“Macroeconometrics - PS 1”

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Exercise 1

- Download the time series for US quarterly real GDP (FRED website).
- Consider an AR(1) process for GDP growth.

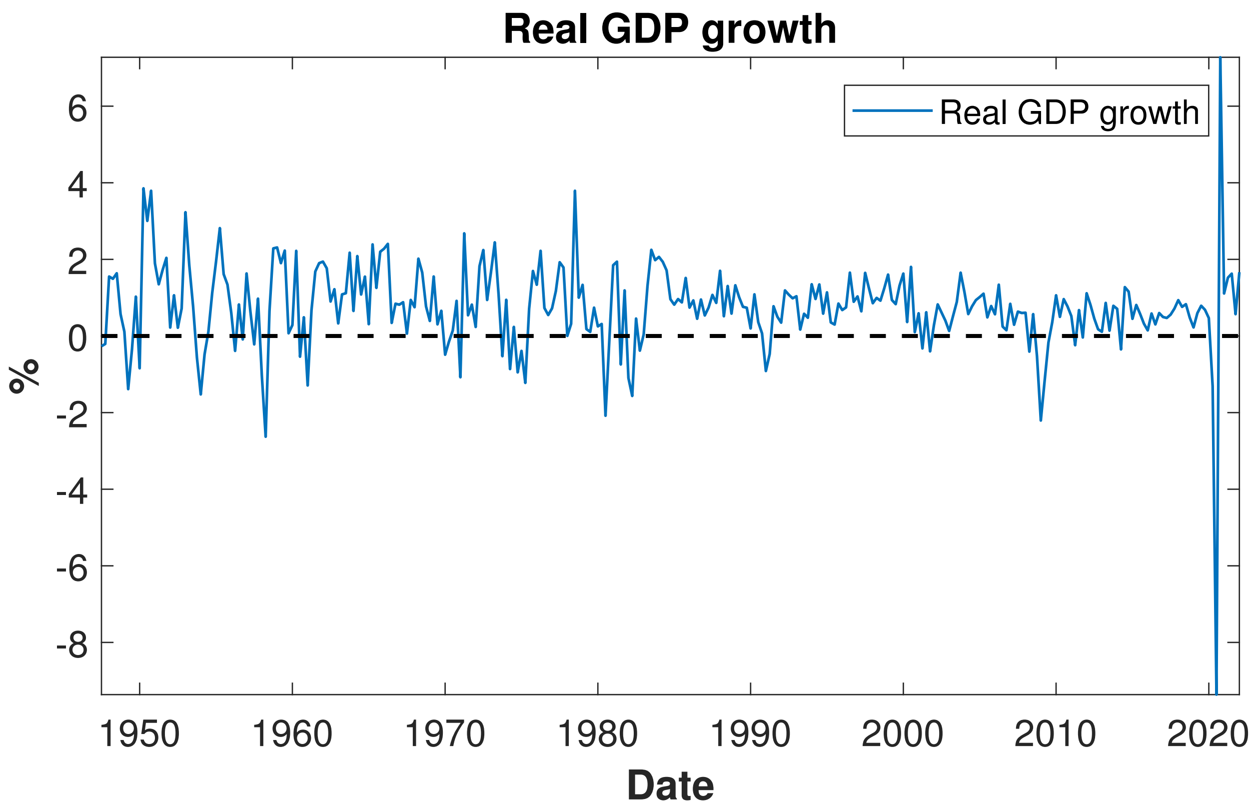
$$y_t = c + \phi y_{t-1} + \epsilon_t,$$

where $\epsilon_t \sim iid(0, \sigma^2)$.

- Transform the series in growth rates using either
    - $100 * (\ln(y_t) - \ln(y_{t-1}))$.
    - $100(y_t - y_{t-1})/y_{t-1}$.
- Since $\ln(1 + x) \approx x$ then

$$\ln(y_t) - \ln(y_{t-1}) = \ln(y_t/y_{t-1}) = \ln(1 + y_t/y_{t-1} - 1) = (y_t - y_{t-1})/y_{t-1}.$$

US GDP Time Series (1a)




```
GDPC1 = csvread('GDPC1.csv',1,1);
GDPgrowth = 100*diff(log(GDPC1));
```

OLS Estimation (1b)

- OLS estimates of c, ϕ can be obtained as follows.

```
% Define lags
Y  = GDPgrowth(2:end);
yL = GDPgrowth(1:end-1);

% Define design matrix
X = [ones(length(yL),1) yL];

% OLS formulae
beta = (X'*X)\X'*Y;
% (Not asked by the exercise) The standard errors and t-stat are given by:
n = length(Y);
k = size(X,2);
err = Y - X*beta;
sigma2 = (err'*err)/(n - k);
stderr_ols = diag(sqrt((sigma2.*inv(X'*X))));
```

- Alternatively, use a numerical routine to optimize.
- $\hat{\beta} = (0.67, 0.12)$ or pre-covid $\hat{\beta} = (0.49, 0.36)$.

Autocorrelations (1c)

- Use sample moments to estimate theoretical moments.

$$\hat{\mu} = \frac{1}{T}\sum_{t=0}^{T} y_t,$$

$$\hat{\gamma(0)} = \frac{1}{T}\sum_{t=0}^{T}(y_t - \hat{\mu})^2,$$

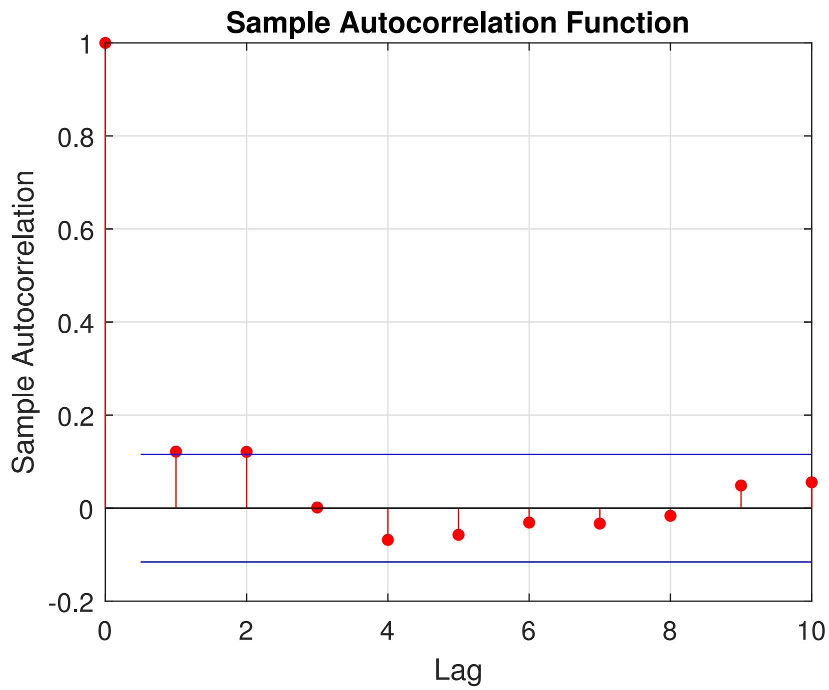
$$\hat{\gamma(k)} = \frac{1}{T}\sum_{t=k+1}^{T}(y_t - \hat{\mu})(y_{t-k} - \hat{\mu}).$$

Autocorrelations (1c)

- To compute the first 10 sample autocorrelations
  - use inbuilt MATLAB function.
  - computes the sample equivalent of $\rho_h = \gamma_h / \gamma_0$.

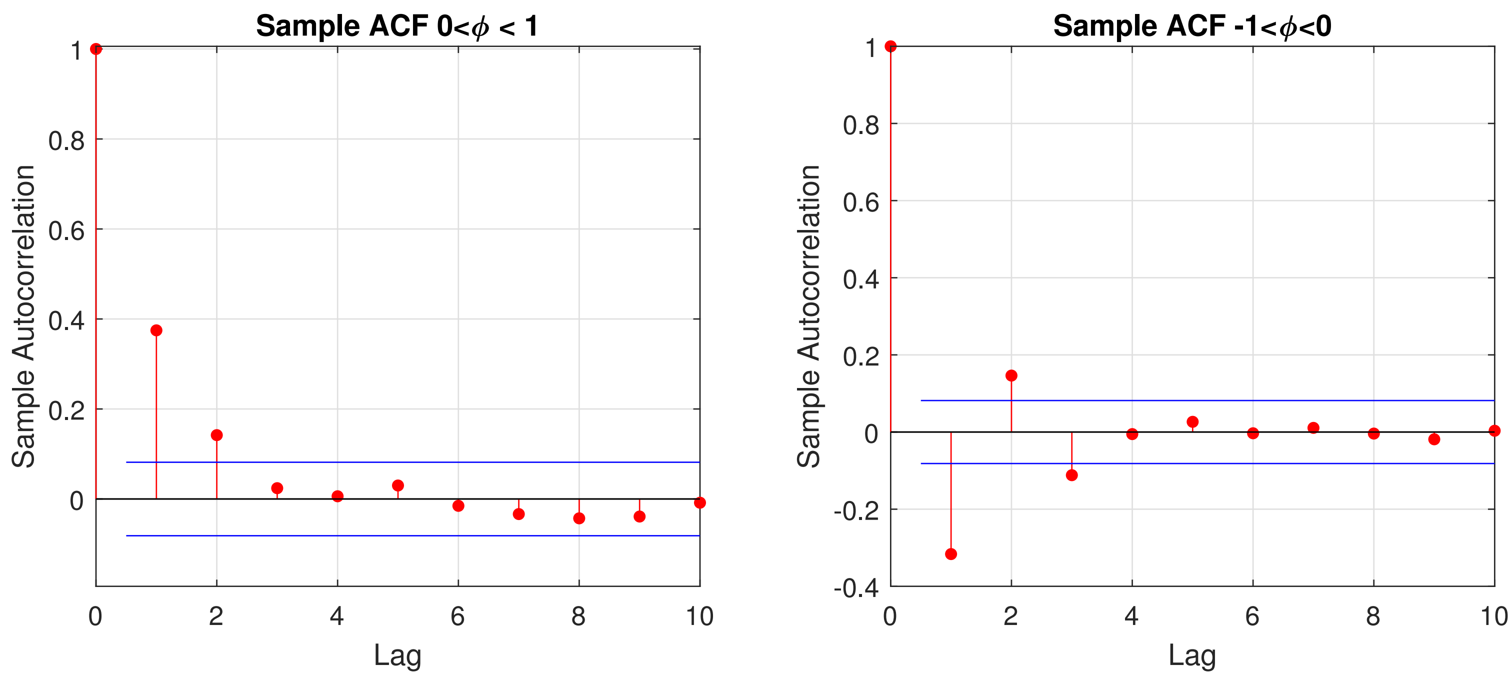
```

y = GDPgrowth;
T = size(y,1);
rho = zeros(11,1);
gamma = zeros(11,1);

for k = 1:11
yt = y(k:end);
yj = y(1:end-k+1);
gamma(k) = sum((yt-mean(y)).*(yj-mean(y)))/T;
rho(k) = gamma(k)/gamma(1); % since gamma(1)=gamma_0=var(y)
end

% Or using the inbuilt function:
figure;
autocorr(y,10)
```

Autocorrelations (1c)



- Autocorrelation decay relatively fast.
- Consistent with low estimate of autoregressive coefficient ϕ.
- The Box-Jenkins approach: Is AR(1) a good model?

Autocorrelations (1c)



- In the AR(1) model $\gamma(k) = \phi^k$ with $|\phi| < 1$ implies a decay.
- Sample ACF computed from a simulations of AR(1) model.

Wold Coefficients (1d)

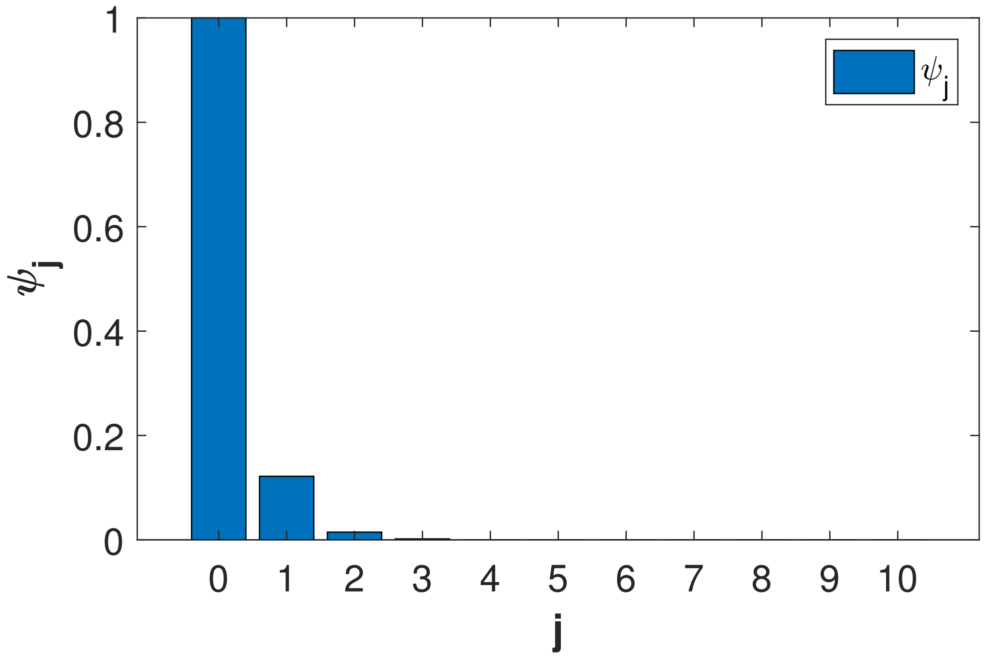
- A stationary AR(1) process admits the wold representation.

$$y_t = c/(1-\phi) + \psi_w(L)\epsilon_t,$$
$$(1-\phi L)y_t = c + \epsilon_t.$$

Since $c/(1-\phi) - (\phi Lc)/(1-\phi) = c$ we have $(1-\phi L)\psi_w(L) = 1$.

$$\psi_0 = 1,$$
$$\psi_1 = \phi,$$
$$\psi_2 = \phi^2,$$
$$\ldots$$

Wold Coefficients (1d)



Exercise 2

- Consider an AR(2) process for GDP growth.

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-1} + \epsilon_t,$$

where $\epsilon_t \sim iid(0, \sigma^2)$.

- OLS estimates: $\hat{\beta} = (0.6, 0.1, 0.1)$ or pre-covid $\hat{\beta} = (0.44, 0.3, 0.1)$.
- Solving $\Phi(z) = (1 - \phi_1 z - \phi_2 z^2) = 0$ we find $|z_1| = 3.5, |z_2| = 2.5$.

```
root_AR2 = abs(roots([-beta(3) -beta(2) 1]));
```

- For the stationarity we need $|z_i| > 1$

Causality and Stationarity (2c)

Consider the AR(1) process

$$y_t = \phi y_{t-1} + \epsilon_t$$

There are two stationary representations:

- Solve backward if $|\phi| < 1$ and obtain $y_t = \sum_{j=0}^{\infty} \phi^j \epsilon_{t-j}$.

$$y_t = \phi(\phi y_{t-2} + \epsilon_{t-1}) + \epsilon_t = \ldots$$

- Solve forward if $|\phi| \geq 1$ and obtain $y_t = -\sum_{j=0}^{\infty}(\phi^{-j})\epsilon_{t+j}$.

$$y_t = \phi^{-1} y_{t+1} - \phi^{-1}\epsilon_{t+1} = \phi^{-1}(\phi^{-1} y_{t+2} - \phi^{-1}\epsilon_{t+2}) - \phi^{-1}\epsilon_{t+1} = \ldots$$

- The first processes are causal or future-independent and stationary.
- To recap AR(2) process is causal if it admits the Wold representation.

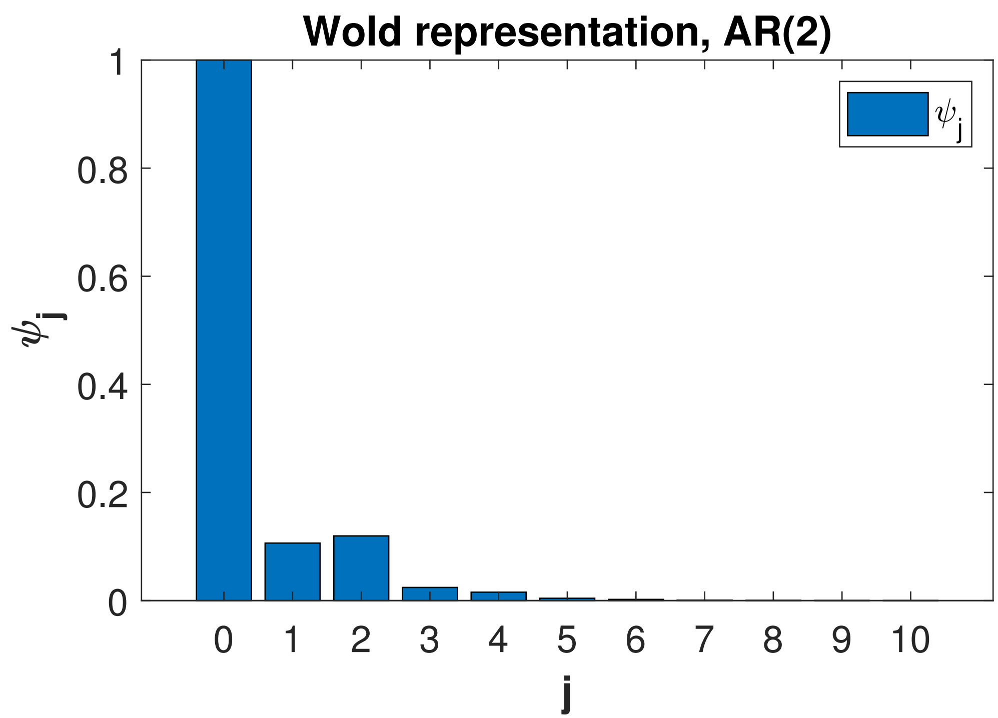
Wold Coefficients (2d)

- We can either proceed as above or we can make use of the companion form.
- Rewrite high-order differential equations as a system of ODE.

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}.$$

$$Y_t = C + FY_{t-1} + \epsilon_t.$$

Wold Coefficients (2d)



- Then the Wold coefficients are the top-left elements of F, F^2, F^3, ...

Exercise 3

- Consider the MA process

$$y_t = c + \epsilon_t + 1.2\epsilon_{t-1} + 2\epsilon_{t-2},$$

where $\epsilon_t \sim iid(0, \sigma^2)$.

- Is the process stationary?
- Find the roots of the MA polynomial. Is it invertible?

Stationarity (3a)

- The moments of the process are finite and time-independent.

$$E(y_t) = E(c + \epsilon_t + 1.2\epsilon_{t-1} + 2\epsilon_{t-2}) = c.$$

$$\begin{aligned} Var(y_t) =& E[(\epsilon_t + 1.2\epsilon_{t-1} + 2\epsilon_{t-2})^2] \\ &= E[(\epsilon_t^2 + 1.44\epsilon_{t-1}^2 + 4\epsilon_{t-2}^2)] \\ &= 6.44\sigma^2. \end{aligned}$$

Stationarity (3a)

$$Cov(y_t, y_{t-h}) = E[(\epsilon_t + 1.2\epsilon_{t-1} + 2\epsilon_{t-2})(\epsilon_{t-h} + 1.2\epsilon_{t-h-1} + 2\epsilon_{t-h-2})]$$

$$h = 0 \Rightarrow Var(y_t) = 6.44\sigma^2,$$
$$h = 1 \Rightarrow E(1.2\epsilon_{t-1}^2 + 2.4\epsilon_{t-2}^2) = 3.6\sigma^2,$$
$$h = 2 \Rightarrow E(2\epsilon_{t-2}^2) = 2\sigma^2,$$
$$h = 3 \Rightarrow Cov(y_t, y_{t-h}) = 0.$$

- Mean and variance are constants
- The covariance only depends on h and not t.

Invertibility (3a)

- Compute the solutions to $(1 + \theta_1 z + \theta_2 z^2) = 0$
- We get (complex) roots less than 1 in absolute value.
- The process is not invertible.