Chapter 4: New Keynesian models

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Quantitative Macroeconomics

Introduction. Price frictions in a neoclassical framework allow us to connect the real side of the economy to nominal variables providing a framework to analyze the macroeconomic effects of monetary and fiscal policies.

I. Neoclassical monetary models

1 Fiscal and monetary policy

Consider the neoclassical growth model with TFP shocks z_t from Chapter 3. We can augment the model with a fiscal policy. Assume a one-period riskless and not state contingent real government bond b_t with price $q_t = R_{t+1}^{-1}$, proportional taxes on consumption τ_t^c and household income τ_t^y, lump-sum transfers τ_t, and government spending g_t. Fiscal policy is a path $\{g_t, \tau_t, \tau_t^c, \tau_t^y\}_{t=0}^{\infty}$ such that $g_t + b_{t-1} = q_t b_t + T_t, \forall t$ where $T_t := \tau_t + \tau_t^c C_t + \tau_t^y (w_t N_t + (1 + r_t - \delta) K_t)$. The integral form of the government budget constraint is given by

$$b_{t-1} = \mathrm{E}_t \sum_{j=0}^{\infty} \beta^j \frac{u_c(c_{t+j})}{u_c(c_t)} (T_{t+j} - g_{t+j}).$$

It is derived solving forward the government budget constraint using a binding no-Ponzi condition and replacing equilibrium prices from the Euler equation under separable preferences $u(c, n)$ and $\tau_t^c = 0$. The government must run a primary surplus in present value, whenever it has outstanding debt $b_{t-1} > 0$. Notice that we have many equilibria each indexed by the exogenous fiscal policy.

Ricardian equivalence. Suppose the government has to finance an exogenous real expenditures $\{g_t\}$ using lump-sum taxes $\{\tau_t\}$. The Ricardian equivalence states that financing government expenditure with lump-sum taxes or debt has the same equilibrium effect. In a simple Arrow-Debreu setting

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t c_t = \sum_{z^t \in Z^t} \sum_{t=0}^{\infty} (p_t w_t - p_t \tau_t).$$

In the sequential markets case $b_t + (1 + r_t - \delta)k_t = c_t + k_{t+1} - w_t + q_t b_{t+1} + \tau_t$ using $1 = \mathrm{E}_t[\beta u_c(c_{t+1})/u_c(c_t)](1 + r_{t+1} - \delta)$ and solving forward

$$b_t + (1 + r_t - \delta)k_t = \mathrm{E}_t \sum_{j=0}^{\infty} \beta^j \frac{u_c(c_{t+j})}{u_c(c_t)} (c_{t+j} - w_{t+j} + \tau_{t+j}), \forall t.$$

Given any two different policy $\{\tau_t\}, \{\hat{\tau}_t\}$ as long as $\mathrm{E}_t \sum_{t=0}^{\infty} \tau_t = \mathrm{E}_t \sum_{t=0}^{\infty} \hat{\tau}_t$ the equilibrium outcome is the same. Agents internalize that public debit will be paid by future taxes and adjust private consumption today increasing private saving to compensate the lower public saving. Under Ricardian expectations there is a crowding-out effect on private consumption in any case. Moreover, under Ricardian equivalence government bonds are not net wealth. To see this rewrite the government budget constraint as $g_t + b_t = q_t b_{t+1} + \tau_t$, subsituting lump-sum taxes in the integral form of the household budget constraint from the government budget constraint we obtain on the right hand side the household net financial wealth at $t = 0$

$$R_0 k_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{u_c(c_t)}{u_c(c_0)} (c_t - w_t + g_t).$$

Monetary neutrality. In the neoclassical growth model with TFP shocks. The Euler equation, the optimal labor supply condition, the first order conditions of the firm, and market clearing determine $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}, \{w_t, r_t\}_{t=0}^{\infty}$ as a function of $\{z_t, k_t\}_{t=0}^{\infty}$. In the model we can determine the equilibrium of the real variables without any reference to nominal variables. It turns out that this is the case also for neoclassical monetary models with nominal variables $\{m_t, p_t, \pi_t, i_t\}_{t=0}^{\infty}$ where m_t is money or nominal currency, p_t the price level, $\pi_t := (p_{t+1} - p_t)/p_t$ the inflation rate, i_t the nominal interest rate. For example, models with money in the utility function, and models with cash-in-advance constraints. In this framework money is neutral and monetary policy has no real effects. Even in models with money in the utility function and nonseparable preferences money is quantitatively almost neutral.

2 Local determinacy and global multiplicity

By linearizing the equilibrium conditions of neoclassical monetary models it is possible to show that an exogenous sequence $\{m_t\}$ implies a locally unique and stable price level $\{p_t\}$. See for example Galì (2015). Alternatively, the central bank can set interest rates $\{i_t\}$. Sargent and Wallace (1975) using a linear model with rational expectations argue that an exogenous path $\{i_t\}$ leads to multiple and stable solutions $\{p_t\}$. Woodford (2003) and Galì (2015) show how interest rates policies $\{i_t\}$ with at least an endogenous component can solve the local multiplicity.

Consider the case in which the central bank sets $\{i_t\}$ exogenously. Among the equilibrium conditions there is a Fisher equation $i_t = \mathrm{E}_t\pi_{t+1} + r_t$. Only expected inflation is pinned down and there are no others equilibrium equations to determine inflation. Rewriting the Fisher equation

$$\hat{p}_{t+1} - \hat{p}_t = i_t - r_t + u_{t+1}$$

where $\{u_t\}$ is the prediction error such that $\mathrm{E}_t u_{t+1} = 0, \forall t$ and $\hat{p}_t := \ln p_t$. The error term u_{t+1} is unrelated to the economic fundamentals therefore is also called sunspot shock. All the equilibria are indexed by this shock. Any sequence $\{p_t\}$ satisfying the previous equation while remaining in a neighborhood of the steady state is an equilibrium. Suppose that instead the central bank sets $\{i_t\}$ endogenously according to $i_t = \phi\pi_t$ where $\phi \geq 0$. Using again the Fisher equation $\phi\pi_t = \mathrm{E}_t\pi_{t+1} + r_t$. If $\phi > 1$ then we can solve the difference equation forward to obtain

$$\pi_t = \sum_{j=0}^{\infty} \phi^{-j-1}\mathrm{E}_t r_{t+j}.$$

The other nominal variables follows. From $1 + \pi_t = p_{t+1}/p_t$ we can determine $\{p_t\}$, then from the money demand and supply equations we can get $\{m_t\}$. The condition for a local determinate and stable price level $\phi > 1$ is known as the Taylor principle. See Taylor (1993). If $\phi < 1$ the equilibrium is again indeterminate. The local indeterminacy of the price level implies that all the nominal variables are indeterminate and subject to sunspot volatility. Alternatively, the fiscal theory of the price level leverage the government budget as additional equation to pin down inflation.

Neoclassical monetary models can feature a global multiplicity of equilibria driven by the nominal block of the model. For example Obstfeld and Kenneth (1983, 1986) show how multiple equilibria characterized by hyperinflation can arise. Moreover, Benhabib, Schmitt-Grohe, and Uribe (2001, 2002) show that by introducing a zero lower bound $i_t \geq 0$ on nominal interest rates deflationary equilibria originating from $\pi_0 < \pi^*$ where $\pi^* > 0$ is the steady state inflation cannot be ruled out using a transversality condition or the Taylor principle.

Monetary and fiscal interactions. Consider both monetary and fiscal policy. If the government adjusts its policy to ensure intertemporal budget balance then monetary policy is free to set the $\{m_t\}$ or $\{i_t\}$. A second policy regime is one in which the government sets $\{g_t, \tau_t\}$ without any attention to intertemporal budget balance as a consequence monetary policy must be adjusted to balance the public sector budget. Sargent (1982) defines the two regimes as Ricardian and non-Ricardian. Leeper (1991) as active monetary policy and active fiscal policy. If government sets the primary deficit to back a fraction $\psi \in [0,1]$ of its debt that is $\tau_t - g_t = \psi((1+r_t)b_{t-1} - b_t)$ or $\sum_{j=0}^{\infty} R_j(\tau_{t+j} - g_{t+j}) = \psi b_{t-1}$, then $\psi = 1$ is the Ricardian regime, $\psi < 1$ is non-Ricardian. These assumptions are not innocuous and affect the dynamics of the model.

II. Monopolistic competition and sticky prices

1 Monopolistic competition models

Continuum of final goods. Instead of a single good there is a continuum of $i \in [0,1]$ varieties of an homogeneous good. Let $\varepsilon > 1$ be the elasticity of substitution among differentiated goods, $p_t = \int_0^1 p_t(i)di$ a price index and c_t a per capita consumption bundle. Then, the representative agent

$$\min_{c_t(i)} \int_0^1 p_t(i)c_t(i)di$$

$$\text{s.t.} \quad c_t = \left[\int_0^1 c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}, \forall t.$$

The first order condition is given by $p_t(i) - \lambda_t \frac{\varepsilon}{\varepsilon-1}\left[\int_0^1 c_t(i)^{1-\varepsilon^{-1}} di\right]^{\frac{1}{\varepsilon-1}} \frac{\varepsilon-1}{\varepsilon} c_t(i)^{-\varepsilon^{-1}} = 0$ That is $\lambda_t c_t(i)^{-\varepsilon^{-1}} c_t^{\varepsilon^{-1}} = p_t(i)$. Multiplying both sides of the first order condition by $c_t(i)$ and integrating over i we have $p_t c_t = \lambda_t c_t \Rightarrow p_t = \lambda_t$. Thus,

$$c_t(i) = c_t \left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon}.$$

In equilibrium $c_t(i) = Y_t(i)$, $c_t = Y_t$. Moreover, $p_t c_t = \int_0^1 p_t(i) c_t (p_t(i)/p_t)^{-\varepsilon} di$. Hence,

$$p_t = \left[\int_0^1 p_t(i)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Alternatively, we can derive the previous results from a utility maximization problem. Assume a CES utility function with elasticity of substitution given by $\varepsilon = 1/(1-\rho)$. The elasticity controls the degree of market power with $\varepsilon = 1$ we have monopoly with $\varepsilon \to \infty$ perfect competition. Define a composite good m and income y. The consumer

$$\max_{c_t(i)} \quad \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\int_0^1 c_t(i)^\rho di \right]^{\frac{1}{\rho}}$$

$$\text{s.t.} \quad \int_0^1 p_t(i) c_t(i) di + m_t \leq y_t, \forall t.$$

Following the same steps as before we arrive to $c_t(i) = (p_t(i)/p_t)^{-\varepsilon} c_t$ or dividing the first order condition evaluated at good i and j yields $c_t(i)/c_t(j) = (p_t(i)/p_t(j))^{-\varepsilon}$. Multiplying both sides by $p_t(i)$ and then integrating over i yields the following Marshallian demand function $c_t(j) = (y_t p_t(j)^{-\varepsilon})/(\int_0^1 p_t(i)^{1-\varepsilon} di).$, which is exactly the demand function above once we define the price index $p_t = (\int_0^1 p_t(i)^{1-\varepsilon})^{\frac{1}{1-\varepsilon}}$.

Continuum of intermediate goods. An equivalent approach is to assume only one final good and a continuum of intermediate goods $i \in [0, 1]$. Given prices a representative firm produces the final good in perfect competition to maximize profits given the production technology. The production function is a CES aggregator. Hence, the firm

$$\max_{y_t(i)} \quad p_t Y_t - \int_0^1 p_t(i) Y_t(i) di,$$

$$\text{s.t.} \quad Y_t = \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

The first order condition is $p_t \frac{\varepsilon}{\varepsilon-1} \left[\int_0^1 Y_t(i)^{1-\varepsilon^{-1}} di \right]^{\frac{1}{(\varepsilon-1)}} \frac{\varepsilon-1}{\varepsilon} Y_t(i)^{-\varepsilon^{-1}} - p_t(i) = 0$ Dividing the first order condition of two intermediate goods i and j yields $p_t(j) = (Y_t(i)/Y_t(j))^{\frac{1}{\varepsilon}} p_t(i)$. Rewrite $p_t(j) Y_t(j) = p_t(i) Y_t(i)^{\varepsilon^{-1}} Y_t(j)^{1-\varepsilon^{-1}}$. Integrating over j by the zero profit $p_t Y_t = p_t(i) Y_t(i)^{\varepsilon^{-1}} \int_0^1 Y_t(j)^{1-\varepsilon^{-1}} dj$. Thus,

$$Y_t(i) = \left(\frac{p_t(i)}{p_t} \right)^{-\varepsilon} Y_t$$

that together with the zero profit condition implies

$$p_t = \left[\int_0^1 p_t(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}.$$

Marginal costs and price setting. There is a continuum of firms indexed by $i \in [0,1]$. They produce either the differentiated goods or intermediate goods under monopolistic competition. The technology is the same across producers and $\alpha \in (0,1)$.

$$\min_{K_t(i),N_t(i)} \quad w_t N_t(i) - r_t K_t(i), \quad \text{s.t.} \quad Y_t(i) = A_t K_t(i)^{\alpha} N_t(i)^{1-\alpha}, \forall t.$$

The first order condition is given by $r_t K_t(i) = \alpha(1-\alpha)^{-1} w_t N_t(i)$. Thus, all firms choose the same capital-labor ratio $K_t(i)/N_t(i) = K_t/N_t$ and the same real marginal costs $mc_t(i) = mc_t$. Let $mc_t := d(w_t N_t(i) - r_t K_t(i))/dY_t(i)$. Substituting the first order condition and production function in the total cost we arrive at

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \left(\frac{1}{A_t}\right) w_t^{1-\alpha} r_t^{\alpha}.$$

If there is a single factor of production. Then firms

$$\min_{N_t(i)} \quad w_t N_t(i), \quad \text{s.t.} \quad Y_t(i) = A_t N_t(i)^{1-\alpha}, \forall t$$

From the first order condition we have $mc_t(i) = \frac{w_t}{A_t(1-\alpha)N_t(i)^{-\alpha}}$. Substituting for $N_t(i)$ from the production function and integrating over i we have mc_t. Thus,

$$mc_t(i) = mc_t \left(\frac{Y_t}{Y_t(i)}\right)^{\frac{-\alpha}{1-\alpha}} = mc_t \left(\frac{p_t(i)}{p_t}\right)^{\frac{\varepsilon\alpha}{1-\alpha}}.$$

Finally, each producer has a market power and set the price facing the entire market demand from households for final good producers or from the final good firm in the case of intermediate producers. Let's do the latter case.

$$\max_{p_t(i)} \quad p_t(i) Y_t(i) - p_t mc_t Y_t(i), \quad \text{s.t.} \quad Y_t(i) = Y_t \left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon}, \forall t.$$

The first order condition $\frac{1}{p_t}\left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon} Y_t - \varepsilon \left(\frac{p_t(i)}{p_t} - mc_t\right)\left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon-1} \frac{1}{p_t} Y_t = 0$ implies

$$p_t(i) = \frac{\varepsilon}{\varepsilon - 1} p_t mc_t.$$

The monopolistic competition model implies an optimal price equal to a constant markup over marginal cost. The producers of the differentiated good or the intermediate producers are earning positive profits. The households claim the monopolistic profits.

2 Sticky price models

Calvo pricing. Either the intermediate good producers or the producers of differentiated goods can only change the price following a Calvo's rule. In the spirit of Calvo (1983), each period a firm adjusts its price with probability $1-\theta$ where $\theta \in [0,1]$. With a continuum of firms θ is the fraction of firms that do not adjust the price. By simmetry let p_t^* be the optimal price chosen by firm i. Let $Y_{t+j|t}$ be the production in $t+j$ of the firm that last set is price in t, ψ_t the cost function. The second part of the firms problem becomes maximizing the present value of real profits with stochastic discount factor given by the Euler equation subject to the entire market demand

$$\max_{p_t^*} \quad \mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \left[\frac{\beta^j \lambda_{t+j}}{\lambda_t} (p_t^* Y_{t+j|t} - \psi_{t+j}(Y_{t+j|t})) \right]$$

$$\text{s.t.} \quad Y_{t+j|t} = \left(\frac{p_t^*}{p_{t+j}} \right)^{-\varepsilon} Y_{t+j}, \forall t.$$

Substituting the constraint in the objective function and using $\partial Y_{t+j|t}/\partial p_t^* = -\varepsilon Y_{t+j|t}/p_t^*$. The first order condition is given by

$$\mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \left[\frac{\beta^j \lambda_{t+j}}{\lambda_t} Y_{t+j|t} \left(1 + \frac{p_t^*}{Y_{t+j|t}} \frac{\partial Y_{t+j|t}}{\partial p_t^*} - \frac{\psi'_{t+j}}{Y_{t+j|t}} \frac{\partial Y_{t+j|t}}{\partial p_t^*} \right) \right] = 0 \Rightarrow$$

$$\mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \left[\frac{\beta^j \lambda_{t+j}}{\lambda_t} Y_{t+j|t} (p_t^* - \mu \psi'_{t+j|t}) \right] = 0$$

where $\mu = \varepsilon/(\varepsilon - 1)$ and $\psi'_{t+j} = p_{t+j} mc_{t+j}$ are the nominal marginal costs. Note that with $\theta = 0$ we are back to the fully flexible price case and the previous expression implies

$$p_t^* = \frac{\varepsilon}{\varepsilon - 1} p_t mc_t.$$

Let $s_t \in [0,1]$ be the number of firms that do not adjust in period t. From the definition of p_t we have

$$p_t = \left[\theta \int_{s_t}^{1} p_{t-1}(i)^{1-\varepsilon} di + (1-\theta)(p_t^*)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} = [\theta p_{t-1}^{1-\varepsilon} + (1-\theta)(p_t^*)^{1-\varepsilon}]^{\frac{1}{1-\varepsilon}}.$$

Dividing by p_{t-1} we have

$$\left(\frac{p_t}{p_{t-1}} \right)^{1-\varepsilon} = \theta + (1-\theta) \left(\frac{p_t^*}{p_{t-1}} \right)^{1-\varepsilon}.$$

Rotemberg pricing. Following Rotemberg (1982) either the intermediate good producers or the producers of differentiated goods face a quadratic cost of price adjustment. Firms solve

$$\max_{\{p_t^*\}} \ \mathrm{E}_t \sum_{t=0}^{\infty} \left[Q_t \left(p_t^* \frac{Y_t(i)}{p_t} - \frac{\psi_t(Y_t(i))}{p_t} - \frac{\Psi}{2} \left(\frac{p_t^*}{p_{t-1}^*} - 1 \right)^2 Y_t \right) \right]$$

$$\text{s.t. } Y_t(i) = \left(\frac{p_t^*}{p_t} \right)^{-\varepsilon} Y_t, \forall t.$$

The first order condition is given by

$$(\varepsilon - 1) \left(\frac{p_t^*}{p_t} \right)^{-\varepsilon} \frac{Y_t}{p_t} = \varepsilon mc_t \left(\frac{p_t^*}{p_t} \right)^{-\varepsilon - 1} \frac{Y_t}{p_t} - \Psi \left(\frac{p_t^*}{p_{t-1}^*} - 1 \right) \frac{Y_t}{p_{t-1}^*} + \Psi \mathrm{E}_t \left[Q_{t+1} \left(\frac{p_{t+1}^*}{p_t^*} - 1 \right) \frac{p_{t+1}^*}{p_t^*} \frac{Y_{t+1}}{p_t^*} \right]$$

or

$$(\varepsilon - 1) = \varepsilon mc_t - \Psi(1 + \pi_t)\pi_t + \beta \mathrm{E}_t[\lambda_{t+1}\lambda_t^{-1}\Psi(1 + \pi_{t+1})\pi_{t+1}Y_{t+1}Y_t^{-1}].$$

The second equation follows from the symmetry conditions $p_t = p_t^*, Y_t(i) = Y_t$, with log-utility and market clearing this expression simplifies even further.

Distortions. Monopolistic competition and price rigidities introduce distorsions in the economy. Consider the case of only one production input. Integratign over i from the production and demand fucntions

$$N_t = \int_0^1 \left(\frac{Y_t(i)}{A_t} \right)^{\frac{1}{(1-\alpha)}} di = \left(\frac{Y_t}{A_t} \right)^{\frac{1}{(1-\alpha)}} \int_0^1 \left(\frac{p_t(i)}{p_t} \right)^{\frac{-\varepsilon}{(1-\alpha)}} di.$$

Solving for Y_t we have

$$Y_t = A_t N_t^{1-\alpha} \left[\int_0^1 \left(\frac{p_t(i)}{p_t} \right)^{\frac{-\varepsilon}{1-\alpha}} di \right]^{\alpha - 1} = A_t N_t^{1-\alpha} \Delta_t^{\alpha - 1}.$$

where $\Delta_t = \int_0^1 (p_t(i)/p_t)^{-\varepsilon} di$ is a measure of price dispersion or efficiency losses implied by inflation. This term appears only if we use the Calvo model. In the Rotemberg model we impose symmetry across firms and $p_t(i) = p_t, \forall i$, with Rotemberg the cost of price rigidities is $(\Psi/2)\pi_t^2 Y_t$. We can add this term to the market clearing condition of aggregate output. If we do not add the cost we can interpret the adjustment costs as being a virtual costs, i.e. they affect decisions but not real resources.

Let Y_t^* be the aggregate output with flexible prices and perfect competition $\mu = 1$, and let Y_t^n be aggregate output with flexibile price and monopolistic competition. One can show that $Y_t^n < Y_t^*$ because the price exceeds marginal costs.

III. New Keynesian models

We are now ready to formulate and analyze New Keynesian (NK) models. I will focus on discrete time and consider first a NK without capital and then a NK model with capital. In contrast with the neoclassical monetary models these models are in their basic versions cashless economies, i.e. do not include money in the analysis and focus only on interest rate policies.

1 The basic NK model

Equilibrium. In the economy there is no capital but a one-period real asset a_t. A seqeuntial competitive equilibrium in the basic NK model is $\{p_t, i_t, \Delta_t, w_t, C_t, N_t, c_t, n_t, a_{t+1}, \Pi_t\}$:

1. Given the prices $\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t, n_t, a_{t+1}\}} \mathrm{E} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} - \chi \frac{n_t^{1+\nu}}{1+\nu} \right]$$

$$\text{s.t.} \quad c_t + a_{t+1}(p_t(1+i_t))^{-1} = w_t n_t + a_t + \Pi_t, \forall t,$$

$$c_t \geq 0, a_t \geq -\phi, \forall t.$$

2. Given the prices firms

$$\max_{y_t(i)} \; p_t Y_t - \int_0^1 p_t(i) Y_t(i) di,$$

$$\text{s.t.} \quad Y_t = \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

$$\min_{N_t(i)} \; w_t N_t(i)$$

$$\text{s.t.} \quad Y_t(i) = A_t N_t(i)^{1-\alpha}, \forall t,$$

$$\max_{p_t^*} \; \mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \left[\frac{\beta^j \lambda_{t+j}}{\lambda_t} (p_t^* Y_{t+j|t} - \psi_{t+j}(Y_{t+j|t})) \right]$$

$$\text{s.t.} \quad Y_{t+j|t} = \left(\frac{p_t^*}{p_{t+j}} \right)^{-\varepsilon} Y_{t+j}, \forall t.$$

3. Prices are such that markets clear

$$C_t = A_t \Delta_t^{\alpha-1} N_t^{1-\alpha}, \forall t$$

$$a_t = 0, \forall t$$

$$N_t = n_t, \forall t.$$

4. Monetary policy choose $\{i_t\}_{t=0}^{\infty}$ such that

$$\ln\left(\frac{1+i_t}{1+i}\right) = \phi_\pi \ln\left(\frac{p_{t+1}}{p_t}\right) + v_t, \forall t.$$

The household budget constraint is obtained dividing the nominal sequential nominal budget $p_t c_t + a_{t+1}(1+i_t)^{-1} = p_t w_t n_t + p_t \Pi_t$, where Π_t are real profits. by the price level p_t. In the Taylor rule $i = \beta^{-1} - 1$ is the nominal interest rate in the steady state and $v_t \overset{iid}{\sim} N(0, \sigma_v^2)$ or $v_t \sim$ AR(1) is a monetary policy shock.

Equilibrium conditions. The equilibrium system can be characterized as

$$mc_t(i) = mc_t(p_t^*/p_t)^{\frac{\varepsilon\alpha}{1-\alpha}},$$

$$Y_{t+j|t} = (p_t^*/p_{t+j})^{-\varepsilon} Y_{t+j},$$

$$p_t^* = \mu(\mathrm{E}_t \textstyle\sum_{j=0}^{\infty} \theta^j \beta^j c_{t+j}^{-\gamma} Y_{t+j|t})^{-1} \mathrm{E}_t \textstyle\sum_{j=0}^{\infty} \theta^j \beta^j c_{t+j}^{-\gamma} Y_{t+j|t} p_{t+j} mc_{t+j|t},$$

$$p_t^{1-\varepsilon} = \theta p_{t-1}^{1-\varepsilon} + (1-\theta)(p_t^*)^{1-\varepsilon},$$

$$c_t^{-\gamma} = \beta \mathrm{E}_t[(p_t/p_{t+1})(1+i_t)c_{t+1}^{-\gamma}],$$

$$\chi n_t^{\nu} = w_t c_t^{-\gamma},$$

$$\lim_{t\to\infty} \mathrm{E}_0 \beta^t c_t^{-\gamma} a_{t+1} = 0.$$

Together with Taylor rule, budget constraints and market clearing conditions. We can define the real interest rate as $1 + r_{t+1} := (p_t/p_{t+1})(1+i_t)$. This is the ex post Fisher equation. The ex ante version holds in expectation $1 + r_{t+1} = \mathrm{E}_t(p_t/p_{t+1})(1+i_t)$.

2 Solving the model locally

We can rewrite the system in log-linear form and solve it by standard methods as in Chapter 3. Now the variables are expressed in deviation from the deterministic steady state.

The IS Curve. Log-linearizing the Euler equation and using the market clearing condition, $\sigma = \gamma^{-1}$ we obtain

$$y_t = \mathrm{E}_t y_{t+1} - \sigma(i_t - \mathrm{E}_t \pi_{t+1} - \rho). \tag{1}$$

Let r_t^n be the natural real interest rate defined by $(Y_t^n)^{-\gamma} = \beta(1 + r_t^n)\mathrm{E}_t(Y_{t+1}^n)^{-\gamma}$, in log-linear form $r_t^n = -\ln\beta + \gamma(\mathrm{E}_t y_{t+1}^n - y_t^n)$. Combining the previous equations yields a forward looking IS curve, which stands for invesment-saving.

$$x_t = \mathrm{E}_t x_{t+1} - \sigma(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n). \tag{2}$$

Solving forward the IS and assuming that $\lim_{T\to\infty} \mathrm{E}_t x_{t+T} = 0$ we have

$$x_t = -\sigma \sum_{j=0}^{\infty} (r_{t+j} - r_{t+j}^n).$$

Forward looking Phillips curve - step 1. Let's start with the price setting equation. Substituting for the stochastic discount factor and $Y_{t+j|t}$ and solving for the optimal price we obtain

$$p_t^* \mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\gamma} p_{t+j}^{\varepsilon-1} = \mu \mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \beta^j Y_{t+j}^{1-\gamma} p_{t+j}^{\varepsilon} mc_{t+j|t}.$$

Log-linearizing each side and equating we get

$$\begin{aligned}
& p^* \sum_j (\theta\beta)^j Y^{1-\gamma} p^{\varepsilon-1} p_t^* \\
& + \mathrm{E}_t \sum_j (\theta\beta)^j p^* Y^{1-\gamma} p^{\varepsilon-1}(1-\gamma) Y_{t+j} + \mathrm{E}_t \sum_j (\theta\beta)^j p^* Y^{1-\gamma} p^{\varepsilon-1}(\varepsilon - 1) p_{t+j} \\
& = \mu \mathrm{E}_t \sum_j (\theta\beta)^j p^* Y^{1-\gamma} p^{\varepsilon} mc[(1-\gamma) Y_{t+j} + \varepsilon p_{t+j} + mc_{t+j|t}].
\end{aligned}$$

Simplifying yields

$$p_t^* = (1 - \beta\theta) \sum_{j=0}^{\infty} (\beta\theta)^j \mathrm{E}_t[mc_{t+j|t} + p_{t+j}]. \tag{3}$$

Forward looking Phillips curve - step 2. The log-linear marginal cost equation is

$$mc_{t+j|t} = mc_{t+j} - b(p_t^* - p_{t+j}).$$

where $b = \varepsilon\alpha(1-\alpha)^{-1}$. Plugging this expression in (3) we have that

$$
\begin{aligned}
p_t^* - \beta\theta \mathrm{E}_t p_{t+1}^* =& (1-\beta\theta)\mathrm{E}_t \Bigg\{ \sum_{j=0}^{\infty} (\beta\theta)^j [mc_{t+j} - b(p_t^* - p_{t+j}) + p_{t+j}] \\
& - \beta\theta \sum_{j=0}^{\infty} (\beta\theta)^j [mc_{t+1+j} - b(p_{t+1}^* - p_{t+1+j}) + p_{t+1+j}] \Bigg\} \\
&= (1-\beta\theta)\{mc_t - b(1-\beta\theta)^{-1} p_t^* + (1+b)p_t + b\beta\theta(1-\beta\theta)^{-1}\mathrm{E}_t p_{t+1}^*\}
\end{aligned}
$$

Thus,

$$p_t^*(1+b) = (1-\beta\theta)mc_t + (1+b)\beta\theta\mathrm{E}_t p_{t+1}^* + (1-\beta\theta)(1+b)p_t$$

or

$$(p_t^* - p_t) = (1-\beta\theta)(1+b)^{-1}mc_t + \beta\theta\mathrm{E}_t(p_{t+1}^* - p_{t+1} + p_{t+1} - p_t).$$

Log-linearizing the price dynamics we obtain

$$\theta\pi_t = (1-\theta)(p_t^* - p_t).$$

Therefore, rearranging terms yields

$$\pi_t = \beta\mathrm{E}_t\pi_{t+1} + \frac{(1-\beta\theta)(1-\theta)}{(1+b)\theta} mc_t. \tag{4}$$

Forward looking Phillips curve - step 3. Log-linearizing marginal costs, labor supply, market clearing, aggregate production function where the price dispersion can be ignored up to a first order approximation and potential output we have

$$
\begin{aligned}
mc_t &= w_t + \frac{\alpha}{1-\alpha}y_t - \frac{1}{1-\alpha}a_t = (\nu n_t + \gamma y_t) + \frac{\alpha}{1-\alpha}y_t - \frac{1}{1-\alpha}a_t \\
&= \frac{\nu+\alpha+\gamma(1-\alpha)}{(1-\alpha)}y_t - \frac{\nu+1}{1-\alpha}a_t = \frac{\nu+\alpha+\gamma(1-\alpha)}{(1-\alpha)}(y_t - y_t^n).
\end{aligned}
$$

Define the output gap as $x_t = y_t - y_t^n = \ln Y_t - \ln Y_t^n$. Finally, we have

$$\pi_t = \beta\mathrm{E}_t\pi_{t+1} + \frac{(1-\beta\theta)(1-\theta)(\nu+\alpha+\gamma(1-\alpha))}{\theta(1-\alpha+\varepsilon\alpha)}x_t = \beta\mathrm{E}_t\pi_{t+1} + \kappa x_t. \tag{5}$$

The log-linear model. Finally, we can rewrite the model using three equations for aggregate demand, supply and monetary policy as in undergraduate courses

$$x_t = \mathrm{E}_t x_{t+1} - \sigma(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n),$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t,$$

$$i_t = \rho + \phi_\pi \pi_t + \phi_y x_t + v_t.$$

where $v_t = \psi v_{t-1} + \varepsilon_t^v$, $\psi \in (0,1)$, $r_t^n = \rho + \sigma^{-1}\psi_{ya}\mathrm{E}_t\Delta a_{t+1}$, a_t ~AR(1). Sometimes, to simplify the exposition, it is assumed a constant natural output. Hence, $y_t = x_t, r_t^n = \rho$. Depending on how we choose to log-linearize the Euler equation we may ignore ρ.

Saddle path stability. Substituting the Taylor rule in the IS curve we obtain

$$\mathrm{E}_t x_{t+1} + \sigma \mathrm{E}_t \pi_{t+1} = (1 + \phi_y \sigma) x_t + \sigma(\phi_\pi \pi_t + v_t - r_t^n + \rho),$$

$$\beta \mathrm{E}_t \pi_{t+1} = \pi_t - \kappa x_t$$

or

$$A\mathrm{E}_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = B \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + C \begin{bmatrix} a_t \\ v_t \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & \sigma \\ 0 & \beta \end{bmatrix}, \qquad B = \begin{bmatrix} (1+\phi_y\sigma) & \phi_\pi\sigma \\ -\kappa & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} \psi_{ya}(1-\rho^a) & \sigma \\ 0 & 0 \end{bmatrix}.$$

Inflation and output gap are two forward looking variables. The matrix $A^{-1}B$ has two eigenvalues outside the unit circle if and only if $\mathrm{tr}(A^{-1}B) = \lambda_1 + \lambda_2 > 2$ and $\det(A^{-1}B) = \lambda_1\lambda_2 > 1$. Because

$$\mathrm{tr}(A^{-1}B) = \beta^{-1} + (1 + \phi_y\sigma) + \kappa\sigma\beta,$$

$$\det(A^{-1}B) = \beta^{-1}(1 + \sigma\phi_y + \sigma\kappa\phi_\pi).$$

Then $\det(A^{-1}B) - \mathrm{tr}(A^{-1}B) > -1 \iff \kappa(\phi_\pi - 1) + (1-\beta)\phi_y > 0$. Bullard and Mitra (2002). Thus, the Taylor principle $\phi_\pi > 1$ implies a locally unique and stable solution for $\{\pi_t, x_t\}$.

3 Monetary policy shocks

Guessing $y_t = \eta_1 v_t, \pi_t = \eta_2 v_t$ yields

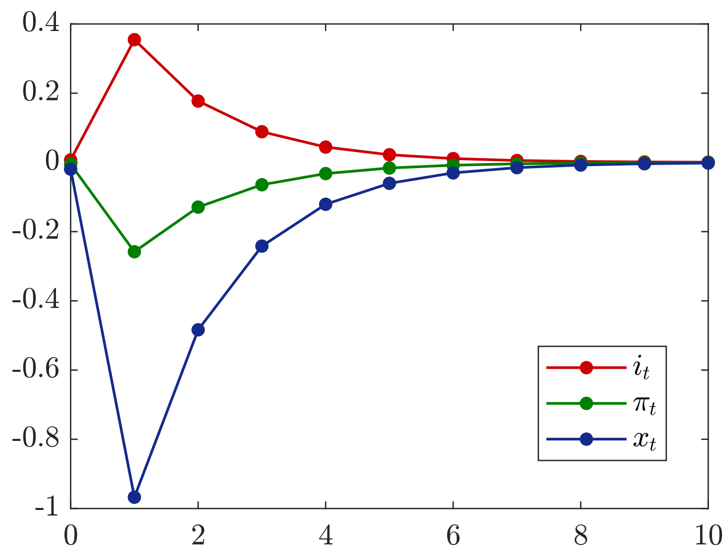
$$y_t = \frac{(\beta\psi - 1)}{(1-\beta\psi)[\gamma(1-\psi)+\phi_y]+\kappa(\phi_\pi - \psi)} v_t, \tag{6}$$

$$\pi_t = \frac{-\kappa}{(1-\beta\psi)[\gamma(1-\psi)+\phi_y]+\kappa(\phi_\pi - \psi)} v_t. \tag{7}$$

A positive realization of ε_t^v increasing the nominal interest rate given inflation and output gap is an exogenous restrictive monetary policy. Each period is a quarter. Setting $\beta = .99, \sigma = 1, \phi_\pi = 1.5, \phi_y = .25, \theta = .8, \alpha = .3, \varepsilon = 6$ and $\psi = 0.5$ from Equations (6)-(7) and the MA representation

$$v_t = \sum_{j=0}^{\infty} \psi^j \varepsilon_{t+j}^v$$

we can compute the impulse-response functions (IRFs) to a one percent deviation from steady state of the nominal interest rate.



In response to a positive monetary policy shock on impact the real interest rate rises, the output gap falls via the IS curve and inflation falls via Phillips curve. The endogenous component in the Taylor rule only partially reduce the increase in the nominal interest rate, the real interest rate rises because the higher i_t and lower $\mathrm{E}_t\pi_{t+1}$. In the simple NK model because of nominal rigidities the central bank can affect the real interest rate and via intertemporal substitution consumption and output. The Phillips curve is the critical link between real and nominal variables in the model. For a great analysis of the interest rate channel in NK models with representative agent see Rupert and Sustek (2019). These results are qualitatively along the line of the VAR literature on monetary policy shocks. Identification problems can explain the lack of evidence on the Phillips curve in the data. In addition to the interest rate channel monetary policy may affect the economy through credit and quantity of money. These chanels are not present in this model.

For simplicity set $\phi_y = 0$. An equivalent way to solve the log-linear model consists of rewriting the model as a second-order difference equation and solve it by factorization and inversion of the lag polynomial. Substituting the Phillips curve in the IS and rearraging yields to

$$\mathrm{E}_t[\beta\pi_{t+2} - (1+\beta+\sigma\kappa)\pi_{t+1} + (1+\sigma\kappa\phi_\pi)\pi_t] = -\sigma\kappa v_t.$$

Using the lag operator factorization lead to

$$\mathrm{E}_t(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})\pi_t = -(1+\sigma\kappa\phi_\pi)^{-1}\sigma\kappa v_t$$

where

$$\lambda_{1,2} = \frac{(1+\beta+\sigma\kappa) \pm \sqrt{(1+\beta+\sigma\kappa)^2 - 4\beta(1+\sigma\phi_\pi\kappa)}}{2(1+\phi_\pi\sigma\kappa)} < 1.$$

Solving forward

$$\pi_t = -\mathrm{E}_t\frac{(1+\sigma\kappa\phi_\pi)^{-1}\sigma\kappa v_t}{(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})}.$$

Using partial fractions and the assumption that $v_t \sim$AR(1) we obtain

$$\begin{aligned}
\pi_t &= \mathrm{E}_t\frac{1}{\lambda_1-\lambda_2}\left[\frac{\lambda_2}{(1-\lambda_2 L^{-1})} - \frac{\lambda_1}{(1-\lambda_1 L^{-1})}\right]\frac{\sigma\kappa}{1+\sigma\kappa\phi_\pi}v_t \\
&= \frac{\sigma\kappa}{1+\sigma\kappa\phi_\pi}\frac{1}{\lambda_1-\lambda_2}\mathrm{E}_t\left[\lambda_2\sum_{j=0}^{\infty}\lambda_2^j v_{t+j} - \lambda_1\sum_{j=0}^{\infty}\lambda_1^j v_{t+j}\right] \\
&= -\frac{\sigma\kappa}{1+\sigma\kappa\phi_\pi}\left[\frac{v_t}{(1-\lambda_1\psi)(1-\lambda_2\psi)}\right].
\end{aligned}$$

Finally, we can solve for x_t from the Phillips curve to obtain

$$x_t = \frac{\sigma}{(1+\sigma\kappa\phi_\pi)(\lambda_1-\lambda_2)}\mathrm{E}_t\left[(1-\beta\lambda_2^{-1})\lambda_2\sum_{j=0}^{\infty}\lambda_2^j v_{t+j} - (1-\beta\lambda_1^{-1})\lambda_1\sum_{j=0}^{\infty}\lambda_1^j v_{t+j}\right].$$

4 A medium-scale NK model

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Equilibrium. A competitive equilibrium in the NK model with money and capital consists of $\{p_t, i_t, \Delta_t, w_t, r_t\}_{t=0}^{\infty}$ and $\{C_t, N_t, I_t, K_{t+1}\}_{t=0}^{\infty}$, $\{c_t, n_t, k_{t+1}, m_{t+1}^d, b_{t+1}^d\}_{t=0}^{\infty}$, $\{m_t^s, b_t^s\}_{t=0}^{\infty}$:

1. Given the prices $\{c_t, n_t, m_{t+1}^d, k_{t+1}, b_{t+1}^d\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t, n_t, m_{t+1}^d, k_{t+1}, b_{t+1}^d\}} \mathrm{E}\sum_{t=0}^{\infty} \beta^t u(c_t, m_t, n_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} + m_t + b_{t+1}^d (p_t(1+i_t))^{-1} = w_t n_t + r_t k_t + (1-\delta)k_t + m_t^d/p_t + b_t - T_t + \Pi_t, \forall t,$$

$$c_t \geq 0, k_t \geq 0, m_t \geq 0, b_t \geq -\phi, \forall t.$$

2. Given the prices firms

$$\max_{y_t(i)} \; p_t Y_t - \int_0^1 p_t(i) Y_t(i) di, \;\; \text{s.t.} \;\; Y_t = \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

$$\min_{K_t(i), N_t(i)} \; w_t N_t(i) - r_t K_t(i), \;\; \text{s.t.} \;\; Y_t(i) = A_t K_t(i)^{\alpha} N_t(i)^{1-\alpha}, \forall t$$

$$\max_{p_t^*} \; \mathrm{E}_t \sum_{j=0}^{\infty} \theta^j \left[\frac{\beta^j \lambda_{t+j}}{\lambda_t} (p_t^* Y_{t+j|t} - \psi_{t+j}(Y_{t+j|t}))\right], \;\; \text{s.t.} \;\; Y_{t+j|t} = \left(\frac{p_t^*}{p_{t+j}}\right)^{-\varepsilon} Y_{t+j}, \forall t.$$

3. Prices are such that markets clear

$$C_t + I_t + G_t = A_t \Delta_t K_t^{\alpha} N_t^{1-\alpha}, \forall t$$

$$b_t^d = b_t^s, \forall t, \;\; K_t = k_t, \forall t, \;\; N_t = n_t, \forall t, \;\; m_t^d = m_t^s, \forall t.$$

4. Given prices and b_{-1}^s fiscal policy $\{G_t, T_t\}$ satisfies

$$\frac{b_{t-1}^s}{p_t} = \mathrm{E}_t \sum_{j=0}^{\infty} R_j (T_{t+j} - G_{t+j}), \forall t.$$

Monetary policy choose $\{i_t\}$ such that

$$\ln\left(\frac{1+i_t}{1+i}\right) = \phi_\pi \ln\left(\frac{p_{t+1}}{p_t}\right) + \phi_y \ln\left(\frac{Y_t}{Y_t^n}\right) + v_t, \forall t.$$

The extension is straightforward. Consider the case the marginal cost with two inputs. By cost minimization we have $K_t(i)/N_t(i) = \alpha(1-\alpha)^{-1}(w_t/r_t)$. Hence $Y_t(i) = A_t(K_t/N_t)^\alpha N_t(i)$ from the demand for the firm $(p_t(i)/p_t)^{-\varepsilon}Y_t = A_t(K_t/N_t)^\alpha N_t(i)$.

$$Y_t \int_0^1 \left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon} di = A_t \left(\frac{K_t}{N_t}\right)^{-\alpha} \int_0^1 N_t(i)di = A_t K_t^\alpha N_t^{1-\alpha}.$$

Therefore,

$$Y_t = A_t \Delta_t K_t^\alpha N_t^{1-\alpha}$$

where $\Delta_t = \int_0^1 (p_t(i)/p_t)^{-\varepsilon} di$.
We often assume an active monetary policy and a passive fiscal policy.

Equilibrium conditions. Assuming a separable utiliy function the Equilibrium system can be characterized as

$$mc_t = (1-\alpha)^{-1+\alpha}\alpha^{-\alpha}(1/A_t)w_t^{1-\alpha}r_t^\alpha,$$

$$Y_{t+j|t} = (p_t^*/p_{t+j})^{-\varepsilon}Y_{t+j},$$

$$p_t^* = \mu(\mathrm{E}_t \textstyle\sum_{j=0}^\infty \theta^j \beta^j u_1'(c_{t+j})Y_{t+j|t})^{-1}\mathrm{E}_t \textstyle\sum_{j=0}^\infty \theta^j \beta^j u_1'(c_{t+j})Y_{t+j|t}p_{t+j}mc_{t+j|t},$$

$$p_t^{1-\varepsilon} = \theta p_{t-1}^{1-\varepsilon} + (1-\theta)(p_t^*)^{1-\varepsilon},$$

$$K_t = \alpha(1-\alpha)^{-1}(w_t/r_t)N_t,$$

$$u_1'(c_t) = \beta\mathrm{E}_t[(1+r_{t+1}^k)u_1'(c_{t+1})],$$

$$\mathrm{E}_t(p_t/p_{t+1})(1+i_t) = (1+r_{t+1}^k),$$

$$u_2'(m_t) = i_t(1+i_t)^{-1}u_1'(c_t),$$

$$u_3'(n_t) = w_t u_1'(c_t),$$

$$\lim_{t\to\infty} \mathrm{E}_0\beta^t u_1'(c_t)x_{t+1} = 0, \;\; x \in \{k, b^d, m^d\}.$$

Together with Taylor rule, budget constraints and market clearing conditions.

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