Chapter 2: Life-cycle and overlapping generations

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Quantitative Macroeconomics

Introduction. Many economic decisions regarding education, carrer paths, consumption and saving depend on age and other life-cycle dimensions. So, we might want to relax the assumption of an infinitely lived economy to study these issues. Moreover, the presence of age cohorts, different generations, and demographic changes can be important for many economic issues such as capital accumulation, growth, pension systems, and taxation. In this chapter I will introduce models with overlapping generations (OLG).

I. Life-cycle models

Survival risk. A simple way to introduce life-cycle features in an infinite horizon model is to consider an economy with a continuum of identical individuals, $\mathcal{H} = [0, 1]$, and assume that in each period a fraction of households leaves the economy and for each individual that exit one is born as in Blanchard (1985), Yaari (1965). This is the perpetual youth or dynastic framework.

More formally, in each period there is a constant probability $0 < p < 1$ of leaving the economy. Let T be a random variable denoting the time at which households exit. The survival function is given by $S(t) = \Pr(T > t) = (1-p)^t$. The expected lifetime is $\sum_{t=0}^{\infty} tp(1-p)^{t-1} = 1/p$ since $\sum_{k=0}^{\infty} kx^{k-1} = 1/(1-x)^2$. Therefore,

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t [(1-p)^t u(c_t) + (1-p)^{t-1} pu(c_t)] = \sum_{t=0}^{\infty} \delta^t u(c_t).$$

The second equality follows from the normalization $u(c_T) = 0$ and the definition $\delta := \beta(1-p)$. Thus, the exit rate p simply decreases the discount factor.

Similarly, in continuous time we have a Poisson process in which households exit the model at rate $p \in (0,1)$ per time units. Namely n_t, the number of shock arrivals in the period $[0,t]$, is a Poisson variable with arrival rate pt. Then, T is exponentially distributed. To see this note that $\Pr(T > t) = \Pr(n_t = 0) = e^{-pt}$. Therefore, $S(t) = \Pr(T > t) = e^{-pt}$ and $\int_0^\infty tpe^{-pt}dt = 1/p$.

$$\mathrm{E}_0 \int_0^\infty e^{-\rho t} u(c_t)dt = \int_0^\infty e^{-(\rho+p)t} u(c_t)dt.$$

As households exit randomly they leave "accidental bequests". The standard approach is to assume that is in each period all the accidental bequests are redistributed. Different schemes are possible. For example, normalizing the total population to one, an equal redistribution yields $a' = (y + (1+r)a - c)(1 + (1-S)/S) = S^{-1}(y + (1+r)a - c)$ in discrete time, or maybe agents are paid pa every period to leave the control of their assets in case they exit so $da = (y + ra - c + pa)dt$ in continuous time.

Finite horizon. In the perpetual youth framework we don't keep track of individuals' age which is simply given by the survival function. Now we introduce a partial equilibrium deterministic life-cycle model and index households' age with an additional idiosyncratic state variable j. Since in this model the economy horizon and household horizon coincide $j = t$. In the next section we will consider an infinite horizon economy where agents have finite horizons and different generations coexist in each period.

To this end, assume that $t = 0, 1, 2, ..., T$. Given the income stream $\{y_t\}$, returns to wealth $\{r_t\}$, and initial assets a_0 the representative agent solves the following program subject to the natural borrowing limit $\bar{a}$ and boundary condition $a_{T+1} = 0$.

$$\max_{c_t} \quad \sum_{t=0}^{T} \beta^t u(c_t),$$

$$\text{s.t.} \;\; a_{t+1} = y_t + (1+r_t)a_t - c_t, \quad \forall t,$$

$$\text{s.t.} \;\; a_{t+1} \geq -\bar{a}, \quad \forall t.$$

In the recursive form value and policy functions depend on time only through the states t, a and the Bellman equation reads

$$v_t(a_t) = \max_{a_{t+1}} \{u(y_t + (1+r_t)a_t - a_{t+1}) + \beta v_{t+1}(a_{t+1})\}.$$

We can simulate life-cycle profiles (j, x_j) for income, saving, and consumption. In the data life-cycle profiles from age say 20 to 80 are typically hump shaped.

II. Overlapping generations models

1 Two-period OLG Models

Time is discrete $t \in \mathbb{N}_+$, the economy has an infinite horizon, and households live for two periods $j = 1, 2$. In each period there are two cohorts in the economy, the young with $j = 1$ and the old with $j = 2$. Each individual supply labor inelastically in the first period when young. In the second period individuals retire and consumes through savings. The aggregate production function is given by $F(K_t, A_t N_t)$ the growth rate of the population and technology are respectively g, n. Rescaling $y_t = f(k_t) := F(K_t/(A_t N_t), 1)$. For simplicity we assume full depreciation $\delta = 1$, and $f' > 0, f'' < 0, f(0) = 0, f'(0) \to \infty, f'(\infty) \to 0$. k_0 is given. Firms maximize profits $y_t - (\delta + r_t)k_t - w_t$. Moreover β is the discount factor adjusted for population growth. The representative household

$$\max_{(c_{1,t}, c_{2,t+1}, s_t)} u(c_{1,t}) + \beta u(c_{2,t+1})$$

$$\text{s.t.} \quad c_{1,t} + s_{1,t} = w_t,$$

$$c_{2,t+1} = (1 + r_{t+1}) s_{1,t}.$$

The old at $t = 0$ simply consume their income y_0. From the first order conditions we have the usual Euler equation

$$u'(w_t - s_t) = \beta u'((1 + r_{t+1}) s_t)(1 + r_{t+1}).$$

Dynamic inefficiency. Let k_g be the stock of capital such that $f'(k_g) = (1+n)(1+g)$, i.e. the golden rule level of capital that maximizes steady state consumption per worker

$$c = f(k) - (1 + g)(1 + n)k.$$

If $k > k_g$ then $dc/dk < 0$ i.e. $1 + r < (1 + n)(1 + g)$ and the economy is said to be dynamically inefficient. It overaccumulates capital. Without infinite lifetime and the transversality condition we cannot rule out dynamic inefficiency. Intuitively, the economy horizon differs from agents planning horizon. With the infinite time horizon in these models the assumption of a bounded set of agents is violated. Therefore, Pareto inefficient situations may appear in OLG models. If T was finite, then the increase of consumption of all generations until period T would be at the cost of consumption in period T. If the olds want to consume they need to accumulate capital when $1 + r < (1 + n)(1 + g)$ a social planner can ensure a return equal to $(1 + n)(1 + g)$ making everyone better off.

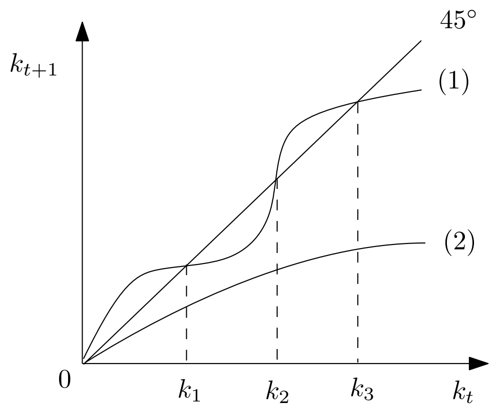
Phase diagrams. The market clearing $(1+g)(1+n)k_{t+1} = s_{1,t}$ implies $k_{t+1} = s(1+r_{t+1}, w_t)(1+n)^{-1}(1+g)^{-1}$ substituting the first order conditions of the representative firm we obtain the nonlinear difference equation

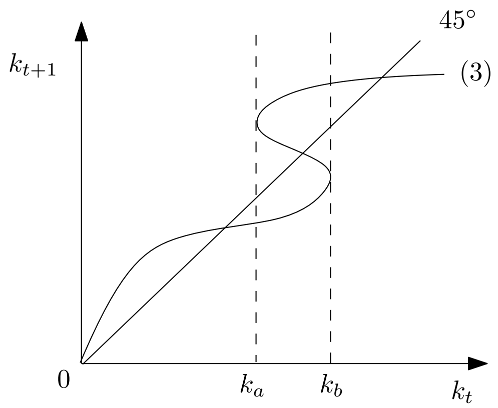
$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}), f(k_t) - k_t f'(k_t)). \tag{1}$$

Next we will construct a phase diagram in discrete time. Writing Equation (1) as $k_{t+1} = g(k_t)$ then $\lim_{k\to\infty} g(k)/k = 0, g(0) = 0$. To see this notice that $(1+n)k_{t+1}/k_t \leq w_t/k_t = f(k_t)/k_t - f'(k_t) \Rightarrow \lim_{k\to\infty} g(k)/k \leq \lim_{k\to\infty}[f(k_t)/k_t - f'(k_t)] = 0$ and the graph of $g(k_t)$ eventually will fall below the $45°$ line $k_{t+1}/k_t = 1$ namely $k_{t+1}/k_t = g(k_t)/k_t < 1$. By the implicit function theorem

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w k_t f''(k_t)}{(1+n)(1+g) - s_r f''(k_{t+1})}. \tag{2}$$

The numerator is always positive. The sign of the denominator depends on the sign of s_r. (1) If $s_r > 0, g'(0) > 1$ then g is increasing in k_t and we could have multiple steady states. (2) If $s_r > 0, g'(0) < 1$ then excluding the origin there is no steady state. (3) If $s_r < 0$ then (1) has multiple solutions for k_{t+1} given k_t. These can be interpreted as sefl-fulfilliing equilibria.





Summarizing, in OLG models a balanced growth path may not exists. The assets in the economy can be subject to endogenous self-fulfilling fluctuations. Some of the equilibria may envolve a steady state capital level above the efficient level.

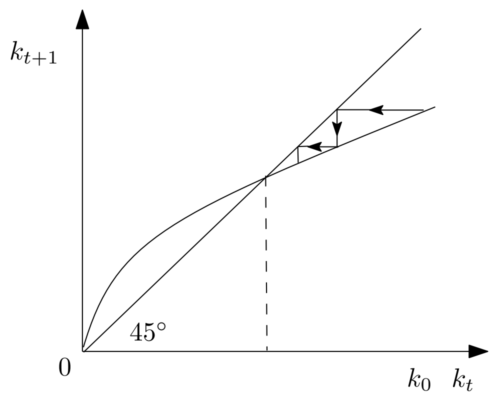
Diamond model. To have a unique steady state Equation (1) needs to satisfy the conditions: $s_r > 0, g'(0) > 1$ and

$$|g'(k)| = \left| \frac{-s_w k_t f''(k_t)}{(1+n)(1+g) - s_r f''(k_{t+1})} \right| < 1.$$

Example 1. If $\theta = 1$ the fraction of income saved is $1/(2+\rho)$. With a Cobb-Douglas production function Equation (1) becomes

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}}{(1+n)(1+g)(2+\rho)}.$$

Under these restrictions there exists a unique balanced growth path globally stable such that $k_{t+1} = k_t = k$ the condition $k_0 > 0$ is sufficient to rule out the uninteresting steady state $k = 0$.



Substituting k^* in the difference equation we obtain

$$k = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)} \right]^{1/(1-\alpha)}.$$

Therefore, aggregate output is

$$y = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)} \right]^{\alpha/(1-\alpha)}.$$

Speculative bubbles. Suppose that in the economy there is an outside asset instrinsically useless if in equilibrium its demand $b_t^d > 0$ we have a rational speculative bubble. Youngs can allocate their savings between capital and the asset $(1+n)(1+g)(k_{t+1} + b_{t+1}) = s(w_t, r_{t+1})$. From the first order conditions we have the condition $(1+n)(1+g)b_{t+1} = (1+r_t)b_t$. Hence, we have a system of difference equations with unknown k_t, b_t

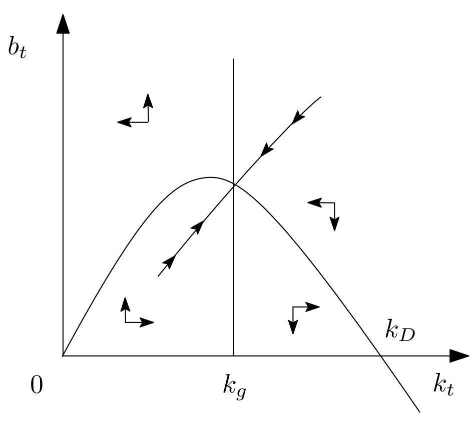
$$k_{t+1} + b_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}), f(k_t) - k_t f'(k_t)),$$

$$(1+n)(1+g)b_{t+1} = (1+r_t)b_t.$$

The steady state solution is given by $(1+r)(k+b) = s(1+f'(k), f(k) - kf'(k))$. Hence, capital is equal to the golden rule capital stock and we can solve the previous equation for b. We can study the stability using a phase diagram in the space (k, b).

$$k_{t+1} - k_t = \frac{1}{(1+n)(1+g)}[s(f'(k_{t+1}), f(k_t) - k_t f'(k_t)) - (1+r_t)b_t] = 0,$$

Solving yields $b_t = [s(w(k_t), r(k_{t+1})) - (1+g)(1+n)k_t]/(1+r(k_t))$ which cross the x-axis at the Diamond steady state. One can check that $\partial b(0)/\partial k > 0, \partial b(k_D)/\partial k < 0$ and $k_{t+1} > k_t \iff b_t < [s(.) - (1+g)(1+n)k_t]/(1+r(k_t))$. Moreover, $b_{t+1} - b_t = ((1+r_t)/[(1+n)(1+g)] - 1)b_t = 0$ implies a line at the golden rule capital stock, $b_{t+1} > b_t \iff (1+r_t) > (1+n)(1+g) \iff k_t < k_g$. To have a steady state with $b_t > 0$ we need $k_g < k_D$ or $(1+r_D) < (1+n)(1+g)$ namely an equilibrium with rational speculative bubble must have a dynamic inefficient steady state and for any k_1 there exists a unique $b_1 = b(k_1)$ on the saddle path such that $\{b_t, k_t\}$ converges to the steady state. The bubble absorbing the excess of saving eliminates the inefficiency.



If $b_1 < b(k_1)$ then $\{b_t, k_t\}$ converges to the Diamond steady state where $b_t = 0$.

2 Multi-period OLG model

Consider an economy in the spirit of Auerbach and Kotlikoff (1987) where time is discrete $t \in \mathbb{N}_+$, the economy has an infinite horizon, and households live for J periods $j = 1, 2, ..., J$. In each period there are J cohorts in the economy. Let $h_{t,j}$ be the number of households in period t with age j. In each period t households entry and exit the model in such a way that the population is constant $\sum_{j=0}^{J} h_{t,j} = 1, \forall t$. To implement this we often assume an exogenous survival probability s_j so that the population size is given by $h_{t,j} = s_{j-1} h_{t,j-1}$ and $h_{t,1}$ is such that the population is stationary. There is perfect insurance for accidental bequests at rate $p_{t,j}$. Alternatively we can assume that $\{h_{0,j} : \sum_{j=0}^{J} h_{0,j} = 1\}, h_{t,1} = h_{t,J}, \forall t > 0$ so newborns simply replace the old.

Sequential equilibrium. A sequential equilibrium in the multi-period OLG model is $\{w_t, r_t\}_{t=0}^{\infty}$,$\{C_t, N_t, I_t, K_{t+1}\}_{t=0}^{\infty}$, $\{c_{t,j}, k_{t+1,j+1}\}_{t,j}$ such that

1. Given the prices N_t, K_t solves in each period

$$\max_{(K_t,N_t)\geq 0} \Pi_t = F(K_t, A_t N_t) - w_t N_t - r_t K_t.$$

2. Given the prices $\{c_{t,j}, k_{t,j+1}\}_{j=0}^{J}$ solves

$$\max_{\{c_{t,j},k_{t+1,j+1}\}_{j=0}^{J}} \sum_{j=0}^{J} \beta^j s_j u(c_{t,j}),$$

$$\text{s.t.} \quad c_{t,j} + k_{t+1,j+1} = w_t + (1 + r_t + p_{t,j} - \delta) k_{t,j} + \Pi_{t,j}, \forall t, \forall j,$$

$$\text{s.t.} \quad k_{t+1,j+1} \geq 0, k_0 \text{ given}, \forall t, \forall j.$$

3. Prices are such that markets clear

$$N_t = 1, \ \forall t,$$

$$K_t = \sum_{j=0}^{J} k_{t,j}, \ \forall t,$$

$$Y_t = \sum_{j=0}^{J} c_{t,j} + K_{t+1} - (1 - \delta) K_t, \ \forall t.$$

References

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