pi_`

(%o104) $\Big(-1.0, \frac{47.25648318042813\,\omega^2}{\omega^2+9.81} - \omega^2 - 2.446483180428134, \frac{15.09683995922528\,\omega^3}{\omega^2+9.81} - 5.0968399592\backslash$
$2528\,\omega\Big)$

(%i105) `Lcontrol_:subst(omega=0.1,Lcontrol)`

(%o105) $\begin{pmatrix} -1.0 & -2.40836048879837 & -0.5081466395112016 \end{pmatrix}$

(%i106) `Lcontrol.D`

(%o106) $d(3)\left(\frac{15.09683995922528\,\omega^3}{\omega^2+9.81} - 5.09683995922528\,\omega\right) + d(2)\left(\frac{47.25648318042813\,\omega^2}{\omega^2+9.81} - \right.$
$\left.\omega^2 - 2.446483180428134\right) - 1.0\,d(1)$

(%i109) `K:[10^8*2.4465,10^8*0.0510, 10^8*2.4500, 10^8*0.0510]`

(%o109) $[2.4465 \times 10^{+8}, 5100000.0, 2.45 \times 10^{+8}, 5100000.0]$

(%i110) `Lcontrol:Gamma_-K.Pi_`

(%o110) $\begin{pmatrix} -1.0 & \frac{2.45\times 10^{+8}\,\omega^2}{\omega^2+9.81} - \omega^2 - 2.4465\times 10^{+8} & \frac{5100000.0\,\omega^3}{\omega^2+9.81} - 5099999.0\,\omega \end{pmatrix}$

(%i111) `K.X`

(%o111) $5100000.0\, x(4) + 2.45\times 10^{+8}\, x(3) + 5100000.0\, x(2) + 2.4465\times 10^{+8}\, x(1)$

(%i112) `Lcontrol.D`

(%o112) $d(3)\left(\frac{5100000.0\,\omega^3}{\omega^2+9.81} - 5099999.0\,\omega\right) + d(2)\left(\frac{2.45\times 10^{+8}\,\omega^2}{\omega^2+9.81} - \omega^2 - 2.4465\times 10^{+8}\right) -$
$1.0\,d(1)$

(%i113)