Chapter 5: Heterogeneous agents

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Quantitative Macroeconomics

Introduction. In this chapter we will introduce several dimensions of economic inequality and the solution methods for macro models with heterogeneous agents and distributions. Household and firm heterogeneity is important to answer key economic questions. For example, how the distribution of income and wealth changes within and across countries, what are the distributional effects of aggregate fluctuations, and how fisal and monetary policy stimulate aggregate demand.

I. Idiosyncratic risk

Consider an intertemporal consumption problem where a_t denote asset. Let w be the wage and h the ammount of labor supplied, e.g. hours worked. Labor income wlh is subject to an idiosyncratic risk l_t. Two interpretations are possible for l. One is that l is an endowment shock affecting the supply of labor lh, an alternative interpretation is that l is a productivity shock affecting the wage wl. We assume that l_t follows a discrete Markov process $l_t : (\Omega, \mathcal{F}, P) \to (S, \mathcal{S})$ where $S = \{l_l, l_h\} \subset \mathbb{R}_+$ with transition function Q and conditional probability distribution $P_{l'|l}(.|l = l_i) : \mathcal{S} \to [0, 1]$. Note that the asset is not state-contingent. $u' > 0, u'' < 0$, Inada conditions hold.

$$\max_{\{c_t, a_{t+1}\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1 + r)a_t + wl_t, \forall t$$

$$\text{s.t.} \quad l_0, a_0 \text{ given}, \; a_t \geq -\phi, \forall t.$$

where $\phi = \min\{A, wl_l/r\}$ if $r > 0$ and $\phi = A$ if $r < 0$. The constant A is an exogenous borrowing limit. The *natural debt limit* $-wl_l/r$ is the highest debt that the household can afford if $c_t = 0, l_t = l_l, \forall t$. By Inada conditions the natural limit is never binding. If $0 \leq A < wl_l/r$ the borrowing constraint may be binding depending on income fluctuations. The household problem in recursive form is

$$v(a, l) = \max_{a' \geq -\phi} \left\{ u(Ra + wl - a') + \beta \int_S v(a', l') dP_{l'|l}(l'|l) \right\}.$$

Let $x_t = Ra_t + wl_t + \phi$ then we can rewrite budget and borrowing constraints as

$$x_{t+1} = R(x_t - c_t) + wl_{t+1} - r\phi,$$

$$c_t + a_{t+1} + \phi = x_t,$$

$$a_{t+1} + \phi \geq 0.$$

If l_t is iid then we can simplify the problem to

$$v(x) = \max_{a' \in [0,x]} \left\{ u(x - (a' + \phi)) + \beta \int_S v(x') dP_{l'}(l') \right\},$$

$$\text{s.t.} \quad x' = (1 + r)(a' + \phi) + wl' - r\phi.$$

The first order conditions are

$$u'(c_t) \geq \beta R \mathrm{E}[u'(c_{t+1})], \text{ with equality if } a_{t+1} + \phi > 0$$

$$\lim_{t \to \infty} \mathrm{E} \beta^t u'(c_t) x_t = 0.$$

Theorem 1. Suppose that l_t is iid.

- If $\beta(1 + r) \geq 1$ then the processes $\{c_t\}, \{a_t\}$ diverge over time almost surely.
- If $\beta(1 + r) < 1$ then $\{c_t\}, \{a_t\}$ are bounded, $\{x_t\}$ has a unique stationary distribution P^*.

See Schechtman and Escudero (1977), Sotomayor (1984), Chamberlain and Wilson (2000) for details. If $\beta R > 1$. Define $M_t = u'(c_t)(\beta R)^t$ the Euler inequality implies $M_t \geq \mathrm{E}_t M_{t+1}$, i.e. M_t is a supermartingale and by the supermartingale convergence theorem M_t has a finite limit. However, $(\beta R)^t \to \infty$ then $u'(c_t) \overset{a.s.}{\to} 0$. Since $u' > 0$ then $c_t \overset{a.s.}{\to} \infty$. Income and debt are bounded. Therefore, $a_t \overset{a.s.}{\to} \infty$. If $\beta R = 1$ using the envelope condition $v'(x) = u'(g^c(x))$ and the Euler inequality we have that $v(x)$

is a supermartingale $v'(x_t) \geq \mathrm{E}_t v'(x_{t+1})$, by the supermartingale convergence theorem $x_t \overset{a.s.}{\to} x^* < \infty$. By the envelope condition and the concavity of v the consumption policy function is strictly increasing in x_t. Thus, $c_t \overset{a.s.}{\to} c^* < \infty$. Using the budget constraint we reach a contradiction since $x^* - R(x^* - c^*) \neq \lim_{t\to\infty} wl_t$. So, $v'(x_t) \overset{a.s.}{\to} 0, x_t \overset{a.s.}{\to} \infty$. For the case $\beta R < 1$ it can be shown that there exists a cutoff level of x below which households are borrowing constrained.

II. The heterogeneous agent model in discrete time

This section introduces the Ayagari-Bewley-Hugget model, namely the basic heterogeneous agents (HA) model. Huggett (1993) proves Theorem 1 for a Markov processes with two states and CRRA utility. Aiyagari (1994) add production to the income fluctuation problem.

1 Equilibrium

Let's establish notation. We use product measures. Let $(X, \mathcal{X})$ be a measurable space where $X = A \times S = [0, \infty) \times \{l_l, l_h\}$ is the state space with product σ-algebra $\mathcal{X} = \sigma(\mathcal{B}(A) \times P(S)) = \mathcal{B}(A) \otimes P(S)$ with $P(S)$ power set of S and $\mathcal{B}(A)$ Borel σ-algebra of A. Let $\mathcal{A} \in \mathcal{B}(A), \mathcal{S} \in P(S)$, and $a_{t+1} = g^a(a_t, l_t)$ be the policy function of assets tomorrow. The transition function $Q : (X, \mathcal{X}) \to [0, 1]$ of $\{a_t, l_t\}$ is

$$\begin{aligned} Q((a,l),(\mathcal{A},\mathcal{S})) &= 1_{[g^a(a,l)\in\mathcal{A}]} Q_l(l,\mathcal{S}) \\ &= 1_{[g^a(a,l)\in\mathcal{A}]} \sum_{l'\in\mathcal{S}} P_{l'|l}(l'|l) = \begin{cases} \sum_{l'\in\mathcal{S}} P_{l'|l}(l'|l) & \text{if } g^a(a,l) \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

for all events $(\mathcal{A}, \mathcal{S}) \in \mathcal{X}$. Intuitively, Q gives the probability of $a' \in \mathcal{A}, l' \in \mathcal{S}$ for someone with current states (a, l). The transition function induces a sequence of probability distributions:

$$P_{a',l'}((\mathcal{A},\mathcal{S})) = \int_X Q((a,l),(\mathcal{A},\mathcal{S})) dP_{a,l}(a,l).$$

Let M be the set of probability measures on $(X, \mathcal{X})$. It is convenient to use an implicit form for the previous law of motion $H : M \to M$,

$$P_{a',l'} = H(P_{a,l}).$$

For any $B \in \mathcal{X}$ the probability distribution $P^*_{a,l}$ is stationary if it is a fixed point of H. There is a continuum $\mathcal{H} = [0, 1]$ of ex ante identical agents subject to idiosyncratic labor income shocks. Hence, the joint probability distribution $P_{a_t,l_t}(a_t, l_t)$ give us the cross-sectional distribution of assets and income risk in period t. The distribution $P_{a_t,l_t} := \psi_t$ is an additional state variable, namely it determines $\{r_{t+1}\}$ via market clearing hence it will be an argument of the policy functions of the agent. The individual states are (a, l) and the aggregate state is ψ. With a continuum of agents ψ is an infinite-dimensional object. This makes the model hard to solve numerically. We start considering stationary equilibria with $P^*_{a,l}$, i.e. a fixed point of H. This is computationally easier. If ψ is constant then the mean of the marginal distribution of capital is constant, i.e. $K' = K$ so K is a sufficient aggregate state variable and r is constant. Moreover, it is interesting to study the steady state properties of cross-sectional income and wealth distributions as in Aiyagari (1994).

Definition 1. A sequential markets competitive equilibrium in the HA model is given by households' allocations $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ and $\{K_t, L_t\}_{t=0}^{\infty}$ for the representative firm:

1. Given prices agents solve

$$\max_{\{c_t, a_{t+1}\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1 + r_t)a_t + w_t l_t, \forall t$$

$$l_0, a_0 \text{ given}, \; a_t \geq -\phi, \forall t.$$

2. Given prices $\{K_t, L_t\}_{t=0}^{\infty}$ satisfies

$$F_K(K_t, L_t) = r_r + \delta, \forall t$$

$$F_L(K_t, L_t) = w_t, \forall t.$$

3. Labor, asset and good markets clear:

$$L_t = \int_X l_t dP_{a_t,l_t}(a_t, l_t) = \sum_{l_t \in S} \int_0^{\infty} l_t f_{a_t,l_t}(a_t, l_t) da_t, \forall t$$

$$K_t = \int_X a_t dP_{a_t,l_t}(a_t, l_t) = \sum_{l_t \in S} \int_0^{\infty} a_t f_{a_t,l_t}(a_t, l_t) da_t, \forall t$$

$$\int_X c_t dP_{a_t,l_t}(a_t, l_t) + \int_X a_{t+1} dP_{a_t,l_t}(a_t, l_t) = F(K_t, L_t) + (1 - \delta)K_t, \forall t$$

$$P_{a_{t+1},l_{t+1}} = H_t(P_{a_t,l_t}), \forall t.$$

Note that given $P(l^t|l_0) = P(l_t|l_{t-1})...P(l_1|l_0)$, one could rewrite

$$K_{t+1} = \int_{X^0} \sum_{l^t \in S^t} a_{t+1}(a_0, l^t) P(l^t|l_0) dP_{a_0,l_0}(a_0, l_0).$$

Definition 2. A recursive competitive equilibrium is a value function $v : X \times M \to \mathbb{R}$, policy functions $g^c : X \times M \to \mathbb{R}_+, g^a : X \times M \to \mathbb{R}$ and firm choices $L : M \to \mathbb{R}, K : M \to \mathbb{R}$, prices $r : M \to \mathbb{R}, w : M \to \mathbb{R}$ and a law of motion $H : M \to M$:

1. Given w, r the functions g^a, g^c solve the household's problem

$$v(l, a; \psi) = \max_{a' \geq 0} \left\{ u((1 + r(\psi))a + w(\psi)l - a') + \beta \int_S v(l', a'; \psi') dP_{l'|l}(l'|l) \right\},$$

2. Given w, r the choices L, K solve

$$F_K(K(\psi), L(\psi)) = r(\psi) + \delta,$$

$$F_L(K(\psi), L(\psi)) = w(\psi).$$

3. At equilibrium prices labor, asset and good markets clear:

$$\psi' = H(\psi),$$

$$L(\psi) = \int_X l d\psi = \sum_{l \in S} \int_0^\infty l f_{a,l}(a, l) da, \forall \psi$$

$$K'(\psi') = \int_X g^a(a, l) d\psi = \sum_{l \in S} \int_0^\infty g^a(a, l) f_{a,l}(a, l) da, \forall \psi$$

$$\int_X g^c(a, l) d\psi + \int_X g^a(a, l) d\psi = F(K(\psi), L(\psi)) + (1 - \delta)K(\psi), \forall \psi.$$

By Walras' law and the exogeneity of l the of existence of the equilibrium boils down to prove that asset market clear $K = A(r)$, namely that the aggregate demand of saving K is downward sloping and the aggregate supply A upward sloping, e.g. substitution effect prevails on the income effect. Theoretical results are limited to the stationary case. In these models $\beta(1 + r) < 1 \Rightarrow (1 + r) < \beta^{-1} = (1 + r_{cm})$. This implies that the capital stock with complete markets is lower than that under incomplete markets. The difference can be interpreted as precautionary saving. Aiyagari (1994) quantitatively showed that the aggregate saving rate with incomplete markets is only slightly higher than the complete markets model by no more than 3%. If a' is linear in a and with same slope over l we would have perfect aggregation. $K' = c + b\int a d\psi = c + bK$. The economy admits a representative agent. The average capital will be sufficient to pin down prices. In this economy a binding borrowing constraint introduces a kink in the saving function, in particular when $l_t = l_l$. However, the savings functions are almost linear, the curvature is present only for low level of (a, l). Borrowing constrained households are few and hold a small fraction of aggregate wealth. Aggregate shocks move the wealth distribution very slightly. Hence, "quasi-aggregation". These features are not present in richer quantitaive models in which typically distributional features and nonlinear decision rules play a crucial role.

2 Numerical solution

Solving the steady state. First of all, we discretize the state space with an asset grid $G_A = \{\phi, ..., \bar{a}\}$ over A of dimension I and idiosyncratic income grid over $G_S = S = \{l_l, l_h\}$ of dimension $J = 2$, clearly in the two states example we already have a discrete set S. Then, the algorithm proceeds with an outer loop iterating over equilibrium prices and an inner loop solving the household optimization problem.

Algorithm 1.

1. Fix an initial guess $r^n \in (-\delta, \beta^{-1} - 1)$, derive $w(r^n), K(r^n)$.

2. Given prices $r^n, w(r^n)$ obtain g^a, g^c from the household's problem.

3. Given g^a, Q compute $P^*_{a,l}(r^n)$ and the aggregate supply of capital $A(r^n)$.

4. Find r such that the asset market clears and if $A(r^n) > K(r^n) \Rightarrow r^1 < r^0$.

5. Update r^n to r^{n+1} using a bisection algorithm and iterate until $|r^{n+1} - r^n| < \varepsilon$.

Algorithm 2.

1. Fix an initial guess K^n and derive the wage rate w^n, r^n.

2. Given prices $r^n, w(r^n)$ obtain g^a, g^c from the household's problem.

3. Given g^a, Q compute $P^*_{a,l}(K^n)$ and the aggregate supply of capital $A(K^n)$.

4. Use the fixed point scheme $K^{n+1} = \alpha A(K^n) + (1 - \alpha)K^n, \alpha \in (0, 1)$.

5. Update the initial guess and iterate until convergence $|K^{n+1} - K^n| < \varepsilon$.

The initial lower bound follows from the capital supply equation $r = F_K - \delta$. The wage rate and capital demand can be derived from the firm's first order conditions. Given $\{l_t\}$ with transition matrix Π and stationary distribution p, if we interpret l as a wage shock then $L = 1$, otherwise the labor supply can be computed exogenously from

$$L = \sum_{j=1}^{J} l_j p_j$$

In step 2 we solve the household's problem with discrete value function or Euler equation iteration. Once we have the policy functions check that $\bar{a}$ is not binding. Check that g^a intersects the 45-degree line for a large enough ruling out explosive wealth dynamics. Then, we can compute the stationary distribution. First, we compute the transition probabilities given by Q from pairs (a, l) to pairs (a', l'). Since we discretized the state space we are working on a grid $G_A \times G_S$ and we can collect these probabilities in a $IJ \times IJ$ matrix A. Second, we solve the Chapman-Kolmogorov equation, i.e. $\psi' = H(\psi)$. There are three equivalent ways to do so. We can use a fixed point iteration scheme

$$\psi^{n+1} = A'\psi^n.$$

and iterate until convergence $|\psi^{n+1} - \psi^n| < \varepsilon$. We can use the eigenvalue problem

$$A'\psi = \lambda\psi$$

The eigenvector associated to the eigenvalue $\lambda = 1$ is the stationary distribution. We can solve this easily as it is very likely that the programming language you are using already has routines for it. We should rescale ψ to make sure that it adds up to 1. Alternatively, simulate a large number of household say 10,000 initialize each individual at (a_0, l_0) and use g^a and a random number generator to replicate the Markov process $\{l_t\}$ and generate the joint Markov process $\{a_t, l_t\}$. In each period compute a set of cross-sectional moments m_t for the distribution of assets like mean, variance, quantiles. When $m_t \approx m_{t+1}$ we can stop. In this case the distribution has converged. Then, compute

$$A(r^n) = \sum_{(a,l)} g^a(a, l) P^*_{a,l}(a, l)$$

In step 5 we have a root-finding problem $A(r) - K(r) = 0$. We can solve it using a bisection os a fixed point iteration.

Example: steady State. Assume A Cobb-Douglas production function $K_t^\alpha L_t^{1-\alpha}$ and a CRRA utility function $c_t^{1-\gamma}/(1-\gamma)$. Figure 1 shows a global solution for steady state policy functions in general equilibrium obtained using bisection and value function iteration. Note the concavity of the consumption function close to the borrowing limit for the low income realization (blue line). The saving function eventually crosses the 45-degree line so that wealth dynamics are not explosive and a stationary wealth distribution exists. Figure2 reports the capital supply and demand in this calibration.

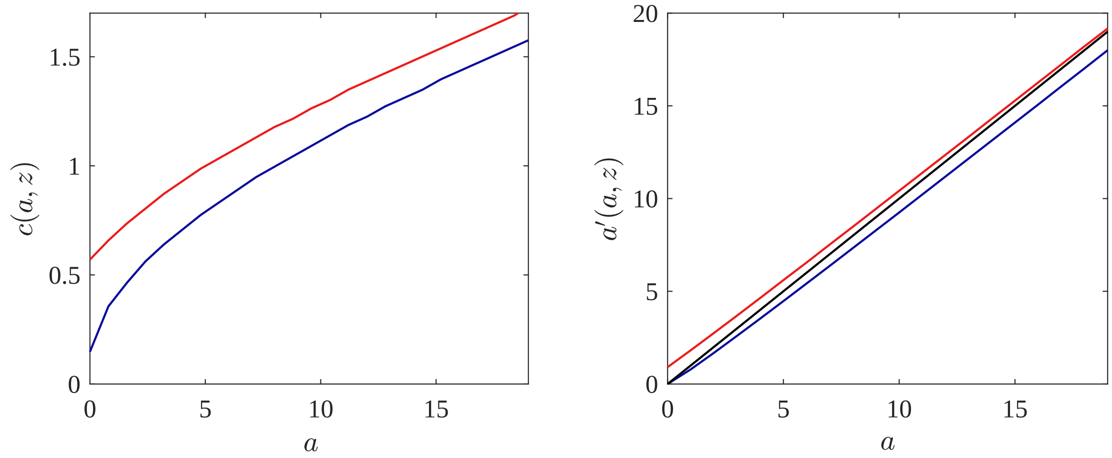

Figure 1: Consumption and saving functions.

Note: I use $\alpha = .3, \gamma = 2, \delta = .05, \rho = .05, \phi = 0$, a two state Markov process for z_t with main diagonal transition probabilities of .8, and $a_{\max} = 20$. The equilibrium prices are $w = 1.4, r = .03 < \rho$. The red line corresponds to the high income realization.

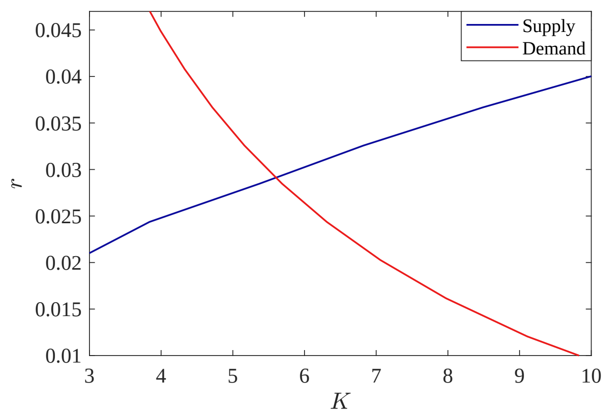

Figure 2: Stationary equilibrium.

3 Transition dynamics and "MIT shocks"

We can combine a sequential and recursive formulation of the model. To do so, write the household problem recursively and a sequential equilibrium in prices and aggregates. Consider an economy without aggregate risk. Prices and aggregates are constant at the steady state level and we can define a recursive equilibrium using K as aggregate state. Suppose that there is an exogenous and unexpected - zero probability event, e.g. a permanent or transitory change in some parameters or government policy. Then, we are interested in studying the deterministic transition of the economy from the initial steady state to the final steady state. This is the impulse response associated to a "MIT shock". The challenge is that now we work with a nonstationary sequential equilibrium, that is we have a new equilibrium path in which the value function depends on time via equilibrium prices, that is $v_t(l_t, a_t) := v(l_t, a_t, \{r_t, w_t\})$ and households solve

$$v_t(l, a) = \max_{a' \geq 0} \left\{ u((1 + r_t)a + w_t l - a') + \beta \int_S v_{t+1}(l', a') dP_{l'|l}(l'|l) \right\},$$

Algorithm 3.

1. Fix T.
2. Compute the initial steady state $\psi_0, v_0, r_0, w_0, K_0$.
3. Compute the final steady state $\psi_T, v_T, r_T, w_T, K_T$.
4. Guess $\{K_t\}_{t=0}^{T-1}$ and find $\{w_t, r_t\}$.
5. Solve the Bellman equation backward for $\{v_t, a_{t+1}, c_t\}_{t=0}^{T-1}$ given v_T.
6. Given $\{a_{t+1}\}_{t=0}^{T}, \{H_t\}_{t=0}^{T}, \psi_0$. Solve the law of motion forward for $\{\psi_{t+1}\}_{t=0}^{T-1}$.
7. Compute $\{A_t\}_{t=0}^{T}$ and check $\max_{0 \leq t \geq T-1} |A_t - K_t| < \epsilon$ if not adjust the guess.
8. If yes check $|A_T - K_T| < \epsilon$ otherwise increse T.

4 Endogenous grid method

To solve the household optimization problem we can use value function iteration. However, the endogenous grid method provides a more efficient alternative. Rather than using a grid $G_A \in A$ over a using a grid $G_A \in A$ over a' allow us to solve the Euler equation analytically. This substantially speed up the computations because we can avoid the inner loop that solves the Euler equation or the maximization in the Bellman equation numerically.

Algorithm 4.

1. Fix a grid over assets $G_A = \{a_1, ..., a_I\}$ and income shocks $G_Y = Y = \{y_1, ..., y_J\}$.
2. Guess a decision rule $c_n(a_i, y_j)$, e.g. $a'_n(a_i, z_j) = 0, c_n(a_i, y_j) = (1 + r)a_i + y_j$.
3. For any pair $(a_i, y_j) \in G_A \times G_Y$ compute the right-hand-side of the Euler equation
$$RHS_{ij} = \beta(1 + r) \sum_{j'=1}^{J} P(y_{j'}|y_j)u'(c_n(a'_i, y_{j'})).$$
4. Compute $c(a'_i, y_j) = (u')^{-1}(RHS_{ij})$ and $a(a'_i, y_j) = (1 + r)^{-1}(y_j + c(a'_i, y_j) - a'_i)$.
5. For each j invert the mapping a to go from $(a'_i, a(a'_i))$ to $(a(a'_i), a'_i)$.
6. For each j interpolate $(a(a'_i), a'_i)$ over $a_i \in G_A$ to get $a'_{n+1}(a_i, y_j)$ and $(a(a'_i), c(a'_i, y_j))$ over $a_i \in G_A$ to get $c_{n+1}(a_i, y_j)$.
7. If $a_i < a(a'_1, y_j)$ the agent will be constrained in the next period and we cannot use the Euler equation. Hence, set $a'_{n+1} = a_1$ and $c_{n+1}(a_i, y_j) = (1 + r)a_i + y_j - a_1$.
8. Iterate from step 3 until convergence $\max_{ij} |c_{n+1} - c_n| < \varepsilon$.

Note that $a(a'_i, y_j)$ are the assets today that will lead the consumer to have a'_i assets tomorrow given the shock today y_j. This function is not necessarily on the grid G_A and is an endogenous asset grid that changes in each iteration. In step 6 we want to move the optimal decisions from the endogenous grid to the exogenous grid G_A, if needed we can always extrapolate. The algorithm can be extended to include a labor supply decision and to finite horizon using a backward solution.

III. The heterogeneous agent model in continuous time

This section requires a basic knowledge of stochastic calculus. We can recast the HA model in continuous time using two partial differential equations, the Hamilton-Jacobi-Bellman (HJB) equation and the Kolmogorov-Forward (KF) or Fokker-Planck equation.

1 Equilibrium

We consider again two individual states, assets and idiosyncratic income, $x_t = (a_t, z_t)$, I denote with $\psi_t(x_t)$ their joint distribution and with $f_t(x_t)$ the associated density function. Finally, $v_t(x_t)$ is the value function, I use the short hand notation $m_a := (r_t a_t + w_t z_t - c_t), m_z := \mu(z_t), s_z := \sigma_z(z_t)$ for the states drifts and standard deviation. In a stationary equilibrium we set $\partial v_t/\partial t = \partial f_t/\partial t = 0$.

Definition 3. A competitive equilibrium in the HA model is given by $(c_t, da_t, K_t, L_t, r_t, w_t)$:

1. Given prices agents solve the HJB equation

$$\rho v_t = \max_{c_t} \left\{ u(c_t) + \frac{\partial v_t}{\partial a} m_a + \frac{\partial v_t}{\partial z} m_z + \frac{1}{2} \frac{\partial^2 v_t}{\partial z^2} s_z^2 + \frac{\partial v_t}{\partial t} \right\},$$

$$da_t = (r_t a_t + w_t z_t - c_t) dt,$$

$$dz_t = \mu(z_t) dt + \sigma(z_t) dw_t.$$

2. Given prices (K_t, L_t) satisfies

$$F_K(K_t, L_t) = r_r + \delta,$$

$$F_L(K_t, L_t) = w_t.$$

3. Labor, asset and good markets clear:

$$\frac{\partial f_t}{\partial t} = -\frac{\partial}{\partial a}(f_t m_a) - \frac{\partial}{\partial z}(f_t m_z) + \frac{1}{2} \frac{\partial^2}{\partial z^2}(f_t s_z^2),$$

$$L_t = \int_X z_t d\psi_t = \int_X f_t(x_t) z_t dx_t,$$

$$K_t = \int_X a_t d\psi_t = \int_X f_t(x_t) a_t dx_t,$$

$$\int_X c_t d\psi_t + I_t = F(K_t, L_t).$$

HJB equation. Let's start from the agent problem. We assume a diffusion process for $\ln z_t$, e.g. Ornstein–Uhlenbeck process. Then, the Ito's formula implies that the income risk z_t follows a geometric Brownian motion.

$$\max_{c_t} \mathrm{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt,$$

$$\text{s.t.} \quad da_t = (r_t a_t + w_t z_t - c_t) dt,$$

$$dz_t = \mu(z_t) z_t dt + \sigma_z(z_t) d\hat{w}_{z,t},$$

$$z_0, a_0 \text{ given}, \ a_t \geq -\phi.$$

The Bellman principle yields to

$$0 = \max_{c_t} \mathbb{E}_t \left\{ \int_t^{t+dt} e^{-\rho(s-t)} \hat{u}(c_s) ds + dv_t(a_t, z_t) \right\}.$$

Notice that $\int_t^{t+dt} f(s)ds = dF(t) = F'(t)dt = f(t)dt$. Hence, $0 = \max_{c_t}\{e^{-\rho t}\hat{u}(c_t)dt + \mathbb{E}_t[dv_t(a_t, z_t)]\}$ Given that $da_t = m_a dt, dz_t = m_z dt + s_z d\hat{w}_{z,t}$, as the coefficients m_z and m_a are non-anticipating functions and under the assumption of uncorrelated shocks we can apply the Ito's lemma to $dv_t(a_t, z_t)$ and taking the conditional expectation yields

$$\mathbb{E}_t[dv_t(a_t, z_t)] = \left(\frac{\partial v_t}{\partial t} + \frac{\partial v_t}{\partial a} m_a + \frac{\partial v_t}{\partial z} m_z + \frac{1}{2} \frac{\partial^2 v_t}{\partial z^2} s_z^2 \right) dt,$$

where we used the fact that $\mathbb{E}(d\hat{w}_t) = 0$. Define a differential operator $D_x := \partial/\partial x$. Given that we have an infinite horizon discounted problem and m_a varies over time because of the equilibrium prices $D_t v_t(a_t, z_t) = -\rho e^{-\rho t}\hat{v}_t + e^{-\rho t} D_t \hat{v}_t$. Therefore, the HJB equation is

$$\rho v_t(x_t) = \max_{c_t} \Big\{ u_t(c_t) + D_a v_t(x_t)(w_t z_t + r_t a_t - c_t) + D_z v_t(x_t)\mu(z_t) + \frac{1}{2} D_{zz} v_t(x_t) \sigma_z(z_t)^2 + e^{-\rho t} D_t \hat{v}_t(x_t) \Big\}.$$

Notice we can rescale everything multiplying both sides by $e^{\rho t}$

$$\rho \hat{v}_t = \max_{c_t} \left\{ \hat{u}(c_t) + \frac{\partial \hat{v}_t}{\partial a} m_a + \frac{\partial \hat{v}_t}{\partial z} m_z + \frac{1}{2} \frac{\partial^2 \hat{v}_t}{\partial z^2} s_z^2 + \frac{\partial \hat{v}_t}{\partial t} \right\}.$$

In a stationary equilibrium $\hat{u}, m_a, m_z, s_z$ are time invariant, i.e. only depend on time via states, then $\hat{v}$ is time invariant and $D_t \hat{v} = 0$.

Kolmogorov-Forward equation. Step 1 - Ito's Formula and Dynkin's Operator. Given f_0, and any twice continuously differentiable function g on $X \subseteq \mathbb{R}^2$ we have

$$\mathbb{E}_t(g(x_t)) = \int_{\mathbb{R}^2} g(x_t) f_t(x_t) dx_t.$$

Using the Ito's formula

$$dg(x) = [g_a m_a + g_z m_z + g_t + .5 g_{aa} s_a^2 + .5 g_{zz} s_z^2] dt + g_a s_a d\hat{w}_a + g_z s_z d\hat{w}_z.$$

Taking the expected value $\mathbb{E}_0(\int_0^t dg(x_s))$ and differentiating with respect to t we have

$$\frac{d}{dt}\mathbb{E}_0(g(x_t)) = \int_{\mathbb{R}^2} [g_a m_a + g_z m_z + g_t + .5 g_{aa} s_a^2 + .5 g_{zz} s_z^2] f_t(x_t) dx_t.$$

Rewriting the first term

$$\int_{\mathbb{R}^2} g(x_t) \frac{\partial}{\partial t} f_t(x_t) dx_t = \int_{\mathbb{R}^2} \left(g_a m_a + g_z m_z + g_t + .5 g_{aa} s_a^2 + .5 g_{zz} s_z^2 \right) f_t(x_t) dx_t.$$

Step 2 - Integration by Parts. If g, g_a, g_z tend to zero as x approach the bounds or infinity and $g_i f, h_i f < \infty$ we can use integration by parts once on each first order term and twice on each second order term to get

$$\int_{\mathbb{R}^2} g(x_t) \frac{\partial}{\partial t} f_t(x_t) dx_t =$$
$$\int_{\mathbb{R}^2} g(x_t) \left(-\frac{\partial}{\partial a}(f(x_t) m_a(x_t)) + \frac{\partial^2}{\partial a^2}(.5 f(x_t) s_a^2) - \frac{\partial}{\partial z}(f(x_t) m_z) + \frac{\partial^2}{\partial z^2}(.5 f(x_t) s_z^2) \right) dx_t.$$

The law of motion for ψ is given by the Fokker-Planck or Kolmogorov Forward equation

$$\frac{\partial}{\partial t} f_t(x_t) = -\frac{\partial}{\partial a}(f(x_t) m_a) + \frac{\partial^2}{\partial a^2}(.5 f(x_t) s_a^2) - \frac{\partial}{\partial z}(f(x_t) m_z) + \frac{\partial^2}{\partial z^2}(.5 f(x_t) s_z^2).$$

2 Numerical solution

The numerical solutions presented in this section are due to Achdou, Han, Lasry, Lions, and Moll (2018). The algorithms to compute steady states and transition dynamics equilibria are the same as in discrete time. The main difference is the solution of the HJB and KF equations, which is the focus of this section. The main idea is to use numerical derivatives to transform these partial differential equations in a linear system with sparse matrices.

Boundary condition. One advantage of continuous time relative to discrete time is that the borrowing constraint $a \geq -\phi$ only give rise to boundary condition and never binds in the interior of the state sapce. As a consequence the first order condition $u'(c) = v_a(a, z)$ holds everywhere. The relevant boundary condition is

$$v_a(\phi, z) \geq u'(wz + r\phi), \ \ \forall z.$$

Notice that since $u'(c) = v_a(\phi, z)$ holds at $a = \phi$ the condition above implies that savings $s(a, z) := (wz + ra - c) \geq 0$ at $a = \phi$ and the constraint is never violated. Let's start from a stationary solution. I will consider a uniform grid for each state and denote with i, j the grid points for a, z.

Implicit scheme. An implicit scheme updates the value function according to

$$\frac{v_{ij}^{k+1} - v_{ij}^{k}}{\Delta} + \rho v_{ij}^{k+1} = u(c_{ij}^{k}) + v_{a,ij}^{k+1}(wz_j + ra_i - c_{ij}^{k}) + v_{z,ij}^{k+1} m_{z,j} + \frac{1}{2} v_{zz,ij}^{k+1} (s_{z,j})^2, \quad (1)$$

where

$$c_{ij}^{k} = u^{-1}(v_{a,ij}^{k}).$$

Upwind schemes. To approximate the partial derivative $v_{a,ij}^{k}$ with a forward or backward difference we use an upwind scheme. The idea is to use the forward difference when the drift of the state variable is positive and the backward difference when it is negative. In particular, let $(x)^+ := \max(x, 0)$, $(x)^- := \min(x, 0)$ and $s_{ij,F}^{k} = wz_j + ra_i - c_{ij,F}^{k}$, $s_{ij,B}^{k} = wz_j + ra_i - c_{ij,B}^{k}$. If v is concave in a then $v_{a,ij,F} < v_{a,ij,B}$ and so $s_{ij,F} < s_{ij,B}$. Therefore, we don't have to worry about the case $s_{ij,F} > 0$ and $s_{ij,B} < 0$, but for some grid points we could have $s_{ij,F} \leq 0 \leq s_{ij,B}$. In this case we set saving to zero and $\tilde{v}_{ij} = u'(wz_j + ra_i)$. This upwind scheme give rise to the following approximation

$$v_{a,ij} = v_{a,ij,F} 1_{[s_{ij,F} > 0]} + v_{a,ij,B} 1_{[s_{ij,B} < 0]} + \tilde{v}_{ij} 1_{[s_{ij,F} \leq 0 \leq s_{ij,B}]}.$$

We impose the boundary condition implied by the borrowing limit by setting

$$v_{a,1j,B} = u'(wz_j + ra_1).$$

So whenever $s_{ij,F} \leq 0$ the boundary condition holds with equality. If $s_{1j,F} > 0$ the forward difference is used $v_{a,1j,F} = u'(c_{1j,F}) > u'(wz_j + ra_1)$. Hence, the constraint is never violated.

Finite difference. Once we have c_{ij}^k we can go back to (1) using the approximation

$$\frac{v_{ij}^{k+1}-v_{ij}^k}{\Delta}+\rho v_{ij}^{k+1}=u(c_{ij}^k)+\frac{v_{i+1j}^{k+1}-v_{ij}^{k+1}}{\Delta a}(s_{ij,F}^k)^+ +\frac{v_{ij}^{k+1}-v_{i-1j}^{k+1}}{\Delta a}(s_{ij,B}^k)^- \\ +\frac{v_{ij+1}^{k+1}-v_{ij}^{k+1}}{\Delta z}m_{z,j}^+ +\frac{v_{ij}^{k+1}-v_{ij-1}^{k+1}}{\Delta z}m_{z,j}^- +\frac{s_{z,j}^2}{2}\frac{v_{ij+1}^{k+1}-2v_{ij}^{k+1}+v_{ij-1}^{k+1}}{(\Delta z)^2}.$$

Collecting coefficients on the right hand side yields

$$\frac{v_{ij}^{k+1}-v_{ij}^k}{\Delta}+\rho v_{ij}^{k+1}=u(c_{ij}^k)+v_{i-1j}^{k+1}x_{ij}+v_{ij}^{k+1}(y_{ij}+\nu_j)+v_{i+1j}^{k+1}z_{ij}+v_{ij-1}^{k+1}\chi_j+v_{ij+1}^{k+1}\zeta_j.$$

This is a system of $I\times J$ equations where $i=1,...,I, j=1,...,J$. We impose $x_{1j}=z_{Ij}=0,\forall j$ so that v_{0j},v_{I+1j} are never used. At the boundaries of the j dimension we use the "ghost nodes" $j=0, j=J+1$ and set $v_{i1}=v_{i0}, v_{iJ+1}=v_{iJ}$ so that $v_{z,iJ,F}=v_{z,i1,B}=0$.

$$\frac{v_{i1}^{k+1}-v_{i1}^k}{\Delta}+\rho v_{i1}^{k+1}=u(c_{i1}^k)+v_{i-11}^{k+1}x_{i1}+v_{i1}^{k+1}(y_{i1}+\nu_1+\chi_1)+v_{i+11}^{k+1}z_{i1}+v_{i2}^{k+1}\zeta_1,$$

$$\frac{v_{iJ}^{k+1}-v_{iJ}^k}{\Delta}+\rho v_{iJ}^{k+1}=u(c_{iJ}^k)+v_{i-1J}^{k+1}x_{iJ}+v_{iJ}^{k+1}(y_{iJ}+\nu_J+\zeta_J)+v_{i+1J}^{k+1}z_{iJ}+v_{iJ-1}^{k+1}\chi_J.$$

This restrict the sign of the z drif imposing a reflecting barrier in the z-dimension at z_1,z_J.

$$\nu_1+\chi_1=\frac{\mu_1^-}{\Delta z}-\frac{\mu_1^+}{\Delta z}-\frac{s_{z,1}^2}{(\Delta z)^2}-\frac{\mu_1^-}{\Delta z}+\frac{s_{z,1}^2}{2(\Delta z)^2}=-\frac{\mu_1^+}{\Delta z}-\frac{1}{2}\frac{s_{z,1}^2}{(\Delta z)^2}.$$

Let $v^k=(v_{11}^k,...,v_{I1}^k,v_{12}^k,...,v_{I2}^k,...,v_{IJ}^k)$ the HJB approximation in matrix notation is

$$\frac{1}{\Delta}(v^{k+1}-v^k)+\rho v^{k+1}=u^k+(C^k+B)v^{k+1}. \tag{2}$$

Therefore, $A^kv^{k+1}=b^n$ where $b^k=u^k+\Delta^{-1}v^k, A^k=(\Delta^{-1}+\rho)I-(C^k+B)$. Note that C^k is given by a main central diagonal $(y_{11},...,y_{I1},y_{12},...,y_{I2},...,y_{1J},...,y_{IJ})$ a main lower diagonal $(x_{21},...,x_{I1},0,x_{22},...,x_{I2},0,...,x_{2J},...,x_{IJ})$ and main upper diagonal of forward coefficients $(z_{11},...,z_{I-11},0,z_{12},...,z_{I-12},0,...,z_{1J},...,z_{I-1J})$, and zero elsewhere. B is given by a main diagonal $(\nu_1+\chi_1,\nu_1+\chi_1,...,\nu_1+\chi_1,\nu_2,\nu_2,...,\nu_3,\nu_3,...,\nu_J+\zeta_J,\nu_J+\zeta_J,...)$ where each repetition is over I positions, a lower diagonal $(\chi_2,\chi_2,...,\chi_J,\chi_J)$ starting in position $I+1\times 1$, an upper diagonal $(\zeta_1,\zeta_1,...,\zeta_{J-1},\zeta_{J-1})$ starting in position $1\times I+2$, and zero elsewhere.

The KF equation. We can approximate the KF equation in a very convenient way

$$
\begin{aligned}
0 = &-\frac{(s_{ij,F}^k)^+ f_{ij} - (s_{i-1j,F}^k)^+ f_{i-1j}}{\Delta a} - \frac{(s_{i+1j,B}^k)^- f_{i+1j} - (s_{ij,F}^k)^- f_{ij}}{\Delta a} \\
&- \frac{m_{zj}^+ f_{ij} - m_{zj-1}^+ f_{ij-1}}{\Delta z} - \frac{m_{zj+1}^- f_{ij+1} - m_{zj}^- f_{ij}}{\Delta z} \\
&+ \frac{1}{2}\frac{s_{z,j+1}^2 f_{ij+1} - 2s_{z,j}^2 f_{ij} + s_{z,j-1}^2 f_{ij-1}}{(\Delta z)^2}.
\end{aligned}
$$

Collecting terms yields

$$
0 = z_{i-1j} f_{i-1j} + (y_{ij} + \nu_j) f_{ij} + x_{i+1j} f_{i+1j} + \zeta_{j-1} f_{ij-1} + \chi_{j+1} f_{ij+1}.
$$

Note that the density outside the state space is zero. Hence,

$$
(A^k)' f = 0,
$$

where A^k is the matrix from the HJB equation. Finally, we impose

$$
\sum_{j=1}^{J} \sum_{i=1}^{I} f_{ij} \Delta a \Delta z = 1.
$$

Pseudocode. Summarizing, the algorithm to solve HJB and KF equations is

Algorithm 5.

1. Guess v_{ij}^k and compute $v_{a,ij}^k$ according to the upwind scheme
2. Compute $c_{ij}^k = u^{-1}(v_{a,ij}^k)$.
3. Solve the linear system (2) for v^{k+1} and iterate until $|v^{k+1} - v^k| < \epsilon$.
4. Use the intensity matrix A^k to find the density function of the states.

An initial guess could be $v_{ij}^0 = u(wz_j + ra_i)/\rho$. In the implicit case the step size Δ can be arbitrarily large. We should also make sure that at the upper bound of the asset grid a backward difference is used. It might help stability to impose $a \leq a_{\max}$ by setting $v_{a,Ij,F} = u'(wz_j + ra_I)$. Note that $(A^k)' f = 0$ is an homogenoeus linear system and $(A^k)'$ has a zero eigenvalue it is singular and not invertible thus the system admits nontrivial solutions. One trick is to discard one equation normalizing one element of f to any scalar, say .1. We solve the modified system $Pg = 0$ inverting the modified coefficient matrix P, and to recover f we use the fact that the density add up to one $f = g/[\sum(g_{ij})\Delta a \Delta z]$.

Extensions. More general applications often features problems with nonconvexities. If the value function is not concave we could have $s_{ij,F} > 0$ and $s_{ij,B} < 0$. One method that works well in practice is to use the Hamiltonians $H_{ij} = u(c_{ij}) + v_{a,ij}s_{ij}$ to move in the direction with larger gain. Let $d_{[both]} = 1_{[s_{ij,B}<0<s_{ij,F}]}$, $d_{[not\ both]} = 1_{[s_{ij,F}>0, s_{ij,B}\leq 0]} + 1_{[s_{ij,F}\leq 0, s_{ij,B}<0]}$ then

$$\begin{aligned}
v_{a,ij} &= v_{a,ij,F}(1_{[s_{ij,F}>0]}d_{[not\ both]} + 1_{[H_{ij,F}\geq H_{ij,B}]}d_{[both]}) \\
&\quad + v_{a,ij,B}(1_{[s_{ij,B}<0]}d_{[not\ both]} + 1_{[H_{ij,F}<H_{ij,B}]}d_{[both]}) \\
&\quad + \tilde{v}_{ij}1_{[s_{ij,F}\leq 0\leq s_{ij,B}]}.
\end{aligned}$$

Transition dynamics can be computed extending the algorithm to the case in which $v_{ij}^k = v(a_i, z_j, t_k)$. In this case we solve backward for v^k the HJB equation

$$\rho v^k = u^{k+1} + A^{k+1}v^k + \frac{1}{\Delta t}(v^{k+1} - v^k),$$

given a terminal condition v^K. Then, solving forward for f^{k+1} the KF equation given an initial condition f^0 with an explicit method

$$\frac{f^{k+1} - f^k}{\Delta t} = (A^k)' f^k,$$

or an implicit method

$$\frac{f^{k+1} - f^k}{\Delta t} = (A^k)' f^{k+1}.$$

Finally, the previous solution can be extended to nonuniform grids. For example we might want to use a power grid to put enough points for low asset levels, i.e. the nonlinear region of the policy functions

$$a_i = \underline{a} + (\bar{a} - \underline{a})\left(\frac{i-1}{I-1}\right)^\eta, \quad \forall i = 1, 2, ..., I.$$

Now we simply use $\Delta a_{i,F} := a_{i+1} - a_i$, $\Delta a_{i,B} := a_i - a_{i-1}$ for first order derivatives and

$$\frac{\Delta a_{i,B}v_{i+1j} + (\Delta a_{i,B} + \Delta a_{i,F})v_{ij} + \Delta a_{i,F}v_{i-1j}}{.5(\Delta a_{i,B} + \Delta a_{i,F})\Delta a_{i,B}\Delta a_{i,F}}$$

for second order derivatives. In order to preserve the mass in the KF we work directly with the vector $g = f\Delta_i$ where from the trapezoidal rule $\Delta_i = .5\Delta a_{i,F}$ if $i = 1$, $\Delta_i = .5\Delta a_{i,B}$ if $i = I$ and $\Delta_i = .5(\Delta a_{i,B} + \Delta a_{i,F})$ otherwise. Once we solved for g as usual we can recover the true density $f_i = g_i/\Delta_i$. The same applies with many income states z_j.

Example: steady state. Assume A Cobb-Douglas production function $K_t^{\alpha} L_t^{1-\alpha}$ and a CRRA utility function $c_t^{1-\gamma}/(1-\gamma)$. Figure 3 shows a global solution for steady state policy and density functions in general equilibrium. One can notice the nonlinearities in the policy functions close to the borrowing limit. From the density we see the positive correlation between wealth and income risk, i.e. wealthier households have higher income.

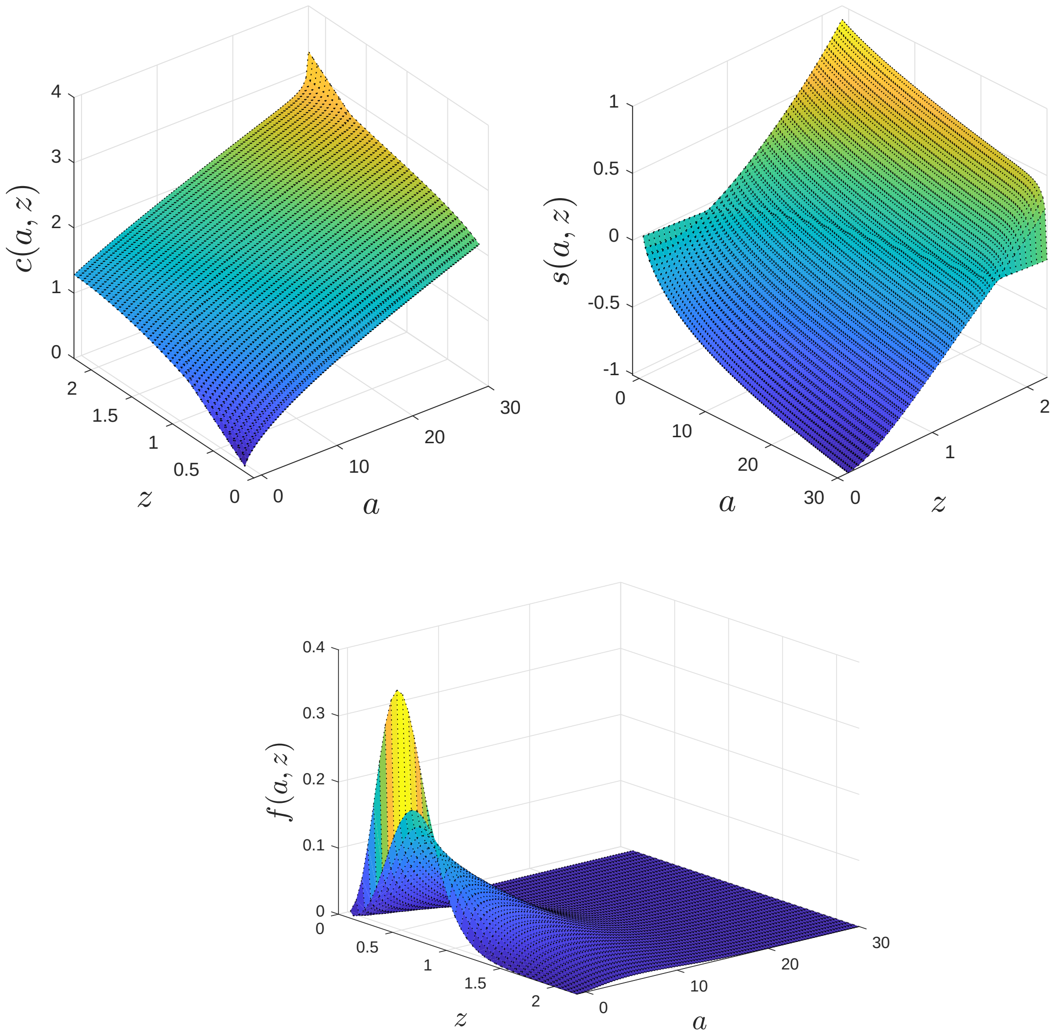

Figure 3: Consumption, saving and density functions.

Note: I use $\alpha = .33, \gamma = 2, \delta = .1, \rho = .05, \phi = -1$, a Ornstein–Uhlenbeck process for $\ln z_t$ with annual autocorrelation of .9 and standard deviation of innovations of .2, and computational parameters $\Delta = 1000, a_{\max} = 30, I = 100, J = 40$. The equilibrium prices are $w = 1, r = .03 < \rho$.

IV. Idiosyncratic and aggregate risk

This section continues in discrete time. Let z_t be an aggregate TFP shock. The production function is $Y_t = z_t F(K_t, L_t)$. Now the joint process $\{z_t, l_t\}$ follows a discrete Markov process, $\{z_t\}$ and $\{l_t\}$ may be correlated, if they are independet we can factorize the transition function $Q_{z,l}((z,l),(.,.)) = Q_l(l,.)Q_z(z,.)$ where Q_l and Q_z are the transition functions for l_t and z_t respectively. We have an infinite horizon discounted problem, u is time invariant, the constraint correspondence is also time invariant since it depends on time only through states. Hence, value and policy functions are time invariant.

Definition 4. A recursive competitive equilibrium consists of a value function $v : X \times (X, \mathcal{X}) \times Z \to \mathbb{R}$, an individual decision function $g^i : X \times (X, \mathcal{X}) \times Z \to \mathbb{R}, i \in \{a, c\}$, factor demands $K, L; (X, \mathcal{X}) \times Z \to \mathbb{R}$ a law of motion $H : (X, \mathcal{X}) \times Z^2 \to (X, \mathcal{X})$ and $r, w : (X, \mathcal{X}) \times Z \to \mathbb{R}$:

1. Given H, r, w, g^i, v solve the Bellman equation:

$$v(a, l; \psi, z) = \max_{a' \geq 0} \left\{ u(c) + \beta \int_{Z \times S} v(a', l'; \psi', z') dP_{z'l'|zl}(z', l'|z, l) \right\},$$
$$\text{s.t.} \quad c + a' = (1 + r(\psi, z))a + w(\psi, z)l.$$

2. Firms maximize profits:

$$zF_K(K, L) = r(\psi, z) + \delta,$$
$$zF_L(K, L) = w(\psi, z).$$

3. Markets clear:

$$K'(\psi', z') = \int_X g^a(a, l; \psi, z) d\psi,$$
$$L(\psi, z) = \int_X l d\psi,$$
$$\int_X g^c(a, l; \psi, z) d\psi + \int_X g^a(a, l; \psi, z) d\psi = F(K(\psi, z), L(\psi, z)) + (1 - \delta)K(\psi, z),$$
$$\psi' = H(\psi, z; z')((\mathcal{A}, \mathcal{S})) = \int_X Q((a, l), (\mathcal{A}, \mathcal{S})) d\psi(a, l),$$
$$Q((a, l), (\mathcal{A}, \mathcal{S})) = 1_{[g(a,l;\psi,z) \in \mathcal{A}]} \sum_{l' \in \mathcal{S}} P_{z'l'|zl}(z', l'|z, l).$$

1 Numerical algorithms

The Krusell-Smith algorithm. Let m be a vector of first k moments of the wealth distribution i.e. the marginal of ψ with respect to a. Krusell and Smith (1998) propose to approximate the distribution by a set of moments, which are finite dimensional objects. The law of motion of m is given by $m' = H_k(m, z)$. This solution method effectively assumes a form of bounded rationality, rather than solving the model for all possible paths of the aggregate states we modify these aggregate states. Then, Krusell and Smith (1998) show that with $k = 1, m_1 = K, \ln K' = \beta_0(z) + \beta_1(z) \ln K$ is possible to obtain accurate solutions. The algorithm hinges on linear law of motion, which is not always the case in richer models.

Algorithm 6.

1. Guess the coefficients in the law of motion $\beta_0(z), \beta_1(z)$ for each $z \in Z$.
2. Solve the household's problem and obtain $g^a(a, l; K, z)$.
3. Simulate the economy for N individuals and T periods and compute the mean $\bar{K}_t = N^{-1} \sum_{i=1}^{N} a_t^i$.
4. Discard the first $T_0 = 500$ periods and run the regression $\ln \bar{K}_{t+1} = \beta_0(z) + \beta_1(z) \ln \bar{K}_t, \forall z$.
5. If $(\hat{\beta}_0(z), \hat{\beta}_1(z)) \neq (\beta_0(z), \beta_1(z))$ try a new guess and repeat from step 1.
6. If $(\hat{\beta}_0(z), \hat{\beta}_1(z)) \approx (\beta_0(z), \beta_1(z))$ for each z compute R^2 of the regression.
7. If the R^2 has improved keep iterating.

In step 3 draw a random sequence $\{z_t\}_{t=0}^{T}$ and $\{l_t^i\}_{t=0}^{T}$ for each $i = 1, ..., N$ and use g to generate $\{a_t^i\}_{i=1}^{N}$ for each $t = 0, 1, 2, ..., N$, in each period. In step 4 we avoid the dependence of the results from the initial conditions. In step 6 we assess the goodness of fit of the approximate law of motion.

The neural networks algorithm. Fernández-Villaverde, Hurtado, and Nuno (2020) use neural networks to approximate the model's aggregate law of motion and solve the model in its original state space.

Perturbation algorithms. For local solutions see Reiter (2009), Ahn, Kaplan, Moll, Winberry and Wolf (2018), Bayer, Luetticke (2018).

V. OLG model with idiosyncratic risk

Equilibrium. Let J be the set of all possible age realizations and $P(J)$ its power set. A sequential equilibrium in the OLG model is $\{w_t, r_t\}_{t=0}^{\infty}$,$\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$, $\{c_{t,j}, a_{t+1,j+1}\}_{t,j}$:

1. Given the prices $\{c_{t,j}, a_{t,j+1}\}_{j=0}^{J}$ solves

$$\max_{\{c_{t,j}, a_{t+1,j+1}\}_{j=0}^{J}} \sum_{j=0}^{J} \beta^j s_j u(c_{t,j}),$$

$$\text{s.t.} \quad c_{t,j} + a_{t+1,j+1} = w_t e_{t,j} + (1 + r_t + p_{t,j} - \delta) a_{t,j}, \forall t, \forall j,$$

$$\text{s.t.} \quad a_{t+1,j+1} \geq -\phi, a_0 \text{ given}, \forall t, \forall j.$$

2. Given the prices N_t, K_t solves in each period

$$\max_{(K_t, N_t) \geq 0} \Pi_t = F(K_t, A_t N_t) - w_t N_t - r_t K_t.$$

3. Prices are such that markets clear

$$L_t = \int_X e_t d\psi_t, \forall t$$

$$K_t = \int_X a_t d\psi_t, \forall t$$

$$\int_X c_t d\psi_t + \int_X a_{t+1} d\psi_t = F(K_t, L_t) + (1 - \delta) K_t, \forall t$$

$$\psi_{t+1}(\mathcal{J} \times \mathcal{A} \times \mathcal{S}) = H_t(\psi_t) = \int_X Q_t((j, a_t, e_t), \mathcal{J} \times \mathcal{A} \times \mathcal{S}) d\psi_t, \forall t$$

$$Q_t((j, a_t, e_t), \mathcal{J} \times \mathcal{A} \times \mathcal{S}) = 1_{[g_{t+1}^a \in \mathcal{A}, j+1 \in \mathcal{J}]} s_j P_{e_{t+1}|e_t}(e_{t+1}|e_t), \forall t,$$

and for newborns

$$\psi_{t+1}(\{1\} \times \mathcal{A} \times \mathcal{S}) = 1_{[0 \in \mathcal{A}]} p(e_t) h_{t,1}, \forall t.$$

The main difference is that now also age j is an idiosyncratic state and $X = J \times A \times S$. See Chapter 2 for further details. The previous algorithms can be easily extended to this case. While the state space is larger, one advantage of finite planning horizons is that we do not need to iterate on the law of motion of the distribution and on the household policy or value functions until convergence.

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