Improving Fidelity in 3 qubit XXX Heisenberg model

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XXX Heisenberg model

- In general a XXX Heisenberg model is defined as follows:

$$H = J\sum_{\langle i,j\rangle}^{N}(\sigma_x^{(i)}\sigma_x^{(j)} + \sigma_y^{(i)}\sigma_y^{(j)} + \sigma_z^{(i)}\sigma_z^{(j)}) + h\sum_{i}^{N}\sigma_z^{(i)} \tag{1}$$

  where N in the number of 1/2 spins, $\sigma^{(i)}$ are the Pauli operators of the i-th spin, J is the coupling constant and h is the external magnetic field.
- We will work in the specific case of 3 qubit in a line in absence of external field so the Hamiltonian is:

$$H = \sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)} + \sigma_x^{(2)}\sigma_x^{(3)} + \sigma_y^{(2)}\sigma_y^{(3)} + \sigma_z^{(2)}\sigma_z^{(3)} \tag{2}$$

- The XXX Heisenberg Hamiltonian in particular commute with the total magnetization $[H, \sum\sigma_z] = 0 \Rightarrow \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$

$$\begin{cases} |000\rangle & m = 0 \\ |001\rangle, |010\rangle, |100\rangle & m = 1 \\ |011\rangle, |101\rangle, |110\rangle & m = 2 \\ |111\rangle & m = 3 \end{cases} \tag{3}$$

Fidelity and State Tomography

- Fidelity is a measure of the "closeness" of two quantum states. Is not a measure in the space of density matrices. Given two density matrices it is define as follows:

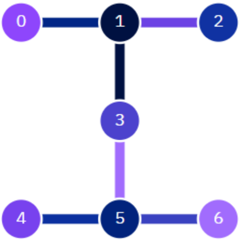
$$F(\rho,\sigma) = (Tr\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2 \tag{4}$$

- if ρ and σ rapresent two pure state the fidelity is the overlap between the two states.

$$F = \left| \langle\psi(0)| \, U^\dagger U^{approx} \, |\psi(0)\rangle \right|^2 \tag{5}$$

- State tomography is a method for determining the quantum state of a qubit, or qubits.In state tomography, a quantum circuit is repeated with measurements in different bases to exhaustively determine the density matrix. For the case of a 3 qubit tomography we need 27 different basis measurements.
- a high fidelity measured by state tomography doesn't gaurentee a high fidelity quantum simulation, but a low fidelity state tomography does imply a low fidelity quantum simulation.

IBM Jakarta Quantum Computer

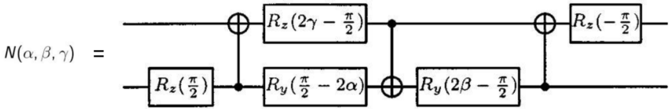


Qubit	T1 (us)	T2 (us)	Frequency (GHz)	Anharmonicity (GHz)	Readout assignment error	Prob meas0 prep1	Prob meas1 prep0
Q0	78.34	43.11	5.236	-0.33988	2.770e-2	0.0388	0.0166
Q1	114.47	21.56	5.014	-0.3432	2.090e-2	0.0336	0.0082
Q2	155.45	26.73	5.108	-0.34162	1.660e-2	0.0236	0.0096
Q3	89.62	43.38	5.178	-0.34112	1.810e-2	0.0224	0.0138
Q4	130.65	55.62	5.213	-0.33925	1.590e-2	0.0216	0.0102
Q5	148.56	46.46	5.063	-0.34129	4.560e-2	0.0696	0.0216
Q6	134.1	22.67	5.3	-0.33836	5.380e-2	0.0668	0.0408

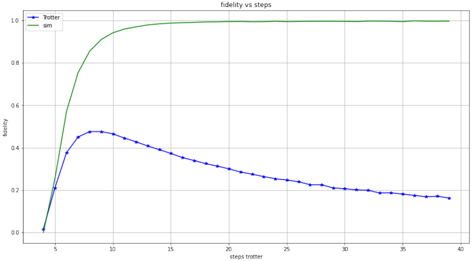
Iterated Trotterization

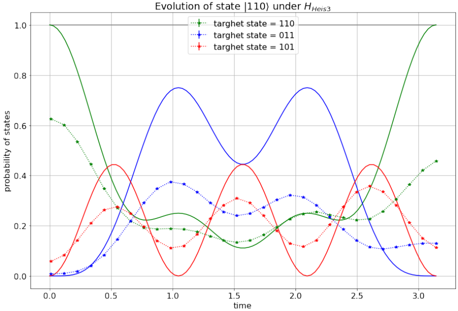
- The evolution of a particular state (ex.$|011\rangle$) is given by the Schroedinger equation $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$.We need to find a quantum circuit for the unitary operator e^{-iHt} in terms of single and two qubit gates.
- To do this we can divide the Hamiltonian in two parts: H_1 relative to the interaction between spins 1-2 and H_2 for spins 2-3.
- However $[H_1, H_2] \neq 0$ so using first order Trotterizzation $e^{\delta(A+B)} = e^{\delta A}e^{\delta B} + O(\delta^2)$

$$e^{-iHt} \approx (e^{-iH_1 t/n}e^{-iH_2 t/n})^n = ((N(-\frac{t}{n}, -\frac{t}{n}, -\frac{t}{n}) \otimes 1)(1 \otimes N(-\frac{t}{n}, -\frac{t}{n}, -\frac{t}{n}))^n \tag{6}$$



Results


fidelity vs steps
Trotter
sim
fidelity
steps trotter
1.0
0.8
0.6
0.4
0.2
0.0
5
10
15
20
25
30
35
40


Evolution of state |110⟩ under H_{Heis3}
targhet state = 110
targhet state = 011
targhet state = 101
probability of states
time
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

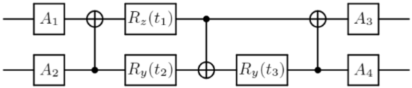
Looking for a minimal Decomposition

- Instead of repeating the previous circuit step by step we numerically calculate $(e^{-iH_1 t/n} e^{-iH_2 t/n})^n$ and look for a minimal decompostion.
- In the B basis $Be^{-i\frac{t}{n}H_1} e^{-i\frac{t}{n}H_2} B^{-1} = I \otimes \tilde{U}$. So this is true also for U^n

$$B(e^{-i\frac{t}{n}H_1} e^{-i\frac{t}{n}H_2})^n B^{-1} = I \otimes \tilde{U}^n \tag{7}$$

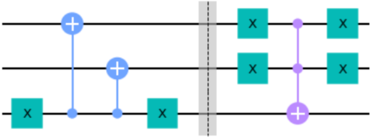
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-iq}\cos(q) & -ie^{-iq}\sin(q) & 0 & 0 & 0 & 0 & 0 & 0 \\ -i\sin(q)\cos(q) & \cos^2(q) & -ie^{-iq}\sin(q) & 0 & 0 & 0 & 0 & 0 \\ -\sin^2(q) & -i\sin(q)\cos(q) & e^{-iq}\cos(q) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2iq} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-iq}\cos(q) & -ie^{-iq}\sin(q) & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\sin(q)\cos(q) & \cos^2(q) & -ie^{-iq}\sin(q) & 0 \\ 0 & 0 & 0 & 0 & -\sin^2(q) & -i\sin(q)\cos(q) & e^{-iq}\cos(q) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2iq} \end{bmatrix}$$

- In conclusion we have $U^n |\psi\rangle = B^{-1}(I \otimes \tilde{U}^n)B |\psi\rangle$. To decrease the number of c-nots we can prepare $|\phi\rangle = B |\psi\rangle$ and is easy because B is just a permutation matrix.
- The $\tilde{U}$ Operator is a two qubit gate so it can be decompose with at most 3 cnots.

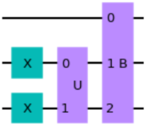


The circuit

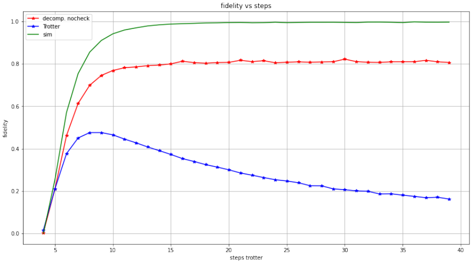
- The B matrix is just a permutation matrix so it can be implemented as a circuit of Toffoli, c-nots and X gate because they perform rows and column moves.



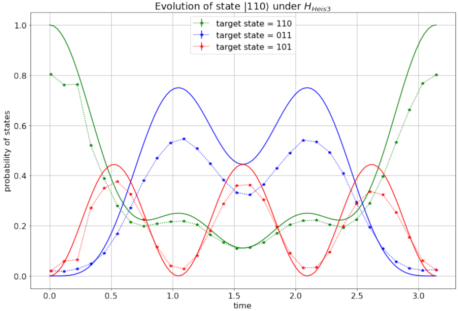
- So in conclusion for every time and every number of Trotter steps the the depth circuit will be fixed to 11 cnots (for the particular case of our circuit 14) instead of 6n



Results



fidelity vs steps
decomp. nocheck
Trotter
sim
fidelity
steps trotter



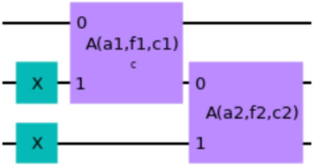
Evolution of state |110⟩ under H_{Heis3}
target state = 110
target state = 011
target state = 101
probability of states
time

Single State Preparation

- The state $|110\rangle$ evolves in the m=2 space so we need 3 complex parameters to generate the evolution
- We define a generic exchange two qubit gate A that preserves the magnetization

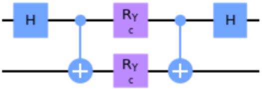
$$\begin{bmatrix} e^{0.5ia}e^{0.5if} & 0 & 0 & 0 \\ 0 & e^{0.5ia}e^{0.5if}\cos(c) & e^{0.5ia}e^{-0.5if}\sin(c) & 0 \\ 0 & -e^{-0.5ia}e^{0.5if}\sin(c) & e^{-0.5ia}e^{-0.5if}\cos(c) & 0 \\ 0 & 0 & 0 & e^{-0.5ia}e^{-0.5if} \end{bmatrix}$$

- The exchange gate depend on 3 parameters so the circuit in figure preserve the magnetization symmetry and depends on 6 real parameters.



The circuit

- The exchange gate can be decompose as $A = (I \otimes R_z(f))G(c)(I \otimes R_z(a))$ where G is the Givens Rotation gate equivalent to a $R_y(c)$ in the subspace $|01\rangle, |10\rangle$



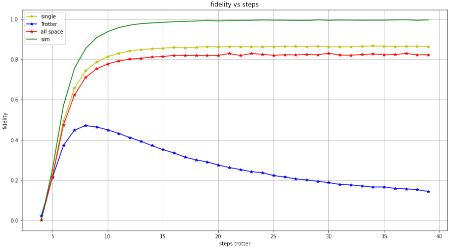
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(c) & \sin(c) & 0 \\ 0 & -\sin(c) & \cos(c) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Imposing that $U^n|110\rangle = A_2 A_1 |110\rangle$ we have six equations and six variables so we can find the parameters of the circuit.

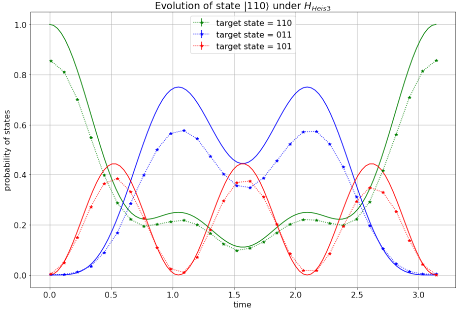
$$\begin{aligned} \alpha_0 &= Re(u_{47}) \\ \alpha_1 &= Re(u_{67}) \\ \alpha_2 &= Re(u_{77}) \\ \beta_0 &= Im(u_{47}) \\ \beta_1 &= Im(u_{67}) = Im(u_{77}) \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{atan(\frac{\beta_1}{\alpha_2}) - atan(\frac{\beta_1}{\alpha_1})}{2} \\ f_1 &= \frac{atan(\frac{\beta_1}{\alpha_2}) - atan(\frac{\beta_1}{\alpha_1})}{2} \\ c_1 &= acos(\sqrt{(\alpha_2^2 + \beta_1^2)}) \\ a_2 &= \frac{-atan(\frac{\beta_1}{\alpha_1}) + atan(\frac{\beta_0}{\alpha_0})}{2} \\ f_2 &= \frac{+atan(\frac{\beta_1}{\alpha_1}) - atan(\frac{\beta_0}{\alpha_0}))}{2} \\ c_2 &= acos(\sqrt{(\alpha_1^2 + \beta_1^2)/sin(r_1)}) \end{aligned}$$

Results



fidelity vs steps
single
Trotter
all space
sim
fidelity
steps trotter



Evolution of state |110⟩ under H_{Heis3}
target state = 110
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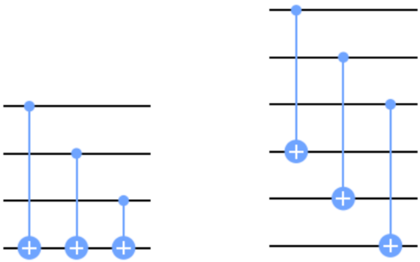
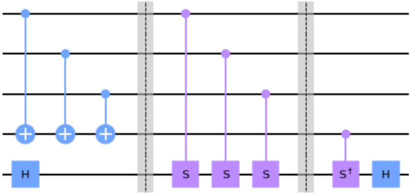
Parity and Copy check

- Because of the symmetries of the Hamiltonian we can use the ancillas to measure some invariant quantity (for example parity or magnetization)
- If we prepare a state with m=2 and after a magnetization invariant circuit we measure $\sum_i \sigma_z^{(i)}$ m=1 we know that an error happened and we can discard this run.
- MEM is also performed for the ancillas. In this case we can choose to separately construct a calibration matrix for the ancillas and for the target qubit or for the whole qubits.



Magnetization check



$$|\psi_0\rangle = \Big(\sum_k a_k |k\rangle\Big) |00\rangle \tag{8}$$

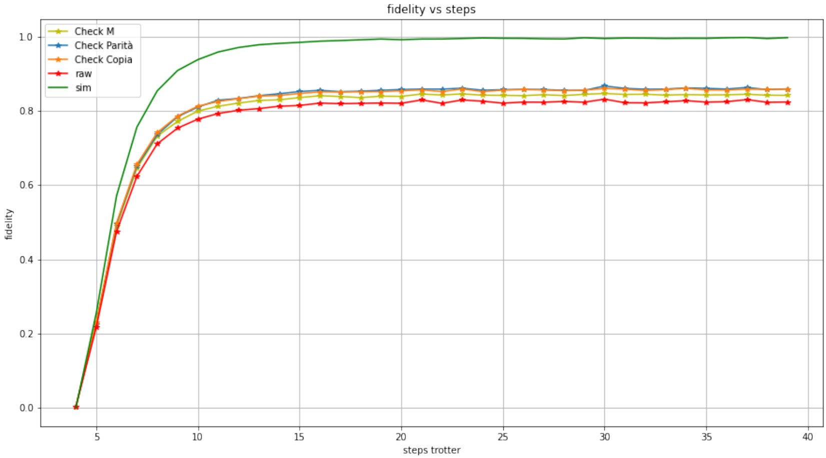
$$|\psi_1\rangle = \Big(\sum_{even} a_j |j\rangle\Big) |0+\rangle + \Big(\sum_{odd} a_i |i\rangle\Big) |1+\rangle \tag{9}$$

$$\begin{aligned} |\psi_2\rangle = {} & a_0 |000\rangle |0+\rangle + (a_1 |001\rangle + a_2 |010\rangle + a_4 |100\rangle) |1\rangle (|0\rangle - i |1\rangle) \\ & + (a_3 |011\rangle + a_5 |101\rangle + a_6 |110\rangle) |0-\rangle + a_7 |111\rangle |1\rangle (|0\rangle + i |1\rangle) \end{aligned} \tag{10}$$

m=0	00
m=1	10
m=2	01
m=3	11

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

Analysis Ancille Mitigation



Measurement Error Mitigation

- The measurement error mitigation (MEM) is used to mitigate the errors introduced by a noisy measurement device.Let $\vec{p}^{ideal}$ and $\vec{p}^{noisy}$ denote the probability vectors of measurement statistics from an ideal and noisy measurement.
- We can imagine that the measurement first selects one of these outputs in a perfect and noiseless manner, and then noise subsequently causes this perfect output to be randomly perturbed before it is returned to the user

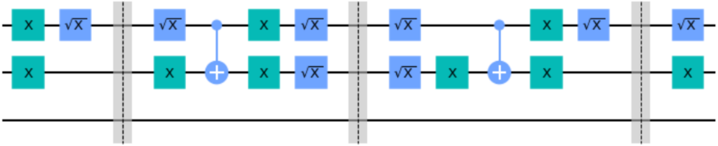
$$\vec{p}^{noisy} = \Lambda \vec{p}^{ideal} \tag{11}$$

where Λ is a left stochastic transformation matrix usually call Calibration Matrix.
- To construct the Calibration Matrix we simply prepare each of the 2^n possible basis states, immediately measure them, and see what probability exists for each outcome assuming and ideal preparation.
- Inverting the Calibration Matrix we have the ideal probability vector.

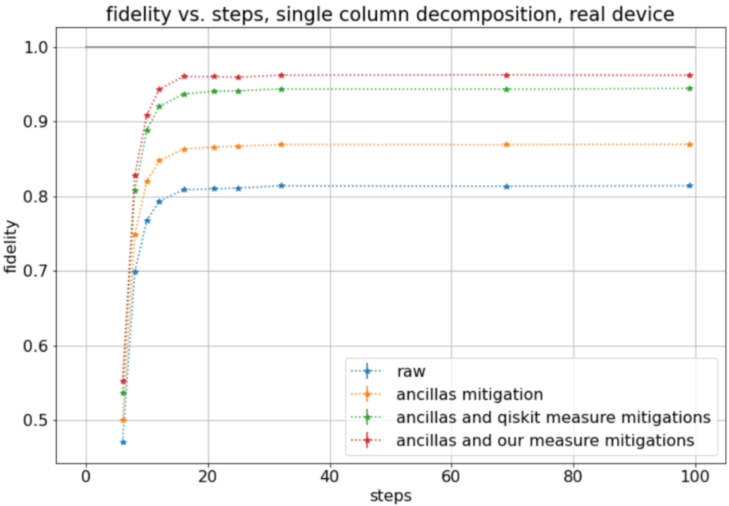
$$\vec{p}^{ideal} = \Lambda^{-1} \vec{p}^{noisy} \tag{12}$$

Improvement of the Calibration Matrix

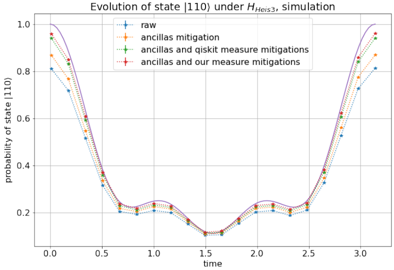
- In addition to the noisy measurement there is the noise do to the quantum channels (mainly decoherence and depolarizing channel). We can try to construct a Calibration Matrix that accounts the effect of interaction with the environment
- In this particular case the form of the circuit is fixed for every time and number of trotter steps. This particular geometry of the circuit allow us to implement an Identity circuit with the same form of $A_1 A_2$, so in first approximation with the same noisy channel.
- As before we construct the Calibration Matrix preparing each of the 2^n and measuring them after the identity circuit.

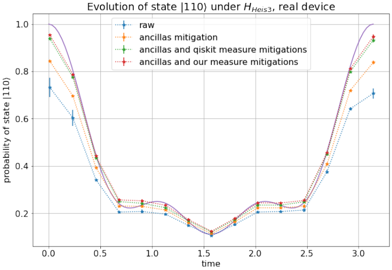


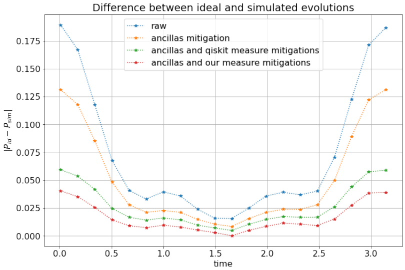
Best Results

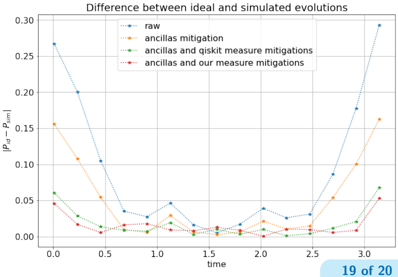


Analysis Measurement Mitigation

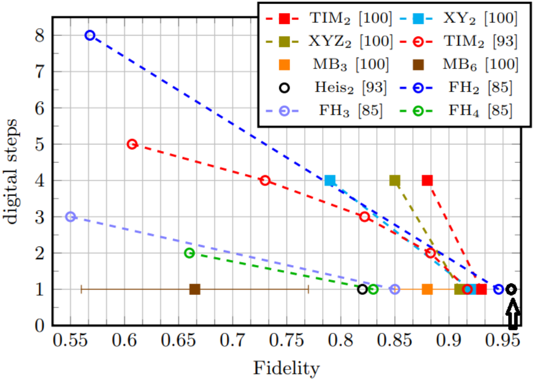








Comparison



Summary of state-of-art experimental digital quantum simulations. Open circles represent results obtained on superconducting circuits quantum processors, while squares correspond to experimental quantum simulations on trapped ions processors, Quantum computers as universal quantum simulators: state-of-art and perspectives(2020)