# Chapter 1: Complete markets and growth

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Quantitative Macroeconomics

*Introduction*. In this chapter I review basic concepts and results that are useful to formulate economic models. I also discuss aggregate risk, consumption insurance, asset pricing, capital accumualtion, and growth.

# I. Complete markets models

## 1 Deterministic models

*The economy*. Time is a discrete variable and the economy has an infinite horizon $t \in \mathbb{N}$. There is a countable set $\mathcal{H} \subseteq \mathbb{N}$ of households or individuals. Each household $i \in \mathcal{H}$ lives forever. Individuals consume $c_{it} \in \mathbb{R}_+$ units of nondurable goods. The preferences of the individuals are represented by the lifetime utility function

$$U_i(c_i) = \sum_{t=0}^{\infty} \beta^t u_i(c_{it})$$

where $u_i : \mathbb{R}_+ \to \mathbb{R}$ is the instantaneous utility function and $\beta = (1+\rho)^{-1} \in (0,1)$ the discount factor. Individuals are impatient, if $c_{it} = c_{it+1} = c$ then $\beta^t u(c) > \beta^{t+1} u(c)$. A consumption allocation for the individual $i$ is an infinite-dimensional vector or sequence $c_i = \{c_{it}\}_{t=0}^{\infty}$ and $x = (c_i)_{i \in \mathcal{H}}$. Each agent has an exogenous endowment of the consumption good or wealth $w_i = \{w_{it}\}_{t=0}^{\infty}$, $w_{it} \in \mathbb{R}_+$. We assume that the utility function $u_i(c_{it})$ is twice differentiable $u_i \in C^2$, strictly increasing $u_i'(c_{it}) > 0$ and strictly concave $u_i''(c_{it}) < 0, \forall i$. Moreover, Inada contions hold: $\lim_{c_{it} \to 0^+} u_i'(c_{it}) = +\infty, \lim_{c_{it} \to +\infty} u_i'(c_{it}) = 0, \forall i$.

An exchange economy is given by $E = (\mathcal{H}, U, e)$ where $\mathcal{H} \subseteq \mathbb{N}$ is a countably finite set of household, $U = \prod_{i \in \mathcal{H}} U_i$ is the set of utility functions, $e = (w_i)_{i \in \mathcal{H}}$ is the exogenous distribution of endowments.

*Centralized economies.* Consider an economy $E = (\mathcal{H}, U, e)$. The social planner dictates allocations to maximize social welfare.

$$\max_{(x,y)} \sum_{i \in \mathcal{H}} \theta_i \sum_{t=0}^{\infty} \beta^t u_i(c_{it}), \tag{SP}$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{H}} c_{it} = \sum_{i \in \mathcal{H}} w_{it}.$$

The parameters $\theta \in \mathbb{R}_+^{card(\mathcal{H})}$ are the social planner's weights. An allocation $(c_i)_{i \in \mathcal{H}}$ is Pareto efficient if it is feasible: $c_{it} \geq 0, \forall i, \forall t$, $\sum_i c_{it} \leq \sum_i w_{it}, \forall t$, and if there is no other feasible $(\hat{c}_i)_{i \in \mathcal{H}} : U_i(\hat{c}_i) \geq U_i(c_i)$, strict inequality for at least one $i$.

*Competitive equilibirum.* A market system is represented by a set of prices $p = \{p_t\}_{t=0}^{\infty}$ one price $p_t \in \mathbb{R}_+$ for each commodity $x_t$. We assume perfect information, competitive and complete markets. All the prices are known to the agents. The agents take prices as given. All the goods $\{x_t\}_{t=0}^{\infty}$ have a market. Even with one-sector we have an infinite-dimensional commodity space $\mathbb{R}_+^{\infty} := \mathbb{R}_+ \times \mathbb{R}_+ \times ....$ Suppose that in $t = 0$ individuals can trade $\{x_t\}_{t=0}^{\infty}$ signing binding contracts which reports all the infinite future trades and then the trades take place over time accordingly. This trade structure is known as the Arrow-Debreu (AD) time-0 trading and $p$ as Arrow-Debreu prices. This allow us to write a lifetime budget constraint

$$\sum_{t=0}^{\infty} p_t c_{it} \leq \sum_{t=0}^{\infty} p_t w_{it}.$$

In the sequential markets (SM) structure agents trade at the beginning of each period the good available in that period. There is a market in which agents can trade a one-period asset $a_{it+1}$ known as Arrow security or contigent claim. The price of the asset is $q_t = (1 + r_{t+1})^{-1}$ units of the good in $t$. If $a_{it+1} < 0$ the individual issued the security. We have an infinite sequence of budget constraints

$$c_{it} + q_t a_{it+1} \leq w_{it} + a_{it} \quad \forall t = 0, 1, 2, ...$$

$$a_{it} \in [-A^i, \infty), \forall t$$

The initial asset holding $a_{i0}$ is exogenous. The debt limit $A^i$ is necessary to avoid Ponzi schemes, that is without debt limit the individual $i$th will be strictly better off borrowing to consume more in the initial period and then rolling over the debt in each future period.

**Definition 1.** An Arrow-Debreu competitive equilibrium for the economy $E = (\mathcal{H}, U, e)$ is an allocation $\hat{x}$ where $\hat{x} = (\hat{c}_i)_{i\in\mathcal{H}}$ and prices $\hat{p}$ such that

1. Given $\hat{p}$ the vector $\hat{c}_i$ solves the utility maximization problem for each $i$:

$$U_i(\hat{c}_i) \geq U_i(c_i), \forall c_i : \sum_{t=0}^{\infty} \hat{p}_t c_{it} \leq \sum_{t=0}^{\infty} \hat{p}_t w_{it}.$$

2. The prices $\hat{p}$ are such that $\hat{x}$ clear the market of the consumption good:

$$\sum_{i\in\mathcal{H}} \hat{c}_{it} \leq \sum_{i\in\mathcal{H}} \hat{w}_{it}, \forall t.$$

**Definition 2.** A sequential trading competitive equilibrium for the economy $E = (\mathcal{H}U, Y, e)$ is an allocation $(\hat{x}, \hat{a})$ where $(\hat{x}, \hat{a}) = (\hat{c}_i, \hat{a}_i)_{i\in\mathcal{H}}$, prices $\hat{q}$ such that

1. Given $\hat{q}, a_0$ the vectors $(\hat{c}_i, \hat{a}_i)$ solve the maximization problem for each $i$:

$$U_i(\hat{c}_i) \geq U_i(c_i), \forall a_i, c_i : c_{it} + \hat{q}_t a_{it+1} \leq w_{it} + a_{it}, a_{it} \in [-A^i, \infty), \forall t.$$

2. The prices $\hat{q}$ are such that $(\hat{x}, \hat{a})$ clear the markets:

$$\sum_{i\in\mathcal{H}} \hat{c}_{it} \leq \sum_{i\in\mathcal{H}} \hat{w}_{it}, \forall t$$

$$\sum_{i\in\mathcal{H}} \hat{a}_{it} = 0.\forall t.$$

In general equilibrium prices are endogenous variables determined by demand and supply. In partial equilibrium we take prices as given and focus on the forces driving demand and supply. For example, in the previous definitions we consider a general equilibrium framework, while point 1 gives the associated partial equilibrium definition. The market clearing condition $\sum_i a_{it} = 0$ assumes that the asset is in zero net supply. With perfect markets, continuous, monotone and convex preferences the competitive equilibrium exists. The existence of a market clearing price $p$ follows by applying a fixed point theorem to the aggregate excess demand correspondence $z(p) := x(p) - w$. See Debreu (1959), Bewley (2007). Remember that the walrasian demands $x(p)$ is homogeneous of degree zero in prices therefore if $p$ are equilibrium prices $\lambda p$ are equilibrium prices as well $\forall\lambda \in \mathbb{R}_{++}$, that is only relative prices can be determined in equilibrium, so we can rescale the prices $1, p_1/p_0, ...$ and normalize $p_0 = 1$, now the other prices are expressed in units of good $x_0$. The Walras' law give us one degree of freedom. Since $U$ is strictly increasing in equilibrium the budget constraints is binding $\forall i$. In each period if $n-1$ markets clear then the $n$-th market also clear. For example in the definition of sequential equilibrium the condition $\sum_t c_{it} \leq \sum_i w_{it}$ is redundant because $\sum_i a_{it} = 0, \sum_i c_{it} + q_t \sum_i a_{it+1} = \sum_i w_{it} + \sum_i a_{it} \Rightarrow \sum_i c_{it} = \sum_i w_{it}$.

*The no-Ponzi condition.* Given the budget constraint in future value $c_{it} + a_{it+1} = w_{it} + (1+\hat{r}_t)a_{it}$ or in present value $c_{it} + (1+\hat{r}_{t+1})^{-1}a_{it+1} = w_{it} + a_{it}$ we can solve it forward substituting recursively for $a_{it+1}$ in $c_{it} + (1+\hat{r}_{t+1})^{-1}a_{it+1} = w_{it} + a_{it}$ from the budget constraint in $t+1$ and so on to obtain

$$a_{it} = \sum_{j=0}^{\infty} R_{t+j}(c_{it+j} - w_{it+j}) + \lim_{T\to\infty} R_{T+1}a_{iT+1}, \;\; \forall t$$

where $R_{t+j} = \prod_{s=0}^{j}(1+\hat{r}_{t+s})^{-1}$. In particular,

$$a_{i0} = \sum_{t=0}^{\infty} R_t(c_{it} - w_{it}) + \lim_{T\to\infty} R_{T+1}a_{iT+1},$$

$R_t = \prod_{s=1}^{t}(1+\hat{r}_s)^{-1}, \prod_{s=1}^{0}(1+\hat{r}_s)^{-1} = 1$. The condition $\lim_{T\to\infty} R_{T+1}a_{iT+1} \geq 0$ is often used as a no-Ponzi condition. This is not a good practice since it only holds at equilibrium prices. Note that $a_{it} \geq -A^i \Rightarrow \lim_{T\to\infty} R_{T+1}a_{iT+1} \geq \lim_{T\to\infty} -R_{T+1}A^i = 0$ if $r_t > 0, \forall t$.

*The natural borrowing limit.* The tightest debt limit $-A_t^i$ such that $a_{it} \geq -A^i$ never binds in equilibrium is known as the natural debt limit. We can derive it solving forward the SM budget constraints. If $\lim_{T\to\infty} R_{T+1}a_{iT+1} \geq 0$ and using the fact that consumption must be nonnegative we have the natural debt limit

$$a_{it} \geq -\sum_{\tau=0}^{\infty} R_{t+\tau}w_{it+\tau}, \forall t \geq 1,$$

where $R_{t+\tau} = \prod_{s=0}^{\tau}(1+\hat{r}_{t+s})^{-1}$. By Inada conditions the natural borrowing limit will never be binding. In models with sustained growth $w_{it} = (1+g)^t w_{i0}, g > 0$ and the natural debt limit is potentially $-\infty$. Suppose that there exists a limiting interest rate $r : \{r_t\} \to r$ as $t \to \infty$ if $r > g$ then the lifetime wealth is finite and the natural debt limit is well defined. Note that if one relates equilibrium prices by

$$1 + r_{t+1} = \frac{p_{t+1}}{p_t}.$$

Then, $\prod_{j=1}^{t}(1+\hat{r}_j) = (\hat{p}_0/\hat{p}_1)(\hat{p}_1/\hat{p}_2)\dots(\hat{p}_{t-1}/\hat{p}_t) = 1/\hat{p}_t$ and the SM budget constraint

$$\sum_{t=0}^{\infty} \hat{p}_t c_{it} + \lim_{T\to\infty} R_{T+1} a_{iT+1} = \sum_{t=0}^{\infty} \hat{p}_t w_{it},$$

is equivalent to the AD budget constraint

$$\sum_{t=0}^{\infty} \hat{p}_t c_{it} \le \sum_{t=0}^{\infty} \hat{p}_t w_{it}$$

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*Continuous time*. In continuous time we can rewrite agents utility as

$$U_i(c_i) = \int_0^{\infty} e^{-\rho t} u_i(c_{it}) dt,$$

where $\rho$ is the subjective discount rate and the law of motion of assets can be written as $\dot{a}_{it} = \hat{r}_t a_{it} + w_{it} - c_{it}$. Solving it forward we arrive to the lifetime budget constraint

$$\int_0^{\infty} e^{-R_t} c_{it} dt + \lim_{T\to\infty} e^{-R_T} a_{iT} = \int_0^{\infty} e^{-R_t} w_{it} dt + a_{i0}.$$

where $R_t = \int_0^t \hat{r}_s ds$.

*The representative household.* An economy admits a representative agent or consumer if aggregate consumption can be obtained as the outcome of the choices of a single consumer. The economy $E = (\mathcal{H}, U, e)$ with walrasian demands $x_i(p, w_i) \in \mathbb{R}^n_+, \forall i, p \in \mathbb{R}^n_+$ and wealth distribution $\omega = (w_i)_{i\in\mathcal{H}}$ admits a representative consumer if there exists a rational preference relation $\succsim$ on $\mathbb{R}^n_+$ such that the aggregate demand

$$x(p, \omega) = \sum_{i\in\mathcal{H}} x_i(p, w_i).$$

is equal to the walrasian demand $x(p, w)$ generated by $\succsim$ given the wealth $\sum_i w_i = w \in \mathbb{R}_+$. The economy admits a representative agent when the aggregate demand depends only on the aggregate wealth $x(p, \sum_i w_i)$ and not on the wealth distribution $\omega$. Namely,

$$\omega, \omega' : \sum_{i \in \mathcal{H}} w_i = \sum_{i \in \mathcal{H}} w_i' \Rightarrow \sum_{i \in \mathcal{H}} x_i(p, w_i) = \sum_{i \in \mathcal{H}} x(p, w_i') \Longleftrightarrow x(p, w).$$

Starting from an initial distribution $\omega$ let's consider a small redistribution $d\omega : \sum_i dw_i = 0$. By the differentiability of the individual demands, $D_w x(p, \sum_i w_i)$ is given by

$$\sum_{i \in \mathcal{H}} \frac{\partial x_{ik}(p, w_i)}{\partial w_i} dw_i = 0 \quad k = 1, ..., n.$$

Thus,

$$\frac{\partial x_{ik}(p, w_i)}{\partial w_i} = \frac{\partial x_{il}(p, w_l)}{\partial w_l}, \forall i, l, i \neq l$$

which means that the marginal propensity to consume must be the same for all agents and wealth levels. With constant marginal propensity to consume any wealth redistribution does not change the individual demands so the individual income expansion path must be linear and parallel for all consumers, e.g. when all the consumers have the same homothetic preferences or quasilinear preferences with respect to the same good.

From general equilibrium theory we know that the demand side of the economy does not admit a representative household since the aggregate demand may not coincide with the individual demand because the Slutsky matrix may not be symmetric and negative semidefinite. See Debreu (1974), Mas-Colell, Whinston, and Green (1995). However, under more strict conditions the economy admits a representative agent.

**Theorem 1. (Gorman's Aggregation)**. All the consumer in an economy have linear and parallel income expansion paths $\forall p \in \mathbb{R}^n_+$ if and only if all the consumers have preferences with indirect utility function

$$v_i(p, w_i) = a_i(p) + b(p) w_i.$$

Note that if $a_i(p), b(p)$ are differentiable Roy's Identity implies that

$$x_{ik}(p, w_i) = -b(p)^{-1} \frac{\partial a_i(p)}{\partial p_k} - b(p)^{-1} \frac{\partial b(p)}{\partial p_k} w_i.$$

The slope of this linear relation does not depend on $i$.

In frictionless production economies a representative firm always exists because of the abscence of income effects when firms only face technological constraints.

*General equilibrium theorems*. The first theorem states an equivalence between the solution of the social planner's problem and efficiency, and an equivalence between Arrow-Debreu and sequential equilibria. The second theorem provides sufficient and necessary conditions for efficiency of competitive equilibria. In practice, these results are useful as in frictionless economies we can find a competitive equilibrium by solving the associated social planner's problem.

**Theorem 2.** An allocation $\hat{x}$ is Pareto efficient if and only if it solves (SP) for some $\theta$. Moreover, an economy $E = (\mathcal{H}, U, e)$ if $(\hat{p}, \hat{x})$ is an Arrow-Debreu equilibrium then there exists a sequential market equilibrium $(\tilde{q}, \tilde{x}, \tilde{a})$ such that $\hat{x} = \tilde{x}, \tilde{q}_t = \hat{p}_{t+1}/\hat{p}_t, \forall t$. If $(\hat{q}, \hat{x}, \hat{a})$ is a sequential market equilibrium then there exists an Arrow-Debreu equilibrium $(\tilde{p}, \tilde{x})$ with $\hat{x} = \tilde{x}, \tilde{p}_{t+1} = \hat{q}_t \tilde{p}_t, \forall t$.

**Theorem 3.** Let $(\hat{x}, \hat{p})$ be an equilibrium with complete and competitive markets. If preferences are continuous and locally nonsatiated then $\hat{x}$ is Pareto efficient. Moreover, if markets are complete and competitive, $u_i$ are continuous, locally nonsatiated and quasi-concave. Then, for any Pareto efficient allocation $\hat{x}$ in the economy $(\mathcal{H}, U, e)$ there exists a strictly positive price vector $\hat{p}$ and a wealth redistribution $\hat{e}$ such that $(\mathcal{H}U, \hat{e})$ has a competitive equilibrium $(\hat{x}, \hat{p})$.

## 2 Risk, consumption insurance, and asset pricing

In each period there are different possible states of the world given by a random variable $s_t \in S$. A stochastic economy is given by a probability space $(S^\infty, \mathcal{S}^\infty, P)$ of complete histories $\{s_t\}_{t=1}^\infty \in S^\infty$ with probability measure $P : \mathcal{S}^\infty \to [0, 1]$. Then, we can consider a filtration $\{\mathcal{S}^t\}_{t \geq 1}$ on $(S^\infty, \mathcal{S}^\infty)$ with $s^t = (s_1, ..., s_t) \in S^t = S \times S \times ... \times S, \forall t \geq 1$ as the history of the exogenous states up to time $t$ given an initial state $s_0$.

Now we can allow some variables $x$ in the model to depend on these shocks $x_t(s^t)$. In particular, the consumption good is different across time and states $c_t^i(s^t)$. Hence, also Arrow-Debreu prices are given by $p_t(s^t)$ and at $t = 0$ individuals plan trades over all possible realizations of the states. Similarly, in the SM equilibrium there is a one period asset $a_{t+1}^i(s^t, s_{t+1})$ with price $q_t(s^t, s_{t+1})$ known as Arrow's security, contingent claim or insurance contract. In economics we typically use the short hand notation $\mathrm{E}_t$ for the expected value conditional on $\mathcal{S}^t$ and $x_t$ for $x_t(s^t)$. Finally, we can write

$$\mathrm{E}_0 \sum_{t=0}^\infty \beta^t u(c_t(s^t)) = \sum_{t=0}^\infty \beta^t \int_{S^t} u(c_t(s^t)) dP(s^t).$$

Relative in the stochastic framework the assumption of perfect information has two components. The assumption of rational expectations requires that all the agents know $P$.

The assumption of full information requires that agents know all the structural equations in the model including other agents preferences and constraints.

**Definition 3.** The AD competitive equilibrium are $\{\hat{p}_t(s^t)\}_{t\in\mathbb{N},s^t\in S^t}$ and $\{\hat{c}_{it}(s^t)\}_{t\in\mathbb{N},s^t\in S^t}$ for each $i \in \mathcal{H}$ such that

1. Given $\{\hat{p}_t(s^t)\}$ the allocation $\{\hat{c}_{it}(s^t)\}$ solves for each $i \in \mathcal{H}$

$$\max_{\{\hat{c}_{it}(s^t)\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}(s^t))$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) c_{it}(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \hat{p}_t(s^t) w_{it}(s^t),$$

2. Markets clear

$$\sum_{i=1}^{n} \hat{c}_{it}(s^t) = \sum_{i=1}^{n} w_{it}(s^t) \quad \forall t, \forall s^t.$$

**Definition 4.** A SM competitive equilibrium is given by $\{\hat{c}_{it}(s^t), \hat{a}_{it+1}(s^t, s_{t+1})\}_{t\in\mathbb{N},s^t\in S^t}$ for each $i \in \mathcal{H}$, $\{\hat{q}_t(s^t, s_{t+1})\}_{t\in\mathbb{N},s^t\in S^t}$ such that

1. Given the prices the sequence $\{\hat{c}_{it}(s^t), \hat{a}_{it+1}(s^t, s_{t+1})\}$ solves for each $i \in \mathcal{H}$

$$\max_{(\hat{c}^i, \hat{a}^i)} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}(s^t))$$

$$\text{s.t.} \quad c_{it}(s^t) + \sum_{s_{t+1} \in S} \hat{q}_t(s^t, s_{t+1}) \hat{a}_{it+1}(s^t, s_{t+1}) \leq w_{it}(s^t) + a_{it}(s^t), \quad \forall t, \forall s^t$$

$$a_{it+1}(s^t, s_{t+1}) \geq -A^i \quad \forall t, \forall s^t, \forall s_{t+1}$$

2. Markets clear

$$\sum_{i=1}^{n} \hat{c}_{it}(s^t) = \sum_{i=1}^{n} w_{it}(s^t) \quad \forall t, \forall s^t,$$

$$\sum_{i=1}^{n} \hat{a}_{it+1}(s^t, s_{t+1}) = 0 \quad \forall t, \forall s^t, \forall s_{t+1}.$$

All the previous definitions and results can be generalized to this setting. As in the deterministic case we can normalize only one price, say $p_0(s_0) = 1$ which holds only for a particular realization of $s_0$.

*Arrow-Debreu equilibrium conditions*. The solution of the household's problem, including the constraints $c_{it}(s^t) \geq 0, \forall t, \forall s^t$, can be derived from the Lagrangian.

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \Big( \beta^t u(c_{it}(s^t))P(s^t) - \lambda(p_t(s^t)c_{it}(s^t) - p_t(s^t)w_{it}(s^t)) - \mu_t(s^t)c_{it}(s^t) \Big).$$

The first order sufficient conditions $\forall t, \forall s^t$ together with complementary slackness conditions and transversality condition are given by

$$\beta^t P(s^t)u'(c_{it}(s^t)) - \lambda p_t(s^t) - \mu_t(s^t) = 0,$$

$$\lambda \Big( \sum_{t=0}^{\infty} \sum_{s^t} p(s^t)c_{it}(s^t) - \sum_{t=0}^{\infty} \sum_{s^t} p(s^t)w_{it}(s^t) \Big) = 0,$$

$$\mu_t(s^t)c_{it}(s^t) = 0,$$

$$\lim_{t \to \infty} \mathrm{E}_0 D_w v(w_t, s_t) w_{it}(s^t) = 0.$$

If $\mu_t(s^t) > 0 \Rightarrow c_{it}(s^t) = 0$ then $u'(0) = \lambda p_t(s^t) - \mu_t(s^t) < +\infty$ violating the Inada condition. Hence, $\mu_t(s^t) = 0$ and $c_{it}(s^t) > 0$. If $\lambda = 0$ then $\sum_{t=0}^{\infty} p_t(s^t)c_{it}(s^t) - \sum_{t=0}^{\infty} p_t(s^t)w_{it}(s^t) \leq 0$. However, $\beta^t u'(c_{it}(s^t)) = 0$ which contradicts monotonocity. Thus, $\lambda > 0$ and the budget constraint will be binding.

Solving for $\lambda$ the first order conditions in $t$ and $t+1$ and using

$$\frac{P(s_{t+1}, s^t)}{P(s^t)} = P(s_{t+1}|s^t)$$

we obtain the Euler equation

$$\frac{p_{t+1}(s^{t+1})}{p_t(s^t)} = \frac{\beta u'(c_{it+1}(s^{t+1}))}{u'(c_{it}(s^t))} P(s^{t+1}|s^t).$$

The Euler equation equates the marginal rate of substitution between consumption today and tomorrow $u'(c_{it})/\beta u'(c_{it+1})$ and the relative price or opportunity cost of consumption today, i.e. the inverse of the gross inflation rate $1/\Pi_{t+1}$ where $\Pi_{t+1} := p_{t+1}/p_t$. This optimality condition equalize the marginal benefit of consumption today $u'(c_{it})/p_t$ with the marginal benefit of consumption tomorrow $\beta u'(c_{it+1})P(s^{t+1}|s^t)/p_{t+1}$.

The Euler equation with the transversality condition, the budget constraint and market clearing conditions chracterize the equilibrium of the model. This is a system of stochastic difference equations for which in general we do not have an analytical solution. In the next chapter we will study how to solve these models numerically.

*Sequantial market equilibrium conditions*. Consider now the sequential markets definition and the associated Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Big( u(c_{it}(s^t))P(s^t) - \lambda_t(s^t)((c_{it}(s^t) \\ + \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1})a_{it+1}(s^t, s_{t+1}) - w_{it}(s^t) - a_{it}(s^t)) \Big).$$

Now the first order conditions $\forall t, \forall s^t, \forall s_{t+1}$ are given by

$$\beta^t P(s^t)u'(c_{it}(s^t)) - \lambda_t(s^t) = 0,$$

$$-q_t(s_{t+1}, s^t)\lambda_t(s^t) + \mu_t(s^t) + \lambda_{t+1}(s_{t+1}, s^t) = 0,$$

$$\lambda_t(s^t)((c_{it}(s^t) + \textstyle\sum_{s_{t+1}\in S} q_t(s^t, s_{t+1})a_{it+1}(s^t, s_{t+1}) - w_{it}(s^t) - a_{it}(s^t)) = 0,$$

$$\mu_t(s^{t+1})(A^i - a_{it+1}(s_{t+1}, s^t)) = 0,$$

$$\lim_{t\to\infty} \mathrm{E}_0 D_x v(x_t, s_t)x_{it}(s^t) = 0. \;\; x \in \{w, a\}.$$

The characterization for the sequential market equilibrium is given by an Euler equation

$$q_t(s_{t+1}, s^t) = \frac{\beta u'(c_{it+1}(s^{t+1}))}{u'(c_{it}(s^t))} P(s^{t+1}|s^t)$$

together with transversality condition, budget constraint, market clearing. By Walras' law we have a redundant equation, usually we ignore the goods market clearing condition.

*Euler equations and asset pricing*. If we have a one period not state contingent asset $a_{t+1}(s^t)$, a form of incomplete markets, instead of rewriting the budget constraints $c_t(s^t) + p_t^a(s^t)a_{t+1}(s^t) = a_t(s^{t-1}) + w_t(s^t), \forall t, \forall s^t$ and taking again the first order conditions we can use the Arrow-Debreu and sequential prices to price this and any other asset in the economy. Let's consider first a general case and then derive some examples. Consider an asset that pays dividends $\{d_t(s^t)\}_{t=0}^{\infty}$, i.e. state contingent claims to $d_t(s^t)$ units of consumption. The price of such asset is given by the present value of all consumption goods the asset delivers at all future states

$$p_0^a(s_0) = \frac{\sum_{t=0}^{\infty} \sum_{s^t\in S^t} p_t(s^t)d_t(s^t)}{p_0(s_0)},$$

or at time $t$

$$p_t^a(s^t) = \frac{\sum_{j=t+1}^{\infty} \sum_{s^j\in S^{j-t}} p_j(s^j, s^t)d_j(s^j)}{p_t(s^t)}.$$

The one period gross return is

$$R_{t+1}^a(s^{t+1}) := \frac{p_{t+1}^a(s^{t+1}) + d_{t+1}(s^{t+1})}{p_t^a(s^t)}.$$

*Example* 1. Consider an Arrow security, i.e. $d_{t+1}(\hat{s}_{t+1}, s^t) = 1, d_{t+1}(s_{t+1}, s^t) = 0, \forall s_{t+1} \neq \hat{s}_{t+1}$. Then the price $p_t^a(s^t) = p_{t+1}(s^{t+1})/p_t(s^t) = q_t(s^t, s_{t+1}), \forall s^t$. The return is $R_{t+1}^a = (0+1)/p_{t+1}(s^{t+1})/p_t(s^t) = 1/q_t(s^t, s_{t+1})$.

*Example* 2. Consider a one period risk free bond $b_t(s^t) = b_t$ that pays one unit of consumption at each state. Then the price $p_t^a(s^t) = \sum_{s_{t+1}\in S} p_{t+1}(s_{t+1}, s^t)/p_t(s^t) = \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1}), \forall s^t$. The return is $R_{t+1}^a(s^{t+1}) = 1/\sum_{s_{t+1}\in S} q_t(s^t, s_{t+1}) = R_{t+1}^a(s^t)$.

These considerations can be summarized by saying that $\beta u'(c_{t+1})/u'(c_t)P(s_{t+1}|s^t)$ is the time $t$ pricing kernel. To ease notation we often omit the argument $s^t$. So, we can always write the Euler equations $\forall t, \forall s^t$ for a one period asset as

$$p_t^a u'(c_t) = \beta \mathrm{E}_t u'(c_{t+1}) d_{t+1},$$

$$u'(c_t) = \beta \mathrm{E}_t R_{t+1}^a u'(c_{t+1}).$$

The second equation follows from the first since $p_{t+1}^a = 0$. Similarly, for an asset without maturity since $p_t^a u'(c_t) = \mathrm{E}_t \beta u'(c_{t+1})(d_{t+1} + p_{t+1}^a)$ we have

$$p_t^a u'(c_t) = \mathrm{E}_t \sum_{j=t+1}^{\infty} \beta^{j-t} u'(c_j) d_j,$$

$$u'(c_t) = \beta \mathrm{E}_t R_{t+1}^a u'(c_{t+1}).$$

*Perfect cross-sectional insurance*. In a frictionless enviroment with complete markets consumers can trade to smooth consumption over time and states of the world. As a result we have perfect risk sharing across households. Intuitively this follows from the convexity of preferences and rational expectations. An allocation $\{\hat{c}_t^i(s^t)\}_{t\in\mathbb{N}, s^t\in S^t}$ for each $i \in \mathcal{H}$ has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time and states. Next I show that the competitive allocations satisfy $c_{it}(s^t) = \alpha_i e_t(s^t), \forall i, \forall s^t, \forall t$, where $\alpha_i \in [0,1], \sum_i \alpha_i = 1$, i.e. the economy admits a representative agent and perfect risk sharing. The Arrow-Debreu prices internalize the probability of each state history and do not depend on the distribution of the aggregate endowment. To see this let's start from the the first order conditions of the consumer $i$ with respect to $c_{it}(s^t)$ given by $\beta^t P(s^t) u'(c_{it}(s^t)) = \lambda_i p_t(s^t), \forall t, \forall s^t$. Dividing by the

same condition for $j$ yields
$$\frac{u'(c_{it}(s^t))}{u'(c_{jt}(s^t))} = \frac{\lambda_j}{\lambda_i}.$$

Since the first order conditions have to hold $\forall i$ we can substitute for all the terms not function of $i$ and using $s^0 = s_0$ at the denominator and $t$ at the numerator we get that $u'(c_{it}(s^t))/u'(c_{i0}(s_0)) = u'(c_{jt+1}(s^t))/u'(c_{j0}(s_0))$ namely $u'(c_{it}(s^t))/u'(c_{jt+1}(s^t)) = u'(c_{i0}(s_0))/u'(c_{j0}(s_0))$. Therefore, the individual demand of consumption have to satisfy the eqaulity of marginal rate of substitutions across time and state. This implies that given the aggregate endowment $e_t(s^t) = \sum_{i=1}^{n} w_{it}(s^t)$ from the resource constraint $c_{it}(s^t) = \alpha_{it}(s^t)e_t(s^t) = \alpha_i e_t(s^t), \forall i, \forall s^t, \forall t$. Therefore, the consumption risk depends only on the risk of the aggregate and not on the fluctuations of $w_{it}(s^t)$. Moreover, from the first order conditions and market clearing we have

$$p_t(s^t) = \beta^t \frac{P(s^t)u'(c_{it}(s^t))}{P(s^0)u'(c_{i0}(s^0))} = \beta^t \frac{P(s^t)u'(e_t(s^t))}{P(s^0)u'(e_0(s^0))},$$
$$\frac{p_t(s^t)}{p_0(s_0)} = \frac{\beta^t P(s^t)u'(e_t(s^t))}{P(s_0)u'(e_0(s_0))}.$$

*The permanent income hypotesis*. Assume that the aggregate endowment is constant over time i.e. there are no aggregate shocks. From perfect risk sharing also consumption will be constant over time. Substituting the first order conditions $\beta^t P(s^t)u'(c_{it}(s^t)) = \lambda_i p(s^t), \forall t, \forall s^t$ in the budget constraint yields

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \lambda_i^{-1} \beta^t P(s^t) u'(c_{it}(s^t))[w_{it}(s^t) - c_{it}(s^t)] = 0.$$

Canceling the multiplier $\lambda$ and solving for $c_{it}$ we have

$$c_{it} = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t P(s^t) w_{it}(s^t)$$

Consumption is a fraction of the lifetime or permanent income. See Friedman (1957), Modigliani and Brumberg (1954) and Hall (1978) who analyze the consumer intertemporal problem in a frictionless economy. Note that the marginal propensity to consume $\partial c_{it}/\partial w_{it}$ is close to zero when $\beta \to 1$ as in most quantitative macro work. Flavin (1981), Hall and Mishkin (1982), Campbell and Mankiw (1989) find evidence of sensitivity of consumption to current income and temporary income fluctuations. More recent empirical work uncovers sizable marginal propensities to consume out of tax rebates, the 2008 US fiscal stimulus payments, and lottery winnings in Norway. See Broda and Parker

(2014), Parker, Souleles, Johnson, and McClelland (2008), Fagereng, Holm, and Natvik (2020). As we will see in the next chapters heterogeneous agent models can account for these findings.

*The martingale property*. Consider the sequential markets in partial equilibrium. To ease the notation I drop the household index $i$. Let $R_{t+1} = (1 + r_{t+1})$ be the gross real interest rate. If $R_{t+1} = R_t = R$, the first order conditions of the household's problem implies the Euler equation

$$\mathrm{E}_t[u'(c_{t+1}(s^{t+1}))] = (\beta R)^{-1} u'(c_t(s^t)).$$

The process $\{u'(c_t(s^t))\}_{t=0}^{\infty}$ is a martingale, i.e.

$$\mathrm{E}_t[u'(c_{t+1})|u'(c_t)] = \mathrm{E}_t[u'(c_{t+1})|u'(c_t), I_t]$$

where $I_t$ is the information set, i.e. no other variable known at time $t$ can improve upon forecast for $u'(c_{t+1}(s^{t+1}))$ based only on $u'(c_t(s^t))$.

*Precautionary saving*. One can consider the effect on consumption of an increase in uncertainty. Set $R_{t+1} = R$. A second order taylor expansion around the steady state $c_{t+1} = c_t$ yields

$$\beta R \mathrm{E}_t \left[ 1 + \frac{c_t u''(c_t)}{u'(c_t)} \left( \frac{c_{t+1} - c_t}{c_t} \right) + .5 \frac{c_t u'''(c_t)}{u''(c_t)} \frac{c_t u''(c_t)}{u'(c_t)} \left( \frac{c_{t+1} - c_t}{c_t} \right)^2 + \epsilon \right] = 1.$$

Let $x_t$ be consumption growth, $1/\theta_t^1 = -c_t u''(c_t)/u'(c_t)$ and $\theta_t^2 = -c_t u'''(c_t)/u''(c_t)$ then

$$\mathrm{E}_t x_{t+1} = \theta_t^1 \alpha + .5\theta_t^2 \mathrm{Var}(x_{t+1}) + \xi.$$

If $u'''(c_t) = 0$ then $\mathrm{Var}(x_t)$ drops out. With constant relative risk aversion we obtain certainty equivalence $\mathrm{E}_t x_{t+1} = x_{t+1}$. As Kimball (1990) pointed out a strictly convex marginal utility $u'''(c_t) > 0$ is sufficient for precautionary saving. Another approach to obtain precautionary saving is to introduce potentially binding borrowing constraints and idiosyncratic income risk, i.e. heterogeneous agent framework.

*General equilibrium asset pricing*. Following Lucas (1978) suppose that in the economy there is a one-period riskless bond $B_t$ with gross return $R_t$ and price $q_t = (1 + r_t)^{-1}$ and a risky stock $N_t$ with price $p_{t+1}$ that pays a stream of dividends $d_{t+1}$, which follows a Markov Process. Let $A_t$ be the total wealth. To save on notation I will omit the argument of $x_t(s^t), x \in \{c, N, p, d\}$. The market clearing condition for equity shares is $\sum_i N_{it} = 1$ and for bonds $\sum_i B_{it} = 0$. The representative agent

$$\max_{\{c_t, B_t, N_t\}} \mathrm{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j})$$

$$\begin{aligned}
\text{s.t.} \quad & c_t + q_t B_t + p_t N_t \leq A_t && \forall t, \forall s^t \\
& A_{t+1} = B_t + (d_{t+1} + p_{t+1}) N_t && \forall t, \forall s^t \\
& B_t \geq -b_B, N_t \geq -b_N && \forall t, \forall s^t.
\end{aligned}$$

The first order and transversality conditions are sufficient to characterize the solution

$$\begin{aligned}
u'(c_t) q_t &= \beta \mathrm{E}_t[u'(c_{t+1})] \\
\lim_{k \to \infty} \mathrm{E}_t[\beta^k u'(c_{t+k}) q_{t+k} B_{t+k}] &= 0 \\
u'(c_t) p_t &= \beta \mathrm{E}_t[u'(c_{t+1})(d_{t+1} + p_{t+1})] \\
\lim_{k \to \infty} \mathrm{E}_t[\beta^k u'(c_{t+k}) p_{t+k} N_{t+k}] &= 0.
\end{aligned}$$

Solving forward the Euler equation yields

$$u'(c_t) p_t = \mathrm{E}_t \sum_{k=1}^{\infty} \beta^k u'(c_{t+k}) d_{t+k} + \mathrm{E}_t \lim_{k \to \infty} \beta^k u'(c_{t+k}) p_{t+k}.$$

Therefore, the price of the stock has two components one related to the foundamentals $d_{t+k}$ and a speculative component or bubble $b_t = \mathrm{E}_t \lim_{k \to \infty} \beta^k u'(c_{t+k}) p_{t+k}$. From the representative agent assumption and market clearing $\sum_i N_{it} = N_t = 1$, substituting this in the transversality condition we can rule out speculative bubbles in general equilibrium $\lim_{k \to \infty} \mathrm{E}_t \beta^k u'(c_{t+k}) p_{t+k} N_{t+k} = \lim_{k \to \infty} \mathrm{E}_t \beta^k u'(c_{t+k}) p_{t+k} = 0$. Intuitively if the limit is positive then $u'(c_t) p_t >$E$_t \sum_{k=1}^{\infty} \beta^k u'(c_{t+k}) d_{t+k}$ that is the marginal utility from selling equity is grater than the expected marginal utility from consuming the dividends. Hence, everyone want to sell and this cannot be an equilibrium. Thus, in equilibrium there cannot be speculative bubbles and $p_t = \mathrm{E}_t \sum_{k=1}^{\infty} \beta^k \frac{u'(c_{t+k})}{u'(c_t)} d_{t+k}$. The price is increasing in expected dividends and in its ability to provide insurance to households. Namely, the lower is Cov$(c_{t+j}, d_{t+j})$ the higher will be the price as the asset pays higher dividends when consumption is low. Finally, from the consumption based asset pricing equation $p_t = \beta \mathrm{E}_t[(u'(c_{t+1})/u'(c_t))(d_{t+1} + p_{t+1})]$. One can derive the efficient markets hypothesis which states that the price of any asset is a martingale.

# II. The neoclassical growth model in discrete time

*The economy*. Time is discrete $t \in \mathbb{N}$. In the economy there is a consumption-investment good and a continuum of identical households $\mathcal{H} = [0,1]$ with identical members. So, the economy trivially admits a representative household. The population within each household $h_t$ grows at the gross rate $h_{t+1}/h_t = (1+g_n)$, hence $h_t = (1+g_n)^t h_0$ and without loss of generality we set $h_0 = 1$. The preferences of the representative household are given by

$$U = \sum_{t=0}^{\infty} \beta^t (1+g_n)^t u(c_t, n_t)$$

where $c_t \in \mathbb{R}_+, n_t \in [0,1], \forall t$ represent a household member's consumption and labor supply. Hence, $c_t$ is the per capita consumption of the final good and $n_t$ is the fraction of working members of each household or per capita hours of work, $\beta(1+g_n) \in (0,1)$. The function $u$ is continuous, two times differentiable, $u_c > 0, u_{cc} < 0, u_n < 0, u_{nn} < 0$ and Inada conditions holds. Disutility from labor is equivalent to utility from leisure $u(c,l)$ where $u_l > 0, u_{ll} < 0$. If we only use $u(c)$ we usually set $h_t = 1$ and assume an inelastic labor supply $n_t = 1, \forall t$.

Let $K_t$ be the aggregate stock capital, $N_t$ the aggregate labor services, $A_t$ the technological level, $C_t = (1+g_n)^t c_t$ aggregate consumption,

$$I_t = K_{t+1} - (1-\delta)K_t$$

aggregate investment with depreciation rate $\delta \in [0,1]$. The economy admits a representative firm. The production side is described by an aggregate production function with labor-augmenting technology

$$Y_t = F(K_t, A_t N_t)$$

which is continuous, differentiable, homogeneous of degree one in capital and labor, i.e. $F(\lambda K_t, \lambda N_t) = \lambda F(K_t, N_t)$,$\forall \lambda > 0$, strictly increasing and strictly concave in both its arguments. Moreover, $F(K,0) = F(0,N) = 0, \forall K, N > 0$ and $F_K(K,1) \to 0, K \to \infty, F_K(K,1) \to \infty, K \to 0$. Let $(1+g_a)$ be the gross growth rate of $A_t$, i.e. the technological progress. Notice that $I_t$ can be negative meaning that the capital stock can be turned into consumption. There are no market frictions. Households own the factors of production and run the firms.

*Balanced growth and technology*. Kaldor (1963) showed that historically GDP per capita and capital-labor ratio tend to increase over time, capital-output ratio is approximately constant, real wages increase over time and real interest rates are constant, labor and capital shares $wN/Y, rK/Y$ are also constant. The neoclassical growth models is built to replicate these empirical patterns. A balanced growth path is a sequence $\{C_t, K_t, Y_t\}_{t=0}^{\infty}$ where each variable grows at a constant rate $g_C, g_K, g_Y$. There are several ways to model productivity: Hicks-neutral $AF(K, N)$, Solow-neutral $F(AK, N)$, Harrod-neutral $F(K, AN)$. Uzawa (1961) shows that constant return to scale and balanced growth imply that the production function must have the Harrod-neutral representation.

**Proposition 1.** Consider an aggregate production function $G : \mathbb{R}^n \times \mathbb{R}_+^2 \to \mathbb{R}_+$ where $Y_t = G(\tilde{A}_t, N_t, K_t)$ with constant return to scale in $K_t, N_t$, aggregate resource constraint

$$K_{t+1} = Y_t - C_t + (1 - \delta)K_t.$$

If $\exists T : \forall t \geq T$ the growth rates for production, capital, consumption and population $g_Y, g_K, g_C, g_n$ are constant. Then $g_Y = g_K = g_C$ and $\forall t \geq T$ the function $G$ can be represented as $Y_t = F(K_t, A_t N_t)$ where $F$ is homogeneous of degree one in its arguments.

*Proof.* We can write the resource constraint as $(g_K + \delta)K_t = Y_t - C_t$. For any $t \geq T$ we can write $(g_K + \delta)(1 + g_K)^t K_T = (1 + g_Y)^t Y_T - (1 + g_C)^t C_T$. Dividing both sides by $(1+g_K)^t$ we have $(g_K+\delta)K_T = (1+g_K)^{-t}(1+g_Y)^t Y_T - (1+g_K)^{-t}(1+g_C)^t C_T$. Notice that $\Delta K_t = g_K K_t$, since $\Delta K_T = Y_T - C_T - \delta K_T$ as long as $g_Y, g_K, g_C > 0$ it must be the case that $g_C = g_K = g_Y$. At time $T$ for any $t \geq T$ can write the production function $(1 + g_Y)^{-t} Y_t = G(A_T, (1 + g_n)^{-t} N_t, (1 + g_K)^{-t} K_t)$. Multiplying both sides by $(1 + g_Y)^t$ by the constant return to scale $Y_t = G(A_T, (1+g_Y)^t(1+g_n)^{-t} N_t, (1+g_Y)^t(1+g_K)^{-t} K_t)$. Since $g_Y = g_K$ we have $Y_t = G(A_T, (1+g_Y)^t(1+g_n)^{-t} N_t, K_t)$ namely $Y_t = F(K_t, A_t N_t)$ where $A_t = (1 + g_Y)^t(1 + g_n)^{-t}$ or $A_{t+1}/A_t = (1 + g_Y)(1 + g_n)^{-1}$. $\square$

If the aggregate production function is Cobb-Douglas then $Y_t = (A_t^K K_t)^{\alpha}(A_t^N N_t)^{1-\alpha}$ is compatible with balanced growth since the Cobb-Douglas function can always be written in a labor augmenting form

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}$$

where $A_t = A_t^{K\alpha/(1-\alpha)} A_t^N$.

*Growth accounting*. Consider the aggregate production function in discrete time $Y_t = G(K_t, N_t, A_t)$. Using elasticities $\eta_X = (\Delta Y/\Delta X)(X/Y)$ and growth rates $g_X = \Delta X/X$ we can write $g_Y = \eta_K g_K + \eta_N g_N + \eta_a g_a$. As we will see in competitive markets factor prices in real terms are given by $r = G_K, w = G_N$ and the elasticities $\eta_K, \eta_N$ correspond to the factor shares $\alpha_K = rK/Y, \alpha_N = wN/Y$. Thus, $\eta_a g_a = g_Y - \alpha_K g_K - \alpha_N g_N$. This accounting equation can be used to estimate the contribution of technological progress to economic growth using data on the right-hand-side variables. The contribution from technologicl progress $\eta_a g_a$ is known as total factor productivity (TFP) or Solow's residual. To make the previous accounting equation useful in applied work we replace the constant factor shares with within period averages, e.g. $(\alpha_t - \alpha_{t+1})/2$. See Solow (1957).

*Arrow-Debreu equilibrium.* An Arrow-Debreu equilibrium in the neoclassical growth model are prices $\{w_t, r_t, p_t\}_{t=0}^{\infty}$, allocations $\{C_t, N_t, I_t, K_{t+1}\}_{t=0}^{\infty}$, $\{c_t, n_t, i_t, k_{t+1}\}_{t=0}^{\infty}$:

1. Given prices $\{K_t, N_t\}_{t=0}^{\infty}$ solves

$$\max_{\{K_t,N_t\}} \Pi = \sum_{t=0}^{\infty} p_t(F(A_t N_t, K_t) - w_t N_t - r_t K_t).$$

2. Given prices $\{c_t, n_t, i_t, k_{t+1}\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t,n_t,i_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t (1+g_n)^t u(c_t, n_t),$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t((1+g_n)^t c_t + (1+g_n)^t i_t) \leq \sum_{t=0}^{\infty} p_t(w_t(1+g_n)^t n_t + r_t(1+g_n)^t k_t) + \Pi,$$

$$\text{s.t.} \quad (1+g_n)k_{t+1} = (1-\delta)k_t + i_t, \forall t,$$

$$\text{s.t.} \quad c_t \geq 0, k_{t+1} \geq 0, k_0 \text{ given}, \forall t.$$

3. Prices are such that markets clear

$$Y_t = C_t + I_t, \ \forall t.$$

$$N_t = (1+g_n)^t n_t, \ \forall t.$$

$$K_t = (1+g_n)^t k_t, \ \forall t.$$

To solve this model we need to solve two infinite-dimensional optimization problems. The firm's first order conditions are given by

$$N_t : w_t = A_t F_N(K_t, A_t N_t),$$

$$K_t : r_t = F_K(K_t, A_t N_t).$$

Substituting these first order conditions in the objective function, by homogeneity of degree one the Euler theorem implies that $Y_t = F_K(K_t, A_tN_t)K_t + F_N(K_t, A_tN_t)A_tN_t$. Hence, $\Pi = 0$. We can characterize the solution of the household problem using first order conditions and a transversality condition:

$$c_t : \beta^t u_c(c_t, n_t) = \lambda p_t,$$

$$n_t : -\beta^t u_n(c_t, n_t) = \lambda p_t w_t,$$

$$k_{t+1} : p_t = p_{t+1}(1 + r_{t+1} - \delta),$$

$$\lim_{t\to\infty} \beta^t u_c(c_t, n_t)k_{t+1} = 0.$$

*Sequential markets equilibrium.* A sequential markets equilibrium in the neoclassical growth model is $\{w_t, r_t\}_{t=0}^{\infty}$,$\{C_t, N_t, I_t, K_{t+1}\}_{t=0}^{\infty}$, $\{c_t, n_t, i_t, k_{t+1}\}_{t=0}^{\infty}$ such that

1. Given the prices $(N_t, K_t)$ solves in each period

$$\max_{(K_t,N_t)\geq 0} \Pi_t = F(K_t, A_tN_t) - w_tN_t - r_tK_t.$$

2. Given the prices $\{c_t, n_t, i_t, k_{t+1}\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t,n_t,i_t,k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t(1 + g_n)^t u(c_t, n_t),$$

$$\text{s.t.} \quad c_t + i_t = w_tn_t + r_tk_t + \Pi_t, \forall t,$$

$$\text{s.t.} \quad (1 + g_n)k_{t+1} = (1 - \delta)k_t + i_t, \forall t,$$

$$\text{s.t.} \quad c_t \geq 0, k_{t+1} \geq 0, k_0 \text{ given}, \forall t.$$

3. Prices are such that markets clear

$$Y_t = C_t + I_t,\ \forall t.$$

$$N_t = (1 + g_n)^t n_t,\ \forall t.$$

$$K_t = (1 + g_n)^t k_t,\ \forall t.$$

The firm first order conditions are the same as before, $\Pi_t = 0, \forall t$.

The first order conditions for the household's problem are given by:

$$c_t : \beta^t(1+g_n)^t u_c(c_t, n_t) = \lambda_t,$$

$$n_t : -\beta^t(1+g_n)^t u_n(c_t, n_t) = \lambda_t w_t,$$

$$k_{t+1} : (1+g_n)\lambda_t = \lambda_{t+1}(1+r_{t+1}-\delta),$$

$$\lim_{t\to\infty} \beta^t u_c(c_t, n_t) k_{t+1} = 0.$$

*Recursive equilibrium.* To ease notation let's assume an inelastic labor supply $n = 1, u(c,1) = u(c)$ and that there is no long run growth, i.e. $g_n = g_a = 0$. The latter assumption is without loss of generality, as we will see later it is always possible to rescale the variables to remove growth. I employ the recursive notation $x' := x_{t+1}, x := x_t$. The state variables are $k, K$ and the controls $c, k'$.

**Definition 5.** A recursive equilibrium is a value function $v(., K) : \mathbb{R}_+ \to \mathbb{R}$, policy functions $g^c(., K) : \mathbb{R}_+ \to \mathbb{R}_+, g^k(., K) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c = g^c(k, K), k' = g^k(k, K)$, pricing functions $w, r : \mathbb{R}_+ \to \mathbb{R}_+$ and an aggregate law of motion for aggregate capital $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that

1. Given $w, r$ in each period firms maximize profits

$$w(K) = F_N(K, N),$$

$$r(K) = F_K(K, N).$$

2. Given $w, r, H$ the value function $v$ solves the Bellman equation

$$v(k, K) = \max_{(c,k')}\{u(c, n) + \beta v(k', K')\},$$

$$\text{s.t. } c + k' = w(K)n + (1 + r(K) - \delta)k.$$

3. The agents's choices are consistent with the law of motion $H$ and markets clear.

$$k = K, n = N,$$

$$K' = H(K),$$

$$H(K) = g^k(K, K),$$

$$g^c(K, K) + g^k(K, K) = F(K, N) + (1-\delta)K.$$

Note that with elastic labor supply elastic we would need $K', N'$ as aggregate states. The household's problem in Definition 5 implies the following first order condition:

$$c : u_c(c,n) = \beta v_{k'}(k',\psi'),$$
$$\lim_{t\to\infty} \beta^t u_c(c_t,n_t)k_{t+1} = 0.$$

At the optimum $v_k(k,K) = u_c(c,n)g_k^c + u_n(c,n)g_k^n + \beta v_{k'}(k',K')[wg_k^n - g_k^c + (1+r-\delta)]$ from the first order conditions above we obtain the envelope condition $v_k(k,K) = \beta v_{k'}(k',K')(1+r-\delta)$. Substituting from the first expression above yields $v_k(k,K) = u_c(c,n)(1+r-\delta)$. Shifting one period ahead $v_{k'}(k',K') = u_c(c',n')(1+r(K')-\delta)$ and substituting it back yields an Euler equation

$$u_c(c,n) = \beta u_c(c',n')(1+r(K')-\delta).$$

*Equilibrium conditions*. All these definitions are equivalent. It is easy to check that all of them imply the same equilibrium system.

$$u_c(c_t,n_t) = \beta u_c(c_{t+1},n_{t+1})(1+r_{t+1}-\delta), \tag{1}$$
$$-u_n(c_t,n_t) = w_t u_c(c_t,n_t), \tag{2}$$
$$r_t = F_K(K_t,(1+g_a)^t N_t), \tag{3}$$
$$w_t = (1+g_a)^t F_N(K_t,(1+g_a)^t N_t), \tag{4}$$
$$(1+g_n)k_{t+1} = (1-\delta)k_t + i_t, \tag{5}$$
$$c_t + i_t = F(k_t,(1+g_a)^t n_t), \tag{6}$$
$$\lim_{t\to\infty} \beta^t u_c(c_t,n_t)k_{t+1} = 0, \tag{7}$$

with $g_n = g_a = 0$ and $n = 1$ instead of (2) for Definition 5. Equation (1) is the intertemporal Euler equation, the intratemporal condition (2) requires that the marginal rate of substitution between consumption and labor supply must be equal to the relative price. We can rewrite (7) using (1)

$$\lim_{t\to\infty} \beta^{t-1} u_c(c_{t-1},n_{t-1})k_t = \lim_{t\to\infty} \beta^{t-1}\beta u_c(c_t,n_t)(1+r_t-\delta)k_t$$
$$= \lim_{t\to\infty} \beta^t u_c(c_t,n_t)(1+r_t-\delta)k_t = 0.$$

Notice that the transversality condition implies that the boundary condition on asset holdings binds. Set $a_t = k_t$ solving backward from the first order conditions with $R_t =$

$\prod_{j=1}^{t}(1+r_j-\delta)^{-1}$ we have

$$\lim_{t\to\infty} \lambda_0(1+g_n)^t R_t a_{t+1} = 0.$$

*CRRA utility function*. Consider the Euler equation. In order to have balanced growth with $r_t \to r$ when $t \to \infty$ we need that the marginale rate of substitution to be constant, i.e. MRS $:= u_c(c_t, n_t)/\beta u_c(c_{t+1}, n_{t+1}) \to \alpha$ when $t \to \infty$. This is possible only with a homothetic utility function $u$, this is the case if for any monotonic transformation $f(u)$ is homogeneous of degree 1. If $u$ is homothetic then $\text{MRS}(c_{t+1}, c_t) = \text{MRS}((1+g_c)^t c_1, (1+g_c)^t c_0) = \text{MRS}(c_1, c_0)$.

King, Plosser, and Rebelo (1988) show that this is the case for utility functions with Constant Relative Risk Aversion (CRRA)

$$u(c,n) = \begin{cases} \dfrac{c^{1-\sigma}}{1-\sigma}v(n) & \sigma \in (0,1) \cup (1,\infty) \\ \\ \ln c + v(n) & \sigma = 1 \end{cases} .$$

If $u'(c)/u'((1+g_c)c) = \alpha \iff u''(c) = (1+g_c)u''((1+g_c)c)\alpha$, we obtain the Arrow-Pratt coefficient of relative risk aversion, i.e. the elasticity of marginal utility with respect to consumption

$$\frac{u''(c)c}{u'(c)} = \frac{u''(c(1+g_c))c(1+g_c)}{u'((1+g_c)c)} = -\sigma.$$

We can solve this differential equation by separation

$$\int \frac{u''(c)}{u'(c)}dc = -\int \frac{\sigma}{c}dc$$

and obtain $\ln u'(c) = c_1 - \sigma \ln c \Rightarrow u'(c) = e^{c_1}c^{-\sigma}$. Integrating again both sides we have the CRRA. Finally, it's easy to check that the intertemporal elasticity of substitution

$$\text{IES}_t(c_t, c_{t+1}) := \left[\frac{d(\text{MRS}(c_t, c_{t+1}))}{d(c_{t+1}/c_t)} \frac{c_{t+1}/c_t}{\text{MRS}(c_t, c_{t+1})}\right]^{-1} = \frac{1}{\sigma}$$

.

*Detrended variables*. Let $\tilde{x}_t = x_t/A_t$ be a detrended variable. To detrend consumption we multiply and divide it by $A_t = (1+g_a)^t A_0$ with $A_0 = 1$. To have a finite utility now we need $\tilde{\beta} := \beta(1+g_n)(1+g_a)^{1-\sigma} \in (0,1) \iff (1+\rho) > (1+g_n)(1+g_a)^{1-\sigma}$. The equilibrium system can be written as a nonlinear system of difference equations with a boundary condition

$$(1+g_a)^\sigma u_c(\tilde{c}_t, n_t) = \beta u_c(\tilde{c}_{t+1}, n_{t+1})(1 + F_K(\tilde{k}_{t+1}, n_{t+1}) - \delta), \tag{8}$$

$$-u_n(\tilde{c}_t, n_t) = F_N(\tilde{k}_t, n_t)u_c(\tilde{c}_t, n_t), \tag{9}$$

$$\tilde{c}_t + (1+g_n)(1+g_a)\tilde{k}_{t+1} = F(\tilde{k}_t, n_t) + (1-\delta)\tilde{k}_t, \tag{10}$$

$$\lim_{t\to\infty} \tilde{\beta}^t u_c(\tilde{c}_t, n_t)\tilde{k}_{t+1} = 0. \tag{11}$$

The steady state of the detrended model (8)-(11) with CRRA utility is given by

$$(1+g_a)^\sigma \beta^{-1} = 1 + F_K(\tilde{k}, n) - \delta, \tag{12}$$

$$-u_n(\tilde{c}, n) = F_N(\tilde{k}, n)u_c(\tilde{c}, n), \tag{13}$$

$$\tilde{c} = F(\tilde{k}, n) - (g_n + g_a + \delta + g_n g_a)\tilde{k}. \tag{14}$$

Equation (12) implicitly define the steady state equilibrium level of $\tilde{k}$, (14) the steady state equilibrium consumption $\tilde{c}$ and (13) the steady state equilibrium labor supply $n$. The steady state has balanced growth in per capita and aggregate variables and match the Kaldor's facts. Indeed $\Delta\tilde{k}_t/\tilde{k}_t = 0 \Rightarrow \Delta k_t/k_t = \Delta A_t/A_t = g_a \Rightarrow (1+g_K) = (1+g_a)(1+g_n)$. Since $g_K = g_Y$ the capital-output ratio is constant over time, the equilibrium capital rental rate is

$$1 + r - \delta = (1+\rho)(1+g_a)^\sigma.$$

Notice that $\tilde{\beta} \in (0,1) \Rightarrow 1 + r - \delta > (1+g_a)(1+g_n)$.

*Social planner's problem.* Let's consider the following social planner's problem who chooses allocations to maximize the representative household's utility.

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t (1+g_n)^t u(c_t, n_t),$$

$$\text{s.t.} \quad c_t + (1+g_n)k_{t+1} = F(k_t, (1+g_a)^t n_t) + (1-\delta)k_t, \ \forall t.$$

Theroem 3 applies. The first order conditions and the transversality condition of the social planner's problem imply the system (8)-(11).

# III. The neoclassical growth model in continuous time

*The economy.* The enviroment is the same as before, but now time is continuous $t \in \mathbb{R}$. Population growth is given by $\dot{h}_t/h_t = g_n$. Therefore, $h_t = \exp(g_n t)$ with $h_0 = 1$. The utility function becomes

$$\int_0^\infty e^{-(\rho-g_n)t} u(c_t, n_t) dt$$

where $\rho > 0$ is the households' discount rate, $c_t \in \mathbb{R}_+$ is the per capita consumption of the final good and $n_t \in [0,1], \forall t$ is the labor supply of each household's member, $\rho > g_n$. The function $u$ is continuous, two times differentiable, $u_c > 0, u_{cc} < 0, u_n < 0, u_{nn} < 0$ and Inada conditions hold. Now $C_t = \exp(g_n t) c_t, I_t := \dot{K}_t + \delta K_t$ with $\delta \in [0,1]$. The economy admits a representative firm, the production side is described by an aggregate production function $Y_t = F(K_t, A_t N_t)$.

*Sequential equilibrium.* A sequential markets competitive equilibrium in the neoclassical growth model are prices $\{w_t, r_t\}$ and allocations $C_t, N_t, K_t, c_t, n_t, k_t$ such that

1. Given the prices $N_t, K_t$ solves in each period

$$\max_{(K_t,N_t)\geq 0} \Pi_t = F(K_t, A_t N_t) - w_t N_t - r_t K_t.$$

2. Given the prices $c_t, n_t, k_t$ solves

$$\max_{(c_t,n_t,k_t)} \int_0^\infty e^{-(\rho-g_n)t} u(c_t, n_t) dt,$$

$$\text{s.t.} \quad c_t + \dot{k}_t = w_t n_t + (r_t - \delta - g_n) k_t + \Pi_t, \forall t,$$

$$\text{s.t.} \quad c_t \geq 0, k_t \geq 0, \forall t, k_0 \text{ given.}$$

3. Prices are such that markets clear

$$Y_t = C_t + I_t, \ \forall t.$$

$$N_t = e^{g_n t} n_t, \ \forall t.$$

$$K_t = e^{g_n t} k_t, \ \forall t.$$

The firm's first order conditions are the usual ones, $\Pi_t = 0, \forall t$.

The first order conditions for the household's problem are given by:

$$u_c(c_t, n_t) = \mu_t,$$

$$-u_n(c_t, n_t) = \mu_t w_t,$$

$$-\dot{\mu}_t + (\rho - g_n)\mu_t = \mu_t(r_t - \delta - g_n),$$

$$\lim_{t\to\infty} e^{-(\rho-g_n)t} u_c(c_t, n_t)k_t = 0.$$

Assuming a log-log CRRA functional form the equilibrium system is given by

$$\dot{c}_t/c_t = (r_t - \rho - \delta),$$

$$-\chi c_t = w_t(1 - n_t),$$

$$r_t = F_K(K_t, e^{g_a t} N_t),$$

$$w_t = F_N(K_t, e^{g_a t} N_t),$$

$$c_t + \dot{k}_t = w_t n_t + (r_t - \delta - g_n)k_t,$$

$$y_t = c_t + i_t,$$

$$\lim_{t\to\infty} e^{R_t} k_t = 0.$$

By separability $\dot{\mu}_t/\mu_t = -(r_t - \rho - \delta) \Rightarrow \mu_t = \mu_0 \exp(-\int_0^t (r_\tau - \rho - \delta)d\tau)$. Substituting into the transversality condition we have a boundary condition on capital where $R_t = \int_0^t (r_\tau - g_n - \delta)d\tau$. Moreover, using the fact that $g_{\tilde{c}} = g_c - g_a$ the detrended equilibrium system is given by

$$\frac{d\tilde{c}_t}{\tilde{c}_t} = (F_K - \rho - \delta - g_a)dt,$$

$$-\chi\tilde{c}_t = (1 - n_t)F_N,$$

$$\tilde{c}_t + \frac{d\tilde{k}_t}{dt} = f(\tilde{k}_t) - (\delta + g_n + g_a)\tilde{k}_t,$$

$$\lim_{t\to\infty} \exp\left(-\int_0^t (F_K - g_n - g_a - \delta)d\tau\right)\tilde{k}_t = 0.$$

In the steady state since consumtpion is constant $F_K = \rho + \delta + g_a$. Therefore, finite lifetime wealth is a necessary condition for finite lifetime utility

$$\rho > g_n \Rightarrow r - \delta > g_n + g_a.$$

*Recursive equilibrium.* A competitive equilibrium is given by a value function $v(., K_t) : \mathbb{R}_+ \to \mathbb{R}$, policy functions $g^c(., K_t) : \mathbb{R}_+ \to \mathbb{R}_+, g^k(., K_t) : \mathbb{R}_+ \to \mathbb{R}$ such that $c_t = g^c(k_t, K_t), dk_t = g^k(k_t, K_t)dt$, pricing functions $w, r : \mathbb{R}_+ \to \mathbb{R}_+$ and an aggregate law of motion $H : \mathbb{R}_+ \to \mathbb{R}_+$:

1. Given $w, r$ in each period firms maximize profits

$$w(K_t) = F_N(K_t, N_t),$$

$$r(K_t) = F_K(K_t, N_t).$$

2. Given $w, r, H$ the value $v$ solves the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho v(k_t, K_t) = \max_{c_t} \Big\{ u(c_t, n_t) + v_k(k_t, K_t)(w(K_t)n_t + (r(K_t) - \delta)k_t - c_t) \Big\},$$

$$\text{s.t. } dk_t = (w(K_t)n_t + (r(K_t) - \delta)k_t - c_t)dt,$$

$$\text{s.t. } \dot{K}_t = H(K_t).$$

3. The agents's choices are consistent with the law of motion $H$ and markets clear.

$$k_t = K_t, n_t = N_t = 1,$$

$$H(K_t) = g^k(K_t, K_t),$$

$$g^c(K_t, K_t) + g^k(K_t, K_t) = F(K_t, N_t) + (1 - \delta)K_t.$$

We can easily derive an Euler equation as in discrete times. Consider the case of a CRRA utility function $c_t^{1-\gamma}/(1-\gamma) - \chi \ln(n_t)$. Differentiating both sides of the HJB equation and rearranging terms we have that at the optimum

$$(\delta + \rho - r_t)v_k = v_{kk}(w_t n_t + (r_t - \delta)k_t - c_t).$$

Let $s_t := w_t n_t + (r_t - \delta)k_t - c_t$. Partially differentiating the consumption first order condition $v_k = u_c$ we have $v_{kk} = u_{cc}c_k$. Moreover, note that $dc = c_k dk = c_k s dt, u_{cc}c/u_c = -\gamma$. Using these results in the previous equation yields the Euler equation

$$\frac{dc_t}{c_t} = \frac{1}{\gamma}(r_t - \delta - \rho)dt.$$

*Example* 3. Consider the growth model without growth, i.e. $g_n = g_a = 0$, $u(c_t)$ is a CRRA utility function, $F(k_t)$ is Cobb-Douglas, and $n_t = 1, \forall t$. Substituting out the prices the equilibrium conditions reduce to $\dot{c}_t/c_t = (\alpha k_t^{\alpha-1} - \rho - \delta)$ and $\dot{k}_t = k_t^{\alpha} - \delta k_t - c_t$. We can use these equations to construct a phase diagram and analyze some of the model's steady state and dynamic properties. Figure 1 shows that there is only one equilibrium given by a saddle path.

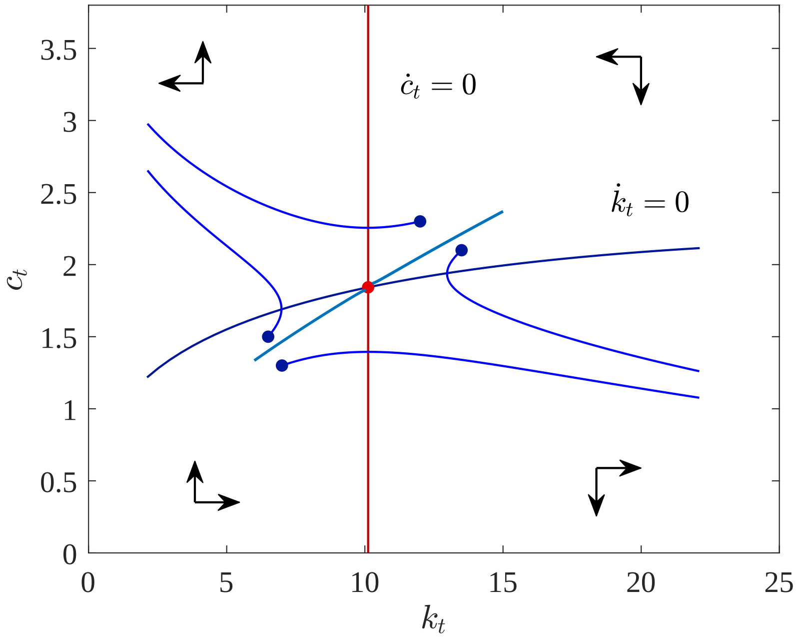


Figure 1: Phase diagram neoclassical growth model

Note: the red dot is the model's steady state, the light blue line is the equilibrium saddle path, the blue lines are explosive paths. In this example $\alpha = .33$, $\rho = .04$, $\delta = .03$.

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