“Macroeconometrics - PS 3”

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Vector Autoregression (VAR)

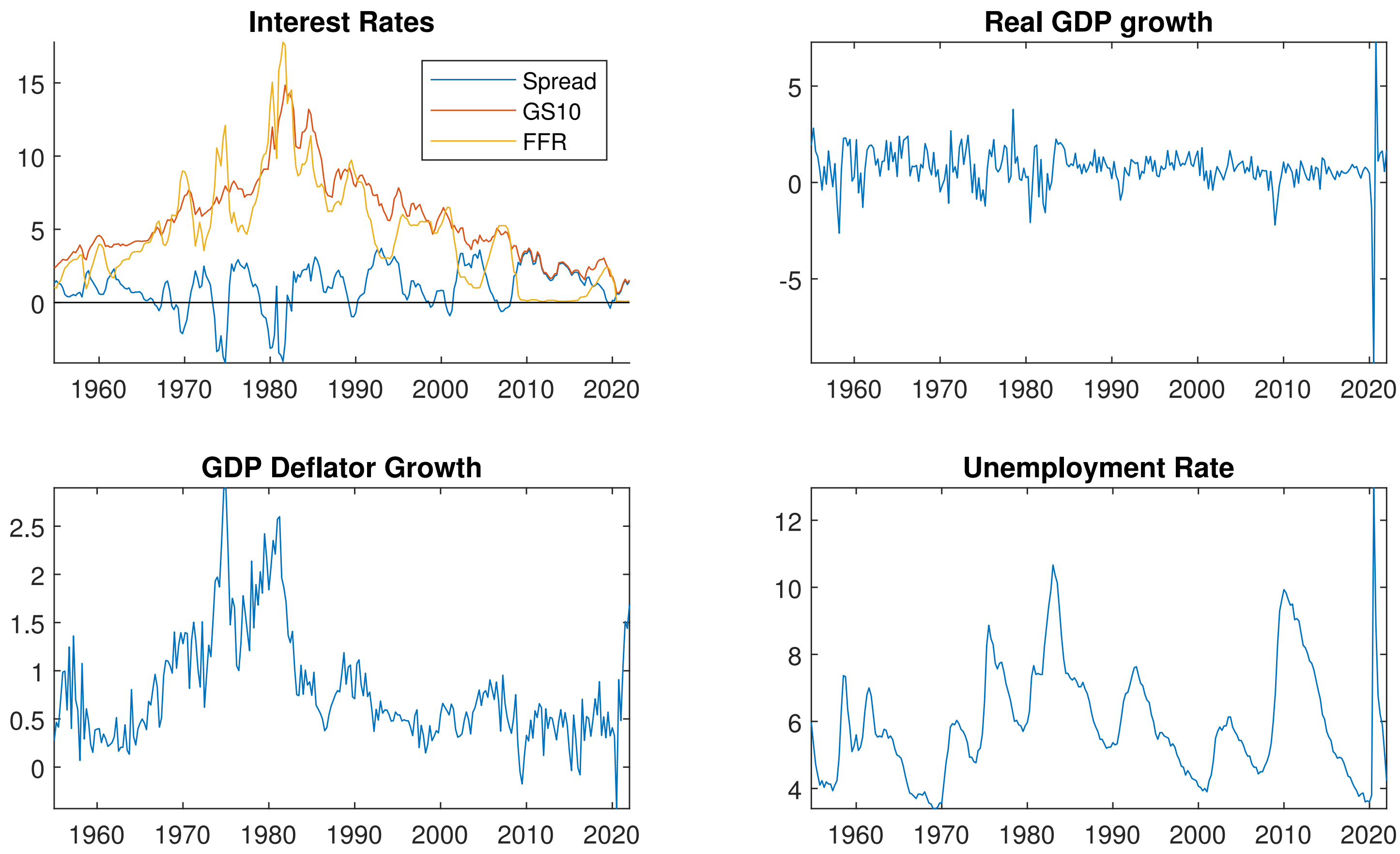
Consider the VAR(4)

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + A_3 Y_{t-3} + A_4 Y_{t-4} + \epsilon_t,$$

where Y_t is a $n \times 1$ vector and A_p are $n \times n$ matrices.

- GDP: GDP growth rate.
- UNR: Unemployment rate.
- INF: Infaltion rate using GDP deflator.
- FFR: Federal funds rate.
- SPR: Spread $i_{10,t} - i_t$.

US data


Interest Rates
Spread
GS10
FFR
15
10
5
0
1960
1970
1980
1990
2000
2010
2020
Real GDP growth
5
0
-5
1960
1970
1980
1990
2000
2010
2020
GDP Deflator Growth
2.5
2
1.5
1
0.5
0
1960
1970
1980
1990
2000
2010
2020
Unemployment Rate
12
10
8
6
4
1960
1970
1980
1990
2000
2010
2020

VAR Estimation

- We have same regressor in each of the $n = 5$ equations.
- Use OLS equation by equation.
- Use OLS on the stacked representation.

Consider the VAR(4)

$$Y_t = c + A_1 Y_{t-1} + A_2 Y_{t-2} + A_3 Y_{t-3} + A_4 Y_{t-4} + \epsilon_t,$$

where Y_t is a $n \times 1$ vector and A_p are $n \times n$ matrices.

$$Y_{T \times n} = X_{T \times np} A_{np \times n} + \varepsilon_{T \times n}$$

VAR Estimation

```
function [beta, residuals] = VAR(y,p,c)

% Function to estimate a VAR(p) with or without constant using OLS
% Inputs:    y = T x N matrix of endogeneous variables
%            p = VAR lag order
%            c = 1 if constant required
% Outputs:   beta = (Np+1 x N) matrix of estimated coefficients (Np x N) if
%                   no constant is included
%            residuals = (T-p x N) matrix of OLS residuals

[T, ~] = size(y);
yfinal = y(p+1:T,:);
if c == 1
    X = [ones(T-p,1), lagmakerMatrix(y,p)];
else
    X = lagmakerMatrix(y,p);
end

beta = (X'*X)\X'*yfinal;
residuals = yfinal - X*beta;

end
```

VAR Estimation

```
function x = lagmakerMatrix(y,p)
% Function to create the matrix of regressors according to the
% SUR representaiton for a VAR
% Inputs:    y = T x N matrix of endogeneous variables
%            p = VAR lag order
% Outputs:   x = T-p x Np matrix of lagged dependent variables as regressors

[T, N] = size(y);
x = zeros(T-p, N*p);
counter = 0;
for i=1:p
    for j=1:N
        counter = counter + 1;
        x(:,counter) = y(p+1-i:T-i,j);
    end
end
end
```

Compute IRFs

- Check stationarity: eigenvalues of the companion matrix $|\lambda_i| < 1$.
- Then the process admits the multivariate Wold representation

$$Y_t = \nu + \Theta(L)\epsilon_t.$$

- Compute IRFs as $\Theta_0, \Theta_1, \Theta_2$, ... using I_n, F_n, F_n^2, ... (why?)
- F_n are the first $n \times n$ entries in the companion form $F_{np \times np}$.

$$\frac{\partial Y_{1t+h}}{\partial \epsilon_{1t}} = \theta_{11,h}$$

Compute IRFs

- Check stationarity: eigenvalues of the companion matrix $|\lambda_i| < 1$.
- Then the process admits the multivariate Wold representation

$$Y_t = \nu + \Theta(L)\epsilon_t.$$

- Rewrite the VAR(p) using the companion form.

$$Y_t^f = FY_{t-1}^f + v_t$$

$$Y_t^f = \begin{bmatrix} Y_t \\ Y_{t-1} \\ Y_{t-2} \\ Y_{t-3} \end{bmatrix} \qquad F_{np\times np} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ & I_{n(p-1)} & & 0_{n(p-1)\times n} \end{bmatrix}.$$

Compute IRFs

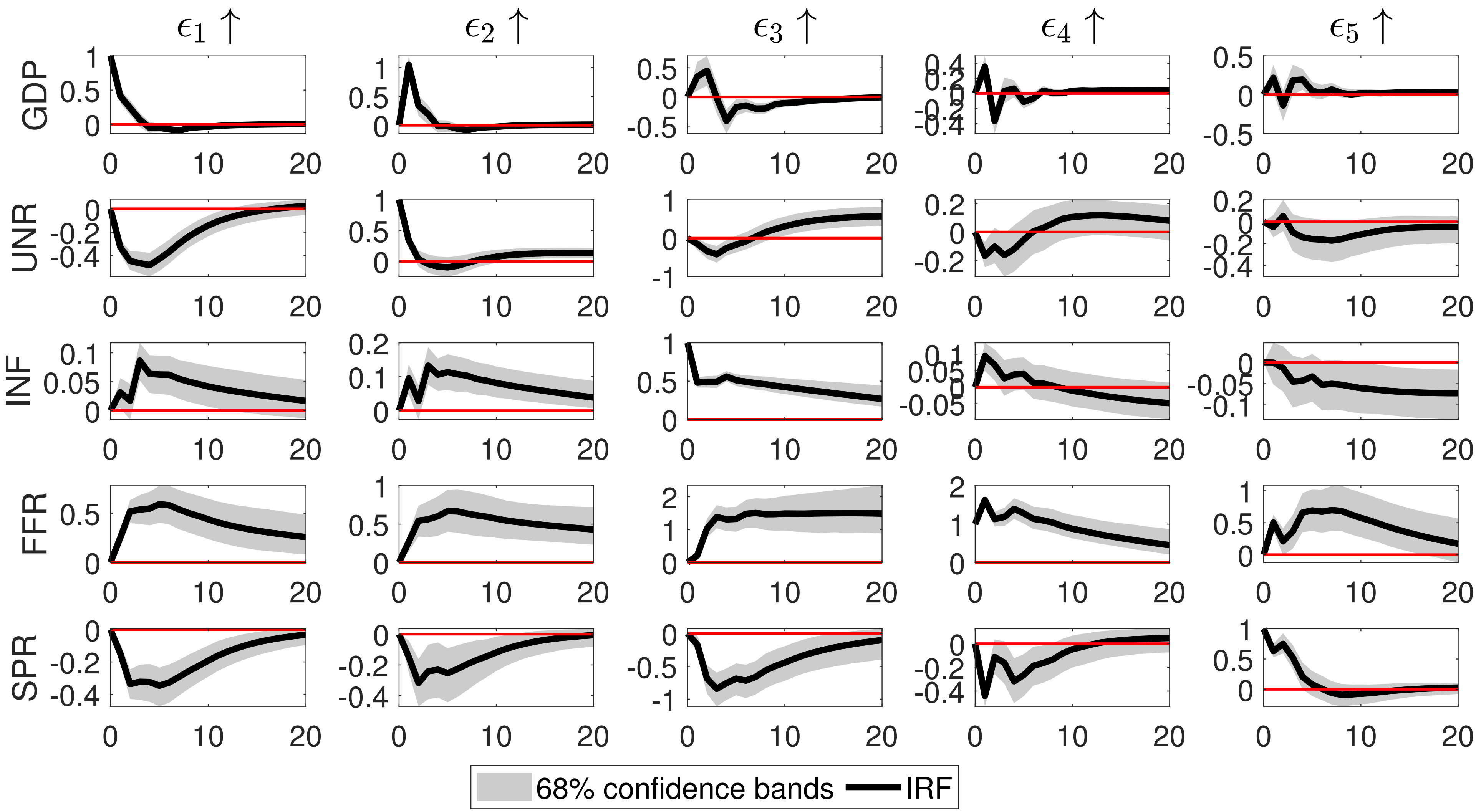
```
% Wold representation impulse responses:

% Populate the companion form matrix
F =[A_hat(2:end,:)'; eye(N*p-N) zeros(N*p-N,N)]; % F companion form, np x np matrix

% Check stability
ev = abs(eig(F));
evmax = ['The maximum eigenvalue is ', num2str(max(ev)), '.'];
disp(evmax)

% Wold IRFs
H = 20;
Theta = zeros(N,N,hor+1);
for j=1:H+1
    Fmat=F^(j-1);
    Theta(:,:,j)=Fmat(1:N,1:N);
end
```

Plot IRFs



$\epsilon_1 \uparrow$
$\epsilon_2 \uparrow$
$\epsilon_3 \uparrow$
$\epsilon_4 \uparrow$
$\epsilon_5 \uparrow$
GDP
UNR
INF
FFR
SPR
68% confidence bands
IRF

Structural VAR

The shocks do not have an economic interpretation:

- Regression residuals include several factors.
- If shocks are correlated cannot separete effects.

Consider the structural VAR

$$B_0 Y_t = \sum_{k=1}^{p} B_k Y_{t-k} + w_t$$

- Allow for contemporaneous effects B_0.
- Structural shocks $E_t(w_t w_t') = I_n$ or any diagonal matrix D.

Structural VAR

Consider the structural VAR

$$B_0 Y_t = \sum_{k=1}^{p} B_k Y_{t-k} + w_t$$

The reduced form representation is

$$Y_t = \sum_{k=1}^{p} B_0^{-1} B_k Y_{t-k} + B_0^{-1} w_t = \sum_{k=1}^{p} A_k Y_{t-k} + \epsilon_t.$$

- $\epsilon_t = B_0^{-1} w_t$.
- $E_t(\epsilon_t \epsilon_t') = B_0^{-1} D (B_0^{-1})'$.

Identification of MP Shocks

- Structural VAR: VAR + parameters restrictions.
- MP shock affects contemporaneously only the spread (partial identification).
- Order the variables as before.
- MA representation of structural shocks $Y_t = \nu + C(L)w_t$.

$$Y = \begin{bmatrix} GDP \\ UNR \\ INF \\ FFR \\ SPR \end{bmatrix}, \qquad C_0 = \begin{bmatrix} c_{11,0} & c_{12,0} & c_{13,0} & 0 & c_{15,0} \\ c_{21,0} & c_{22,0} & c_{23,0} & 0 & c_{25,0} \\ c_{31,0} & c_{32,0} & c_{33,0} & 0 & c_{35,0} \\ c_{41,0} & c_{42,0} & c_{43,0} & c_{44,0} & c_{45,0} \\ c_{51,0} & c_{52,0} & c_{53,0} & c_{54,0} & c_{55,0} \end{bmatrix}$$

Identification of MP Shocks

- Compute the variance covariance matrix $\Sigma_\epsilon = (T - np - p)^{-1} \sum_t \epsilon_t \epsilon_t'$.
- Choleski factorization $\Sigma_\epsilon = PP'$.
- Identify the structural shocks w_t and compute IRFs

$$Y_t = \nu + \Theta(L) P P^{-1} \epsilon_t = \nu + \Theta(L) P w_t$$

- By construction $E(w_t w_t') = P^{-1} \Sigma_\epsilon (P^{-1})' = I_n \Rightarrow$ shocks are orthogonal.
- Recall $\epsilon_t = B_0^{-1} w_t$ we identify $\epsilon_t = P w_t$.
- P sets parameter restrictions (zeros) in the matrix B_0.
- Under stability conditions $\Theta_0 = I_n$, $\Theta_k = F_n^k$.
- To recover structural IRFs $C_0 = P, C_k = \Theta_k P$.

Identification of MP Shocks

```


omega=(err'*err)./(T - N*p-1-p); % Estimate of omega
P=chol(omega,'lower'); % Cholesky factorization, lower triangular matrix

C=zeros(N,N,hor+1);
for i=1:hor+1
    C(:,:,i)=Theta(:,:,i)*P; % Cholesky wold respesentation
end

% Alternatively use the function choleskyIRF
CC = choleskyIRF(Theta, S);
```

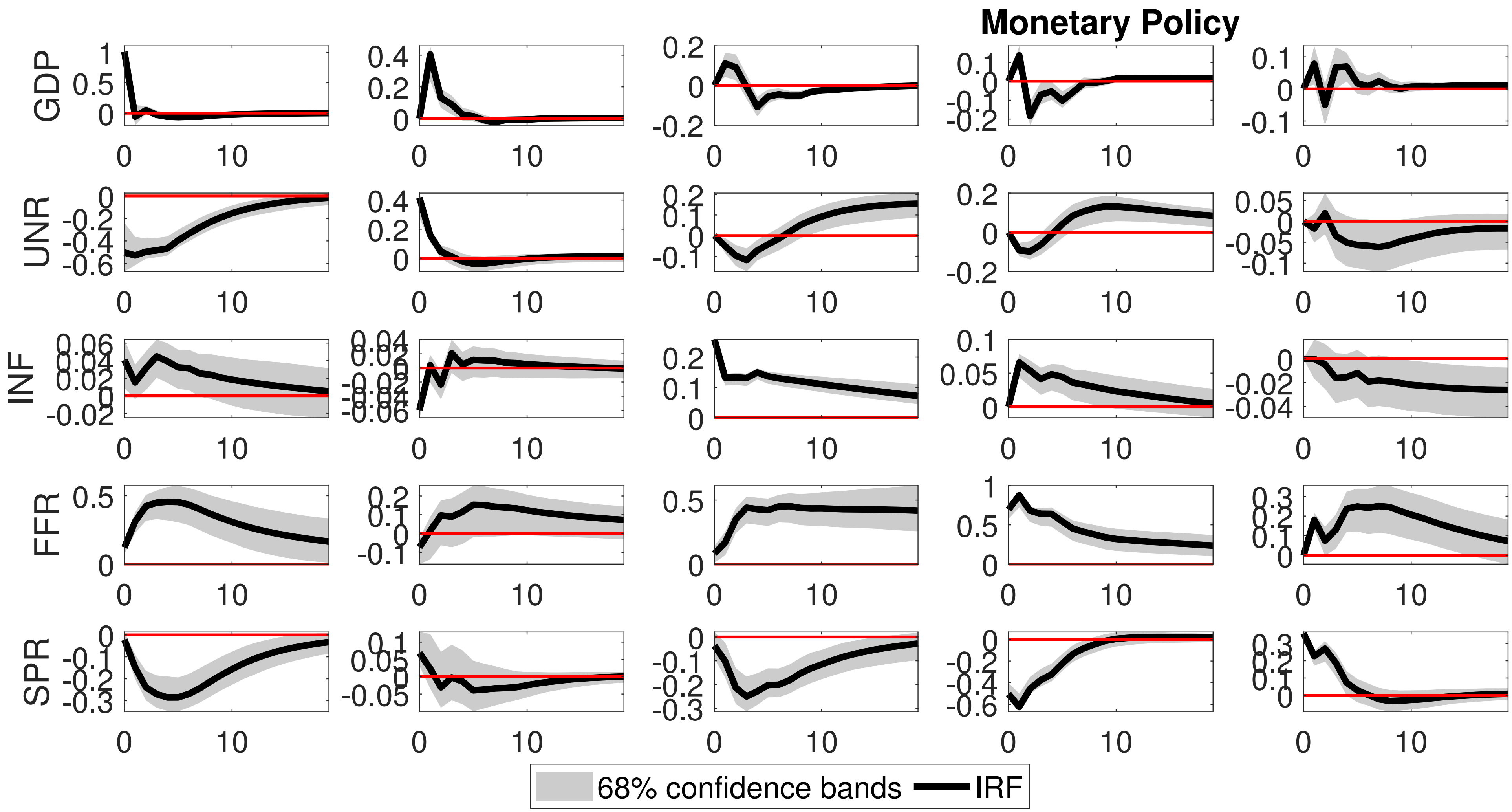
Identification of MP Shocks

```
function [cholirf] = choleskyIRF(wold, S, scaling)
% Function to compute the point estimate of the IRF of a VAR identified
% Inputs:    wold    = (N x N x horizon +1) array of Wold IRFs
%            S       = N x N lower triangular matrix Cholesky factor
%            scaling = 2 x 1 vector where the first argument is the variable
%                      to be used for scaleing and the second is the shock size
% Outputs:   chol = N x N x horizon + 1 array of Cholesky identified IRFs

[N,~, horizon] = size(wold);
cholirf = zeros(N,N,horizon);
for h=1:horizon
    cholirf(:,:,h) = wold(:,:,h) * S;
end

if nargin > 2
    cholirf = cholirf ./ (cholirf(scaling(1),scaling(1),1) * 1/scaling(2));
end
end
```

Plot Structural IRFs



Identification with Long Run Restrictions

- Identify MP shocks using Cholesky and long run restrictions.
- Compute IRFs to the structural MP shocks.

- The long-run level effects are given by $D := I_n + F_n + F_n^2 + ...$
- If the eigenvalues of F_n are such that $|\lambda_i| < 1$

$$D = (I_n - F_n)^{-1}.$$

Identification with Long Run Restrictions

Recall that $E_t(\epsilon_t \epsilon_t') = B_0^{-1}(B_0^{-1})'$.

Therefore note that the long-run IRF to the structural shocks is

$$D = (I_n - F_n)^{-1} B_0^{-1}$$

Moreover,

$$DD' = (I_n - F_n)^{-1} B_0^{-1} (B_0^{-1})' ((I_n - F_n)^{-1})'$$

- Estimate Σ_ϵ use Choleski decomposition to restrict D
- Recover the selection matrix $P = (I_n - F_n)D$.

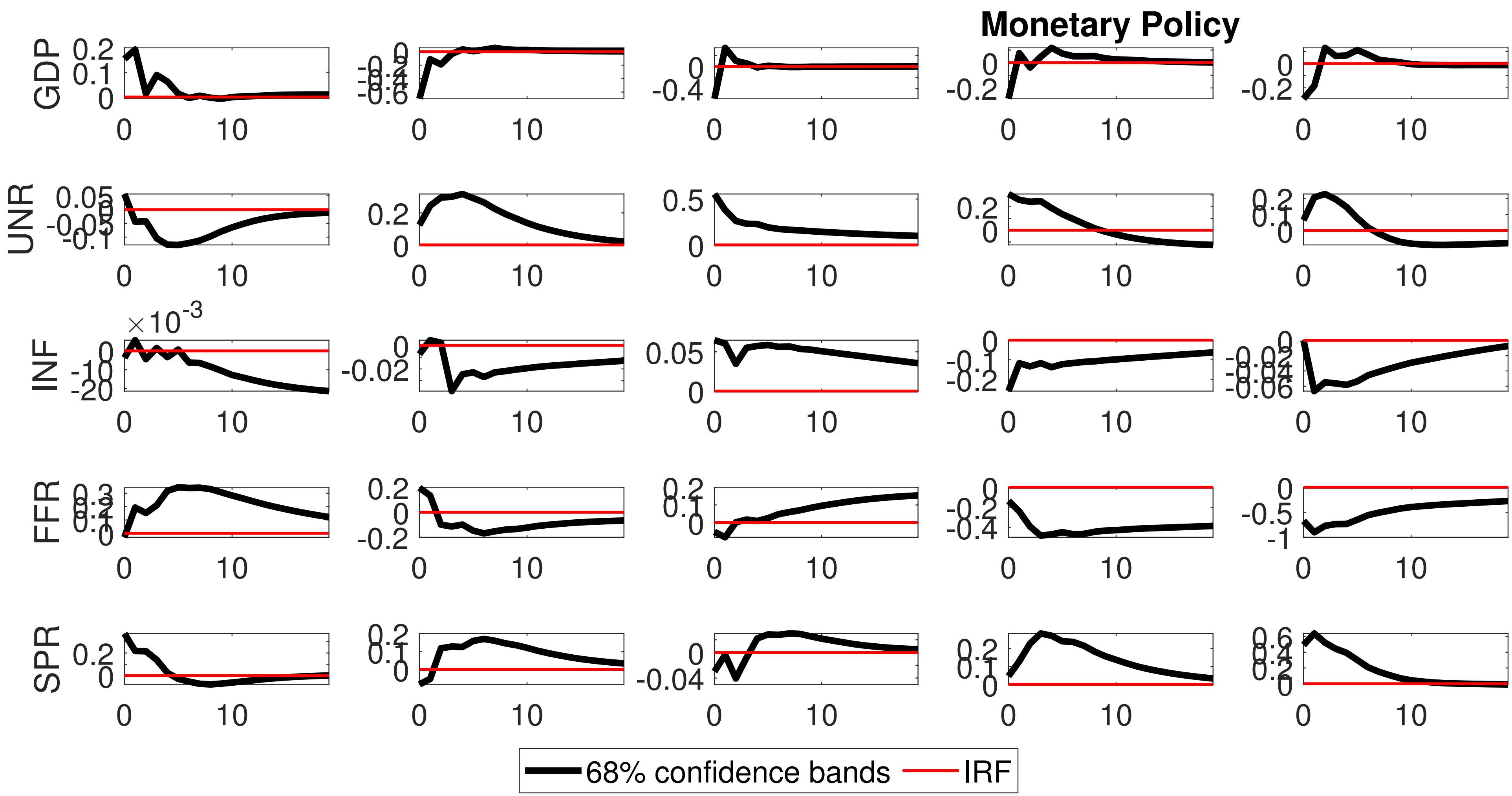
Identification with Long Run Restrictions

```

omega=(err'*err)./(T - N*p-1-p); % Estimate of omega

Finv = inv(eye(N) - F(1:N,1:N));
D = chol(Finv*omega*Finv','lower'); % Cholesky factorization
P = Finv\D;
C=zeros(N,N,hor+1);
for i=1:hor+1
    C(:,:,i) = Theta(:,:,i)*P; % Cholesky wold respesentation
end
```

Identification with Long Run Restrictions



Forecast Error Variance Decomposition

Consider the strucutral Wold representation

$$Y_{t+s} = \nu + C(L)w_{t+s}.$$

Taking conditional expectations of both sides

$$\hat{Y}_{t+s|t} = \nu + \sum_{j=0}^{\infty} C_j w_{t-j}.$$

Therefore,

$$e_t(s) := Y_{t+s} - \hat{Y}_{t+s|t} = \sum_{h=1}^{s} C_h w_{t+h}$$

Forecast Error Variance Decomposition

Proportion of variability in the forecast errors of Y_{it} due to variability in w_t.

$$e_{it}(s) = \sum_{h=1}^{s} \left(\sum_{k=1}^{n} c_{ik,h} w_{k,t+h} \right).$$

Since $w_t \sim iid(0, D)$ the error variance of the ith variable

$$Var(e_{it}(s)) = \sum_{k=1}^{n} \sigma_k^2 (c_{ik,1}^2 + ... + c_{ik,s}^2)$$

If $D = I_n$ the forecast error variance of the ith variable to the jth shock is

$$FEVD_{ij}^{s} = \frac{\sum_{h=0}^{s} c_{ij,h}^2}{\sum_{k=1}^{n} \sum_{h=0}^{s} c_{ik,h}^2}.$$

Forecast Error Variance Decomposition

```
% First square all the elements of the IRF
squared_irfs = D.^2; % For Cholesky D(var, shock, horizon)

% Now construct the numerator at all horizon s = 1,2,...,20
% Sum along all horizons but cumulated to get the evolution over H
numerators = cumsum(squared_irfs,3);
% Sum the sqaured sums along all the shocks to get the denominator
denominators = sum(numerators,2);
denominators_tot = squeeze(denominators(:,1,:));

fevd = zeros(hor+1,N);
for h=1:hor+1 % Pick a horizon
    for nn=1:N % Pick a variable
    % Compute the variance of the variable due to Mon.pol. shock
    num_monpol(h,nn) = numerators(nn,4,h);
    % Compute the total variance of the variable
    denom_monpol(h,nn) = denominators_tot(nn,h);
    end
end

% Compute the ratios of variance explained by monpol shock
fevd = num_monpol./denom_monpol;
```

Forecast Error Variance Decomposition